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Comments on "Time-Varying Lyapunov Functions for Tracking Control of Mechanical Systems With and Without Frictions"

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ABSTRACT In the article[a], the authors introduced a time-varying Lyapunov function for the stability analysis of nonlinear mechanical systems whose motion is governed by standard Newton-Euler equations. Exponential stability in the sense of Lyapunov using integrator backstepping and Lyapunov redesign for the tracking control of mechanical systems is established in this note. This improves upon the asymptotic stability of tracking errors reported in the article[a].

[a]Ren, W., Zhang, B, Li, H, and Yan L. *IEEE Access.* vol. 8. pp. 51510-51517. 2020.

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I. MAIN

By now, it is well-known that the well-established method of Lyapunov analysis can be used to synthesize a stable controller for nonlinear robot manipulators, space flexures, parallel robots, and magnetic bearings whose kinetics are governed by the standard time-invariant Euler-Lagrange equations. When the underlying nonlinear system is time-varying, however, stability is difficult to establish with traditional time-invariant Lyapunov functions.

Considering frictionless mechanical systems, the authors of the article[a] proposed a time-varying Lyapunov function, provided sufficient conditions for their practical stability, and under a zero friction situation, the authors established an *asymptotic convergence rate of the filtered velocity and position tracking errors*.

In this note, we will *prove exponential error convergence when friction is absent from the dynamics*. In addition, only one feedback matrix (here the symmetric positive definite matrix P) is needed for the establishment of our results – as opposed to two symmetric positive definite matrices (P and K).

The non-negative function used to prove asymptotic stability in the paper[a] is given as

$$Q = \frac{1}{2} q_r^T M(q) q_r + \frac{1}{2} \tilde{q}^T P \tilde{q} \tag{1}$$

where all variables are as defined in the paper[a] i.e. q is the joint space variable, q_r is the filtered velocity tracking error, defined as

$$q_r(t) = \dot{\tilde{q}}(t) + \lambda \tilde{q}(t), \tag{2}$$

for a $\lambda > 0$; $\tilde{q}(t)$ is the position tracking error given as $\tilde{q}(t) = q(t) - q_d(t)$, for a desired joint position, q_d. The matrix $M(q)$ is the symmetric positive definite inertia matrix of the system.

The model of the system to be controlled is given by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + F(t,\dot{q}) = \tau \tag{3}$$

where $C(q,\dot{q})$ is the Coriolis and centrifugal forces that govern the system's motion, $G(q)$ models the gravity and non-conservative external forces that affect the system's motion, $F(t,\dot{q})$ is the time-varying friction and τ is the vector of actuator torques. It is well-known that the matrix $(\dot{M} - 2C)$ is skew-symmetric for a suitable choice of $C(q,\dot{q})$.

Proposition 1 (Asymptotic stability without frictions): For the nonlinear manipulator (3) with a zero time-varying friction i.e. $F(t,\dot{q}) = 0$, every motion of the mechanical system $q_d(t)$ is asymptotically stable when $F(t,\dot{q}) \equiv 0$, i.e. $q(t) \to q_d(t)$ as $t \to \infty$ provided we choose a virtual input, $\eta(t)$, which is backstepped as an integrator as follows

$$\begin{aligned} \eta(t) &= \dot{q}(t) && \text{(4a)} \\ &\triangleq \dot{q}_d(t) - \tilde{q}(t) - \frac{1}{2} M^{-1}(q) \dot{M}(q) \tilde{q}(t). && \text{(4b)} \end{aligned}$$

Proof 1: The tracking error dynamics for a desired joint space variable, q_d, is

$$\dot{\tilde{q}}(t) = \eta(t) - \dot{q}_d(t). \tag{5}$$

Now, consider the following Lyapunov function candidate

$$Q_1(t) = \frac{1}{2} \tilde{q}^T(t) M(q) \tilde{q}(t) \succ 0. \tag{6}$$

We find that

$$\begin{aligned} \dot{Q}_1(t) &= \tilde{q}^T(t) M(q) \dot{\tilde{q}}(t) + \frac{1}{2} \tilde{q}^T(t) \dot{M}(q) \tilde{q}(t) && (7) \\ &= \tilde{q}^T(t) M(q) (\eta(t) - \dot{q}_d(t)) && (8) \\ &= -\tilde{q}^T(t) M(q) \tilde{q}(t). && (9) \end{aligned}$$

Equation (9) implies that the system must be *asymptotically stable*.

Corollary 1 (Exponential stability of the system): Consider the entire nonlinear system (3), suppose that there exists a $P = P^T \succ 0$, then trajectories that emanate from the system are **exponentially stable** in the sense of Lyapunov if we choose the control law, τ, according to

$$\tau = M(q) \left(\ddot{q}_d + \lambda \dot{\tilde{q}} \right) + C(q, \dot{q}) \left(\dot{q} + q_r \right) + G(q) + \frac{1}{2} q_r. \tag{10}$$

Proof 2: Consider the Lyapunov function

$$Q(t) = Q_1(t) + \frac{1}{2} q_r^T(t) P M q_r(t) \tag{11}$$

so that the time derivative of $Q(t)$ (dropping the r.h.s templated arguments for notation terseness) is

$$\begin{aligned} \dot{Q}(t) &= \dot{Q}_1 + q_r^T P M \dot{q}_r + \frac{1}{2} q_r^T P \dot{M} q_r && \text{(12a)} \\ &= \dot{Q}_1(t) + q_r^T P M (\ddot{\tilde{q}} + \lambda \dot{\tilde{q}}) + \frac{1}{2} q_r^T P \dot{M} q_r && \text{(12b)} \\ &= \dot{Q}_1(t) + q_r^T P M (\ddot{q}_d - \ddot{q} + \lambda \dot{\tilde{q}}) + \frac{1}{2} q_r^T P \dot{M} q_r && \text{(12c)} \\ &= \dot{Q}_1(t) + q_r^T P M (\ddot{q}_d + \lambda \dot{\tilde{q}}) + q_r^T P \left[C(\dot{q} + \dot{q}_r) + G(q) - \tau \right]. && \text{(12d)} \end{aligned}$$

Where we have used the skew-symmetric property $\dot{M} - 2C$ in arriving at (12d). Let τ be as defined in (10), then we have,

$$\begin{aligned} \dot{Q}(t) &= -\tilde{q}(t) M(q) \tilde{q}(t) - \frac{1}{2} q_r^T(t) M(q) P q_r(t) && \text{(13a)} \\ \dot{Q}(t) &= -2Q(t). && \text{(13b)} \end{aligned}$$

A fortiori, the system is in fact exponentially stable.

II. CONCLUSION

Through a careful choice of virtual input design, the convergence rate of system tracking errors can be significantly improved upon. In addition, the number of parameters to be introduced into the torque control equation is reduced from two to one. This is particularly of importance when designing fast/real-time learning-based optimization for suitable controllers for systems that obey Newton-Euler equations of motion when friction is absent.

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