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On Finding the Source of a Signal

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If an airplane crashes, an emergency locator transmitter (ELT) is activated. The crash site can then be located by taking bearings on the ELT signal. The statistical problem consists of estimating a two-dimensional location parameter, where the data consist of directional bearings observed, with some error, from several known positions. We develop methods for estimating location for several variations of the basic problem: unidirectional and bidirectional observations, biased observations, and finally, techniques that are insensitive to outliers. These methods can easily be implemented on a portable microcomputer, and seem to perform quite well when applied to some actual data.

KEY WORDS: Directional data; Direction finding; Robust procedures; M-estimation; Outliers; Repeated median regression.

1. INTRODUCTION

This paper investigates some problems related to "direction-finding," that is, the estimation of a two-dimensional location parameter based on directional readings from various points. The particular application motivating this work is the problem of finding the site of an airplane crash. Bearings are taken on the plane's emergency locator transmitter (ELT) using specialized receivers and directional antennas, and we wish to find the plane as quickly as possible. In Section 2, we develop an estimator and corresponding computing algorithm, based on the assumption of a suitable error structure for the bearings. Certain generalizations of the basic technique are given in Section 3.

A look at the problem, however, reveals some complications. If a plane crashes in a flat, open area, it is usually easy to find, regardless of the estimation procedure being used. The interesting case arises when the plane is in a mountainous area. In such situations, it may be impossible to detect the ELT signal from some locations, and worse yet, false bearings may occur because of reflections from neighboring obstructions. Such aberrant observations can have a pronounced effect on the estimated location and its associated confidence region. In Section 4, we propose some methods that are fairly insensitive to such outliers. An example is given in Section 5.

2. TECHNIQUES FOR "CLEAN" DATA

The following notations are used in this and all subsequent sections. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ denote n points from which observations are taken. We assume that these coordinates are known precisely. The true location of the target is denoted by (x, y). Let $\theta_1, \theta_2, \ldots, \theta_n$ be the observed bearings from the n observation points. These are assumed to be independent random variables from angular distributions having location parameters $\mu_1, \mu_2, \ldots, \mu_n$ respectively. At this juncture we note that there is a discrepancy between angles considered as compass points (0° = north, 90° = east, etc.), and mathematical convention (0° = positive x axis = east, 90° = north, etc.). To avoid confusion we assume that the θ_i and μ_i are expressed in radian measure and in the mathematical sense. That is, $\theta_i = (\pi/180) \times (90° - i\text{th compass reading})$.

A simple method for estimating (x, y) would be to use the componentwise average of the $n(n-1)/2$ bearing intersections (assuming the observation points are distinct and that each bearing intersects all other bearings). However, these points of intersection are not stochastically independent. Moreover, their error structures depend on the distances from the corresponding observation points as well as the angle of intersection. Thus it would be difficult to determine the statistical properties of such an estimator.

By specifying a probability model for the bearings $\theta_1, \theta_2, \ldots, \theta_n$, we may derive the maximum likelihood estimate for (x, y). In particular, a natural choice for a model is the Von Mises distribution, having the probability density function

$$f(\theta_i; \mu_i, \kappa) = [2\pi I_0(\kappa)]^{-1} \exp[\kappa \cos(\theta_i - \mu_i)], \quad (2.1)$$

where I_0 is a modified Bessel function and κ is the concentration parameter. We assume that κ is the same for all bearings. Extensive discussion of the Von Mises and other distributions on the circle is given in Mardia (1972).

The log-likelihood is then given by

$$L = \text{const.} - n \ln I_0(\kappa) + \kappa \sum_{i=1}^{n} \cos(\theta_i - \mu_i). \quad (2.2)$$

Differentiating with respect to x and y, we obtain

$$\frac{\partial L}{\partial x} = \kappa \sum_{i=1}^{n} \sin(\theta_i - \mu_i) \frac{\partial \mu_i}{\partial x}, \quad (2.3)$$

$$\frac{\partial L}{\partial y} = \kappa \sum_{i=1}^{n} \sin(\theta_i - \mu_i) \frac{\partial \mu_i}{\partial y}. \quad (2.4)$$

Note that $\mu_i = \tan^{-1}[(y - y_i)/(x - x_i)]$, so that (2.2), (2.3), and (2.4) are expressible in terms of x and y. The MLE of (x, y) can be obtained by setting (2.3) and (2.4) equal to zero. Actually, there are some complications here, as L is undefined at each of the observation points (x_i, y_i) and is also not convex. In realistic situations, however, it is unlikely that this would pose a serious problem.

Now, let $s_i = \sin\ \theta_i$, $c_i = \cos\ \theta_i$, and $d_i = [(x - x_i)^2 + (y - y_i)^2]^{1/2}$. We then have $\partial \mu_i / \partial x = -(y - y_i)/d_i^2$, $\partial \mu_i / \partial y = (x - x_i)/d_i^2$, and $\sin(\theta_i - \mu_i) = [s_i(x - x_i) - c_i(y - y_i)]/d_i$. Thus the MLE of (x, y) (if it exists) is the solution to the system:

$$L_x = -\sum_{i=1}^{n} (y - y_i)[s_i(x - x_i) - c_i(y - y_i)]/d_i^3 = 0,$$

$$L_y = \sum_{i=1}^{n} (x - x_i)[s_i(x - x_i) - c_i(y - y_i)]/d_i^3 = 0. \quad (2.5)$$

Note that the solution does not depend on κ.

Because L is not convex, certain algorithms, such as Newton's method, are made rather unreliable for solving (2.5). An algorithm that has worked quite well is as follows. Define $s_i^* = (y - y_i)/d_i^3$, $c_i^* = (x - x_i)/d_i^3$, and $z_i = s_i x_i - c_i y_i$. Then (2.5) can be written in the form:

$$\begin{bmatrix} \sum s_i s_i^* & -\sum c_i s_i^* \\ -\sum s_i c_i^* & \sum c_i c_i^* \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sum s_i^* z_i \\ -\sum c_i^* z_i \end{bmatrix}. \quad (2.6)$$

By noting that $d_i^2 s_i^* = \sin\ \mu_i \doteq s_i$ and similarly $d_i^2 c_i^* \doteq c_i$, we can obtain a starting estimate $(\hat{x}_0, \hat{y}_0)$ by pretending that all the d_i are equal and ignoring the asterisks in (2.6). This solution is then used to compute interim estimates of the d, s_i^*, and c_i^*, and hence a revised solution $(\hat{x}_1, \hat{y}_1)$. Iterations are carried out in this manner until none of the parameters change by more than a negligible amount.

The final solution $(\hat{x}, \hat{y})$, along with some estimate $\hat{\kappa}$ of concentration, can be used to estimate the covariance matrix Q for mean location. Specifically, the estimated information matrix provides the covariance estimate

$$\hat{Q} = -\hat{\kappa}^{-1} \begin{bmatrix} \hat{L}_{xx} & \hat{L}_{xy} \\ \hat{L}_{yx} & \hat{L}_{yy} \end{bmatrix}^{-1}, \quad (2.7)$$

where $L_{uv} = \partial L_u / \partial v$ are obtained from (2.5) and the circumflex ($\hat{}$) denotes evaluation at $x = \hat{x}$, $y = \hat{y}$. Note that

$$\begin{aligned} L_{xy} = L_{yx} &= \tfrac{1}{2}(L_{xy} + L_{yx}) \\ &= \tfrac{1}{2}[\textstyle\sum s_i^*(c_i + 3 \sin \mu_i \sin(\theta_i - \mu_i)) \\ &\quad + \textstyle\sum c_i^*(s_i - 3 \cos \mu_i \sin(\theta_i - \mu_i))] \\ &\doteq \tfrac{1}{2}(\textstyle\sum s_i^* c_i + \sum c_i^* s_i), \end{aligned} \quad (2.8)$$

since the values of $\sin(\theta_i - \mu_i)$ should be small. Similar expressions can be written for L_{xx} and L_{yy}. Thus,

$$\hat{Q} \doteq \hat{\kappa}^{-1} \begin{bmatrix} \sum \hat{s}_i^* s_i & -\frac{1}{2}\sum (\hat{s}_i^* c_i + \hat{c}_i^* s_i) \\ -\frac{1}{2}\sum (\hat{s}_i^* c_i + \hat{c}_i^* s_i) & \sum \hat{c}_i^* c_i \end{bmatrix}^{-1} \quad (2.9)$$

gives us a convenient approximation for (2.7).

There is no guarantee that either (2.7) or (2.9) will be positive definite. However, if κ is reasonably large (i.e., the data are not too noisy), we have $s_i s_i^* = \sin \theta_i \sin \mu_i / d_i^2 \doteq \sin^2 \mu_i / d_i^2 > 0$, while the $s_i c_i^* \doteq c_i s_i^* \doteq \sin \mu_i \cos \mu_i / d_i^2$ are of the same order of magnitude as the $s_i s_i^*$ and $c_i c_i^*$, but could be either positive or negative. Thus, if there is sufficient diversity in the θ_i (bearings taken from several different sides of the target), we would intuitively expect (2.9) to be diagonally dominant, and hence positive definite. These arguments formalize, to some degree, the idea that a "bad" estimate would be obtained when the data are extremely noisy and/or observations are taken from only one side of the signal source.

All that remains is to estimate κ. From (2.2), we obtain the MLE $\hat{\kappa} = A^{-1}(\bar{C})$, where $A(t) = (d/dt)\ln I_0(t)$ and $\bar{C} = \sum \cos(\theta_i - \hat{\mu}_i)/n$. Tables of A^{-1} are given in Mardia (1972), but the approximation

$$\begin{aligned} \hat{\kappa}^{-1} &\doteq 2(1 - \bar{C}) \\ &\quad + (1 - \bar{C})^2[.48794 - .82905\bar{C} - 1.3915\bar{C}^2]/\bar{C} \end{aligned} \quad (2.10)$$

works well when $\bar{C}$ is not too small (see Lenth 1981).

3. SOME VARIATIONS

In some situations, there is a possibility of systematic error in the bearings. For example, one method of determining direction is based on measuring doppler shifts in the received signal using a rotating antenna. If the decoding equipment is out of phase with the antenna, the observations are biased. In this case, the log-likelihood becomes

$$L = \text{constant} + \kappa \sum_{i=1}^{n} \cos(\theta_i - \mu_i - \beta) - n \ln I_0(\kappa), \tag{3.1}$$

where β is the bias term. Given the estimated directions $\hat{\mu}_i$, the MLE $\hat{\beta}$ of β satisfies $R \cos \hat{\beta} = C$ and $R \sin \hat{\beta} = S$, where $C = \sum \cos(\theta_i - \hat{\mu}_i)$, $S = \sum \sin(\theta_i - \hat{\mu}_i)$, and $R^2 = C^2 + S^2$. By applying the sum of angles formula, (3.1) becomes simply $L = \text{constant} + \kappa R - n \ln I_0(\kappa)$. The location estimate can be obtained using an adaptation of the original algorithm. This entails replacing s_i by $s_i - c_i$ and c_i by $s_i + c_i$ in (2.6). The concentration estimate is given by $\hat{\kappa} = A^{-1}(R/n)$.

The other variation we consider arises from the fact that some direction-finding antennas are bidirectional. In this case we need to use a bimodal version of the Von Mises distribution, but this is easily accomplished by doubling all angles (see Mardia 1972). We now have the log-likelihood

$$L = \text{constant} + \kappa \sum_{i=1}^{n} \cos 2(\theta_i - \mu_i) - n \ln I_0(\kappa). \tag{3.2}$$

Again, the solution may be obtained by modifying the original algorithm, in this case multiplying each term of each sum in (2.6) by the factor $r_i = 4 \cos(\theta_i - \mu_i) = 4d_i^2(c_i c_i^* + s_i s_i^*)$.

It is also possible, of course, to combine these two variations in order to estimate (x, y) from axial data when there may be bias. In practice, the axial data situation is probably the more important of these two variations, since data are usually collected by more than one observer. Hence if bias were present, it would not be the same for all observations, and cannot be estimated. Furthermore, the layout of observation points becomes quite critical for the model with bias, as is demonstrated in Section 5 below.

4. ROBUST ESTIMATION

As mentioned earlier, the situations in which direction finding techniques are necessary are also those in which there is a hazard of obtaining outliers in the observed bearings. The Von Mises model is then inappropriate, and a method is needed that is resistant to such outliers.

One technique can be developed using an adaptation of the "repeated median regression" (RMR) method given in Siegel (1979). The method for direction finding is as follows. Consider each location and bearing $(x_i, y_i; \theta_i)$ as a ray in two-dimensional space, and compute the median (componentwise) of all points of intersection of this ray and all others that intersect it. Then compute the median of the n medians so obtained. (If the bearings are axial, use lines instead of rays.) This method has the advantage, because of the breakdown properties of medians, of being highly insensitive to outliers.

Another approach is that of M-estimation. The objective here, as in the case of estimating a conventional location parameter (see Huber 1964, for example), is to define an estimator that performs fairly efficiently whether or not outliers are present. Lenth (1981) gives an M-estimation technique for estimating the location parameter of a distribution on a circle. The technique adapts to the present problem as follows. Instead of maximizing (2.2), we wish to find $\hat{x}$ and $\hat{y}$ satisfying

$$\sum_{i=1}^{n} \rho[t(\theta_i - \hat{\mu}_i; \kappa)] = \text{minimum}, \tag{4.1}$$

where t is a standardization function,

$$t(\phi; \kappa) = [2\kappa(1 - \cos \phi)]^{1/2}. \tag{4.2}$$

Here ρ is some even function chosen so as to make the estimator "robust." Taking partial derivatives, we obtain the system of equations

$$\sum_{i=1}^{n} w_i \sin(\theta_i - \mu_i) \frac{\partial \mu_i}{\partial x} = 0, \tag{4.3}$$

$$\sum_{i=1}^{n} w_i \sin(\theta_i - \mu_i) \frac{\partial u_i}{\partial y} = 0, \tag{4.4}$$

where $w_i = \psi[t(\theta_i - \mu_i; \kappa)]/t(\theta_i - \mu_i; \kappa)$ and $\psi(t) = d\rho(t)/dt$. Equations (4.3) and (4.4) are simply weighted versions of (2.3) and (2.4). Note that if we take $\rho(t) = \frac{1}{2}t^2$, we have $w_i \equiv 1$ and the result is the "clean data" technique given in Section 2. The M-estimate we propose uses the same ρ (or ψ) function one would ordinarily use in conventional M-estimation, for example,

$$\text{Huber: } \psi_H(t) = \text{sign}(t) \cdot \min(|t|, c) \tag{4.5}$$

$$\text{Andrews: } \psi_A(t) = c \sin(t/c) \cdot I_{[|t| < c\pi]}, \tag{4.6}$$

where in each case c is a specified tuning constant (typically, $c = 1.5$). A variety of such ψ functions are discussed in Andrews et al. (1972).

Note that, in general, the solution to (4.3) and (4.4) also involves κ, so this parameter must be estimated. To this end, we suggest the weighted estimate $\hat{\kappa}_w^{-1} = 1/A^{-1}(\bar{C}_w)$, where

$$\bar{C}_w = \sum w_i \cos(\theta_i - \hat{\mu}_i)/\sum w_i.$$

This in turn may be approximated by substituting $\bar{C}_w$ into (2.10). The location estimate may be computed in much the same way as before: Start with $w_i = 1$,

$\hat{s}_i^* = s_i$, and $\hat{c}_i^* = c_i$, then iteratively solve the linear system

$$\begin{bmatrix} \sum w_i s_i \hat{s}_i^* & -\sum w_i c_i \hat{s}_i^* \\ -\sum w_i s_i \hat{c}_i^* & \sum w_i c_i \hat{c}_i^* \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \sum w_i \hat{s}_i^* z_i \\ -\sum w_i \hat{c}_i^* z_i \end{bmatrix}, \tag{4.7}$$

revising the w_i, $\hat{\kappa}_w$, $\hat{s}_i^*$, and $\hat{c}_i^*$ between iterations. In addition, if ψ is scaled so that $w_i = 1$ when $t = 0$, the weighted version of (2.9) provides some semblance of a covariance matrix estimate.

The adaptation of this technique to other variations as described in Section 3 is straightforward, since we need simply to use weighted sums and modify (4.2) to account for the bias parameter and/or axial bearings, as the case may be.

5. AN EXAMPLE

The data given in Table 1, and plotted in Figure 1, were collected in some field experimentation. The actual location of the target is (7.5, 2.2), and is shown as a dark square in Figure 1. If we use the seven available bearings (no reading was obtained from location number 6), the unidirectional "clean data" technique yields the estimate (7.23, 1.98), indicated by a small "x" in the figure. Only seven iterations were required.

If the data are treated as though the bearings are axial, the estimate is the same, to two decimal places, as the unidirectional estimate. However, the bias model results in a quite different estimate: (3.44, 2.69). It turns out that, by rotating each bearing clockwise by approximately 30°, a tighter cluster of bearing intersections is obtained, as is shown in Figure 2. The reason for this radical change is that all the observation points are north of the actual target. Such a phenomenon is much less likely to occur if bearings are taken from points completely surrounding the target. However, these conditions can be hard to meet in a practical situation. It is also worth noting that the algorithm converged very slowly (hundreds of iterations were required), and this is another indication that the design is inadequate for the bias model.

Table 2 and Figure 3 give results based on the maximum likelihood estimator and some robust procedures. Specifically, the estimators considered are Huber (4.5) with $c = 1.5$, Andrews (4.6) with $c = 1.5$, and the repeated median regression estimate. All estimates use the assumption of unidirectional unbiased bearings. Estimated standard errors and correlations for MLE, H 1.5, and A 1.5 were obtained from a weighted version of (2.9), and by jackknifing for the RMR method. The jackknifed RMR estimates are also included. Note that for the original

Table 1. *Some Data Collected in Field Experimentation*

Obs. no.	Obs. point (map coordinates)		Bearing (compass reading)
1	9.0	3.2	234°
2	9.4	5.5	215°
3	9.95	9.05	196°
4	8.7	8.7	193°
5	8.07	9.7	188°
6	5.6	9.0	(no reading)
7	4.4	8.9	160°
8	1.85	5.25	118°

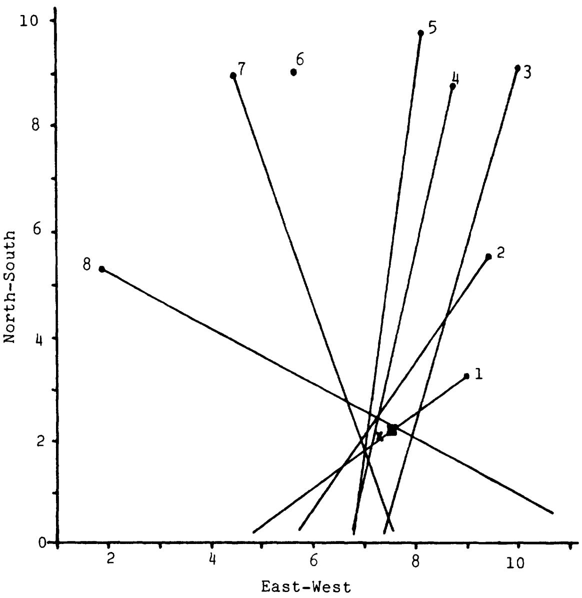

Figure 1. *Plotted Data From Table 1, With the Actual Target and the MLE*

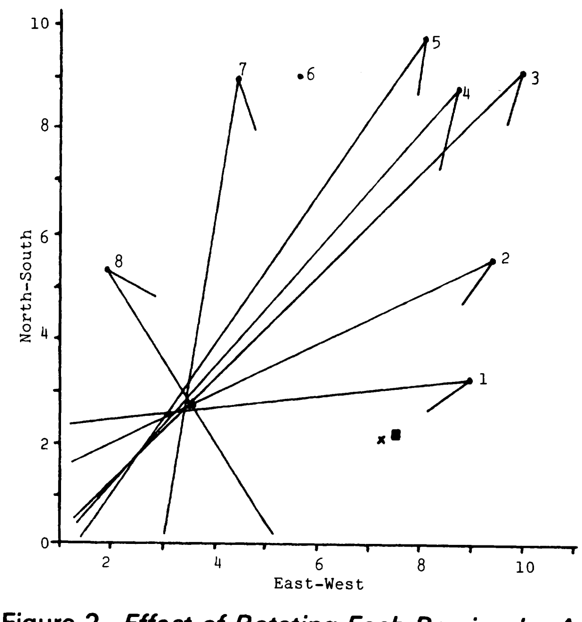

Figure 2. *Effect of Rotating Each Bearing by Approximately 30° clockwise. Also Shown Are the Target, the MLE, and the Bias Model MLE*

Table 2. *Results of Various Estimation Procedures for the Data in Table 1. Estimates Based on Eight Observations Include $\theta_6 = 250°$. A Jackknife Technique Was Used to Estimate Standard Errors for the RMR Method, and the Resulting Location Estimates (JRMR) Are Also Given*

No. of observations	Estimator	Location $\hat{x}$	estimate $\hat{y}$	Distance from target	SE($\hat{x}$)	SE($\hat{y}$)	Corr($\hat{x}$,$\hat{y}$)
7	MLE	7.23	1.98	.349	.156	.156	.664
7	H 1.5	7.21	1.98	.363	.152	.152	.662
7	A 1.5	7.21	1.97	.373	.156	.155	.669
7	RMR	7.09	1.79	.583	.181	.200	.354
7	JRMR	7.12	1.44	.850	.181	.200	.354
8	MLE	5.87	1.13	1.947	1.490	1.733	.509
8	H 1.5	6.78	1.66	.899	.883	.945	.600
8	A 1.5	7.21	1.97	.373	.156	.155	.669
8	RMR	7.07	1.82	.573	.125	.274	.291
8	JRMR	7.21	1.25	.990	.125	.274	.291

seven observations the M-estimates are almost identical to the MLE. The RMR estimate is fairly close to the others.

If observer number 6, however, had picked up a strong reflection from a westerly direction, say 250°, the situation changes. This observation is about 86° off, and its effect on the MLE and H 1.5 is substantial. Incidentally, the MLE based on the assumption of axial data is (7.01, 1.82), quite different from the unidirectional estimate. The Huber estimate offers some improvement over the MLE, and assigns weight .32 to the sixth observation. Andrews' procedure is capable of assigning zero weight and rejects the sixth bearing completely, yielding a result identical to the Andrews estimate with just seven observations. The RMR method shows a similar lack of sensitivity. Jackknifing does not seem to provide very good estimates of location or standard errors when applied to the RMR procedure.

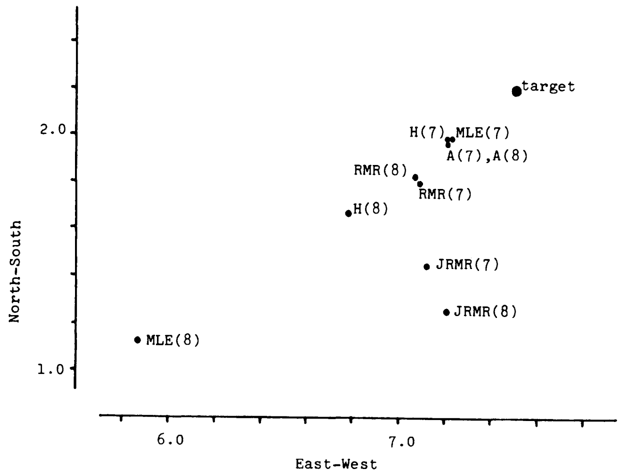

Figure 3. *Estimates From Data in Table 1*

A global view of the behavior of these estimates is obtained by varying θ_6 and plotting the distance between the location estimate and the target. This gives us something similar to sensitivity curves for the estimators, and they are shown in Figure 4. Note that the most influential outlier for purposes of location estimation is one that is perpendicular to the target. The M-estimates behave in a manner similar to the MLE in the vicinity of the true bearing from the sixth observation point. Outside of this range, the Andrews procedure rejects the observation, and the Huber estimator is influenced less strongly than is the MLE. The RMR estimate is the least sensitive of the four, with its most significant changes being jumps in the direction of the target and in the opposite direction of the target. This behavior is characteristic of medians. In this example, the RMR estimate never outperforms Andrews's procedure, but the situation could be different for a different set of data.

Of some interest is local shift sensitivity, which is related to the modulus of continuity of the curves in Figure 4. The MLE has a very smooth sine curve (it "bounces" off the horizontal axis only because distance is an unsigned quantity), and the Huber procedure is also fairly sinusoidal. However, the curves for the RMR and Andrews methods have some rather rapid transitions, indicating that a small change in one bearing can produce a relatively large change in ($\hat{x}$, $\hat{y}$). It would perhaps be desirable to choose a ψ function that redescends at a somewhat slower rate than does the Andrews procedure.

6. SUMMARY AND CONCLUSIONS

Several different methods have been given for estimating a bivariate location parameter from directional data, as well as variations for cases in which the

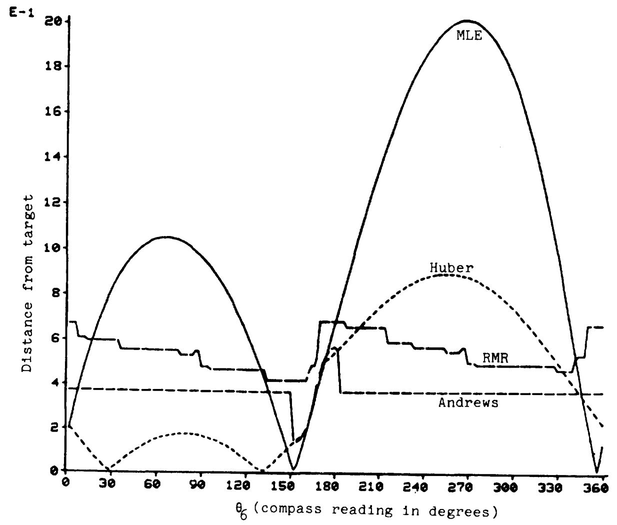

Figure 4. *"Sensitivity" Curves Obtained by Varying θ_6*

data are subject to systematic error and/or the bearings are axial. Extreme care is required in the case of biased observations, even when the data are "clean," and one may be better off assuming no systematic error when it may in fact be present.

Conventional M-estimation techniques can be adapted to this problem in a straightforward manner for cases in which outliers are possible. Furthermore, these estimates, and their associated confidence regions, can be computed efficiently using a portable microcomputer, making it suitable for field use. (A short BASIC program for this purpose is currently being used in direction-finding work.) Estimates based on repeated medians are also fairly easy to compute, but require much more computer memory unless the pairwise bearing intersections are recomputed each time they are needed. There are also bookkeeping problems because two bearings may be parallel, originate from the same point, or fail to intersect.

In practical applications, such as locating an ELT, there is a hazard of wildly outlying data, and it is advisable that an estimator having low sensitivity be used, such as the RMR method or an M-estimate having hard rejection properties. The Andrews estimator appears to perform quite well, though a ψ function that redescends a bit more slowly may be preferable to improve the local shift sensitivity.

Finally, all of the techniques given in this paper assume that the positions from which observations are taken are known precisely, and that they are close enough together that the effect of the curvature of the earth is negligible. Further work needs to be done regarding variations on these assumptions and generalizations to a three-dimensional setting.

7. ACKNOWLEDGMENTS

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