Analysis 3

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ODEs

$ay''+by'+cy=f \rightarrow y=y_h+y_p$

y_h: ch.Polynom $a\lambda^2+b\lambda+c=0 \rightarrow \lambda_{1/2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\lambda_1\neq\lambda_2\in\mathbb{R}: y=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$; $\lambda_1=\lambda_2\in\mathbb{R}: y=(C_1+C_2\cdot t)e^{\lambda t}$

$\lambda_{1,2}=\alpha\pm i\omega: y=e^{\alpha t}(C_1\sin(\omega t)+C_2\cos(\omega t))$

y_p: $f\rightarrow$ Ansatz : $Ce^{at}\rightarrow Ae^{at}$; $C\,{}^{\sin}_{\cos}(bt)\rightarrow A\cos(bt)+B\sin(bt)$; $\sum c_i x^i \rightarrow \sum A_i x^i$;

$C\,{}^{\sin}_{\cos}(bt)\,e^{at} \rightarrow (A\sin(bt)+B\cos(bt))e^{at}$

Classification

Linear: linear in all derivatives. Coeffs may depend on $x,y,\ldots$

Quasi-linear: linear w.r.t. the highest order derivative

Homogeneous: Every term that doesn't depend on u is equal to zero

Vector space of solutions / superposition principle: Let u_{h_1} and u_{h_2} be the solutions of the PDE $L[u]=0$ and u_p of $L[u]=f(x)$.

Then $\alpha u_{h_1}+\beta u_{h_2}+u_p$ solves $L[u]=f(x)$ too.

⚠ Do not add multiple particular solutions.

Well-posedness: 1) Existence 2) Uniqueness 3) Stability (Small perturbation in IC / BC $\Rightarrow$ Small change in sol)

Strong solutions: all derivatives exist and are continuous.

Gradient $\nabla f=\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\ldots\right)^T \in\mathbb{R}^n$

Laplace-Operator $\Delta u=\nabla^2 u=\sum_i u_{x_i x_i} \in\mathbb{R}$

Cauchy problem PDE coupled with a set of init conditions

Method of Characteristics (quasilinear)

$a(x,y,u)u_x+b(x,y,u)u_y=c(x,y,u)$, $u(x_0(s),y_0(s))=u_0(s)$

1. $a=x_t(t,s)$, $b=y_t(t,s)$, $c=\tilde{u}_t(t,s)$, $\Gamma(s)=(x_0(s),y_0(s),u_0(s)) \rightarrow \tilde{u}(s,t)$

2. Find $t(x,y)$, $s(x,y)$ and solve $u(x,y)=\tilde{u}(t(x,y),s(x,y))$

Unique solution in neighbourhood of initial curve $\Leftrightarrow$ transversality cond.

$$J=\det\begin{bmatrix}x_t & y_t\\ x_s & y_s\end{bmatrix}=\det\begin{bmatrix}a(0,s) & b(0,s)\\ \frac{d}{ds}x(0,s) & \frac{d}{ds}y(0,s)\end{bmatrix}\neq 0$$

"init curve not tangential to characteristics"

don't forget

Conservation laws & shock waves

$$\begin{cases} u_y+c(u)u_x=u_y+\frac{\partial}{\partial x}F(u)=0 & (F'=c)\\ u(x,0)=h(x) & (x \text{ Raum}, y \text{ Zeit})\end{cases}$$

$c \overset{<}{>} 0$: flow left $\overset{\leftarrow}{\rightarrow}$ right

MoC: $x_t=c(u)$, $y_t=1$, $\tilde{u}_t=0$, $x(0,s)=s$, $y(0,s)=0$, $\tilde{u}(0,s)=h(s)$

Characteristics: $y(s,t)=t$, $x(s,t)=s+c(h(s))\cdot t$, u const along characteristics

Implicit solution: $u(x,y)=h(x-c(u(x,y))y)$

Nach critical time y_c entweder nur noch schwache oder keine Lösung mehr.

$y_c=\inf_{s\in\mathbb{R}}\left\{-\frac{1}{c(h(s))_s} \,\middle|\, c(h(s))_s<0\right\}$ Falls $c(h(s))>0\ \forall s\in\mathbb{R}$ $\Rightarrow y_c=\infty \Leftrightarrow h$ monoton steigend

integral formulation: $\int_{x_0}^{x_1}u(x,y_1)-u(x,y_0)\,dx=-\int_{y_0}^{y_1}F(u(x_1,y))-F(u(x_0,y))$

$\forall x_0<x_1, 0<y_0<y_1$ every classical solution is also a weak solution

Shockwave (Rankine-Hugoniot): $\sigma'(y)=\frac{F(u^+)-F(u^-)}{u^+-u^-}$ $\Rightarrow x=\sigma(y)$ ist shockwave

Entropy condition: Weak sol. nicht unique $\rightarrow$ welche macht phys. Sinn? $c(u^+)<\sigma'<c(u^-)$

(Charakteristiken in Stoss rein, nicht raus) (sol. that satisfies cond. is unique)

Tips for checking weak solutions: must be p.w. continuous

slope of shockwave (σ') = slope of discontinuity

MoC tips

$x_t=x \Rightarrow \frac{dx}{dt}=x \Leftrightarrow \int\frac{dx}{x}=\int dt \Rightarrow x=e^t\cdot\alpha(s)$

$\tilde{u}_t=2\tilde{u} \Rightarrow \tilde{u}=e^{2t}\cdot\gamma(s)$

or maybe add up $w=x+y+\tilde{u}=\ldots$

$yu_x+uu_y=x \Rightarrow w=w_t$ + use init curve for more information about w

$\Rightarrow$ if $w\equiv 0 \Rightarrow x+y+\tilde{u}=0 \Leftrightarrow \tilde{u}=-x-y$

2nd order PDEs $au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y+fu=g$

$\delta=b^2-ac$ (local property)

- $\delta<0$: elliptic Laplace/Poisson $u_{xx}+u_{yy}=0$
- $\delta=0$: parabolic Heat $u_t-ku_{xx}=0$
- $\delta>0$: hyperbolic Wave $u_{tt}-c^2u_{xx}=0$

Wave equation $u_{tt} - c^2 u_{xx} = 0$ $\quad (x,t) \in \mathbb{R} \times (0,\infty)$

c = wave speed

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$ } initial conditions

$\begin{cases} u(0,t) = u(L,t) = 0 & \text{Dirichlet} \\ u_x(0,t) = u_x(L,t) = 0 & \text{van Neumann} \end{cases}$ } boundary conditions

fwd bwd

decompose u in 2 waves: $u(x,y) = F(x-ct) + F(x+ct)$

$\xi = x+ct,\ \eta = x-ct \Rightarrow w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta)) = F(\xi) + G(\eta)$

Characteristics: $x+ct = \alpha$, $x-ct = \beta$ $(\alpha,\beta \in \mathbb{R})$.

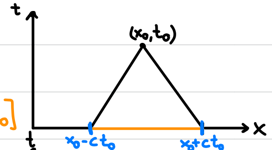
u is constant on the charact. lines, singularities propagate along

d'Alembert $u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$ (unique!)

Domain of dependence

- Sol. in (x_0,t_0) depends on $f(x_0-ct_0)$

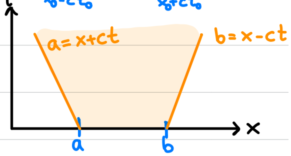
and $f(x_0+ct_0)$ and g in $[x_0-ct_0, x_0+ct_0]$



Region of influence

- All points satisfying $x-ct \le b$, $x+ct \ge a$

are dependent on the init conditions in $[a,b]$.



Symmetry: f, g, F even / odd / periodic $\Rightarrow u$ even / odd / periodic

$\exists$ boundary condition?

- No: Homogeneous?
  - Yes: d'Alembert
  - No: d'Alembert or Superposition
- Yes: Can use symmetry to eliminate boundary conditions?
  - Yes: Extend problem with suitable f, g, then homog. d'Alembert.
  - No: Separation of variables

Inhomogeneous wave equation $u_{tt} - c^2 u_{xx} = F(x,t)$

- d'Alembert: $u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)\,ds\,d\tau$
- Superposition: Find $v(x,t)$ s.t. $v_{tt} - c^2 v_{xx} = F(x,t)$. Define $w = u - v$

$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0) = f(x,0) - v(x,0) \\ w_t(x,0) = g(x,0) - v_t(x,0) \end{cases}$

$\Rightarrow$ Solve homog. problem for w

$\Rightarrow u = w + v$

Good to use if F only

Heat equation $u_t - k u_{xx} = 0$ $\quad (x,t) \in [0,L] \times [0,\infty[$

$u(x,0) = f(x)$ } initial condition

$\begin{cases} u(0,t) = u(L,t) = 0 & \text{Dirichlet} \\ u_x(0,t) = u_x(L,t) = 0 & \text{Van Neumann} \\ \text{or mixed} \end{cases}$ } boundary condition

Dirichlet problem has a unique solution

- falls $F(x,t) = t^n$ oder andere einfache Funktion, dann besser partikuläre Lösung finden.

Separation of variables (for homogeneous equations)

Assume u is of the form $u(x,t) = X(x) \cdot T(t)$

Substitute in equation $\Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda$ or $\frac{X''}{X} = \frac{T'}{kT} = -\lambda$

ODEs: $X'' = -\lambda X$:
- $\lambda > 0$: $X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$
- $\lambda = 0$: $X = A + Bx$
- $\lambda < 0$: $X = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x)$

D.B.C. $X_n = \sin(\frac{n\pi}{L}x)$ $\quad n = 1,2,3,\dots$

V.N.B.C $X_n = \cos(\frac{n\pi}{L}x)$ $\quad n = 0,1,2,3,\dots$

} $\lambda_n = (\frac{n\pi}{L})^2$

heat:
- Using λ_n, we find $T_n = e^{-k(\frac{n\pi}{L})^2 t}$ $\quad (T' = -\lambda k T)$
- $u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t}$
- $u(x,t) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t}$

wave:
- Using λ_n, we find $T_n = A_n\cos(\frac{n\pi}{L}ct) + B_n\sin(\frac{n\pi}{L}ct)$ $\quad (T'' = -c^2\lambda T)$
- $u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)\left(A_n\cos(\frac{n\pi}{L}ct) + B_n\sin(\frac{n\pi}{L}ct)\right)$
- $u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos(\frac{n\pi}{L}x)\left(A_n\cos(\frac{n\pi}{L}ct) + B_n\sin(\frac{n\pi}{L}ct)\right)$

Use initial conditions to determine coefficients.

If boundary conditions inhomogeneous: Find w that solves inhomog., subtract it from PDE $(v = u - w)$, solve for v, finally $u = v + w$

S.o.v. for inhomogeneous equations:

Apply s.o.v. to the homog. PDE using the Ansatz $u = X(x)T(t)$. Find general solution for X, make distinction for λ. Formulate general solution $u = XT$ with basis found in prev. step. Insert in inhomog. PDE and use init. conds. to determine coeffs.

$u_{tt} - c^2 u_{xx} = \sum_n T_n'' X_n - c^2 T_n X_n'' = F(x,t)$

$u_t - k u_{xx} = \sum_n T_n' X_n - k T_n X_n'' = F(x,t)$

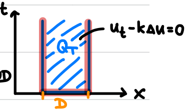
⚠ V.N.B.C don't forget $n=0$! Might be a linear term $\frac{A_0 t + B_0}{2}$

Max. principle for homogeneous heat equation

Let u solve $u_t = k\Delta u$ in Q_T for some $k > 0$. Assume D bounded

Then u achieves its maximum (and minimum) on $\partial_p Q_T$.

$\{0\} \times D \cup [0,t_0] \times \partial D$



Proof: Uniqueness of inhomogeneous 1D wave equation.

Existence given by d'Alembert. Suppose u_1 and u_2 are solutions that solve $u_{tt} - c^2 u_{xx} = F(x,t)$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $(x,t) \in \mathbb{R} \times (0,\infty)$

Let $w := u_1 - u_2$. w solves

$\begin{cases} w_{tt} - c^2 w_{xx} = u_{1tt} - c^2 u_{1xx} - [u_{2tt} - c^2 u_{2xx}] = 0 \\ w(x,0) = u_1(x,0) - u_2(x,0) = f(x) - f(x) = 0 \\ w_t(x,0) = u_{1t}(x,0) - u_{2t}(x,0) = g(x) - g(x) = 0 \end{cases}$

We use d'Alembert's formula to solve this cauchy problem and get $w \equiv 0$ and thus $u_1 = u_2$ ■

2^{nd} order, linear

Laplace equation $\Delta u = 0$ ($\Leftrightarrow u$ is a harmonic function)

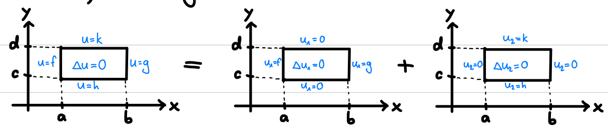
Poisson equation $\Delta u = u_{xx} + u_{yy} = \rho(x,y)$ $(x,y) \in D \subset \mathbb{R}^2$

$$\begin{cases} u(x,y) = g(x,y) & (x,y) \in \partial D \quad \text{Dirichlet} \\ \partial_n u(x,y) = \vec{n} \cdot \nabla u = g(x,y) & (x,y) \in \partial D \quad \text{Van Neumann} \\ u(x,y) + \alpha\, \partial_n u(x,y) = g(x,y) & (x,y) \in \partial D \quad \text{Mixed} \end{cases}$$

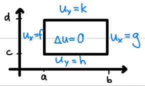
Necessary condition for existence of solution

Neumann: $\int_{\partial D} g(x(s), y(s))\, ds = \int_D \rho(x,y)\, dx\, dy$ | Dirichlet: continuity of boundary

Boundary splitting



Laplace



V.N.B.C: Verify $\int_{\partial D} \partial_n u = 0 = \int_c^d g - \int_c^d f + \int_a^b k - \int_a^b h$

D.B.C: Ensure continuity: $f(a,c) = h(a,c)$, $h(b,c) = g(b,c)$, $g(b,d) = k(b,d)$, $k(a,d) = f(a,d)$

If not fulfilled: Use linearity, add harmonic polynomial s.t. coeffs. meet boundary conditions. Solve for $\tilde{u} = u + P_H$.

Homogeneous direction: (X for u_2, Y for u_1) $\quad \lambda_n = \left(\frac{n\pi}{b-a}\right)^2$

D.B.C.: $X_n = A_n \sin(\sqrt{\lambda_n}(x-a))$ $\quad Y_n = A_n \sin(\sqrt{\lambda_n}(y-c))$

V.N.B.C.: $X_n = A_n \cos(\sqrt{\lambda_n}(x-a))$ $\quad Y_n = A_n \cos(\sqrt{\lambda_n}(y-c))$

Other direction:

D.B.C.: $Y_n = C_n \sinh(\sqrt{\lambda_n}(y-c)) + D_n \sinh(\sqrt{\lambda_n}(y-d))$

$X_n = C_n \sinh(\sqrt{\lambda_n}(x-a)) + D_n \sinh(\sqrt{\lambda_n}(x-b))$

V.N.B.C.: $Y_n = C_n \cosh(\sqrt{\lambda_n}(y-c)) + D_n \cosh(\sqrt{\lambda_n}(y-d))$

$X_n = C_n \cosh(\sqrt{\lambda_n}(x-a)) + D_n \cosh(\sqrt{\lambda_n}(x-b))$

Poisson + Dirichlet have unique solution

$\rightarrow$ Proof on next page

Homogeneous?

- No: Find particular solution, subtract from PDE and b.c. $\rightarrow$ homog. Problem
- Yes: $\rightarrow$ Laplace. Domain?
  - Rectangular: If necessary: boundary splitting / add harmonic polynomial to fulfill existence condition
  - Solve in both homogeneous directions, add up solutions
  - Solve for u! Not for $v = u - w$ or $\tilde{u} = u - P_h$
  - Circular: Ball / Ring } full v section $\rightarrow$ summary

Some harmonic functions

$a_0 + a_1 x + a_2 y + a_3 xy + a_4(x^2 - y^2)$, $\quad \log(x^2+y^2) + c$

Weak max/min principle

Let D be a bounded domain and $u(x,y) \in C^2(D) \cap C(\bar{D})$ a harmonic function. u will take its max/min on ∂D.

$$\max_{\bar{D}} u = \max_{\partial D} u \qquad \min_{\bar{D}} u = \min_{\partial D} u$$

Proof: Let $u_\varepsilon(x,y) = u + \varepsilon(x^2+y^2)$ with $\varepsilon > 0$, u harmonic.

If $\max u_\varepsilon = u(x_0,y_0) \in D$, then $\Delta u_\varepsilon(x_0,y_0) \overset{!}{\leq} 0$. However,

$\Delta u_\varepsilon(x_0,y_0) = \Delta u + \Delta \varepsilon(x^2+y^2)\big|_{(x_0,y_0)} = 4\varepsilon > 0$ ↯ $\Rightarrow \max u_\varepsilon \in \partial D$

Mean value theorem

Let u be harmonic in D and $B_R(x_0,y_0) \subseteq D$. Then:

$$u(x_0,y_0) = \frac{1}{2\pi R} \int_{\partial B_r(x_0,y_0)} u(x(s),y(s))\, ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\, d\theta$$

(theorem holds $\Leftrightarrow$ u is a harmonic function.)

Strong maximum principle

Let u be harmonic in D and u reaches its max inside D, then u is constant on all D.

Proof: Use mean value thm. (x_0,y_0) is the avg. of all points around a circle centered in (x_0,y_0). All points must be equal to $u(x_0,y_0)$ since $u(x_0,y_0)$ is the maximum.

Extremum: $\nabla u(x_0,y_0) \overset{!}{=} 0$ $\quad$ Max $\Delta u(x_0,y_0) \leq 0$, $u_{xx}, u_{yy}\big|_{(x_0,y_0)} \leq 0$ $\quad$ Min $\Delta u(x_0,y_0) \leq 0$, $u_{xx}, u_{yy}\big|_{(x_0,y_0)} \leq 0$

Circular domains $\quad x = r\cos\theta$, $y = r\sin\theta$

Laplace: $\Delta w = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}$ $\quad w(r,\theta) = u(r\cos(\theta), r\sin(\theta))$

B.C. $w(R,\theta) = g(x(R,\theta), y(R,\theta)) = h(\theta)$

Ansatz: $w(r,\theta) = R(r)\Theta(\theta)$

Use s.o.v. and periodicity to get

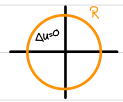
$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad r^2 R''(r) + rR' - \lambda R(r) = 0 \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$$

- $\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$
- $R_n = \begin{cases} C_0 + D_0 \log(r) & n = 0 \\ C_n r^n + D_n r^{-n} & n \neq 0 \end{cases}$ $\quad$ • ignore if $(0,0) \in D$ (because term undefined)

$w(r,\theta) = A_0 + B_0 \log(r) + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)) + r^{-n}(C_n \cos(n\theta) + D_n \sin(n\theta))$

Insert boundary conditions to determine coefficients.

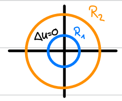
Circle $\bar{D} = \{0 \leq r \leq R, 0 \leq \theta \leq 2\pi\}$



B.C. $\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$

$w(R,\theta) = f(\theta)$

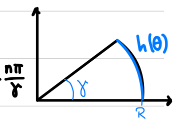
Ring $\bar{D} = \{r_1 \leq r \leq r_2, 0 \leq \theta \leq 2\pi\}$



B.C. $\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$

$w(r_1,\theta) = f(\theta)$, $w(r_2,\theta) = g(\theta)$

Section (circle) $\bar{D} = \{0 \leq r \leq R, 0 \leq \theta \leq \gamma\}$

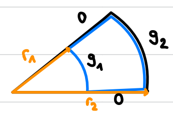


Dirichlet $\Theta(0) = 0 = \Theta(\gamma)$ $\quad w = \sum_{n \geq 1} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}$

van Neumann $\Theta'(0) = 0 = \Theta'(\gamma)$ $\quad w = A_0 + \sum_{n \geq 1} A_n \cos\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}$

$w(R,\theta) = h(\theta)$

Section (ring) $\bar{D} = \{r_1 \leq r \leq r_2, 0 \leq \theta \leq \gamma\}$



$w(r_1,\theta) = k(\theta)$, $w(r_2,\theta) = h(\theta)$

Dirichlet $\Theta(0) = 0 = \Theta(\gamma)$

$w(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}} + B_n \sin\left(\frac{n\pi}{\gamma}\right) r^{-\frac{n\pi}{\gamma}}$

van Neumann $\Theta'(0) = 0 = \Theta'(\gamma)$

$w(r,\theta) = A_0 + B_0 \log(r) + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\gamma}\right) r^{\frac{n\pi}{\gamma}} + B_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{-\frac{n\pi}{\gamma}}$

Trigonometric identities

$\sinh(x) = -i\sin(ix)$ $\qquad$ $\cosh(x) = \cos(ix)$

$\sin^2(x)+\cos^2(x) = 1$ $\qquad$ $\cosh^2(x) - \sinh^2(x) = 1$

$\sin(\operatorname{acos}(x)) = \cos(\operatorname{asin}(x)) = \sqrt{1-x^2}$ $\qquad$ $\sinh(\operatorname{acosh}(x)) = \sqrt{x^2-1}$, $\cosh(\operatorname{asinh}(x)) = \sqrt{x^2+1}$

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ $\qquad$ $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

$\sin(x\pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

$\cos(x\pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

$\sin(nt)\sin(mt) = \frac{1}{2}(\cos((n-m)t) - \cos((n+m)t))$

$\cos(nt)\cos(mt) = \frac{1}{2}(\cos((n+m)t) + \cos((n-m)t))$

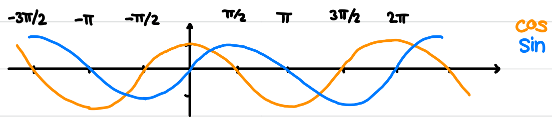
$\sin(nt)\cos(mt) = \frac{1}{2}(\sin((n+m)t) + \sin((n-m)t))$

$\sin(x+y)\sin(x-y) = \cos^2(y) - \cos^2(x) = \sin^2(x) - \sin^2(y)$

$\cos(x+y)\cos(x-y) = \cos^2(y) - \sin^2(x) = \cos^2(x) - \sin^2(y)$

$\sin(x+\frac{\pi}{2}) \equiv \cos(x)$ $\qquad$ $\cos(x-\frac{\pi}{2}) \equiv \sin(x)$

$\sin(x\pm\pi) \equiv -\sin(x)$ $\qquad$ $\cos(x\pm\pi) \equiv -\cos(x)$



Showing uniqueness of solution to Poisson equation

$$\begin{cases}\Delta u = u & \text{in } D\\ u = f & \text{on } \partial D\end{cases}$$

Let u_1 and u_2 be solutions.
Let $v_1 = u_1 - u_2$ and $v_2 = -v_1 = u_2 - u_1$

$$\begin{cases}\Delta v_i = v_i & \text{in } D\\ v_i = 0 & \end{cases}$$

v_i satisfies the eqs. on the left. Assume $v_1 > 0$ somewhere in D. Let $(x,y) = \max_D v_1$

We have $v_1(x,y) > 0$, therefore $\Delta v_1 \leq 0$ which contradicts $\Delta v_1 = v_1$

Hence $v_1 \leq 0$. Same reasoning to get $v_2 \leq 0 \Leftrightarrow v_1 \geq 0$ $\Rightarrow v_1 = v_2 \equiv 0$
$\Rightarrow u_1 = u_2$

Integrals

part. Integration $\quad \int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$

$f(x)$	$F(x)+C$
$\frac{1}{x}$	$\log\lvert x\rvert$
e^x, a^x	e^x $\quad \frac{a^x}{\log\lvert a\rvert}$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\frac{1}{\cos^2 x}$	$\tan(x)$
$\frac{1}{\sin^2 x}$	$-\frac{1}{\tan x}$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\sinh(x)$	$\cosh(x)$
$\log(x)$	$x\cdot\log(x) - x$

$(uv)' = u'v + v'u$

$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

$$\int_{\partial B_R(0,0)} f(x,y)\,d\mu = \int_0^R\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

$\int_0^{\pi/2}\sin = 1$ $\qquad$ $\int_0^{\pi}\sin = 2$ $\qquad$ $\int_0^{2\pi}\sin = 0$

$\int_0^{\pi/2}\sin^2 = \frac{\pi}{4}$ $\qquad$ $\int_0^{\pi}\sin^2 = \frac{\pi}{2}$ $\qquad$ $\int_0^{2\pi}\sin^2 = \pi$

$\int_0^{\pi/2}\cos = 1$ $\qquad$ $\int_0^{\pi}\cos = 0$ $\qquad$ $\int_0^{2\pi}\cos = 0$

$\int_0^{\pi/2}\cos^2 = \frac{\pi}{4}$ $\qquad$ $\int_0^{\pi}\cos^2 = \frac{\pi}{2}$ $\qquad$ $\int_0^{2\pi}\cos^2 = \pi$

Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right)$$

$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx$ $\qquad$ $b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx$