# Reinforcement Learning: States Representation, Policy Robustness and Convergence.

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Presented by **Lekan Molu** (Lay-con Mo-lu)

September 2, 2025

# Talk Outline

- Towards an innovation ecosystem;
- System Identification in Reinforcement Learning (RL);
- Robustness of Deep RL Policies:
  - Iterative Dynamic Game;
  - Convergence and Robustness in Deep RL Policy Optimization.
- Safety in Dynamical Systems;
  - The Hamilton-Jacobi-Isaacs Equation;
  - Solution to HJI Problems as a safety verification bulwark;
  - Scalability and Examples.

# Technical Overview

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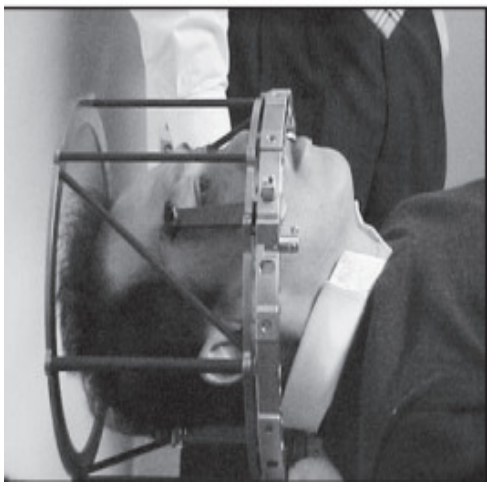
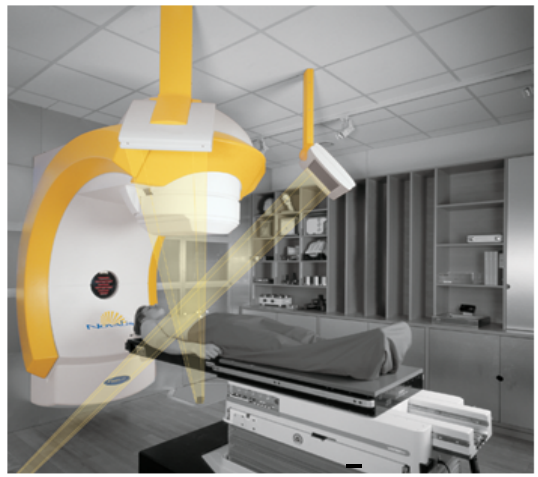
# Foundation Models, Large Behavior Models

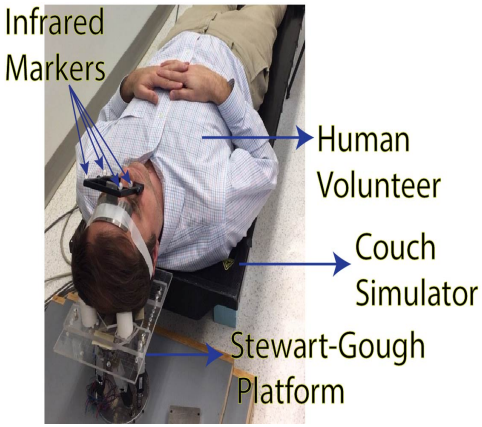
- Large-scale transfer learning, behavior cloning, unsupervised pre-training etc. a new scientific invention.

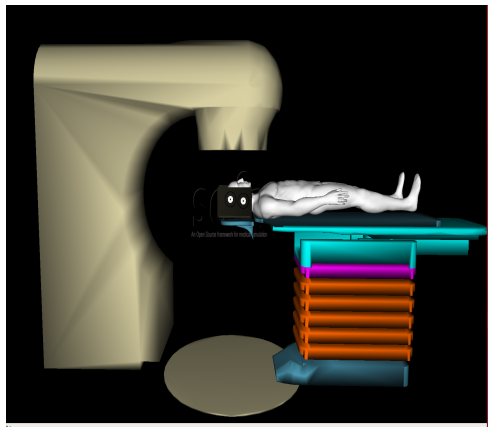

Credit: $\pi_0$: A VLA Flow Model for General Robot Control.

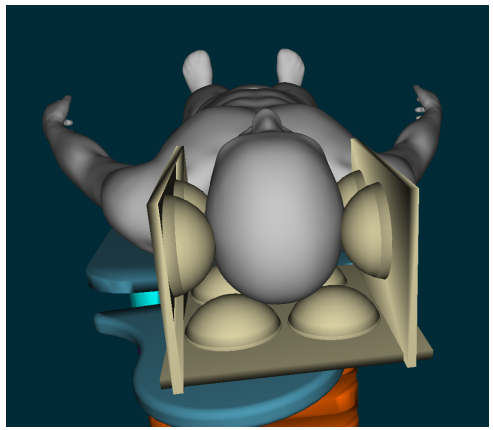
# Patient Head Stabilization in IGRT





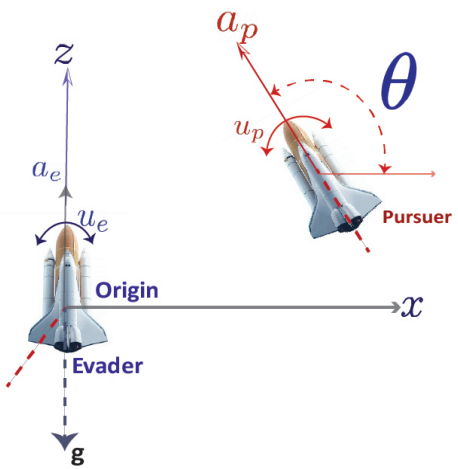


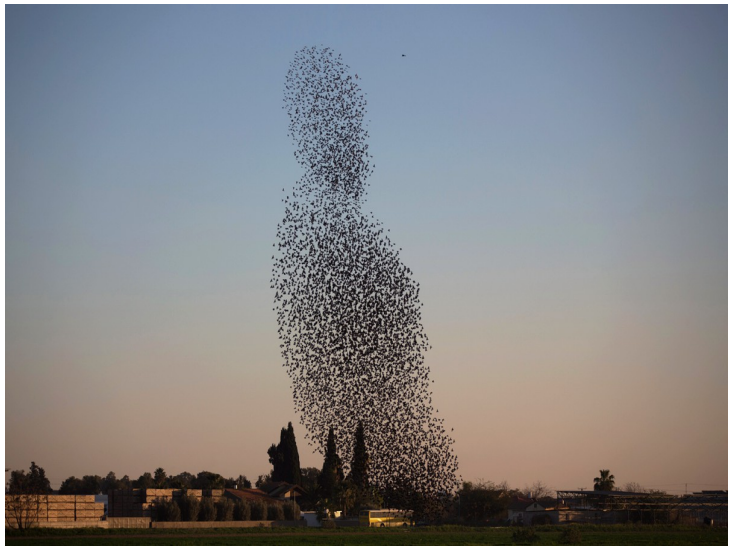





# States Representation

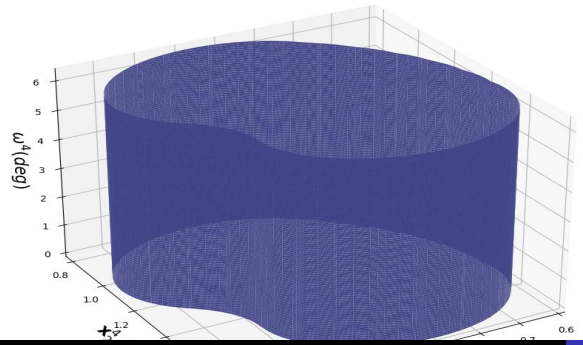
# Numerical safety analysis in dynamical Systems

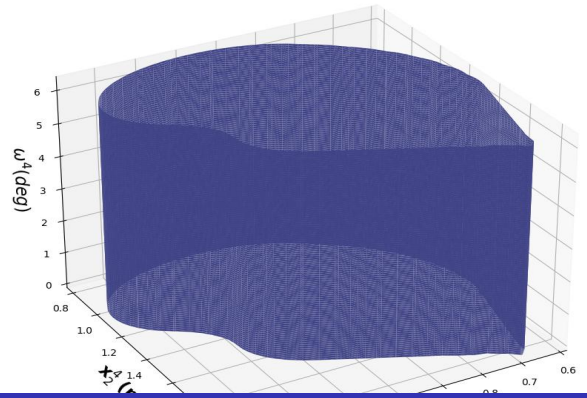


Flock 4's BRAT.

Flock 4's BRAT.



# Numerical safety analysis in dynamical Systems

## The Python LevelSet Toolbox (LevelSetPy)

Lekan Molu
https://github.com/robotsorcerer/levelsetpy

***Abstract*— This paper describes open-source scientific computing contributions in python surrounding the numerical solutions to hyperbolic HJ PDEs viz., their implicit representation on co-dimension one domains; dynamics evolution with levelsets; upwinding spatial derivatives; total variation diminishing Runge-Kutta integration schemes; and their applications to the theory of reachable sets and safety-critical systems. These procedures are increasingly finding interest in multiple research domains including analyzing safety-critical problems in reinforcement learning, robotics and automation; and control engineering among others. We describe a hierarchy of library components, and a representative numerical example included in the online package. Our GPU-accelerated package allows for easy portability to many modern libraries for the numerical analyses of the Bellman and Isaacs equations.**

Finally, extensions to reachability analyses for continuous and hybrid systems, formulated as optimal control or game theory problems using viscosity solutions to HJ PDE's is described. While our emphasis is on the resolution of safe sets in a reachability context for verification settings, the applications of this package extend beyond control engineering applications.

The GPU package, implemented in CuPy [6] and Python, is available on the author's github repository: `LevelSetPy`. Extensions to other python GPU programming language is straightforward (as detailed in the CuPy `interoperability document`). The CPU implementation (in Python) can be found at `on the cpu-numpy`

# Research Plug: Foundation Models' Emergent Capabilities

- Established: The scale of foundation models provides emergent capabilities.
- Opportunity: Its cross-task efficacy stimulates homogenization.
- Plug: Homogenization appealing, but tricky.
- Goal: Working/failure modes, language steering, safety modules etc.

# Innovation in the Age of Foundation Models

## Why am I Here?

If an idea begets a discovery, and if a discovery begets an invention, I am interested in riding the complete **innovation** circuit for intelligence:

- The thorough and wholesale transformation of fundamental scientific ideas in RL and automation into technological products (or processes) capable of widespread practical use.

# Credits

S. Chen

A. Koul

Y. Efroni

D. Misra

D. Foster

R. Islam

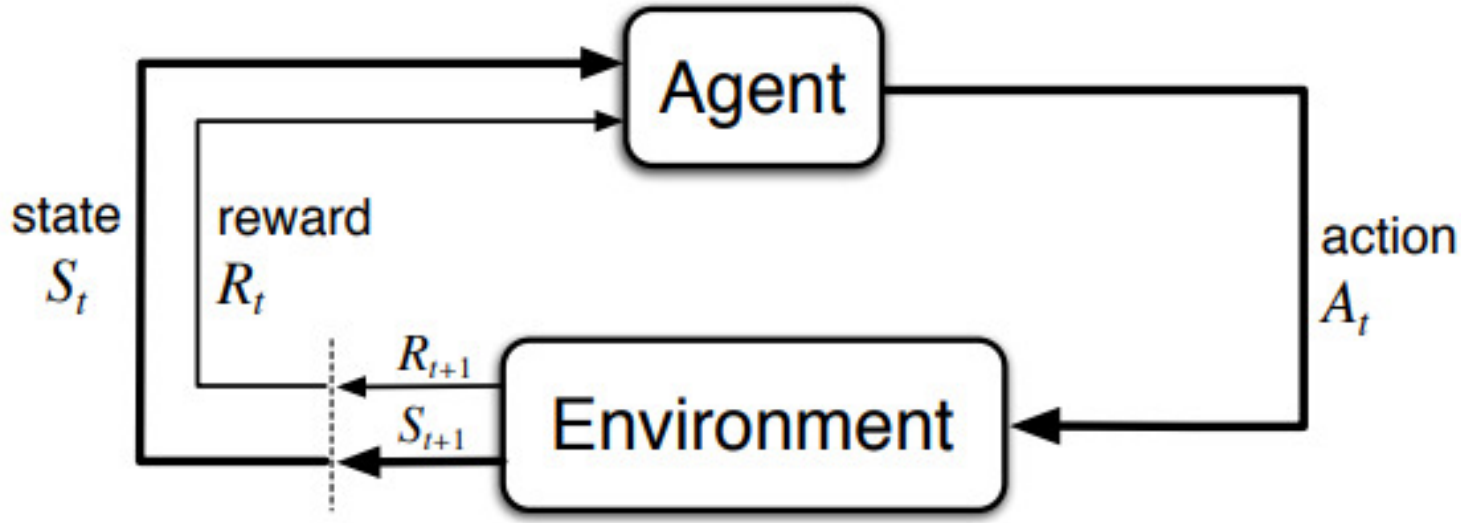
A. Lamb

M. Dudik

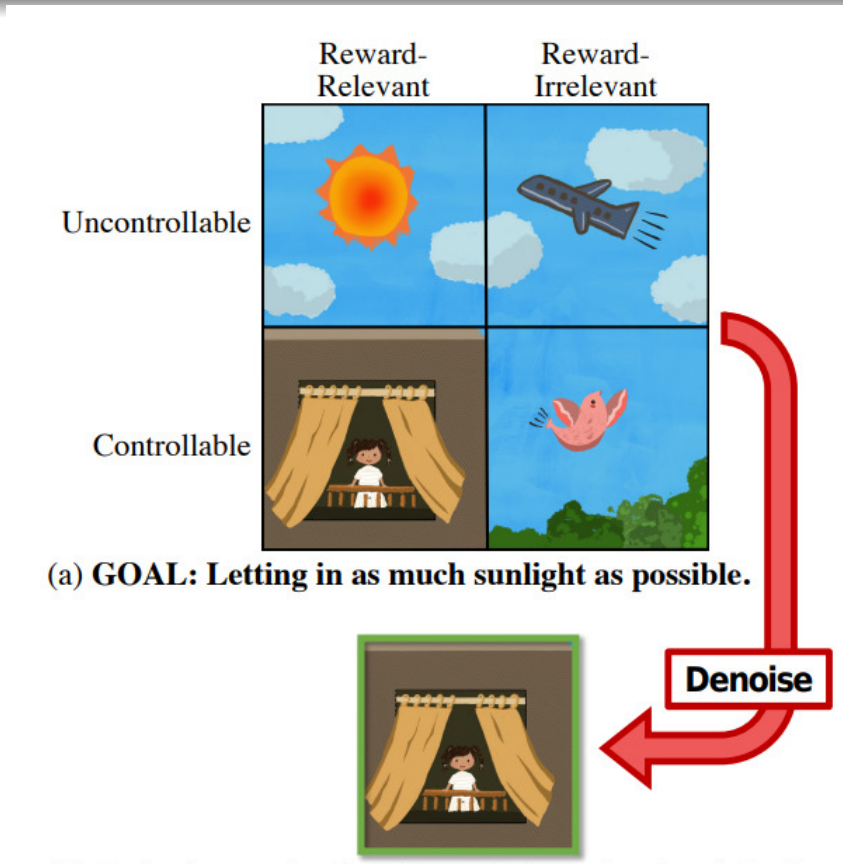
A. Krish.

J. Langford

# Standard Reinforcement Learning



# Compact States without Exogenous Distractors

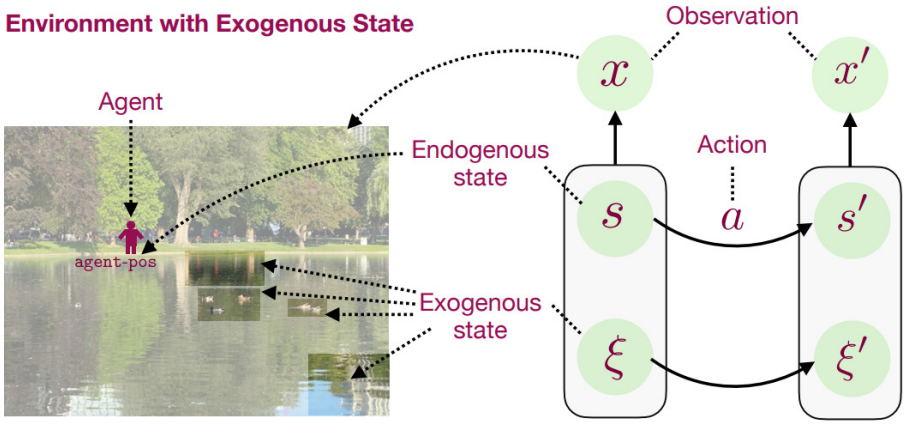



(a) **GOAL: Letting in as much sunlight as possible.**

(b) Optimal control only relies on information that is **both controllable and reward-relevant**. Good world models should ignore other factors as noisy distractors.

Denoised MDPs: Learning World Models Better Than the World Itself [5]

# Compact States without Exogenous Distractors




**Generalized Inverse Dynamics**

Train a model to predict the index of roll-in path

$$f_\theta(\texttt{idx}\,(\nu \circ a) \mid x')$$

$$\nu \rightsquigarrow x \xrightarrow{a} x'$$

$$\nu \sim \texttt{Uniform}(\Psi_{h-1}) \qquad a \sim \texttt{Uniform}(\mathcal{A})$$

Policy cover for the last time step — Action space

Learning $s$ with $[S]$ whilst ignoring temporally correlated $\xi$? Source: [3, Fig. 1].

# Exo-MDP Machinery

- Consider the tuple $\mathcal{M} := (\mathcal{X}, \mathcal{Z}, \mathcal{A}, T, R, H)$
  - Starting distribution $\mu \in \Delta(\mathcal{Z})$;
  - Agent receives observations $\{x_h\}_{h=1}^{H} \in \mathcal{X}$ from the emission function $q : \mathcal{Z} \to \Delta(\mathcal{X})$;
  - Agent transitions between latent states via $T : \mathcal{Z} \times \mathcal{A} \to \Delta(\mathcal{S})$;
    - And rewards by $R : \mathcal{X} \times \mathcal{A} \to \Delta([0, 1])$
- Trajectories: $(z_1, x_1, a_1, r_1, \cdots, z_H, a_H, r_H)$ from repeated interactions;
  - $z_1 \sim \mu_1(\cdot)$, $z_{h+1} \sim T(\cdot|z_h, a_h)$, $x_h \sim q(\cdot|z_h)$ and $r_h \sim R(x_h, a_h, x_{h+1})$ for all $h \in [H]$.
- Define $supp(q(\cdot|z)) = \{x \in \mathcal{X} | q(x|z) > 0\}$ for any $z$.

# Exo-MDP Machinery

Block MDP assumption $supp(q(\cdot|z_1)) \cap sup(q(\cdot|z_2)) = \emptyset$ for all $z_1 \neq z_2$.

- Agent chooses $a \sim \pi(z_h|x_h)$
- There exists non-stationary episodic policies $\Pi_{NS} := \Pi^H \supseteq (\pi_1, \cdots, \pi_H)$;
- Optimal policy $\pi^\star = \text{argmax} V_{\pi \in \Pi_{NS}}(\pi)$;
  - For $V_{\pi \in \Pi_{NS}} = \sum_h = 1^H r_h$.
- EXO-BMDP: Essentially a Block MDP [1] such that the latent states admits the form $z = (s, e)$, where $s \in \mathcal{S}$, $e \in E$.
- $\mu(z) = \mu(s)\mu\xi$ and $T(z'|z, a) = T(s'|s, a)T_e(e'|e)$

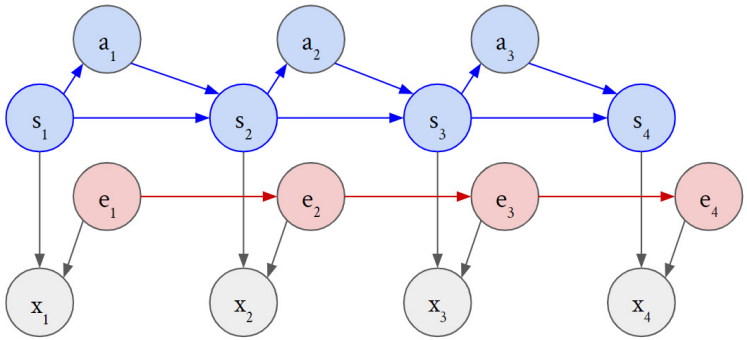
# Literature comparison

| Algorithms | PPE | OSSR | DBC | CDL | Denoised-MDP | 1-Step Inverse | AC-State (Ours) |
|---|---|---|---|---|---|---|---|
| Exogenous Invariant State | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Exogenous Invariant Learning | ✓ | ✓ | ✗ | ✗ | ✗ | ✓ | ✓ |
| Flexible Encoder | ✓ | ✗ | ✓ | ✗ | ✓ | ✓ | ✓ |
| YOLO (No Resets) Setting | ✗ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Reward Free | ✓ | ✓ | ✗ | ✓ | ✓ | ✓ | ✓ |
| Control-Endogenous Rep. | ✓ | ✓ | ✗ | ✓ | ✓ | ✗ | ✓ |

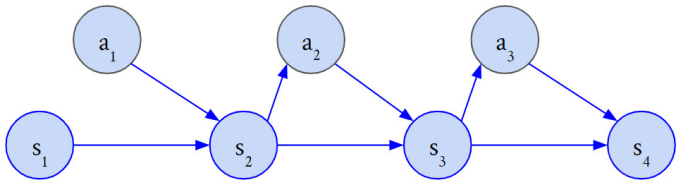
Emphasis on robustness to exogenous information. Comparison with baselines including PPE [3], OSSR [2], DBC [6] , Denoised MDP [5] and One-Step Inverse Models [4].

# Rewards-agnostic Exogenous State Invariance in RL

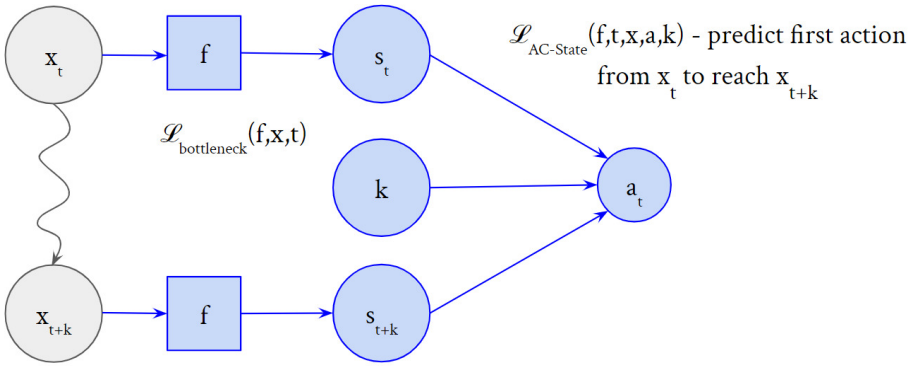
AC-State Discovers the smallest control-endogenous state s assuming factorized dynamics




AC-State collects data with a single random action followed by a high-coverage endogenous policy for k-1 steps




AC-State learns an encoder f for s = f(x) by optimizing a multi-step inverse model with a bottleneck



# Latent States Discovery – Multi-step Inverse Dynamics

- $\hat{f} \approx \arg\min_{f \in \mathcal{F}} \mathbb{E}_{t,k} \Big[ \mathcal{L}_{\mathrm{ACS}}(f, x, a, t, k) + \mathcal{L}_{\mathrm{B}}(f, x_t) + \mathcal{L}_{\mathrm{B}}(f, x_{t+k}) \Big]$

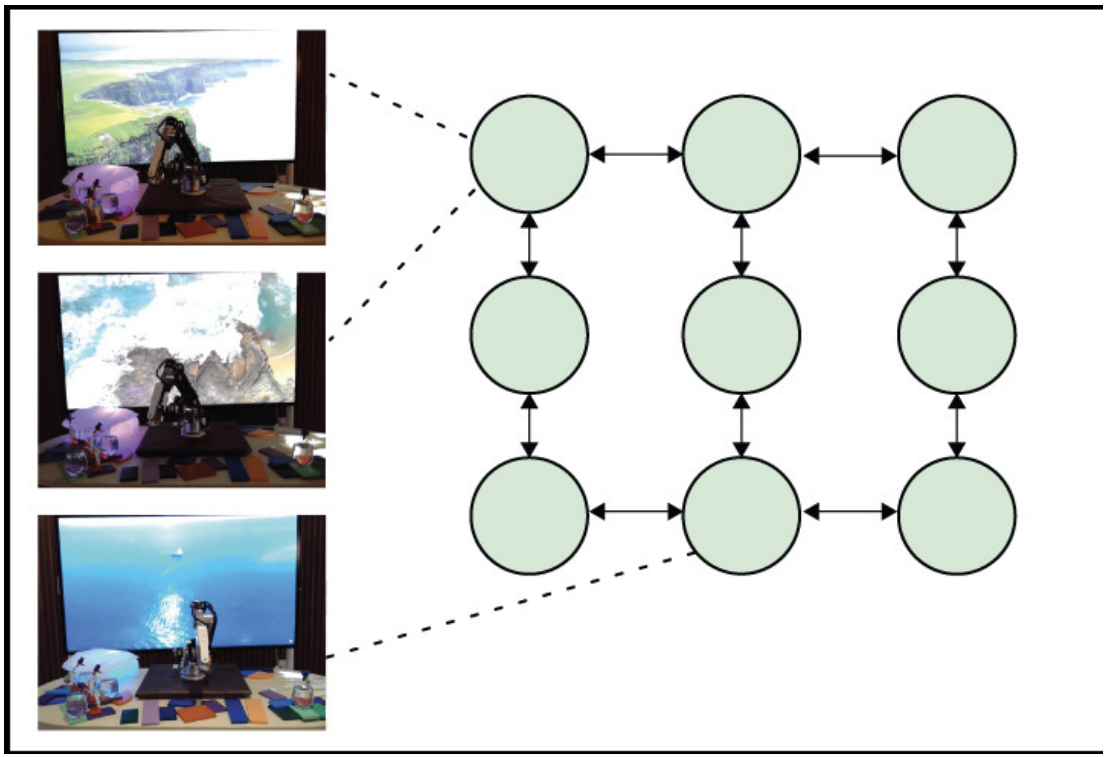
$$\mathcal{L}_{\mathrm{ACS}}(f, x, a, t; k) = -\log(\mathbb{P}(a_t | f(x_t), f(x_{t+k}); k)) \quad (1)$$

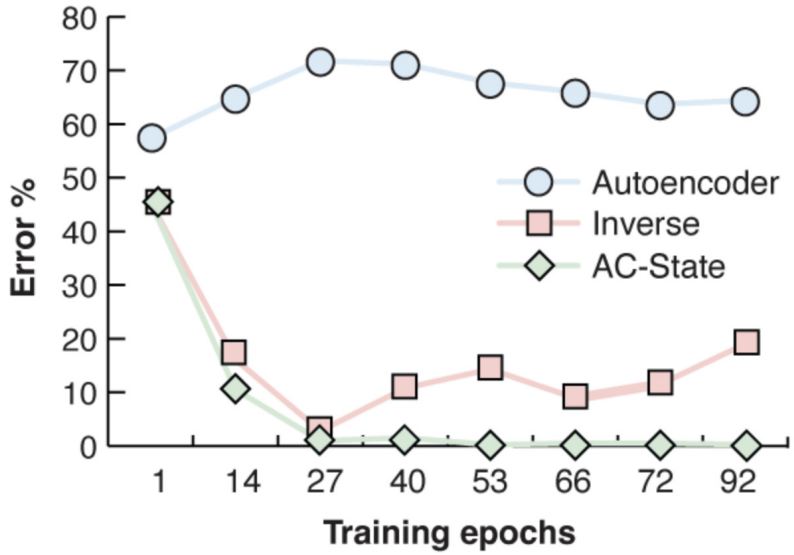
# AC State Algorithm

**Algorithm 1** AC-State Algorithm for Latent State Discovery Using a Uniform Random Policy

Initialize observation trajectory $x$ and action trajectory $a$. Initialize encoder $f_\theta$. Assume any pair of states are reachable within exactly $K$ steps and a number of samples to collect $T$, and a set of actions $\mathcal{A}$, and a number of training iterations $N$.
$x_1 \sim U(\mu(x))$
**for** t = 1, 2, ..., $T$ **do**
    $a_t \sim U(\mathcal{A})$
    $x_{t+1} \sim \mathbb{P}(x'|x_t, a_t)$
**for** n = 1, 2, ..., $N$ **do**
    $t \sim U(1, T)$ and $k \sim U(1, K)$
    $\mathcal{L} = \mathcal{L}_{\texttt{AC-State}}(f_\theta, t, x, a, k) + \mathcal{L}_{\text{Bottleneck}}(f_\theta, x_t) + \mathcal{L}_{\text{Bottleneck}}(f_\theta, x_{t+k})$
    Update $\theta$ to minimize $\mathcal{L}$ by gradient descent.

# AC State in Action

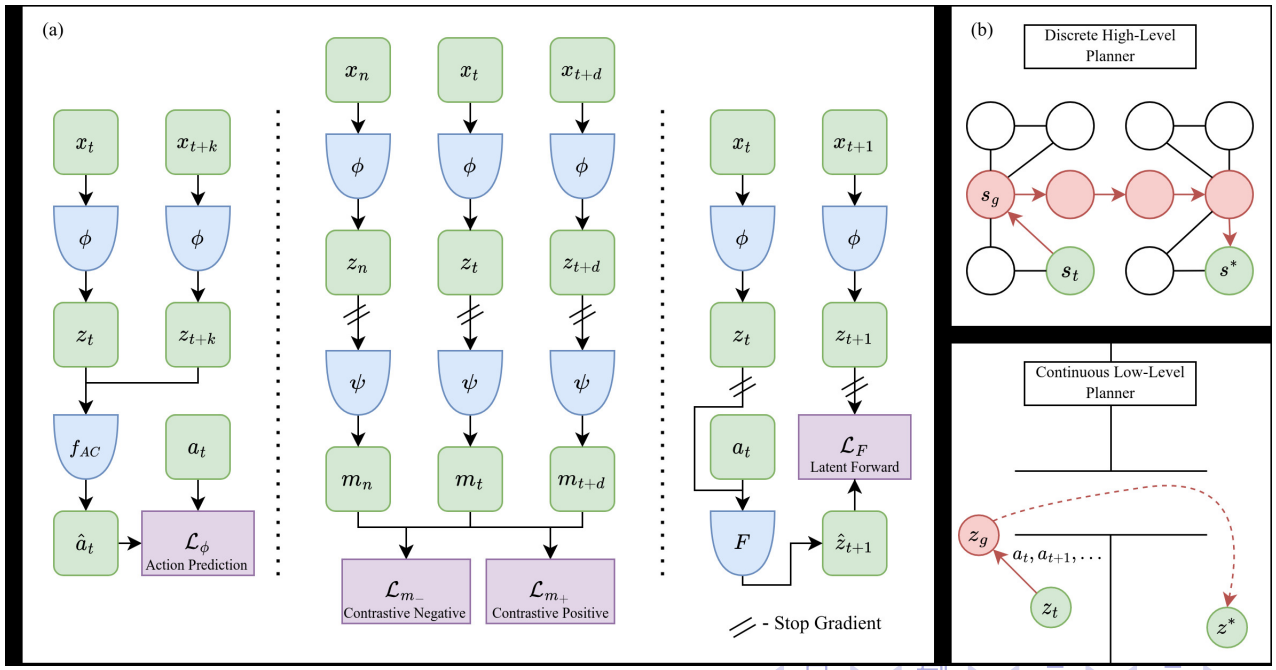




# PCLAST: Agent Plannable Continuous Latent States

**PcLast: Discovering Plannable Continuous Latent States**

Anurag Koul [*1] Shivakanth Sujit [*2 3 4] Shaoru Chen [1] Ben Evans [5] Lili Wu [1] Byron Xu [1] Rajan Chari [1]
Riashat Islam [3 6] Raihan Seraj [3 6] Yonathan Efroni [7] Lekan Molu [1] Miro Dudik [1] John Langford [1] Alex Lamb [1]



# PCLAST Algorithm

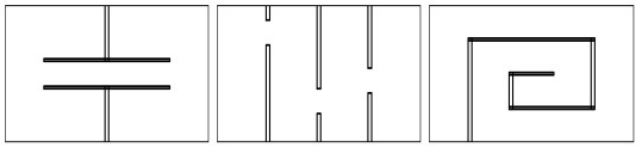
**Algorithm 1** $n$-Level Planner

**Require:**

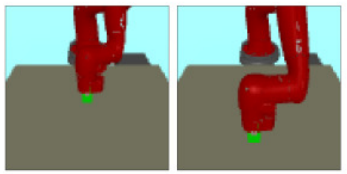
Current observation $x_t$
Goal observation $x_{goal}$
Planning horizon $H$
Encoder $\phi(\cdot)$
PCLAST map $\psi(\cdot)$
Latent forward dynamics $\delta(\cdot, \cdot)$
Multi-Level discrete transition graphs $\{\mathcal{G}_i\}_{i=2}^{n}$

**Ensure:** Action sequence $\{a_i\}_{i=0}^{H-1}$

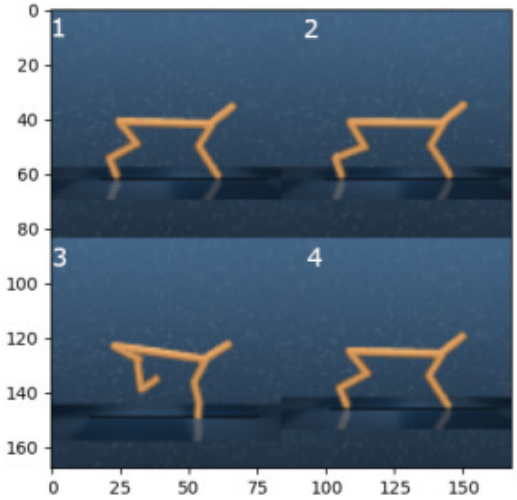
1: Compute current continuous latent state $\hat{s}_t = \phi(x_t)$ and target latent state $\hat{s}^* = \phi(x_{goal})$.
{See Appendix E for details of high-level planner and low-level planner.}
2: **for** $i = n, n-1, \ldots, 2$ **do**
3: $\hat{s}^*$ = high-level planner$(\hat{s}_t, \hat{s}^*, \mathcal{G}_i)$
{Update waypoint using a hierarchy of abstraction.}
4: **end for**
5: $\{a_i\}_{i=0}^{H-1}$ = low-level planner$(\hat{s}_t, \hat{s}^*, H, \delta, \psi)$
{Solve the trajectory optimization problem.}



(a) Hallway (b) Rooms (c) Spiral



(d) Sawyer Reach Environment



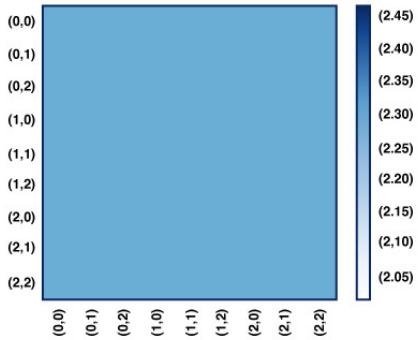
# PCLAST Results

| METHOD | REWARD TYPE | HALLWAY | ROOMS | SPIRAL | SAWYER-REACH |
|---|---|---|---|---|---|
| PPO | DENSE | $6.7 \pm 0.6$ | $7.5 \pm 7.1$ | $11.2 \pm 7.7$ | $\mathbf{86.00 \pm 5.367}$ |
| PPO + ACRO | DENSE | $10.0 \pm 4.1$ | $23.3 \pm 9.4$ | $23.3 \pm 11.8$ | $84.00 \pm 6.066$ |
| PPO + PCLAST | DENSE | $\mathbf{66.7 \pm 18.9}$ | $\mathbf{43.3 \pm 19.3}$ | $\mathbf{61.7 \pm 6.2}$ | $78.00 \pm 3.347$ |
| PPO | SPARSE | $1.7 \pm 2.4$ | $0.0 \pm 0.0$ | $0.0 \pm 0.0$ | $68.00 \pm 8.198$ |
| PPO + ACRO | SPARSE | $21.7 \pm 8.5$ | $5.0 \pm 4.1$ | $11.7 \pm 8.5$ | $\mathbf{92.00 \pm 4.382}$ |
| PPO + PCLAST | SPARSE | $\mathbf{50.0 \pm 18.7}$ | $\mathbf{6.7 \pm 6.2}$ | $\mathbf{46.7 \pm 26.2}$ | $82.00 \pm 5.933$ |
| CQL | SPARSE | $3.3 \pm 4.7$ | $0.0 \pm 0.0$ | $0.0 \pm 0.0$ | $32.00 \pm 5.93$ |
| CQL + ACRO | SPARSE | $15.0 \pm 7.1$ | $\mathbf{33.3 \pm 12.5}$ | $\mathbf{21.7 \pm 10.3}$ | $68.00 \pm 5.22$ |
| CQL + PCLAST | SPARSE | $\mathbf{40.0 \pm 0.5}$ | $23.3 \pm 12.5$ | $20.0 \pm 8.2$ | $\mathbf{74.00 \pm 4.56}$ |
| RIG | NONE | $0.0 \pm 0.0$ | $0.0 \pm 0.0$ | $3.0 \pm 0.2$ | $\mathbf{100.0 \pm 0.0}$ |
| RIG + ACRO | NONE | $\mathbf{15.0 \pm 3.5}$ | $4.0 \pm 1.$ | $\mathbf{12.0 \pm 0.2}$ | $100.0 \pm 0.0$ |
| RIG + PCLAST | NONE | $10.0 \pm 0.5$ | $4.0 \pm 1.8$ | $10.0 \pm 0.1$ | $90.0 \pm 5$ |
| LOW-LEVEL PLANNER + PCLAST | NONE | $86.7 \pm 3.4$ | $69.3 \pm 3.4$ | $50.0 \pm 4.3$ | $\pm$ |
| $n$-LEVEL PLANNER + PCLAST | NONE | $\mathbf{97.78 \pm 4.91}$ | $\mathbf{89.52 \pm 10.21}$ | $\mathbf{89.11 \pm 10.38}$ | $95.0 \pm 1.54$ |

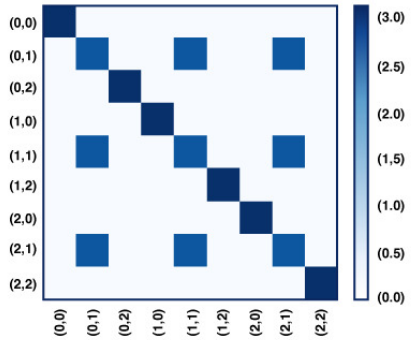
# AC State in Action

Exogenous distractors riddance.

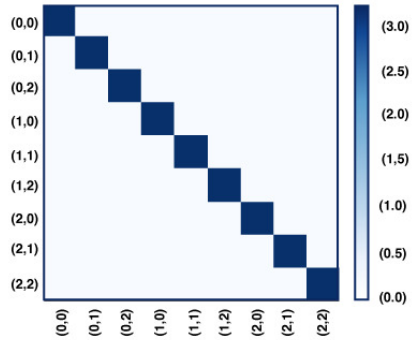
# Agent Controllable States Representation



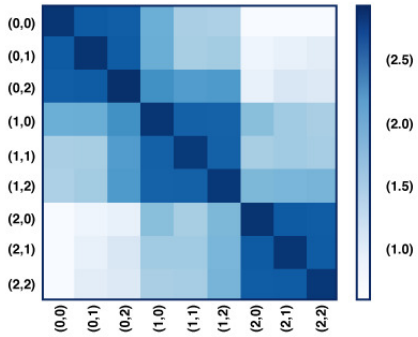

(a) Autoencoder (Theory worst-case)



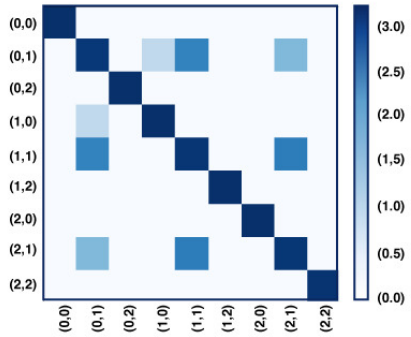

(b) Inverse (Theory worst-case)



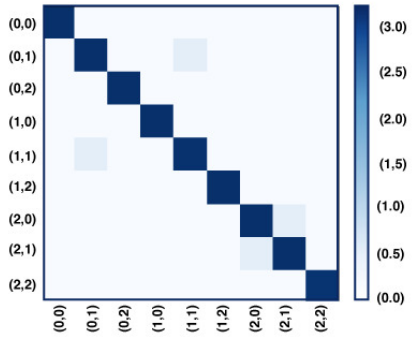

(c) AC-State (Theory worst-case)




(d) Autoencoder (Empirical)

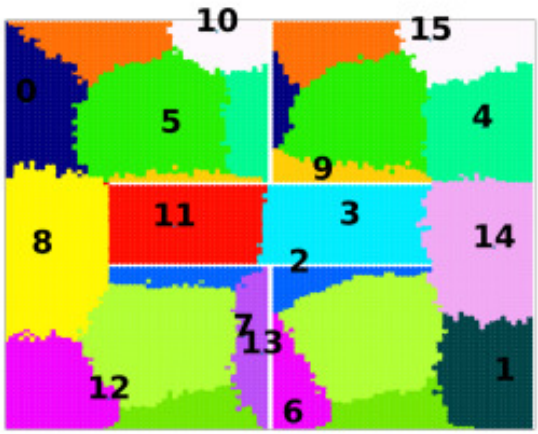



(e) Inverse (Empirical)



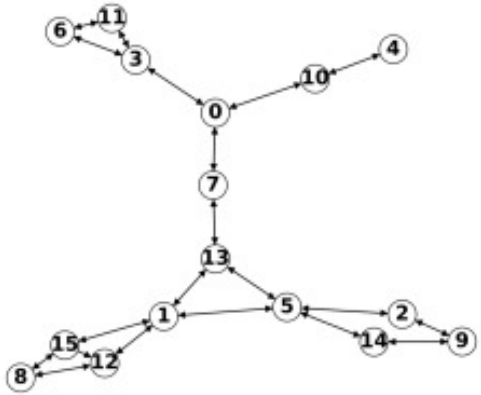

(f) AC-State (Empirical)

# PCLAST Segmentation Results

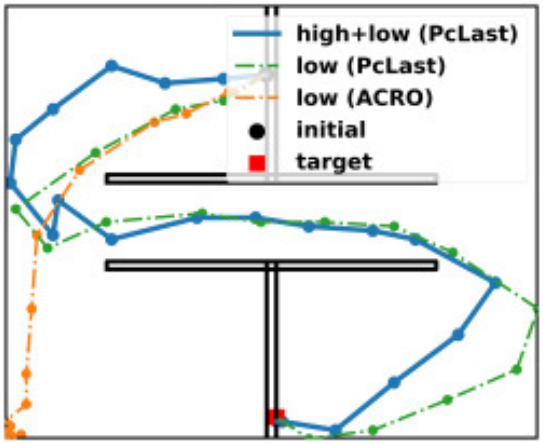



(a) Clusters ACRO


(b) Clusters PCLAST

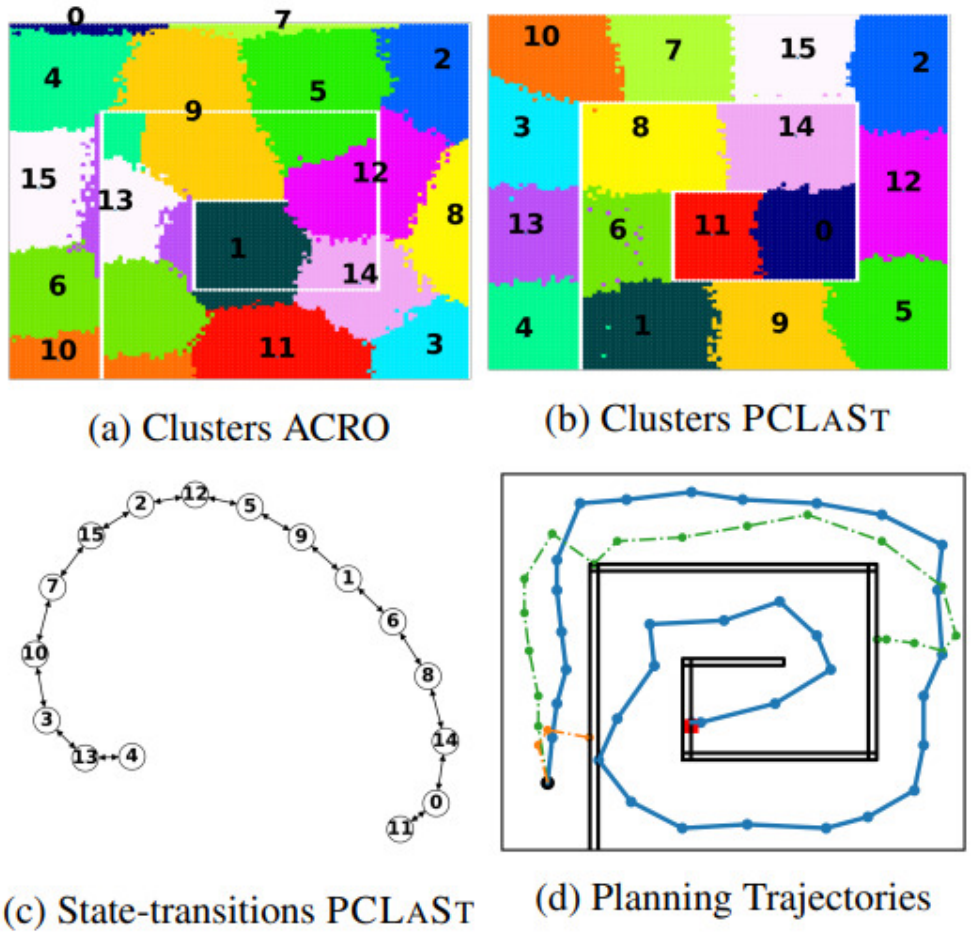



(c) State-transitions PCLAST




(d) Planning Trajectories

# PCLAST Segmentation Results




(a) Clusters ACRO

(b) Clusters PCLAST

(c) State-transitions PCLAST

(d) Planning Trajectories

*Figure 6.* Clustering, Abstract-MDP, and Planning are shown for

# Mixed $H_2/H_\infty$ Policy Optimization in RL

*"The scientist's problem is to recognize basic facts even though they are obscured by a wealth of extraneous material, and then to apply creative imagination in their interpretation. This Karl Jansky did."* – Cyril Jansky.

# On the Robustness and Convergence of Policy Optimization in Continuous-Time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Stochastic Control

Lekan Molu

New York City, NY 10012

Presented by **Lekan Molu** (Lay-con Mo-lu)

September 2, 2025

# Talk Outline and Overview

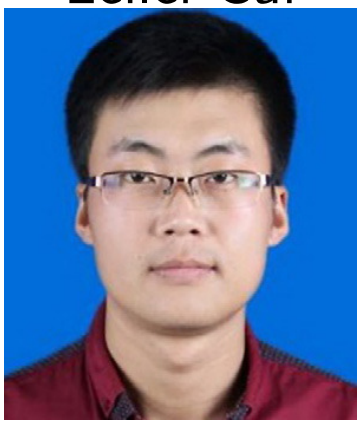


- Policy Optimization and Stochastic Linear Control
  - Connections to risk-sensitive control;
  - Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control theory.
- The case for convergence analysis in stochastic PO.
  - Kleinman's algorithm, *redux.*
  - Kleiman's algorithm in an iterative best response setting;
  - PO Convergence in best response settings.
- Robustness margins in model- and sampling- settings.
  - PO as a discrete-time nonlinear system;
  - Kleiman and input-to-state-stability;
  - Robust policy optimization as a small-input stable state optimization algorithm

# Credits



Leilei Cui



Postdoc, MIT

Zhong-Ping Jiang

Professor, NYU

# Research Significance



- (Deep) RL and modern AI
  - Robotic manipulation (Levine et al., 2016), text-to-visual processing (DALL-E), Atari games (Mnih et al., 2013), e.t.c.
  - Policy optimization (PO) is fundamental to modern AI algorithms' success.
  - Major success story: functional mapping of observations to policies.
  - But how does it work?

# Policy Optimization – General Framework



- PO encapsulates policy gradients (Kakade, 2001) or PG, actor-critic methods (Vrabie and Lewis, 2011), trust region PO Schulman et al. (2015), and proximal PO methods (Schulman et al., 2017).
- PG particularly suitable for complex systems.

$$\begin{aligned} &\min J(K) \\ &\text{subject to } K \in \mathcal{K} \end{aligned} \tag{1}$$

where $\mathcal{K} = \{K_1, K_2, \cdots, K_n\}$.

- $J(K)$ could be tracking error, safety assurance, goal-reaching measure of performance e.t.c. required to be satisfied.

# Continuous-time RL control applications



- A little randomness in a system's mathematical model coefficients?
  - Population growth model: $dN/dt = a(t)N(t),\ N(0) = N_0$; growth rate $a(t)$ subject to random effects e.g. $a(t) = r(t)+$ "noise".
  - We only know the distribution of "noise".
- Filtering and state estimation problems where the nature of the noise is unknown, but it is observed via sensor measurements.
  - Kalman + Bucy Filters – aerospace (Apollo, Mariner etc.).

# Continuous-time RL control applications



- Semielliptic P.D.E.s with Dirichlet boundary value problems e.g. slender flexible rods, Cosserat dynamics etc:
$$\Delta q = \sum_{i=1}^{n} \frac{\partial^2 q}{\partial \xi_i^2} = 0 \in \Omega,\ q = q_{\rightarrow} \text{ on } \partial\Omega,\ \Omega \subset \mathbb{R}^n$$
- An economic portfolio problem where the price, $p(t)$, of a stock satisfies a stochastic differential equation e.g. $dp/dt = (a + \alpha \cdot \text{“}noise\text{”})p$ for $a > 0$, $\alpha \in reline$.
- Call options pricing: The *Black-Scholes option price formula*.

# Policy Optimization – Open questions



- Gradient-based data-driven methods: prone to divergence from true system gradients.
  - Challenge I: Optimization occurs in non-convex objective landscapes.
    - Get performance certificates as a mainstay for control design: Coerciveness property (Hu et al., 2023).
  - Challenge II: Taming PG's characteristic high-variance gradient estimates (REINFORCE, NPG, Zeroth-order approx.).
    - Hello, (linear) robust ($\mathcal{H}_\infty$-synthesis) control!

# Policy Optimization – Open questions



- Challenge III: Under what circumstances do we have convergence to a desired equilibrium in RL settings?
- Challenge IV: Stochastic control, not deterministic control settings.
  - models involving round-off error computations in floating point arithmetic calculations; the stock market; protein kinetics.
- Challenge V: Continuous-time RL control.
  - Very little theory. Lots of potential applications encompassing rigid and soft robotics, aerospace or finance engineering, protein kinetics.

# $\mathcal{H}_\infty$-Control Under Model Mismatch



$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + Bu(t)\mathrm{d}t + D\mathrm{d}w(t),$$
$$z(t) = Cx(t) + Eu(t),\ \alpha > 0;$$

**Algorithm 1** Search for the closed-loop $\mathcal{H}_\infty$-norm

Given a user-defined step size $\eta > 0$
Set the initial upper bound on $\gamma$ as $\gamma_{ub} = \infty$.
Initialize a buffer for possible $\mathcal{H}_\infty$ norms for each $K_1$ to be found, $\Gamma_{buf} = \{\}$.
Initialize ordered **poles** $\mathcal{P} = \{p_i \in Re(s) < 0 \,|\, i = 1, 2, \}$ ▷ $p_1 < p_2 < \cdots$
**for** $p_i \in \mathcal{P}$ **do**
    Place $p_i$ on (2); ▷ (Tits and Yang, 1996)
    Compute stabilizing $K_1^{p_i}$
    Find lower bound $\gamma_{lb}$ for $H(\gamma, K_1^{p_i})$; ▷ using (22)
    $\Gamma_{buf}(i) =$ **get_hinf_norm**$(T_{zw}, \gamma_{lb}, K_1^{p_i})$.
**end for**
**function** **get_hinf_norm**$(T_{zw}, \gamma_{lb}, K_1^{p_i})$
    **while** $\gamma_{ub} = \infty$ **do**
        $\gamma := (1 + 2\eta)\,\gamma_{lb}$;
        Get $\lambda_i(H(\gamma, K_1^{p_i}))$ ▷ c.f. (14)
        **if** $\mathtt{Re}(\Lambda) \neq \emptyset$ for $\Lambda = \{\lambda_1, \cdots \lambda_n\}$ **then**
            Set $\gamma_{ub} = \gamma$; exit
        **else**
            Set buffer $\Gamma_{lb} = \{\}$
            **for** $\lambda_k \in \{\mathtt{Imag}(\Lambda)_{:p-1}\}$ **do** ▷ $k = 1$ to $K$
                Set $m_k = \frac{1}{2}(\omega_k + \omega_{k+1})$
                Set $\Gamma_{lb}(k) = \max\{\sigma\,[T_{zw}(jm_k)]\}$;
            **end for**
            $\gamma_{lb} = \max(\Gamma_{lb})$
        **end if**
        Set $\gamma_{ub} = \frac{1}{2}(\gamma_{lb} + \gamma_{ub})$.
    **end while**
    **return** $\gamma_{ub}$
**end function**

# Tools: Complexity, Convergence, Robustness.



- Risk-sensitive $\mathcal{H}_\infty$-control (Glover, 1989) and discrete- and continuous-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design (Khargonekar et al., 1988; Hu et al., 2023):
  - min. upper bound on $\mathcal{H}_2$ cost subject to satisfying a set of risk-sensitive (often $\mathcal{H}_\infty$) constraints (Basar, 1990):

$$\begin{aligned} &min_{K\in\mathcal{K}} J(K) := Tr(P_K DD^\top) \\ &\text{subject to } \mathcal{K} := \{K | \rho(A - BK) < 1, \|T_{zw}(K)\|_\infty < \gamma\} \end{aligned} \tag{2}$$

  - $P_K$: solution to the generalized algebraic Riccati equation (GARE);
  - $A, B, D, K$: standard closed-loop system matrices;
  - $\|T_{zw}(K)\|_\infty$: $\mathcal{H}_\infty$-norm of the closed-loop transfer function from a disturbance input $w$ to output $z$.

# Tools: Complexity, Convergence, Robustness.



Infinite-horizon

- discrete-time deterministic LQR settings (Fazel et al., 2018):

$$\min_{K\in\mathcal{K}} \mathbb{E}\sum_{t=0}^{\infty}(x_t^\top Q x_t + u_t^\top R u_t) \text{ s.t. } x_{t+1} = Ax_t + Bu_t, x_0 \sim \mathcal{P}_0$$

- discrete-time LQ problems under multiplicative noise (Gravell et al., 2021):
$\min_{\pi\in\Pi} \mathbb{E}_{x_0,\{\delta_i\},\{\gamma_i\}} \sum_{t=0}^{\infty}(x_t^\top Q x_t + u_t^\top R u_t)$
subject to $x_{t+1} = (A+\sum_{i=1}^{p}\delta_{ti}A_i)x_t + (B+\sum_{i=1}^{q}\gamma_{ti}B_i)u_t;$

# (Non-exhaustive) Lit. Landscape on PO Theory

| Literature landscape | Cont. time (Kalman '61, Luenberger '63) | Stochastic. LQR (Kalman '60) | Cont. Phase | LEQG or Mixed $H_2/H_\infty$ | Finite/Infinite Horizon |
|---|---|---|---|---|---|
| Fazel (2018) | No | No | Yes | No | Finite-horizon |
| Mohammadi (TAC -- 2020) | Yes | No | Yes | No | Finite-Horizon |
| Zhang (2019) | Yes | Yes (Gaussian) | Yes | Yes | Inf-horizon |
| Gravell (2021) | No | Multiplicative | Yes | No | Inf-horizon |
| Zhang (2020) | No | No | Yes | Yes | Rand-horizon |
| Molu (2022) | Yes | Yes (Brownian) | Yes | Yes | Inf-Horizon |
| Cui & Molu (2023) | Yes | Yes (Brownian) | Yes | Yes | Inf-Horizon |

# Mainstay

- Continuous-time infinite-dimensional linear systems.
  - Disturbances enter additively as random stochastic Wiener processes.
  - Many natural systems admit uncertain additive Brownian noise as diffusion processes.
    - Theoretical analysis machinery: Îto's stochastic calculus.
- Goal: keep controlled process, $z$, small i.e.

$$\|z\|_2 = \left( \int |z(t)|^2 dt \right)^{1/2},$$

  - Under a minimizing $u(x(t)) \in \mathcal{U}$ in spite of unforeseen $w(t) \in \mathcal{W} \subseteq \mathbb{R}^q$.

# Minimization Objective and Risk-Sensitive Control



- Risk-sensitive linear exponential quadratic Gaussian objective functional (Jacobson, 1973):

$$\min_{u\in\mathcal{U}} \mathcal{J}_{exp}(x_0, u, w) = \mathbb{E}\Big|_{x_0\in\mathcal{P}_0} \exp\left[\frac{\alpha}{2}\int_0^\infty z^\top(t)z(t)\mathrm{d}t\right],$$
$$\text{subject to } \mathrm{d}x(t) = Ax(t)\mathrm{d}t + Bu(t)\mathrm{d}t + D\mathrm{d}w(t),$$
$$z(t) = Cx(t) + Eu(t),\ \alpha > 0; \tag{3}$$

- where $dw/dt = \mathcal{N}(0, W)$, $x_0 = \mathcal{N}(0, \mu)$, and $(x_0, w(t)) \subseteq (\Omega, \mathcal{F}, \mathcal{P})$.

# Minimization Objective and Risk-Sensitive Control



- A Taylor series expansion of (3) reveals:

$$\mathcal{J}_{exp}(x_0, u, w) = \lim_{T\to\infty} \mathbb{E}\Big|_{x_0\in\mathcal{P}_0} \left[\frac{\alpha}{2}\sum_{t=0}^{T} z^\top(t)z(t)\right] + \frac{\alpha^2}{4} var\left[\sum_{t=0}^{T} z^\top(t)z(t)\right]. \tag{4}$$

- Consider the variance term $\frac{\alpha^2}{4} var\left[\sum_{t=0}^{T} z^\top(t)z(t)\right] \to \epsilon$.
  - $\alpha$ a measure of risk-propensity if $\alpha > 0$;
  - $\alpha$ a measure of risk-aversion if $\alpha < 0$;
  - $\alpha = 0$ implies solving a classic LQP.

# RL PO as a Risk-Sensitive Control Problem



- RL (via PG) computes high-variance gradient estimates from Monte-Carlo trajectory roll-outs and bootstrapping.
- If we set $\alpha > 0$ in the LEQG problem (3), we have a controlled setting where we can study the theoretical properties of RL-based PO.
- Framework: an ADP policy iteration (PI) in a continuous PO setting.
- LEQG also interprets as a risk-attenuation algorithm.

# Contributions



- A two-loop iterative alternating best-response procedure for computing the optimal mixed-design policy;
- Rigorous convergence analyses follow for the model-based loop updates;
- In the absence of exact system models, we provide an input-to-state-stable hybrid robust stabilization scheme.

# Transition Slide



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# Problem Setup

For $\alpha > 0$, the cost
$\mathcal{J}_{exp}(x_0, u) = \mathbb{E}\Big|_{x_0 \in \mathcal{P}_0} \exp\left[\frac{\alpha}{2}\int_0^\infty z^\top(t)z(t)\mathrm{d}t\right]$, becomes

$$\mathbb{E}\Big|_{x_0 \in \mathcal{P}_0} \exp\left\{\frac{\alpha}{2}\int_0^\infty \left[x^\top(t)Qx(t) + u^\top(t)Ru(t)\right]\mathrm{d}t\right\}, \quad (5)$$

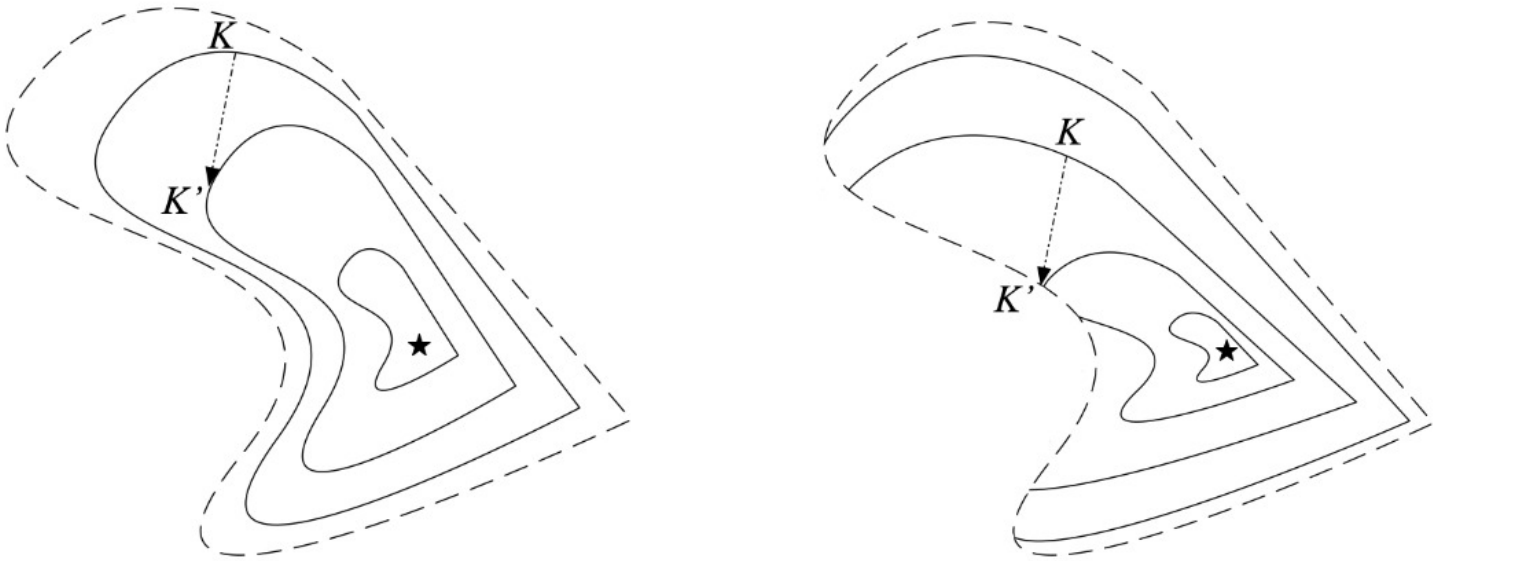
with the associated closed loop transfer function,

$$T_{zw}(K) = (C - EK)(sI - A + BK)^{-1}D. \quad (6)$$

# Nonconvexity and Coercivity in PG



- Coercivity: iterates remain feasible and strictly separated from the infeasible set as the cost decreases.




(a) Landscape of LQR (b) Landscape of Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control

Figure: Coercivity property of PG on LQR and in mixed-design settings. Credit: (Zhang et al., 2019).

# Assumptions



- $C^\top C = Q \succ 0,\ E^T\,(C,\ E) = (0,\ R)$ for some $R \succ 0$.
- Coercivity satisfaction: $(A, B)$ is stabilizable;
- Optimization satisfaction: $(\sqrt{Q}, A)$ is detectable.

# Transition Slide



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# PO and Dynamic Games: Finite-horizon Gain



- Coercivity: feasibility set of optimization iterates

$$\mathcal{K} = \{ K : \ \lambda_i(A - B_1 K) < 0, \ \|T_{zw}(K)\|_\infty < \gamma \}. \quad (7)$$

- Finite-horizon optimization $u^\star(t) = -K^\star_{leqg}\hat{x}(t)$.
- $K^\star_{leqg} = R^{-1}B^\top P_\tau$, and $P_\tau$ is the unique, symmetric, positive definite solution to the algebraic Riccati equation (ARE)

$$A^\top P_\tau + P_\tau A - P_\tau(BR^{-1}B^\top - \alpha^{-2}DD^\top)P_\tau = -Q. \quad (8)$$

(Cui and Molu, 2023a, Proposition I), (Duncan, 2013) .

- $\infty$-horizon case: $P^\star \triangleq P_\infty = \lim_{\tau\to\infty} P_\tau$, and $K^\star_{leqg} \triangleq K_\infty = \lim_{\tau\to\infty} K_\tau$[Theorem on limit of monotonic operators (Kan, 1964)].

# Solving the LEQG Problem



- Directly solving the LEQG problem (3) in policy-gradient frameworks incurs biased gradient estimates during iterations;
- Affects risk-sensitivity preservation in infinite-horizon LTI settings (see (Zhang et al., 2021; Zhang et al., 2019));
- Workaround: an equivalent dynamic game formulation to the stochastic LQ PO problem.

# Two-Player Zero-Sum Game and LEQG

- An equivalent closed-loop two-player game connection (Cui and Molu, 2023b, Lemma 1):

$$\min_{u\in\mathcal{U}} \max_{\xi\in W} \bar{\mathcal{J}}_\gamma(x_0, u, \xi)$$
$$\text{subject to } \mathrm{d}x(t) = Ax(t)\mathrm{d}t + Bu(t)\mathrm{d}t + D\mathrm{d}w(t),$$
$$z(t) = Cx(t) + Eu(t) \tag{9}$$

$$\bar{\mathcal{J}}_\gamma(x_0, u, \xi) = \mathbb{E}_{x_0\sim\mathcal{P}_0,\,\xi(t)} \int_0^\infty \left[x^\top(t)Qx(t) + u^\top(t)Ru(t)\right] \mathrm{d}t$$
$$-\mathbb{E}_{x_0\sim\mathcal{P}_0,\,\xi(t)} \int_0^\infty \left[\gamma^2\xi^\top(t)\xi(t)\right] \mathrm{d}t$$

, $\xi(\equiv dw) \sim \mathcal{N}(0, \Sigma)$, and $\gamma \equiv \alpha$.

# Proof Sketch (Cui and Molu, 2023b, Lemma 1)



- If a non-negative definite (n.n.d ) GARE (8)'s solution exists, then a minimal realization $P^\star$ must exist.
  - Existence: the bounded real Lemma (Zhou et al., 1996).
- If $(A, Q^{\frac{1}{2}})$ is observable, then every n.n.d solution of (8), *i.e.* $P^\star$, is positive definite.
- For a n.n.d $P^\star$, we essentially have a Nash (equivalently a Saddle) equilibrium with $\bar{\mathcal{J}}_\gamma = \underline{\mathcal{J}}_\gamma$.

# Proof Sketch (Cui and Molu, 2023b, Lemma 1)



- If $\bar{\mathcal{J}}_\gamma$ is finite for some $\gamma = \hat{\gamma} > 0$, then $\bar{\mathcal{J}}_\gamma$ is bounded (if and only if the pair $(A, B)$ is stabilizable).
- For a bounded $\bar{\mathcal{J}}_\gamma$ for some $\gamma = \hat{\gamma}$ and for optimal $K^\star = R^{-1}B^\top P_{K,L}$, $L^\star = \gamma^{-2}D^\top P_{K,L}$ and all $\gamma > \hat{\gamma}$, $\bar{\mathcal{J}}_\gamma$ admits the closed-loop matrices

$$A_K^\star = A - BK^\star,\ A_{K,L}^\star = A_K^\star + DL^\star. \tag{10}$$

- Whence, the saddle-point optimal controllers are

$$u^\star(x(t)) = -K^\star x(t),\ \xi^\star(x(t)) = L^\star x(t). \tag{11}$$

# Transition Slide



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# Model-based PO

- Define $\{p, q\}_{p=1,q=1}^{\bar{p},\bar{q}}$ where $(\bar{p}, \bar{q}) \in \mathbb{N}_+$ as nested iteration indices for a gain $K_p$ (in an outer loop) and an alternating gain $L_q(K_p)$ (in an inner-loop).

**Problem 1 (Model-Based Policy Iteration)**

*Given system matrices $A, B, C, D, E$, find the optimal controller gains $K_p$, $L_q(K_p)$ that robustly stabilizes (3) such that the controller gains do not leave the set of all suboptimal controllers denoted by*

$$\breve{\mathcal{K}} = \{(K_p, L_q(K_p)) : \lambda_i(A_K^p) < 0, \lambda_i(A_{K,L}^{p,q}) < 0, \\ \|T_{zw}(K_p, L_q(K_p))\|_\infty < \gamma \text{ for all } (p, q) \in \mathbb{N}\}. \quad (12)$$

# Model-based Policy Optimization



- Further, define the following closed-loop matrix identities

$$A_K^p = A - BK_p, \quad A_{K,L}^{p,q} = A_K^p + DL_q(K_p),$$
$$Q_K^p = Q + K_p^\top R K_p, \; A_K^\gamma = A_K^p + \gamma^{-2} DD^\top P_K^p. \tag{13}$$

- Equation (13) informs the value iterations of the Riccati equations for the outer and inner loops.

$$A_K^{p\top} P_K^p + P_K^p A_K^p + Q_K^p + \gamma^{-2} P_K^p DD^\top P_K^p = 0, \tag{14a}$$
$$K_{p+1} = R^{-1} B^\top P_K^p. \tag{14b}$$

$$A_{K,L}^{(p,q)^\top} P_{K,L}^{p,q} + P_{K,L}^{p,q} A_{K,L}^{p,q} + Q_K^p - \gamma^2 L_q^\top(K_p) L_q(K_p) = 0 \tag{15a}$$
$$K_{p+1} = R^{-1} B^\top P_K^{p,q}, \; L_{q+1}(K_p) = \gamma^{-2} D^\top P_{K,L}^{p,q}. \tag{15b}$$

# Kleinman's Algorithm



- An iterative algorithm for solving infinite-time Riccati equations (Kleinman, 1968).
- Based on a successive substitution method.
- For a *deterministic LTI system's* cost matrix $P_d$, the value iterations of $P_d^k$ are monotonically convergent to $P_d^\star$.
- Kleinman's algorithm as policy iteration
  - Choose a stabilizing control gain $K_0$, and let $p = 0$.
  - (Policy evaluation) Evaluate the performance of $K_p$ from the GARE's solution.
  - (Policy improvement) Improve the policy: $K_p = -R^{-1}B^\top P_d^p$.
  - Advance iteration $p \leftarrow p + 1$.

# Model-based Policy Iteration



**Algorithm 1:** (Model-Based) PO via Policy Iteration

**Input:** Max. outer iteration $\bar{p}$, $q = 0$, and an $\epsilon > 0$;
**Input:** Desired risk attenuation level $\gamma > 0$;
**Input:** Minimizing player's control matrix $R \succ 0$.

Compute $(K_0, L_0) \in \mathcal{K}$; ▷ From [24, Alg. 1];
Set $P^{0,0}_{K,L} = Q^0_K$; ▷ See equation (9);
**for** $p = 0, \ldots, \bar{p}$ **do**
  Compute $Q^p_K$ and $A^p_K$ ▷ See equation (9);
  Obtain $P^p_K$ by evaluating $K_p$ on (10);
  **while** $\|P^p_K - P^{p,q}_{K,L}\|_F \leq \epsilon$ **do**
    Compute $L_{q+1}(K_p) := \gamma^{-2} D^\top P^{p,q}_{K,L}$;
    Solve (11) until $\|P^p_K - P^{p,q}_{K,L}\|_F \leq \epsilon$;
    $\bar{q} \leftarrow q + 1$
  **end**
  Compute $K_{p+1} = R^{-1} B^\top P^{p,\bar{q}}_{K,L}$ ▷ See (11b) ;
**end**

# Transition Slide



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# Convergence Analyses: Outer Loops



## Lemma 1

*Under our assumptions and for the ARE* (14)*, if* $K_0 \in \mathcal{K}$*, then for any* $p \in \mathbb{N}_+$*, we must have the following conditions for the optimal* $K^\star$ *and* $P^\star$*,*

(1) $K_p \in \mathcal{K}$;

(2) $P_K^0 \succeq P_K^1 \succeq \cdots P_K^p \succeq \cdots \succeq P^\star$;

(3) $\lim_{p\to\infty} \|K_p - K^*\|_F = 0$, $\lim_{p\to\infty} \|P_K^p - P^*\|_F = 0$.

# Proof Sketch: The Bounded Real Lemma



Under our standard stabilizability and observability assumptions, for a stabilizing gain $K$, the following conditions are equivalent

- $$\|\mathcal{T}(K)\|_\infty < \gamma;$$
- The Riccati equation
$$A_K^\top P_K + P_K A_K + C^\top C + K^\top R K + \gamma^{-2} P_K D D^\top P_K = 0, \tag{16}$$
admits a unique positive definite solution $P_K \succeq 0$ for a Hurwitz matrix $(A_K + \gamma^{-2} D D^\top P_K)$;
- There exists $P_K \succ 0$ such that
$$A_K^\top P_K + P_K A_K + Q + K^\top R K + \gamma^{-2} P_K D D^\top P_K \prec 0. \tag{17}$$

# Stabilizing Proof Sketch



- At an iteration 0, find a $K_0$ that is stabilizing (Molu, 2023, Alg. 1), so that $K_0 \in \mathcal{K}$ by the bounded real Lemma.
- For $p > 0$, set $Q_K^{p+1} = C^\top C + K_{p+1}^\top R K_{p+1}$, the outer loop GARE is

$$A_K^{(p+1)^\top} P_K^p + P_K^p A_K^{(p+1)} + \gamma^{-2} P_K^p D D^\top P_K^p + C^\top C + K_{p+1}^\top R K_{p+1} + (K_{p+1} - K_p)^\top R (K_{p+1} - K_p) = 0. \quad \text{(A.2)}$$

Thus, for a stabilizing $K_{p+1} (\neq K_p)$ we must have $(K_{p+1} - K_p)^\top R (K_{p+1} - K_p) \succ 0$ so that

$$A_K^{(p+1)^\top} P_K^p + P_K^p A_K^{(p+1)} + \gamma^{-2} P_K^p D D^\top P_K^p + Q_K^{p+1} \prec 0. \quad \text{(A.3)}$$

- For $p > 1$, $K_p \in \mathcal{K}$. Rest: completion of squares, the bounded real Lemma, and the theorem on the "limit of monotonic operators." (Kan, 1964).

# Convergence Analysis



- In (Zhang et al., 2019, Theorem A.7 and A.8), the authors showed that this controller update in the outer-loop has a global sub-linear and local quadratic convergence rates.
- We now show that the outer-loop iteration has a global linear convergence rate.

# Transition Slide



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# Convergence Analysis: Outer Loop



## Lemma 2

*Let* $\Psi = (K_{p+1} - K_p)^\top R(K_{p+1} - K_p)$; *and* $\Psi = \Psi^\top \succeq 0$. *Furthermore, let* $\Phi \in \mathbb{R}^{n\times n}$ *be Hurwitz so that* $\Theta = \int_0^\infty e^{(\Phi^\top t)}\Psi e^{(\Phi t)}dt$ *and define* $c(\Phi) = \log(5/4)\|\Phi\|^{-1}$. *Then,* $\|\Theta\| \geq \frac{1}{2}c(\Phi)\|\Psi\|$.

# Convergence Analysis: Outer Loop



## Remark 1

*For $A_K = A - BK$, we know from the bounded real Lemma (Zhang et al., 2019, Lemma A.1) that the Riccati equation*

$$A_K^\top P_K + P_K A_K + Q_K + \gamma^{-2} P_K D D^\top P_K = 0 \tag{18}$$

*admits a unique positive definite solution $P_K \succ 0$ with a Hurwitz $(A_K + \gamma^{-2} D D^\top P_K)$.*

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# Optimality of the Iteration



**Lemma 3 (Optimality of the iteration)**

*Consider any $K \in \mathcal{K}$, let $K' = R^{-1}B^\top P_K$ (where $P_K$ is the solution to (18), and $\Psi_K = (K - K')^\top R(K - K')$. If $\Psi_K = 0$, then $K = K^\star$.*

**Proof.**

Since $R \succ 0$, $\Psi_K = 0$ implies $K = K'$. Therefore at $\Psi_K = 0$, we must have $K = K'$ which implies that $P_K = P'_K$. If $K = K'$ and $P_K = P'_K$, it suffices to conclude that $K' = K \triangleq K^\star$ where $K^\star = R^{-1}B^\top P^\star$. Hence, $\Psi_K = 0$ is tantamount to $P_K = P^\star$ and $K = K^\star$. □

# Bound on Cost Difference Matrix



## Lemma 4 (Bound on Cost Difference Matrix)

*For any $h > 0$, define $\mathcal{K}_h := \{K \in \mathcal{K} | \, Tr(P_K^p - P^\star) \leq h\}$. For any $K \in \mathcal{K}_h$, let $K' := R^{-1}B^\top P_K^p$, where $P_K^p$ is the $p$'th iterate's solution to (18), and $\Psi_{K_p} = (K_p - K'_p)^\top R(K_p - K'_p)$. Then, there exists $b(h) > 0$, such that $\|P_K^p - P^\star\|_F \leq b(h)\|\Psi_{K_p}\|_F$.*

# Bound on Cost Difference Matrix



- For $A^\star = A - BR^{-1}B^\top P^\star + \gamma^{-2}DD^\top P^\star$, rewrite the closed-loop Riccati equation as

$$\begin{aligned} &A^{\star\top}P_K^p + P_K^p A^\star + Q_{K_p} + (K^\star - K_p)^\top RK_p' \\ &+ K_p'^\top R(K^\star - K_p) - \gamma^{-2}P^\star DD^\top P_K^p - \gamma^{-2}P_K^p DD^\top P^\star \\ &+ \gamma^{-2}P_K^p DD^\top P_K^p = 0. \end{aligned} \tag{19}$$

- Then do completion of squares so that

$$\begin{aligned} &A^{\star\top}(P_K^p - P^\star) + (P_K^p - P^\star)A^\star + \Psi_{K_p} \\ &+ \gamma^{-2}(P_K^p - P^\star)DD^\top(P_K^p - P^\star) \\ &- (K_p' - K^\star)^\top R(K_p' - K^\star) = 0. \end{aligned} \tag{20}$$

# Proof



- Implicit function theorem: $P_K^p = f(K_p \in \mathcal{K})$, $f(\cdot) \in \mathcal{C}^n$.
- There exists a ball $\mathcal{B}_\delta(K^\star) := \{K \in \mathcal{K} | \|K - K^\star\|_F \leq \delta\}$, such that $\mathcal{A}(K)$ is invertible for any $K \in \mathcal{K}_h \cap \mathcal{B}_\delta(K^\star)$.
  - $\mathcal{A}(K_p) = I_n \otimes A^{\star\top} + (A - BR^{-1}B^\top P_K^p + \gamma^{-2} DD^\top P_K^p)^\top \otimes I_n$.
- Therefore, for any $K \in \mathcal{K}_h \cap \mathcal{B}_\delta(K^\star)$,
  - $\|\tilde{P}_K^p\|_F \leq \underline{\sigma}^{-1}(\mathcal{A}(K_p))\|\Psi_{K_p}\|_F$.
- Similarly, for any $K \in \mathcal{K}_h \cap \mathcal{B}_\delta^c(K^\star)$, where $\mathcal{B}^c$ is a complement of $\mathcal{B}$, $\Psi_{K_p} \neq 0$ and there exists a constant $b_1 > 0$ such that $\|\Psi_{K_p}\| \geq b_1$.
- Set $b_2 = \max_{K \in \mathcal{K}_h \cap \mathcal{B}_\delta(K^\star)} \underline{\sigma}^{-1}(\mathcal{A}(K))$ and $b(h) = \max\{b_2, \frac{h + Tr(P^\star)}{b_1}\}$, then the proof follows immediately.

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# Outer Loop Convergence: Exponential Stability of $P_K^p$



**Theorem 2**

*For any $h > 0$ and $K_0 \in \mathcal{K}_h$, there exists $\alpha(h) \in \mathbb{R}$ such that $Tr(P_K^{p+1} - P^\star) \leq \alpha(h) Tr(P_K^p - P^\star)$. That is, $P^\star$ is an exponentially stable equilibrium.*

# Transition Slide



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# Convergence Analysis: Inner Loop



- Now, we analyze the monotonic convergence rate of the inner loop.
- Given arbitrary gains $K_p \in \mathcal{K}$ and $L_q(K_p) \in \mathcal{L}$, and $P^{p,q}_{K,L} \succ 0$ solution of the inner-loop Lyapunov equation, the cost matrix $P^{p,q}_{K,L}$ monotonically converges to the solution of (15).

$$A^{(p,q)^\top}_{K,L} P^{p,q}_{K,L} + P^{p,q}_{K,L} A^{p,q}_{K,L} + Q^p_K - \gamma^2 L_q^\top(K_p) L_q(K_p) = 0 \tag{21a}$$

$$K_{p+1} = R^{-1} B^\top P^{p,q}_K, \;\; L_{q+1}(K_p) = \gamma^{-2} D^\top P^{p,q}_{K,L}. \tag{21b}$$

# Convergence Analysis: Inner Loop I



## Lemma 5

*Suppose that $L_0(K_0)$ is stabilizing, then for any $q \in \mathbb{N}_+$ (with $P^{p,\bar{q}}_{K,L}$ as the solution to (15)), i.e.*

$$A^{(p,q)^\top}_{K,L} P^{p,q}_{K,L} + P^{p,q}_{K,L} A^{p,q}_{K,L} + Q^p_K - \gamma^2 L_q^\top(K_p) L_q(K_p) = 0 \quad (22a)$$

$$K_{p+1} = R^{-1} B^\top P^{p,q}_K, \; L_{q+1}(K_p) = \gamma^{-2} D^\top P^{p,q}_{K,L}. \quad (22b)$$

*Then, the following statements hold*

1. $A^{p,q}_{K,L}$ *is Hurwitz;*
2. $P^{p,\bar{q}}_{K,L} \succeq \cdots \succeq P^{(p,q+1)}_K \succeq P^{p,q}_K \succeq \cdots \succeq P^{p,0}_{K,L}$; *and*
3. $\lim_{q\to\infty} \| P^{p,q}_{K,L} - P^{p,\bar{q}}_{K,L} \|_F = 0.$

# Convergence Rate – Inner Loop

## Lemma 6 (Monotonic Convergence of the Inner-Loop)

*For any $K \in \mathcal{K}$, let $L(K)$ be the control gain for the player $w$ such that $A_K + DL(K)$ is Hurwitz. Let $P_K^L$ be the solution of*

$$(A_K + DL(K))^\top P_K^L + P_K^L (A_K + DL(K)) + Q_K - \gamma^2 L(K)^\top L(K) = 0. \tag{23}$$

*Let $L'(K) = \gamma^{-2} D^\top P_K^L$ and $\Psi_K^L = \gamma^{-2}(L'(K) - L(K))^\top (L'(K) - L(K))$. Then, for a $c(K) = Tr\left(\int_0^\infty e^{(A_K + DL(K^\star))t} e^{(A_K + DL(K^\star))^\top t} \mathrm{d}t\right)$, the following inequality holds $Tr(P_K - P_K^L) \leq \|\Psi_K^L\| c(K)$.*

# Convergence of the Inner Loop Iteration



## Theorem 3

*For a $K \in \check{\mathcal{K}}$, and for any $(p, q) \in \mathbb{N}_+$, there exists $\beta(K) \in \mathbb{R}$ such that*

$$Tr(P_K^p - P_{K,L}^{p,q+1}) \leq \beta(K)\, Tr(P_K^p - P_{K,L}^{p,q}). \tag{24}$$

## Remark 2

*As seen from Lemma 5, $P_K^p - P_{K,L}^{p,q} \succeq 0$. By the norm on a matrix trace (Cui and Molu, 2023a, Lemma 13) and the result of Theorem 3, we have $\|P_K - P_{K,L}^{p,q}\|_F \leq Tr(P_K - P_{K,L}^{p,q}) \leq \beta(K)\, Tr(P_K)$, i.e. $P_{K,L}^{p,q}$ exponentially converges to $P_K$ in the Frobenius norm.*

# Algorithm as a Policy Iteration Scheme



- Choosing a stabilizing $K_p$ we first evaluate $u$'s performance by solving (14).
  - This is the policy evaluation step in PI.
- The policy is then improved in a following iteration by solving for the cost matrix in (15b);
  - This is the policy improvement step.
- Essentially, a policy iteration algorithm whereupon
  - Performance of an initial control gain $K_p$ is first evaluated against a cost function.
  - A newer evaluation of the cost matrix $P_{K,L}^{p,q}$ is then used to improve the controller gain $K_{p+1}$ in the outer loop.

# Transition Slide



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# Sampling-based PO Scheme



- $A, B, C, D, E$ are often unavailable so that the policy evaluation step will result in biased estimates.
- There is the possibility for a divergence from the stability-robustness feasibility set $\breve{\mathcal{K}}$:
  - When errors are present from I/O or state data;
  - Residuals from early termination of numerically solving Riccati equations;
  - Using an approximate cost function owing to inexact values of $Q$ and $R$;
  - Since the inner loop is computed in a finite number of steps;
  - In a data sampling scheme, we must guarantee the stability and robustness of the closed-loop system.

# Sampling-based PO: Statement of the Problem



**Problem 4 (Sampling-based Policy Optimization)**

*If $A, B, C, D, E, P$ are all replaced by approximate matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{P}$, under what conditions will the sequences $\{\hat{P}^{p,q}_{K,L}\}^{(p,q)=\infty}_{(p,q)=1}$, $\{\hat{K}_p\}^{\infty}_{p=0}$, $\{\hat{L}_q\}^{\infty}_{q=0}$ converge to a small neighborhood of the optimal values $\{P^{\star}_{K,L}\}^{(p,q)=\infty}_{(p,q)=0}$, $\{K^{\star}_p\}^{\infty}_{p=0}$, and $\{L^{\star}_q\}^{\infty}_{q=0}$?*

# Discrete-Time Nonlinear System Interpretation



- From assumptions, a $P_K^0 \in \mathbb{S}^n$ exists such that when applied to find a $K_0$ such a $K_0$ will be stabilizing.
- Approximation errors between the nested iteration steps yield a hybrid of a continuous-time policy gain pair $(\hat{K}_p, \hat{L}_q(\hat{K}_p))$ and a learning scheme.
  - This learning scheme is essentially a discrete sampled data from a nonlinear system (owing to errors from various sources).
- Task: under inexact loop updates, lump iterates of gain errors into system inputs to the online PO scheme;

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# Discrete-Time Nonlinear System Interpretation



- How do we converge to the optimal solution and preserve closed-loop dynamic stability?
- What does input-to-state stability (ISS) Sontag (2008) have to do with it?

# Online Model-free Reparameterization



- Suppose that $\hat{P}_K^0 \in \mathbb{S}^n$ is chosen following the controllability and stabilizability assumptions.
  - Then $\hat{K}_k^1 = R^{-1}B^\top \hat{P}_K^0$ will be stabilizing since $\tilde{K}_k^1 = \hat{K}_k^1 - K_k^1 \triangleq 0$.
- Ditto argument for $L_1$.

**Problem 5**

*For $(p, q) > 0$, show that for $\tilde{K}_k^p = \hat{K}_k^p - K_k^p \triangleq 0$ so that the sequence $\{P_{K,L}^{p,q}\}_{(p,q)=0}^{\infty}$ converges to the locally exponentially stable $\hat{P}_{K,L}^{\star}$.*

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# Hybrid System Reparameterization



- Lump estimate errors as an input into the gain terms to be computed in the PO algorithm.
- With inexact outer loop update, $K_{p+1}$ becomes biased so that the inexact outer-loop GARE value iteration involves the recursions

$$\hat{A}_K^{p\top} \hat{P}_K^p + \hat{P}_K^p \hat{A}_K^p + \hat{Q}_K^p + \gamma^{-2} \hat{P}_K^p DD^\top \hat{P}_K^p = 0, \quad (25a)$$

$$\hat{K}_{p+1} = R^{-1} B^\top \hat{P}_K^p + \tilde{K}_{p+1} \triangleq \bar{K}_{p+1} + \tilde{K}_{p+1}, \quad (25b)$$

  - NB: $\hat{A}_K^p = A - B\hat{K}_p$ and $\hat{Q}_K^p = Q + \hat{K}_p^\top R \hat{K}_p$.

# Discrete-Time System Closed-loop System



- Same argument for the inner-loop inexact GARE value iteration updates:

$$\hat{A}_{K,L}^{p,q\top}\hat{P}_{K,L}^{p,q} + \hat{P}_{K,L}^{p,q}\hat{A}_{K,L}^{p,q} + \hat{Q}_K^p - \gamma^2\hat{L}_q^\top\hat{L}_q(\hat{K}_p) = 0 \quad (26a)$$

$$\hat{K}_{p+1} = R^{-1}B^\top\hat{P}_K^{p,q} + \tilde{K}_p, \quad (26b)$$

$$\hat{L}_{q+1}(\hat{K}_p) = \gamma^{-2}D^\top\hat{P}_{K,L}^{p,q} + \tilde{L}_{q+1}(\tilde{K}_p) \quad (26c)$$

$$\triangleq \bar{L}_{q+1}(\bar{K}_p) + \tilde{L}_{q+1}(\tilde{K}_p). \quad (26d)$$

- Rewrite the infinite-dimensional stochastic differential equation as the discrete-time system (for iterates $(p, q) > 0$):

$$\mathrm{d}x = [\hat{A}_{K,L}^{p,q}x + B(\hat{K}_p x - D\hat{L}_q(K_p) + u)]\mathrm{d}t + D\mathrm{d}w. \quad (27)$$

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# System Trajectories from HJB Interpretation



- On a time interval $[s, s+\delta s]$, it follows from Itô's stochastic calculus and the Hamilton-Jacobi-Bellman equation that

$$d\left[x^\top(s+\delta s)\hat{P}^{p,q}_{K,L}x(s+\delta s) - x^\top(s)\hat{P}^{p,q}_{K,L}x(s)\right] = (\mathrm{d}x)^\top \hat{P}^{p,q}_{K,L}x + x^\top \hat{P}^{p,q}_{K,L}\mathrm{d}x + (\mathrm{d}x)^\top \hat{P}^{p,q}_{K,L}(\mathrm{d}x). \quad (28)$$

- Along the trajectories of equation (27) and using the gains in (15), *i.e.*

$$K_{p+1} = R^{-1}B^\top P^{p,q}_K,\ L_{q+1}(K_p) = \gamma^{-2}D^\top P^{p,q}_{K,L}.$$

# System Trajectories

- The r.h.s. in (28) becomes

$$x^\top \left[\hat{A}_{K,L}^{p,q\top} \hat{P}_{K,L}^{p,q} + \hat{P}_{K,L}^{p,q} \hat{A}_{K,L}^{p,q}\right] x\mathrm{d}t + 2x^\top \hat{P}_{K,L}^{p,q} D\mathrm{d}w \tag{29}$$

$$+ 2x^\top \hat{P}_{K,L}^{p,q} B(K_p x - D\hat{L}_q(K_p) + u)dt + Tr(D^\top PD),$$

$$= -x^\top \hat{Q}_K^p x\mathrm{d}t - \gamma^{-2} x^\top \hat{P}_{K,L}^{p,q} DD^\top \hat{P}_{K,L}^{p,q} x dt + Tr(D^\top \hat{P}_{K,L}^{p,q}$$

$$D) + 2x^\top \hat{P}_{K,L}^{p,q} B\left[\hat{K}_p x - D\hat{L}_q(K_p) + u\right]\mathrm{d}t + 2x^\top \hat{P}_{K,L}^{p,q} D\mathrm{d}w \tag{30}$$

# System Trajectories via HJB Expansions

- So that

$$
\begin{aligned}
& x^\top(s+\delta s)\hat{P}^{p,q}_{K,L}(s+\delta s) - x^\top(s)\hat{P}^{p,q}_{K,L}x(s) \\
&= \int_s^{s+\delta s} \left[(-x^\top \hat{Q}^p_K x - \gamma^2 w^\top w)\mathrm{d}t + 2\gamma^2 x^\top \hat{L}^\top_{q+1}(K_p)\mathrm{d}w\right] \\
&+ \int_s^{s+\delta s} 2x^\top \hat{K}^\top_{p+1} R\left[\hat{K}_p x - D\hat{L}_q(\hat{K}_p) + u\right]\mathrm{d}t \\
&\quad + \int_s^{s+\delta s} Tr(D^\top \hat{P}^{p,q}_{K,L} D)\mathrm{d}t. \qquad (31)
\end{aligned}
$$

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# Input To State System Interpretation



- System matrices $\hat{A}^{p,q}_{K,L}, B, C, D$ now embedded within input and state terms: $\hat{Q}^{p}_{K}$, $\hat{K}_{p+1}$, and $\hat{L}_{q+1}$;
- Retrievable via online measurements.
- We essentially end up with an input-to-state system!
- The price that we pay is that the noise feedthrough matrix $D$ must be known precisely.
  - No marvel: in many linear stochastic system with Brownian motion, $D$ is identity (Duncan et al., 2011; Duncan and Pasik-Duncan, 2010).

# Sampling-based Scheme



- Explore system model until we achieve exact equality in $\hat{A}^{p,q}_{K,L} \equiv A^{p,q}_{K,L}$, $\hat{P}^{p,q}_{K,L}$, $\hat{K}_{p+1} \equiv K_{p+1}$, and $\hat{L}_{q+1}(K_p) \equiv L_{q+1}(K_p)$.
  - Choose $u = -K_0 x + \eta_p$ and $w = -L_0 x + \eta_q$ where $(\eta_p, \eta_q)$ is drawn uniformly at random over matrices with a Frobenium norm $r$ similar to (Gravell et al., 2021; Fazel et al., 2018).

# Sampled System Parameterization



- Consider the identities

$$
\begin{aligned}
&x^\top \hat{Q}_K^p x = (x^\top \otimes x^\top)\,\mathrm{vec}(\hat{Q}_K^p),\\
&\gamma^2 w^\top w = \gamma^2 (w^\top \otimes w^\top)\ \mathrm{vec}(I_v),\\
&2\gamma^2 x^\top \hat{L}_{q+1}^\top(\hat{K}_p)\mathrm{d}w = 2\gamma^2 (I_n \otimes x^\top)\mathrm{d}w\ \mathrm{vec}(\hat{L}_{q+1}^\top(\hat{K}_p)),\\
&2x^\top \hat{K}_{p+1}^\top R\hat{K}_p x = 2(x^\top \otimes x^\top)(I_n \otimes \hat{K}_p^\top)\,\mathrm{vec}(\hat{K}_{p+1}^\top R),\\
&2x^\top \hat{K}_{p+1}^\top RD\hat{L}_q(\hat{K}_p) = 2(\hat{L}_q^\top(\hat{K}_p)D^\top \otimes x^\top)\,\mathrm{vec}(\hat{K}_{p+1}^\top R),\\
&2x^\top \hat{K}_{p+1}^\top Ru = 2(u^\top \otimes x^\top)\,\mathrm{vec}(\hat{K}_{p+1}^\top R),\\
&Tr(D^\top \hat{P}_{K,L}^{p,q} D) = \mathrm{vec}^\top(D)\,\mathrm{vec}(\hat{P}_{K,L}^{p,q} D). \qquad (32)
\end{aligned}
$$

# Sampled System Parameterization I



- Let $\Delta_{xx} \in \mathbb{R}^{\frac{n(n+1)}{2} l}$, $\Delta_{ww} \in \mathbb{R}^{\frac{v(v+1)}{2} l}$, $I_{xx} \in \mathbb{R}^{l \times n^2}$, and $I_{ux} \in \mathbb{R}^{l \times mn}$ for $l \in \mathbb{N}_+$
- It follows that

$$\Delta_{xx} = \left[\mathrm{vecv}(x_1), \ldots, \mathrm{vecv}(x_l)\right]^\top, \; x_l = x_{l+1} - x_l,$$
$$\Delta_{ww} = \left[\mathrm{vecv}(w_1), \ldots, \mathrm{vecv}(w_l)\right]^\top, \; w_l = w_{l+1} - w_l,$$
$$I_{xx} = \left[\int_{s_0}^{s_1} x \otimes x \, \mathrm{d}t, \ldots, \int_{s_{l-1}}^{s_l} x \otimes x \, \mathrm{d}t\right]^\top,$$

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# Sampled System Parameterization

$$I_{xw} = \left[\int_{s_0}^{s_1} (I_n \otimes x)\mathrm{d}w, \ldots, \int_{s_{l-1}}^{s_l} (I_n \otimes x)\mathrm{d}w\right]^\top,$$

$$I_{ux} = \left[\int_{s_0}^{s_1} u \otimes x\,\mathrm{d}t, \ldots, \int_{s_{l-1}}^{s_l} u \otimes x\,\mathrm{d}t\right]^\top. \tag{33}$$

Next, set

$$\Theta_{K,L}^{p,q} = \Big[\Delta_{xx}, -2I_{xx}(I_n \otimes \hat{K}_p^\top) + 2(\hat{L}_q^\top(\hat{K}_p)D^\top \otimes x^\top)$$
$$-2I_{ux}, -2\gamma^2 I_{xw}, -\mathrm{vec}^\top(D)\mathrm{vec}(\hat{P}_{K,L}^{p,q} D)\Big], \tag{34a}$$

$$\Upsilon_{K,L}^{p,q} = \left[-I_{xx}\mathrm{vec}(\hat{Q}_K^p),\ -\gamma^2 I_{ww}\mathrm{vec}(I_v)\right]. \tag{34b}$$

# Sampled System Parameterization



Define $\mathbf{1}_{q^2}$ as a one-vector with dimension $q^2$. Thus,

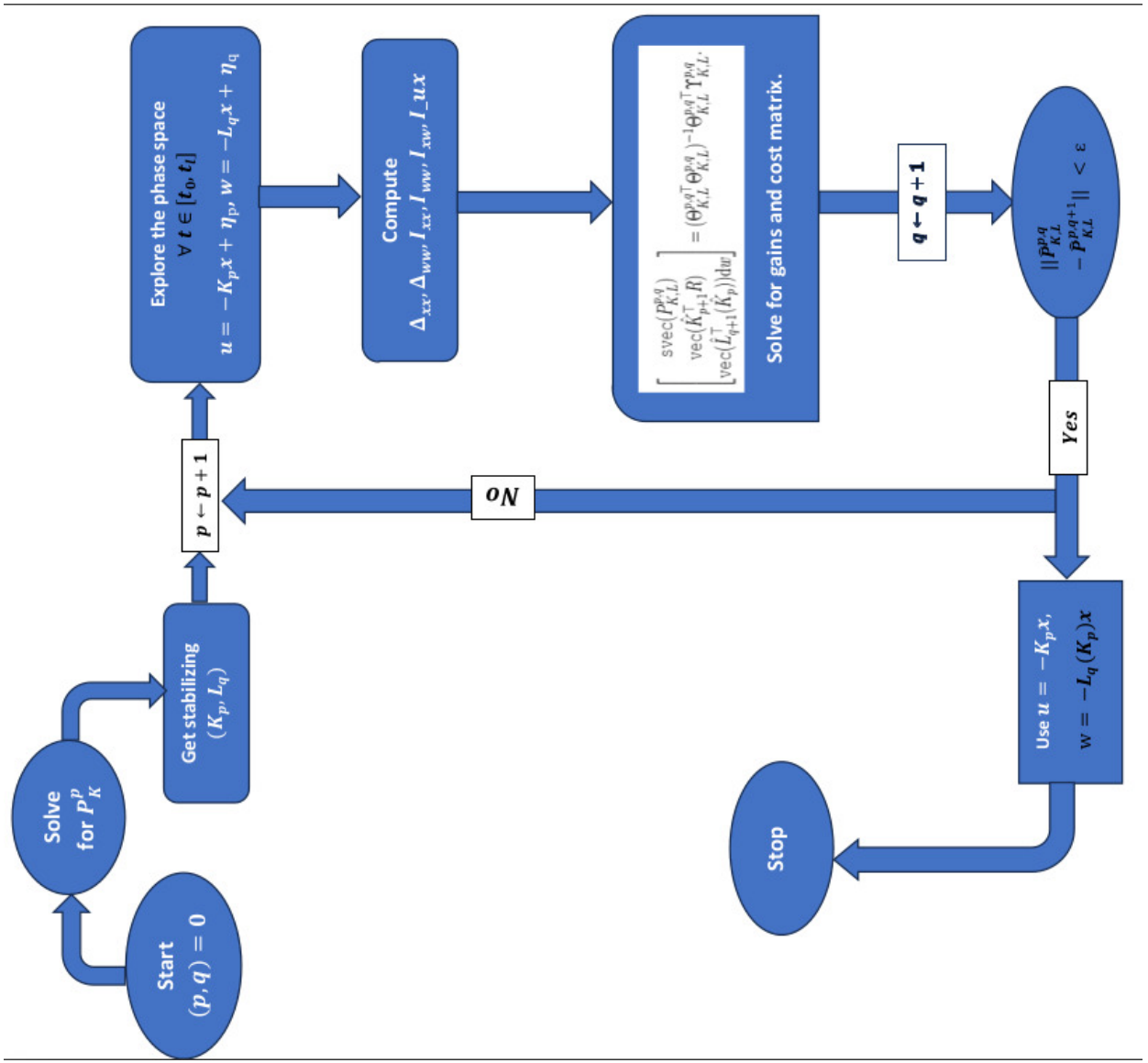
$$\Theta_{K,L}^{p,q}\begin{bmatrix}\mathrm{svec}(P_{K,L}^{p,q}) & \mathrm{vec}(\hat{K}_{p+1}^{\top}R) & \mathrm{vec}(\hat{L}_{q+1}^{\top}(\hat{K}_p)) & \mathbf{1}_{q^2}\end{bmatrix}^{\top} = \Upsilon_{K,L}^{p,q}. \tag{35}$$

Suppose that $\Theta_{K,L}^{p,q}$ is of full rank, then we can retrieve the unknown matrices via least squares estimation *i.e.*

$$\begin{bmatrix}\mathrm{svec}(P_{K,L}^{p,q}) \\ \mathrm{vec}(\hat{K}_{p+1}^{\top}R) \\ \mathrm{vec}(\hat{L}_{q+1}^{\top}(\hat{K}_p))\mathrm{d}w \\ \mathbf{1}_{q^2}\end{bmatrix} = (\Theta_{K,L}^{p,q\top}\Theta_{K,L}^{p,q})^{-1}\Theta_{K,L}^{p,q\top}\Upsilon_{K,L}^{p,q}. \tag{36}$$

# Sampling-based Algorithm





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# Robustness Analyses



- Define $\tilde{P} = P_K - \hat{P}_K$ and $\tilde{K} = K - \hat{K}$.
- Keep $|\tilde{K}| < \epsilon$, start with a $K \in \mathcal{K}$: iterates stay in $\mathcal{K}$.

**Lemma 7 (Lemma 10, C&M, '23)**

*For any $K \in \mathcal{K}$, there exists an $e(K) > 0$ such that for a perturbation $\tilde{K}$, $K + \tilde{K} \in \mathcal{K}$, as long as $\|\tilde{K}\| < e(K)$.*

**Theorem 6**

*The inexact outer loop is small-disturbance ISS. That is, for any $h > 0$ and $\hat{K}_0 \in \mathcal{K}_h$, if $\|\tilde{K}\| < f(h)$, there exist a $\mathcal{KL}$-function $\beta_1(\cdot,\cdot)$ and a $\mathcal{K}_\infty$-function $\gamma_1(\cdot)$ such that*

$$\|P^p_{\hat{K}} - P^\star\| \leq \beta_1(\|P^0_{\hat{K}} - P^*\|, p) + \gamma_1(\|\tilde{K}\|). \tag{37}$$

# ISS Outer Loop Robustness Proof



- Prelim result (Lemma 12, C&M, '23): For any $h > 0$ and $K \in \mathcal{K}_h$, let $K' = R^{-1}B^\top P_K$, where $P_K$ is the solution of (18), and $\hat{K}' = K' + \tilde{K}$. Then, there exists $f(h) > 0$, such that $\hat{K}' \in \mathcal{K}_h$ as long as $\|\tilde{K}\| < f(h)$.
- Therefore, $\hat{K}_K^p \in \mathcal{K}_h$ for any $p \in \mathbb{N}_+$.
- Let

$$f_1(\hat{K}') = \frac{\log(5/4)b(h)}{2n\|A^\star_{\hat{K}'}\|}, f_2(\hat{K}') = Tr\left(\int_0^\infty e^{A^{\star\top}_{\hat{K}'}t} e^{A^{\star}_{\hat{K}'}t} \mathrm{d}t\right).$$

# ISS Outer Loop Robustness Proof



- $$\underline{f}_1(h) = \inf_{\hat{K}' \in \mathcal{K}_h} f_1(\hat{K}') > 0, \bar{f}_2(h) = \sup_{\hat{K}' \in \mathcal{K}_h} f_2(\hat{K}') < \infty. \tag{38}$$
- This implies
$$Tr(P^p_{\hat{K}} - P^\star) \leq [1 - \underline{f}_1(h)]\, Tr(P^{p-1}_{\hat{K}} - P^\star) + \bar{f}_2(h)\|R\|\|\tilde{K}^p_K\|^2. \tag{39}$$
- Repeating (39) for $p, p-1, \cdots, 1$,
$$Tr[P^p_{\hat{K}} - P^\star] \leq (1 - \underline{f}_1)^p\, Tr(P^1_{\hat{K}} - P^\star) + \frac{\bar{f}_2\|R\|\|\tilde{K}\|^2_\infty}{\underline{f}_1(h)}. \tag{40}$$

# Outer Loop Robustness



It follows from (40) and (Mori, 1988, Theorem 2) that

$$\|P^p_{\hat{K}} - P^\star\|_F \leq (1-\underline{f}_1)^p\sqrt{n}\|P^1_{\hat{K}} - P^\star\|_F + \frac{\bar{f}_2\|R\|\|\tilde{K}\|^2_\infty}{\underline{f}_1}. \tag{41}$$

As $p \to \infty$, $P^p_{\hat{K}} \to P^\star$. Whence, a radius of $P^\star$'s neighbor is proportional to $\|\tilde{K}\|^2_\infty$.

# Inner Loop Robustness



The perturbed inner-loop iteration (26) has inexact matrix $\hat{A}^{p,q}_{K,L}$, and sequences $\{\hat{L}_{q+1}(K_p)\}_{q=0}^{\infty}$, and $\{\hat{P}^{p,q}_{K,L}\}_{q=0}^{\infty}$.

**Lemma 8 (Stability of the Inner-Loop's System Matrix)**

*Given $K \in \check{\mathcal{K}}$, there exists a $g \in \mathbb{R}_+$, such that if $\|\tilde{L}_{q+1}(K_p)\|_F \leq g$, $\hat{A}^{p,q}_{K,L}$ is Hurwitz for all $q \in \mathbb{N}_+$.*

# Inner Loop Robustness

## Theorem 7

*Assume* $\|\tilde{L}_q(K_p)\| < e$ *for all* $q \in \mathbb{N}_+$. *There exists* $\hat{\beta}(K) \in [0,1)$, *and* $\lambda(\cdot) \in \check{\mathcal{K}}_\infty$, *such that*

$$\|\hat{P}^{p,q}_{K,L} - P^{p,q}_{K,L}\|_F \leq \hat{\beta}^{q-1}(K)\, Tr(P^{p,q}_{K,L}) + \lambda(\|\tilde{L}\|_\infty). \tag{42}$$

- From Theorem 7, as $q \to \infty$, $\hat{P}^{p,q}_{K,L}$ approaches the solution $P_K$ and enters the ball centered at $P^{p,q}_{K,L}$ with radius proportional to $\|\tilde{L}\|_\infty$.
- The proposed inner-loop iterative algorithm well approximates $P^{p,q}_{K,L}$.

# Transition Slide



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# Numerical Results – Car Cruise Control System



- (Åström and Murray, 2021, §3.1):

$$m\frac{dv}{dt} = \alpha_n u \tau(\alpha_n v) - mgC_r sgn(u) - \frac{1}{2}\rho C_d A|v|v - mg\sin\theta \tag{43}$$

- $u(x(t)) = [u_1(t), u_2(t)]$ must maintain a constant velocity $v$ (the state), whilst automatically adjusting the car's throttle, $u_1(t), t \in [0, T]$
  - despite disturbances characterized by road slope changes ($u_3 = \theta$),
  - rolling friction ($F_r$), and
  - aerodynamic drag forces ($F_d$).

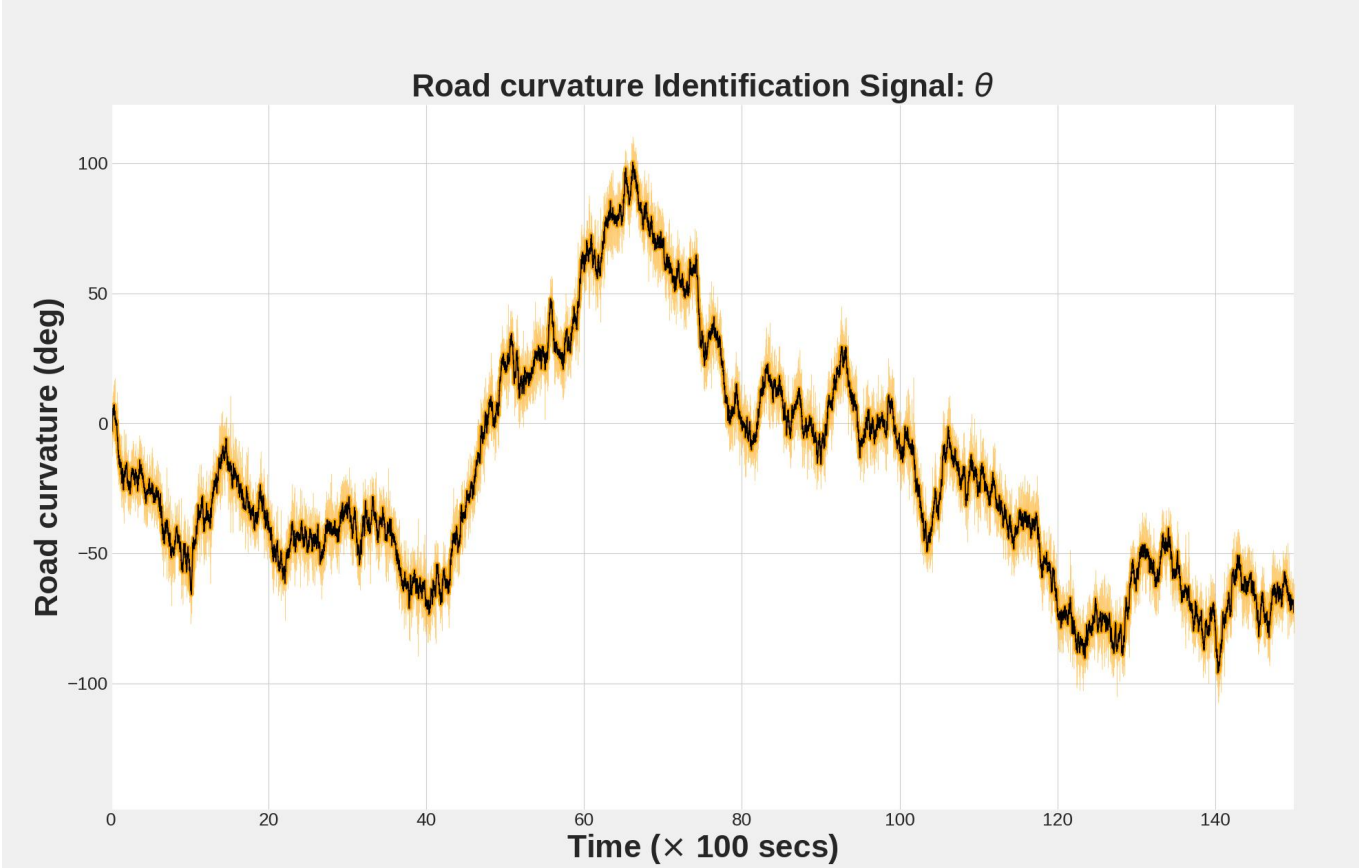
# Numerical Results – Car Cruise Control System



- Well-suited to our robust control formulation because
  - the disturbances and state variables are separable and can be lumped into the form of the stochastic differential equations;
  - it is a multiple-input (throttle, gear, vehicle speed) single-output (vehicle acceleration) system that introduces modeling challenges;
  - the entire operating range of the system is nonlinear though there is a reasonable linear bandwidth that characterize the input/output (I/O) system as we will see shortly.

# Road (Disturbance) Profile





# Search for initial stabilizing gain and $\mathcal{H}_\infty$-norm bound.

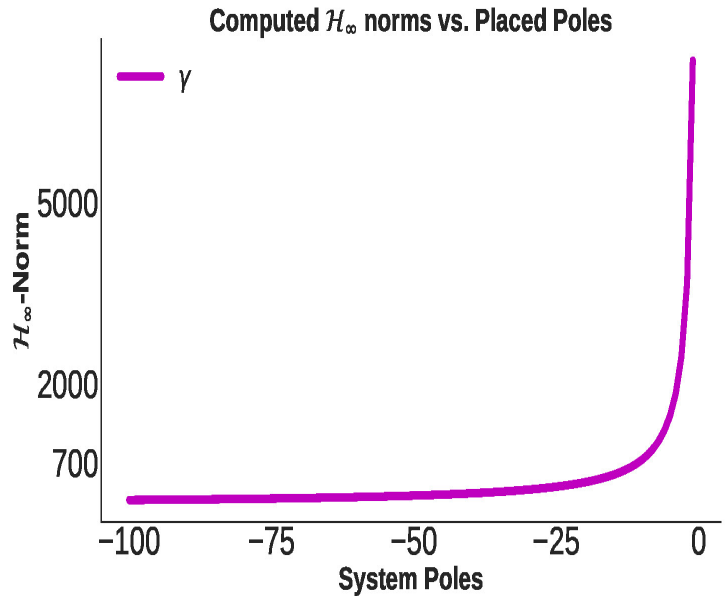


## Proposition 1

*(?) For all $\omega_p \in \mathbb{R}$, we have that $j\omega_p$ is an eigenvalue of the Hamiltonian $H(\gamma_1)$ if and only if $\gamma_1$ is a singular value of $T_{zw}(j\omega_p)$.*

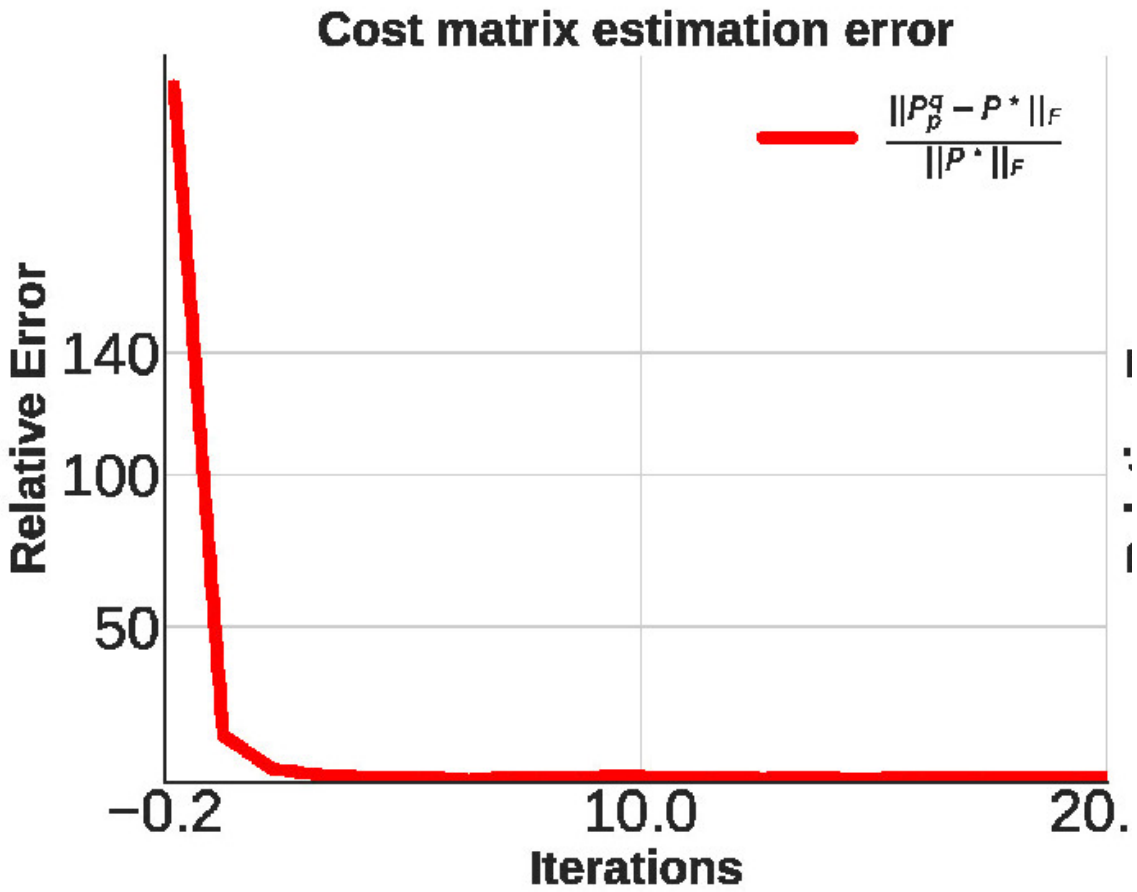
**Algorithm 1** Search for the closed-loop $\mathcal{H}_\infty$-norm

Given a user-defined step size $\eta > 0$
Set the initial upper bound on $\gamma$ as $\gamma_{ub} = \infty$.
Initialize a buffer for possible $\mathcal{H}_\infty$ norms for each $K_1$ to be found, $\Gamma_{buf} = \{\}$.
Initialize ordered `poles` $\mathcal{P} = \{p_i \in Re(s) < 0 \,|\, i = 1, 2, \}$ ▷ $p_1 < p_2 < \cdots$
**for** $p_i \in \mathcal{P}$ **do**
    Place $p_i$ on (2); ▷ (Tits and Yang, 1996)
    Compute stabilizing $K_1^{p_i}$
    Find lower bound $\gamma_{lb}$ for $H(\gamma, K_1^{p_i})$; ▷ using (22)
    $\Gamma_{buf}(i) =$ `get_hinf_norm`$(T_{zw}, \gamma_{lb}, K_1^{p_i})$.
**end for**
**function** `get_hinf_norm`$(T_{zw}, \gamma_{lb}, K_1^{p_i})$
    **while** $\gamma_{ub} = \infty$ **do**
        $\gamma := (1 + 2\eta)\,\gamma_{lb}$;
        Get $\lambda_i(H(\gamma, K_1^{p_i}))$ ▷ c.f. (14)
        **if** `Re`$(\Lambda) \neq \emptyset$ for $\Lambda = \{\lambda_1, \cdots \lambda_n\}$ **then**
            Set $\gamma_{ub} = \gamma$; exit
        **else**
            Set buffer $\Gamma_{lb} = \{\}$
            **for** $\lambda_k \in \{$`Imag`$(\Lambda)_{:p-1}\}$ **do** ▷ $k = 1$ to $K$
                Set $m_k = \frac{1}{2}(\omega_k + \omega_{k+1})$
                Set $\Gamma_{lb}(k) = \max\{\sigma\,[T_{zw}(jm_k)]\}$;
            **end for**
            $\gamma_{lb} = \max(\Gamma_{lb})$



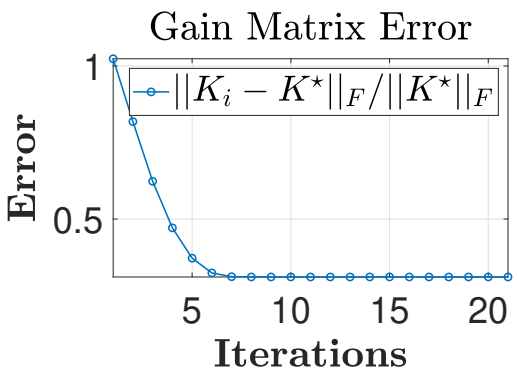
# Cost Matrix and Gains Convergence

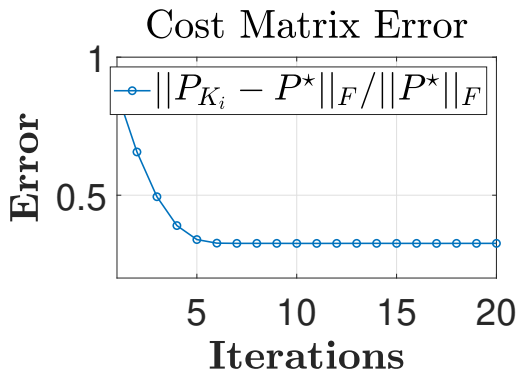


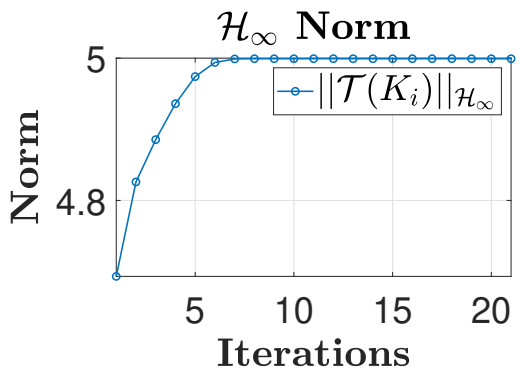


# Pendulums Experiment – Comparison to NPG





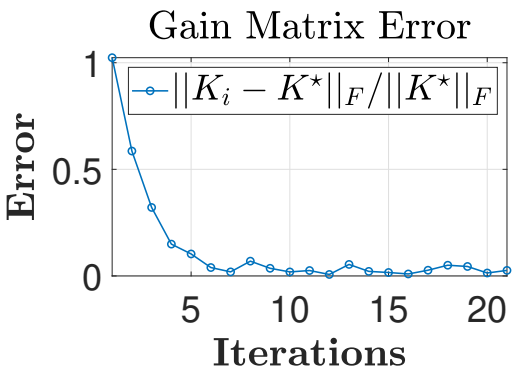



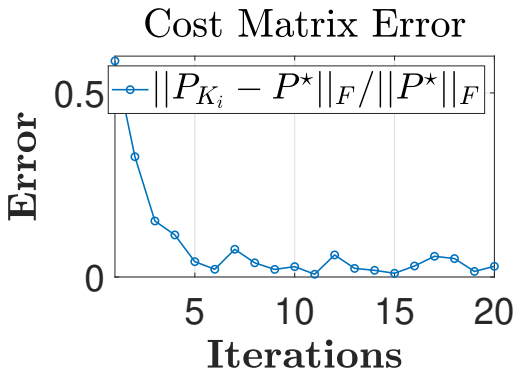



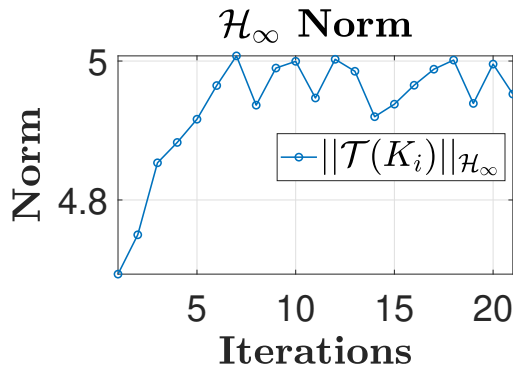

Model-free design: $\|\tilde{K}\|_\infty = 0.15$.

# Pendulums Experiment – Comparison to NPG






Model-based design: $\|\tilde{K}\|_\infty = 0.15$.

# Double Pendulum and Acrobot Experiment – Comparison to NPG



Table: Computational Time: Model-based PO vs. Model-free PO vs. NPG.

| Policy Optimization Computational time (secs) | | | | | |
|---|---|---|---|---|---|
| Double Inverted Pendulum | | | Triple Inverted Pendulum | | |
| Model-based | Model-free | NPG | Model-based | Model-free | NPG |
| 0.0901 | 0.3061 | 2.1649 | 0.1455 | 0.7829 | 2.3209 |

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# Iterative Dynamic Game in RL

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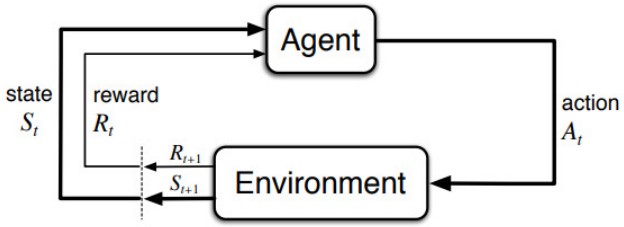
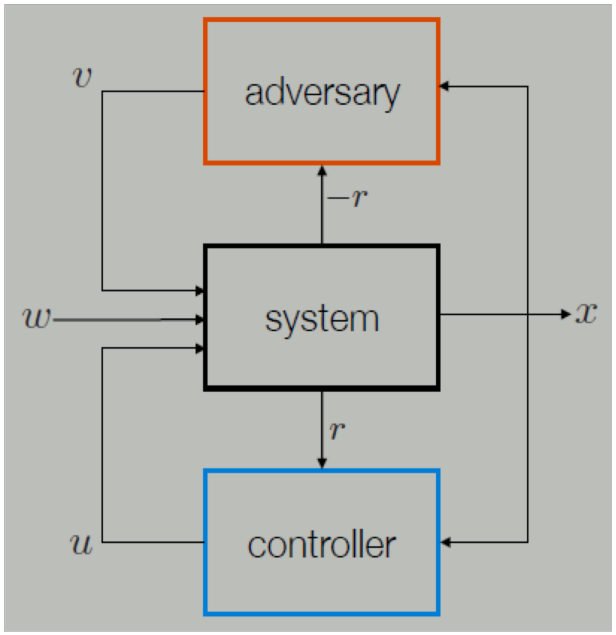
# Inculcating robustness into multistage decision policies



# Problem Setup

- To quantify the brittleness, we optimize the stage cost

$$\max_{\boldsymbol{v}_t \sim \psi \in \Psi} \left[ \sum_{t=0}^{T} \underbrace{c(\boldsymbol{x}_t, \boldsymbol{u}_t)}_{\text{nominal}} - \gamma \underbrace{g(\boldsymbol{v}_t)}_{\text{adversarial}} \right]$$

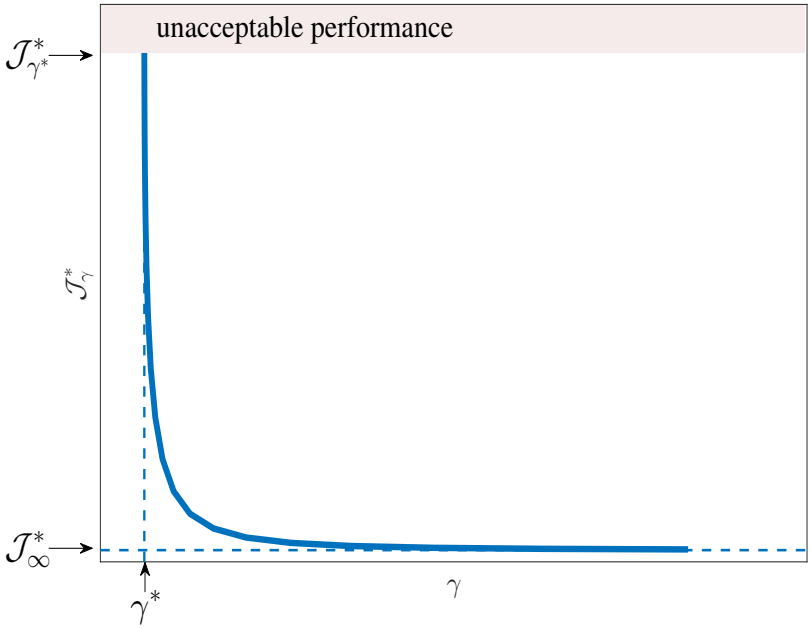
- To mitigate lack of robustness, we optimize the *cost-to-go*

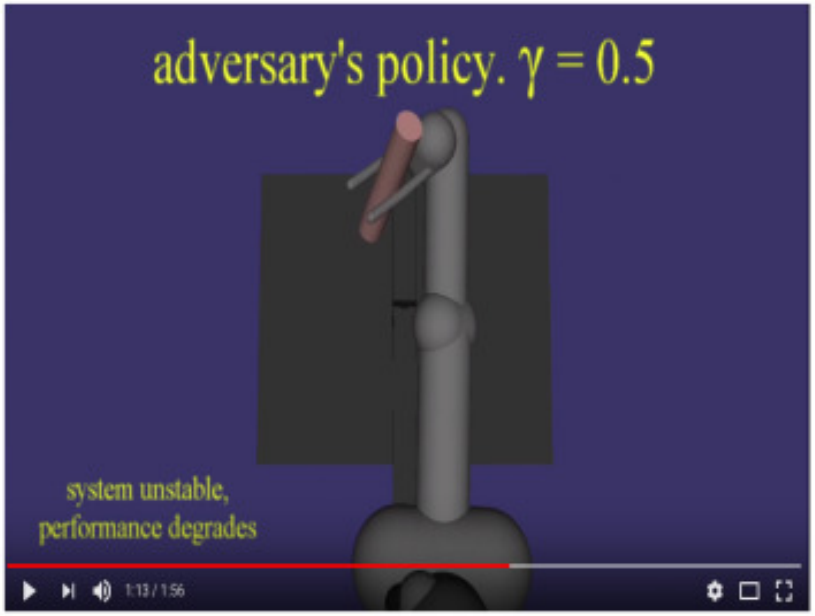
$$c_t(\boldsymbol{x}_t, \pi, \psi) = \min_{\boldsymbol{u}_t \sim \pi} \max_{\boldsymbol{v}_t \sim \psi} \left( \sum_{t=0}^{T-1} \ell_t(\boldsymbol{x}_t, \boldsymbol{u}_t, \boldsymbol{v}_t) + L_T(\boldsymbol{x}_T) \right),$$

- and seek a saddle point equilibrium policy that satisfies

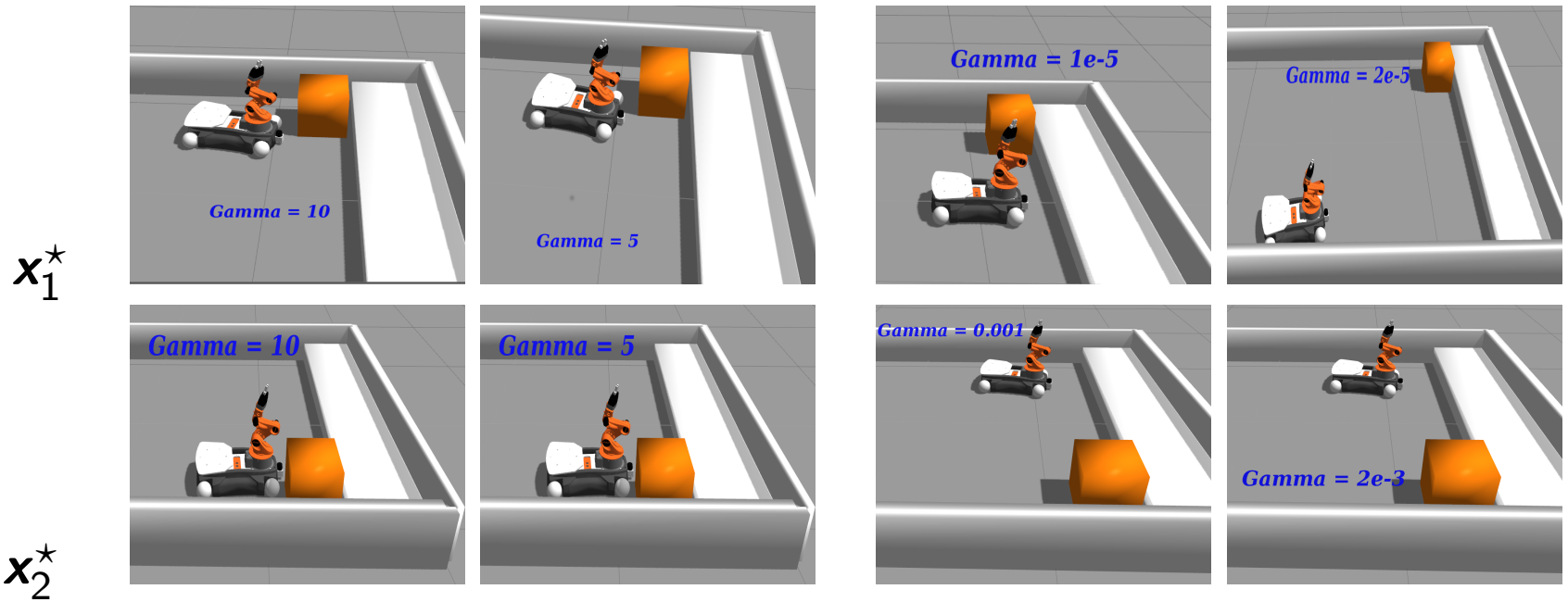
$$c_t(\boldsymbol{x}_t, \pi^*, \psi) \leq c_t(\boldsymbol{x}_t, \pi^*, \psi^*) \leq c_t(\boldsymbol{x}_t, \pi, \psi^*),$$

# Results: Brittleness Quantification



# Results: Iterative Dynamic Game




End pose of the KUKA platform with our iDG formulation given different goal states and $\gamma$-values.

# Innovation in the Age of Foundation Models

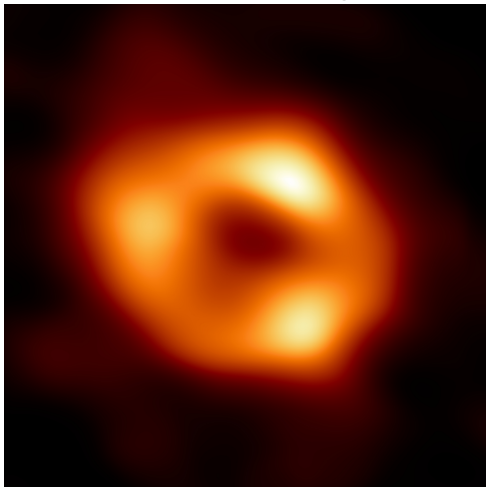
## Why am I Here?

If an idea begets a discovery, and if a discovery begets an invention, I am interested in riding the complete **innovation** circuit for intelligence:

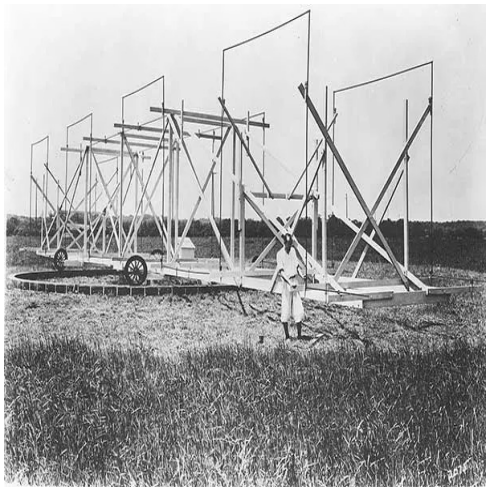
- The thorough and wholesale transformation of fundamental scientific ideas in RL and automation into technological products (or processes) capable of widespread practical use.

# Discovery for Physical Autonomy

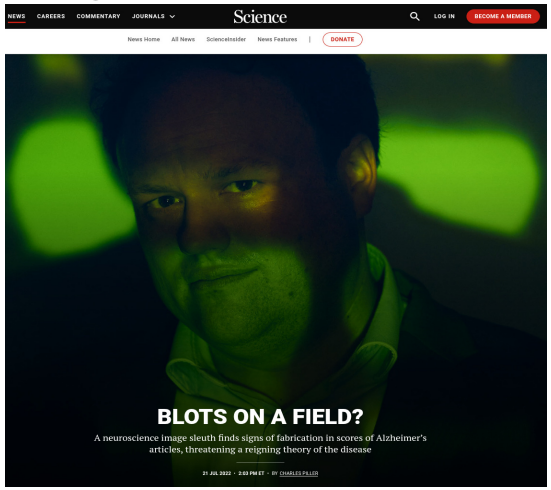
Discovery: The fundamental unit of human progress.



Sagittarius A*, EHT



Karl Jansky, Bell Labs

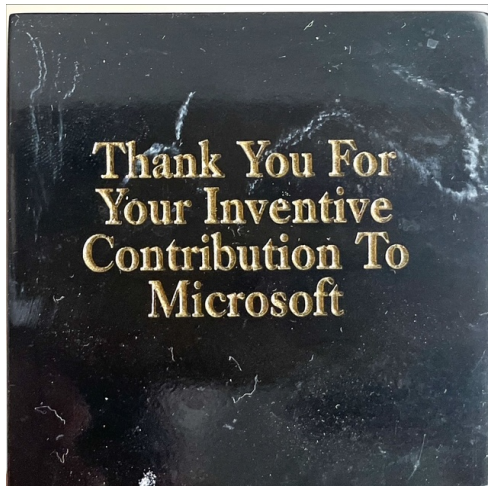



$A\beta^{\star}56$ "undiscovery"

- To wend straight and narrow path between discovery and invention.

# Discovery & Invention for Physical Autonomy

Discovery: The fundamental unit of human progress.


# Foundation Models, Large Behavior Models

- Large-scale transfer learning, behavior cloning, unsupervised pre-training etc. a new scientific invention.


Credit: $\pi_0$: A VLA Flow Model for General Robot Control.

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