ANT 261 HOMEWORK, CHAPTER 4

1. Mistakes and TFT. In Chapter 4, we mainly wave our hands a lot about the effects of mistakes, without showing you how to make the models. Here you will solve what is probably the simplest model of mistakes in the iterated prisoner's dilemma (IPD).

Consider the IPD with the strategies TFT and ALLD, but with one modification. Now when any TFT individual intends to cooperate (play C), there is a chance a that the individual instead defects (an implementation error). Assume that the opposite mistake, cooperating when one intends to defect, is impossible.

First, show that the payoff to TFT when common is:

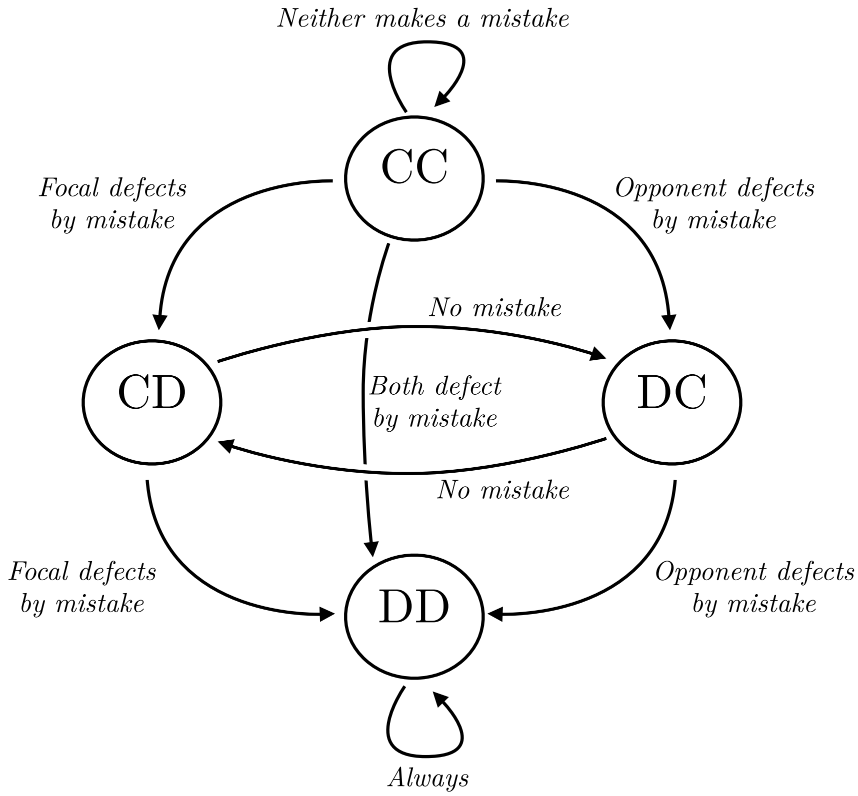
$$V(TFT|TFT) = \frac{(1-a)(b-c)}{1-w(1-a)}.$$

Then show that the condition for TFT to be an ESS against ALLD is now:

$$(1-a)wb > c.$$

You haven't quite seen a model of this kind in class, yet. To help you model this, think of the system in terms of the four different intentional states the pair of TFT players can be in: CC, CD, DC, and DD. On any given round, they will have one of these four pairs of intentions. This means, if you know the chance each makes a mistake, you can write an exact expression for the average payoff in that round. Then, if interactions continue, they will move to a set of intentions for the next round. The payoff for that round will also be easy to write, given you know the intentions at the start and the chance each makes a mistake. Your task then is to write payoffs for each pair of intentions and link these together.

One might in fact diagram the system like this:



Your task is to translate this diagram into expressions for the payoffs at each node—CC, CD, DC, or DD—linking them together with the proper probability expressions. Then you will solve for the expected payoff when the system starts at the CC node. To get you started, here's the DC node, without the explicit probability expressions:

$$\begin{aligned} V(DC) = {} & \Pr(\text{no mistake})\big(b + \Pr(\text{interaction continues})V(CD)\big) \\ & + \Pr(\text{opponent defects by mistake})\big(0 + \Pr(\text{interaction continues})V(DD)\big). \end{aligned}$$

Fill in the probabilities with the proper expressions, write one of these equations for each node, then solve for $V(CC)$. Easy!

2. Forgiveness doesn't pay. Assume the IPD with errors and TFT common, as in Problem 1 above. Now prove that the payoff to a rare invading ALLC is:

$$V(ALLC|TFT) = (1-a)\frac{b(1-aw)-c}{1-w}.$$

Use this to show that the condition for ALLC to invade a population of TFT is $(1-a)wb > c$, just like ALLD. Why does ALLC suffer more from rare errors than TFT does, even though pairs of TFT get into feuds, while ALLC always "makes up"? Justify your answer with inspection of the payoff expressions.

3. Kin selection and the evolution of punishment. Re-analyze the punishment model in section 4.5.2 to include the assumption that social groups comprise kin of average relatedness r. Show that the condition for AP to invade a population of RC is now:

$$\frac{w}{1-w}(b-c) - k(n-1)(1-r) > c - \frac{b}{n}\big(1+(n-1)r\big).$$

Interpret this expression.

ANT 261 HOMEWORK, CHAPTER 4
SOLUTIONS

2. Mistakes and TFT.

SOLUTION. To find $V(TFT|TFT)$, we need to account for all the possible paths the two players could take, as interactions proceed. This can seem daunting at first, but it's actually very simple. An intuitive approach is to write payoffs for each pair of *intended* actions. Each will branch to the others with certain probabilities. So we can write exact expressions for the expected payoffs. Then since we know both players start by intending to cooperate (CC), we can figure out the expected payoff over many such interactions.

The first state to consider is when both intend to cooperate (CC). Four things can happen: (1) both players cooperate, (2) focal makes a mistake and defects, (3) opponent makes a mistake and defects, and (4) both make mistakes. Now, to figure out the expected payoff, we need to average over these events. We also account for the chance interactions continue (w) and link to the payoff for the next pair of intended actions. So if both intend to cooperate, the payoff will be:

$$\begin{aligned} V(CC) = (1-a)^2\{(b-c) + wV(CC)\} + a(1-a)\{b + wV(CD)\} \\ + (1-a)a\{-c + wV(DC)\} + a^2(0). \end{aligned}$$

When both players defect, there is no way to escape continued mutual defection, so the interaction effectively ends. There is a similar expression for each of the other pairs of intentions:

$$\begin{aligned} V(CD) &= (1-a)\{-c + wV(DC)\} + a\{0\} \\ V(DC) &= (1-a)\{b + wV(CD)\} + a\{0\}. \end{aligned}$$

If the pair enters with intentions CD, and no mistake is made, then they will exit with intentions DC, because each copies the previous move of the other. Likewise for entering DC and exiting CD. If the C player makes a mistake, though, then both defect and interactions end.

Now, to solve this thing, recognize that we have a set of three simultaneous equations and three unknowns ($V(CC), V(CD), V(DC)$). So we can solve by isolating each unknown and then substituting it into the next equation, and so on. Let's start by substituting $V(DC)$ into

$V(CD)$:

$$V(DC) = (1-a)(b + wV(CD)),$$
$$V(CD) = (1-a)(-c + wV(DC)).$$

So:

$$V(CD) = (1-a)(-c + w\{(1-a)(b + wV(CD))\}).$$

Solving for $V(CD)$:

$$V(CD) = (1-a)\frac{w(1-a)b - c}{1 - w^2(1-a)^2}.$$

Now we have an expression for $V(CD)$, without any unknowns on the right side. So we can insert this into $V(DC)$:

$$\begin{aligned} V(DC) &= (1-a)\left(b + w\frac{w(1-a)^2 b - (1-a)c}{1 - w^2(1-a)^2}\right) \\ &= (1-a)\left(\frac{b - w(1-a)c}{1 - w^2(1-a)^2}\right). \end{aligned}$$

Now we have pure expression for $V(CD), V(DC)$. Each of these says: if the cooperating player doesn't make a mistake, then the two will alternate cooperation and defection for an average of $1/(1-w^2(1-a)^2)$ rounds. The only reason these two expressions differ is because in the first, CD, the focal individual cooperates first. In the second, DC, the focal cooperates second. So one is just the offset of the other.

It'll make the algebra in the next step easier if you notice that the denominator $1 - w^2(1-a)^2$ is a difference of squares. You learned in high school that any polynomial of the form $a^2 - b^2$ is equal to $(a-b)(a+b)$. In this case, $a = 1$ and $b = w(1-a)$. So:

$$V(CD) = (1-a)\frac{w(1-a)b - c}{(1 - w(1-a))(1 + w(1-a))}$$
$$V(DC) = (1-a)\frac{b - w(1-a)c}{(1 - w(1-a))(1 + w(1-a))}.$$

All that remains is to substitute these into $V(CC)$ and simplify, solving for $V(CC)$. After some algebra:

$$V(CC) = \frac{(1-a)(b-c)}{1 - w(1-a)}.$$

There's our answer. When both TFT players start off intending to cooperate, the expected payoff will be $V(TFT|TFT) = (1-a)(b-c)/(1-w(1-a))$. Errors reduce both the expected payoff and the duration of interactions.

Now, the expression $V(ALLD|TFT)$ is derived the same way, but it's a lot simpler. In the first round, ALLD defects and TFT cooperates, unless it makes a mistake. If it doesn't make a mistake, the focal ALLD receives b, and then interactions end. If TFT makes a mistake, then ALLD receives nothing, and then interactions end all the same. So:

$$V(ALLD|TFT) = (1-a)b + a(0) = (1-a)b.$$

When this is less than $V(TFT|TFT)$, TFT is an ESS against ALLD.

$$\frac{(1-a)(b-c)}{1-w(1-a)} > (1-a)b.$$

After a little algebra, this reduces to:

$$(1-a)wb > c.$$

■

3. Forgiveness doesn't pay.

SOLUTION. Just as before, we write payoffs for each node. There are only two nodes relevant in this case, because ALLC will never defect on purpose, so DC and DD are impossible. This leaves us:

$$\begin{aligned} V(CC) &= (1-a)^2(b-c+wV(CC)) + a(1-a)(-c+wV(CD)) \\ &\quad + (1-a)a(b+wV(CC)) + a^2(0+wV(CD)) \\ V(CD) &= (1-a)(-c+wV(CC)) + a(0+wV(CD)). \end{aligned}$$

Solving for $V(CD)$:

$$V(CD) = (1-a)\frac{wV(CC)-c}{1-aw}.$$

Substituting this into the right side of $V(CC)$ above and solving for $V(CC)$ yields the answer, $V(ALLC|TFT) = (1-a)\frac{b(1-aw)-c}{1-w}$.

Now we ask when TFT is an ESS against ALLC:

$$\begin{gathered} V(TFT|TFT) > V(ALLC|TFT) \\ \frac{(1-a)(b-c)}{1-w(1-a)} > (1-a)\frac{b(1-aw)-c}{1-w}. \end{gathered}$$

After a little algebra, this provides the answer.

The problem with ALLC in this environment is that, while it does make up with its opponent, it gets taken advantage of. Each time TFT mistakenly defects, it gains a benefit b without helping the ALLC. But when ALLC mistakenly defects, the TFT responds by defecting, making sure it receives a b for each b it provides.

This is why, in $V(TFT|TFT)$, errors reduce a factor $b-c$: pairs of TFT keep balanced books, because they trade defections. Errors reduce both aid received and costs of providing aid, equally. Errors also reduce the length of interactions, as seen in the denominator, $1-w(1-a)$.

ALLC, however, doesn't keep the books balanced. When it makes a mistake, it gets a turn of free benefits, b. On the next turn, TFT recriminates and withholds b. When TFT mistakenly defects, the TFT gets a free b, too. But the ALLC doesn't withhold anything on the next round. So as long as errors aren't too common, it's better to get into a feud (be unforgiving) than to always cooperate.

If errors are common enough, though, then it is better to get some benefits from the TFT partner, while being fleeced, than to spin off quickly into a feud and then mutual defection, like pairs of TFT do. By making up, ALLC can at least sustain interaction, when there are mistakes.

Of course, since ALLD will also invade when $(1-a)wb > c$, if ALLC can invade, ALLD will do even better. ALLD will farm the ALLC individuals, because:

$$\begin{aligned}V(ALLD|ALLC) &= (1-a)(b+wV(ALLD|ALLC)) + a(0+wV(ALLD|TFT))\\ &= \frac{(1-a)b}{1-w}.\end{aligned}$$

If ALLC is common:

$$\begin{aligned}V(ALLC|ALLC) = V &= (1-a)^2(b-c+wV) + a(1-a)(b+wV)\\ &\quad + (1-a)a(-c+wV) + a^2wV\\ &= \frac{(1-a)(b-c)}{1-w}.\end{aligned}$$

So ALLD always does better, as you might expect.

Punchline: If implementation errors are common, TFT reciprocity can't be an ESS. In that case, some strategy that keeps track of justified and unjustified defections, like CTFT (see the book), is needed. ■

3. EXTRA CREDIT: Kin selection and the evolution of punishment.

SOLUTION. The payoffs stay the same:

$$V(AP|x) = (x+1)\frac{b}{n} - c - (n-x-1)k + \frac{w}{1-w}(b-c),$$

$$V(RC|x) = \begin{cases} x>0 & xb/n - xh + \frac{w}{1-w}(b-c) \\ x=0 & 0 \end{cases}$$

The fitness of AP is therefore:

$$W(AP) = w_0 + \sum_{i=0}^{n-1} \Pr(i) V(AP|i).$$

Since $V(AP|x)$ is linear in x, this is easy to expand:

$$\begin{aligned}
W(AP) &= w_0 + \sum_{i=0}^{n-1} \Pr(i) \left((i+1)\frac{b}{n} - c - (n-i-1)k + \frac{w}{1-w}(b-c) \right), \\
&= w_0 + \frac{b}{n}\,\mathrm{E}(i+1) - c - k\,\mathrm{E}(n-i-1) + \frac{w}{1-w}(b-c), \\
&= w_0 + \frac{b}{n}\,\mathrm{E}(i) + \frac{b}{n} - c - k(n-1) + k\,\mathrm{E}(i) + \frac{w}{1-w}(b-c).
\end{aligned}$$

Now group together all the $\mathrm{E}(i)$ terms, so we can make it easy to insert the haploid kin selection probabilities of groups:

$$W(AP) = w_0 + \mathrm{E}(i)\left(\frac{b}{n} + k\right) + \frac{b}{n} - c - k(n-1) + \frac{w}{1-w}(b-c).$$

We know that under the simple binomial group formation pattern in the book, we compute the expected number of AP with:

$$\mathrm{E}(i) = (n-1)\big(r + (1-r)p\big).$$

This gives us, when $p \approx 0$, $\mathrm{E}(i) \approx (n-1)r$. Slipping this into our expression:

$$W(AP) = w_0 + (n-1)r\left(\frac{b}{n} + k\right) + \frac{b}{n} - c - k(n-1) + \frac{w}{1-w}(b-c).$$

We want to compare this to the average fitness of RC, which is still:

$$W(RC) = w_0 + V(RC|0) = w_0 + 0.$$

AP will increase if $W(AP) > 0$, which simplifies to:

$$\frac{w}{1-w}(b-c) - k(n-1)(1-r) > c - \frac{b}{n}(1 + (n-1)r).$$

That is what I told you to prove.

Now, what does this mean? The left side is the net benefit of long term cooperation. The first term on the left is the long-run benefits of cooperation after the first round of interaction. The second term on the left is the cost of punishing on the first round. You can think of this punishment cost as the price of getting cooperation going. The right side is the net cost of cooperation in the first round.

Now notice that r appears on both sides. On the left, its role is to reduce the cost of punishment. When $r = 1$, everyone is AP, so there is in fact no cost of punishment. So AP invades more easily, under kin selection, because there are fewer individuals to punish, not because there is more punishment imposed on the RC. Also notice that on the right, r effectively increases the benefits received in the first round.

These two effects together increase the left side and reduce the right side, making it easier to satisfy the condition. In fact, when $r = 1$, the condition reduces to:

$$\frac{w}{1-w}(b-c) > c-b.$$

Since $b > c$ by assumption (otherwise cooperation is a lost cause), this is always satisfied, because the right side is negative, and the left side is always positive, even if $w = 0$! As you'll see in Chapter 6, what has happened here is that by making $r = 1$, we've made the group level selection everything, because there is no variation within groups.

Kinship makes it easier for AP to invade, both because it reduces the amount of punishment needed on the first round, and because it increases the expected benefits received on the first round. It has nothing to do with imposing more punishment on RC, because remember that the majority of RC individuals are in groups with no AP individuals, so there mean fitness is unaffected.

If you also calculate the stability condition, however, you'll find that different factors matter in that case. ■