

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Policy Learning Synthesis

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Abstract: A *robustly stabilizing optimal control policy* in a model-free mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control setting is here put forward for counterbalancing the slow convergence and non-robustness of traditional high-variance *policy optimization* (and by extension *policy gradient*) algorithms. Leveraging Itô’s stochastic differential calculus, we iteratively solve the system’s *continuous-time (closed-loop) generalized algebraic Riccati equation* (GARE) whilst updating its admissible controllers in a two-player, zero-sum differential game setting. Our new results gather previously disseminated results in this field in one holistic data-driven presentation with greater simplification, improvement, and clarity. Our source code, data, and examples are available at <https://github.com/robotsorcerer/robust-design>.

Keywords: Robust Control; Optimal Control; Adaptive and Learning Systems; Mechatronic Systems; Linear Control Systems.

1. INTRODUCTION

We consider system stabilization together with Zames’ sensitivity compensation in linear plants disturbed by additive Wiener process and uncertainties (Zames, 1981) in completely model-free policy optimization and policy gradient settings. We pose our solution within the classical mixed $\mathcal{H}_2/\mathcal{H}_\infty$ linear quadratic (LQ) optimal control problem (OCP) (Khargonekar et al., 1988; Rotea and Khargonekar, 1991) within the family of stochastic policy optimization (PO) schemes. Connecting this mixed design synthesis to modern policy optimization algorithms in machine learning, we optimize a performance index that is the upper bound on the \mathcal{H}_2 -norm of the plant transfer function subject to \mathcal{H}_∞ -norm constraints: we must find feasible stabilizing policies whilst guaranteeing robustness to a measure of disturbance (Zhang et al., 2019; Cui and Molu, 2022).

Data-driven sequential decision-making and control have become essential ingredients for many engineering solutions in modern business value chains. This is all the more pronounced in robotics, reinforcement learning, games, and other engineering systems. In these settings, often in continuous time, the *separation principle* (Joseph and Tou, 1961) of feedback theory allows the optimal control and state estimation problems to be decoupled so that in an optimization setting, a controller or policy can be found for a setpoint regulation or trajectory-following goal. Indeed, most of the large-scale success stories of *learning-based* stochastic optimal control arise within PO settings. These include (i) offline reinforcement learning (RL) (Wang et al., 2020); (ii) online continuous control for RL (Lillicrap et al., 2015); (iii) deep visuomotor policies for robot manipulation (Levine et al., 2016); (iv) learning reaching motions from robot demonstrations (Khansari-Zadeh and Billard, 2014); and (v) policy search for robot

manipulation, navigation, and control (Deisenroth, 2011); among others.

PO algorithms, which encapsulate *policy gradient* (PG) methods (Kakade, 2001; Agarwal et al., 2021), are attractive for modern data-driven problems since they (i) admit continuously differentiable policy parameterization; (ii) are easily extensible to function approximation settings; and (iii) admit structured state and control spaces. As such, PG algorithms are increasingly becoming integral to modern engineering solutions, recommender systems, finance, and critical infrastructure given the growing complexity of the systems that we build and the massive availability of datasets. A major drawback of PG algorithms, however, is that they compute *high-variance* gradient estimates of the LQR costs from Monte-Carlo *trajectory rollouts* and *bootstrapping*. As such, they tend to possess slow convergence guarantees.

To address PG’s non-robustness to uncertainty, and its characteristic slow convergence, recent efforts have proposed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control proposal (Zhang et al., 2019; Zhang et al., 2019; Cui and Molu, 2022) as a risk-mitigation design tool: imposing an additional \mathcal{H}_∞ -norm constraint on the \mathcal{H}_2 cost to be minimized, one guarantees *robust stability and performance* in the presence of unforeseen uncertainties, noise, worst-case disturbance or incorrectly estimated dynamics – signatures of PG algorithms.

Under *stabilizable* and *observable* system parameter conditions, (Zhang et al., 2019) established *globally sub-linear* and *locally super-linear* convergence rates in linear quadratic (LQ) zero-sum dynamic game settings. We improved upon these convergence rates in (Cui and Molu, 2022) by solving the PO problem recursively *given an initial stabilizing feedback gain that also preserved the \mathcal{H}_∞ robustness metric*. In many modern engineering systems that employ PO, however, stochastic system parameters often have to be identified from nonlinear system trajec-

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tory data. For these schemes to work, the control designer must linearize nonlinear trajectories about successive equilibrium points (whilst imposing the standard stabilizability and observability constraints on system parameters to be identified). In this paper, we take steps to curb these assumptions in model-free settings.

Contributions: We here present a holistic synthesis of our previous dissemination, initiate a vector based search in a convex optimization setting for the initial stabilizing and \mathcal{H}_∞ -control constraints-preserving feedback gain, prove the existence of a solution to the convex search problem, and demonstrate the efficacy of our results on numerical problems. The rest of this paper is structured as follows: in section 2, we introduce notations and contextualize the problem; in section 3, we present our methods; and we present our results in 4.

2. PRELIMINARIES

We now introduce common notations used throughout the rest of this article.

2.1 Notations

We adopt standard vector-matrix notations throughout. Conventions: Capital and lower-case Roman letters are respectively matrices and vectors; calligraphic letters are sets. Exceptions: time variables e.g. t, t_0, t_f, T will always be real numbers.

The n -dimensional Euclidean space is \mathbb{R}^n . The singular values of A are represented $\sigma_i(A), i = 1, 2, \dots$. The standard \mathcal{H}_∞ norm of a system with transfer function matrix $G(j\omega)$ is defined over analytical real or matrix-valued functions in the right half plane and is given by $\|T_{zw}\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(j\omega))$ where $\sigma_{\max}(\cdot)$ denotes the maximum singular value. The \mathcal{L}_2 norm or induced matrix norm is $\|\cdot\|_2$; and $\{\lambda_i(X)\}_{i=1}^n$ denotes the n -eigenvalues of $X \in \mathbb{R}^{n \times n}$ with $\lambda_1 < \lambda_2 < \dots < \lambda_n$. When an optimized variable e.g. u is optimal with respect to an index of performance, it shall be denoted u^* .

All vectors are column-stacked. The Kronecker product of $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is $A \otimes B$. Symmetric n -dimensional matrices shall belong in \mathbb{S}^n . A positive definite (resp. negative definite) A is written $A \succ 0$ (resp. $A \prec 0$). We denote the index of a given matrix or vector by subscripts. Colon notation denotes the full range of a given index. Indexing ranges over $1, \dots, n$ for an n -dimensional vector. The i th row of a given matrix A is $A_{[i,:]}$, while the j th column of A is $A_{[:,j]}$. Blockwise indexing follows similar conventions.

Denote by x_{ij} the (ij) 'th entry of $X \in \mathbb{R}^{m \times n}$ and by x_i the i 'th element of $x \in \mathbb{R}^n$. The full vectorization of $X \in \mathbb{R}^{m \times n}$ is the $mn \times 1$ vector obtained by stacking the columns of X on top of one another i.e. $\text{vec}(X) = [x_{11}, x_{21}, \dots, x_{m1}, x_{12}, x_{22}, \dots, x_{mn}]^T$. Let $P \in \mathbb{S}^n$, then the half-vectorization of P is obtained by stacking the upper-triangular part of its columns on top of one another i.e. $\text{svec}(P) = [p_{11}, p_{12}, p_{22}, \dots, p_{1n}, p_{2n}, \dots, p_{nn}]^T$.

The vectorization of the dot product $\langle x, x^T \rangle$, where $x \in \mathbb{R}^n$, is $\text{vecv}(x) := [x_1^2, \dots, x_1 x_n, x_2 x_1, x_2^2, x_2 x_3, \dots, x_n^2]^T$.

The inverse of $\text{vec}(x)$ and $\text{svec}(y)$ are the full and symmetric matricizations: $\text{mat}_{m \times n}(x) = (\text{vec}(I_n)^T \otimes I_m)(I_n \otimes x)$, and $\text{smat}_m(y)$ respectively so that $\text{smat}(\text{svec}(P)) = P$. Here, $x \in \mathbb{R}^{mn}$ and $y \in \mathbb{R}^{m(m+1)/2}$ for $n, m \in \mathbb{R}_{\geq 0}$.

2.2 System Description

Consider the following nonlinear system

$$\dot{x}(t) = f(x; t) + g(x)u(t) + h(x)w(t), \quad x(0) = x_0 \quad (1a)$$

$$z(t) = \mathcal{G}(x, u; t), \quad z(0) = z_0 \quad (1b)$$

where $f(\cdot), g(\cdot), h(\cdot)$, and $\mathcal{G}(\cdot)$ are nonlinear functions with appropriate dimensions. The state process is $x \in \mathbb{R}^n$, the controlled output process is $z \in \mathbb{R}^m$, the control input is $u \in \mathcal{U} \subseteq \mathbb{R}^p$, and the vector-valued (stochastic) Wiener process is $w \in \mathcal{W} \subseteq \mathbb{R}^q$. Let the following finite-dimensional linear time-invariant (FDLTI) system describe the resulting stochastic differential equation

$$dx(t) = Ax(t)dt + B_1 u(t)dt + B_2 dw(t), \quad x(0) = x_0 \quad (2a)$$

$$z(t) = Cx(t) + Du(t), \quad z(0) = 0, \quad (2b)$$

where dw is the Gaussian white noise; $x(0)$ is an arbitrary zero-mean Gaussian random vector independent of $w(t)$; and A, B_1, B_2, C, D are real matrix-valued functions of appropriate dimensions. The random signal, $x(0)$, and process $w(t)$ are defined over a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is w 's sample space, \mathcal{F} is the σ -algebra i.e. the filtration generated by w , and \mathcal{P} is the probability measure on which $w(t)$ is drawn for a $t \in [0, T]$ (where $T > 0$ is fixed).

Assumption 1. We impose the following conditions in the algorithm to be presented. We take $C^T C \triangleq Q \succ 0$, $D^T(C, D) = (0, R)$ for some $R \succ 0$ on the interval $[0, T]$; and should one wish that the noise process in (2) be statistically independent, then we may take $B_2 B_2^T = 0$. Seeing we are seeking a linear feedback controller for (2), we require that the pair (A, B_1) be *stabilizable*. We expect to compute solutions via an optimization process, therefore we require that *unstable modes of A must be observable through Q* . Whence (\sqrt{Q}, A) must be *detectable*.

Problem 1. (Problem Statement). The goal is to keep the controlled process, z , small in an infinite-horizon LTI constrained optimization problem under a minimizing control u in spite of unforeseen disturbances w .

Let the closed-loop operator (under an arbitrary negative feedback gain $K \in \mathcal{K}, u(x(t)) = -Kx(t)$) mapping w to z be $\|T_{zw}(K)\|_2$. Then,

$$T_{zw}(K) = (C - DK)(sI - A + B_1 K)^{-1} B_2. \quad (3)$$

Or in (Zhou and Doyle, 1998)'s packed representation,

$$P := T_{zw}(K) \triangleq \left[\begin{array}{c|c} A - B_1 K & B_2 \\ \hline C - DK & 0 \end{array} \right]. \quad (4)$$

Design principles in linear control theory exist for solving problem (2) when the covariance of the noise model has a small magnitude. For stochastic \mathcal{H}_2 control problems with large noise intensities (such as PG methods), it suffices to solve a *linear exponential quadratic control problem* under a robustness constraint. To further contextualize the problem, let us introduce the control design problem.

2.3 Risk-Sensitive LEQG as a Mixed Design Problem

In (Zhang et al., 2020), the authors established that the risk-sensitive infinite-horizon linear exponential quadratic Gaussian (LEQG) state-feedback control problem (Jacobson, 1973; Whittle, 1981) is an equivalent mixed- $\mathcal{H}_2/\mathcal{H}_\infty$ control design problem for *linear time-invariant* systems with additive noise of the form (2). We iterate upon this contribution since it introduces a measure of risk-design as an implicit robustness metric when the process noise has a large covariance intensity. And this is typical for policy gradient settings. The state evolves according to (2a) and without loss of generality, the stochastic linear system's performance criterion is

$$\mathcal{J}(K) = \limsup_{t_f \rightarrow \infty} \frac{2\gamma^2}{t_f} \log \mathbb{E} \exp \left[\frac{1}{2\gamma^2} \int_{t=0}^{t_f-1} \langle z(t), z(t) \rangle dt \right], \quad (5)$$

Suppose that the variance term $\gamma^{-2} \text{var}(z^T z)$ is small, then γ is a measure of *risk-propensity* if $\gamma > 0$; similarly, γ is a measure of *risk-aversion* if $\gamma < 0$; and γ is a measure of *risk-neutrality* if $\gamma = 0$ (this also corresponds to the standard state-feedback LQP). Given LEQG's connection under risk-propensity to the high-variance associated with PG algorithms, throughout the rest of this paper we take $\gamma > 0$ to be a given constant in our optimization process.

2.4 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Policy Optimization Synthesis

We now define the standard mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control theory problem: given the system (2) and a real number $\gamma > 0$, find an FDLTI *admissible controller* K that exponentially¹ stabilizes (3) and renders $\|T_{zw}\|_\infty < \gamma$. The set of all *suboptimal* controllers that robustly stabilizes (2) against all (finite gain) stable perturbations Δ , interconnected to the system by $w_1 = \Delta z$, such that $\|\Delta\|_\infty \leq 1/\gamma$ can be succinctly denoted as

$$\mathcal{K} = \{K : \lambda_i(A - B_1 K) < 0, \|T_{zw}\|_\infty < \gamma\} \quad (6)$$

for $i = 1, \dots, n$. We say $\mathcal{K} \neq \emptyset$ if and only if the pair (A, B_1) is stabilizable and (C, A) is detectable cf. (2).

Aside from the constraint (6), the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure can be framed as minimizing an “upper-bound” on the \mathcal{H}_2 -norm of the cost subject to the constraint $\|T_{zw}\|_\infty < \gamma$ (Bernstein and Haddad, 1989) for a $\gamma > 0$. Abusing notation, let $\mathcal{J}(T_{zw})$ denote the (closed-loop) mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control performance measure for the LTI system (2).

Lemma 1. For the control problem (2) with (a slightly abused) quadratic performance measure (5) i.e.

$$\mathcal{J}(T_{zw}) = \mathbb{E} \left\{ \limsup_{t_f \rightarrow \infty} \left[\exp \left(2\gamma^{-2} \int_{t=0}^{t_f-1} \langle z(t), z(t) \rangle dt \right) \right] \right\} \quad (7)$$

so that $x(t)$ is the unique solution of (2) after optimizing $\min_{K \in \mathcal{K}} \mathcal{J}(\cdot)$ with the *unique, optimal controller*,

$$u^*(x(t)) = -R^{-1}B_1^T P(t)x(t), \quad t \in [0, t_f]. \quad (8)$$

¹ Concerning matters relating to linear systems, we take exponential stability to mean internal stability, so that the transfer matrix belongs in the real-rational \mathcal{H}_∞ space i.e. $T_{zw} \in \mathcal{RH}_\infty$.

In (8), $P(t)$ is the unique, symmetric positive solution to the continuous-time (closed-loop) generalized algebraic Riccati equation (GARE)

$$PA + A^T P - P(B_1 R^{-1} B_1^T - \gamma^{-2} B_2 B_2^T)P + Q = 0 \quad (9)$$

if $Q \succ 0$ for a $\gamma > 0$.

Proof 1. This Lemma is the infinite-horizon retrofitting of Duncan's solution to the LEQG control value function based on a standard completion of squares and a Radon-Nikodym derivative (Duncan, 2013, Th II.1).

Corollary 1. (Th 9.7 Başar (2008)). The closed-loop GARE (9) in an infinite-horizon LTI setting admits an equivalent *LQ two-player zero-sum differential game* with the following *upper value*

$$\mathcal{J}(T_{zw}) = \limsup_{t_f \rightarrow \infty} \inf_{u \in \mathcal{U}} \sup_{w \in \mathcal{W}} \int_{t=0}^{t_f-1} [x^T(t)Qx(t) + u(\cdot)^T R u(\cdot) - \gamma^{-2} w^T(t)w(t)] dt, \quad \forall x \in \mathbb{R}^n \quad (10)$$

subject to assumption 1 (Başar, 2008, §. 9.7). Note that $\gamma > 0$ can be interpreted as an upper bound on the L_2 gain disturbance attenuation or the \mathcal{H}_∞ -norm of the system. In addition, let a finite scalar $\gamma^\infty > 0$ exist, then for all $\Gamma \triangleq \inf\{\gamma > \gamma^\infty\}$, (9) has a unique, finite, and positive definite solution if (C, A) is observable.

Corollary 2. (Th 4.8, (Başar, 2008)). If $\Gamma \neq \emptyset$, and if the LQ zero-sum differential game has a closed-loop perfect-state information structure defined on $[0, t_f]$, $t_f \rightarrow \infty$, then (10) *admits a unique solution* with feedback controls

$$u^*(t) = -R^{-1}B_1^T P_\gamma x(t), \quad w^*(t) = -\gamma^{-2}B_2^T P_\gamma x(t) \quad (11)$$

for a $t \geq 0$, $\gamma > \gamma^\infty$, P_γ is the unique solution to (9) in the class of positive definite feedback matrices² which makes the following feedback matrix Hurwitz,

$$A_\gamma = A - (B_1 R^{-1} B_1^T - \gamma^{-2} B_2 B_2^T) P_\gamma. \quad (12)$$

Remark 1. Clearly, (10) is a minimax problem whose controller admits the form

$$\min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} \mathcal{J}(T_{zw}) = \limsup_{t_f \rightarrow \infty} \int_{t=0}^{t_f-1} [x^T(t)Qx(t) + u^T(\cdot) R u(\cdot) - \gamma^{-2} w^T(t)w(t)] dt, \quad \forall x \in \mathbb{R}^n. \quad (13)$$

Another common form of $\mathcal{J}(T_{zw})$ easily amenable to policy gradient algorithms is $\mathcal{J}(T_{zw}) = \text{Tr}(P_\gamma B_2 B_2^T)$ (Mustafa, 1989).

Remark 2. The cost (13) is nonconvex and not coercive (Zhang et al., 2019). However, our iterative solver (Cui and Molu, 2022) guarantees uniform convergence of the iterates during optimization.

Remark 3. The objective (13) is differentiable for any $K \in \mathcal{K}$, and its policy gradient $\nabla \mathcal{J}(T_{zw}) := 2(RK - B^T P_\gamma) \Lambda_\gamma$ where Λ_γ admits a form amenable to a (continuous-time) closed-loop Lyapunov equation (Zhang et al., 2019, Lemma A.4)

$$\Lambda_\gamma(A - B_1 K + \gamma^{-2} B_2 B_2^T P_\gamma)^T + (A - B_1 K + \gamma^{-2} B_2 B_2^T P_\gamma) \Lambda_\gamma + B_2 B_2^T = 0. \quad (14)$$

For $\gamma > 0$ and $\gamma \neq \sigma_i(B_2)$, $i = 1, 2$, and following the packed form (4), we define the following closed-loop Hamiltonian matrix for γ at iteration p as

² We have used the subscript γ on P to denote its direct dependence on γ .

$$H_p(\gamma, K_p) = \begin{bmatrix} A - B_1 K_p & -\gamma^{-1} B_2 B_2^T \\ -\gamma^{-1} (C^T C + K_p^T R K_p) & -(A - B_1 K_p)^T \end{bmatrix} \quad (15)$$

where we have used $R = (D^T D - \gamma^2 I)$ and $S = (D D^T - \gamma^2 I)$ as in (Bruinsma and Steinbuch, 1990, Eq. 2.2).

3. METHODS

We now introduce the the identification procedure, linearization and the iterative solver for the (generalized) algebraic Riccati equation (9) which is needed for computing the minimizing (cf. (8)) and maximizing control (cf. (11)) for the value function (10). We refer interested readers to the convergence and robustness analyses of the solver introduced to (Cui and Molu, 2022).

3.1 Nonlinear Identification and Linearization

The condition stipulated in assumption 1 must be realized before we can implement a learning-based procedure. The parsimonious **Nonlinear Auto-Regressive Moving Average with eXogeneous input** (NARMAX) (Chen et al., 1989) has powerful yet simple parsimonious representation capability on real systems (Chen et al., 1989). We first identified a suitable NARMAX structure and model parameters, compute equilibrium points – about which we linearized to a form of cf. (2), before we estimate the *robustly stabilizing and optimal control policy* for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control problem.

Suppose that an input-output data from a real system (1) has been collected. Denote this as $D^N = \{z_1, \dots, z_m, u_1, \dots, u_p, w_1, \dots, w_q\}$. Let the maximum lags in the input, disturbance, and output data be denoted by n_u, n_w , and n_y respectively. We fit a polynomial NARMAX model to D^N with the power-form ℓ -degree polynomial,

$$\begin{aligned} z(t) = & \theta_0 + \sum_{i_1=1}^n \theta_{i_1} x_{i_1}(t) + \sum_{i_1=1}^n \sum_{i_2=i_1}^n \theta_{i_1} \theta_{i_2} x_{i_1}(t) u_{i_2}(t) \dots \\ & + \sum_{i_1=1}^n \dots \sum_{i_\ell=i_{\ell-1}}^n \theta_{i_1} \dots \theta_{i_\ell} x_{i_1}(t) \dots x_{i_\ell}(t) + e(t) \end{aligned} \quad (16)$$

whose parameters $\theta_{i_1 \dots i_m}, m \in [1, \ell]$ are to be identified. The model structure has order $n = n_z + n_u + n_w + n_e$, where n_e is the maximum order for a *pseudo-random binary sequence* $e(k)$ that aids identification robustness. The state variables are explicitly,

$$x_m(k) = \begin{cases} z(k-m), & 1 \leq m \leq n_z \\ u(k-(m-n_z)), & n_z+1 \leq m \leq n_z+n_u \\ w(k-(m-n_z-n_u)), & n_z+n_u+1 \leq m \leq n_z+n_u+n_w \\ e(k-(m-n-n_e)), & n-n_e+1 \leq m \leq n. \end{cases} \quad (17)$$

Equation 16 admits a linear regression model of the form $z(t) = \sum_{i=1}^M \phi_i(t) \theta_i + e(t)$ for $t = 1, \dots, N$ and a process noise $e(t)$. Or in matrix form: $Z = \Phi \Theta + \Xi$, where $\Phi = [\phi_1 \dots \phi_M]$ denotes the regression matrix, and $\Theta = [\theta_1, \dots, \theta_M]$ are parameters to be learned, typically in a regression process. The solution to the least squares cost $\min_{\Theta} \|z - \Phi \Theta\|_2$ yields the parameter estimates Θ for the nonlinear model. We adopt the computationally efficient Householder transformation in transforming the

information matrix, $\Phi^T \Phi$ into a well-conditioned QZ-matrix partition. Afterwards, we recover the NARMAX parameters by solving the resulting triangular system of linear equations in a least squares sense.

Given that the nonlinear structure is unknown ahead of time, we start with large values of n_z, n_u , and n_w in P – adding regression variables that capture natural properties such as damping and friction e.t.c. in order to capture as many nonlinear variation that exist in the data as possible. We then iteratively pruned the parameters using the *error reduction ratio* algorithm (Billings, 2010) within the forward orthogonal regression least square algorithm (Chen et al., 1989).

The identified NARMAX model is then linearized about a suitable equilibrium point to obtain a form of (2) in state space form. In our experience, we have found assumption 1 to be satisfied after linearization. Suppose that the pair (A, B) is still not controllable after linearization (we have not seen this in practice), a perturbation can be made of the multi-input LTI system as follows: Suppose that there exists a diagonalizable matrix T such that $\bar{A} = T^T A T$, and $\bar{B} = T^T B$. Then (A, B) can be reduced to (\bar{A}, \bar{B}) as follows:

$$\bar{A} = \begin{bmatrix} A_{cnt} & * \\ 0 & \bar{A}_{cnt} \end{bmatrix}, \quad \bar{B} = [B_{cnt} \dots 0]^T \quad (18)$$

for³

$$A_{cnt} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1,p-1} & A_{1p} \\ A_{21} & A_{22} & \dots & A_{2,p-1} & A_{2p} \\ 0 & A_{32} & \dots & A_{3,p-1} & A_{3p} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & & A_{pp} \end{bmatrix}, \quad B_{cnt} = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (19)$$

where p is a pair's controllability index, blocks $B_1, A_{21}, \dots, A_{p,p-1}$ possess full row ranks, and $\dim(\bar{A}_{cnt}) = \dim[(A, B)_{cnt}]$.

3.2 LEQG/LQ Differential Game

In (Cui and Molu, 2022, Algorithm II), we introduced an iterative solver for the closed-loop controls u and w respectively. A key drawback is the need for the first $K_1 \in \mathcal{K}$ to be known. This limits the practicality of the algorithm to data-driven PO schemes. In addition, our examples did not illustrate a means of respecting the *stabilizability* and *detectability* assumptions needed to guarantee a solution to the minimax problem (13). We now provide an all-encompassing learning scheme for obtaining the solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control problem in a purely data-driven setting compatible with modern model-free policy optimization schemes.

Let $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ denote the iteration indices at which the controllers $u_p(t) = K_p x(t)$ and $w_q(t) = L_p^q x(t)$ are updated in (11), where K_p and L_p^q are feedback gains

$$K_p = R^{-1} B_1^T P_p^q, \quad L_p^q = -\gamma^{-2} B_2^T P_p^q. \quad (20)$$

Furthermore, let the closed-loop transition matrix under the gains of (20), and quadratic matrix term in (13) be (see (Cui and Molu, 2022, Equation 12))

$$A_\gamma = A - B_1 K_p + B_2 L_p^q, \quad Q_\gamma = C^T C + K_p^T R K_p. \quad (21)$$

³ \bar{X}_{cnt} signifies the non-controllable part of X .

Observe: P_p^q is finite if and only if the closed-loop system matrix A_γ possesses eigenvalues with negative real parts. In this case, P_p^q , for $p, q = 0, 1, 2, \dots$ is the unique positive definite solution to the closed-loop GARE

$$A_\gamma^T P_p^q + P_p^q A_\gamma + Q_\gamma - \gamma^{-2} L_p^{qT} L_p^q = 0 \quad (22)$$

where recursively, (20) holds for $p, q = 1, 2, \dots$. Note that K_1 must be chosen such that $A_\gamma^1 = A - B_1 K_1 + B_2^T L_1^1$ is Hurwitz and its closed-loop \mathcal{H}_∞ norm transfer function is bounded from above by a user-defined $\gamma > 0$ Kleinman (1968). Then (i) $K_1 \leq P_{p+1}^q \leq P_p^q \leq \dots$, $p, q = 1, 2$, (ii) $\lim_{(p,q) \rightarrow \infty} P_p^q = P$. See proof in (Cui and Molu, 2022) (Kleinman, 1968; Cui and Molu, 2022).

3.3 Iterative Dynamic Game

We now give the full form of the dynamic game in vectorized form. Equation (9) can be written as

$$\begin{aligned} & \text{svec}(A_\gamma^T P_p^q + P_p^q A_\gamma) - \text{mat}_{n \times m}(K_p^T) \text{vec}(B_1^T P_p^q) \\ & - \text{vec}(P_p^q B_1) \text{mat}_{m \times n}(K_p) + \text{svec}(Q_p - \gamma^{-2} L_p^{qT} L_p^q) \\ & + [\text{mat}_{n \times m}(L_p^{qT} B_2^T) + \text{mat}_{m \times n}(B_2 L_p^q)] \text{svec}(P_p^q) = 0. \end{aligned} \quad (23)$$

Or

$$\begin{aligned} & \text{svec}(A_\gamma^T P_p^q + P_p^q A_\gamma) + \text{svec}(Q_p - \gamma^{-2} L_p^{qT} L_p^q) \\ & - [\text{mat}_{n \times m}(K_p^T) + \text{mat}_{m \times n}^T(K_p)] \text{vec}(B_1^T P_p^q) \\ & + [\text{mat}_{n \times m}(L_p^{qT} B_2^T) + \text{mat}_{m \times n}(B_2 L_p^q)] \text{svec}(P_p^q) = 0. \end{aligned} \quad (24)$$

Since the differential game (13) admits equal upper and lower *optimal* values owing to the GARE (10) having a positive definite solution i.e. $\mathcal{J}^* = x^T P x$ (Başar, 2008, Th 4.8 (iii)), control laws must be computed along the trajectories of (2), using the derivative of $\mathcal{J}^* = x^T P x$. By Itô's differential rule, $d(\mathcal{J}^*(\cdot))$ admits the solution

$$\begin{aligned} d(x^T P x) &= x^T (A_\gamma^T P + P A_\gamma) x dt + 2x^T P B_1 u dt \\ &+ 2x^T P B_2 dw + \text{Tr}(B_2^T P B_2) dt \end{aligned} \quad (25)$$

where $\text{Tr}(M)$ denotes the trace of matrix M . Letting $\phi(t) = [\text{vec}^T(x), 2(x^T \otimes u^T), 1]^T$, $\Phi = \mathbb{E}(\phi \phi^T)$, and $\Psi = \lim_{t_f \rightarrow \infty} \int_0^{t_f} \phi d(\text{vec}^T(x))$, it can be ascertained⁴ from (25) that

$$\begin{bmatrix} \text{svec}(A_\gamma^T P + P A_\gamma) \\ \text{vec}(B_1^T P) \\ \text{Tr}(B_2^T P B_2) \end{bmatrix} = \Phi^{-1} \Psi \text{svec}(P). \quad (26)$$

Let $n_1 := n(n+1)/2$ and $n_2 := n_1 + mn$. Then,

$$\text{svec}(A_\gamma^T P + P A_\gamma) = [\Phi^{-1}]_{[1:n_1]} \Psi \text{svec}(P) \quad (27a)$$

$$\text{vec}(B_1^T P) = [\Phi^{-1}]_{[1+n_1:n_2]} \Psi \text{svec}(P). \quad (27b)$$

Whence, (24) in light of (27) becomes

$$\begin{aligned} & [\Phi^{-1}]_{[1:n_1]} \Psi \text{svec}(P_p^q) + \text{svec}(Q_p - \gamma^{-2} L_p^{qT} L_p^q) \\ & - [\text{mat}_{n \times m}(K_p^T) + \text{mat}_{m \times n}^T(K_p)] [\Phi^{-1}]_{[1+n_1:n_2]} \Psi \text{svec}(P_p^q) \\ & + [\text{mat}_{n \times m}(L_p^{qT} B_2^T) + \text{mat}_{m \times n}(B_2 L_p^q)] \text{svec}(P_p^q) = 0. \end{aligned} \quad (28)$$

Define $\bar{\Phi}_1 = [\Phi^{-1}]_{[1:n_1]}$, $\bar{\Phi}_2 = [\Phi^{-1}]_{[1+n_1:n_2]}$, so that

$$\begin{aligned} \Upsilon_p^q &= \bar{\Phi}_1 \Psi - [\text{mat}_{n \times m}(K_p^T) + \text{mat}_{m \times n}^T(K_p)] \bar{\Phi}_2 \Psi \\ &+ [\text{mat}_{n \times m}(L_p^{qT} B_2^T) + \text{mat}_{m \times n}(B_2 L_p^q)]. \end{aligned} \quad (29)$$

⁴ For the reasoning behind these equations, we refer readers to (Cui and Molu, 2022, §IV)

Algorithm 1 Search for the closed-loop \mathcal{H}_∞ -norm

```

1: Given a user-defined step size  $\eta > 0$ 
2: Set the initial upper bound on  $\gamma$  as  $\gamma_{ub} = \infty$ .
3: Initialize a buffer for possible  $\mathcal{H}_\infty$  norms for each  $K_1$ 
   to be found,  $\Gamma_{buf} = \{\}$ .
4: Initialize ordered poles  $\mathcal{P} = \{p_i \in \text{Re}(s) < 0 \mid i =$ 
    $1, 2, \dots\} \triangleright p_1 < p_2 < \dots$ 
5: for  $p_i \in \mathcal{P}$  do
6:   Place  $p_i$  on (2);  $\triangleright$  (Tits and Yang, 1996)
7:   Compute stabilizing  $K_1^{p_i}$ 
8:   Find lower bound  $\gamma_{lb}$  for  $H(\gamma, K_1^{p_i})$ ;  $\triangleright$  using (31)
9:    $\Gamma_{buf}(i) = \text{get\_hinf\_norm}(T_{zw}, \gamma_{lb}, K_1^{p_i})$ .
10: end for
11: function  $\text{get\_hinf\_norm}(T_{zw}, \gamma_{lb}, K_1^{p_i})$ 
12:   while  $\gamma_{ub} = \infty$  do
13:      $\gamma := (1 + 2\eta) \gamma_{lb}$ ;
14:     Get eigenvalues,  $\lambda_i(H(\gamma, K_1^{p_i}))$   $\triangleright$  cf. (15)
15:     if all  $\Lambda := \{\lambda_1, \dots, \lambda_n\} \in \mathbb{R}_-$  then
16:       Set  $\gamma_{ub} = \gamma$ ; exit
17:     else
18:       Set buffer  $\Gamma_{lb} = \{\}$ 
19:       for  $\lambda_k \in \{\text{Im}(\Lambda)_{:p-1}\}$  do  $\triangleright k = 1$  to  $K$ 
20:         Set  $m_k = \frac{1}{2}(\omega_k + \omega_{k+1})$ 
21:         Set  $\Gamma_{lb}(k) = \max\{\sigma[T_{zw}(jm_k)]\}$ ;
22:       end for
23:        $\gamma_{lb} = \max(\Gamma_{lb})$ 
24:     end if
25:     Set  $\gamma_{ub} = \frac{1}{2}(\gamma_{lb} + \gamma_{ub})$ .
26:   end while
27:   return  $\gamma_{ub}$ 
28: end function

```

Then, it follows that (28) admits the following form

$$\text{svec}(P_p^q) = -(\Upsilon_p^q)^{-1} \text{svec}(Q_p - \gamma^{-2} L_p^{qT} L_p^q). \quad (30)$$

3.4 Mixed Sensitivity Initialization

Recall that in order for the algorithm to work in a purely data-driven setting, K_1 must be in \mathcal{K} . The means for finding a K_1 that satisfies the constraints equation (6) is itemized in algorithm 1. Following Corollary 2, we must first find the upper bound of γ i.e. $\gamma^\infty := \gamma_{ub}$ whereupon the unique, finite, and positive definite solution to the GARE is satisfied. Consider the following proposition.

Proposition 1. (Bruinsma and Steinbuch (1990)). For all $\omega_p \in \mathbb{R}$, we have that $j\omega_p$ is an eigenvalue of the Hamiltonian $H(\gamma_1)$ if and only if γ_1 is a singular value of $T_{zw}(j\omega_p)$.

Remark 4. Singular value computation is easily obtainable given a frequency of a system. Proposition 1 allows us to obtain all frequencies that correspond to a singular eigenvalue.

Procedure: We search for stabilizing gains $K_1^{p_i}$ over the space of negative reals (or range matrix inequalities for multiple input, multiple output systems) such that each $K_1^{p_i}$ on line 7 of Algorithm 1 is stabilizing. A starting lower bound for each γ_i corresponding to gain $K_1^{p_i}$ can be set

$$\gamma_{lb} := \max\{\sigma_{\max}(G(0)), \sigma_{\max}(G(j\omega_p)), \sigma_{\max}(B_2)\} \quad (31)$$

where ω_p is chosen as specified in (Bruinsma and Steinbuch, 1990, Eq. (4.4)). The \mathcal{H}_∞ norm computation scheme is based on the singular values of (3) and the eigenvalues

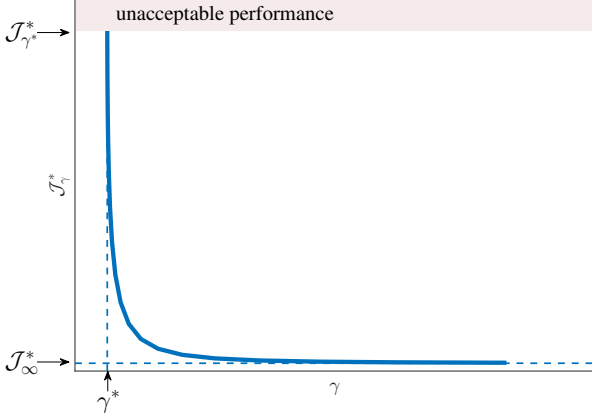


Fig. 1. Performance Robustness Trade-off Curve. Amongst all computed γ , one can start the optimization algorithm with the

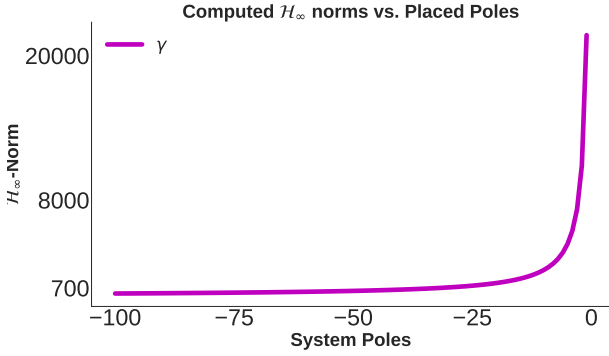


Fig. 2. Computed \mathcal{H}_∞ -norm for searched poles. We see that the closer to the origin, the greater the value of $\|T_{zw}\|_{\mathcal{H}_\infty}$

of (15) – adopted from (Bruinsma and Steinbuch, 1990) and woven into our search scheme. For poles far away from the origin, γ will be small. However, as $\lambda \rightarrow 0$, $\gamma \rightarrow \infty$. The \mathcal{H}_∞ computation scheme is fast and has a guaranteed quadratic convergence as long as the poles or $K_1^{p_i}$'s are closed-loop stabilizing since $\gamma_{lb}(i+1) > \gamma_{lb}(i)$. Given a user-defined step size, η , the rest of the algorithm consists in iteratively increasing the value of γ_{lb} until all eigenvalues of the closed-loop system Hamiltonian cf. (15) have no imaginary part. For all computed \mathcal{H}_∞ -norms, a critical value of γ^* can be obtained (see 1) below which the cost becomes infinite. One can then use this heuristic to ensure that a K_1 that fulfills (6) is chosen.

The entire procedure is described in Algorithm 2.

4. RESULTS

We now present numerical results of the algorithm described in the foregoing.

4.1 Car Cruise Control Environment

We consider a car cruise control system (Åström and Murray, 2021, §3.1) whereupon a controller $u(x(t)) = [u_1(t), u_2(t), u_3(t)]$ must maintain a constant velocity, v , whilst automatically adjusting the car's throttle, u_1 , at each time step $t \in [0, T]$ despite disturbances characterized by

Algorithm 2 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Control Synthesis

- 1: Pick a terminal horizon for the control-disturb iteration pair (\bar{i}, \bar{j}) .
- 2: Given an I/O data, identify a nonlinear model by expanding (16).
- 3: Linearize the dynamics about appropriate equilibrium points: x_{eq}, u_{eq} . \triangleright Obtain $Z^L = \{A, B_1, B_2, C, D\}$.
- 4: Using Z^L , optimize for a $\hat{K}_1 \in \mathcal{K}$.
- 5: Construct matrices $\hat{\Phi}(t_f)$ and $\hat{\Psi}(t_f)$ from Z^L ;
- 6: **while** $i \leq \bar{i}$ **do**
- 7: **while** $j \leq \bar{j}$ **do**
- 8: Form $\hat{\Upsilon}_p^q(t_f)$ using (29);
- 9: Update $\hat{P}_{K_p}^q$ and $\hat{L}_{K_p}^q$ cf. (20);
- 10: Form $\widehat{vec^T(B^T P_{K_i}^j)}$ as $[\hat{\Phi}^{-1}(t_f)]_{n_1+1:n_2} \hat{\Xi}(t_f) \widehat{vecs(\hat{P}_{K_i}^j)}$;
- 11: Calculate $\hat{K}_{i+1} = R^{-1} B^T P_{K_i}^j$;
- 12: **end while**
- 13: **end while**

road slope changes ($u_3 = \theta$), rolling friction (F_r), and aerodynamic drag forces (F_d). In addition, an external driver interface can disengage the cruise controller, set acceleration or deceleration, or engage the cruise controller.

This control design problem is well-suited to our robust control formulation because (i) the disturbances and state variables are separable and can be lumped into the form of the stochastic differential equations (1) and (2); (ii) it is a multiple-input (throttle, gear, vehicle speed) single-output (vehicle acceleration) system that introduces modeling challenges; (iii) the entire operating range of the system is nonlinear though there is a reasonable linear bandwidth that characterize the input/output (I/O) system as we will see shortly. The model is

$$m \frac{dv}{dt} = \alpha_n u \tau (\alpha_n v) - mg C_r \text{sgn}(u) - \frac{1}{2} \rho C_d A |v| v - mg \sin \theta \quad (32)$$

where v is the velocity profile of the vehicle (taken as the system's state), m is vehicle's mass, α_n is the inverse of the vehicle's effective wheel radius, τ is the vehicle's torque – it is controlled by the throttle $u := u_1$. The rolling friction coefficient is C_r and C_d is the aerodynamic drag constant for a vehicle of area A . The road curvature, θ , is modeled as a Wiener process cf. (2) with $w_i(t) \sim \mathcal{N}(0, 1)$ where for $i \in [1, \dots, q]$, $dw_i = \sum_{j=1}^i w_j$. If we let $x := v$, $u_1 := u$, $u_2 := \alpha_n$, and $u_3 := \theta$. where we set the constants following (Åström and Murray, 2021) as $C_r = 0.01$, $C_d = 0.32$, $\rho = 1.3 \text{ kg/m}^3$, $A = 2.4 \text{ m}^2$. The torque τ is given as $\tau = \tau_m - \tau_m \beta (\omega/\omega_m - 1)^2$ and $\beta = 0.4$, $\omega_m = 420$ and $\tau_m = 190$, so that

$$\tau = 190 - 76 \left(\frac{39x}{420} - 1 \right)^2.$$

4.2 Nonlinear Systems Identification and Linearization

We employ the polynomial NARMAX identification described in the previous section in first identifying the nonlinear system from I/O data using (32).

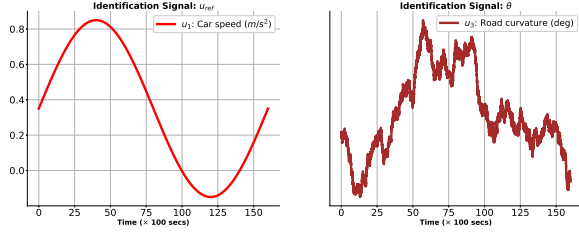


Fig. 3. Identification I/O Signals.

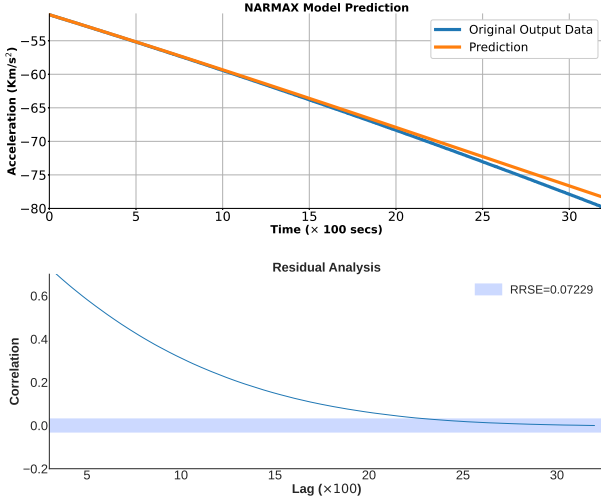


Fig. 4. Model prediction and correlation residues.

In our structure selection and model estimation scheme, we first start with a large number of parameters and regressors that consists of the polynomial expansion in (16), `sinuoisal`, and `signum` functions following (32). We choose a polynomial degree of 3 and the inputs u and state lags x were chosen as $[1, 1, 1]$ and $[1]$ respectively. We employed the forward regression orthogonal least squares algorithm (Chen et al., 1989) in estimating the parametric terms of the model. We then employed the **error reduction ratio** Chen et al. (1989); Billings (2010) algorithm in pruning extraneous terms obtained from the identification step. This whittled down the eventual model to the following parsimonious representation

$$\begin{aligned} \dot{x}(t) = & 0.062518u_2(t)x(t) - 0.12051u_1(t)u_2^2(t) \\ & + 0.00081339u_2^3(t) + 0.9767\sin(u_3(t)). \end{aligned} \quad (33)$$

We found that our NARMAX structure selection and model estimation step produced a good fit with the test data (with a) and subsequent linearization fulfills the assumption 1, making this identification method a preferred means of guaranteeing the practicality of (Cui and Molu, 2022).

Using (32) with u_3 as additive stochastic Gaussian noise signal with variance $\sigma = 0.05$, a constant gear ratio of 40 and a car mass of $1600kg$, we collect I/O data with 40,000 samples as shown in Fig. 4.2. whose prediction based on held-out validation data is shown in Fig. 4. For a root relative squared test error defined as

$$RRSE = \sqrt{\sum_{i=1}^N (\hat{x}_i - x_i)^2 / \sum_{i=1}^N (\hat{x}_i - \bar{x})^2}, \text{ where } \hat{x}, \bar{x}$$

respectively denote the prediction and mean of x , the model has an RRSE of 0.07229.

To amend the nonlinear control problem (1) to the setup (2), we first compute the values of the states, inputs, and outputs for the system (16)'s equilibrium points about $x_{eq} := v_{ref}$, $u_{eq} := [u_1, u_2]$, and $z_{eq} \triangleq v_{ref}$ given the initial state, input, and output values $x(0) = 20$, $u(0) = [0, 40]$, and $z(0) = 20$. We employ a modification of the Powell hybrid method in MINPACK (Moré et al., 1980) in solving for all points that characterize the constraints and state and input equations for linearization. We leave the additive Wiener process so that we end up with the form of (2) in the following equation

$$A = [100.0288], \quad B_1 = [-193.072, 137.3123],$$

$$B_2 = [-17014.7221, -10557.48189] \quad C = [1, 0] \quad (34)$$

and D is $[1, 1]$. As seen, the pairs (A, B_1) is stabilizable while the pair (C, A) is observable as well. One attractive property of NARMAX models is their immense capability to faithfully capture the parsimonious model of a system.

5. CONCLUSION

REFERENCES

- Agarwal, A., Kakade, S.M., Lee, J.D., and Mahajan, G. (2021). On the Theory of Policy Gradient Methods: Optimality, Approximation, and Distribution Shift. *J. Mach. Learn. Res.*, 22(98), 1–76.
- Åström, K.J. and Murray, R.M. (2021). *Feedback Systems: An Introduction for Scientists and Engineers*. Princeton University Press.
- Başar, T. (2008). *H_∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*. Springer.
- Bernstein, D. and Haddad, W. (1989). LQG control with an H_{∞} performance bound: a Riccati equation approach. *IEEE Transactions on Automatic Control*, 34(3), 293–305. doi:10.1109/9.16419.
- Billings, S. (2010). *Nonlinear System Identification*, volume 39.
- Bruinsma, N. and Steinbuch, M. (1990). A Fast Algorithm to Compute the H_{∞} -norm of a Transfer Function Matrix. *Systems & Control Letters*, 14(4), 287–293.
- Chen, S., Billings, S.A., and Luo, W. (1989). Orthogonal Least Squares Methods and Their Application to Nonlinear System Identification. *International Journal of Control*, 50(5), 1873–1896.
- Cui, L. and Molu, L. (2022). Mixed H_2/H_{∞} Control for Robust Policy Optimization. (Under review at) *Transactions in Automatic Control*. URL <https://arxiv.org/abs/2209.04477>.
- Deisenroth, M.P. (2011). A Survey on Policy Search for Robotics. *Foundations and Trends in Robotics*, 2(1), 1–142.
- Duncan, T.E. (2013). Linear-Exponential-Quadratic Gaussian control. *IEEE Transactions on Automatic Control*, 58(11), 2910–2911. doi: 10.1109/TAC.2013.2257610.
- Jacobson, D. (1973). Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games. *IEEE Transactions on Automatic Control*, 18(2), 124–131. doi: 10.1109/TAC.1973.1100265.

- Joseph, D.P. and Tou, T.J. (1961). On linear control theory. *Transactions of the American Institute of Electrical Engineers, Part II: Applications and Industry*, 80(4), 193–196.
- Kakade, S.M. (2001). A Natural Policy Gradient.
- Khansari-Zadeh, S.M. and Billard, A. (2014). Learning Control Lyapunov Function to Ensure Stability of Dynamical System-based Robot Reaching Motions. *Robotics and Autonomous Systems*, 62(6), 752–765.
- Khargonekar, P., Petersen, I., and Rotea, M. (1988). \mathcal{H}_∞ optimal control with state-feedback. *IEEE Transactions on Automatic Control*, 33(8), 786–788. doi: 10.1109/9.1301.
- Kleinman, D.Z. (1968). On an iterative technique for riccati equation computations. *IEEE Transactions on Automatic Control*, 13, 114–115.
- Levine, S., Finn, C., Darrell, T., and Abbeel, P. (2016). End-to-end training of deep visuomotor policies. *The Journal of Machine Learning Research*, 17(1), 1334–1373.
- Lillicrap, T.P., Hunt, J.J., Pritzel, A., Heess, N., Erez, T., Tassa, Y., Silver, D., and Wierstra, D. (2015). Continuous control with deep reinforcement learning. *arXiv preprint arXiv:1509.02971*.
- Moré, J.J., Garbow, B.S., and Hillstom, K.E. (1980). User guide for minpack-1. Technical report, CM-P00068642.
- Mustafa, D. (1989). Relations between maximum-entropy/ \mathcal{H}_∞ control and combined \mathcal{H}_∞ /LQG control. *Systems and Control Letters*, 12(3), 193–203.
- Rotea, M. and Khargonekar, P. (1991). Mixed H_2/H_∞ Control: A Convex Optimization Approach. *IEEE Trans. Automat. Control*, 36(5), 824–837.
- Tits, A. and Yang, Y. (1996). Globally convergent algorithms for robust pole assignment by state feedback. *IEEE Transactions on Automatic Control*, 41, 1432–1452.
- Wang, R., Foster, D.P., and Kakade, S.M. (2020). What are the statistical limits of offline RL with linear function approximation? *arXiv preprint arXiv:2010.11895*.
- Whittle, P. (1981). Risk-sensitive linear/quadratic/gaussian control. *Advances in Applied Probability*, 13(4), 764–777.
- Zames, G. (1981). Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control*, 26(2), 301–320. doi: 10.1109/TAC.1981.1102603.
- Zhang, K., Hu, B., and Başar, T. (2019). Policy Optimization for \mathcal{H}_2 Linear Control with \mathcal{H}_∞ Robustness Guarantee: Implicit Regularization and Global Convergence. *arXiv e-prints*, arXiv:1910.09496.
- Zhang, K., Hu, B., and Basar, T. (2020). Policy optimization for \mathcal{H}_2 linear control with \mathcal{H}_∞ robustness guarantee: Implicit regularization and global convergence. In *Learning for Dynamics and Control*, 179–190. PMLR.
- Zhang, K., Yang, Z., and Basar, T. (2019). Policy optimization provably converges to nash equilibria in zero-sum linear quadratic games. In *Advances in Neural Information Processing Systems*, volume 32.
- Zhou, K. and Doyle, J.C. (1998). *Essentials of robust control*, volume 104. Prentice hall Upper Saddle River, NJ.