A PRACTITIONER'S INTRODUCTION TO STOCHASTIC RESERVING: THE ONE-YEAR VIEW

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Abstract

The aim of this paper is to build on the Pragmatic Stochastic Reserving Working Party's first paper (Carrato, et al., 2016) and present an overview of stochastic reserving used with a one-year view of the risk, which is suitable both for those working at the coalface of reserving and capital modelling as well as more senior actuaries tasked with oversight of the process. We discuss in detail the one-year view of risk, and how it relates to non-life claims reserves in particular. We describe and discuss three commonly use methods for calculating one-year reserve risk: the Merz-Wüthrich formula, the Actuary-in-the-Box, and Emergence Patterns. For the Actuary-in-the-Box method we describe the method in detail for Mack's model, the Over-Dispersed Poisson model, and the stochastic Bornhuetter-Ferguson model described in the working party's first paper (Carrato, et al., 2016). We develop the theory for Emergence Patterns in detail as this material has not be published in a unified form before. We also briefly describe some other methods that can be used to estimate one-year reserve risk. Some numerical examples are provided to illustrate the concepts discussed in the paper.

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Disclaimer

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We would like to thank all the members of the Pragmatic Stochastic Reserving Working Party who gave lots of interesting market insights and valuable contributions. In particular, we would like to explicitly thank:

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- Henry Chan, for helping create the diagrams in sections 2 and 3

1 INTRODUCTION AND SCOPE

1.1 OVERVIEW

1.1.1 SCOPE

The aim of this paper is to build on the work done in the working party's first paper (Carrato, et al., 2016). In the first paper we restricted ourselves to look at the ultimate view of reserve risk, in this paper we now look at the one-year view of reserve risk. Other than that change of focus our ambition remains the same: to smooth the path for general insurance actuaries, regardless of experience, to engage with and understand the commonly used stochastic reserving methods.

This paper also has the same target audience as the first:

- Actuaries tasked, through calculations and analysis, with assessing reserve variability;
- More senior, experienced, actuaries with responsibility for the oversight and, most likely, review of such reserve variability assessments.

Although this paper builds on the first, we have not assumed that the reader has a detailed knowledge of it. Instead, where relevant in this paper, we refer the reader to specific sections of the first paper. However, we encourage readers of this paper to also read the first paper in full, as it will give them a broader understanding of the topic. We strongly believe that neither the ultimate nor one-year view of risk is definitively correct or superior to the other. They take different views of the risk, and both provide valuable insights, and have limitations. No understanding of reserve risk is complete unless you understand both.

We have chosen to focus on the following three methods for estimating one-year reserve risk:

- The Merz-Wüthrich formula
- The Actuary-in-the-Box method
- Emergence Patterns

In addition to these three methods we give brief descriptions of seven other methods.

We chose these three methods as they are by far the most commonly used methods, and they also allow us to build on the material in the first paper. The Merz-Wüthrich formula gives the one-year view within Mack's model, and for the Actuary-in-the-Box method we describe in detail how to apply it to bootstrapped versions of the three models discussed in the first paper.

Emergence Patterns are a deceptively simple method. There has been scattered discussion of them, mainly in various conference presentations, but until now, there has been no unified development and discussion of the ideas. We give a non-technical account in the main body of the paper, and develop the theory in detail in an appendix. We make clear that there are multiple different interpretations of the basic idea, so the user needs to be clear which interpretation is being used. We also discuss methods of parameterising emergence factors, and argue that this is an intrinsically hard problem, with no wholly satisfactory solution in sight.

As in the first paper, we limit ourselves to looking at the application of the methods to gross data, without allowance for reinsurance recoveries. We have also focussed on "Accident Year" data, instead of "Underwriting Year", accepting that the latter is in common use. Our main reason for this is to preserve the relative independence of the resulting claims cohorts, which is a common assumption within the methods.

Finally, we have restricted our attention to the consideration to independent error which is amenable to quantitative measurement. Model error, or systemic risk which requires a more qualitative, judgemental approach, is out of scope of the paper. However, this scope limitation is not intended to imply that model error may be ignored. In many cases, it represents a material component of the overall prediction error. In particular we do not discuss ENIDs (Events Not In Data) in this paper, although they should be included in any full assessment of risk. Some further discussion on this point may be found in the first paper in section 3.5 "Sources of Uncertainty".

1.1.2 GUIDE TO THE PAPER

The paper opens with discussion comparing the one-year and ultimate view of reserve risk, and gives some of the reasons why it is useful to consider the one-year view. We then discuss the one-year view in more detail, both in general and for reserve risk in particular.

In section 3 we describe the three main methods of estimating one-year reserve risk: the Merz-Wüthrich formula, the Actuary-in-the-Box, and Emergence Patterns. We also discuss the strengths and limitations of each method. We round-off this section with brief descriptions of some other methods of estimating one-year reserve risk.

In section 4 we discuss additional validation that can be done for one-year reserve risk. These methods are in addition to any validation done for the underlying models discussed in the first paper.

In section 5 we illustrate the use of the methods with the same two example data sets as used in the first paper. These examples extend what was done in section 7 of the first paper.

There are two appendices where we develop some of the material discussed in the main body of the paper in more technical detail. In appendix A we develop the concepts and notation needed to discuss one-year reserve risk. In appendix B we develop the ideas about emergence patterns discussed in the main body of the paper.

As a number of acronyms are used in this paper, we have included a glossary for ease of reference.

1.1.3 Other Relevant Working Parties

The Pragmatic Stochastic Reserving Working Party has focussed on quantitative methods of assessing reserve risk. The following two working parties have taken a more qualitative approach, and we encourage readers of this paper to also study their outputs available at the Institute and Faculty of Actuaries website:

- Managing Uncertainty Qualitatively
  https://www.actuaries.org.uk/practice-areas/general-insurance/research-working-parties/measuring-uncertainty-qualitatively-muq
- Managing Uncertainty with Professionalism
  https://www.actuaries.org.uk/practice-areas/general-insurance/research-working-parties/managing-uncertainty-professionalism

1.1.4 Supporting Files on GitHub

Files containing example implementations of some of the methods discussed in this paper, and the data used in the numerical examples in section 5 are available in the working party's repository in GitHub here:

https://github.com/robertmscarth/stochastic-reserving-wp

2 Introduction to One-Year risk

2.1 Ultimate view and one-year view

Traditionally actuaries analysing claims reserves sought a point estimate of the future claims payments arising from prior periods of exposure (if working on an accident year basis), or from policies already written (if working on an underwriting year basis). This was often calculated using a fairly simple method such as the basic chain ladder or the Bornhuetter-Ferguson method; judgement was frequently used to augment the results or adjust the output of the methods. As early as 1975 attempts were made to put these methods onto a firm statistical footing (Hachemeister, et al., 1975). In 1993 Thomas Mack published his now well-known model of the chain ladder (Mack, 1993), and in 1998 Renshaw and Verrall (Renshaw, et al., 1998) published the over-dispersed Poisson model. See section 4 of (Carrato, et al., 2016) for some more details of this history. However, these models were little used in practice until the turn of the century, when regulations such as the UK's ICAS regime required insurers to calculate capital requirements. (See sections 2 and 3 of (Taylor, et al., 1983) for a discussion of some of the reasons why these models were not used much in practice.)

The regulations and the initial theoretical work both took an ultimate view of the claims reserve risk. This means that they considered all possible variation in the claims payments between the point in time that the reserve exercise was carried out and the final settlement of all the claims arising from the prior periods (either exposure or underwriting, depending on the basis). This is a justifiable approach, as the insurance company needs to hold sufficient funds to ensure it can meet its obligations, no matter how long it takes to settle them. Claims can be reported with some delay, and settlement of claims can take time. This is especially true of long-tail classes of business such as liability classes, where in some cases settlement of claims can involve legal action taking years. Furthermore, the final total amount paid is uncertain. The insurance company will therefore want to have an understanding of how much the final total amount paid could vary from the current best estimate, so that it can hold sufficient funds over and above the best estimate, to reduce the chance of being unable to meet its obligations to an acceptably small amount[1].

However, there are limitations with the ultimate view of risk. Insurance companies report profits on an annual basis, and movements in the claims reserves make a contribution to the profit. Insurance companies also make business plans annually for the next year. Longer term plans are also made, but usually for a shorter period than the time taken to settle all claims for some long-tail liability classes. Decisions about how to deploy capital – how much premium to write for each class, how to invest

[1] This is true even without regulation. Potential customers are unlikely to want to buy insurance from a company that has a high chance of not being able to pay claims. The principle of *uberrima fides* would oblige an insurer to tell potential policy holders if there was a reasonable chance that their claims could go unpaid. Although the possibility that some potential policy holders might not understand what this means, a lack of clarity about exactly how likely insolvency needs to be before insurers are obliged to tell policy holders, and exactly what the insurance company is obliged to say, would provide arguments in favour of solvency regulation that specifies insurers' obligations more clearly.

assets, how much capital to return to shareholders, and how much solvency capital to hold – are usually made annually. Furthermore, these decisions rely on taking an integrated view of the risks that the insurance company faces. This means that all the risks the insurance company faces need to be considered in a consistent manner so that they can be compared and aggregated. For some risks, such as equity holdings or operational risk, there is no equivalent to the ultimate view taken for insurance risks. Because of this it is useful to the management of the insurance company to have a good understanding of shorter-term fluctuations in the claims reserves, particularly over a one-year time horizon.

Finally, recent regulation – in particular Solvency II and the Swiss Solvency Test (SST) – require insurance companies to take a one-year view of risk. This was discussed in section 2 of (Carrato, et al., 2016).

The ultimate view and the one-year view are different ways of viewing the same risk. To make the distinction between them more concrete consider the case of a single accident year, and consider the full life-time of the risk from the start of the period of exposure (at time 0) to the final claims payment (at time n). This is represented in the Figure 2-1 below.

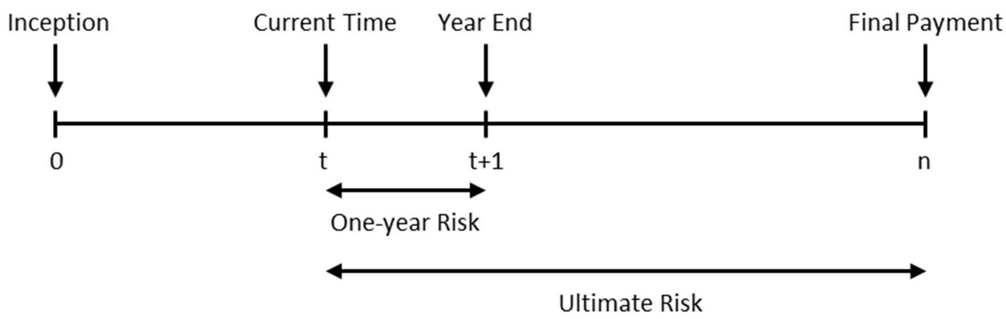

FIGURE 2-1

The ultimate risk at time t is the full distribution of claims paid after time t, taking account of all the information known at time t. From the full distribution a risk measure can be calculated. The risk measure used will depend on the purpose it is needed for. For example, for a reserve range a standard deviation might be used, whereas for capital requirements a 99.5th percentile might be used. The best estimate claims reserve at time t is then the expected value of the distribution of all claims paid after time t, conditional on all the information known at time t. The key point is that the ultimate risk considers all possible variability in the claims payments between time t and time n.

The one-year risk is the distribution of the movement from the best estimate claims reserve at time t to the best estimate claims reserve at time $t + 1$, making allowance for the claims paid between time t and time $t + 1$. The best estimate claims reserve at time t is known at time t. The best estimate claims reserve at time $t + 1$ is not known at time t, but will vary depending on the information learnt between time t and time $t + 1$. The amount of claims paid between time t and time $t + 1$ is not known at time t, and is part of the information learnt between time t and time $t+1$. The key point is that the one-year risk considers all possible variability due to information learnt between time t and time $t+1$. This is discussed in more detail in section 2.3 and section 9 (appendix A).

The contrast between the ultimate view and the one-year view is therefore that the ultimate view considers variability due to all information learnt between time t and time n, whereas the one-year view considers variability due to information learnt between time t and time $t+1$. Since the information learnt between time t and time $t+1$ is a subset of the information learnt between time t and time n, it would be reasonable to conclude that the one-year risk is lower than the ultimate risk. This is true if variance is used as the risk measure (see section 9 appendix A). In (Papachristou, 2016) it is claimed, without proof or a reference, that this is true for all coherent risk measures.

It is sometimes argued that in practice it is possible for the one-year view of risk to be greater than the ultimate view. One possible situation this might happen is where it is known that the reserving actuary has a tendency to over react to information learnt between time t and time $t+1$. In such a situation the best estimate claims reserves set at time $t+1$ would be increased or decreased by too much given the information learnt between time t and time $t+1$, and so the variability over one year could be greater than the ultimate variability. However, if it is known that the over-reaction is happening in a systematic or predictable way then it should be allowed for as part of the model used for one-year risk. This point was discussed by Dimitris Papachristou in a 2016 GIRO presentation (Papachristou, 2016).

It is important to note that neither the ultimate nor the one-year view is definitively correct, or superior to the other. They take different views of the risk, and both provide valuable insight, and have limitations. It is however important that the one-year view and the ultimate view are consistent. Ensuring this consistency is one of the challenges we discuss in this paper.

2.2 THE ONE-YEAR VIEW OF RISK

In this sub-section we give an overview of the one-year view of risk in an insurance company. We take an integrated view and consider all the risks faced by the insurance company. In the following sub-section, we discuss reserve risk specifically.

An insurance company accepts risk by writing insurance policies which it sells to its customers, the policy holders, in exchange for the payment of a premium. In selling these policies the insurance company incurs a liability, first for future periods of exposure which the insurer has contracted to provide cover for, and then for claims which occurred during prior periods of exposure, but which have not yet been settled. So that it can pay these liabilities the insurer holds assets, and as the value of the liabilities and the assets can fluctuate it also holds risk capital to ensure that the risk that it cannot pay the liabilities is reduced to an acceptably small amount. This can be represented as in Figure 2-2 which shows the economic balance sheet at time 0.

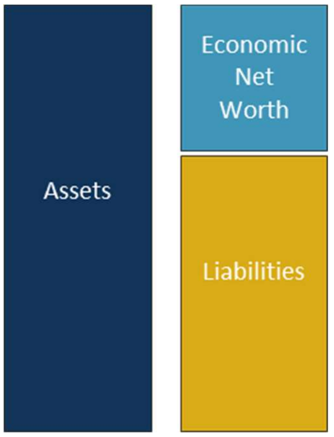

FIGURE 2-2

The Economic Net Worth is calculated as Assets minus Liabilities, this is the capital that is available to absorb fluctuations in the value of the assets and liabilities. The values shown are economic values. For assets or liabilities which are traded on financial markets this means that the market values are used. For assets or liabilities that are not traded on financial markets we use market consistent values.

For insurance liabilities we calculate market consistent values as follows. First, we project all future cashflows, discount these using market interest rates and then take the expected value. This gives us the discounted best estimate of the liabilities. However, as the ultimate cost of the liabilities is not known, additional risk capital needs to be held to ensure that the liabilities can be paid even if they are greater than the best estimate. Holding this capital has a cost, and so we add the cost of the capital to the discounted best estimate to get the value of the liabilities. In Solvency II this additional amount is called the *Risk Margin*, and in the Swiss Solvency Test it is called the *Market Value Margin*. For further discussion of market consistent valuation (Wüthrich, et al., 2008), and for a discussion of the calculation of the risk margin see (Czernuszewicz, et al., 2009).

The cost of capital arises because the capital is provided by shareholders, and the insurance company incurs costs in holding the capital that the shareholders would not. These costs are:

- *Double taxation*. The insurance company is taxed on any return the investment earns, and the shareholder is taxed again when the profit is paid out as a dividend.
- *Agency costs*. The shareholders have hired the management of the insurance company to invest their capital; management is the agent of the shareholders. The management and the shareholders do not have the same interests and so the management might not invest the shareholders' capital in the best way for the shareholders.

- *Regulatory costs*. The insurance company's regulator might require the insurance company to invest its capital in a certain way, thus costing the company lost investment return.
- *Financial distress costs*. If the company has to use the risk capital it will have to do so at a time not of its choosing and might therefore have to sell assets for a poor price. Very extreme losses might be caused by an event that also reduces the value of assets, for example a lethal pandemic. Also financial distress can lead to loss of brand or institutional value.

The economic balance sheet diagram can therefore be refined by splitting the Liabilities in to the Best Estimate Liabilities and the Cost of Capital (Coc), as shown in Figure 2-3.

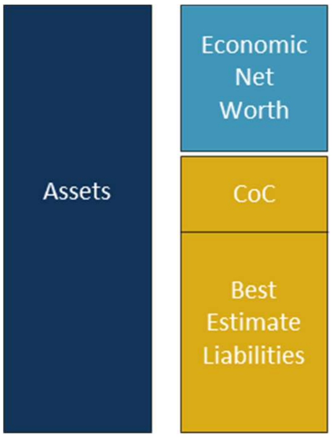

FIGURE 2-3

The Economic Net Worth is the capital that is available to absorb fluctuations in the value of assets and liabilities. The overall one-year risk is the distribution of the change in value of the Economic Net Worth over a one-year time horizon. We define this change as the Economic Net Worth at time 1 minus the Economic Net Worth at time 0. This is then the profit or loss over the one-year period. Figure 2-4 shows an example where the assets have grown from time 0 to time 1, but the liabilities have grown by even more, so the economic net worth has gone down, and the insurer has made an economic loss.

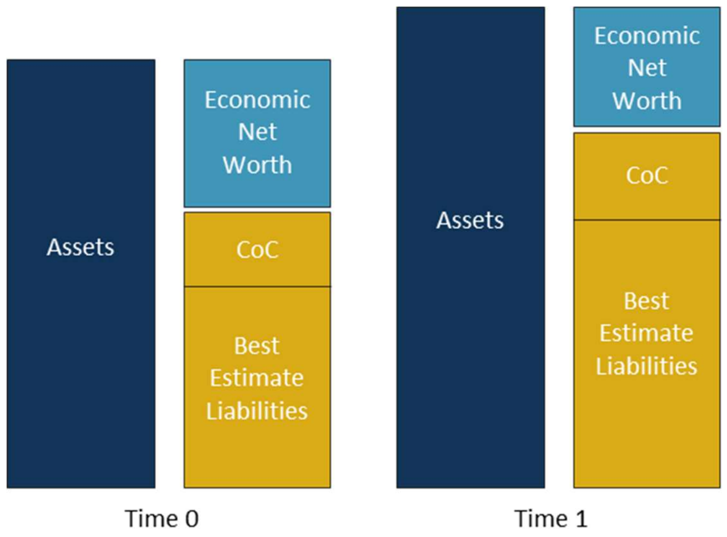

FIGURE 2-4

Holding more capital reduces the risk that the insurance company will not be able to pay its liabilities. However as discussed above holding capital has a cost. There is therefore a trade-off between increasing security and reducing the cost. How these are traded-off depends on the risk appetite of the insurance company, and is a management decision[1]. The amount of capital that the management decides to hold is called the Economic Capital, and capital held in addition is called Excess Capital.

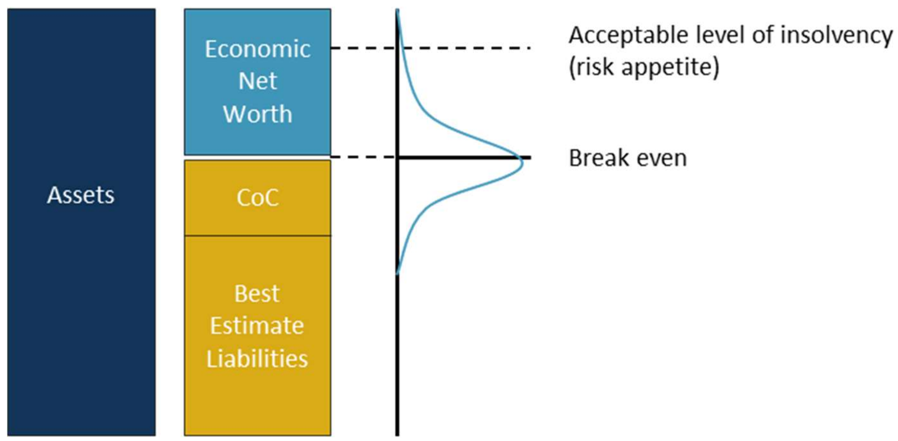

FIGURE 2-5

Figure 2-5 shows the time 0 economic balance sheet, and the distribution of the economic profit between time 0 and time 1, with “Break even” denoting an economic profit of zero. If the Economic Net Worth increases then the company has made an economic profit, if it decreases it has made an economic loss.

[1] There will of course be constraints, both implicit and explicit, from shareholders, bond holders, policyholders, regulators, and other stakeholders.

2.3 The one-year view of reserve risk

The reserves are part of the liabilities and one-year reserve risk is the part of one-year risk arising from movements in the reserves. This includes movements in the risk margin and potentially movements due to discounting. However, in this paper we concentrate on movements in the undiscounted best estimate claims reserves. In this sub-section we give an intuitive overview of how to consider reserve risk within a one-year view. In section 9 appendix A we discuss in technical detail the concepts and notation needed to consider one-year reserve risk.

For one-year reserve risk we need to consider the movement from the claims reserves set up at the start of the year (or the opening claims reserves) to the claims reserves set up at the end of the year (or the closing claims reserves). The opening claims reserves are known with certainty at the start of the year, whereas the closing claims reserves are not, and can be consider as a random variable. We also need to consider the claims paid out from the reserves during the year. This is because an allowance for them is included in the opening reserves, but not the closing reserves. The claims paid during the year are also not known at the start of the year, and can be consider as a random variable. We call the movement from the opening position to the closing position the *claims development result* (CDR). This has the following definition:

Claims Development Result = Opening Reserve – Closing Reserve – Claims Paid During the Year

An alternative equivalent way of defining the CDR is as the movement from the opening estimate of ultimate claims to the closing estimate of ultimate claims.

Claims Development Result = Opening Estimate of Ultimate – Closing Estimate of Ultimate

If the claims reserves are set on a best estimate basis, then the sequence of estimates of the ultimate claims is unbiased – that is the expected value of the closing estimate of ultimate claims is equal to the opening estimate of the ultimate claims. This means that the expected value of the CDR is zero.

As we described in section 2.1 above the opening claims reserve at time t is the best estimate of claims payments made after time t. We interpret this as the expected value of the claims payments made after time t, conditional on the information known at time t. The closing claims reserve is likewise the expected value of claims payments made after time $t+1$, conditional on the information known at time $t+1$. Similarly, the opening estimate of ultimate claims is the expected value of all claims payments, conditional on the information known at time t, and the closing estimate of ultimate claims is the expected value of all claims payments, conditional on the information known at time $t+1$.

The opening claims reserve, and the opening estimate of ultimate claims are both known with certainty at time t, whereas the closing claims reserve, the claims paid between time t and time $t+1$, and the closing estimate of ultimate clams are all random variables. To get a distribution of the claims development result we therefore need to get a distribution of the closing estimate of ultimate claims, or distributions for the closing claims reserve and the claims paid between time t and time $t+1$.

In this paper we discuss the following three methods for calculating a distribution of the closing claims reserve, or closing estimate of the ultimate claims:

- The Merz-Wüthrich formula
- Actuary-in-the-box
- Emergence Patterns

All these methods depend on, to a greater or lesser degree, a model of the claims development which takes an ultimate view of the risk. A distribution for the claims paid between time t and time $t+1$ can often, but not always, be got from the ultimate model. This is discussed further in the relevant sub-sections of section 3.

These are not the only methods that have been proposed for calculating one-year reserve risk. Many other methods have been proposed, and we discuss some of these briefly in section 3.4.

In this paper we focus on reserve risk. However, when taking an aggregated one-year view of the risks facing an insurance company for all items on the balance sheet which are estimates of payments stretching into the future we need to calculate a distribution of the amount on the closing balance sheet. These items include the following:

- Gross Outstanding Claims Provisions
    - Claims
    - Premiums
    - Expenses
- Reinsurance Outstanding Claims Provisions
    - Claims
    - Premiums
    - Expenses
- Bad Debt Outstanding Claims Provisions
- Gross Premium Provisions
    - Claims
    - Premiums
    - Expenses
- Reinsurance Premium Provisions
    - Claims
    - Premiums

    - Expenses
- Bad Debt Premium Provisions

In this paper we only explicitly consider the gross outstanding claims provisions excluding premiums and expenses. However, the methods we describe can be used to calculate closing distributions for the other items.

Furthermore, for claims reserves we do not consider claims inflation or discounting.

3 METHODS

3.1 THE MERZ-WÜTHRICH FORMULA

3.1.1 METHOD INTRODUCTION

Merz and Wüthrich ((Merz, et al., 2008), (Wüthrich, et al., 2009), and (Merz, et al., 2015)) studied the claims development result (CDR) within Mack's model (Mack, 1993). They calculated analytic formulae within Mack's model for the mean squared error of prediction (MSEP) of the CDR for an individual accident year, and for all accident years in total. In (Merz, et al., 2008) and (Wüthrich, et al., 2009) this was done for the first future time period, in (Merz, et al., 2015) this was extended to all future time periods. In this paper we describe only the formula for the first future time period.

The Merz-Wüthrich formula is based on Mack's model and the discussion in this section builds on the discussion of Mack's model in section 4.1 of the working party's first paper (Carrato, et al., 2016).

3.1.2 DESCRIPTION OF METHOD

In this subsection we describe the assumptions of Mack's model, we state the Merz-Wüthrich formula for the MSEP of the CDR for an individual accident year, and for the total of all accident years, and we compare the Merz-Wüthrich formula with Mack's formula for the MSEP of the reserve estimate. In this subsection we have drawn on sections 4.2 and 4.6 of (Hindley, 2018).

First, recall the notation used in section 4.1 of the working party's first paper. We assume that we have a triangle of cumulative claims, with n origin periods and development periods:

$$\{C_{ij} : i = 1, \dots, n, j = 1, \dots, n - i + 1\}$$

Merz and Wüthrich make the following assumptions which are slightly stronger than (and so imply) the assumptions made by Mack:

1. Origin periods are independent i.e. for each $s \neq t$, $\{C_{s1}, \dots, C_{sn}\}$ and $\{C_{t1}, \dots., C_{tn}\}$ are independent.
2. The time series of cumulative claims for each origin period is a Markov process, and there exist constants f_j and σ_j such that for $i = 1, \dots, n$ and $j = 2, \dots, n$

$$E\left[C_{ij} \mid C_{i,j-1}\right] = f_{j-1} C_{i,j-1}$$

$$Var\left(C_{ij} \mid C_{i,j-1}\right) = \sigma_{j-1}^2 C_{i,j-1}$$

The estimators for the parameters f_j and σ_j are the same as in Mack's model:

$$\hat{f}_j = \frac{\sum_{i=1}^{n-j} C_{i,j+1}}{\sum_{i=1}^{n-j} C_{ij}}$$

And

$$\hat{\sigma}_j^2 = \frac{1}{n-j-1}\sum_{i=1}^{n-j} C_{ij}\left(f_{ij} - \hat{f}_j\right)^2$$

Where the f_{ij} are the individual development factors:

$$f_{ij} = \frac{C_{i,j+1}}{C_{ij}}$$

Before stating the formulas for MSEP of the CDR we define some variables for some of the subparts of the formulas. There are two reasons for doing this. First it makes it easier to see the structure of the formulas, and to compare the formulas for Mack and Merz-Wüthrich. Secondly, by breaking the formula up into sub-calculations, it makes it easier to see how to implement it in a spreadsheet or other software.

First, we define variables for the column sums used in the basic chain ladder

$$S_j^k = \sum_{m=1}^{k-j} C_{mj}$$

With this definition, the basic chain ladder development factors are given by $\hat{f}_j = S_{j+1}^{n+1}/S_j^n$.

Now we define four variables that are key components of the formulas for the MSEP. The four variables are represented by the Greek letters $\omega, \Omega, \lambda, \Lambda$. The upper-case forms are sums of the lower-case forms. The omegas are used in the formulas for the process error, whereas the lambdas are used in the formulas for the parameter error. The lower-case variables are used in the Merz-Wüthrich formulas and the upper-case variables are used in Mack's formulas. We present the formulas in the 2×2 Table 3-1 to emphasise these connections, and the similarity of the definitions.

$\widehat{\omega}_{ij} = \dfrac{\hat{\sigma}_j^2/\hat{f}_j^2}{\hat{C}_{ij}}$	$\widehat{\Omega}_i = \displaystyle\sum_{k=n+1-i}^{n-1} \widehat{\omega}_{ik}$
$\hat{\lambda}_j = \dfrac{\hat{\sigma}_j^2/\hat{f}_j^2}{S_j^n}$	$\widehat{\Lambda}_i = \displaystyle\sum_{k=n+1-i}^{n-1} \hat{\lambda}_k$

TABLE 3-1

In a similar way to Mack, Merz and Wüthrich present estimators for the MSEP of the CDR for individual origin periods, and for the total CDR over all origin periods. The MSEP for the individual origin periods is split into a process error component and a parameter error component. The estimator for the MSEP of the total CDR must also take account of the fact that the estimators of the CDR of each origin period are correlated due to the fact that they rely on the same estimated parameters $\hat{f}_j$ and $\hat{\sigma}_j^2$. We call this term for the correlation the covariance term. The MSEP of the total CDR is then the sum of the MSEPs for the individual origin periods plus the covariance term.

It is important to note that the break-down of the MSEP into the process error and parameter error that is given by Merz and Wüthrich depends on a linear approximation in the derivation. See (Wüthrich, et al., 2008) chapter 3 for a detailed discussion of this.

We present the formulas for the three components – process, parameter, and covariance – in Table 3-2, showing the corresponding Merz-Wüthrich and Mack formulas side by side.

	Merz-Wüthrich	**Mack**
Process	$\hat{C}_{in}^2 \widehat{\omega}_{i,n+1-i}$	$\hat{C}_{in}^2 \widehat{\Omega}_i$
Parameter	$\hat{C}_{in}^2 \left(\hat{\lambda}_{n+1-i} + \sum_{j=n+2-i}^{n-1} \frac{C_{n-j,j+1}}{S_j^{n+1}} \hat{\lambda}_j \right)$	$\hat{C}_{in}^2 \widehat{\Lambda}_i$
Covariance	$2 \sum_{1 \le i < k \le n} \hat{C}_{in} \hat{C}_{kn} \left(\hat{\lambda}_{n+1-i} + \sum_{j=n+2-i}^{n-1} \frac{C_{n-j,j+1}}{S_j^{n+1}} \hat{\lambda}_j \right)$	$2 \sum_{1 \le i < k \le n} \hat{C}_{in} \hat{C}_{kn} \widehat{\Lambda}_i$

TABLE 3-2

We now compare and contrast the formulas for Mack and Merz-Wüthrich.

First note that all the formulas have a term for the ultimate claims. For the process and parameter error this is $\hat{C}_{in}^2$ whereas for the covariance term it is $\hat{C}_{in}\hat{C}_{kn}$. This is means that the RMSEP of both the CDR (Merz-Wüthrich), and the claims reserve (Mack) is proportional to the estimate of ultimate claims.

For the process error term, the term $\widehat{\omega}_{i,n+1-i}$ in the Merz-Wüthrich formula is the first term in the sum giving the corresponding term $\widehat{\Omega}_i$ in the Mack formula. This makes intuitive sense, as the Merz-Wüthrich formula is an estimator for the one-year risk, and the Mack formula is an estimator for the ultimate risk. For the one-year process error we'd expect only the term corresponding to the first year's development to contribute, as subsequent development is uncorrelated with it.

For the parameter error term, the lambda term in the Merz-Wüthrich formula corresponding to the term $\widehat{\Lambda}_i$ in the Mack formula is more complicated:

$$\hat{\lambda}_{n+1-i} + \sum_{j=n+2-i}^{n-1} \frac{C_{n-j,j+1}}{S_j^{n+1}} \hat{\lambda}_j$$

The term $\hat{\lambda}_{n+1-i}$ is the first term in the sum giving $\widehat{\Lambda}_i$. The term $\sum_{j=n+2-i}^{n-1} \frac{C_{n-j,j+1}}{S_j^{n+1}} \hat{\lambda}_j$ is made up of the second and subsequent terms in the sum giving $\widehat{\Lambda}_i$ with the $\hat{\lambda}_j$ factors scaled down. This makes intuitive sense. For the parameter error we need to consider the parameter error in all future development years; we will get the full amount for the first development year, and a reduced amount for the subsequent development years.

For the covariance term, for both the Mack and Merz-Wüthrich formulas, the lambda term is the same as the lambda term for the parameter error. This makes sense because there is only correlation between the estimates because of the common parameters used.

3.1.3 Discussion of Method

The Merz and Wüthrich formulas give analytic expression for the MSEP of the CDR within Mack's model. They are therefore based on a well-established model of the chain ladder, and give an estimate of the one-year risk that is consistent with the estimate of the ultimate risk given by Mack's formulas. The formulas are simple enough to be implemented in a spreadsheet, and can be implemented as a fairly straight-forward extension of the calculations of Mack's formulas. In (Merz, et al., 2015) the formulas were extended to all future time periods, and so give the MSEP for multi-year CDRs, which allows us to see how the risk runs-off year on year.

Analytic expressions have the advantage over simulation methods in that they allow for explicit interpretations in terms of the parameters involved, and allow for sensitivity analysis with respect to parameter changes. See the remark at end of section 3.1 of (Wüthrich, et al., 2008). In these remarks Wüthrich and Merz also say "these estimates are very easy to interpret". However, the formulas are fairly complex, and it is not prima facie obvious how, in general, changes in the data or parameters would affect the resulting MSEP. In (Gisler, 2019) Gisler derives the uncertainty estimators in Mack's model in a new way. It is claimed that the derivation is more easily understandable, and gives equivalent but simpler and more interpretable formulas than Mack and Merz-Wüthrich.

The formulas only apply to Mack's model of the chain ladder. If any alterations are made to the model, such as curve fitting, or adding a tail factor, then the formulas no longer apply. If a model other than Mack's model has been used to estimate the ultimate risk then the one-year estimates from the Merz-Wüthrich formulas will not be consistent. If Mack's model is not a good fit to the triangle of claims data then the Merz-Wüthrich formulas are liable to give unreasonable or misleading results.

The formulas only give one statistic of the CDR – the MSEP. If other statistics are wanted (e.g. the 99.5th percentile) then we need to do something else such as fit a distribution, and read the desired statistic from that. This introduces more degrees of freedom, and so greater uncertainty to the final results.

The formulas also only give one-year reserve risk, and cannot give one-year premium risk.

Both Mack and Merz-Wüthrich's formulas apply only to a single claims triangle, and therefore they cannot allow for dependence between two or more claims triangles. In (Braun, 2004) Braun extends Mack's model to multiple correlated claims triangles, and in (Appert-Raullin, et al., 2013) the corresponding one-year formulas are derived.

3.2 ACTUARY-IN-THE-BOX

3.2.1 METHOD INTRODUCTION

Actuary-in-the-box is a general procedure. The term was coined by Esbjorn Ohlsson, and was first described in the literature in (Ohlsson, et al., 2009), although it was not new at this point. It was known and likely reasonably widely used for several years before 2009. A version of the method applied to a bootstrapped ODP model was described in 2003 in section 9 of (De Felice, et al., 2003), a short description of the method in a bootstrap context was described in (Björkwall, et al., 2009), and a version of the method was also described in (Diers, 2009). A version was first described to the first author by colleagues at Swiss Re around about 2003, and was in use at Swiss Re shortly thereafter.

We first describe the more general procedure described in (Ohlsson, et al., 2009) and then describe the method applied to the bootstrapped models of claims data such as the Mack, and ODP models, and Alai, Merz and Wüthrich's stochastic Bornhuetter-Ferguson model.

The method described by Ohlsson and Lauzeningks is the following:

1. Obtain the Best Estimate of the opening reserve. It is assumed that this is done according to a well-defined algorithm, and that it does not include any risk margin.
2. Extend the input data needed for the algorithm used in step 1 by simulating one further year of data.
3. Apply exactly the same algorithm as is step 1 to the extended data set generated in step 2 to produce a distribution of the closing claims reserve.

Following the above method gives us the opening claims reserve R_0, the claims amount paid during the year, P, and the closing claims reserve, R_1. This allows us to calculate the claims development result (CDR) using

$$CDR = R_0 - P - R_1$$

In well behaved situations the expected value of the CDR is zero. However, there are some common situations where this is not the case. We discuss this further in section 3.2.5.

Step 1 above is to obtain the best estimate opening reserves. This is a natural first step, however it is not necessary to do precisely this. If the goal is to get a distribution of the claims development result, then the claims development result can also be calculated as the movement in the estimate of ultimate claims:

$$CDR = U_0 - U_1$$

It would therefore be possible for the first step to be to obtain the opening best estimate of the ultimate claims – this is just the best estimate opening reserves plus the claims paid to date. The advantage of this is that in step 2 we would not need the simulated data to include the claims paid during the year. This allows us to apply the actuary-in-the-box to bootstrapped triangles of incurred claims. However, note that in some cases it will be necessary to output the closing reserve, or the claims paid during the year, for example if projecting a balance sheet. In this case the model that the actuary-in-the-box method is applied to will need to include paid claims.

For Step 2, Ohlsson and Lauzeningks don't go into any details of how the further year of data should be simulated. They also do not state that the way that this is done needs to be consistent with the algorithm used in step 1, although it clearly should. Ohlsson and Lauzeningks consider the method above only for calculating one-year reserve risk, in step two they therefore do not include the claims data arising from the period of exposure between time 0 and time 1. The one-year premium risk does need to be considered, and Ohlsson and Lauzeningks do discuss it (in section 3), however they discuss other methods of calculating it. It is possible to extend the actuary-in-the-box method to include premium risk. We discuss this briefly in section 3.2.5 below. In the current section we follow Ohlsson and Lauzeningks, and do not consider claims arising from the exposure between time 0 and time 1.

In their discussion of step 1 Ohlsson and Lauzeningks appear to make very few assumptions about the algorithm used to set the reserves. However, in their discussion of step 2 they consider only triangle-based models, and describe the step as simulating a new diagonal for the development triangle. The actuary-in-the-box method does not require a triangle-based model. However, to apply it we do need to assume that the algorithm used in step 1 considers claims development information in some manner, so that in step 2 it is possible to simulate the claims paid in the year in a way consistent with the algorithm used in step 1. If the reserves are set using a method that does not do this then the actuary-in-the-box method cannot be applied.

It is worth emphasising that, while the extended data set produced in step 2 is stochastic, the algorithm applied in step 3 is a deterministic algorithm; it is applied deterministically to each simulated value. The output is therefore stochastic, but all the randomness comes from simulated additional year in step 2, no further randomness is added is step 3.

The actuary-in-the-box procedure described by Ohlsson and Lauzeningks is a very general procedure, and they do not say very much about the details of how it should be implemented. However not all models used for calculating ultimate reserve risk are amenable to the actuary-in-the-box. We discuss this further in section 3.2.5 below. The most common situation where the actuary-in-the-box is applied are bootstrapped models. We now describe how to apply the method in these cases.

3.2.2 Actuary-in-the-Box and Bootstrapped Models

It is clear that the actuary-in-the-box procedure described above is a very general procedure and can be applied in any case where the claims reserving algorithm can be clearly specified, and where an additional year of data can be simulated in a way that is consistent with this algorithm.

We now describe the procedure as it would be applied to bootstrapped models of claims data such as the Mack, ODP and stochastic Bornhuetter-Ferguson models. The method is the following:

1. Carry out the bootstrap procedure
2. Extend the claims data by one year using the bootstrap output
3. Re-fit the underlying deterministic model to the extended claims data
4. Calculate the ultimate claims for the extended claims data using the underlying deterministic model

Steps 1 and 2 of the above procedure correspond to steps 1 and 2 of Ohlsson and Lauzeningks' procedure, and steps 3 and 4 correspond to step 3 of Ohlsson and Lauzeningks' procedure.

To carry out the bootstrap procedure in step 1 we need to have a well-defined model, and we therefore meet the requirement in Ohlsson and Lauzeningks' step 1 that the opening reserves be calculated using a well-defined algorithm.

In step 2, by extending the claims data using the bootstrap output we ensure that the simulated data for the new risk year is consistent with the algorithm used in step 1.

As discussed in section 2.3 the claims development result can be calculated either as the movement in the estimates of ultimate claims, or as the movement in the reserves (allowing for the claims paid in the year). In the latter case it is necessary to have a simulation of the claims paid. This means that the bootstrapped model of claims data must include the paid claims data. In the case of the Mack, ODP, or stochastic Bornhuetter-Ferguson model, this means that the model must be applied to paid claims triangles. Alternatively, the actuary-in-the-box could be applied to a method which uses both paid and incurred claims data, such as the Double Chain Ladder (Martinez-Miranda, et al., 2012) or the Munich Chain Ladder (Quarg, et al., 2008).

3.2.3 Mack's model and the ODP model

In this section we describe how to apply the Actuary-in-the-box procedure to a bootstrapped Mack or ODP model. The procedure for both models is exactly the same.

Both Mack's model and the ODP model are discussed in (Carrato, et al., 2016). Mack's model is described in section 4.1 and the application of the bootstrap is discussed in section 6.4.1. The ODP model is described in section 4.2 and the application of the bootstrap is discussed in section 6.4.2.

The claims data input for the Mack and ODP models is the triangle of claims:

$$T = \{C_{ij} : i = 1, \dots, n, j = 1, \dots, n - i + 1\}$$

We represent the triangle of observed claims in Figure 3-1

FIGURE 3-1

Step 1 of the actuary-in-the-box procedure is to carry out a bootstrap based on the model. The bootstrap procedure fills out the bottom right of the triangle to get

$$T^* = \{C^*_{ij} : i = 2, \dots, n, j = n - i + 2, \dots, n\}$$

We represent the triangle of observed claims and the bootstrap projection in Figure 3-2

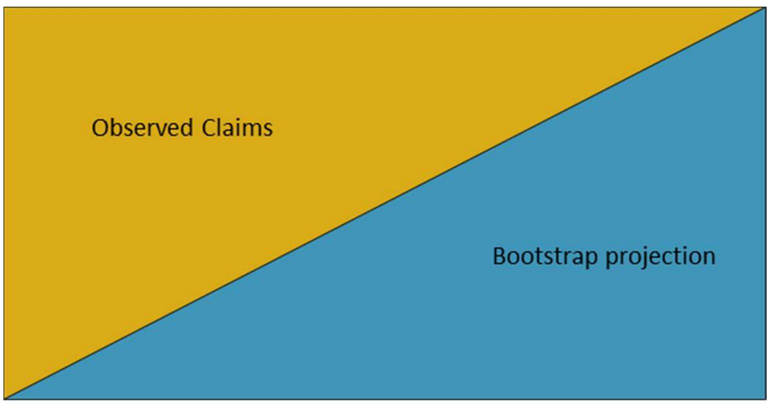

FIGURE 3-2

Step 2 is to extend the input claims data using the bootstrap output. The bootstrap output corresponding to the next year is the diagonal:

$$D^* = \{C^*_{i,n-i+2} : i = 2, \dots, n\}$$

The original triangle is extended by appending this diagonal:

$$T' = \{C'_{ij} : i = 1, \dots, n, j = 1, \dots, n-i+2\}$$

where

$$C'_{ij} = \begin{cases} C_{ij} & \text{if } j \leq n-i+1 \\ C^*_{ij} & \text{otherwise} \end{cases}$$

We represent the extended triangle in Figure 3-3

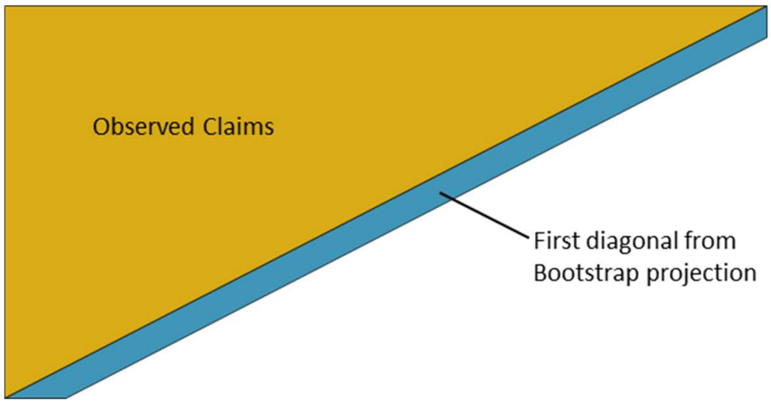

FIGURE 3-3

Step 3 is to re-fit the underlying deterministic model to the extended claims data. For both Mack's model and the ODP model, the underlying deterministic model is the chain ladder. We fit this by calculating the development factors with a slight change to the usual formulae to allow for the extra diagonal:

$$\hat{f}'_j = \frac{\sum_{i=1}^{n-j+1} C'_{i,j+1}}{\sum_{i=1}^{n-j+1} C'_{ij}}$$

Note that since $C'_{i,n-i+2}$ is stochastic, the fitted development factors $\hat{f}'_i$ are also stochastic.

Step 4 is to calculate the estimate of ultimate claims for the extended claims data using the underlying deterministic model. This just means applying the development factors in the usual way to get the estimate of the ultimate claims, taking account of the extra diagonal:

$$\hat{C}'_{in} = C'_{i,n-i+2} \prod_{k=n-i+2}^{n-1} \hat{f}'_k$$

We represent the deterministic projection of the extended triangle in Figure 3-4

FIGURE 3-4

If the triangle contains paid claims data then the closing reserve can be calculated using:

$$R_i = \hat{C}'_{in} - C'_{i,n-i+2}$$

Finally, the claims development result can be calculated.

To hopefully make the above procedure clearer, consider a single accident year. The graph in Figure 3-5 shows the cumulative claims development for a single accident year. The accident year has seen three years of development, and so the first three periods are deterministic. The development after year three is projected using a bootstrap, and so is stochastic (step 1 is to carry out the bootstrap). In the graph shown there are 1000 simulations, so 1000 possible paths for claims development after development year three.

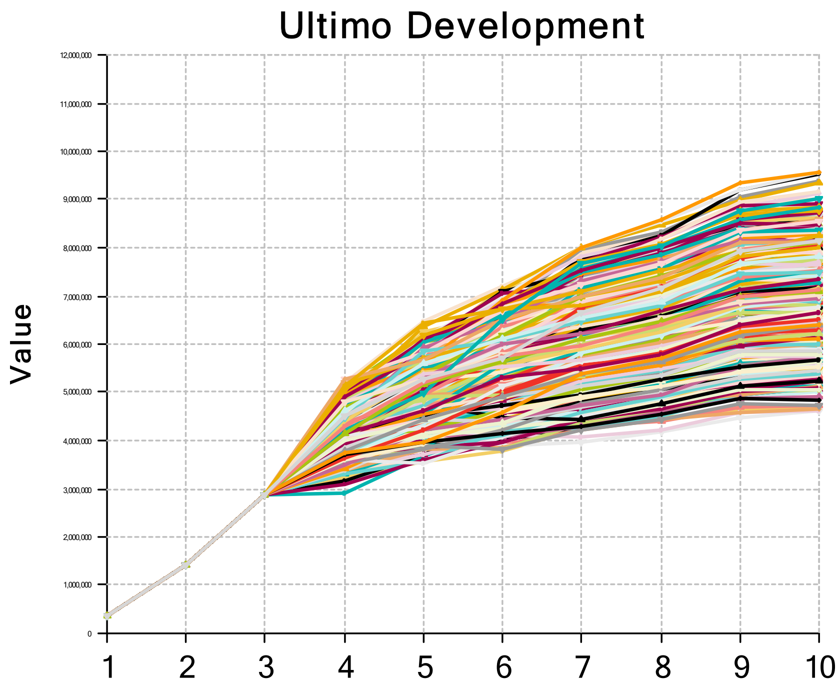

FIGURE 3-5

Step 2 is to extend the claims triangle by one further year of development, using the bootstrap output. For the single accident year under consideration, this means extending the observed claims development by one further year using the bootstrap simulations. The resulting graph is shown in Figure 3-6

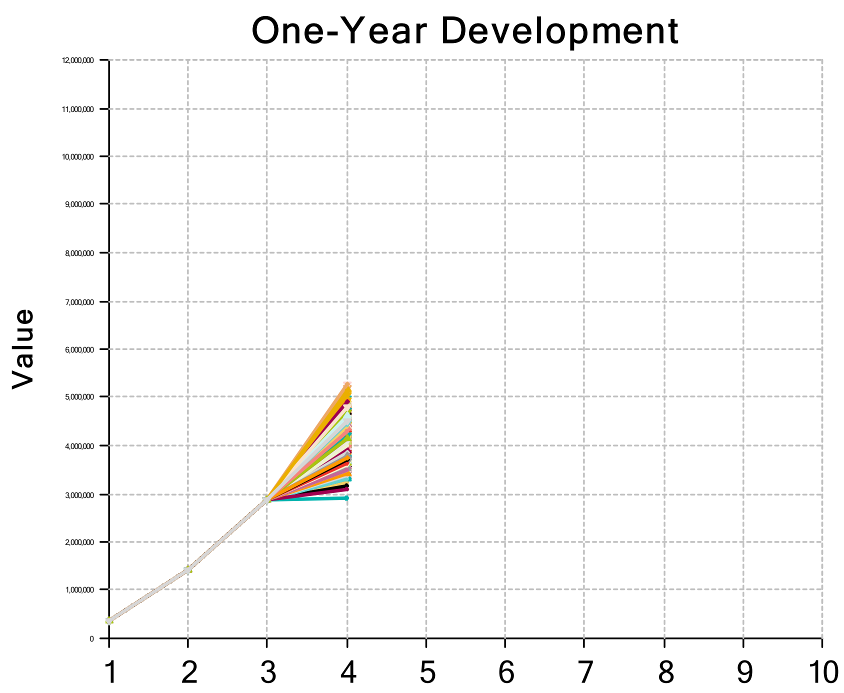

FIGURE 3-6

Step 3 is to refit the model – in this case the basic chain ladder – and then deterministically project to ultimate. When we do this we get projected future cumulative development as shown in Figure 3-7

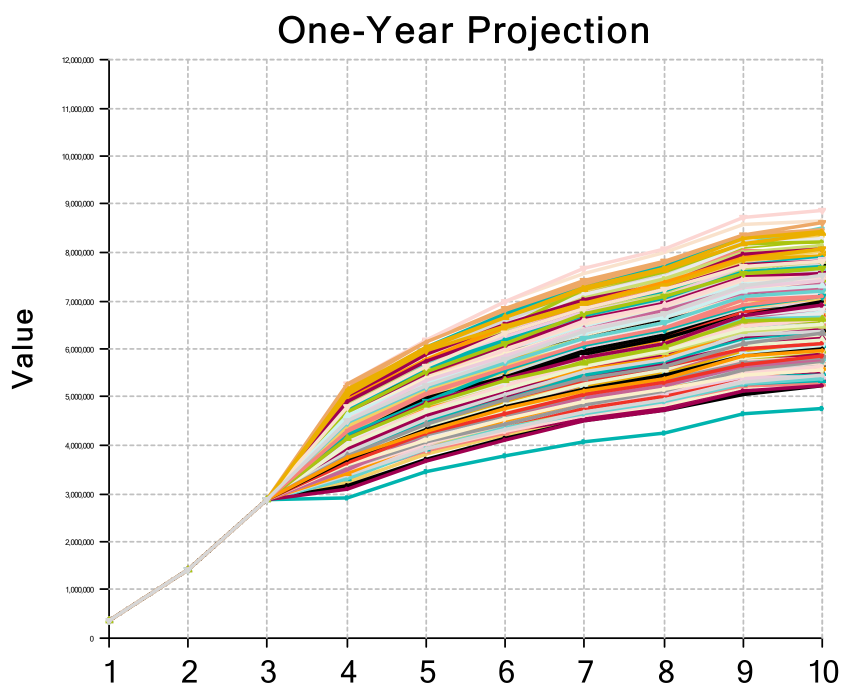

FIGURE 3-7

It is worthwhile to closely compare the Ultimate Development graph (Figure 3-5) and the One-Year Projection graph (Figure 3-7). Both show stochastic projections after development year three. However, the Ultimate Development graph shows more volatility in the future development than the One-Year Projection graph – the future development paths are more likely to cross over one another, and the range of final values is much wider. The reason for this is that in the one-year projection, the only source of randomness is the first year of development (between time three and time four), after the fourth development year the projection is deterministic, given the development between time three and time four. Whereas for the ultimate projection there is randomness at all the future development periods.

The graph in Figure 3-8 shows the PDFs for the ultimate view of the ultimate claims (in grey) and the closing estimate of the ultimate claims (in red) for the same example as the graphs above. As can be seen the ultimate distribution is more spread out than the closing distribution, while the mean of the two distributions appears to be very similar. In section 5 below we look more closely at two numerical examples.

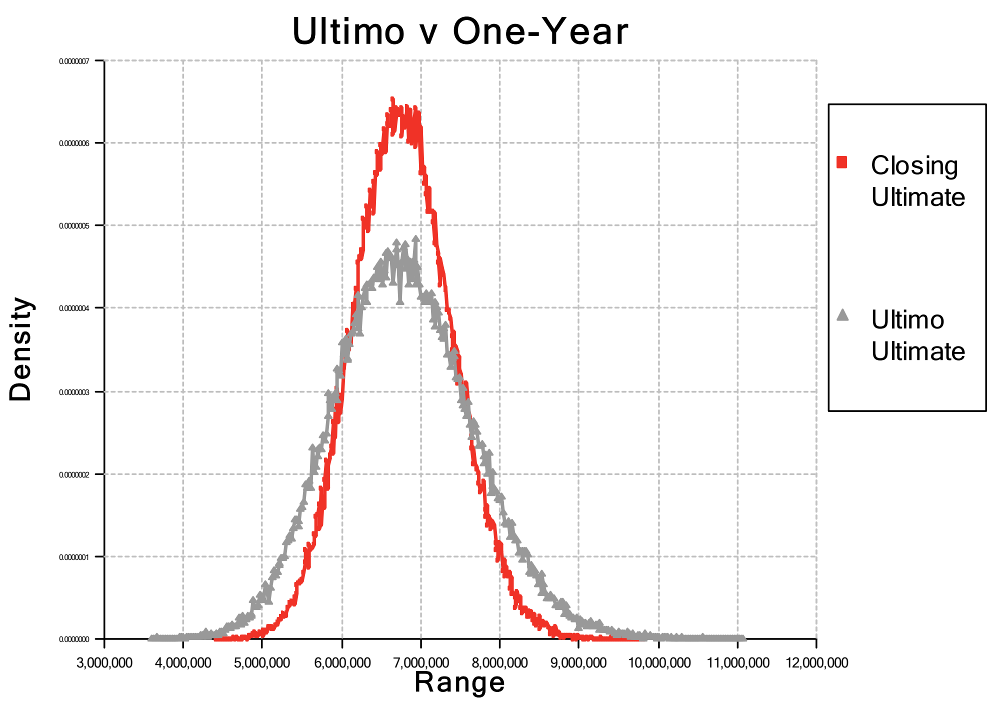

FIGURE 3-8

3.2.4 THE AMW-BORNHUETTER-FERGUSON MODEL

In this section we describe how to apply the actuary-in-the-box procedure to a bootstrapped Alai, Merz, and Wüthrich Bornhuetter-Ferguson model (see (Alai, et al., 2009) and (Alai, et al., 2011)). The procedure is similar to that for Mack's model and the ODP model, with some alterations to allow for the features specific to the Bornhuetter-Ferguson. However, there are a number of issues associated with the prior estimate of ultimate claims which we discuss below.

The Alai, Merz, and Wüthrich's stochastic Bornhuetter-Ferguson is also discussed in (Carrato, et al., 2016). The model is described in section 4.3, and the application of the bootstrap is discussed in section 6.4.3. The model is also discussed in (Scarth, 2015).

It is worth noting that Ohlsson and Lauzeningks are sceptical about applying the actuary-in-the-box procedure to a Bornhuetter-Ferguson model. In section 2.1.1 of (Ohlsson, et al., 2009) they say:

> In our opinion the quite popular Bornhuetter-Ferguson (BF) method does not fulfil the requirement of being algorithmic. The reason is that it uses some a priori loss ratios, whose calculation is outside the method. In practice, these loss ratios will probably be estimated from loss data and in order for BF to be algorithmic, this estimation should be made explicit. The [Generalised Cape Cod] could be seen as a way of making BF algorithmic and is hence preferred here.

This is a good point; however, they go on to say

> If [the actuary's subjective] judgement is still necessary, we have to find an approximate algorithm A_1, capturing the main features of the best estimate, for use in the simulation, Step 3.

It is not clear why, despite the fact that the prior estimate of the ultimate claims used in the Bornhuetter-Ferguson comes from outwith the model, that it is not possible in at least some cases to model (or find an "approximate algorithm" for) the process of setting the prior estimate of the ultimate claims, and so apply the actuary-in-the-box procedure to the AMW-BF model. We discuss this further below.

The claims data input for the AMW-BF model is the triangle of claims:

$$T = \{C_{ij} : i = 1, \dots, n, j = 1, \dots, n - i + 1\}$$

and the mean and coefficient of variation (CoV) of the prior estimate of ultimate claims.

We represent the triangle of observed claims and the prior ultimate assumptions in Figure 3-9

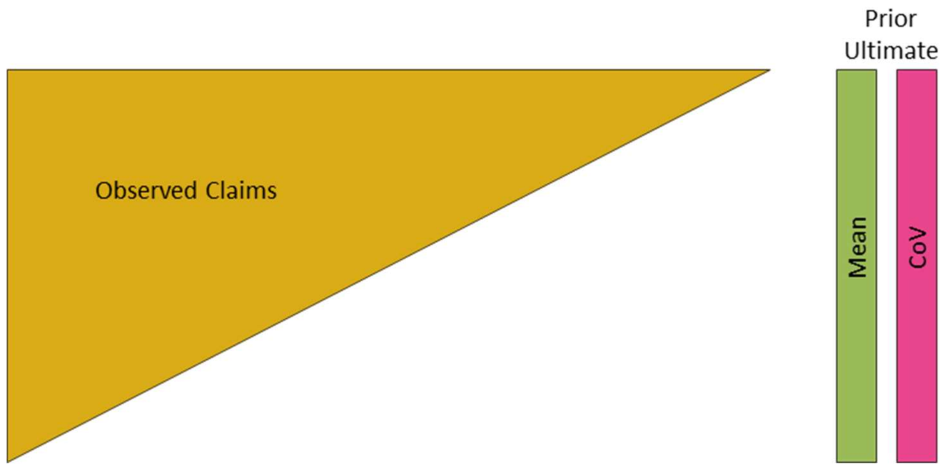

FIGURE 3-9

Step 1 of the actuary-in-the-box procedure is to carry out a bootstrap based on the model. The bootstrap procedure fills out the bottom right of the triangle to get

$$T^* = \{C^*_{ij} : i = 2, \dots, n, j = n - i + 2, \dots, n\}$$

We represent the triangle of observed claims, the bootstrap projection and the simulated prior ultimates used in the bootstrap in Figure 3-10

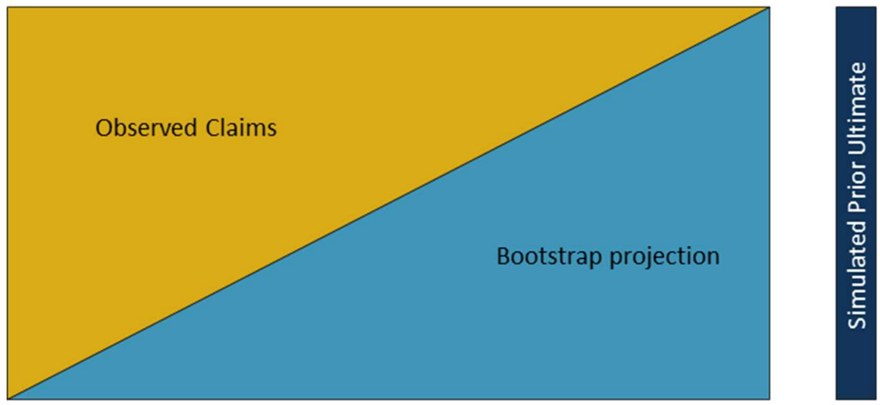

FIGURE 3-10

Step 2 is to extend the input claims data using the bootstrap output. The triangle can be extended in exactly the same way as for Mack's model and the ODP model (see above). We also need to generate new prior estimates of the ultimate claims. However, it is not clear exactly how to do this. We will discuss this further below, but for now we assume that we have simulated new prior estimates of the ultimate claims, ν'_i.

We represent the extended triangle and the simulated prior ultimates in Figure 3-11

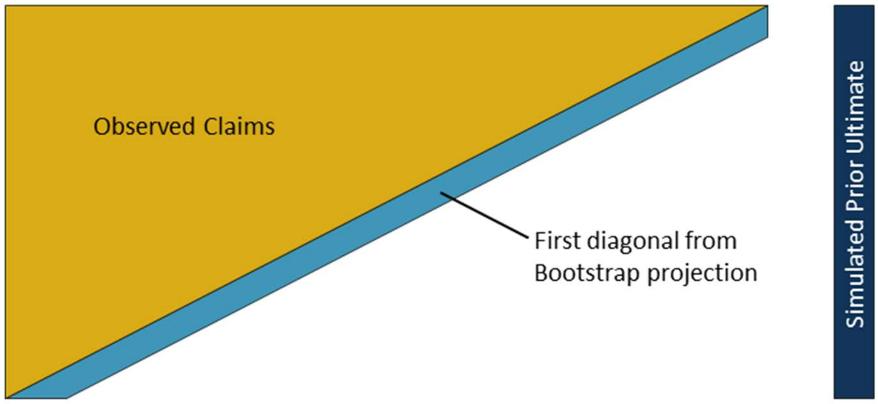

FIGURE 3-11

Step 3 is to re-fit the underlying deterministic model to the extended claims data. For the AMW-BF model this means re-fitting the chain-ladder model to calculate the cumulative development pattern. To do this calculate the development factors $\hat{f}_j'$ as for the Mack and ODP models and then calculate the cumulative development proportions using

$$\hat{\beta}_j' = \frac{1}{\prod_{k=j}^{n-1} \hat{f}_k'}$$

Note that the $\hat{\beta}_j'$ are stochastic.

Step 4 is to calculate the reserves for the extended claims data using the underlying deterministic model. The underlying deterministic model is the Bornhuetter-Ferguson method, so this just means applying that method to calculate the ultimate claims, taking account of the extra year of development, using the following formula:

$$\hat{C}_{in}' = C_{i,n-i+2}' + \left(1 - \hat{\beta}_{n-i+2}'\right) \nu_i'$$

We represent the deterministic projection of the extended triangle in Figure 3-12

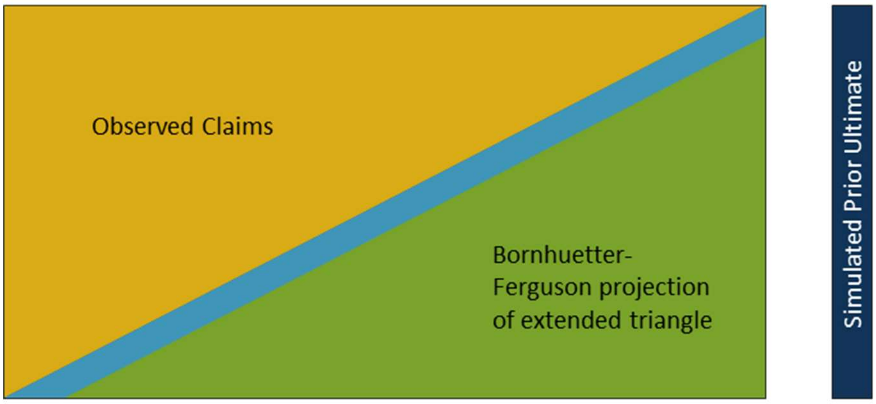

FIGURE 3-12

The closing reserve is then calculated in exactly the same way as for the Mack and ODP models.

We now discuss some of the issues with simulating the prior estimate of ultimate claims. First recall that what we are simulating is the prior estimate of the ultimate claims that will be used at the end of the next period. How this will be selected at the end of the period is a decision of the reserving actuary. Therefore, answers to the questions we raise below must necessarily come from outwith the model, and can only be answered by considering the processes the reserving actuary will follow when deciding on the prior estimate of the ultimate claims. This is the reason for Ohlsson and Lauzeningks' opinion that the Bornhuetter-Ferguson is not suitable for use with the actuary-in-the-box procedure. However, we believe that in a well-run company, with a reserving actuary following good practice, there is likely to be sufficient regularity to allow this to be modelled, to an acceptable degree of accuracy.

The first question to consider is whether the prior estimate of ultimate claims will change at all. If it is believed that the prior estimate of ultimate claims will not change over the period then the closing prior estimate of ultimate claims should be set to the opening prior estimate of ultimate claims.

If it is believed that the prior estimate of the ultimate claims will change over the period, then the next question is what distribution it should have, including what the mean and CoV should be. Two possible answers for what the mean should be are the following. It could either be equal to the mean of the prior estimate of ultimate claims used for the ultimate view bootstrap, or it could be equal to the simulated value of the prior estimate of ultimate claims used for the ultimate view bootstrap.

It is less clear what the CoV should be. The variability of the prior estimate of ultimate claims over a single period used in the actuary-in-the-box is fundamentally different from the variability of the prior estimate of the ultimate claims used in an ultimate view model. In the ultimate view model the variability is essentially parameter error in a prior estimation of the mean of the ultimate claims. In the actuary-in-the-box the variability is due to a combination of the process followed by the reserving actuary in setting the prior estimate of the ultimate claims, changes in the information considered by the actuary, and changes in the actuary's judgement.

A further choice that needs to be made is the distribution used. Possible choices are normal, gamma, or lognormal.

The lack of clarity over how to simulate the closing prior estimate of ultimate claims is an unsatisfactory feature of the AMW-BF model. How it should be done can only be answered by considering the processes that are likely to be followed by the reserving actuary in selecting what the prior estimates will be. The simplest and most easily communicated assumption is that the prior estimates will not change, and so in the absence of any strong reason to make a different assumption this is perhaps the assumption that should be used.

The AMW-BF model assumes that the prior estimates of ultimate claims are independent of each other and of the data in the claims triangle. This is a very strong assumption, and likely does not hold in reality. In particular if the prior estimates used in the Bornhuetter-Ferguson method are periodically reviewed then it seems unlikely that they will be completely independent of each other or of the data in the claims triangle. If the model is being bootstrapped this is not a fatal problem as dependence assumptions can be built into the bootstrap and the actuary-in-the-box process. However, the problem then becomes to specify and parameterise the dependence structure used. This cannot be done without considering the processes that are likely to be followed by the reserving actuary in selecting the prior estimates. This is another consideration when deciding how to model the closing prior estimates, and another unsatisfactory feature of the AMW-BF model.

3.2.5 Discussion of method

The actuary-in-the-box method is a very general procedure with a wide range of applicability. It can be applied to a wide variety of models, and it is important not to ascribe the limitations of those models to the actuary-in-the-box method. However, the actuary-in-the-box method does have its limitations.

A fundamental limitation of the actuary-in-the-box method is that it cannot adequately capture the judgement used by a real-world actuary in setting reserves, or many of the other subtle aspects of a complex reserving process. This is particularly a problem with volatile classes of business or classes where there is little claims data and so where the reserving actuary is likely to apply a lot of judgement, and where the actuary-in-the-box can sometimes give completely unreasonable results. Unfortunately, this is a limitation that cannot be overcome by any method of estimating the one-year risk, short of developing an artificial intelligence capable of replacing a human actuary – discussion of whether this is possible or not is beyond the scope of this paper.

Another fundamental limitation is that the actuary-in-the-box method cannot make use of information not contained in the claims data used by the underlying model, which the reserving actuary would likely consider when setting reserves. Such information would include, for example losses which have not yet breached the layer of cover, or information about claims other than the case reserves which are included in the incurred claims data. It also includes information arising from changes in the claims environment, such as changes in claims inflation. These are likely to give rise to the largest changes in the claims estimate. The actuary-in-the-box might therefore underestimate the potential extent of changes in the reserves over one-year.

In some situations, it might be necessary to output, the closing reserve or claims paid during the year. For example, if projecting the balance sheet in a capital model. If the actuary-in-the-box is based on a Mack, ODP or AMW-BF model then this means that the underlying model has to use paid claims triangles. Whereas the reserving actuary might use incurred claims triangles when setting the reserves. This is particularly likely to be the case with long-tailed lines of business. Alternative models which use both paid and incurred claims data, such as the DCL or MCL, are not widely used when setting reserves. The use of paid claims data, when incurred data is used in setting the reserves means that, first, a different model is being used by the actuary-in-the-box and the real-world actuary, and second that information in the incurred claims data is not used by the actuary-in-the-box, when the information is being used by the real-world actuary.

The actuary-in-the-box method is computationally relatively expensive, and is relatively complex to implement. However, the actuary-in-the-box is often used on top of a bootstrapped model, and it is relatively less computationally expensive than the underlying bootstrap, and also relatively straight-forward to build on top of a bootstrapped model. Although it does require coding skills, and cannot easily be built in a spreadsheet. The method is also available in several popular reserving software products.

The relatively high computational cost, together with the fact that the actuary-in-the-box can sometimes give unreasonable results means that it is often not suitable for use in the main part of the calculation kernel of a capital model. Instead the actuary-in-the-box is usually run outside the main part of the calculation kernel to estimate parameters which are then fed into the main calculation kernel. These parameters can be reviewed for reasonableness before being fed into the main calculation kernel. Emergence patterns are one of the possible ways to feed this information into the main calculation kernel.

The actuary-in-the-box method cannot be applied to models where claims development is not modelled, for example if the reserves are modelled in aggregate, as there is no way to produce the claims paid during the year (without extending the model).

The actuary-in-the-box method cannot be applied to a model that has parameters that are calculated outwith the model. For example, if the reserves are modelled in aggregate with the mean and standard deviation set using an external benchmark, then there is no way to meaningfully re-fit the model as step 3 of Ohlsson and Lauzeningks method requires.

Frequency-severity models are sometimes used to model reserves for large claims. However, in their usual form, these models do not consider information about claims development. This is in contrast to models of triangles of claims data which inherently contain information about claims development. Furthermore, the parameters come from outwith the model, unlike the Mack, or ODP where the parameters are fitted using a definite algorithm to a defined set of data. Because of this it is not possible to apply an actuary-in-the-box procedure without extending the model. The model would have to be extended to include claims development data and include a definite algorithm for fitting the parameters to that data. A model for large claims which considers claims development information is the Murphy-McLennan model (Murphy, et al., 2006). However, while this model is sometimes used, it is not the most commonly used model.

The actuary-in-the-box method can contribute to the calculation of premium risk. However, its uses are limited, and it is rarely used in practice for this purpose.

As we discussed in section 2.3 if the claims reserves are set on a best estimate basis, then the sequence of estimates of the ultimate claims is unbiased. This implies that the expected value of the claims development result is zero. However, there are some situations in which this does not happen. In particular if the actuary-in-the-box method is applied to the AMW-BF model then the expected value of the claims development result is not zero. The reason for this is that the AWM-BF model assumes an ODP model for the triangle of observed claims, but then via the prior estimate of ultimate claims, assumes a mean for the future incremental claims different from that implied by the chain ladder. This difference means that the expected value of the claims development result is not zero. Specifically, if the prior estimate of ultimate claims is greater than the chain ladder estimate of the ultimate claims then the expected value of the claims development result is negative, as the loss relative to the chain ladder implied by the greater prior estimate of ultimate claims emerges. Similarly, if the prior estimate of the ultimate claims is less than the chain ladder estimate then the expected value of the claims development result is positive. For similar reasons, in the Mack or ODP model, if a curve is fitted to the chain ladder development factors, then the expected value of the claims development result is not zero.

The sequence of estimates of the ultimate claims in the AMW-BF model and a Mack or ODP model with a curve fit, fails to have the martingale property because the model of the past and future claims is not the same. We say that these models are not “time-symmetric”. More generally, if there is no bias in the claims reserving algorithm, the algorithm is time-symmetric, and the new year is simulated consistently with the algorithm, then the expected value of the claims development result will be zero.

One very nice feature of the actuary-in-the-box method is that it can be iterated to calculate the distribution of the claims development result over several periods, and indeed all the way to ultimate. This can be used to show how the ultimate risk emerges over time, which can give valuable insight into the risk. For long tail classes it is sometimes the case that very little risk emerges over the first years of development, instead most of the risk emerges over the middle years of development. This can lead to the paradoxical result that on a one-year basis the most recent cohorts are apparently not very risky, whereas on an ultimate basis they are very risky. This is sometimes called the Time Horizon Paradox.

One nice feature of the claims development result is that the sum of the variances of the claims development results up to ultimate is equal to the variance of the distribution of the ultimate view of the risk. This result is quite general and follows because the time series of estimates of the ultimate claims forms a martingale, and in any martingale the sequence of differences has this property. See section 9 appendix A for the technical details of this result. Note this does not hold if the expected value of the claims development result is zero, as happens with the AMW-BF model.

3.3 Emergence Patterns

3.3.1 Method introduction

"Emergence Patterns" are a family of methods rather than a single method. All the different versions are based on the simple idea of applying a scaling factor to the ultimate risk to derive the one-year risk. Emergence patterns can be applied in cases where the actuary-in-the-box cannot be applied, or does not give reasonable results. However, the problem then becomes how to parameterise the emergence patterns, and which version of the family to apply. Unfortunately, it is not always clear how to solve these problems, and usually a degree of judgement is required.

The scaling of the distribution of the ultimate view of risk should only affect the volatility, not the mean. This is because, as discussed in section 2.3, the sequence of best estimates is unbiased, so the expected value of future best estimates is equal to the current best estimate, which is the mean of the distribution of the ultimate view of risk. The general formula applying emergence factors therefore has the form:

$$\hat{X} = \alpha(X - E[X]) + E[X]$$

Where X denotes the distribution of the ultimate view of risk, α denotes the emergence factor applied, and $\hat{X}$ denotes the one-year distribution. The emergence factor α takes a value between zero and one. It is easy to see that $E[\hat{X}] = E[X]$ and $\sigma(\hat{X}) = \alpha\sigma(X)$. The graph in Figure 3-13 gives a visual representation of this formula. The red line shows the PDF of X and the grey line shows the PDF of $\hat{X}$. As can be seen the one-year distribution is less spread out than the distribution of the ultimate view of risk. The emergence factor α controls how spread out the one-year distribution is, relative to the distribution of the ultimate view of risk. The closer α gets to one, the closer the one-year distribution gets to the distribution of the ultimate view of risk. As α gets closer to zero, the one-year distribution becomes more tightly concentrated around the mean. The means of the two distributions are identical, although because of the positive skew the mode of the one-year distribution is slightly greater than the mode of the distribution of the ultimate view of risk. Compare the graph in Figure 3-13 with the graph in Figure 3-8 at the end of section 3.2.3.

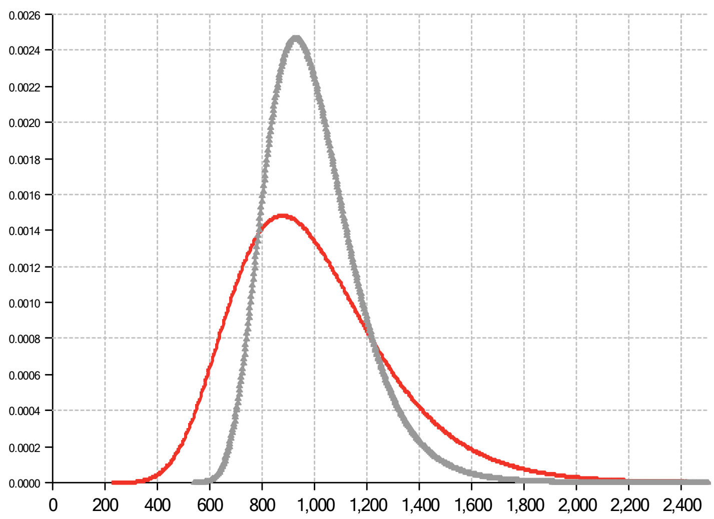

FIGURE 3-13

There are different varieties of emergence pattern. The first difference is determined by which distribution of the ultimate view of risk is used. Either the distribution of the ultimate claims can be used, or the distribution of the claims outstanding at time 1 (so not including the claims paid during the year). The second difference is determined by whether the pattern is expressed as a risk decay pattern or as a life-time risk emergence pattern. The difference between these is that a risk decay pattern is an answer to the question: of the risk remaining how much will emerge over the next year? Whereas a life-time risk emergence pattern is an answer to the question: what proportion of the life-time risk (i.e. the total risk from the start of the relevant origin period to the final claim payment) will emerge during the n^{th} development year? We explain this in more detail in the sections below, and in section 10 appendix B. Different origin period bases could be used (either accident periods, underwriting periods, or reporting periods), and these will give rise to different emergence patterns.

Another possible variation in emergence patterns is the level of granularity that they are applied at, in particular they can be applied to reserves by origin year (with a different emergence factor applied to each origin year), or they can be applied to reserves aggregated over origin years.

Yet another possible variation in emergence patterns is the statistic to which the scaling factor is applied. The natural statistic to apply the scaling factor to is the standard deviation, and this is what we consider in what follows. However, in principle it is possible to apply the scaling factor to another statistic, such as a specified percentile. We discuss this briefly in section 10 appendix B.

One of the biggest challenges with emergence patterns is calibration. A common way to do this is to use the actuary-in-the-box. Emergence patterns are often used in cases where the actuary-in-the-box method cannot be applied. In this case the actuary-in-the-box can be used to calibrate benchmark emergence patterns, which can then be used as the basis for emergence patterns to be applied, or can be applied directly. In section 3.2.5 above we made the point that, due to the fact that it is relatively computationally expensive, the actuary-in-the-box is not suitable for use in the main part of the calculation kernel of a capital model. Instead the actuary-in-the-box is run to create parameters for input into the calculation kernel. The form of these inputs is often an emergence pattern.

In principle emergence patterns could be stochastic. However, there are problems with stochastic emergence patterns, and so we do not consider them in the main body of the paper. We briefly discuss them, and the problems with them in section 10 appendix B.

3.3.2 DIFFERENT VARIETIES OF EMERGENCE PATTERN

In this section we attempt to give an intuitive description of the different varieties of emergence pattern that we introduced in the previous section, and of the relationships between them. Very little material on emergence patterns has appeared in the literature. We therefore develop this material in more detail in section 10 appendix B.

The first case we consider is perhaps the most intuitive way to consider risk emergence. In this case we consider a single origin period from its start to the final payment, the risk over this whole period we call the “life-time risk”. As we move through time, we learn more about the total amount to be paid, and more of the claims are actually paid. The risk therefore emerges as the origin period develops. At the start of the origin period none of the risk has emerged, by the time of the final payment all of the risk has emerged. We can consider the proportion of the total risk that has emerged at each point in time between start of the origin period and the final payment. This starts at zero, and increases until it takes the value one by the time of the final payment, as shown in the graph in Figure 3-14.

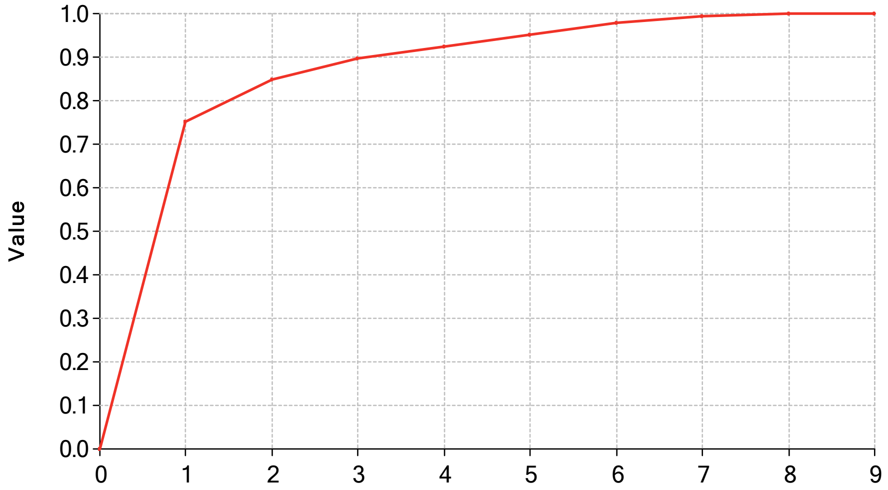

FIGURE 3-14

We call this the "Cumulative ultimate life-time risk emergence". We denote these emergence factors by α_k where k is the number of periods of development (i.e. the horizontal axis of the graph). So $0 = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n = 1$.

However, in practice, if we are applying emergence factors to individual origin periods then we will be applying them to partially developed periods, and therefore need to consider not the life-time ultimate risk, but the remaining ultimate risk. We therefore consider how, for a partially developed origin period, the remaining risk emerges between the current time (which is after the start of the origin period) and the time of the final payment. As in the case of the life-time risk emergence as we move through time, we learn more about the total amount remaining to be paid. The proportion of the remaining risk that has emerged at each point in time therefore increases from zero to one as shown in the graph in Figure 3-15.

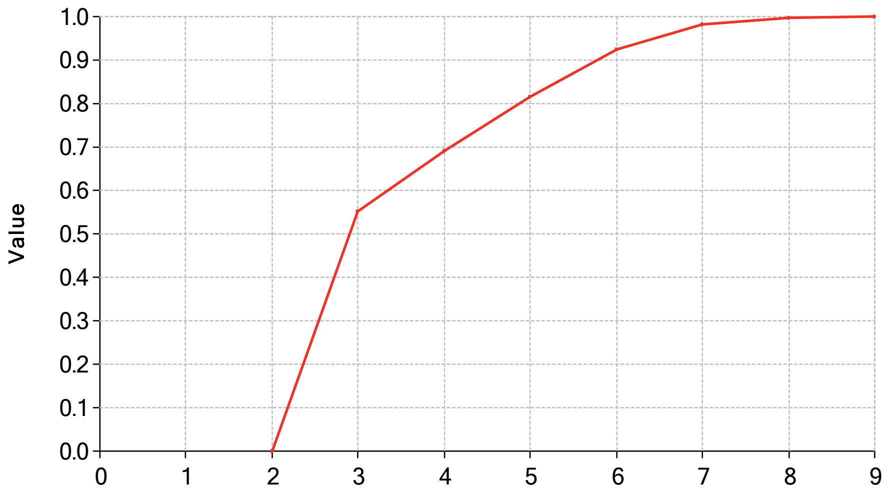

FIGURE 3-15

We call this the "Conditional cumulative ultimate emergence". We denote these emergence factors by $\alpha_{t,k}$ where t is the number of periods of prior development, and k is the number of periods of future development. So the graph above shows the values of $\alpha_{2,0}, \alpha_{2,1}, \dots, \alpha_{2,7}$. More generally $0 = \alpha_{t,0} \leq \alpha_{t,1} \leq \cdots \leq \alpha_{t,n-t} = 1$.

However, we would like to apply emergence factors to estimate the one–period risk. We are therefore interested in the proportion of remaining risk that will emerge over the next year for a succession of origin periods. We therefore need to consider the succession of conditional cumulative ultimate emergence graphs as shown in Figure 3-16.

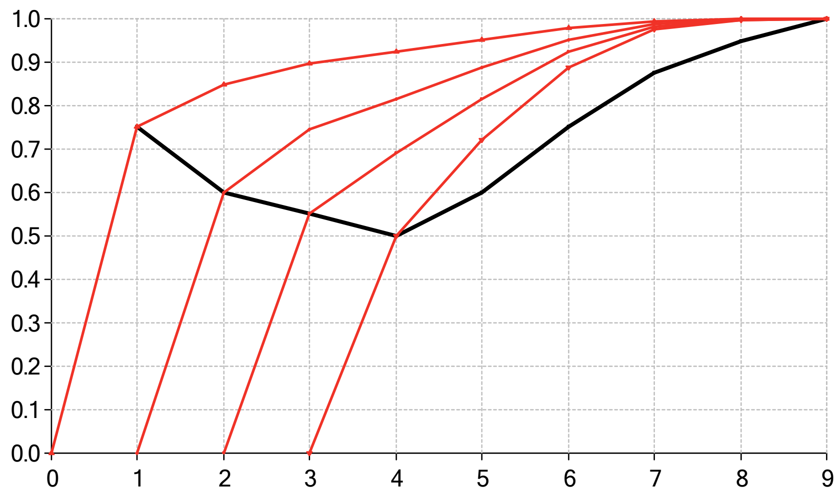

FIGURE 3-16

The emergence pattern to apply to the successive origin periods to get the one-period view of risk is that shown by the black line. In this case the emergence factors do not need to form an increasing sequence, although the final value does need to be one. We call this the "ultimate one-period risk-decay pattern". Using the notation from above the emergence pattern indicated by the black line on the graph consists of the factors $(\alpha_{0,1}, \alpha_{1,1}, \dots, \alpha_{n,1})$.

So far, we have discussed emergence factors applied to the distribution of ultimate claims. We now discuss an issue with this, and introduce another interpretation of emergence factors to address the issue. At time t the remaining risk includes risk stemming from claims paid between time t and time $t+1$, but by time $t+1$ these claims will have been paid, and so the amount paid will be known with certainty – all the risk associated with payments between time t and time $t+1$ will have emerged by time $t+1$. However if we apply an emergence factor to the distribution of ultimate claims then we are applying an emergence factor to these payments, and we are assuming that only a proportion of the risk emerges before time $t+1$. The diagram in Figure 3-17 shows this.

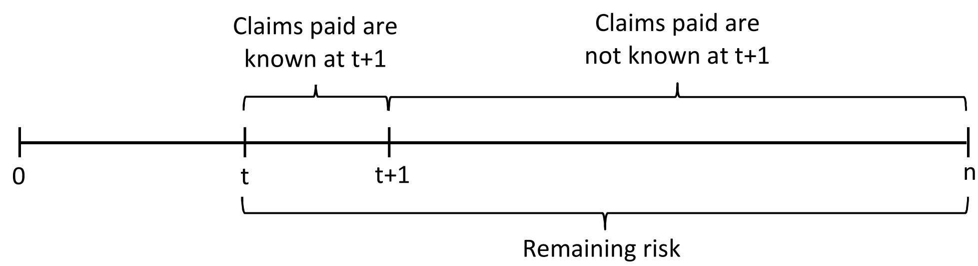

FIGURE 3-17

We therefore need to adjust the definition of the emergence factors that we apply, so that we take account of this. Emergence factors should be applied to the distribution of the ultimate view of risk of claims paid after time $t+1$, and no adjustment should be made to the distribution of claims paid between time t and time $t+1$. We call this the "conditional outstanding emergence pattern".

As with the ultimate emergence factors we would like to use them to estimate the one-period risk. We therefore need to consider the succession of conditional outstanding emergence patterns for accident years with successive periods of prior development. Doing so, in exactly the same way as with the ultimate emergence factors, gives us an "outstanding one-period risk-decay pattern".

We use β to denote outstanding emergence factors, and we use subscripts in the same way and with the same meaning as for the ultimate emergence factors denoted by α. So β_k denotes the outstanding life-time emergence factor to apply to estimate the distribution of closing claims reserves after k years of development. And $\beta_{t,k}$ denotes the conditional outstanding emergence factor to apply to an origin year that has already seen t years of prior development to estimate the distribution of closing claims reserves after a further k years of development. Similarly to the ultimate emergence factors, the emergence pattern to apply to successive accident years to estimate the one-year risk would be $(\beta_{0,1}, \beta_{1,1}, \dots, \beta_{n-2,1})$. Note that $\beta_{n-1,1}$ is not defined as there are no outstanding claims at time point n.

We now collect the different varieties of emergence pattern discussed above into a more systematic framework. The three main characteristics that differ are the following:

1. Which distribution of the ultimate view of risk is used
2. Whether there has been any prior development
3. Whether expressed on a life-time or risk-decay basis

There are two possibilities for each characteristic, but as we will see some combinations are not possible, and there are in total six possibilities. We now describe these.

For the first question, the distribution of the ultimate view of risk used can either be the distribution of ultimate claims, or the distribution of the claims outstanding at the end of the period. We describe the first case with "Ultimate", and the second case with "Outstanding".

For the second question we can either have no prior development, or some prior development. We describe the first case with "Unconditional", and the second case with "Conditional". This is because if there has been some development then the emergence factors are conditional on the development so far.

For the third question we can either have "Life-time", or "Risk-decay". This makes no difference to the emergence factors calculated, the difference is in how the emergence factors are collected to form emergence patterns.

This gives us eight possibilities, which we list in Table 3-3. We have already seen five of these, and two do not make sense.

	1	2	3	Emergence Factor	Emergence Pattern
1	Ultimate	Unconditional	Life-time	α_k	$(\alpha_0, \dots, \alpha_n)$
2	Ultimate	Unconditional	Risk-decay	Does not make sense	
3	Ultimate	Conditional	Life-time	$\alpha_{t,k}$	$(\alpha_{t,0}, \dots, \alpha_{t,n-t})$
4	Ultimate	Conditional	Risk-decay	$\alpha_{t,k}$	$(\alpha_{0,1}, \alpha_{1,1}, \dots, \alpha_{n-1,1})$
5	Outstanding	Unconditional	Life-time	β_k	$(\beta_0, \dots, \beta_{n-1})$
6	Outstanding	Unconditional	Risk-decay	Does not make sense	
7	Outstanding	Conditional	Life-time	$\beta_{t,k}$	$(\beta_{t,0}, \dots, \beta_{t,n-t-1})$
8	Outstanding	Conditional	Risk-decay	$\beta_{t,k}$	$(\beta_{0,1}, \beta_{1,1}, \dots, \beta_{n-2,1})$

TABLE 3-3

Note that the emergence factors for lines 3 and 4, and lines 7 and 8, are the same. The difference is in how they are combined to form the emergence pattern. Also note that unconditional emergence factors can be considered as a special case of conditional emergence factors, as $\alpha_k = \alpha_{0,k}$ and $\beta_k = \beta_{0,k}$. Unconditional Risk-decay emergence patterns do not make sense, so number 2 and 6 on the list are not filled in.

In the above exposition we have discussed the emergence factors as representing a proportion of the ultimate risk without specifying how the risk was measured. As discussed in section 3.3.1 above applying an emergence factor scales the standard deviation of the ultimate risk down by that factor. Because of this standard deviation is the natural risk measure to use for emergence factors, and this is what we use when we develop the theory in section 10 appendix B. In this case we define an emergence factor to be the ratio of the standard deviations of the relevant distributions. It is possible to use other risk measures, and we briefly discuss this in section 10 appendix B.

In general, the emergence factors discussed above vary by origin year. We can ensure that emergence factors are constant across origin years by assuming that corresponding ultimate risk and one-period distributions vary in the same proportion to the same function of the mean across origin periods. This works as the ultimate risk and one-period distributions have the same mean, and the emergence factor is defined as the ratio of the two standard deviations. For example, if the ultimate risk distributions had the same coefficient of variation (CoV) across origin years, and the one-period distributions had the same CoV across origin periods then the emergence factor would be the ratio of the one-period CoV to the ultimate risk CoV. This is discussed in more detail in section 10 appendix B.

In general, the emergence factors discussed above are stochastic. We discuss this more in section 10 appendix B. If we assume that the ultimate emergence factors are deterministic then we can derive the following relationship among them:

$$1 - \alpha_{t+k_1,k_2}^2 = \frac{1 - \alpha_{t,k_1+k_2}^2}{1 - \alpha_{t,k_1}^2}$$

Intuitively this formula can be understood as showing how to calculate "forward" emergence factors, in a way similar to how forward interest rates are calculated. The equation shows a relationship between the factors $1 - \alpha_*$ because it expresses a relationship between the risk which remains at various time periods. The formula says that the risk which emerges between time $t + k_1$ and time $t + k_1 + k_2$ (given by α_{t+k_1,k_2}) can be calculated by taking the risk which emerges between time t and time $t + k_1 + k_2$ (given by α_{t,k_1+k_2}) and "factoring out" the risk which emerges between time time t and time $t + k_1$ (given by α_{t,k_1}). This is illustrated in the diagram in Figure 3-18.

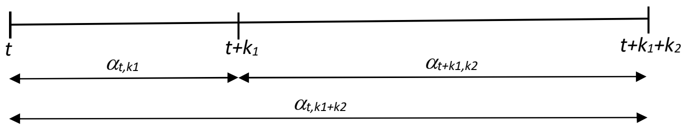

FIGURE 3-18

No similarly elegant relationship exists among the outstanding emergence factors, due to the complicating need to consider the claims payments.

So far in this sub-section we have discussed emergence factors which are applied to reserves by origin year. It is also possible to consider emergence factors applied to reserves aggregated over origin years. Similar to above we can consider ultimate emergence factors or outstanding emergence factors. However, a significant problem with aggregate emergence factors is that they confound risk emergence with claims run-off.

3.3.3 PARAMETERISATION OF EMERGENCE PATTERNS

In this section we discuss how one might parameterise emergence factors and emergence patterns. This is not an easy thing to do, and there is no generally accepted way of doing it. We describe two methods of parameterising emergence patterns, we discuss the limitations of these methods, and we discuss some general reasons why parameterising emergence patterns is difficult.

Recall that there are different varieties of emergence factors and emergence patterns. In this section we discuss parameterising risk-decay emergence patterns (ultimate and outstanding). We also discuss parameterising aggregate emergence factors.

The first method is to use the actuary-in-the-box. In section 3.3.1 we said that emergence patterns can be applied in cases where the actuary-in-the-box cannot be applied, or does not give reasonable results. It might therefore seem pointless to use the actuary-in-the-box to parameterise an emergence pattern. There are however two cases where it might make sense. In section 3.2.5 we made the point that, due to the relatively high computational cost of the actuary-in-the-box, and the fact that it sometimes gives unreasonable results, it is often not suitable for use in the main part of the calculation kernel of a capital model. The actuary-in-the-box is usually run outside the main part of the calculation kernel to estimate parameters which are then fed into the main calculation kernel. These parameters can be reviewed for reasonableness before being fed into the main calculation kernel. Emergence patterns are one of the possible ways to feed this information into the main calculation kernel. It also might make sense to use the actuary-in-the-box to parameterise an emergence pattern for a similar class where we have enough data to use the actuary-in-the-box, and then appropriately adjust the pattern for the class we wish to apply it to. This latter approach was explored in (England, et al., 2012) with a view to finding some general behaviour that could be used to guide the process, but with limited success.

We now describe how to parameterise risk-decay emergence patterns using the actuary-in-the-box. In this section we try to do this with minimal use of equations, and also to make the discussion more concrete we describe the method in the context of the actuary-in-the-box applied to a bootstrapped chain-ladder model (either Mack, or ODP). A detailed discussion of the calculations can be found in appendix B in section 10.

First consider the ultimate one-year emergence factors, $\alpha_{t,1}$. These are defined as the proportion of the ultimate risk for an origin year with t years of development that will emerge over the next year. Bootstrapping the model gives us the distributions we need to calculate the ultimate risk for all prior origin years. The actuary-in-the-box gives us the distribution of the closing estimate of ultimate claims that we need to calculate the one-year risk for all prior origin years. The ultimate one-year emergence factors are then the ratio of the one-year risk to the ultimate risk for each origin year.

More precisely suppose we have n origin years $i = 1, \dots, n$. Let ${}_iU$ denote the ultimate risk distribution of ultimate claims for origin year i, and let ${}_iU_1$ denote the distribution of the closing estimate of ultimate claims. Then, if we're using standard deviation as our risk measure, the ratio of the standard deviation of ${}_iU_1$ to the standard deviation of ${}_iU$ gives the ultimate one-year emergence factor $\alpha_{n-i+1,1}$, where $i = 2, \dots, n$. This then gives us the ultimate one-year risk decay emergence pattern

$$(\alpha_{1,1}, \dots, \alpha_{n-1,1})$$

In a similar way we can use a combination of Mack's model and the Merz-Wüthrich formula to parameterise the ultimate one-year risk decay emergence pattern. Mack gives us a formula for the standard error of the estimate of the claims reserve for each origin year. This corresponds to the standard deviation of ${}_iU$ described above. The Merz-Wüthrich formula can be used to calculate the standard error of the claims development result over the next year for each origin period. This corresponds to the standard deviation of ${}_iU_1$ described above. We can therefore take the ratio of the Merz-Wüthrich formula to Mack's formula for origin year i to calculate the emergence factor $\alpha_{n-i+1,1}$, where $i = 2, \dots, n$. As above this then gives us the ultimate one-year risk decay emergence pattern.

Now consider the outstanding one-year emergence factors, $\beta_{t,1}$. These are defined as the proportion of the ultimate risk of claims paid after the first future year for an origin year with t years of development that will emerge over the next year. The bootstrapped model gives us the distributions we need to calculate the ultimate risk for all prior origin years. The actuary-in-the-box gives us the distribution of the closing reserves that we need to calculate the one-year risk for all prior origin years. The outstanding one-year emergence factors are then the ratio of the one-year risk (calculated from the closing reserves) to the ultimate risk of claims paid after the first future year for each origin year.

More precisely suppose we have n origin years $i = 1, \dots, n$. Let ${}_iOC$ denote the ultimate risk distribution of claims paid after the first future year for origin year i, and let ${}_iR_1$ denote the distribution of the closing reserves. Then, if we're using standard deviation as our risk measure, the ratio of the standard deviation of ${}_iR_1$ to the standard deviation of ${}_iOC$ gives the outstanding one-year emergence factor $\beta_{n-i+1,1}$, where $i = 3, \dots, n$. This then gives us the ultimate one-year risk decay emergence pattern

$$(\beta_{1,1}, \dots, \beta_{n-2,1})$$

Note that for both the ultimate and outstanding risk-decay emergence pattern we are missing the first emergence factor, $\alpha_{0,1}$ and $\beta_{0,1}$ respectively. This is unavoidable as the most recent origin year in a claims triangle has already seen one year of development.

There is a further issue, when we apply the emergence factors we apply them to ultimate risk distributions to get estimates of one-year distributions. As a result of doing this the correlations between the estimated one-year distributions is the same as the correlation between the corresponding ultimate risk distributions used to derive them. In general, this is not the same correlation as that between the one-year distributions got from the actuary-in-the-box. Therefore, the standard deviation of the claims development result totalled over all origin years will be different from that got from the actuary-in-the-box. We might therefore want to adjust the emergence factors got using the above method of allow for this.

Aggregate emergence factors can be calibrated in a similar way. However, instead of using random variables split by origin periods, you would use the corresponding random variables aggregated over origin periods. So, for example, if parameterising ultimate one-year aggregate emergence factors, then you would consider the ultimate risk distribution of aggregate ultimate claims, $U = {}_1U + \cdots + {}_nU$, and the distribution of the closing estimate of aggregate ultimate claims, $U_1 = {}_1U_1 + \cdots + {}_nU_1$, and calculate the emergence factor as the ratio of the standard deviations.

We now describe another method for estimating emergence factors. This method depends only on having ultimate risk distributions for ultimate claims for several consecutive origin years, and does not depend on the actuary-in-the-box. The basic idea is to compare the ultimate risk for adjacent origin years, and assuming a degree of similarity between the years, back-out the one-year risk emergence implied by the two ultimate risks. This method is less used in practice than the one described above based on the actuary-in-the-box, but it is nonetheless an interesting approach.

Suppose that we have n origin years $i = 1, \ldots, n$, and we have already estimated ultimate risk distributions for the ultimate claims ${}_iU$ for each origin year. We estimate the ultimate one-year emergence factors using the coefficient of variation (CoV) of the ultimate distributions. The CoV is the ratio of the standard deviation to the mean:

$$CoV_t({}_iU) = \frac{\sigma_t({}_iU)}{E_t[{}_iU]}$$

We then estimate the ultimate one-year emergence factors using

$$\alpha_{n-i+1,1} = \sqrt{1 - \left(\frac{CoV_t({}_{i-1}U)}{CoV_t({}_iU)}\right)^2}$$

The full derivation of this equation is in appendix B (section 10.7). The key assumption in deriving it is that the CoV of origin year $i - 1$ is good estimate for the ultimate risk CoV of origin year i in one year's time. We also discuss a slight generalisation of the formula in section 10 appendix B.

We now discuss why, in general, parameterising emergence factors is difficult, and the consequent limitations of the above methods of parameterisation.

All three methods described above depend on other methods. The first method depends on the actuary-in-the-box, the second method depends on Mack and Merz-Wüthrich, and the third method requires a method for estimating ultimate risk. In all three cases if no appropriate method can be applied to the claims data then no emergence factors can be calculated. An alternative is to apply the methods to claims data from a class which is believed to have similar characteristics to the class to which the emergence patterns will be applied. Another similar alternative is to use benchmark emergence patterns. However, there are issues with this, as we discuss below.

There are more fundamental problems with the methods described above for parameterising emergence factors. We first describe these in the context of using the actuary-in-the-box, we then discuss the implications of this for any method which attempts to parameterise emergence factors.

When using the actuary-in-the-box to parameterise emergence factors the calculation done is to take the ratio of two standard deviations. One such calculation is done for each origin period comprising the claims triangle. Each origin period has undergone a number of periods of development and there is a one-to-one relationship between the origin periods and the development periods. This gives us an emergence factor $\alpha_{i,1}$ for each of the development periods $i = 1, \dots, n-1$. The problem here is that, as described in the appendices, emergence factors are stochastic, and in the method described above we have made exactly one observation of each emergence factor, and have used that single observation as the estimate of the emergence factor.

This is ok when the emergence factors are calculated for a specific claims triangle at a specific point in time outside of a capital calculation kernel and then fed into the calculation kernel and applied to the same claims data as they were calculated from. In this case all that has been done is that information about the risk emergence has been compressed into the emergence factor. However, if the method is used to calculate emergence factors that are then applied to a different class or at a different point in time then we are at best using a single observation of a stochastic variable made at a different time.

To overcome this problem we need to be able to make multiple observations of each emergence factor, to find some way of combining the observations in a way that gives an unbiased estimator of the expected value and variance of the emergence factor and to understand how these might change from data set to data set so that appropriate adjustments can be made to allow emergence factors to be applied to data sets from which they cannot be calculated.

We can make multiple observations of emergence factors as follows. As discussed in section 3.2.5 the actuary-in-the-box can be iterated all the way to ultimate to get estimates of the one-year risk for each future year for each origin year, all the way to ultimate. This gives the emergence factors α_{tk} where $t = 1, \dots, n-1$ and $k = 1, \dots, n-t$. The equation at the end of section 3.3.2 above can then be used to calculate multiple values of each one-year emergence factor.

However, the equation from section 3.3.2 only applies in the restricted case that emergence factors are deterministic. Furthermore, it is not clear how to combine the factors so derived to get an unbiased estimate of the expected value or variance of the emergence factors.

The problems outlined above for the actuary-in-the-box method of parameterising emergence factors also apply to the Mack/Merz-Wüthrich method. For the CoV method multiple observations of the emergence factors can be got by comparing origin periods more than one period apart, but the same problems still apply. In general, any method proposed must show how to make multiple observations of emergence factors, and how to combine them to get unbiased estimates of the expected value and variance of the emergence factors.

3.3.4 Discussion of method

Emergence factors are an apparently simple method, which can easily be explained. However, as the above discussion shows there are many hidden complexities. Furthermore, there is no widely accepted method of parameterising emergence factors and the methods which have been proposed all have significant limitations.

The main advantage of using emergence factors is that the calculations are quick and easy to perform, and are significantly simpler than those needed for the actuary-in-the-box method or the Merz-Wüthrich formula. This makes them suitable to use in situations where speed is important, for example inside the calculation kernel of a capital model. Emergence factors can be used in combination with another method (for example the actuary-in-the-box), where the emergence factors can be calculated outside the calculation kernel and reviewed for reasonableness before being applied. However as discussed in section 3.3.3 above there are limitations to doing this.

The biggest drawback to using emergence factors is the difficulty of parameterising them. Emergence factors are stochastic, and all methods of parameterising emergence factors proposed so far, essentially rely on just a single observation of the emergence factor, taking that as the estimate. A satisfactory method would have to make use of multiple observations of emergence factors, and provide a way of combining them to get an unbiased estimate of the expected value and variance of the emergence factors. No proposed method has come close to doing this, and it is not at all clear how it might be achieved.

The simplicity of emergence factors is an advantage in cases where we have an estimate of ultimate risk, but where other methods of estimating one-year risk cannot be applied. As discussed above it is difficult to parameterise emergence factors, however their simplicity makes it possible to use judgement to select values for the emergence factors to apply. The selection can incorporate the expert judgement of those outside the capital modelling and reserving teams. Because emergence factors are can be explained in a straight-forward non-technical way, these experts can meaningfully contribute to a discussion on what values should be selected, and can confidently challenge values proposed by capital modelling actuaries.

There are multiple different interpretations of the idea of emergence factors. The main difference is whether the claims payments during the future period(s) are included or not. If the claims payments are included, we call the emergence factors "ultimate emergence factors", and denote the factor with α, whereas if the claims payments are not included, we call the emergence factors "outstanding emergence factors", and denote the factor with β. The distinction matters if we are interested in getting a distribution of the closing reserves, as we would be if projecting the balance sheet in a capital model. In this case applying an ultimate emergence factor will scale down the variance of the claims paid during the year. This is not correct, as the claims paid during the year will be known with certainty at the end of the year, and so the full variance needs to be included. This can be achieved using outstanding emergence factors. However outstanding emergence factors are less tractable than ultimate emergence factors, and are consequently even harder to parameterise.

An issue with emergence factors to be aware of is how they affect dependence. Because of the way emergence factors are applied the resultant one-year distributions inherit the ultimate dependence. That is the correlations between the one-year distributions between different origin periods will be the same as the correlations between the corresponding ultimate distributions. In general, however these dependencies are different. This then affects the standard deviation of the total one-year distribution. As discussed in section 3.3.3 above the emergence factors can be adjusted so that the total distribution is more correct, there is always however a trade-off, and it is not possible for both the origin period one-year distributions, and the total one-year distribution to be correct.

The simplest version of emergence factors is probably aggregate emergence factors. These have the great advantage of simplicity. However, as they are applied to the aggregate over multiple origin periods, they confound risk emergence with claims run-off. This might not be a big problem with a static book of business, however if this approach is used with a growing or shrinking book of business this might be a problem. In this case the emergence factor applied would need to be carefully reviewed and probably changed regularly to reflect changes in the book of business.

One further criticism of emergence factors is that they approach the problem back-to-front. Emergence factors are applied to the ultimate distribution to derive the corresponding one-year distribution – that is the direction is from the ultimate distribution to the one-year distribution. However, it could be argued that it is easier to validate a one-year movement than an ultimate movement. To validate a one-year movement, you can look back at history, and see how reserves (or estimates of ultimate claims) have moved from one year to the next. It would also be easier to apply experience and judgement to assess whether the one-year distribution is reasonable; it is much harder to do this for ultimate risk. This is a reasonable criticism. However, in part for historical reasons, the usual practice is to estimate the ultimate distribution and then the one-year distribution.

It would be interesting to discuss further how one might reverse this, but it is beyond the scope of this paper.

3.4 Other Methods

3.4.1 Introduction

The three sections above discuss the three most commonly used methods for estimating one-year reserve risk. However, several other methods have also been used or proposed. In this section we briefly describe some of these.

3.4.2 Solvency II Undertaking Specific Parameters

In the technical specifications of the Quantitative Impact Studies 5 (QIS5 – see (CEIOPS, 2010b)) three standardized methods are specified for calculating Undertaking Specific Parameters (USP) for reserve risk. For each of the three methods a closed, analytic formula is used to calculate the USPs. The strengths and weaknesses of the QIS5 approaches are discussed in (Bulmer, 2012). In (CEIOPS, 2010a) a comprehensive revision of the calibration of the premium and reserve risk factors in the Non-Life and Health underwriting risk module of the standard formula was carried out. Annex XVII of the European Commission’s Solvency II Delegated Act (European Commission, 2014) contains only two calibration methods. The loss reserving method underlying method 2 is the Merz-Wüthrich formula (see section 3.1 and (Cerchiara, et al., 2016) for discussions of the strengths and weaknesses of this method).

In the Delegated Act EIOPA proposed another formula for calculating the reserve risk USPs. This formula is also a closed, analytic formula, and requires only claims data as an input. The theoretical method underlying this approach is one of the four methods tested by the “Joint Working Group – JWG - on Non-Life and Health NSLT Calibration” in (EIOPA, 2011) in the subsection Lognormal Models with Second Variance Parametrisation. The strengths and weaknesses of this method are also discussed in (Cerchiara, et al., 2016).

3.4.3 Simulation based approaches

In this sub-section we discuss some simulation-based approaches. In (Wacek, 2007) Wacek presented a framework for stochastically modelling the path of the ultimate loss ratio estimate through time from the inception of exposure to the payment of all claims. The author showed how to use information implicit in Hayne's lognormal model (Hayne, 1985) to determine the distribution of future estimates derived from stochastic versions of the chain ladder and Bornhuetter-Ferguson methods, with particular attention to the loss ratio estimate one year out. Wacek adjusted Hayne's model to allow for parameter uncertainty, and illustrated the effect. Because the adjusted distribution does not have the multiplicative properties of the lognormal distribution, Wacek illustrated the use of Monte Carlo simulation to model the distribution of future ultimate loss ratio estimates.

Other simulation approaches, based on collective risk theory (and simulation) have been discussed. In (Ricotta, et al., 2016), Ricotta and Clemente extended a proposal of the International Actuarial Association Insurer Solvency Assessment Working Party (International Actuarial Association, 2004), which assumes a collective risk model to analyse outstanding claims reserve with the goal of assessing the capital required for the one year reserve risk. Ricotta and Clemente assume that the incremental payments are a compound mixed Poisson process where the uncertainty in the claim size is measured via a multiplicative structure variable. Two structure variables, on claim count and average cost, are considered in order to describe parameter uncertainty on both random variables. By adapting the "re-reserving" method they estimated both the variability of the claims development result, and the extreme quantiles of its simulated probability distribution. The main advantage of this proposal is that it directly considers the parameter uncertainty of the claim size estimation which is neglected by other models.

In (Dal Moro, et al., 2014) the authors point out that for several common methods, including the Bornhuetter-Ferguson, Cape-Cod, and Benktander-Hovinen methods, there are no closed-form formulas for the one-year volatility, and that the only alternative is to use simulation methods. In section 3.2.4 above we showed how to use the actuary-in-the-box method to get the one-year volatility (within the AMW-BF model) for the Bornhuetter-Ferguson method. It is possible to adapt the AMW-BF model and apply the bootstrap and actuary-in-the-box to get the ultimate and one-year volatility for the Cape-Cod and Benktander-Hovinen methods.

In some simulation approaches the ultimate risk is determined first, and then the one-year risk is determined from the ultimate risk. This approach might be preferred for portfolios where information from outwith the claims triangle is significant, for example: changes in the external environment, non-proportional claims below a threshold, or complex reinsurance programs. In such cases it is better to focus first on getting the ultimate risk right, and then using a simple approach to get the one-year risk. This was discussed in (Möhr, et al., 2013) (slide 21). Emergence patterns can be considered a simple method used to derive the one-year risk from the ultimate risk. In (White, et al., 2010) other methods of deriving the one-year risk from the ultimate risk are discussed.

3.4.4 Bayesian methods

In (Mack, 1993) Mack showed how to estimate the (deterministic) chain ladder parameters from the claims triangle data. In a Bayesian approach we would assume that the unknown parameters follow a (specified) prior distribution, and then using the claims triangle data and the prior distribution calculate the posterior distribution of the parameters, and the future claims amounts. In (Bühlmann, et al., 2009) the authors present a recursive credibility formula for the calculation of the Bayesian estimators and derive formulas for the conditional MSEP of the one-year claims development result. In most of the cases the calculation of Bayesian estimators can't be done via analytic formula and so numerical methods like Markov Chain Monte Carlo method must be used instead (see (Verrall, 2004) in the framework of generalized linear models).

3.4.5 Robbin's Method

In (Robbin, 2012) Robbin proposed a new and relatively simple way to calculate one-year reserve risk. The method is practical and robust; it does not need claims triangles, or stochastic simulations. Instead it uses estimates of the mean and ultimate CoV of the reserves, payment patterns, and reporting patterns. The method could be considered as an emergence factor (Robbin calls them "recognition factors") approach with a systematic way of deriving emergence factors based on the reserve run-off.

The key idea behind Robbin's method is that IBNR and case reserves have different levels of volatility, and that the CoVs for these two types of reserve remains the same throughout the run-off, but that as the mix between them changes, the CoV of the total reserve changes correspondingly. The method therefore assumes that projections of the run off of the IBNR and case reserves are available. The method first projects the ultimate CoVs of the reserves in future calendar years as they run-off, and from these calculates the one-year CoVs over each future calendar year.

Robbin then assumes that the one-year distribution of the unpaid claims are lognormally distributed, and from the mean and CoV previously calculated, calculates the 99.5th percentile of the one-year distribution to get the SCR, and then the risk margin.

A further important assumption that the method makes is that the IBNR and case reserves are uncorrelated.

The assumption that the CoVs of the IBNR and case reserves remain constant throughout the run-off is open to question. Instead the CoV might increase as the reserve decreases – this is often seen in practice with claims reserves. The assumption that the IBNR and case reserves are uncorrelated is also open to question. In particular since there is a natural progression from IBNR to case reserves, a reduction in IBNR might be more likely to occur alongside an increase in the case reserve than the opposite.

Robbin's method is a pragmatic way to calculate one-year reserve risk, and the risk margin. It has low data requirements, and for many of the inputs, estimates or benchmarks figures could be used. It is therefore a method that can be used in cases where there is little reliable data. It could also be used to validate more sophisticated methods.

3.4.6 Hindsight re-estimation

The Hindsight re-estimation method was proposed by Andrew Houltram in 2005 (Houltram, 2005). One of the main aims of this method is to allow for the impact of actuarial judgement on the volatility of reserve estimates. The basic idea is to use previous best estimates of the ultimate claims to estimate how volatile the current best estimate is. First a triangle of previous best estimates for each origin year is formed. These are then adjusted for exposure. Hindsight development factors are calculated, as ratio of one best estimate to the previous one. A bootstrap process is then used to get a distribution of the next year's best estimate.

The hindsight re-estimate method is widely used in the Australian market (see (Bruce, et al., 2008) section 4.2.4).

The method relies on there being several prior years of consistent best estimates available. This is unlikely to be the case if there have been recent changes to the business such as mergers or take-overs, changes in the volume or mix of business written, or changes to the level of granularity that the reserves are analysed at. It also assumes that past patterns of deviation will be repeated in future. This is quite a strong assumption as over several years, different individuals, and different methods are likely to be used.

3.4.7 Complementary loss ratio method

In (Quarg, et al., 2008) Quarg and Mack proposed a method, called the Munich Chain Ladder, which combines incurred and paid claims data to get a best estimate for the ultimate claims amount. However, this method is deterministic, and there is no known analytic formula for the prediction error within it. In (Dahms, 2007) Dahms extended the complementary loss ratio method (CLRM) which derives predictions for the ultimate claims by combining incurred and paid claims data. For this extended model Dahms presents an unbiased estimator for the reserves, and the corresponding conditional mean squared error. This work is built on in (Dahms, et al., 2009) where they consider the CDR in Dahms' extended CLRM, and derive formulas for the MSEP of the CDR.

3.4.8 Perfect Foresight

The final method we discuss is the simplest. In this method it is assumed that the one-year risk is equal to the ultimate risk. This is equivalent to assuming that the closing estimate of ultimate claims will be exactly equal to the ultimate claims payment, in other words the closing estimate of ultimate claims will show "perfect foresight" of the ultimate claims payment. This method has the advantage of simplicity. In some situations, it might be suitable to use for short-tailed classes of business, where almost all the risk is expected to emerge over the one-year time-horizon (for example, natural catastrophe claims). It might also be suitable for mature origin years which do not make a material contribution to the total risk for the class. It would not be suitable for longer-tailed classes, or for more recent origin years where a material amount of the risk emerges after one-year.

4 VALIDATION

4.1 INTRODUCTION

Before applying the results of any model, or using that model for further inference, it is essential to check that the model provides a reasonable description of the data. Failure to do so leaves the modeller at risk of using an ill-fitting model. In the specific example of stochastic reserving with a one-year time-horizon, consequences could include materially incorrect estimates of the reserve uncertainty over the one-year time-horizon, which would lead to a material misestimate of the capital required for the relevant book of business. This in turn can lead to misinformed management decisions. For example, an overestimate of capital might lead to writing a less than optimal amount of a profitable line, whereas an underestimate of capital might lead to writing a more than optimal amount of a line that is riskier than believed. Such misallocation of capital can lead to the insurer being less profitable than its competitors. More straight-forwardly an underestimate of capital can increase the chance of regulatory intervention or insolvency.

Validating the model used to analyse the one-year risk has a lot of overlap with validating the model for the corresponding ultimate risk. However, the modeller needs to remain aware that using a model for estimating one-year risk is different from using the same model with the same data for estimating ultimate risk, and so further validation is needed to check that the model remains suitable for estimating one-year risk. In this paper we discuss only the further validation needed. Validation of the models for ultimate risk is discussed in section 5 of (Carrato, et al., 2016), and the reader should read that section alongside this one.

4.2 COMPARING MEASURES OF UNCERTAINTY

A key validation is to compare the one-year risk with the ultimate risk. To do this we need to compare the same measure of uncertainty from the corresponding one-year and ultimate distributions, calculated in a consistent way.

The following comparison is common and useful. For each origin period, the two distributions compared are the ultimate distribution of ultimate claims, and the distribution of the one-year claims development result. For both of these distributions, calculate the standard deviation. There are two different means that can be used to divide the standard deviation. These are the opening estimate of the ultimate claims, and the ultimate estimate of the claims reserve. In general, using the opening estimate of the ultimate claims gives results that are more stable and easier to interpret.

These statistics are the coefficient of variation of the distribution of the ultimate claims, and of the closing reserves plus the claims paid in the year. They are therefore sometimes called the “ultimate CoV”, and the “reserve CoV”. Although note that the name “reserve CoV” is potentially slightly misleading as it is not the CoV of the closing reserve, but the CoV of the closing reserve plus the claims paid during the year. We need to include the claims paid during the year to get a distribution that can be consistently compared with the ultimate distribution.

Once these statistics have been calculated they can be graphed. Three key things to look at are:

- The general level of the CoVs
- The pattern of the CoVs across origin periods
- The relative level of the CoVs for the one-year CDR and the ultimate claims

The graph in Figure 4-1 shows ultimate CoVs. This graph is typical of a short-tail class of business. The general level of the CoVs is quite low – 3.5% is the highest value. This reflects the low volatility of the class. The ultimate CoV is highest for the most recent origin periods, and drops rapidly to zero as the origin periods age. The CoV of the one-year CDR is slightly lower than the CoV of the ultimate risk. Both these features reflect the rapid emergence of the risk in a short-tailed class.

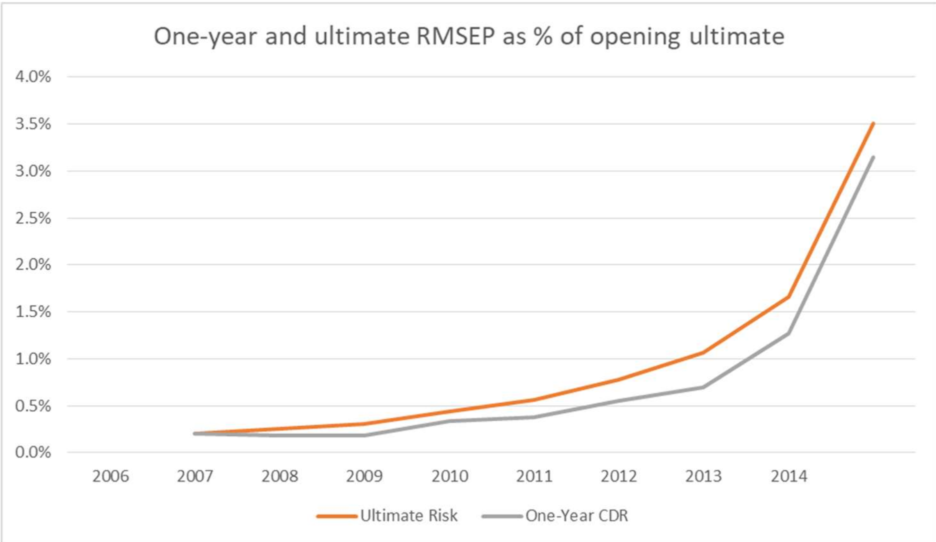

FIGURE 4-1

For comparison the graph in Figure 4-2 shows the same statistics, but for a long-tailed and more volatile class of business. The general level of the CoVs is much higher than the class above – several are over 20%. This reflects the higher volatility of the class. The ultimate CoV is highest for the most recent origin periods, but it declines much more slowly than the short-tail class as the origin periods age. Also, the gap between the CoV of the one-year CDR and the CoV of the ultimate risk is larger, particularly for the most recent origin periods. Both these features reflect the slower emergence of the risk in a long-tailed class.

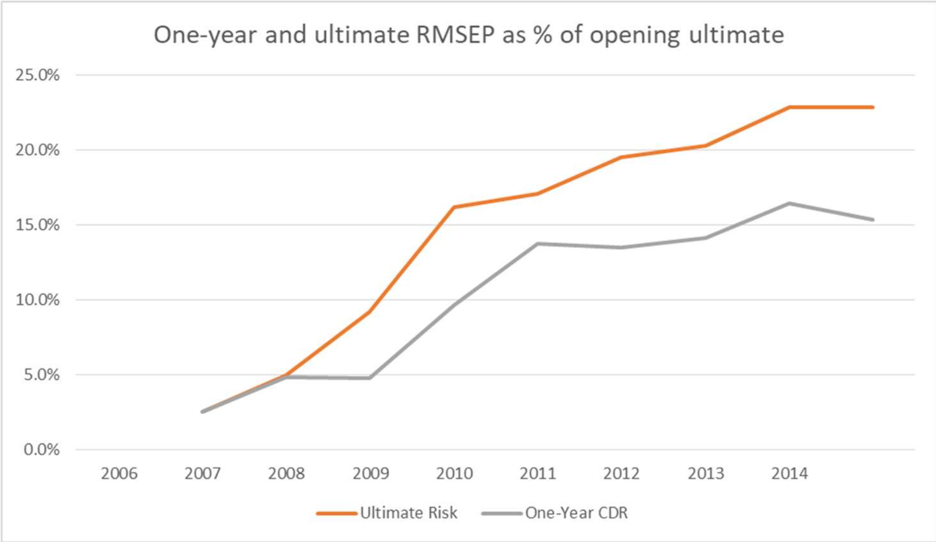

FIGURE 4-2

When calculating the ultimate CoV and reserve CoV, the standard deviations used are the same, the CoVs are different because the means are different. The two CoVs are therefore related as follows:

Reserve CoV = Ultimate CoV x Mean Ultimate / Mean Reserve

The CoV measure of uncertainty focuses the validation on the variation of the distributions around the mean. An analysis similar to the above can be done with other measures of uncertainty, targeted at other parts of features of the distribution. For example, TVaR and TVaR-contribution could be used, which would focus the validation on the tail of the distribution.

4.3 Validation of Risk Emergence

Another key validation is to calculate and graph the one-year risk as a proportion of the ultimate risk. As in the previous section, the two measures must be consistent. For example, we can calculate the proportion of risk emerging for each origin period as the standard deviation of the one-year CDR divided by the standard deviation of the ultimate claims. The proportion of risk emerging should always be between 0% and 100%. The risk emergence percentage can be graphed as shown in Figure 4-3.

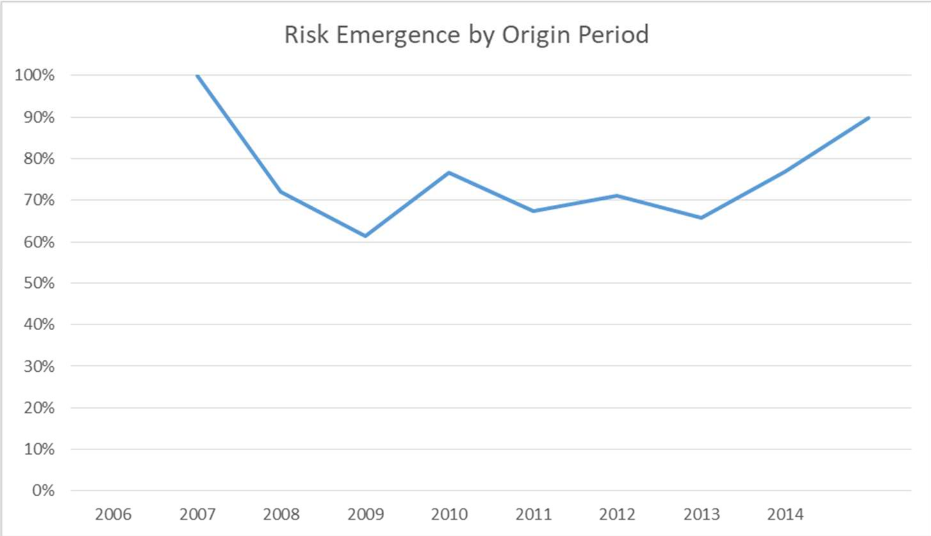

FIGURE 4-3

In the graph in Figure 4-3 the risk emergence is high in the most recent origin periods and the oldest origin periods, and lowest in the middle-aged origin periods. This is sometimes called the “smile curve”. It is fairly typical if a chain ladder model (e.g. the Mack or ODP) is used. The reason we might expect a higher proportion of the outstanding risk to emerge in the most recent years when using a chain ladder is that most of the development happens in the first few development periods, and the chain ladder, in contrast to the Bornhuetter-Ferguson model, is fully responsive to the movements in this early development. See (England, et al., 2012) for some more discussion of this.

The Bornhuetter-Ferguson model, on the other hand, explicitly gives the movements in the early development less weight than the prior estimates, and so the risk emergence in the most recent origin periods, is often less than the risk emergence of older origin periods. The Bornhuetter-Ferguson model will typically be applied to a long-tailed class of business. In this case very little weight will be given to the most recent origin periods, and so there can be almost no risk emergence for these cohorts over the first year of development. However, as the claims develop, the middle-aged origin periods see more volatility in the claims development, and the Bornhuetter-Ferguson model gives more weight to the claims experience, so the one-year risk is a larger proportion of the ultimate risk. For the oldest origin periods, the claims are almost fully developed so the proportion of risk emerging is lower again. The shape of the risk emergence graph is therefore peaked in the middle, and lower for the oldest and youngest origin periods. See (Scarth, 2015) for further discussion of this.

The shape of the risk emergence graph therefore depends on both the underlying claims data and the model used. The following characteristics will affect the shape of the risk emergence, and should be considered when carrying out validation. However, they will interact with one another, and so the validator must carefully think through exactly what reasonable results would look like. A key thing is to remember that the risk emergence proportion shows the proportion of the remaining risk – not the total life-time risk – that will emerge over the coming year.

- Length of tail – the risk will emerge more quickly for a short tail class than a long tail class.
- Volatility of class – how the risk emerges depends on how the volatility interacts with the length of tail, and the model used. For short tail volatile classes like terrorism, we would expect the risk emerge quickly. For long tail volatile classes like casualty modelled using a Bornhuetter-Ferguson model we might expect a higher risk emergence proportion for the middle-aged origin periods.
- Origin period – both the age and relative size of the claims experience of an origin period will affect the risk emergence.
- Volume of business – a smaller volume will be more volatile, and we might expect the claims emergence to be slower, although this will also depend on other features of the business.
- Model used – as discussed above whether a chain-ladder or Bornhuetter-Ferguson model is used will affect the speed of risk emergence.

In addition to looking at the emergence percentages, we can also look at the whole distribution. This would be available if the model is bootstrapped. A graph of the PDFs of the one-year CDR and the mean-centered ultimate claims can be examined. We need to use the mean-centered ultimate claims so that it has a mean of zero, like the CDR, and so can be more easily visually compared. The graph in Figure 4-4 shows an example of this. This shows that the one-year risk (the blue line) is more tightly concentrated around the mean than the ultimate risk (the red line), and more extreme values are less likely. This is what we would expect to see.

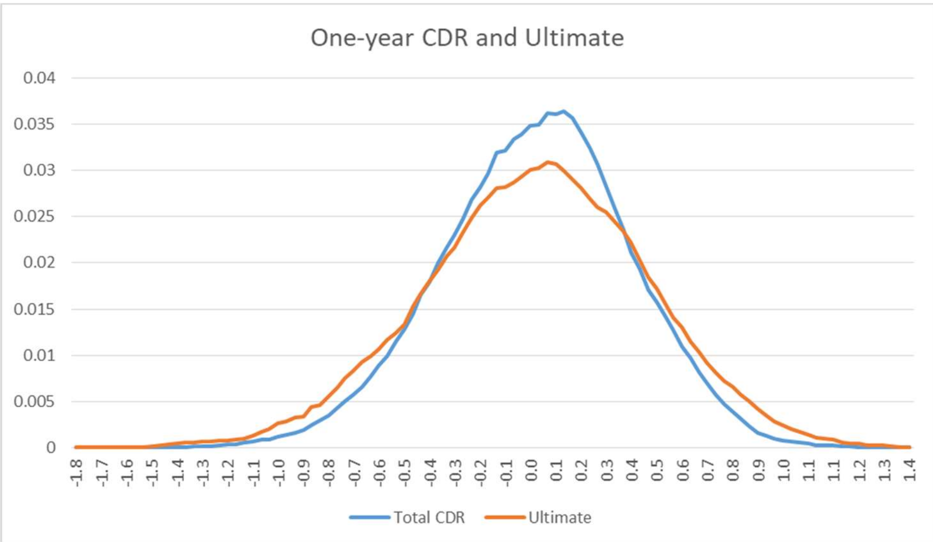

FIGURE 4-4

4.4 BENCHMARKING

The ultimate CoVs, reserve CoVs, and emergence percentages can be compared against suitable benchmarks. It is important when doing this that the benchmarks are an appropriate comparison to the business being validated. In particular they should be expected to be similar in terms of length of tail, volatility of business, measure of uncertainty used, and the model used should at least have similar behavior to that used for the class being benchmarked.

Different classes of business can also be compared to one another to check whether any are out of line. Again, the main features to consider when doing this are the length of tail, volatility, measure of uncertainty, and the model used.

Premium risk CoVs can also be compared against reserve risk CoVs. The premium risk CoV can be decayed out using a pattern for back years. The pattern should be representative of how we believe risk emerges over its lifetime.

4.5 Scenario Testing

Scenario testing is another technique which can help calibrate the reserve uncertainty distribution thereby validating it. For example, a worst case one-year forward scenario, say, at a 1 in 200 level can be deterministically worked out and then be compared with the 99.5th percentile of the one-year reserve risk distribution. Other return periods could also be considered.

4.6 Backtesting

If historic loss ratios are available then the following backtesting exercise can be performed. Suppose that we have historic loss ratios for each origin period as shown in Table 4-1

Accident	Historical Loss Ratios									
Year	0	1	2	3	4	5	6	7	8	9
2005	89%	90%	90%	90%	90%	90%	90%	90%	90%	89%
2006	88%	87%	87%	86%	86%	85%	85%	85%	85%	
2007	85%	85%	86%	86%	85%	85%	85%	85%		
2008	77%	76%	77%	77%	78%	78%	78%			
2009	80%	79%	79%	79%	79%	79%				
2010	76%	78%	79%	80%	81%					
2011	78%	78%	77%	77%						
2012	68%	69%	70%							
2013	68%	70%								
2014	77%									

TABLE 4-1

The idea of the method is to assume that for each origin period, the loss ratio at each development period is sampled from a distribution with mean equal to the loss ratio of the previous development period, and CoV given by the one-year CoV for the origin period having seen the same amount of development.

For example, for origin period 2013, development period 1, we assume that the loss ratio is sampled from a distribution with mean 68% (i.e. the 2013 loss ratio at development period 0), and CoV given by the origin period with one-year of development – in this case 2014. Note the 2014 CoV will be used for development period 1 for all the origin periods.

Any reasonable distribution can be assumed, but a typically and appropriate choice would be the lognormal distribution.

Given the distributional assumptions we can then calculate the percentile of the actual observed loss ratios. We do this for all origin periods and development periods except for development period 0. We expect the percentiles to be uniformly distributed between 0 and 1, and to show no trends with respect to origin period, development period, or calendar period. Table 4-2 shows an example of this.

Accident	Percentile of Historical Loss Ratios									
Year	0	1	2	3	4	5	6	7	8	9
2005		0.674	0.517	0.566	0.547	0.256	0.871	0.057	0.070	0.031
2006		0.302	0.631	0.114	0.273	0.027	0.212	0.928	0.706	
2007		0.514	0.819	0.525	0.195	0.728	0.315	0.582		
2008		0.311	0.891	0.602	0.852	0.258	0.925			
2009		0.407	0.636	0.465	0.315	0.318				
2010		0.776	0.797	0.950	0.958					
2011		0.433	0.163	0.262						
2012		0.689	0.805							
2013		0.735								
2014										

TABLE 4-2

To check that the percentiles are uniformly distributed we can plot the empirical CDF against the expected straight-line CDF for the uniform distribution. The graph in Figure 4-5 shows an example of this.

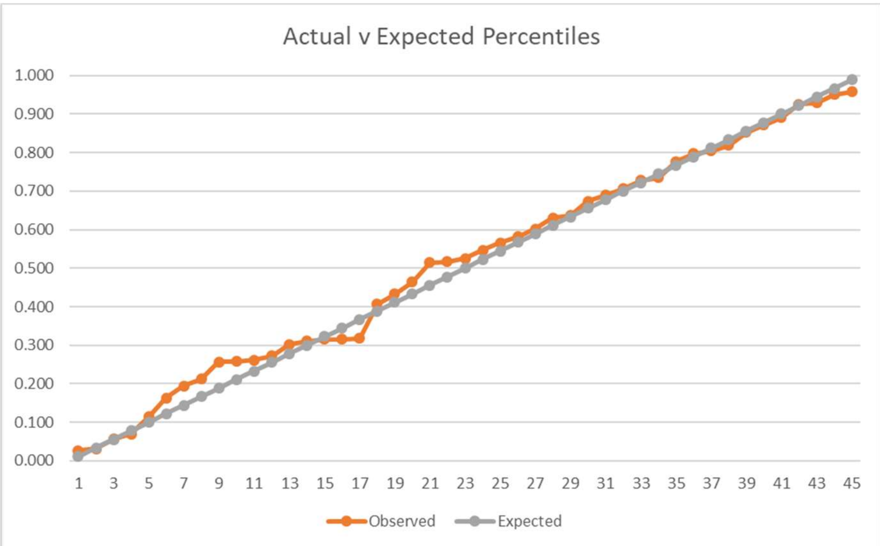

FIGURE 4-5

The shortcoming of this method is that it requires consistent historic ultimate loss ratios to have been recorded and to be available. It also does not work well if there have been material changes in the volume of business between origin periods, as higher volume origin periods would be expected to have lower CoVs than those with lower volume.

5 NUMERICAL EXAMPLES

We now apply the methods discussed in this paper to some real-life data sets to help illustrate their use in practice. We use exactly the same data sets and models as in the working party's first paper (Carrato, et al., 2016), and extend the analysis to the one-year case. This section therefore builds on the numerical examples from the first paper, and to gain a full understanding of the application of the methods to these data sets the reader should read section 7 of the first paper before reading this section. Note that some of the results in this paper may differ slightly from those in the first paper due to simulation error.

5.1 EXAMPLE 1

The first dataset is taken from (Alai, et al., 2009) and (Alai, et al., 2011). This is a reasonably well-behaved claims triangle, with a relatively short tail, and no unusual features immediately apparent. The data is presented in Table 5-1 as a triangle of incremental claim amounts.

Accident	Cumulative claim payments in development year (£000)									
Year	0	1	2	3	4	5	6	7	8	9
2005	5,947	9,668	10,564	10,772	10,978	11,041	11,106	11,121	11,132	11,148
2006	6,347	9,593	10,316	10,468	10,536	10,573	10,625	10,637	10,648	
2007	6,269	9,245	10,092	10,355	10,508	10,573	10,627	10,636		
2008	5,863	8,546	9,269	9,459	9,592	9,681	9,724			
2009	5,779	8,524	9,178	9,451	9,682	9,787				
2010	6,185	9,013	9,586	9,831	9,936					
2011	5,600	8,493	9,057	9,282						
2012	5,288	7,728	8,256							
2013	5,291	7,649								
2014	5,676									

TABLE 5-1

5.1.1 ANALYSIS OF DATA SET 1

In (Carrato, et al., 2016) this data set was analysed and a model validation carried out for the Mack and ODP models. The conclusion was the ODP model is preferable to Mack's model for this data set. We now want to extend the analysis to the case of one-year risk, which is a different model use that analysing ultimate risk. We therefore now carry out additional validation to check that the ODP model is suitable for this new use.

First, we compare one-year results with corresponding ultimate results. Table 5-2 shows, for each accident period, the opening estimates of the ultimate claims and reserves, the ultimate RMSEP, the RMSEP of the one-year CDR, and the risk emergence percentage, which is calculated as the one-year RMSEP divided by the ultimate RMSEP. The graphs in Figure 5-1 and Figure 5-2 are based on same numbers as in Table 5-2. These show, for each accident year, the one-year and ultimate RMSEP as a % of the opening ultimate, the risk emergence %. The graph in Figure 5-3 shows the distributions of the one-year claims development result, and the mean-centred ultimate claims distribution from the bootstrap output of the model.

The claims data is for a short-tail low volatility class of claims. In light of this, the emergence % figures all look reasonable – they are all less than 100% (excepting the oldest accident year), range from 60% to 90%, and the total emergence percent is about 84%. The graph of risk emergence by origin period shows the "smile curve" that is typical for chain-ladder models. The graph of RMSEP as a percentage of the opening ultimate shows that risk declines quickly as accident years mature, and that the one-year risk follows the same pattern, but is slightly lower than the ultimate risk. The graph of the CDR and ultimate risk distributions shows that the one-year risk is less than the ultimate risk (as more of the probability is in the centre of the distribution), although the difference is not large. All this is reasonable for a short-tail, low volatility class.

Accident Year	Ultimate	Reserve	Ultimate Risk RMSEP	One-Year CDR RMSEP	Emergence %
2005	11,148	0			
2006	10,663	15	22	22	100.0%
2007	10,662	26	27	20	72.1%
2008	9,759	34	30	18	61.4%
2009	9,872	85	43	33	76.8%
2010	10,092	156	57	38	67.5%
2011	9,568	286	74	53	71.0%
2012	8,705	449	93	61	65.8%
2013	8,692	1,043	144	111	76.9%
2014	9,626	3,951	337	303	89.8%
Total	98,787	6,046	440	369	83.9%

TABLE 5-2

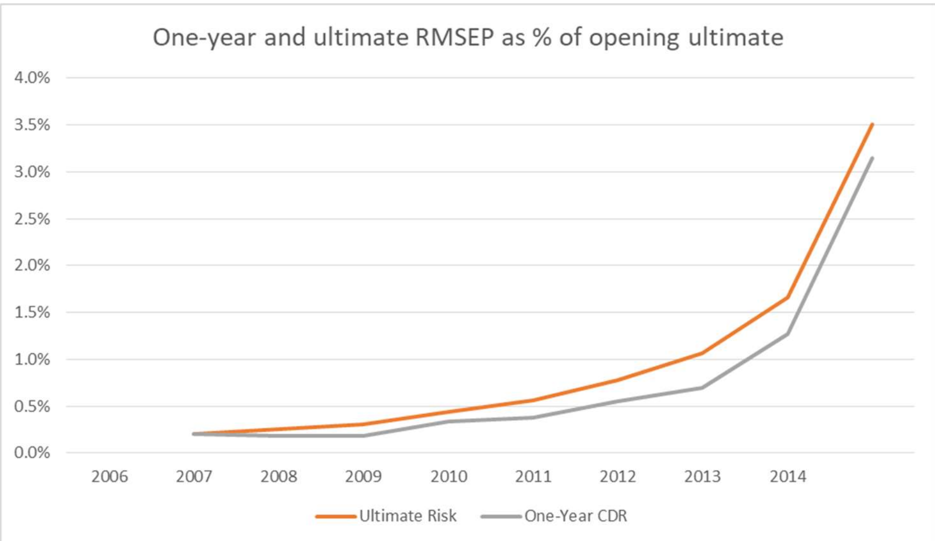

FIGURE 5-1

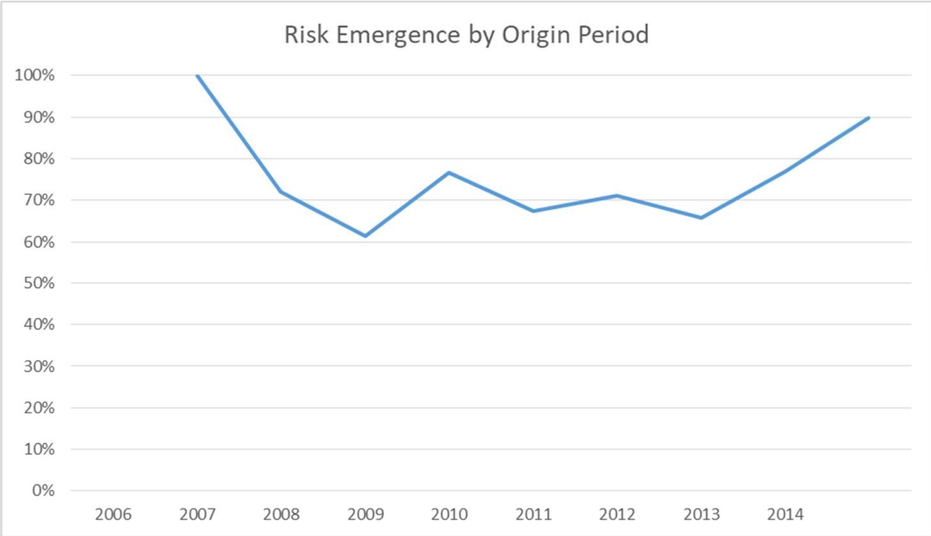

FIGURE 5-2

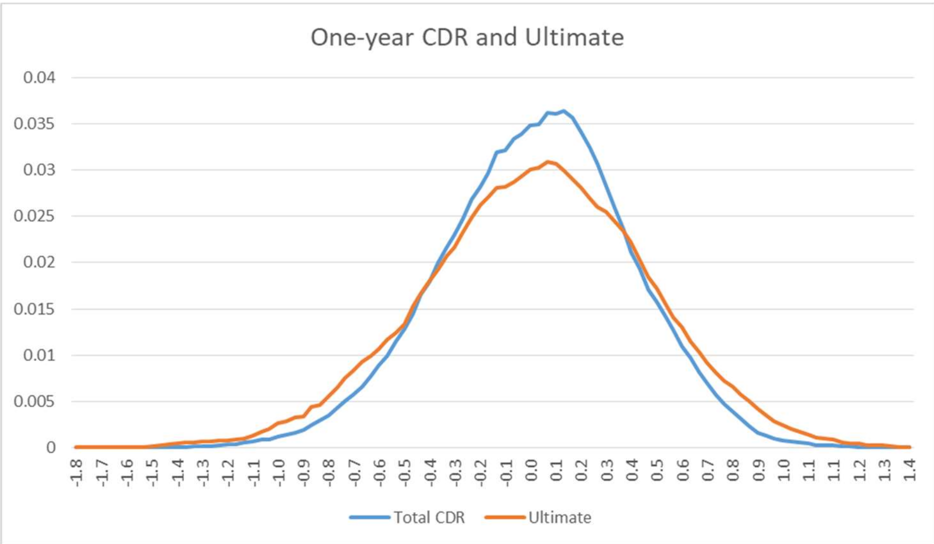

FIGURE 5-3

Next we back test the one-year results using historical loss ratios for this triangle of claims. The original data set in (Alai, et al., 2009) and (Alai, et al., 2011) did not contain historical loss ratios, so we have created some fictional loss ratios to illustrate the method, these are shown in Table 5-3.

Accident	Historical Loss Ratios									
Year	0	1	2	3	4	5	6	7	8	9
2005	89%	90%	90%	90%	90%	90%	90%	90%	90%	89%
2006	88%	87%	87%	86%	86%	85%	85%	85%	85%	
2007	85%	85%	86%	86%	85%	85%	85%	85%		
2008	77%	76%	77%	77%	78%	78%	78%			
2009	80%	79%	79%	79%	79%	79%				
2010	76%	78%	79%	80%	81%					
2011	78%	78%	77%	77%						
2012	68%	69%	70%							
2013	68%	70%								
2014	77%									

TABLE 5-3

As described in section 4.6, we assume that loss ratios have a lognormal distribution with mean given by the previous year's loss ratio, and CoV implied by the one-year CDR RMSEP. We then calculate the percentiles of the historic loss ratios. These are shown in Table 5-4.

Accident	Percentile of Historical Loss Ratios									
Year	0	1	2	3	4	5	6	7	8	9
2005		0.674	0.517	0.566	0.547	0.256	0.871	0.057	0.070	0.031
2006		0.302	0.631	0.114	0.273	0.027	0.212	0.928	0.706	
2007		0.514	0.819	0.525	0.195	0.728	0.315	0.582		
2008		0.311	0.891	0.602	0.852	0.258	0.925			
2009		0.407	0.636	0.465	0.315	0.318				
2010		0.776	0.797	0.950	0.958					
2011		0.433	0.163	0.262						
2012		0.689	0.805							
2013		0.735								
2014										

TABLE 5-4

There is no obvious calendar year, development year, or accident year trend in the percentiles. We expect the percentiles to be uniformly distributed. The graph in Figure 5-4 shows the empirical CDF of the percentiles. This shows that the percentiles are uniformly distributed.

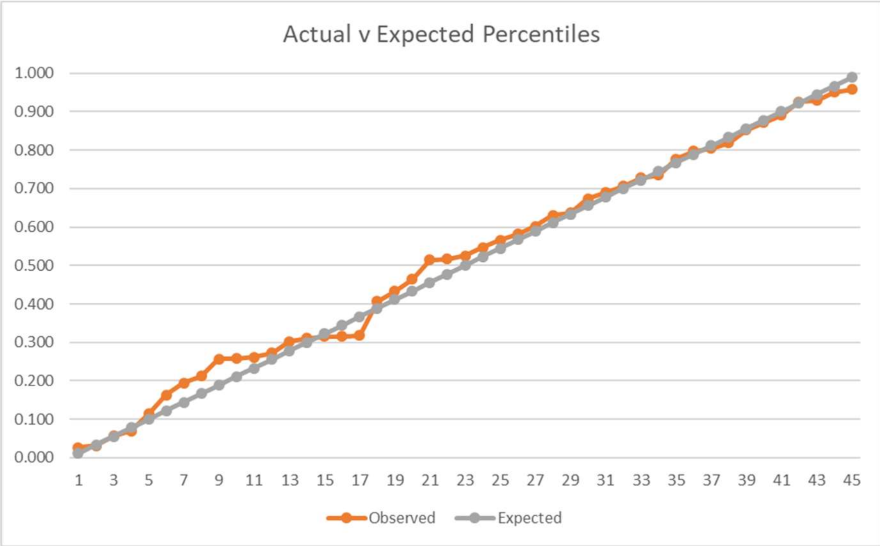

FIGURE 5-4

The initial validation concluded that the ODP model should be preferred for this data set. The above validation has found no issues with extending the analysis done with this model to the one-year case.

5.1.2 Comparison of Methods

In this subsection we compare the results of the ODP presented in the previous section with the Mack model and the AMW-BF model. The tables in this subsection present the results; Table 5-5 shows the results for Mack's model, Table 5-6 for the AMW-BF model (with a constant scale parameter).

Accident Year	Ultimate	Reserve	Ultimate Risk RMSEP	One-Year CDR RMSEP	Emergence %
2005	11,148	0			
2006	10,663	15	0	0	100.0%
2007	10,662	26	1	1	100.0%
2008	9,759	35	3	3	96.5%
2009	9,872	85	8	7	91.8%
2010	10,092	156	34	33	97.2%
2011	9,568	286	74	66	89.5%
2012	8,705	449	84	50	59.4%
2013	8,692	1,043	134	104	77.8%
2014	9,626	3,950	411	386	93.9%
Total	98,787	6,045	461	420	91.1%

TABLE 5-5

Accident Year	Ultimate	Reserve	Ultimate Risk RMSEP	One-Year CDR RMSEP	Emergence %
2005	11,148	0			
2006	10,664	16	23	23	100.0%
2007	10,663	27	28	21	73.2%
2008	9,762	37	31	20	62.9%
2009	9,882	95	47	36	77.4%
2010	10,114	178	62	42	68.6%
2011	9,623	341	83	60	72.6%
2012	8,830	574	107	73	68.4%
2013	8,968	1,319	166	133	80.4%
2014	10,441	4,765	367	337	91.6%
Total	100,094	7,352	478	412	86.0%

TABLE 5-6

Mack's model gives the same central estimates as the ODP, but makes different distributional assumptions. For this data set, Mack's model gives a total ultimate RMSEP similar to the ODP model, however, more of the uncertainty is in the most recent accident periods, and less in the older accident periods – so it appears to have the risk emerging quicker. Consistent with this, the emergence percentages are higher for Mack's model.

The AMW-BF model is based on the ODP model, but gives the same central estimate as the Bornhuetter-Ferguson model. For this data set the Bornhuetter-Ferguson estimate for the reserve is higher than the chain-ladder estimate. The ultimate and one-year RMSEPs are therefore a little higher than for the ODP model, but the distribution between the accident periods, and the emergence percentages are similar. This reflects the fact that the underlying distribution assumptions are the same, and the assumption that the prior ultimates have low volatility.

5.2 EXAMPLE 2

The second dataset is taken from (Liu, et al., 2008). It is an aggregate data set from Lloyd's syndicates. It is long tail, and much more volatile that the first dataset. The data consists of triangles of paid and incurred claims. These are presented below, Table 5-7 shows the paid claims, Table 5-8 the incurred claims.

Accident	Cumulative claim payments in development year (£)									
Year	0	1	2	3	4	5	6	7	8	9
2005	184	1,845	3,748	5,400	6,231	9,006	9,699	10,008	10,035	10,068
2006	155	1,483	3,768	7,899	8,858	13,795	15,360	15,895	19,333	
2007	676	2,287	10,635	16,102	22,177	28,825	29,828	30,700		
2008	67	367	2,038	2,879	6,329	14,366	16,201			
2009	922	1,693	3,523	4,641	6,431	8,325				
2010	22	488	3,424	5,649	7,813					
2011	76	435	1,980	5,062						
2012	24	1,782	3,881							
2013	39	745								
2014	306									

TABLE 5-7

Accident	Cumulative incurred claims in development year (£)									
Year	0	1	2	3	4	5	6	7	8	9
2005	1,530	8,238	10,564	12,332	12,173	10,576	10,630	10,316	10,325	10,280
2006	1,505	6,247	8,728	10,500	15,241	16,720	16,845	16,829	19,675	
2007	2,505	6,150	17,937	22,143	29,511	33,336	32,162	31,500		
2008	204	2,748	9,984	13,167	16,523	17,807	18,959			
2009	2,285	4,361	6,432	8,834	12,092	15,309				
2010	269	5,549	7,214	12,422	13,581					
2011	1,271	2,657	6,187	11,004						
2012	298	3,533	6,423							
2013	2,023	5,415								
2014	1,779									

TABLE 5-8

5.2.1 Analysis of Data Set 2

In (Carrato, et al., 2016) this data set was analysed and a model validation carried out for the Mack, ODP, and AMW-BF models. The conclusion was the Mack and ODP models are not suitable for this data set, and that the AMW-BF model gives more reasonable results. This is because of the long-tailed and volatile nature of the claims data. We now want to extend the analysis to the case of one-year risk, which is a different model use that analysing ultimate risk. We therefore now carry out additional validation to check whether the AMW-BF model is suitable for this new use.

First, we compare one-year results with corresponding ultimate results. Table 5-9 shows, for each accident period, the opening estimates of the ultimate claims and reserves, the ultimate RMSEP, the RMSEP of the one-year CDR, and the risk emergence percentage, which is calculated as the one-year RMSEP divided by the ultimate RMSEP. The graphs in Figure 5-5 and Figure 5-6 show some of the figures in the table. These show, for each accident year, the one-year and ultimate RMSEP as a % of the opening ultimate, the risk emergence %. The graph in Figure 5-7 shows the distributions of the one-year claims development result, and the mean-centred ultimate claims distribution from the bootstrap output of the model.

The claims data is for a long-tail high volatility class of claims. In light of this, the emergence % figures all look reasonable – they are all less than 100% (excepting the oldest accident year), range from 50%

to 70% (except for the second oldest accident year), and the total emergence percent is about 73%. The four most recent accident years contain three quarters of the reserves, and the emergence percentage for all of them is about 70%. The graph of RMSEP as a percentage of the opening ultimate shows that risk declines much more slowly as accident years mature than in example 1 (reflecting the long-tail nature of the triangle), and the one-year risk follows the same pattern, but is slightly lower than the ultimate risk. The graph of the CDR and ultimate risk distributions in Figure 5-7 shows that the one-year risk is less than the ultimate risk (as more of the probability is in the centre of the distribution). This is not unreasonable for a long-tail, high volatility class.

Accident Year	Ultimate	Reserve	Ultimate Risk RMSEP	One-Year CDR RMSEP	Emergence %
2005	10,068				
2006	19,404	71	491	491	100.0%
2007	33,138	2,438	1,647	1,616	98.1%
2008	19,182	2,981	1,772	916	51.7%
2009	12,548	4,223	2,035	1,214	59.7%
2010	17,201	9,388	2,943	2,366	80.4%
2011	17,276	12,214	3,373	2,332	69.1%
2012	19,175	15,294	3,900	2,720	69.7%
2013	19,363	18,618	4,430	3,193	72.1%
2014	20,095	19,789	4,593	3,090	67.3%
Total	187,450	85,016	10,241	7,462	72.9%

TABLE 5-9

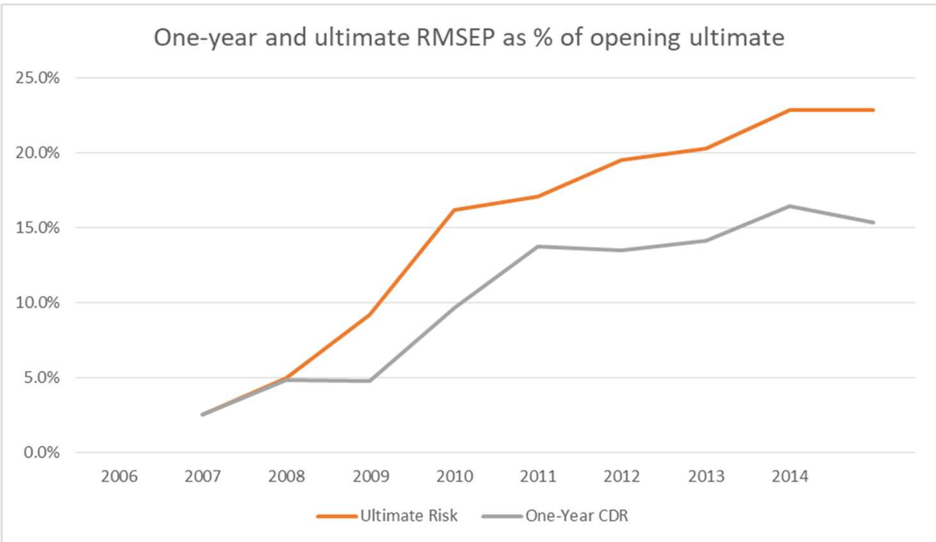

FIGURE 5-5

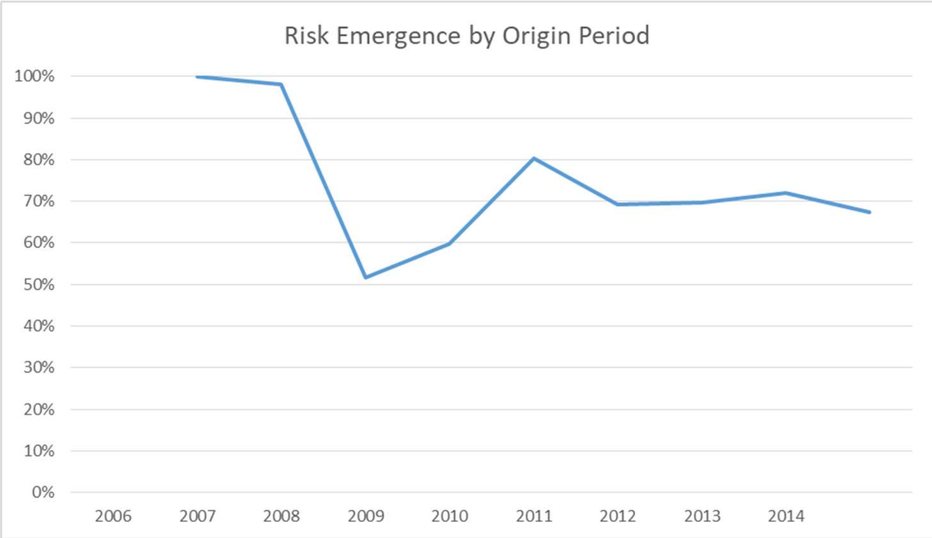

FIGURE 5-6

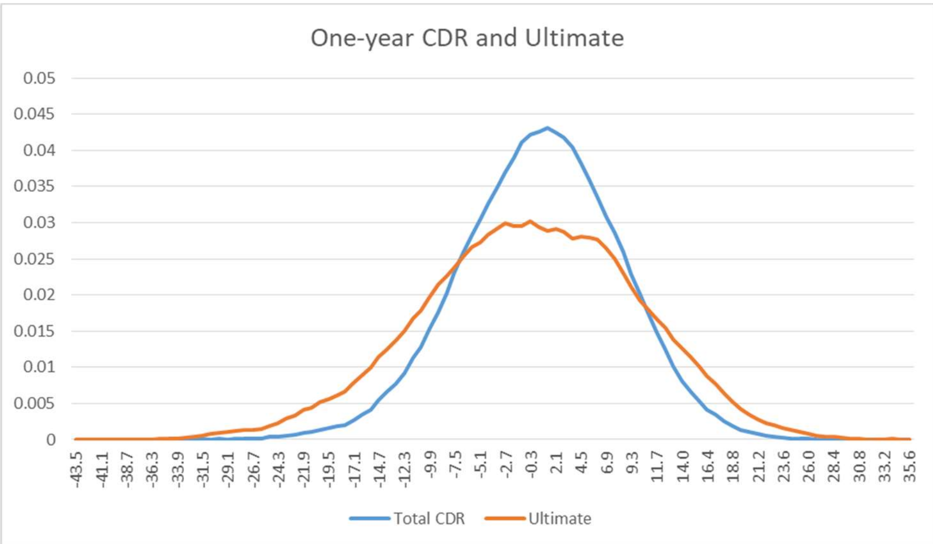

FIGURE 5-7

Next, we back test the one-year results using historical loss ratios for this triangle of claims. The original data set in (Alai, et al., 2009) and (Alai, et al., 2011) did not contain historical loss ratios, so we have created some fictional loss ratios to illustrate the method, these are shown in Table 5-10.

Accident	Historical Loss Ratios									
Year	0	1	2	3	4	5	6	7	8	9
2005	112%	102%	105%	81%	84%	95%	95%	92%	86%	89%
2006	109%	113%	117%	123%	99%	89%	77%	84%	88%	
2007	116%	132%	137%	181%	139%	136%	152%	151%		
2008	94%	88%	109%	84%	81%	88%	87%			
2009	104%	112%	101%	74%	60%	57%				
2010	116%	96%	107%	92%	78%					
2011	54%	70%	80%	79%						
2012	109%	85%	87%							
2013	78%	88%								
2014	91%									

TABLE 5-10

As described in section 4.6, we assume that loss ratios have a lognormal distribution with mean given by the previous year's loss ratio, and CoV implied by the one-year CDR RMSEP. We then calculate the percentiles of the historic loss ratios. These are shown in Table 5-11.

Accident	Percentile of Historical Loss Ratios									
Year	0	1	2	3	4	5	6	7	8	9
2005		0.297	0.593	0.040	0.644	0.832	0.487	0.328	0.078	0.913
2006		0.633	0.615	0.672	0.054	0.253	0.075	0.955	0.868	
2007		0.827	0.617	0.979	0.028	0.463	0.891	0.434		
2008		0.370	0.914	0.038	0.406	0.761	0.478			
2009		0.726	0.291	0.016	0.073	0.366				
2010		0.121	0.777	0.157	0.126					
2011		0.963	0.798	0.493						
2012		0.061	0.585							
2013		0.808								
2014										

TABLE 5-11

There is no obvious calendar year, development year, or accident year trend in the percentiles. We expect the percentiles to be uniformly distributed. The graph in Figure 5-8 shows the empirical CDF of the percentiles. This shows that the percentiles are uniformly distributed.

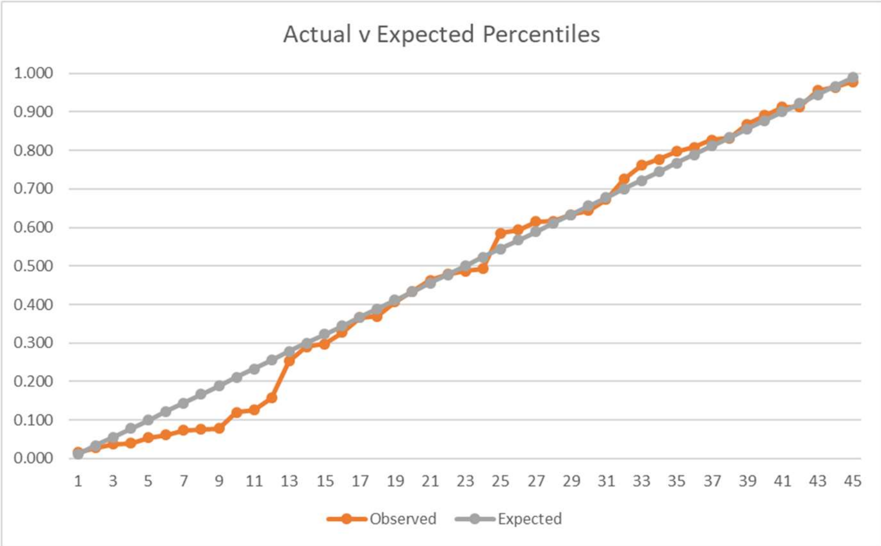

FIGURE 5-8

The initial validation concluded that neither the Mack nor ODP models were appropriate, and was somewhat ambivalent about the AMW-BF model, concluding that “the actuary should then consider whether an additional loading for model error is required.” The above validation has found no further issues with extending the analysis done with this model to the one-year case, however the caveat from the ultimate analysis still applies to the one-year analysis.

5.2.2 Comparison of Methods

In this subsection we compare the results of the AMW-BF presented in the previous section with the Mack and ODP models. The ODP model gives completely unreasonable results, and Mack’s model gives much higher RMSEP than the AMW-BF model. This gives further reason for rejecting the use of these models. Since the AMW-BF model is based on the ODP model, the unreasonable results for this model cast doubt on whether the AMW-BF is a suitable model for this data set.

The tables below present the results; Table 5-12 shows the results for Mack’s model, Table 5-13 for the ODP model (with a constant scale parameter).

Accident Year	Ultimate	Reserve	Ultimate Risk RMSEP	One-Year CDR RMSEP	Emergence %
2005	10,068	0			
2006	19,421	88	94	94	100%
2007	36,339	5,639	4,302	4,300	100%
2008	19,906	3,705	2,741	1,274	46%
2009	11,309	2,984	2,010	995	50%
2010	17,082	9,269	5,129	4,441	87%
2011	16,273	11,211	6,182	3,546	57%
2012	21,677	17,796	8,542	4,323	51%
2013	16,221	15,476	11,104	6,970	63%
2014	46,906	46,600	50,353	35,657	71%
Total	215,201	112,767	56,881	39,971	70%

TABLE 5-12

Accident Year	Ultimate	Reserve	Ultimate Risk RMSEP	One-Year CDR RMSEP	Emergence %
2005	10,068				
2006	19,408	75	493	493	100%
2007	35,074	4,374	2,697	2,666	99%
2008	19,117	2,916	1,891	1,114	59%
2009	10,617	2,292	1,533	949	62%
2010	14,872	7,059	3,165	2,623	83%
2011	13,158	8,096	3,865	2,607	67%
2012	16,689	12,808	6,362	4,519	71%
2013	10,772	10,027	10,282	8,750	85%
2014	102,034	101,728	6,240,351	6,371,840	102%
Total	251,810	149,376	6,240,613	6,372,058	102%

TABLE 5-13

6 CONCLUSION

The aim of this paper has been to build on the working party's first paper, by presenting an introduction to common methods used to estimate one-year reserve risk. For two of the methods described – the Merz-Wüthrich formula, and the Actuary-in-the-Box – it is intended as a guide to the existing literature. For the third method – Emergence Patterns – the intention is to present, for the first time, a unified systematic treatment of the topic. Much of this has been developed in practice, but what has appeared to date has been piecemeal and scattered across various conference presentations.

The working party has also developed implementations of some of the methods described in this paper, and the working party's first paper. These are available in the working party's repository in GitHub.

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8 GLOSSARY

For convenience, a list of common acronyms or abbreviations used in the paper is presented here.

AMW-BF	Alai, Merz, Wüthrich, Bornhuetter-Ferguson stochastic model
BF	Bornhuetter-Ferguson model
CL	Chain Ladder
CDF	Cumulative Distribution Function
CDR	Claims Development Result
CLRM	Complementary Loss Ratio Method
CoC	Cost of Capital
CoV	Coefficient of Variation
CY	Calendar Year
DA	Delegated Act
DCL	Double Chain Ladder model
DY	Development Year
EIOPA	European Insurance and Occupational Pensions Authority
IBNR	Incurred But Not Reported
IBNER	Incurred But Not Enough Reported
ICAS	Individual Capital Adequacy Standards
LoB	Line of Business
MCL	Munich Chain Ladder model
MSEP	Mean Square Error of Prediction
ODP	usually refers to the Over-dispersed Poisson model, but may occasionally refer to the Over-Dispersed Poisson probability distribution.
PDF	Probability Density Function

QIS	Quantitative Impact Study
RMSEP	Root Mean Square Error of Prediction (i.e. the square root of the MSEP)
SCR	Solvency Capital Requirement
SST	Swiss Solvency Test
USP	Undertaking Specific Parameters

9 APPENDIX A – ONE-YEAR RESERVE RISK

9.1 INTRODUCTION

In this appendix we develop in detail the concepts and notation needed to discuss one-year reserve risk. There are two further subsections, the first contains the main part of the discussion, and the other contains supporting results.

9.2 MAIN RESULTS

For $t = 1, 2, 3, \dots$ by the *time period* t we mean the interval $(t - 1, t]$, so the time period t includes all times s where $t - 1 < s \leq t$. By the *time point* t we mean precisely the instant at time t, so the time point t is the very last instant of the time period t.

In what follows we will discuss conditional expectation and conditional variance. We use the following notation for this. First let I_t denote the information known at time point t. For a random variable X the expectation of X, conditional on the information known at time point t is denoted $E[U \mid I_t]$, and we usually abbreviate this to $E_t[U]$. Similarly, we denote the variance of X, conditional on the information known at time point t by $Var_t(X)$.

We denote the claims paid during time period t by P_t. We also suppose that all claims are paid by time point n, so if $t > n$ then $P_t = 0$. Since we do not know the amount of claims that will be paid during future time periods, the P_t are random variables. The *ultimate claims* paid is given by

$$U = \sum_{t=1}^{n} P_t$$

The distribution of U gives us the *ultimate risk of the ultimate claims*. However, we need to take account of the time point that the estimate of the ultimate risk is made at. For example, if the risk measure we are using is variance then the ultimate risk of the ultimate claims at time point t is the variance of U conditional on the information known at time point t i.e. $Var_t(U)$.

At each time point $t = 1, \dots, n$ an estimate will be made of the value of the ultimate claims. The estimate of the ultimate claims at time point t is denoted by U_t and is given by the conditional expectation.

$$U_t = E[U \mid I_t] = E_t[U]$$

The sequence of estimates of the ultimate claims is $U_1, \dots, U_n$. The successive estimates are unbiased as the current value is the best estimate of any future value i.e. $E_t[\, U_{t+1} \,] = U_t$, or more generally if $s < t$ then $E_s[\, U_t \,] = U_s$. In other words, the sequence forms a *martingale*. This has some interesting consequences as we will shortly see.

For one-year risk we need to consider the movement from one estimate of the ultimate claims to the next. This movement is called the *claims development result*, or CDR, and is given by

$$CDR_{t,1} = U_t - U_{t+1}$$

We can extend this to consider the movement in the estimate of ultimate claims over any period of time to get

$$CDR_{t,k} = U_t - U_{t+k}$$

We sometimes denote $CDR_{t,1}$ by CDR_t. Because the sequence $U_1, \dots, U_n$ forms a martingale the expected value of the claims development result is zero:

$$E_t\left[CDR_{t,k}\right] = E_t[U_t] - E_t[U_{t+k}] = U_t - U_t = 0$$

And then

$$E\left[CDR_{t,k}\right] = E\left[E_t\left[CDR_{t,k}\right]\right] = 0$$

A further consequence is that the linear correlation between non-overlapping CDRs is also zero. Specifically if $t_1 + k_1 \le t_2$ then the linear correlation between CDR_{t_1,k_1} and CDR_{t_2,k_2} is zero. This result is proved in general in section 9.3 below.

Following from this we get the nice fact that the variance of the ultimate risk decomposes as the sum of the variances of the claims development results. More precisely

$$Var_t(U) = \sum_{k=0}^{n-t-1} Var_t(CDR_{t+k})$$

This follows easily from the observation that

$$U = U_t - \sum_{k=0}^{n-t-1} CDR_{t+k}$$

and the fact that the linear correlation between the CDRs is zero, and that U_t is known with certainty at time t. It further follows that $Var_t(CDR_t) \le Var_t(U)$ and so the one-year risk is less than or equal to the ultimate risk when variance is used as the risk measure.

In fact, we can generalize the above a little. If $s < t$ then

$$U_t = U_s - \sum_{k=0}^{t-s-1} CDR_{s+k}$$

and so

$$Var_s(U_t) = \sum_{k=0}^{t-s-1} Var_s(CDR_{s+k})$$

From this it is easy to see that if $s < l < t$ then $Var_s(U_l) \leq Var_s(U_t)$. This is just saying that if we extend our time horizon further into the future then the risk does not reduce.

Note that we can use the law of total variance (see section 9.3 below) together with the fact that the expected value of the CDR is zero to see that for $s \leq l \leq t$ we have $E_s[Var_l(CDR_t)] = Var_s(CDR_t)$. Similarly we can show that $E_s[Var_l(U_t)] = \sum_{k=0}^{t-l-1} Var_s(CDR_{l+k})$. The current estimates of the ultimate risk are therefore best estimates of future estimates of ultimate risk.

We can also derive results similar to those above about the covariance between the ultimate risks for different classes or origin periods, and the component claims development results. Let i and j denote two different classes or origin periods, and use the bottom left sub-script to denote this in the relevant quantities. Then, if $s < t$

$$Cov_s({}_iU_t, {}_jU_t) = \sum_{k=0}^{t-s-1} Cov_s({}_iCDR_{s+k}, {}_jCDR_{s+k})$$

We can prove this as follows. First

$$Cov_s({}_iU_t, {}_jU_t) = Cov_s\left({}_iU_s - \sum\nolimits_k {}_iCDR_{s+k}\,, {}_jU_s - \sum\nolimits_k {}_jCDR_{s+k}\right)$$

Since ${}_iU_s$ and ${}_jU_s$ are known with certainty at time s, the above simplifies to

$$Cov_s({}_iU_t, {}_jU_t) = Cov_s\left(\sum\nolimits_k {}_iCDR_{s+k}\,, \sum\nolimits_k {}_jCDR_{s+k}\right)$$

Since covariance is linear in both arguments this becomes

$$Cov_s({}_iU_t, {}_jU_t) = \sum\nolimits_k \sum\nolimits_l Cov_s({}_iCDR_{s+k}, {}_jCDR_{s+l})$$

We then apply theorem 9.3.2, from section 9.3 below to see that $Cov_s({}_iCDR_{s+k}, {}_jCDR_{s+l}) = 0$ unless $k = l$, which gives us the result.

We can also express the claims development result as a movement in the reserves, in which case we also need to consider the claims paid over the period. To make this more precise we first need to make a couple more definitions.

The *outstanding claims* at time point t is the sum of claims paid after time point t and is therefore given by the sum

$$OC_t = \sum_{s=t+1}^{n} P_s$$

The *claims reserve* at time point t is denoted by R_t and is given by

$$R_t = E_t[OC_t] = U_t - \sum_{s=1}^{t} P_s$$

The one-year claims development result can then be expressed

$$CDR_t = R_t - P_{t+1} - R_{t+1}$$

To extend this a multi-period claims development result, first define for $k = 0, 1, 2, \dots$ $P_{t,k} = \sum_{s=t}^{t+k-1} P_s$ then we can express the claims development result over more than one period, as follows

$$CDR_{t,k} = R_t - P_{t+1,k} - R_{t+k}$$

It is a straight-forward application of the definitions above to show that the two different ways of expressing the CDR are the same.

9.3 Supporting Results

In this sub-section we state and prove a few results used in the appendices. These results are well known and we prove them here for completeness. The results are that the linear correlation of non-overlapping claims development results is zero, and the "Law of Total Variance".

A time series of random variables $\{X_t : t = 0,\ 1, 2, \dots\}$ is a martingale if for all $t = 1, 2, \dots$ $E[X_t \mid I_{t-1}] = X_{t-1}$, where I_t denotes the information known at time t. It then follows that for s$< t$ $E[X_t \mid I_s] = X_s$, since $E[X_{t-1} \mid I_s] = E[E[X_t \mid I_{t-1}] \mid I_s] = E[X_t \mid I_s]$.

Define the difference between terms of the time series by $\Delta X_{s,t} = X_t - X_s$. Then for a martingale $E\left[\Delta X_{s,t}\right] = 0$. The expectation is unconditional since $E\left[\Delta X_{s,t}\right] = E\left[E\left[\Delta X_{s,t} \mid I_s\right]\right] = E\left[E[X_t - X_s \mid I_s]\right] = E[E[X_t \mid I_s] - X_s] = E[X_s - X_s] = 0$.

We then have the following.

Theorem 9.3.1. If $\{X_t : t = 0,\ 1, 2, \dots\}$ is a martingale, and $s_1 < t_1 \leq s_2 < t_2$ then the linear correlation between $\Delta X_{s_1,t_1}$ and $\Delta X_{s_2,t_2}$ is zero.

Proof. We show that the covariance is zero, from this it follows that the correlation is zero. The covariance between $\Delta X_{s_1,t_1}$ and $\Delta X_{s_2,t_2}$ is given by

$$Cov\left(\Delta X_{s_1,t_1}, \Delta X_{s_2,t_2}\right) = E\left[\Delta X_{s_1,t_1} \Delta X_{s_2,t_2}\right] - E\left[\Delta X_{s_1,t_1}\right] E\left[\Delta X_{s_2,t_2}\right]$$

Since $E\left[\Delta X_{s_1,t_1}\right] = E\left[\Delta X_{s_2,t_2}\right] = 0$, we consider $E\left[\Delta X_{s_1,t_1}\Delta X_{s_2,t_2}\right]$. Now $E\left[\Delta X_{s_1,t_1}\Delta X_{s_2,t_2}\right] = E\left[E\left[\Delta X_{s_1,t_1}\Delta X_{s_2,t_2} \mid I_{s_2}\right]\right] = E\left[\Delta X_{s_1,t_1} E\left[\Delta X_{s_2,t_2} \mid I_{s_2}\right]\right] = E\left[\Delta X_{s_1,t_1} 0\right] = 0$. Hence the covariance, and so the correlation is zero. QED.

We can apply the above argument to get a similar result between two different martingales:

Theorem 9.3.2. If $\{X_t : t = 0,\ 1, 2, \dots\}$ and $\{Y_t : t = 0,\ 1, 2, \dots\}$ are martingales, and $s_1 < t_1 \leq s_2 < t_2$ then the linear correlation between $\Delta X_{s_1,t_1}$ and $\Delta Y_{s_2,t_2}$ is zero.

Proof. We show that the covariance is zero, from this it follows that the correlation is zero. The covariance between $\Delta X_{s_1,t_1}$ and $\Delta Y_{s_2,t_2}$ is given by

$$Cov\left(\Delta X_{s_1,t_1}, \Delta Y_{s_2,t_2}\right) = E\left[\Delta X_{s_1,t_1}\Delta Y_{s_2,t_2}\right] - E\left[\Delta X_{s_1,t_1}\right]E\left[\Delta Y_{s_2,t_2}\right]$$

Since $E\left[\Delta X_{s_1,t_1}\right] = E\left[\Delta Y_{s_2,t_2}\right] = 0$, we consider $E\left[\Delta X_{s_1,t_1}\Delta Y_{s_2,t_2}\right]$. Now $E\left[\Delta X_{s_1,t_1}\Delta Y_{s_2,t_2}\right] = E\left[E\left[\Delta X_{s_1,t_1}\Delta Y_{s_2,t_2} \mid I_{s_2}\right]\right] = E\left[\Delta X_{s_1,t_1} E\left[\Delta Y_{s_2,t_2} \mid I_{s_2}\right]\right] = E\left[\Delta X_{s_1,t_1} 0\right] = 0$. Hence the covariance, and so the correlation is zero. QED.

The "Law of Total Variance" is the following:

Theorem 9.3.3. Suppose that $s > t$ then

$$Var_t(X) = E_t[Var_s(X)] + Var_t(E_s[X])$$

Proof. Recall that for any random variable Z, $Var(Z) = E[Z^2] - E[Z]^2$. So

$$E_t[Var_s(X)] = E_t[E_s[X^2] - E_s[X]^2] = E_t[X^2] - E_t[E_s[X]^2]$$

and

$$Var_t(E_s[X]) = E_t[E_s[X]^2] - E_t\left[E_s[X]\right]^2 = E_t[E_s[X]^2] - E_t[X]^2$$

Adding the two above expressions together gives

$$E_t[X^2] - E_t[E_s[X]^2] + E_t[E_s[X]^2] - E_t[X]^2 = E_t[X^2] - E_t[X]^2 = Var_t(X)$$

Which gives the result.

10 APPENDIX B – EMERGENCE FACTORS AND EMERGENCE PATTERNS

10.1 INTRODUCTION

In this appendix we develop in technical detail the ideas about emergence factors and emergence patterns discussed in section 3.3 above. To do this we build on the concepts and notation from section 9 appendix A. We assume that the reader of this appendix has read section 3.3.

As discussed in section 3.3.1 the scaling of the ultimate risk distribution should only affect the volatility, not the mean. The general formula applying emergence factors therefore has the form:

$$\hat{X} = \alpha(X - E[X]) + E[X]$$

Where X denotes the ultimate risk distribution, α denotes the emergence factor applied, and $\hat{X}$ denotes the one-year distribution. The emergence factor scales down the standard deviation, it is therefore natural to develop the theory of emergence patterns using standard deviation as the risk measure, and to define an emergence factor as a ratio of two standard deviations. We use the notation $\sigma(X)$ to denote the standard deviation of a random variable X, and $\sigma_t(X)$ to denote the standard deviation of X conditional on I_t.

The title of this appendix is "Emergence Factors and Emergence Patterns" we now describe what is meant by "factor" and "pattern". In the formula above the symbol α denotes an emergence factor. This is a single variable. As we discussed in section 3.3 there are several different types of emergence factor. An emergence pattern is a list of emergence factors. As we saw in section 3.3 there are different ways to list emergence factors, and so different emergence patterns. In particular the same emergence factors can be listed in different ways to give different emergence patterns. We discuss this in more detail below.

10.2 DIFFERENT VARIETIES OF EMERGENCE FACTORS AND EMERGENCE PATTERNS

In this subsection we state the definition of the various different types of emergence factors and emergence patterns introduced in section 3.3.2, and we also describe some of their properties. We do this in more technical detail than was appropriate for section 3.3.2

Table 10-1 was first presented in section 3.3.2 as Table 3-3

	1	2	3	Emergence Factor	Emergence Pattern
1	Ultimate	Unconditional	Life-time	α_k	$(\alpha_0, \dots, \alpha_n)$
2	Ultimate	Unconditional	Risk-decay	Does not make sense	
3	Ultimate	Conditional	Life-time	$\alpha_{t,k}$	$(\alpha_{t,0}, \dots, \alpha_{t,n-t})$
4	Ultimate	Conditional	Risk-decay	$\alpha_{t,k}$	$(\alpha_{0,1}, \alpha_{1,1}, \dots, \alpha_{n-1,1})$
5	Outstanding	Unconditional	Life-time	β_k	$(\beta_0, \dots, \beta_{n-1})$
6	Outstanding	Unconditional	Risk-decay	Does not make sense	

7	Outstanding	Conditional	Life-time	$\beta_{t,k}$	$(\beta_{t,0}, \dots, \beta_{t,n-t-1})$
8	Outstanding	Conditional	Risk-decay	$\beta_{t,k}$	$(\beta_{0,1}, \beta_{1,1}, \dots, \beta_{n-2,1})$

TABLE 10-1

We now give precise definitions of each of the emergence factors. First we discuss the ultimate factors then we discuss the outstanding factors.

10.2.1 The ultimate emergence factors

The ultimate unconditional emergence factor α_k is defined as the ratio of the standard deviation of the estimate of the ultimate claims at time point k to the standard deviation of the ultimate claims:

$$\alpha_k = \frac{\sigma(U_k)}{\sigma(U)}$$

The factor α_k can be interpreted as the cumulative proportion of the total life-time risk which will have emerged by time point k. All α_k are deterministic, and since standard deviations are all positive $\alpha_k > 0$. We can show that $\alpha_0 = 0$, $\alpha_n = 1$, and $0 \leq \alpha_k \leq 1$. We can also show that if $l < k$ then $\alpha_l \leq \alpha_k$. These results are proved below.

The ultimate conditional emergence factor $\alpha_{t,k}$ is defined as the ratio of the conditional standard deviation of the estimate of the ultimate claims at time point t to the conditional standard deviation of the ultimate claims; the standard deviations are conditional on the information known at time point t.

$$\alpha_{t,k} = \frac{\sigma_t(U_{t+k})}{\sigma_t(U)}$$

for $k = 0, 1, \dots, n-t$. The factor $\alpha_{t,k}$ can be interpreted as the cumulative proportion of the total life-time risk remaining at time point t which will have emerged by time point $t+k$. The factors $\alpha_{t,k}$ are not known with certainty until time point t. Similarly, to the unconditional patterns above we can show that $0 = \alpha_{t,0} \leq \alpha_{t,1} \leq \cdots \leq \alpha_{t,n-t} = 1$.

It is clear that the ultimate unconditional emergence factors are a special case of the ultimate conditional emergence factors as $\alpha_k = \alpha_{0,k}$.

We now show that $0 = \alpha_{t,0} \leq \alpha_{t,1} \leq \cdots \leq \alpha_{t,n-t} = 1$. By definition the ultimate conditional emergence factors are

$$\alpha_{t,k} = \frac{\sigma_t(U_{t+k})}{\sigma_t(U)}$$

Since U_t is known with certainty at time point t we have $\sigma_t(U_t) = 0$ and so $\alpha_{t,0} = 0$. Since standard deviations are positive it is clear that $\alpha_{t,k} \geq 0$. Now

$$\alpha_{t,k}^2 = \frac{Var_t(U_{t+k})}{Var_t(U)}$$

and recall from subsection 9.2 that if $s < t$ then

$$Var_s(U_t) = \sum_{k=0}^{t-s-1} Var_s(CDR_{s+k})$$

Therefore

$$\alpha_{t,k}^2 = \frac{\sum_{l=0}^{k-1} Var_t(CDR_{t+l})}{\sum_{l=0}^{n-t-1} Var_t(CDR_{t+l})}$$

From which it is clear that $\alpha_{t,k} \leq 1$, that if $k_1 < k_2$ then $\alpha_{t,k_1} \leq \alpha_{t,k_2}$, and that $\alpha_{t,n-t} = 1$.

10.2.2 The ultimate emergence patterns

An emergence pattern is a list of emergence factors. The ultimate unconditional life-time emergence pattern is a list of the ultimate unconditional emergence factors.

$$(\alpha_0 , \dots , \alpha_n)$$

This pattern expresses how the life-time risk emerges. The values start at $\alpha_0 = 0$ and increase to $\alpha_n = 1$. It can also be expressed as a graph as shown in Figure 10-1

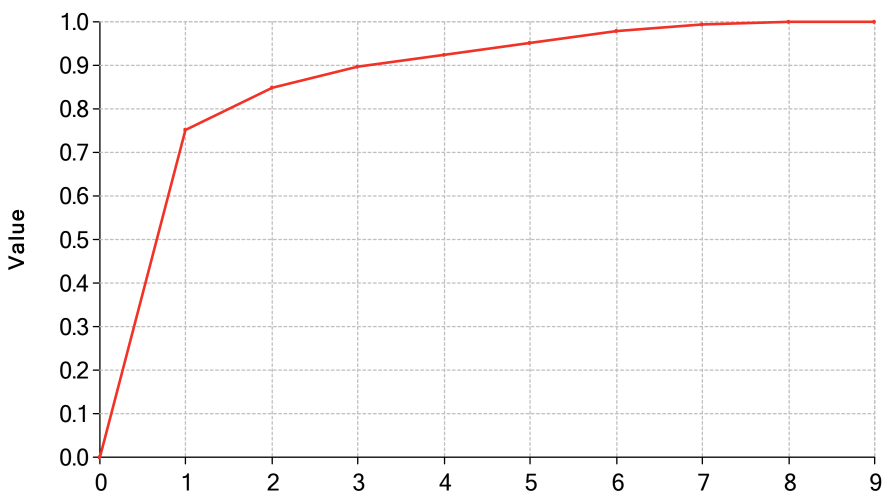

FIGURE 10-1

The ultimate conditional life-time emergence pattern is a list of the ultimate conditional emergence factors, arranged so that it expresses how the remaining risk of a partially developed cohort emerges from the current time point t until it is fully run off by time point n. There are $n - t$ future periods, and so the pattern contains $n - t + 1$ factors.

$$\left(\alpha_{t,0}\,,\dots,\alpha_{t,n-t}\right)$$

The values start at $\alpha_{t,0} = 0$ and increase to $\alpha_{t,n-t} = 1$. It can also be expressed in a graph as in Figure 10-2

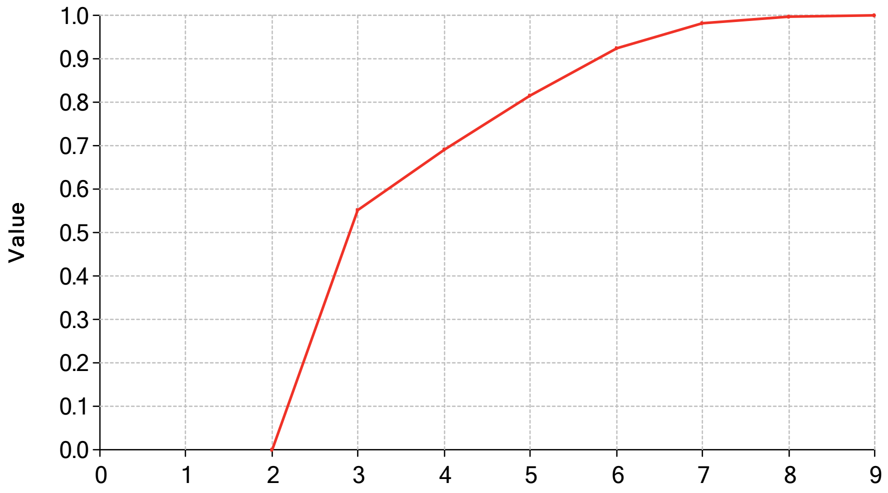

FIGURE 10-2

The ultimate conditional risk-decay emergence pattern is a list of the ultimate conditional emergence factors, arranged so that it expresses, for a sequence of origin periods, the proportion of the remaining risk that will emerge over the next period.

$$\left(\alpha_{0,1}\,,\dots,\alpha_{n-1,1}\right)$$

There is no reason to suppose that $\alpha_{0,1} = 0$, $\alpha_{n-1,1} = 1$, or that the sequence is increasing. It can be thought of as a cross-section of the ultimate conditional life-time emergence patterns. This is shown in the graph Figure 10-3 where the black line shows the risk-decay emergence pattern, and the red lines show the life-time emergence patterns.

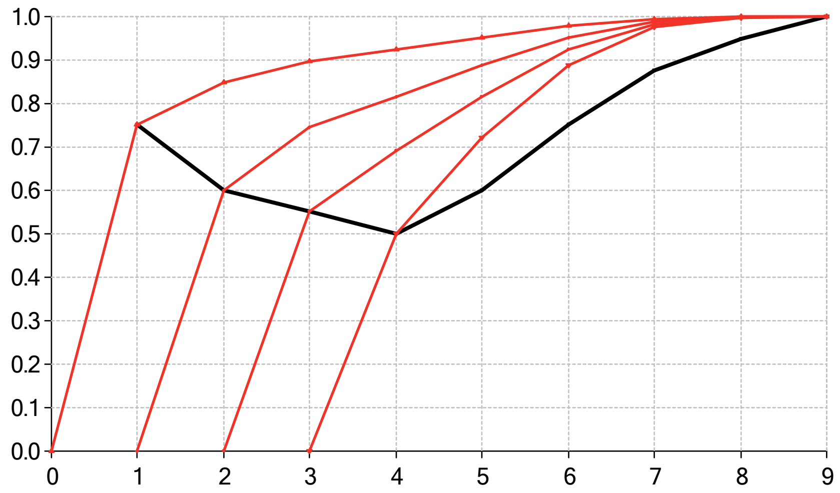

FIGURE 10-3

10.2.3 THE OUTSTANDING EMERGENCE FACTORS

The outstanding unconditional emergence factor β_k is defined for $k = 0, \dots, n-1$ as the ratio of the standard deviation of the claims reserve at time point k to the standard deviation of the claims outstanding at time point k:

$$\beta_k = \frac{\sigma(R_k)}{\sigma(OC_k)}$$

The factor β_k can be interpreted as the proportion of the life-time risk in the claims outstanding at time point k that will emerge by time point k. All the β_k are deterministic, and since standard deviations are all positive $\beta_k \geq 0$. As a consequence of the law of total variance (see section 9.3) $\beta_k \leq 1$. We can also show that $\beta_0 = 0$. However, it is not the case in general that $(\beta_0, \dots, \beta_{n-1})$ is an increasing sequence, since the denominator varies from factor to factor.

The outstanding conditional emergence factor $\beta_{t,k}$ is defined for $t = 0, \dots, n-1$ and $k = 0, \dots, n-t-1$ as the ratio of the conditional standard deviation of the claims reserve at time point $t+k$ to the conditional standard deviation of the claims outstanding at time point $t+k$; the standard deviations are conditional on the information known at time point t:

$$\beta_{t,k} = \frac{\sigma_t(R_{t+k})}{\sigma_t(OC_{t+k})}$$

The factor $\beta_{t,k}$ can be interpreted as the proportion of the risk at time point t in the claims outstanding at time point $t+k$ that will emerge between time point t and time point $t+k$. The factors $\beta_{t,k}$ are not known with certainty until time point t. We can show that $\beta_{t,0} = 0$, and that $0 \leq \beta_{t,k} \leq 1$ (see below). However, it is not the case in general that $\left(\beta_{t,0}, \dots, \beta_{t,n-t-1}\right)$ is an increasing sequence, since the denominator varies from factor to factor.

It is clear that the outstanding unconditional emergence factors are a special case of the outstanding conditional emergence factors as $\beta_k = \beta_{0,k}$.

We now show that that $\beta_{t,0} = 0$, and that $0 \leq \beta_{t,k} \leq 1$. We use the *Law of Total Variance*, which is stated in subsection 9.3 above. By definition the conditional cumulative ultimate emergence factors are

$$\beta_{t,k} = \frac{\sigma_t(R_{t+k})}{\sigma_t(OC_{t+k})}$$

Since standard deviations are positive, we immediately have that $\beta_{t,k} \geq 0$. Now

$$\beta_{t,k}^2 = \frac{Var_t(R_{t+k})}{Var_t(OC_{t+k})} = \frac{Var_t(E_{t+k}[OC_{t+k}])}{Var_t(OC_{t+k})}$$

Since $Var_t(E_{t+k}[OC_{t+k}]) > 0$ and, by the law of total variance, $Var_t(E_{t+k}[OC_{t+k}]) = Var_t(OC_{t+k}) - E_t[Var_{t+k}(OC_{t+k})]$, it follows that $\beta_{t,k} \leq 1$.

10.2.4 THE OUTSTANDING EMERGENCE PATTERNS

An emergence pattern is a list of emergence factors. The outstanding unconditional life-time emergence pattern is a list of the outstanding unconditional emergence factors

$$(\beta_0, \dots, \beta_{n-1})$$

The pattern expresses how outstanding life-time risk emerges. The values start at $\beta_0 = 0$, however as discussed in section 10.2.3 above the sequence is not necessarily increasing, and does not end at one.

The outstanding conditional life-time emergence pattern is a list of the outstanding conditional emergence factors, arranged so that it expresses how the remaining outstanding risk of a partially developed cohort emerges from the from the current time point t until it is fully run off by time point n. There are $n - t$ future periods, and so the pattern contains $n - t$ factors, one fewer than the corresponding ultimate emergence pattern.

$$(\beta_{t,0}, \dots, \beta_{t,n-t-1})$$

The values start at $\beta_{t,0} = 0$, however as discussed in section 10.2.3 above the sequence is not necessarily increasing, and does not end at one.

The outstanding conditional risk-decay emergence pattern is a list of the outstanding conditional emergence factors, arranged so that it expresses, for a sequence of origin periods, the proportion of the remaining outstanding risk that will emerge over the next period.

$$(\beta_{0,1}, \dots, \beta_{n-2,1})$$

There is no reason to suppose that $\beta_{0,1} = 0$, $\beta_{n-2,1} = 1$, or that the sequence is increasing. It can be thought of as a cross-section of the outstanding conditional life-time emergence patterns.

10.3 Calculating the Claims Development Result

In this subsection we discuss some issues that arise when using emergence factors to calculate the Claims Development Result. We consider how to use emergence patterns to calculate the CDR by origin period, and then some of the issues with aggregating these to get the total CDR. For the CDR the appropriate emergence patterns to consider are the risk-decay emergence patterns. For ease of exposition we consider only the one-year CDR in this section.

First consider the ultimate emergence patterns. Suppose we have n origin periods $i = 1, \dots, n$. Let ${}_iU$ denote the ultimate risk distribution of the ultimate claims for origin period i. Suppose that at the current time origin period i has undergone $n - i + 1$ periods of development. So the current, or opening, best estimate for the ultimate claims for origin period i is ${}_iU_{n-i+1}$, and the closing best estimate of the ultimate claims is ${}_iU_{n-i+2}$. We use the ultimate conditional risk-decay emergence pattern to estimate the closing best estimate. The ultimate conditional risk-decay emergence pattern is

$$(\alpha_{0,1}, \dots, \alpha_{n-1,1})$$

The closing best estimate of ultimate claims for origin period i is then estimated using the following formula

$${}_i\widehat{U}_{n-i+2} = \alpha_{n-i+1,1}({}_iU - {}_iU_{n-i+1}) + {}_iU_{n-i+1}$$

We can then calculate the CDR for origin period i using the formula

$${}_i\widehat{CDR}_{n-i+1} = {}_iU_{n-i+1} - {}_i\widehat{U}_{n-i+2}$$

Now consider the outstanding emergence patterns. Again suppose that we have n origin periods $i = 1, \dots, n$, and that at the current time origin period i has undergone $n - i + 1$ periods of development. The current, or opening, best estimate reserve for origin period i is therefore ${}_iR_{n-i+1}$, and the closing best estimate reserve is ${}_iR_{n-i+2}$. We use the outstanding conditional risk-decay emergence pattern to estimate the closing best estimate reserve. The outstanding conditional risk-decay emergence pattern is

$$(\beta_{0,1}, \dots, \beta_{n-2,1})$$

To estimate the closing best estimate reserve, we apply these emergence factors to the closing outstanding claims. For origin period i this is ${}_iOC_{n-i+2}$. The closing best estimate reserve for origin period i is then estimated using the following formula

$$ {}_{i}\hat{R}_{n-i+2} = {}_{i}\beta_{n-i+1,1}({}_{i}OC_{n-i+2} - E_{n-i+1}[{}_{i}OC_{n-i+2}]) + E_{n-i+1}[{}_{i}OC_{n-i+2}] $$

We can then calculate the CDR for origin period i using the formula

$$ {}_{i}\widehat{CDR}_{n-i+1} = {}_{i}R_{n-i+1} - {}_{i}P_{n-i+2} - {}_{i}\hat{R}_{n-i+2} $$

Note that the distinction between the two approaches is that the CDR calculated using the outstanding emergence factors includes the full risk of the claims paid over the next period, where as in the CDR calculated using the ultimate emergence factors the risk from the claims paid over the next period is scaled down by the emergence factor applied. This is a reason for preferring the outstanding emergence factors over the ultimate emergence factors.

However, there is another issue with the CDR calculated using the outstanding emergence factors. The emergence factor is defined so that the standard deviation of ${}_{i}\hat{R}_{n-i+2}$ is equal to the standard deviation of ${}_{i}R_{n-i+2}$. However as ${}_{i}\hat{R}_{n-i+2}$ is a rescaling of the distribution of ${}_{i}OC_{n-i+2}$, the linear correlation between ${}_{i}\hat{R}_{n-i+2}$ and ${}_{i}P_{n-i+2}$ is equal to the linear correlation between ${}_{i}OC_{n-i+2}$ and ${}_{i}P_{n-i+2}$. In general, this will be different from the linear correlation between ${}_{i}R_{n-i+2}$ and ${}_{i}P_{n-i+2}$. The standard deviation of ${}_{i}\widehat{CDR}_{n-i+1}$ will therefore in general be different from the standard deviation of ${}_{i}CDR_{n-i+1}$. This is not desirable, and the problem does not arise with the CDR calculated using the ultimate emergence factors. However, in practice the difference in the standard deviations is usually small, and it is possible to parameterise the outstanding emergence factors so that the standard deviations of the CDRs agree. This is discussed further in subsection 10.7 below.

A similar problem occurs when we aggregate the CDRs over different origin periods. In this case the problem affects both the ultimate emergence factors and the outstanding emergence factors. In the case of the CDR calculated using the ultimate emergence factors, because ${}_{i}\hat{U}_{n-i+2}$ is a rescaling of ${}_{i}U$ the linear correlation between the ${}_{i}\widehat{CDR}_{n-i+1}$ for different origin periods i is equal to the linear correlation between the ${}_{i}U$, conditional on the information currently known. This is different in general from the linear correlation between the ${}_{i}CDR_{n-i+1}$ for different origin period i and so the standard deviation of the total CDR will be different. However, in practice the difference in the standard deviations is usually small, and it is possible to adjust the emergence factors so that the standard deviations of the total CDRs agree. This is discussed further in subsection 10.7 below.

10.4 Emergence Factors and Origin Periods

The emergence patterns described above in general vary by origin period. Ideally, we would like the emergence factors to not vary by origin period. We now discuss assumptions which ensure that this happens.

All the emergence factors defined above are the ratio of the standard deviation of two random variables, which have the same mean. So, if the relevant standard deviations vary in the same way with the mean across origin periods, then the emergence factor won't vary across origin periods. For example, consider the ultimate conditional emergence factors (where the left subscript i denotes the origin period as in section 10.3 above)

$$ {}_i\alpha_{t,k} = \frac{\sigma_t({}_iU_{t+k})}{\sigma_t({}_iU)} $$

If both standard deviations vary with the mean in the same proportion across origin periods i.e. for all origin periods i, $\sigma_t({}_iU_{t+k}) = CV_{t,k}\,E_t[{}_iU_{t+k}]$ and $\sigma_t({}_iU) = CV_t E_t[{}_iU]$, then since $E_t[{}_iU_{t+k}] = E_t[{}_iU]$ we get

$$ {}_i\alpha_{t,k} = \frac{CV_{t,k}}{CV_t} $$

Which does not vary by origin period. In this example the coefficient of variation is constant across origin periods, however it is clear that other assumptions are possible. For example, we could suppose that the standard deviation varies with the square root of the mean in the same proportion across origin periods. In general, all we need to assume is that the standard deviation varies in the same proportion to the same function of the mean across origin periods.

10.5 Deterministic Emergence Factors

The emergence patterns described above are in general stochastic, as they are defined as the ratio of two conditional standard deviations, and in general are not known with certainty until the start of the period over which they apply. Ideally, we would like the emergence factors to be unconditionally deterministic. We now discuss assumptions which ensure that this happens.

By the *Law of Total Variance* (see subsection 9.3 above) $E[Var_s(CDR_t)] = Var(CDR_t)$. It is therefore not unreasonable to make the assumption that the conditional variances of the CDR are deterministic, and so that $Var_s(CDR_t) = Var(CDR_t) = \sigma_t^2$. Recall from subsection 10.2.1 above that

$$ \alpha_{t,k}^2 = \frac{\sum_{l=0}^{k-1} Var_t(CDR_{t+l})}{\sum_{l=0}^{n-t-1} Var_t(CDR_{t+l})} $$

Therefore, if $Var_s(CDR_t)$ is deterministic for all s and t then the ultimate emergence factors $\alpha_{t,k}$ are also deterministic.

A more general assumption leading to the same conclusion is to assume that $Var_s(CDR_t)$ vary in deterministic proportion to the same random variable. Specifically suppose for some random variables X_s and positive real numbers $\sigma_{s,t}^2$ we have $Var_s(CDR_t) = \sigma_{s,t}^2 X_s$ then

$$ \alpha_{t,k}^2 = \frac{\sigma_{tt}^2 X_t + \cdots + \sigma_{t,t+k-1}^2 X_t}{\sigma_{tt}^2 X_t + \cdots + \sigma_{t,n-1}^2 X_t} = \frac{\sigma_{tt}^2 + \cdots + \sigma_{t,t+k-1}^2}{\sigma_{tt}^2 + \cdots + \sigma_{t,n-1}^2} $$

In fact, this latter condition is both necessary and sufficient for the ultimate emergence factors to be deterministic, and furthermore we can take $X_s = Var_s(CDR_s)$. The above argument proves that the condition is sufficient, we now prove that it is necessary.

First, to make the notation less cumbersome, write $V_{s,t}$ for $Var_s(CDR_t)$. Then

$$\alpha_{t,k}^2 = \frac{\sum_{l=0}^{k-1} V_{t,t+l}}{\sum_{l=0}^{n-t-1} V_{t,t+l}} = \frac{V_{tt} + \cdots + V_{t,t+k-1}}{V_{tt} + \cdots + V_{t,n-1}}$$

Now suppose that all the ultimate emergence factors $\alpha_{t,k}$ are deterministic. First divide the numerator and denominator of the above expression for $\alpha_{t,k}^2$ by the numerator to get

$$\alpha_{t,k}^2 = \frac{1}{1 + \frac{V_{t,t+k} + \cdots + V_{t,n-1}}{V_{tt} + \cdots + V_{t,t+k-1}}}$$

Thus

$$\frac{V_{t,t+k} + \cdots + V_{t,n-1}}{V_{tt} + \cdots + V_{t,t+k-1}} = C_{tk}$$

For some real number C_{tk}. Rearranging we get the following system of $n - t - 1$ equations (where $k = 1, \dots, n - t - 1$)

$$V_{t,t+k} + \cdots + V_{t,n-1} = C_{tk}\left(V_{tt} + \cdots + V_{t,t+k-1}\right)$$

Subtracting the $(k+1)^{th}$ equation from the k^{th} equation gives us the following $n - t - 2$ equations (where $k = 1, \dots, n - t - 2$)

$$V_{t,t+k} = \left(C_{tk} - C_{t,k+1}\right)\left(V_{tt} + \cdots + V_{t,t+k-1}\right) - C_{t,k+1}V_{t,t+k}$$

Rearranging gives us

$$V_{t,t+k} = \frac{C_{tk} - C_{t,k+1}}{1 + C_{t,k+1}}\left(V_{tt} + \cdots + V_{t,t+k-1}\right)$$

We can then use induction to show that $V_{t,t+k}$ is a constant multiple of V_{tt} for all $k = 1, \dots, n - t - 2$. Finally, the $(n - t - 1)^{th}$ equation from the first set of equations asserts that

$$V_{t,n-1} = C_{t,n-t-1}\left(V_{tt} + \cdots + V_{t,n-2}\right)$$

And so $V_{t,t+k}$ is a constant multiple of V_{tt} for $k = n - t - 1$ as well.

10.6 Relationships Between Emergence Factors

In this subsection we state and prove some relationships among the different varieties of emergence factors.

If the ultimate emergence factors are deterministic then

$$1-\alpha_{t,k_1+k_2}^2 = \left(1-\alpha_{t,k_1}^2\right)\left(1-\alpha_{t+k_1,k_2}^2\right)$$

Where $0 \le t \le n-2$, $1 \le k_1 \le n-t-1$ and $1 \le k_2 \le n-t-k_1$. This then allows us to calculate the conditional ultimate emergence factors from the unconditional ultimate emergence factors as rearranging

$$1-\alpha_{0,t+k}^2 = \left(1-\alpha_{0,t}^2\right)\left(1-\alpha_{t,k}^2\right)$$

gives us

$$1-\ \alpha_{t,k}^2 = \frac{1-\ \alpha_{0,t+k}^2}{1-\ \alpha_{0,t}^2}$$

In fact, the first equation holds if and only if the ultimate emergence factors are deterministic. First we prove the "if" part. Suppose that all the ultimate emergence factors are deterministic, then the results from subsection 10.5 tell us that there are real numbers $\sigma_{tt}^2, \dots, \sigma_{t,n-1}^2$ such that

$$\alpha_{t,k}^2 = \frac{\sigma_{tt}^2 + \cdots +\ \sigma_{t,t+k-1}^2}{\sigma_{tt}^2 + \cdots +\ \sigma_{t,n-1}^2}$$

and so

$$1-\ \alpha_{t,k}^2 = \frac{\sigma_{t,t+k}^2 + \cdots +\ \sigma_{t,n-1}^2}{\sigma_{tt}^2 + \cdots +\ \sigma_{t,n-1}^2}$$

Thus

$$1-\alpha_{t,k_1}^2 = \frac{\sigma_{t,t+k_1}^2 + \cdots +\ \sigma_{t,n-1}^2}{\sigma_{tt}^2 + \cdots +\ \sigma_{t,n-1}^2}$$

$$1-\alpha_{t,k_1+k_2}^2 = \frac{\sigma_{t,t+k_1+k_2}^2 + \cdots +\ \sigma_{t,n-1}^2}{\sigma_{tt}^2 + \cdots +\ \sigma_{t,n-1}^2}$$

and

$$1-\alpha_{t+k_1,k_2}^2 = \frac{\sigma_{t+k_1,t+k_1+k_2}^2 + \cdots +\ \sigma_{t+k_1,n-1}^2}{\sigma_{t+k_1,t+k_1}^2 + \cdots +\ \sigma_{t+k_1,n-1}^2}$$

We can quickly see that

$$\frac{1-\ \alpha_{t,k_1+k_2}^2}{1-\ \alpha_{t,k_1}^2} = \frac{\sigma_{t,t+k_1+k_2}^2 + \cdots +\ \sigma_{t,n-1}^2}{\sigma_{t,t+k_1}^2 + \cdots +\ \sigma_{t,n-1}^2}$$

To prove that this is equal to $1-\alpha_{t+k_1,k_2}^2$ we need the following result.

If the emergence factors α_{tk} are all deterministic the for $s \leq l \leq t$ we have $\sigma_{st}^2 = \sigma_{sl}^2 \sigma_{lt}^2$. To prove this, suppose that the emergence factors are all deterministic. Then $V_{st} = \sigma_{st}^2 V_{ss}$ and $V_{lt} = \sigma_{lt}^2 V_{ll}$. Also $V_{st} = E_s[V_{lt}] = E_s[\sigma_{lt}^2 V_{ll}] = \sigma_{lt}^2 E_s[V_{ll}] = \sigma_{lt}^2 V_{sl} = \sigma_{lt}^2 \sigma_{sl}^2 V_{ss}$. Dividing by V_{ss} gives the result.

Applying this result gives, for $s = 0, \dots, n - t - k_1 - 1$

$$\sigma_{t+k_1,t+k_1+s}^2 = \frac{\sigma_{t,t+k_1+s}^2}{\sigma_{t,t+k_1}^2}$$

Substituting this into the appropriate equation above we get

$$1 - \alpha_{t+k_1,k_2}^2 = \frac{\sigma_{t,t+k_1+k_2}^2 + \cdots + \sigma_{t,n-1}^2}{\sigma_{t,t+k_1}^2 + \cdots + \sigma_{t,n-1}^2} = \frac{1 - \alpha_{t,k_1+k_2}^2}{1 - \alpha_{t,k_1}^2}$$

We now prove the "only if" part. Suppose that for all t, k_1, and k_2 where $0 \leq t \leq n-2$, $1 \leq k_1 \leq n - t - 1$ and $1 \leq k_2 \leq n - t - k_1$ the following equations hold

$$1 - \alpha_{t,k_1+k_2}^2 = \left(1 - \alpha_{t,k_1}^2\right)\left(1 - \alpha_{t+k_1,k_2}^2\right)$$

Now recall from subsection 10.5 that

$$\alpha_{t,k}^2 = \frac{\sum_{l=0}^{k-1} V_{t,t+l}}{\sum_{l=0}^{n-t-1} V_{t,t+l}} = \frac{V_{tt} + \cdots + V_{t,t+k-1}}{V_{tt} + \cdots + V_{t,n-1}}$$

And so

$$1 - \alpha_{t,k}^2 = \frac{V_{t,t+k} + \cdots + V_{t,n-1}}{V_{tt} + \cdots + V_{t,n-1}}$$

Therefore, if

$$1 - \alpha_{t,k_1+k_2}^2 = \left(1 - \alpha_{t,k_1}^2\right)\left(1 - \alpha_{t+k_1,k_2}^2\right)$$

then

$$\frac{V_{t,t+k_1+k_2} + \cdots + V_{t,n-1}}{V_{tt} + \cdots + V_{t,n-1}} = \frac{V_{t,t+k_1} + \cdots + V_{t,n-1}}{V_{tt} + \cdots + V_{t,n-1}} \cdot \frac{V_{t+k_1,t+k_1+k_2} + \cdots + V_{t+k_1,n-1}}{V_{t+k_1,t+k_1} + \cdots + V_{t+k_1,n-1}}$$

Simplifying and rearranging gives us

$$\frac{V_{t,t+k_1+k_2} + \cdots + V_{t,n-1}}{V_{t,t+k_1} + \cdots + V_{t,n-1}} = \frac{V_{t+k_1,t+k_1+k_2} + \cdots + V_{t+k_1,n-1}}{V_{t+k_1,t+k_1} + \cdots + V_{t+k_1,n-1}}$$

Notice that the LHS is known with certainty at time t whereas the RHS is ostensibly known with certainly only at time $t + k_1$, although as it is equal to the LHS it must also be known with certainty at time t. Following this observation let $t = 0$, $s = k_1$, and $k = k_2$. Then $1 \leq s \leq n - 1$, and $1 \leq k \leq n - s$ and

$$\frac{V_{0,s+k} + \cdots + V_{0,n-1}}{V_{0,s} + \cdots + V_{0,n-1}} = \frac{V_{s,s+k} + \cdots + V_{s,n-1}}{V_{ss} + \cdots + V_{s,n-1}}$$

The LHS is known with certainty at time 0, and so is deterministic. Therefore, there is a real number D_k such that

$$V_{s,s+k} + \cdots + V_{s,n-1} = D_k\left(V_{ss} + \cdots + V_{s,n-1}\right)$$

From which it follows that there is a real number $C_k = D_k/(1 - D_k)$ such that

$$V_{s,s+k} + \cdots + V_{s,n-1} = C_k\left(V_{ss} + \cdots + V_{s,s+k-1}\right)$$

Where $1 \leq s \leq n-1$, and $1 \leq k \leq n-s-1$. We found this exact same equation in subsection 10.5, it therefore follows from this equation that $V_{s,s+k}$ is a constant multiple of V_{ss} and so that the ultimate emergence factors are deterministic.

10.7 Parameterisation of Emergence Factors and Emergence Patterns

In this subsection we discuss the parameterisation of emergence factors and emergence patterns, stating and proving any results that we need. Calibrating emergence patterns is not easy, and there is no generally accepted way of doing it. In this section we describe and discuss two methods of calibrating emergence patterns.

The first method is to use the actuary-in-the-box. In this section we consider the case of the actuary-in-the-box applied to a bootstrapped triangle-based model such as the chain ladder. We describe in detail how to parameterise risk-decay emergence patterns using the actuary-in-the-box, we do this for both ultimate and outstanding patterns. These methods were outlined in section 3.3.3 above. For the ultimate patterns we can use a bootstrapped triangle of paid or incurred claims, as the calculations depend only on the ultimate claims, however for the outstanding patterns we need to use a bootstrapped triangle of paid claims.

First consider the ultimate one-year emergence factors $\alpha_{t,1}$. These are defined by

$$\alpha_{t,1} = \frac{\sigma_t(U_{t+1})}{\sigma_t(U)}$$

Suppose that the triangle of claims data has n origin years $i = 1, \ldots, n$, and that each origin year is fully run off after n development years. The bootstrapped model gives us the distribution of the ultimate claims for each origin year ${}_iU$. In the above definition of $\alpha_{t,1}$ the subscript t denotes how many years of development the relevant origin year has undergone. For the claims triangle that has been bootstrapped origin year i has undergone $n-i+1$ years of development. The bootstrap output therefore allows us to calculate the standard deviations $\sigma_{n-i+1}({}_iU)$ for $i = 2, \ldots, n$. Applying the actuary-in-the-box to the bootstrapped model gives us the distribution of the closing estimate of ultimate claims for each origin year ${}_iU_{n-i+2}$, from which we can calculate $\sigma_{n-i+1}({}_iU_{n-i+2})$. It is then simple to calculate

$$\alpha_{n-i+1,1} = \frac{\sigma_{n-i+1}({}_iU_{n-i+2})}{\sigma_{n-i+1}({}_iU)}$$

For $i = 2, \dots, n$, which gives us the ultimate risk-decay emergence pattern

$$(\alpha_{1,1}, \dots, \alpha_{n-1,1})$$

Calculating the outstanding risk-decay emergence pattern requires more calculations. The outstanding one-year emergence factors are defined by

$$\beta_{t,1} = \frac{\sigma_t(R_{t+1})}{\sigma_t(OC_{t+1})}$$

Where R_{t+1} denotes the closing claims reserve, and OC_{t+1} denotes the ultimate risk distribution of the claims paid after time $t+1$. (This means that $R_{t+1} = E_{t+1}[OC_{t+1}]$.) A bootstrapped triangle of paid claims gives us the ultimate risk distribution of the future incremental paid claims amounts for each origin year and each future development year, i.e. it gives us ${}_iP_t$ for $i = 2, \dots, n$ and $t = n-i+2, \dots, n$. We can then calculate ${}_iOC_{n-i+2}$ for $i = 3, \dots, n$ using

$${}_iOC_{n-i+2} = \sum_{k=3}^{i} {}_iP_{n-i+k}$$

And then we can calculate $\sigma_{n-i+1}({}_iOC_{n-i+2})$ for $i = 3, \dots, n$. Applying the actuary-in-the-box to the bootstrapped model gives us the distribution of the closing reserves ${}_iR_{n-i+2}$ for $i = 3, \dots, n$, which allows us to calculate $\sigma_{n-i+1}({}_iR_{n-i+2})$. It is then simple to calculate

$$\beta_{n-i+1,1} = \frac{\sigma_{n-i+1}({}_iR_{n-i+2})}{\sigma_{n-i+1}({}_iOC_{n-i+2})}$$

For $i = 3, \dots, n$, which gives us the outstanding risk-decay emergence pattern

$$(\beta_{1,1}, \dots, \beta_{n-2,1})$$

We now describe the CoV method. Suppose that we have n origin years $i = 1, \dots, n$, and we have already estimated ultimate risk distributions for the ultimate claims ${}_iU$ for each origin year. By definition, the ultimate one-year emergence factors are given by

$$\alpha^2_{n-i+1,1} = \frac{Var_{n-i+1}({}_iU_{n-i+2})}{Var_{n-i+1}({}_iU)}$$

By applying the law of total variance (section 9.3) we see that

$$1-\alpha_{n-i+1,1}^2=\frac{E_{n-i+1}[Var_{n-i+2}({}_iU)]}{Var_{n-i+1}({}_iU)}$$

Now consider the coefficient of variation (CoV) of the ultimate risk distributions. The CoV is the ratio of the standard deviation to the mean:

$$CoV_{n-i+1}({}_iU)=\frac{\sigma_{n-i+1}({}_iU)}{E_{n-i+1}[{}_iU]}$$

We suppose that

$$E_{n-i+1}[Var_{n-i+2}({}_iU)]=\ (CoV_{n-i+1}({}_{i-1}U)E_{n-i+1}[{}_iU])^2$$

In words this assumption is saying that the current CoV of origin year $i-1$ is a good estimator of the CoV of origin year i in one year's time. We then substitute this assumption into the equation above to get

$$1-\alpha_{n-i+1,1}^2=\frac{(CoV_{n-i+1}({}_{i-1}U)E_{n-i+1}[{}_iU])^2}{Var_{n-i+1}({}_iU)}$$

Now observe that $Var_{n-i+1}({}_iU)=(CoV_{n-i+1}({}_iU)E_{n-i+1}[{}_iU])^2$, and so cancelling and rearranging we get

$$\alpha_{n-i+1,1}=\sqrt{1-\left(\frac{CoV_{n-i+1}({}_{i-1}U)}{CoV_{n-i+1}({}_iU)}\right)^2}$$

The key assumption in deriving the above is that

$$E_{n-i+1}[Var_{n-i+2}({}_iU)]=\ (CoV_{n-i+1}({}_{i-1}U)E_{n-i+1}[{}_iU])^2$$

We can generalise the result by noting that it is not necessary to use the CoV in this assumption, we can make the more general assumption that the variance is a constant multiple of some function of the mean:

$$Var_{n-i+1}({}_iU)=\eta_i^2f(E_{n-i+1}[{}_iU])$$

and then assume that

$$E_{n-i+1}[Var_{n-i+2}({}_iU)]=\ \left(\eta_{i-1}f(E_{n-i+1}[{}_iU])\right)^2$$

We then get

$$\alpha_{n-i+1,1}=\sqrt{1-\left(\frac{\eta_{i-1}}{\eta_i}\right)^2}$$

As an example, we might apply the above with $f(x)=\sqrt{x}$.

In section 3.3.3 above we briefly described how to make multiple observations of emergence factors from the output of an iterated actuary-in-the-box. We now describe how to do this in detail by extending the calculations described above. We then describe how to extend the CoV method to get multiple observations of emergence factors.

The actuary-in-the-box can be iterated and so used to get a distribution of the ultimate claims, and successive best estimate ultimate claims. If the underlying model includes paid claims amounts for each future period then we can also get successive best estimate reserves, and future claims payments. It is then straight-forward to calculate the relevant standard deviations, and so emergence factors. We need, however to be clear about exactly which emergence factors we are calculating.

Suppose that there are n origin periods $i = 1, \dots, n$. In the following, the lower-left subscript is used to denote the origin period, and the lower right subscript denotes the development period within the origin period. The iterated actuary-in-the-box, applied to a triangle of paid claims, gives us simulated values for the following quantities for $i = 2, \dots, n$ and $t = n - i + 2, \dots, n$

- The future paid claims amounts ${}_iP_t$
- The future best estimate reserves ${}_iR_t$
- The future best estimate ultimate claims ${}_iU_t$

We can then calculate the following ultimate emergence factors for $i = 2, \dots, n$ and $k = 1, \dots, i - 1$

$${}_i\alpha_{n-i+1,k} = \frac{\sigma_{n-i+1}({}_iU_{n-i+1+k})}{\sigma_{n-i+1}({}_iU)}$$

And the following outstanding emergence factors for $i = 3, \dots, n$ and $k = 1, \dots, i - 2$

$${}_i\beta_{n-i+1,k} = \frac{\sigma_{n-i+1}({}_iR_{n-i+1+k})}{\sigma_{n-i+1}({}_iOC_{n-i+1+k})}$$

We can then form the emergence patterns as described in sections 10.2.2, and 10.2.4 above.

We can also extend the CoV method described above to calculate multi-period emergence factors. By definition the ultimate multi-year emergence factors are given by

$$\alpha_{n-i+1,k}^2 = \frac{Var_{n-i+1}({}_iU_{n-i+1+k})}{Var_{n-i+1}({}_iU)}$$

By applying the law of total variance (section 9.3) we see that

$$1 - \alpha_{n-i+1,k}^2 = \frac{E_{n-i+1}[Var_{n-i+1+k}({}_iU)]}{Var_{n-i+1}({}_iU)}$$

We then assume, exactly as before, that the variance is a constant multiple of some function of the mean:

$$Var_{n-i+1}({}_iU) = \eta_i^2 f(E_{n-i+1}[{}_iU])$$

and then adjust the key assumption to allow for the multi-period context, and assume that

$$E_{n-i+1}[Var_{n-i+1+k}({}_iU)] = \left(\eta_{i-k} f(E_{n-i+1}[{}_iU])\right)^2$$

We then get

$$\alpha_{n-i+1,k} = \sqrt{1 - \left(\frac{\eta_{i-k}}{\eta_i}\right)^2}$$

Note the difference in the key assumption for multi-period emergence factors. In words the assumption in the one year case is saying that the current η-factor of origin year $i-1$ is a good estimator of the η-factor of origin year i in one year's time. The assumption in the multi-period case is saying that the current η-factor of origin year $i-k$ is a good estimator of the η-factor of origin year i in k year's time.

10.8 Emergence Factors and Other Risk Measures

Throughout this paper we've developed emergence patterns using standard deviation as the risk measure. This is natural, however it is possible to use other risk measures, and we briefly outline how one might do this in this subsection.

For a risk measure ρ define an ultimate emergence factor as above using

$$\alpha_{t,k} = \frac{\rho_t(U_{t+k})}{\rho_t(U)}$$

Where ρ_t denotes the risk measure calculated conditional on the information known at time point t. We would then apply the emergence factor as follows:

$$\widehat{U}_{t+k} = \alpha_{t,k}(U - U_t) + U_t$$

We would like $\rho_t(\widehat{U}_{t+k}) = \rho_t(U_{t+k}) = \alpha_{t,k}\rho_t(U)$. But $\rho_t(\widehat{U}_{t+k}) = \rho_t(\alpha_{t,k}U + (1-\alpha_{t,k})U_t)$. So we need to require that the risk measure obeys $\rho_t(X+k) = \rho_t(X)$ and $\rho_t(kX) = k\rho_t(X)$, where X is a random variable and k is known with certainty at time point t. In particular this means that we would need to use the mean-centred versions of VaR and TVaR.

10.9 Stochastic Emergence Factors

In this subsection we discuss what it might mean for emergence factors to be stochastic rather than deterministic. Note that as discussed above emergence factors defined using standard deviation are in general stochastic, but are always known with certainty at the time that they need to be applied.

Here we consider emergence factors without any intermediating risk measure, which would then be stochastic when applied.

Consider the ultimate emergence factors. We have the requirement that $E_t[U_{t+k}] = E_t[U]$, which suggests that we assume that $\alpha_{t,k}$ and U are uncorrelated, and we should apply the stochastic emergence factor with the equation:

$$\widehat{U}_{t+k} = \alpha_{t,k}(U - U_t) + U_t$$

Which suggests the following definition of a stochastic emergence factor:

$$\alpha_{t,k} = \frac{U_{t+k} - U_t}{U - U_t}$$

However, the problem here is that as the distribution $U - U_t$ is centred around zero, the distribution of $\alpha_{t,k}$ will be extremely heavy tailed, and can take values much greater than one. In simulation contexts this can completely swamp any useful information which might otherwise be contained in the distribution. The same issue arises with outstanding emergence factors.

When applying emergence factors, it would be reasonable to consider stochastic emergence factors so as to allow for parameter error. Exactly how this should be done depends on the model used to parameterise the emergence factors.