

# Kinematics and Kinetics of an In-Parallel-Actuated Soft Manipulator

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*A soft parallel multi-degree of freedom robot is here put forward to counterbalance the present preoccupation with in-parallel-actuated rigid mechanisms, currently prevalent in precision manipulation tasks. We study the kinematics of the entirely soft mechanism using continuum mechanics and differential geometry. We then derive the Newton-Euler system of equations using elasticity theory. Relating the solutions to the dynamic boundary value problem of the soft actuators' total Cauchy stress tensor to the force wrenches on the object/load, we write out the manipulation map and construct the associated Jacobian for its direct positioning analysis. Within the bounds here set, the mechanism is intended to provide precise manipulation in lieu of the rigid parallel platforms that are used today. Our goal is to edge open the door a little further towards in-parallel soft actuation mechanisms that provide an equivalent or better manipulation precision as well as compliance needed for sensitive medical application tasks.*

## 1 INTRODUCTION

We present the systematic analysis of an in-parallel-actuated soft continuum manipulator. Possible applications could be the real-time closed-loop automatic motion deviation correction, particularly during beam-on time, in robotic radiosurgery; or as a magnetic resonance imaging (MRI)-compatible soft robotic patient motion correction system in frameless and maskless cranial manipulation in radiation therapy (RT) [1, 2]. It may hasten the current treatment time in clinics, minimize patient discomfort post-treatment, or drastically improve the dose efficacy so that the patient's treatment can be effectively fractionated. Other possible applications are in the animation industry where animated pluses are required to be accurately actuated to specific configurations [3, 4] or the real-time control of hyperelastic materials [5, 6].

**Contributions:** We present a parallel multi-degree of freedom soft robot made out of deformable shells. The individual shells are constrained to strain along the circumferential direction and their kinematic arrangement have actua-

tion freedoms [7] that yield the desired range of motions for the complete manipulation of an object located within their workspace.

We first provide a constitutive model that governs their deformation given the baked deformation constraints into the material properties of the individual actuators. We then analyze the kinematics and kinetics of the mechanism with the minimal number of linkages that achieves 6-DOF manipulability.

Iterating from [2, 8–11], we expand upon the kinematics and dynamics of the multi-DOF soft actuation system. The rest of this paper is organized as follows: in § 3, we review the hardware setup and system configuration, prescribe the finite elastic deformation model and solve the dynamic boundary value problem. We then analyze the contact kinematics with the head in § 4 and derive the dynamic equations in § 5. In § 6, we derive the end-effector velocities and forces; and conclude the paper in § 8. Proofs and derivations are provided in the appendices.

## 2 Actuator Description

We call the individual soft robots in our mechanism inflatable air bladders (IABs) owing to their hollow internal chambers that admit or release air based on an applied internal pressurization. Inspired by the behavior of the skin papillae of certain cephalopods (bivalves, mollusks, octopus and cuttlefish) which can change their planar and smooth physical texture into 3D textures up to a specific maximum size [12] in less than 3 seconds [13], we pattern the design of our soft actuators similar to the skin papillae of these organisms where the elasticity of the skin papillae is controlled by a muscular hydrostatic mechanism: an elastomeric dermis antagonizes the muscle's fibers – causing uniaxial shape erection.

### 2.1 Soft Actuator Design

The soft actuator fabrication methodology is illustrated in Fig. 1. A thin-layered fabric is laser cut into circular patterns (a), the cut meshes are removed and laid onto

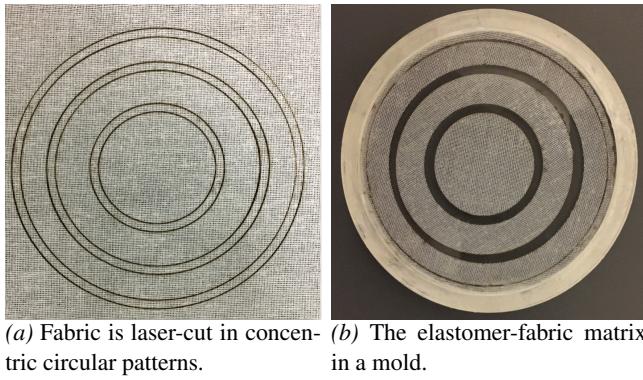


Fig. 1: Actuators fabrication

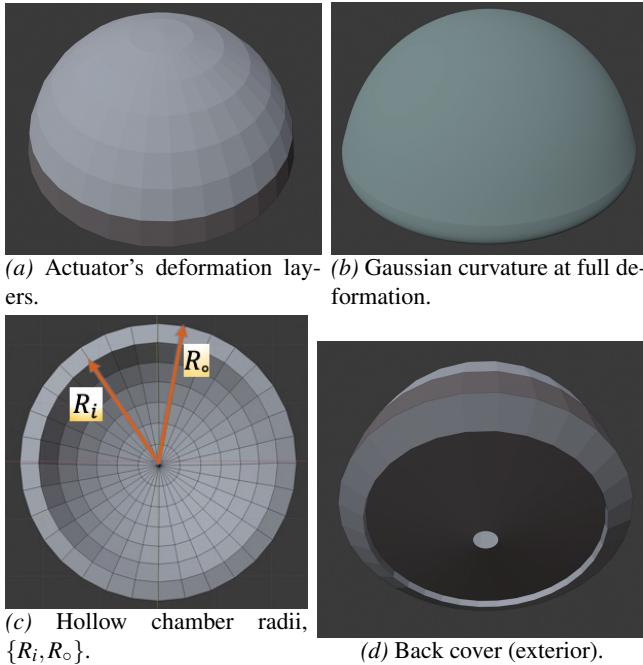


Fig. 2: IAB Geometrical Model

uncured silicone (b) which has been poured into a mold (cast with soft lithography) [14]. We further add a silicone topcoat layer to the fabric before we allow it to cure. Upon low pneumatic pressurization, the cured rubber deforms, obeying a Circumferentially-COnstrained And Radially Stretched fiber-Elastomer (CCOARSE) property [15] as seen in Fig. 2a. This unique deformation pattern is similar to the way a balloon would stretch along its axial direction if a rope were tied around its circumference. The soft actuator, after cure, is laid onto an impact-resistant, low-temperature rigid PVC insulation foam sheet, encased in a carbon nanofiber material (Fig. 2b). This aids radiation transparency, an important requirement in robotic radiation therapy and stereotactic radiosurgery [16] where the immobilization mechanism must not attenuate dose radiation. The actuator has shells with radii  $2.75 \pm 0.5\text{cm}$  and  $3.0 \pm 0.5\text{cm}$  respectively and there exists constant volume within their wall. The wall thickness is  $0.25\text{cm}$  in the reference configuration, to accommodate flex and shrinking from repeated use as well as exhibit enough tensile strength for robust manipulation. This proposed fabrication method allows users to rapidly it-

erate different designs with compressed low air pressure (at 3-15 psi) which is advantageous because air is (i) cheaply available, (ii) environmentally-friendly, (iii) avoids electrical wirings, (iv) lightweight, and (v) inviscid. This aids a clean and safe human-robot workspace suitable for medical robotics applications such as in emerging magnetic resonance imaging-linear accelerators (MRI-Linacs) [17-19].

We fabricate the IAB's elastic membranes from elastomeric rubber with a shore hardness of 10A (Dragon Skin 10, Smooth-on Inc.) and 475 psi tensile strength. We control the circumferential strain with concentric circular fabric meshes (Fleishman Fabrics & Supplies, Philadelphia, PA) similar to some of the designs of [13, 15]. Deformation can be sensed with electrical grade Kapton polyimide sheets cut into specific kirigami geometries [20], [21] but we leave this implementation to a future treatment.

In their reference configuration, the actuators are planar while in the current configuration, they deform, creating an internal cavity that contributes a single degree of freedom (DOF) to a soft actuator linkage in the system – in part holding the object in place and moving the load along a desired axis of motion. To enforce the single DOF during deformation, a fabric material was laser-cut in concentric circular patterns (Fig. 1a). Uncured silicone (with equal parts of Dragon Skin A and B) are then mixed, degassed at 29in Hg, and then poured into a mold (additively manufactured). We further laid the cut fabric membrane onto the uncured rubber, add silicone topcoat and the admixture was allowed to cure at room temperature. After room-temperature vulcanization, we post-cured the admixture at  $176^{\circ}\text{F}$  for 2 hours and at  $212^{\circ}\text{F}$  for 1 hour respectively in order to allow the rubber material to attain its maximum physical properties. Each IAB is linked to the object – each supporting only a part of the total load, redistributing link-loads and solving some of the problems with serial links. The actuator's internal chambers are constrained to deform only along the radial strain in order to exert appropriate push on an object.

The experiments of Fig. 3 illustrates the deformation behavior of the manufactured actuator with two different designs. The cured rubber material is screwed onto a bottom PVC foam sheet using **nylon** Phillips screws on a laser-cut acrylic ring. The behavior at different levels of pressurization are indicated in each column of the figure. The top row shows the cured silicone without fabric while the bottom row shows the cured elastomer with the entrenched fiber matrix. As seen, the fiber-free material exhibits a circumferential strain as well as radial strain while the fiber-constrained elastomer only exhibits a radial strain. As a result, we can generate a full *Gaussian deformation* and return to the reference planar configuration in 2-3 seconds (see more images and videos in [scriptedonachip.com/soro](http://scriptedonachip.com/soro)) similar to the spikes produced by the skin papillae of the Octopus. These quick Gaussian spikes are useful for rapid manipulation, grasp, push and release of objects when the actuators are interconnected as linkages to fit a kinematician's desired mechanisms.

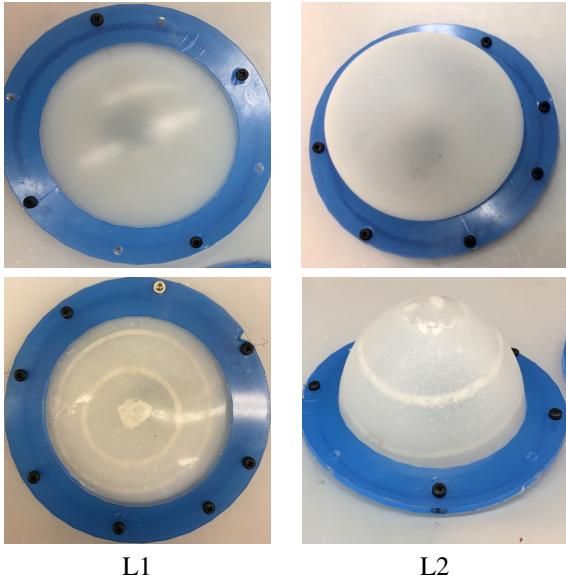


Fig. 3: Deformation of Elastomer-only (top) and Elastomeric-Fabric Matrix (bottom) under Low Air Pressurization (3-15psi).

## 2.2 Mechanism Setup

While most of the treatment in this section applies to head motion compensation in RT, we note that the formulation which we have provided are applicable to any soft robot mechanism a kinematician may choose to design. We merely provide this mechanism for didactic purposes. This mechanism may have medical applications such as magnetic resonance imaging or emerging MRI-Linac radiation therapy (RT) applications. Explorative robotic positioning research studies have demonstrated the feasibility of maintaining stable patient cranial motion consistent with treatment plans using rigid Stewart-Gough platforms [27–29]. These achieve a  $\leq 0.5\text{mm}$  and  $\leq 0.5^\circ$  positioning accuracy 90% of the time. While aiding better clinical accuracy, they utilize rigid metallic components, electric motors and linear actuators which are not suitable for MRI imaging: they interfere with the magnets of the MRI machine, and can lead to patient fatality or significant damage to the MRI machine [30]. Time-resolved MRI techniques, which provide superior soft tissues image scans, can provide soft tissues delineation for use in brain or head and neck (H&N) radiation therapy (RT) [17–19, 31]. Existing frame-based and frameless and maskless robotic motion correction mechanisms are not suitable for this because of their electro-mechanical parts that introduce radiation-attenuation and magnetic compliance concerns.

We position eight IABs around the patient's cranium as illustrated in Fig. 4. The IABs are held in place around the head by a low-temperature rigid PVC foam insulation sheet, encased in carbon fiber to prevent radiation beam attenuation. Velcro stickers (not shown) hold the IABs in place. The freedoms provided by each IAB within the setup in Fig. 4b are described as follows: the side actuators correct head motion along the *left-right* axis of the head anatomy, including the yaw and roll motions, while the base IABs correct the

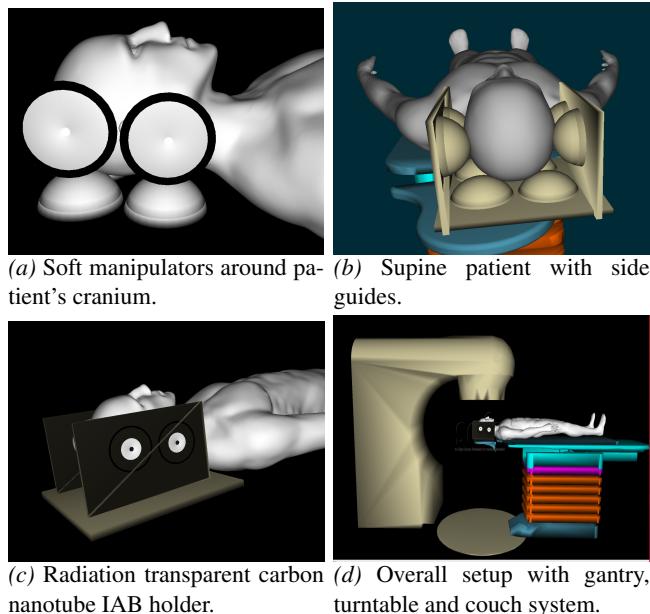


Fig. 4: System setup.

head motion along the *anterior-posterior* axis [7, Ch. 2]. This arrangement offers prehensile manipulation via sensor-based motion manipulation strategies with flexible and electroelastic proprioceptive sensor plans [20,21,32,33]. By this, we mean the mechanical interactions of pushing or releasing by the IABs may be harnessed to further improve head manipulation robustness [34–36].

## 3 Deformation Analysis

Suppose that at rest, the IAB occupies a stress-free reference configuration  $\mathcal{B}_0$  with boundary  $\delta\mathcal{B}_0$  when no mechanical loads or electric fields within the sensor are present. Upon the application of a mechanical load and an electric field, deformation occurs so that the body is transformed into a new configuration  $\mathcal{B}$  with boundary  $\delta\mathcal{B}$ . Material points in  $\mathcal{B}_0$  are denoted by position vectors  $\mathbf{X}$  and they have a correspondence in configuration  $\mathcal{B}$  as  $\mathbf{x}$ . We assume regularity in the deformation from  $\mathcal{B}_0$  to  $\mathcal{B}$ , denoted by  $\chi$  such that  $\mathbf{x} = \chi(\mathbf{X}, t)$ . In the Lagrangean configuration, we define the deformation tensor to be  $\mathbf{F} = \text{Grad } \chi$  with  $\text{Grad}$  being the gradient operator acting on the material points,  $\mathbf{X}$ , in the configuration  $\mathcal{B}_0$ . The common notations throughout the rest of this article are as defined in Table 1. We work from a continuum mechanics framework, whereby we consider only final configurations of the soft actuators; thus we drop the explicit dependence of a configuration on time and rather write it as  $\chi(\mathbf{X})$ .

### 3.1 Nonlinear Elastic Deformation Model

As we are dealing with fluids in the chambers of the IABs, we use the Eulerian specification in our stress field equations. Forces that produce deformations are derived using the strain energy-invariants relationship. For readers implementing these principles on solids or hyperelastic materials, a Lagrangean formulation may be appropriate since a refer-

Table 1: Common notations

Notation	Definition
$B$	An open set of particles on a Body, $B$
$\mathbf{F}$	The deformation gradient tensor, $\mathbf{F} = \nabla \chi_k(\mathbf{X}) = \dot{\chi}_k(\mathbf{X})$ .
$\boldsymbol{\sigma}$	The Cauchy stress tensor.
$\mathbf{C}$	The right Cauchy-Green tensor, $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ .
$\mathbf{B}$	The left Cauchy-Green tensor, $\mathbf{B} = \mathbf{F} \mathbf{F}^T$ .
$\mathbf{v}(\mathbf{x})$	The velocity field $\dot{\chi}(\mathbf{x}) = \dot{\chi}(\chi^{-1}(\mathbf{X}))$ , where without loss of generality, we have taken $X = \mathbf{X}$ , i.e. the place of $X$ .

ence configuration may be chosen to coincide with the geometry of the solid initially.

Circular fibers are embedded within the elastomeric IAB volume as a reinforcement for conformal radially symmetric deformation, and the elastic properties of the elastomer and fiber reinforcing are described in terms of the strain-energy functions  $\Psi_{\text{iso}}$  and  $\Psi_{\text{mesh}}$  respectively. Choosing a Mooney-Rivlin formulation [37,38], we let the elastomer energy,  $\Psi_{\text{iso}}$ , depend on the invariants  $I_1$ , and  $I_2$ , functions of the principal stretches,  $\lambda_r, \lambda_\phi, \lambda_\theta$  i.e.,

$$I_1 = \text{tr}(\mathbf{C}) = \lambda_r^2 + \lambda_\phi^2 + \lambda_\theta^2, \quad \text{and} \\ I_2 = \text{tr}(\mathbf{C}^{-1}) = \lambda_r^{-2} + \lambda_\phi^{-2} + \lambda_\theta^{-2}, \quad (1)$$

where  $\lambda_r \lambda_\phi \lambda_\theta = 1$  following the incompressibility assumption of the elastomeric IAB material. The Mooney-Rivlin strain energy for small deformations as a function of the strain invariants (1) is

$$\Psi_{\text{iso}}(I_1, I_2) = \frac{1}{2} C_1(I_1 - 3) + \frac{1}{2} C_2(I_2 - 3). \quad (2)$$

The fiber membrane's strain energy is constrained along the circumferential direction  $\mathbf{M}$  so that it only applies along the radial,  $r$ , and zenith,  $\phi$ , axes (see Fig. 5). We set the strain energy invariant as  $I_4 = \mathbf{M} \cdot \mathbf{CM}$ , where the symbol “.” denotes the dot product and we choose the standard reinforcing model i.e.,

$$\Psi_{\text{mesh}}(I_4) = \frac{C_4}{2}(I_4 - 1)^2 \quad (3)$$

where  $C_4$  is the fiber stiffness. We consider the material to be incompressible ( $J \equiv \det \mathbf{F} = 1$ ) with the total energy of the

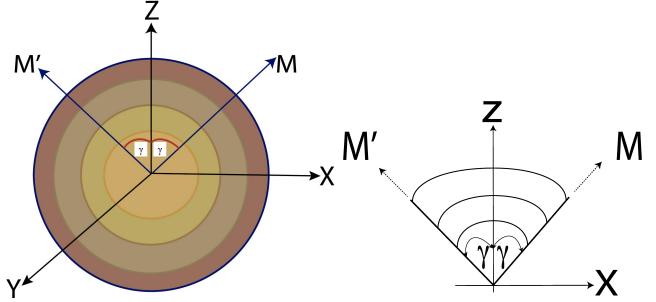


Fig. 5: Sketch of an IAB strip reinforced by radially symmetric circular membranes at angles,  $\gamma$ , along directions  $\mathbf{M}$  and  $\mathbf{M}'$ .

elastomer-fiber composite being

$$\Psi(I_1, I_2, I_4) = \Psi_{\text{iso}}(I_1, I_2) + \Psi_{\text{mesh}}(I_4). \quad (4)$$

To find a closed-form expression for  $I_4$ , we consider a strip of the IAB material in the  $XY$  plane with vectors  $\mathbf{M}$  and  $\mathbf{M}'$  in the reference configuration forming circular segments with angles  $\gamma$  as shown in Fig. 5. The matrix forms of  $\mathbf{M}$  and  $\mathbf{M}'$  are

$$\mathbf{M} = [\cos \gamma, \sin \gamma, 0]^T, \quad \mathbf{M}' = [\cos \gamma, -\sin \gamma, 0]^T. \quad (5)$$

The anisotropy term is only active if  $I_4 > 1$  so that the matrix bears the stress under compression in the fiber direction. If there is a deformation, we must have

$$\mathbf{m} = \mathbf{FM} = \begin{bmatrix} \lambda_r \cos \gamma & \lambda_\phi \sin \gamma & 0 \end{bmatrix}^T \quad (6)$$

where  $\lambda_r$  and  $\lambda_\phi$  are the principal stretches along the axial and zenith directions, respectively. The fiber invariant is,

$$I_4 = \mathbf{M} \cdot \mathbf{CM} = \left[ \lambda_r^2 \cos^2 \gamma, \lambda_\phi^2 \sin^2 \gamma, 0 \right]^T \quad (7)$$

and by symmetry we have  $\mathbf{M}' \cdot \mathbf{CM}' = I_4$ .

### 3.2 Strain Analysis: The Deformation Gradient

Since the physical texture of the deformed IAB material is a hemisphere, we choose spherical polar coordinates  $(r, \phi, \theta)$ , where  $r$  represents the radial distance of the particle from a fixed origin,  $\theta$  is the azimuth angle on a reference plane through the origin and orthogonal to the polar angle,  $\phi$ . Denote the internal and external radii as  $r_i$ , and  $r_o$  respectively with current/reference configuration constraints (since radial



Reference Configuration:  $(R, \Phi, \Theta)$  Current Configuration:  $(r, \phi, \theta)$

Fig. 6: Radii change under deformation.

symmetry is preserved throughout deformation as seen in Fig. 6),

$$\begin{aligned} r_i \leq r \leq r_o, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi/2 \\ R_i \leq R \leq R_o, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq \Phi \leq \pi/2. \end{aligned} \quad (8)$$

The position vectors  $\mathbf{R}$  and  $\mathbf{r}$  in either configurations are

$$\mathbf{R} = \begin{pmatrix} R \cos \Theta \sin \Phi, \\ R \sin \Theta \sin \Phi, \\ R \cos \Phi \end{pmatrix}, \mathbf{r} = \begin{pmatrix} r \cos \theta \sin \phi, \\ r \sin \theta \sin \phi, \\ r \cos \phi \end{pmatrix}. \quad (9)$$

Given the incompressibility of the IAB material, the material volume contained between the IAB walls (of radii  $R_o$  and  $R_i$  respectively) remains constant throughout deformation, *i.e.*  $\frac{2}{3}\pi(R^3 - R_i^3)$ , equal in volume to  $\frac{2}{3}\pi(r^3 - r_i^3)$  so that

$$r^3 = R^3 + r_i^3 - R_i^3, \quad \theta = \Theta, \quad \phi = \Phi. \quad (10)$$

For bases  $\{\mathbf{e}_x\}$  and  $\{\mathbf{e}_X\}$  (with respect to an observer) in the current and reference configurations respectively, we introduce the gradient operator,  $\nabla$ , (in the Lagrangean axes) for a fiber element  $\mathbf{dx} = \mathbf{dx}_r \mathbf{e}_r + \mathbf{dx}_\phi \mathbf{e}_\phi + \mathbf{dx}_\Theta \mathbf{e}_\Theta$  (in Eulerian axes) so that

$$\nabla = \mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\Phi \frac{1}{R} \frac{\partial}{\partial \Phi} + \mathbf{e}_\Theta \frac{1}{R \sin \Phi} \frac{\partial}{\partial \Theta}. \quad (11)$$

The deformation gradient,  $\mathbf{F}$ , as a dyadic product of a material line element in the current configuration and the gradient operator (11) is

$$\begin{aligned} \mathbf{F} &= \mathbf{dx} \otimes \nabla \\ &= (\mathbf{dx}_r \mathbf{e}_r + \mathbf{dx}_\phi \mathbf{e}_\phi + \mathbf{dx}_\Theta \mathbf{e}_\Theta) \otimes \\ &\quad \left( \mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\Phi \frac{1}{R} \frac{\partial}{\partial \Phi} + \mathbf{e}_\Theta \frac{1}{R \sin \Phi} \frac{\partial}{\partial \Theta} \right) \end{aligned} \quad (12)$$

where  $\otimes$  denotes the dyadic product between two tensor fields. We can verify that the following deformation gradient relation holds (using (10) as a representative configuration change relation)

$$\mathbf{F} = \begin{pmatrix} \frac{R^2}{r^2} & -\frac{\phi}{R} & -\frac{\theta}{R} \\ 0 & \frac{r}{R} + \frac{1}{R} & -\frac{\theta}{R} \cot \phi \\ 0 & 0 & \frac{r}{R} + \frac{\phi}{R} \cot \phi + \frac{1}{R \sin \Phi} \end{pmatrix} \quad (13)$$

(see Appendix A for the derivation). Given the fiber constraints along the  $\theta$  direction, the deformation gradient actu-

ally turns out to be

$$\mathbf{F} = \begin{pmatrix} \frac{R^2}{r^2} & -\frac{\phi}{R} & 0 \\ 0 & \frac{r}{R} + \frac{1}{R} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (14)$$

with principal stretches,

$$\lambda_r = \frac{R^2}{r^2}; \quad \lambda_\phi = \frac{r}{R} + \frac{1}{R}; \quad \lambda_\theta = 1 \quad (15)$$

and associated right and left Cauchy-Green tensors

$$\mathbf{C} = \mathbf{B} = \begin{pmatrix} \frac{R^4}{r^4} & 0 & 0 \\ 0 & \left(\frac{1}{R} + \frac{r}{R}\right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

### 3.3 Macroscopic Representation of Deformation

At a material point  $\mathbf{X}$  of the IAB surface in a configuration  $\mathcal{B}$ , it can be verified that the tensor  $\mathbf{C} - \mathbf{I}$  represents a change in length of an arbitrary line element of the material. For the material not to be strained, we must have the Lagrangean strain tensor (strain rate) as zero *i.e.*  $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = 0$ . A particle  $\mathbf{X}$ 's displacement from the reference to the current configurations must be such that the point difference  $d = \mathbf{x} - \mathbf{X}$  is  $d(\mathbf{X}) = \chi(\mathbf{X}) - \mathbf{X}$ , where  $\chi(\mathbf{X})$  follows the notations of [1, 2, 39]. We therefore characterize the point differences by the two-point *displacement gradient*,

$$\mathbf{D} = \text{Grad } d(\mathbf{X}) = \mathbf{F} - \mathbf{I}. \quad (17)$$

For the unit vectors  $\mathbf{m}$  and  $\mathbf{M}$  for material line elements  $\mathbf{dx}$  and  $\mathbf{dX}$  in the current and reference configurations respectively<sup>1</sup>, it follows from the invariant of deformation is

$$\mathbf{m}|\mathbf{dx}| = \mathbf{FM}|\mathbf{dX}| \implies |\mathbf{dx}|^2 = \mathbf{M} \cdot (\mathbf{CM})|\mathbf{dX}|^2. \quad (18)$$

Hence, the *stretch* between line elements  $\mathbf{dx}$  and  $\mathbf{dX}$  is

$$\lambda(\mathbf{M}) = \frac{|\mathbf{dx}|}{|\mathbf{dX}|} = |\mathbf{FM}| \equiv (\mathbf{M} \cdot (\mathbf{CM}))^{\frac{1}{2}}. \quad (19)$$

Similarly, for line elements  $\mathbf{dX}$  and  $\mathbf{dX}'$  with unit tangent vectors  $\mathbf{M}$  and  $\mathbf{M}'$  at points  $\mathbf{X}$  and  $\mathbf{X}'$  in the reference configuration; these fibers correspond to  $\mathbf{dx}$  and  $\mathbf{dx}'$  with unit vectors  $\mathbf{m}$  and  $\mathbf{m}'$  at points  $\mathbf{x}$  and  $\mathbf{x}'$  in the current configuration. Let  $\beta$  denote the angle between the directions  $\mathbf{M}$  and

<sup>1</sup> $\mathbf{dx}$  and  $\mathbf{dX}$  are respectively the material line elements at the points  $\mathbf{x}$  and  $\mathbf{X}$  in their respective configurations *i.e.*  $\mathcal{B}_0, \mathcal{B}$ .

$\mathbf{M}'$  and let  $\alpha$  be the angle between directions  $\mathbf{m}$  and  $\mathbf{m}'$ ; then we must have

$$\mathbf{d}\mathbf{x} = \mathbf{F}\mathbf{d}\mathbf{X}, \quad \mathbf{d}\mathbf{x}' = \mathbf{F}\mathbf{d}\mathbf{X}', \quad \beta = \cos^{-1}(\mathbf{M} \cdot \mathbf{M}'), \quad (20a)$$

$$\text{and } \alpha = \cos^{-1}(\mathbf{m} \cdot \mathbf{m}') = \cos^{-1}(\mathbf{M} \cdot (\mathbf{C}\mathbf{M}')) / \lambda(\mathbf{M}) \cdot \lambda(\mathbf{M}'). \quad (20b)$$

In the plane of shear of  $\mathbf{M}$  and  $\mathbf{M}'$ , the *angle of shear* between the material line elements is the reduction in angles

$$\phi = \beta - \alpha. \quad (21)$$

Notice that both (17), and (19) are characterized by the stretch  $\lambda = r/R$  and the radius in the current configuration (c.f. [1]). Therefore, we take the kinematic quantities that characterize the IAB deformation as

$$\mathbf{r} = \begin{bmatrix} \lambda \\ \phi \end{bmatrix} = \begin{bmatrix} r/R \\ \beta - \alpha \end{bmatrix} \quad (22)$$

i.e.  $\mathbf{r}$  is a function of material stretch, and the angle of shear between material line elements.

### 3.4 Stress Response from Strain Energy

We are concerned with the magnitudes of the differential stress on the IAB shells from a mechanical point of view and *our approach is based on a continuum mechanics viewpoint which is independent of finite element methods*. Since the IAB deforms at ambient temperature, we take thermodynamic properties such as temperature and entropy to have negligible contribution. The IAB material stress response,  $\mathbf{G}$ , at any point on the IAB's boundary at time  $t$  determines the Cauchy stress,  $\boldsymbol{\sigma}$ , as well as the history of the motion up to and at the time  $t$  [39]. The *constitutive equation* that relates the stress to an arbitrary motion will be determined using [40]'s *determinism for the stress principle*. The constitutive relation for the nominal stress deformation for an elastic IAB material is given by

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{pmatrix} = \mathbf{G}(\mathbf{F}) + q\mathbf{F} \frac{\partial \mathbf{\Lambda}}{\partial \mathbf{F}}(\mathbf{F}), \quad (23)$$

where  $\mathbf{G}$  is a functional with respect to the configuration  $\chi$ ,  $q$  acts as a Lagrange multiplier, and  $\mathbf{\Lambda}$  denotes the internal (incompressibility) constraints of the IAB system. For an incompressible material, the indeterminate Lagrange multiplier becomes the hydrostatic pressure i.e.  $q = -p$  [41]. The incompressibility of the IAB material properties imply that  $\mathbf{\Lambda} \equiv \det \mathbf{F} - 1$ . We can verify that

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) - p\mathbf{I} \quad (24)$$

since  $\det(\mathbf{F}) = 1$ . In terms of the stored strain energy, we find that

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^T - p\mathbf{I} \quad (25)$$

where  $\mathbf{I}$  is the identity tensor and  $p$  represents an arbitrary hydrostatic pressure. It follows that the constitutive law that governs the Cauchy stress tensor is

$$\begin{aligned} \boldsymbol{\sigma} &= \frac{\partial \Psi_{\text{iso}}}{\partial \mathbf{I}_1} \frac{\partial \mathbf{I}_1}{\partial \mathbf{F}} \mathbf{F}^T + \frac{\partial \Psi_{\text{iso}}}{\partial \mathbf{I}_2} \frac{\partial \mathbf{I}_2}{\partial \mathbf{F}} \mathbf{F}^T + \frac{\partial \Psi_{\text{mesh}}}{\partial \mathbf{I}_4} \frac{\partial \mathbf{I}_4}{\partial \mathbf{F}} \mathbf{F}^T - p\mathbf{I} \\ &= \frac{1}{2} C_1 \frac{\partial \mathbf{tr}(\mathbf{FF}^T)}{\partial \mathbf{F}} \mathbf{F}^T + \frac{1}{2} C_2 \frac{\partial \mathbf{tr}([\mathbf{F}^T \mathbf{F}]^{-1})}{\partial \mathbf{F}} \mathbf{F}^T \\ &\quad + C_4(I_4 - 1) \left[ \frac{\partial(\mathbf{M} \cdot \mathbf{CM})}{\partial \mathbf{F}} + \frac{\partial(\mathbf{M}' \cdot \mathbf{CM}')}{{\partial \mathbf{F}}} \right] - p\mathbf{I} \\ &= C_1 \mathbf{FF}^T - C_2 (\mathbf{F}^T \mathbf{F})^{-1} \\ &\quad + 2C_4(I_4 - 1) [(\mathbf{FM})(\mathbf{FM})^T + (\mathbf{FM}')(\mathbf{FM}')^T] - p\mathbf{I} \\ \boldsymbol{\sigma} &= C_1 \mathbf{B} - C_2 \mathbf{C}^{-1} + 2C_4(I_4 - 1) [\mathbf{m} \otimes \mathbf{m} + \mathbf{m}' \otimes \mathbf{m}'] - p\mathbf{I}. \end{aligned} \quad (26)$$

Expanding (26), we see that the shear stresses vanish and we are left with the normal stresses, given by

$$\begin{aligned} \sigma_{rr} &= -p + \frac{R^4}{r^4} C_1 - \frac{r^4}{R^4} C_2 - 2C_4 \frac{R^8}{r^8} \cos^2 \gamma \\ &\quad + 4C_4 \frac{R^{16}}{r^{16}} \cos^4 \gamma \end{aligned} \quad (27a)$$

$$\begin{aligned} \sigma_{\phi\phi} &= -p + \frac{(1+r)^2}{R^2} C_1 - \frac{R^2}{(1+r)^2} C_2 \\ &\quad - 2C_4(1+r)^4 \sin^2 \gamma \frac{(R^4 - 2(1+r)^4 \sin^2 \gamma)}{R^8} \end{aligned} \quad (27b)$$

$$\sigma_{\theta\theta} = -p + C_1 - C_2. \quad (27c)$$

A visualization of the component stresses of (27) on the outer shells of the IAB material is illustrated in Fig. 7.

### 3.5 Boundary-Value Problem of Traction

The dynamic problem is to find the stress at every point in the robot's body subjected to external forces under suitable boundary conditions. The soft elastic materials are normally positioned around the object so that body forces are applied only along the axial direction; the fixed direction of axial loading implies that the deformation is a function of conservative forces only so that uniqueness of solution of stress field equations are preserved owing to Kirchhoff's theorem [42, §7.4]. Furthermore, we assume that the applied pressure does not exceed a threshold that makes the rubber material yield to the point of buckling – thus, simplifying the solutions of the ensuing partial differential equations. The equilibrium equations for the physical component vectors of

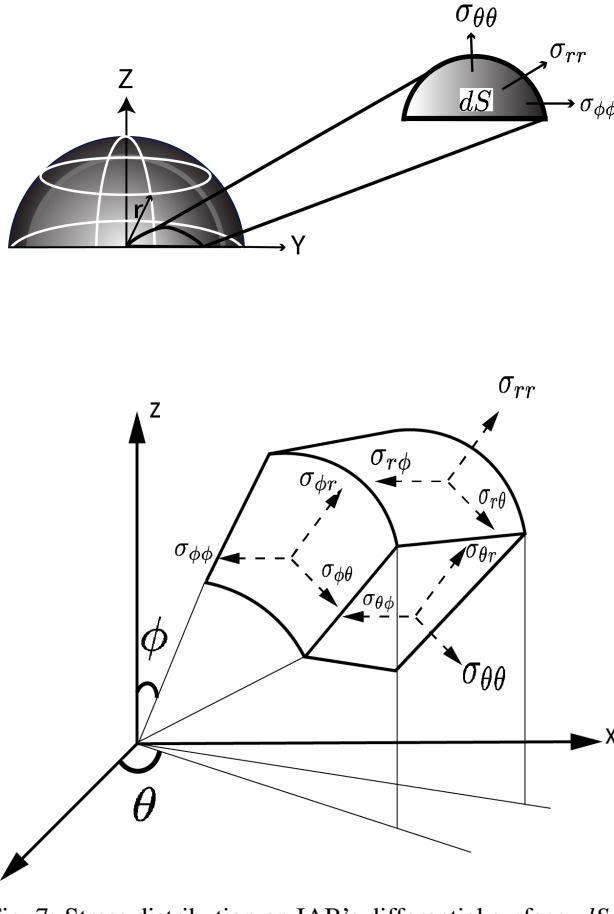


Fig. 7: Stress distribution on IAB's differential surface,  $dS$ .

the body force,  $\mathbf{b} = \{b_r, b_\phi, b_\theta\}$  are (see [1])

$$-b_r = \frac{1}{r^2} \frac{\partial(r^2 \sigma_{rr})}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial(\sin \phi \sigma_{r\phi})}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta} \sigma_{r\theta} - \frac{1}{r} (\sigma_{\theta\theta} + \sigma_{\phi\phi}) \quad (28a)$$

$$-b_\phi = \frac{1}{r^3} \frac{\partial(r^3 \sigma_{r\phi})}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial(\sin \phi \sigma_{\phi\phi})}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial(\sigma_{\theta\phi})}{\partial \theta} - \frac{\cot \phi}{r} (\sigma_{\theta\theta}) \quad (28b)$$

$$-b_\theta = \frac{1}{r^3} \frac{\partial(r^3 \sigma_{\theta\theta})}{\partial r} + \frac{1}{r \sin^2 \phi} \frac{\partial(\sin^2 \phi \sigma_{\theta\phi})}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta} \sigma_{\theta\theta}. \quad (28c)$$

Owing to the CCOARSE symmetry of the elastic IAB materials, all shearing stresses *i.e.*  $\sigma_{r\theta}, \sigma_{\phi\theta}, \sigma_{r\phi}$  etc., vanish. We prescribe the following boundary conditions for the radial normal stress

$$\sigma_{rr}|_{r=r_o} = -P_{\text{atm}}, \quad \sigma_{rr}|_{r=r_i} = -P_{\text{atm}} - P \quad (29)$$

where  $P_{\text{atm}}$  is the atmospheric pressure and  $P > 0$  is the internal pressure exerted on the internal walls of the IAB above  $P_{\text{atm}}$  *i.e.*,  $P > P_{\text{atm}}$ . Furthermore, we note that  $\sigma_{\theta\theta} = 0$  owing to the CCOARSE circumferential constraint. Thus, (28)

becomes

$$-b_r = \frac{1}{r^2} \frac{\partial(r^2 \sigma_{rr})}{\partial r} - \frac{\sigma_{\phi\phi}}{r} \quad (30a)$$

$$-b_\phi = \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} (\sin \phi \sigma_{\phi\phi}), \quad -b_\theta = 0. \quad (30b)$$

Expanding (30a) and substituting (29), we can verify that

### Internal Pressure and Radial Body Force

$$P = \int_{r_i}^{r_o} \frac{1}{r} (rb_r + \sigma_{\phi\phi} - 2\sigma_{rr}) dr \quad (31)$$

where  $r_i$  and  $b_r$  are known. Under the incompressibility properties of the IAB material we have,

$$r^3 = R^3 + r_i^3 - R_i^3, \text{ and } r_o^3 = R_o^3 + r_i^3 - R_i^3. \quad (32)$$

Equations (31) and (32) completely determine the inverse kinematics (SOFT IK) of the IAB material: for a required IAB deformation, it determines the internal pressurization or normal Cauchy stress required.

### 4 Analysis of Contact Kinematics

The interactions among the manipulators and head are considered as a classical case of elastic bodies in contact. Using the soft finger contact primitive [43], the IAB forces and torques are modeled within a “cone of forces” about the direction of the surface normal from the object (in this case the head) (see Fig. 8). The trajectory of the head under the influence of motion of an IAB is influenced by the position vector  $\mathbf{r}$  of (22). When the IAB deforms, body forces in its current configuration and the *traction* over its boundary  $\partial\mathcal{B}$  impact motion on the head. Constrained by the frictional coefficient, we define the soft contact force inside the friction cone as

$$\tilde{F}_{c_i} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & n_{c_i} \end{pmatrix} \begin{pmatrix} f_{c_i} \\ \tau_{c_i} \end{pmatrix}, \quad (33)$$

where  $f_{c_i} \in \mathbb{R}^3$  denotes the amount of force exerted by the IAB along the direction of contact,  $\tau_{c_i} \in \mathbb{R}$  is the moment of the contact force, and  $n_{c_i}$  is the *normal* or *Gauss map*<sup>2</sup> for a manifold  $S \subset \mathbb{R}^3$  of a head surface. For contact models with friction, we require that all contact forces lie within the friction cone, determined by the frictional coefficient. The set of forces within or on the boundary of the friction cone is

$$FC = \{f_c \in \mathbb{R}^n : \|f_{c_{ij}}^t\| \leq \mu_{ij} \|f_{c_i}^n\|, i = 1, \dots, k, j = 1, \dots, m_i\} \quad (34)$$

<sup>2</sup>A normal map for a manifold  $S$  is a continuous map  $g : S \rightarrow S^2 \subset \mathbb{R}^3$  such that for every  $s \in S$ ,  $g(s)$  is orthogonal to  $S$  at  $s$  [44].

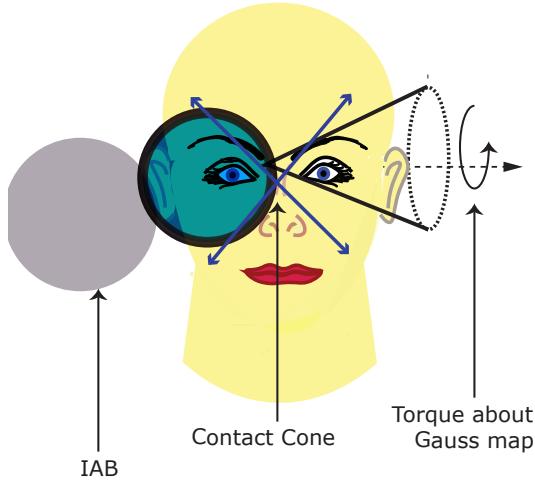


Fig. 8: Soft Contact Illustration

where  $f_{cij}^t$  is the tangent component of the  $j^{th}$  element of the contact force,  $f_{cij}^n$  is  $i^{th}$  contact's normal force, and  $\mu_{ij}$  is  $f_{cij}$ 's coefficient of friction.

#### 4.1 Contact/Traction Forces and Stress Equivalence

Here, we relate the microscopic contact stress of the previous section with the macroscopic descriptions of the contact friction to enable us treat different material combinations for the manipulator and head. We assume that the stress vector  $\sigma$  at a point on the IAB surface is uniform and continuous throughout the IAB boundary so that it linearly depends on the normal map (this follows from Cauchy's theorem; readers may see the proof in [39, §3.3.1]). The correspondence between material line,  $\{\mathbf{dx}, \mathbf{dX}\}$ , elements in the reference and current configuration is

$$\mathbf{dx} = \mathbf{F} \mathbf{dX} \implies \mathbf{F}^{-T} \mathbf{dx} = \mathbf{dX}. \quad (35)$$

where  $\mathbf{F}$  is the deformation gradient.

Let  $\mathbf{H} = \mathbf{F}^{-T}$  and  $\mathbf{da}$  represent an infinitesimal vector element on the material surface at a neighborhood of point  $\mathbf{X}$  in  $\mathcal{B}$  such that  $\mathbf{da} = \mathbf{n} da$ , where  $\mathbf{n}$  is the unit outward normal to the IAB surface. On the bounding surface of the IAB the exerted force on a surface area  $\mathbf{da}$  in global form is

$$\mathbf{f}_c = \int_{\partial\mathcal{B}} \boldsymbol{\sigma} \mathbf{n} da. \quad (36)$$

At the region of contact, we have the  $i^{th}$  contact force within

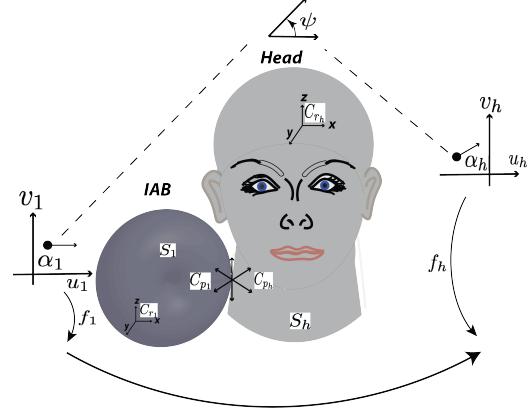


Fig. 9: Sliding and rolling contact illustration of a single IAB and the Head. [Image best visualized in colored ink].

the friction cone as

$$\begin{aligned} f_{ci} &= (\sigma_{rr} + \sigma_{\phi\phi}) n_{ci} da_i \\ f_{ci} &= \left[ -2p + \frac{r^4 + 2r^5 + r^6 + R^6}{r^2 R^2} \left( \frac{C_1}{r^2} - \frac{r^2}{(1+r)^2 R^2} C_2 \right) \right. \\ &\quad + 2\cos^2\gamma \frac{R^8}{r^8} \left( \cos^2\gamma \frac{R^8}{r^8} - 1 \right) C_4 \\ &\quad \left. - \frac{(1+r)^4}{R^8} \sin^4\gamma \cos(2\gamma) \right. \\ &\quad \left. (-1 - 4r - 6r^2 - 4r^3 - r^4 + 2R^4 + (1+r)^4) C_4 \right] n_{ci} da_i. \end{aligned} \quad (37)$$

with the outward normal map  $n_{ci}$  from (33). The torque is the moment of the contact force on the  $i^{th}$  IAB, given as

$$\tau_{ci} = f_{ci} \times r_{ci} \quad (38)$$

where  $r_{ci} \in \mathbb{R}^3$  is the unit vector between the head reference point and the contact. Therefore, we have (33) as

Friction Cones' Contact Force
$\tilde{F}_{ci} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & n_{ci} \end{bmatrix} \begin{bmatrix} (\sigma_{rr} + \sigma_{\phi\phi}) n_{ci} da_i \\ (\sigma_{rr} + \sigma_{\phi\phi}) n_{ci} da_i \times r_{ci} \end{bmatrix}. \quad (39)$

#### 4.2 Contact Coordinates and Head Velocity

The head will make contact with the IAB at multiple points on its surface, so we describe the kinematics of these contact points using an atlas<sup>3</sup> of contact coordinate charts. In this sentiment, let  $C_{r1}$  and  $C_{rh}$  respectively represent a fixed

<sup>3</sup>An atlas  $\tilde{\mathcal{S}}$  is a set of surfaces where each surface  $S \in \tilde{\mathcal{S}}$  has an invertible map  $f(\mathbf{u})$  from an open subset  $U$  of  $\mathbb{R}^2$  to a surface  $S \subset \mathbb{R}^3$  such that the partial derivatives  $\frac{\partial f}{\partial u}(\mathbf{u}), \frac{\partial f}{\partial v}(\mathbf{v})$  are linearly independent for all  $\mathbf{u} = (u, v) \in U$ .

reference frame with respect to the IAB and head,  $H$  (see Fig. 9). Furthermore, let  $S_1 \subset \mathbb{R}^3$  and  $S_h \subset \mathbb{R}^3$  denote the respective *orientable manifold*<sup>4</sup> embeddings of the IAB and head surfaces with respect to frames  $C_{r_1}$  and  $C_{r_h}$ . We shall let  $S_1$  and  $S_h$  belong to the *atlases*  $\{S_{1_i}\}_{i=1}^{n_1}$ ,  $\{S_{h_i}\}_{i=1}^{n_h}$  respectively. Suppose  $(f_1, U_1)$  and  $(f_h, U_h)$  are *coordinate systems* for the IAB and the head respectively, where  $f_i$  is an invertible map,  $f_i(u_i, v_i) : U \rightarrow S_i \subset \mathbb{R}^3$

$$f_i(u_i, v_i) : \{U \rightarrow S_i \subset \mathbb{R}^3 | i = 1, h\},$$

from an open subset  $U$  of  $\mathbb{R}^2$  to a *coordinate patch*  $S_i \subset \mathbb{R}^3$  such that the partial derivatives  $\frac{\partial f_i}{\partial u_i}$  and  $\frac{\partial f_i}{\partial v_i}$  are linearly independent. Let  $p_1(t) \in S_1$  and  $p_h(t) \in S_h$  represent the positions of the contact points with respect to frames  $C_{r_1}$  and  $C_{r_h}$  respectively at time  $t$ . In general, the contact points  $p_1(t)$  and  $p_h(t)$  will not remain in the coordinate systems  $S_1$  and  $S_h$  for all time. Thus, we choose an interval  $I$  where  $p_1(t) \in S_{1_i}$  and  $p_h(t) \in S_{h_j}$  for all  $t \in I$  and some  $i$  and  $j$ . As seen in Fig. 9,  $C_{p_1}$  and  $C_{p_h}$  denote the contact frames that coincide with the *normalized Gauss frames* at  $p_1$  and  $p_h$  for all  $t \in I$ , and  $\alpha_1, \alpha_h$  are local coordinate frames that describe the IAB motion with respect to the head such that

$$\alpha_1 = (u_1, v_1) \in U_1, \text{ and } \alpha_h = (u_h, v_h) \in U_h. \quad (40)$$

Let the angle between the tangent planes of  $\alpha_1$ , and  $\alpha_h$  be  $\psi$ . The transformation matrix  $g \in \Omega \subset SE(3)$  encodes the relative orientation and position of the IAB with respect to the head where  $\Omega$  is the set of all relative positions and orientations in the atlases  $\{S_{1_i}\}_{i=1}^{n_1}$ ,  $\{S_{h_i}\}_{i=1}^{n_h}$  for which the IAB and head remain in contact. We let the *contact coordinates* be described by  $\eta = (\alpha_1, \alpha_h, \psi)$ . The head's motion is governed by traction forces arising from the friction tangential to the IAB surface and the pressure normal to the IAB surface. Thus, at the points of contact, if  $R \in SO(3)$  is the rotatory component of  $g$ ,  $\eta$  must satisfy

$$g \circ f_1(\alpha_1) = f_h(\alpha_h) \quad (41a)$$

$$R n_1(\alpha_1) = -n_h(\alpha_h) \quad (41b)$$

since the contact locations must coincide for the IAB and the head, and the tangent planes must coincide so that the outward normal maps  $n_1 : S_1 \rightarrow S^2 \subset \mathbb{R}^3$  and  $n_h : S_h \rightarrow S^2 \subset \mathbb{R}^3$  agree. Furthermore, the orientation of the tangent planes of  $\alpha_1$  and  $\alpha_h$  is the unique angle  $\psi \in [0, 2\pi)$  between the  $x$ -axes of  $C_{p_1}$  and  $C_{p_h}$  such that

$$R \frac{\partial f_1}{\partial \alpha_1} M_1^{-1} R_\Psi = \frac{\partial f_h}{\partial \alpha_h} M_h^{-1} \quad (42)$$

where  $M_i$  is a  $2 \times 2$  square root of the Riemannian metric tensor [45] that normalizes the columns of  $\frac{\partial f}{\partial \alpha}$ , *i.e.*

$$M_i = \begin{bmatrix} \|\frac{\partial f_i}{\partial u_i}\| & 0 \\ 0 & \|\frac{\partial f_i}{\partial v_i}\| \end{bmatrix} \quad (43)$$

and  $R_\Psi$  is chosen such that a rotation of  $C_{p_1}$  about its  $z$ -axis through  $-\psi$  radians aligns the  $x$ -axes of the local coordinate system  $\alpha_1$  to that of the head's local coordinate system  $\alpha_h$  *i.e.*

$$R_\Psi = \begin{bmatrix} \cos \psi & -\sin \psi \\ -\sin \psi & \cos \psi \end{bmatrix} \quad (44)$$

with the special property that  $R_\Psi = R_\Psi^T = R_\Psi^{-1}$ . We define the normalized Gauss frame at a point  $u$  on the surface  $U$  of the orthogonal coordinate system  $(f, U)$  as,

$$[x_u \ y_u \ z_u] = \left[ \frac{\partial f}{\partial u} / \|\frac{\partial f}{\partial u}\| \ \frac{\partial f}{\partial v} / \|\frac{\partial f}{\partial v}\| \ n_u(f(u)) \right] \quad (45)$$

where  $x_u, y_u$ , and  $z_u$  are functions mapping  $U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  and  $n_u$  is the continuous Gauss map  $n_u : S \rightarrow S^2 \subset \mathbb{R}^3$ . The motion of the contacts  $\dot{\eta}$  as a function of components of the twist vector  $\xi = (v, w)^T$  is given in (46) as the respective *first*, *second*, and *third equations of contact*. We can verify that (see proof in Appendix B of [2])

$$\dot{\alpha}_h = M_h^{-1} (\mathcal{K}_h + \tilde{\mathcal{K}}_1)^{-1} (\omega_t - \tilde{\mathcal{K}}_1 v_t) \quad (46a)$$

$$\dot{\alpha}_1 = M_1^{-1} R_\Psi (\mathcal{K}_h + \tilde{\mathcal{K}}_1)^{-1} (\omega_t - \mathcal{K}_h v_t) \quad (46b)$$

$$\dot{\psi} = \omega_n + T_h M_h \dot{\alpha}_h + T_1 M_1 \dot{\alpha}_1 \quad (46c)$$

where

$$\begin{aligned} T_h &= y_h^T \frac{\partial x_h}{\partial \alpha_h} M_h^{-1}, & T_1 &= y_1^T \frac{\partial x_1}{\partial \alpha_1} M_1^{-1}, \\ \mathcal{K}_h &= [x_h^T, y_h^T]^T \frac{\partial n_h^T}{\partial \alpha_h} M_h^{-1}, & \omega_n &= z_h^T \omega \\ \mathcal{K}_1 &= R_\Psi [x_1^T, y_1^T]^T \frac{\partial n_1^T}{\partial \alpha_1} M_1^{-1} R_\Psi, \\ \omega_t &= [x_h^T, y_h^T]^T [n_h \times \omega]^T, \\ \text{and } v_t &= [x_h^T, y_h^T]^T [(-f_h \times \omega + v)]^T. \end{aligned} \quad (47)$$

Note that  $\omega_t$  is the rolling velocity of the head projected onto the tangent plane of the contact and  $v_t$  is the sliding velocity;  $\omega_n$  is the relative rotational velocity projected to the contact's surface normal, and  $\tilde{\mathcal{K}}_1 = R_\Psi \mathcal{K}_1 R_\Psi$  is the curvature of the IAB with respect to the contact frame that coincides with the normalized Gauss frame at  $p_1(t)$ . The matrix  $(\mathcal{K}_h + \tilde{\mathcal{K}}_1)^{-1}$  is the so-called *relative curvature* originally coined by [44].

<sup>4</sup>An orientable manifold is a manifold  $S$  for which the Gauss map exists.

Simplifying (47), we find that

$$\begin{aligned} x_h &= \frac{\partial f}{\partial u_h} / \left\| \frac{\partial f}{\partial u_h} \right\|, \quad y_h = \frac{\partial f}{\partial v_h} / \left\| \frac{\partial f}{\partial v_h} \right\|, \quad z_h = n_u(f(u)) \\ T_h &= y_h \left[ \frac{\partial x_h^T}{\partial u_h} / \left\| \frac{\partial f}{\partial u_h} \right\|, \frac{\partial x_h^T}{\partial v_h} / \left\| \frac{\partial f}{\partial v_h} \right\| \right], \\ T_1 &= y_1 \left[ \frac{\partial x_1^T}{\partial u_1} / \left\| \frac{\partial f}{\partial u_1} \right\|, \frac{\partial x_1^T}{\partial v_1} / \left\| \frac{\partial f}{\partial v_1} \right\| \right], \\ K_b &= [x_h^T, y_h^T]^T \left[ \frac{\partial n_h^T}{\partial u_h} / \left\| \frac{\partial f}{\partial u_h} \right\|, \frac{\partial n_h^T}{\partial v_h} / \left\| \frac{\partial f}{\partial v_h} \right\| \right], \\ K_1 &= [x_1^T, y_1^T]^T \left[ \frac{\partial n_1^T}{\partial u_1} / \left\| \frac{\partial f}{\partial u_1} \right\|, \frac{\partial n_1^T}{\partial v_1} / \left\| \frac{\partial f}{\partial v_1} \right\| \right]. \end{aligned} \quad (48)$$

We see that for the contact interaction between an IAB and the head, for a  $U \subset \mathbb{R}^2$  we must choose an appropriate  $f_i : U_i \rightarrow S_i \subset \mathbb{R}^3$  in order to characterize the setup.

## 5 System's Newton-Euler Equations

Following § 3.3, we are only interested in the final position and orientation of the IAB as a whole rather than the system of particles that characterize a deformation at every time  $t$ . When the head exerts a reactive wrench on an IAB, it is natural to expect a dent. The shear angle in (22) captures the amount of angular deformation.

### 5.1 Euler-Lagrange Equations

For a kinetic energy  $T$  and a potential energy  $V$ , the *Lagrangian*,  $L$ , of the system in generalized coordinates is

$$L(\mathbf{r}, \dot{\mathbf{r}}) = T(\mathbf{r}, \dot{\mathbf{r}}) - V(\mathbf{r}). \quad (49)$$

The equations of motion for the pneumatic IAB system is of the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}_i} - \frac{\partial L}{\partial \mathbf{r}_i} = \boldsymbol{\tau}_i, \quad i = 1, \dots, m \quad (50)$$

where  $\boldsymbol{\tau}_i$  is the torque acting on the  $i^{th}$  generalized coordinate. Written in matrix form, we can write the Euler-Lagrange of (50) as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{\partial L}{\partial \mathbf{r}} = \boldsymbol{\tau}. \quad (51)$$

It now remains to derive the kinetic and potential energies for the IAB material. Let the velocity of an IAB material particle  $\mathbf{x}$  in the current configuration at time  $t$  be  $\mathbf{v}(\mathbf{r}, t)$ , then the Eulerian velocity gradient tensor can be defined as

$$\boldsymbol{\Gamma} = \text{grad } \mathbf{v}(\mathbf{r}, t). \quad (52)$$

Using Cauchy's first law of motion [1, eq. 19], we can write the balance equation of mechanical energy of the system

such that after a little manipulation, we find that

$$\begin{aligned} \text{div}(\boldsymbol{\sigma}^T \cdot \mathbf{v}) + \rho \mathbf{b} \cdot \mathbf{v} &= \rho \mathbf{v} \cdot \dot{\mathbf{v}} \\ \text{or } \text{div}(\boldsymbol{\sigma}^T \mathbf{v}) - \text{tr}(\boldsymbol{\sigma} \boldsymbol{\Gamma}) + \rho \mathbf{b} \cdot \mathbf{v} &= \rho \mathbf{v} \cdot \dot{\mathbf{v}}. \end{aligned} \quad (53)$$

where  $\rho$  is the IAB's mass density. Following mass conservation, we integrate over volume  $\mathcal{B}$  and employ the divergence theorem, so that the above relation yields the *balance of mechanical energy*

$$\begin{aligned} \int_{\mathcal{B}} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial \mathcal{B}} f_p \cdot \mathbf{v} da &= \frac{d}{dt} \int_{\mathcal{B}} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} dv \\ &\quad + \int_{\mathcal{B}} \text{tr}(\boldsymbol{\sigma} \boldsymbol{\Gamma}) dv \end{aligned} \quad (54)$$

where  $f_p$  is the IAB body force density, and the left hand side of the foregoing is the so-called *rate of working of the applied forces*. The symmetry of the stress tensor  $\boldsymbol{\sigma}$  implies that  $\text{tr}(\boldsymbol{\sigma} \boldsymbol{\Gamma}) = \text{tr}(\boldsymbol{\sigma} \boldsymbol{\Sigma})$  where  $\boldsymbol{\Sigma}$  is given in terms of the Eulerian-strain rate tensor,  $\boldsymbol{\Gamma}$  *i.e.*,

$$\boldsymbol{\Sigma} = \frac{1}{2} (\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^T) \quad (55)$$

so that the kinetic energy density and stress power are

$$T(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v}, \quad V(\mathbf{r}) = \text{tr}(\boldsymbol{\sigma} \boldsymbol{\Sigma}). \quad (56)$$

### 5.2 Case I: Euler-Lagrange Equation for Cauchy-Elastic IAB Material

The stress-strain relation for the IAB we have presented are only related through the deformation tensor, implying that the material is Cauchy elastic. For Cauchy elastic materials, the stress power term is not conserved during deformation making integration over the material body  $\mathcal{B}$  physically unrealistic [39]. For such materials, we may set the stored strain energy  $V$  to an arbitrary constant (*e.g.*  $V(I) = 0$ ). We can derive the torque dynamics of an IAB actuator as (see proof in Appendix B)

$$\boldsymbol{\tau} = \underbrace{\begin{bmatrix} \rho/R^2 & 0 \\ 0 & \rho \end{bmatrix}}_{M_{iab}} \ddot{\mathbf{r}} + \underbrace{\begin{bmatrix} \rho \dot{r}/R^3 & 0 \\ 0 & 0 \end{bmatrix}}_{C_{iab}} \dot{\mathbf{r}}. \quad (57)$$

Rewriting equation (57) in terms of the torque for each soft actuator, we have the dynamics for IAB  $j$  as

$$M_{iab_j}(\mathbf{r}_j) \ddot{\mathbf{r}}_j + C_{iab_j}(\mathbf{r}_j, \dot{\mathbf{r}}_j) \dot{\mathbf{r}}_j = \boldsymbol{\tau}_j \quad (58)$$

where  $M_{iab_j}$  and  $C_{iab_j}$  contain the respective inertia and Coriolis forces for actuator  $j$ . Since the IAB material is incompressible, the mass density is uniform throughout the body

of the material. In general, we write equation (58) as

$$\mathbf{M}_{iab}(\tilde{\mathbf{r}})\ddot{\tilde{\mathbf{r}}} + \mathbf{C}_{iab}(\tilde{\mathbf{r}}, \dot{\tilde{\mathbf{r}}})\dot{\tilde{\mathbf{r}}} = \tilde{\boldsymbol{\tau}} \quad (59)$$

where  $\tilde{\mathbf{r}} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \dots \mathbb{R}^{n_8}$  gives the generalized coordinates for all the IABs and  $\tilde{\boldsymbol{\tau}}$  are the vectorized torques of the individual robots.

### 5.3 Case II: Euler-Lagrange Equation for Green Elastic IAB Material

For completeness, we treat the case where the IAB material body is *hyperelastic* (*i.e.*, *Green elastic*)<sup>5</sup>, the eulerian form of the stress power expression is

$$V(\mathbf{r}) = \mathbf{tr}(\boldsymbol{\sigma}\boldsymbol{\Sigma}). \quad (60)$$

We are mostly interested in the mechanical energy in the current configuration, however, it is worthwhile to note that the equivalent relation in the Lagrangean form is

$$\begin{aligned} \int_{\mathcal{B}_o} \rho_o \mathbf{b}_o \dot{\chi} dV + \int_{\partial \mathcal{B}_o} (\boldsymbol{\sigma}^T \mathbf{N}) \dot{\chi} dA = \\ \frac{d}{dt} \int_{\mathcal{B}_o} \frac{1}{2} \rho_o \dot{\chi} \cdot \dot{\chi} dV + \int_{\mathcal{B}_o} \mathbf{tr}(\boldsymbol{\sigma} \dot{\mathbf{F}}) dV. \end{aligned} \quad (61)$$

It follows that,

$$V(\mathbf{r}) = \mathbf{tr}(\boldsymbol{\sigma} \dot{\mathbf{F}}) \quad (62)$$

for a Green elastic material. Similar to the arguments in § 5.2, we find the torque as (see derivation in Appendix B)

#### Green-Elastic IAB Material Torque

$$\begin{aligned} \boldsymbol{\tau} = & \frac{\rho \ddot{r}}{R^2} + \rho \ddot{\alpha} + \frac{\rho \dot{r}^2}{R^3} + 4C_1 \left( \frac{2R^3}{r^5} + \frac{r}{R^3} \right) \\ & + 4C_2 \left( \frac{2r^3}{R^5} + \frac{R}{r^3} \right). \end{aligned} \quad (63)$$

## 6 Statics and Kinetics

### 6.1 End Effector Forces

From the derived relationship between the load contact coordinates and the relative motion  $(v_t, \omega_t)$  of each IAB *i.e.* equation (46), we can associate a Jacobian that maps IAB velocities to load pose. A fundamental assumption in our formulation is that the IABs make contact with the load throughout manipulation, and the manipulation is stable and prehensile. A forward kinematic map from the configuration of the  $i^{th}$

IAB,  $\chi_{iab_i}$  maps from respective IAB configurations to load position and orientation *i.e.*  $K_{iab_i} : \chi_{iab_i} \rightarrow SE(3)$ . The velocity of the load with respect to a fixed base frame in terms of IAB velocities can be written in terms of the forward kinematics Jacobian:

$$\begin{pmatrix} v_{iab_i} \\ \omega_{iab_i} \end{pmatrix} = \frac{\partial K_{iab_i}}{\partial \mathbf{r}_i} \frac{d\mathbf{r}_i}{dt} K_{iab_i}^{-1} = \mathbf{J}_i(\mathbf{r}_i) \dot{\mathbf{r}}_i \quad (64)$$

where  $\mathbf{r}_i$  is the spatial position of IAB  $i$ , and  $(v_{iab_i}^T, \omega_{iab_i}^T) \in \mathbb{R}^6$  represents the linear and angular velocity of the  $i^{th}$  IAB about its screw basis. In essence,  $\mathbf{r}_i \in \mathbb{R}^3$  with its rows of mapped to scalars by an appropriate choice of norm. The contact between the head and the IABs is mapped by the Jacobian

$$\mathbf{J}_{c_i}(\xi_h, \xi_{iab_i}) = \begin{bmatrix} \mathbf{I} & \hat{\mathbf{w}}(r_{c_i}) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} J_{r_i}, \quad (65)$$

where  $\mathbf{J}_{c_i} : \xi_{r_i} \rightarrow [v_{c_i}^T, w_{c_i}^T]^T$ ,  $r_{c_i} \in \mathbb{R}^3$  is a vector between the head reference point (e.g. the center of mass) and the contact with the  $i^{th}$  IAB,  $\xi_h$  is the position and relative orientation of the head,  $\xi_{iab_i}$  is the position and relative orientation of the  $i^{th}$  soft robot in world coordinates,  $\hat{\mathbf{w}}(r_{c_i})$  is an anti-symmetric matrix for the vector  $r_{c_i}$ , and  $\xi_r = (\xi_{r_1}, \xi_{r_2}, \dots, \xi_{r_8})$  are the positions and orientations for each of the 8 IABs. The manipulation map,  $G_i$  is made up of matrices of the form

$$G_i(\xi_h, \xi_r) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{w}}(r_{c_i}) & \mathbf{I} \end{bmatrix} B_i(\xi_h, \xi_r), \quad (66)$$

where  $B_i(\xi_h, \xi_r)$  is the selection map as defined in [46] for the desired manipulation. The net force on the head is a sum of the individual forces arising from each IAB. Owing to the linearity of each individual IAB's contact force, the resultant head force can be stitched together to form  $G$ , *i.e.*

$$\tilde{F}_h = [G_1, \dots, G_8] \begin{pmatrix} \tilde{F}_{c_1} \\ \vdots \\ \tilde{F}_{c_8} \end{pmatrix} = G \tilde{F}_c, \quad (67)$$

where  $F_h \in \mathbb{R}^6$  and  $F_c \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \dots \times \mathbb{R}^{m_8}$ . The *internal* or *null forces* is captured by the null space  $\mathcal{N}(G)$  of the manipulation map  $G$ ; these forces correspond to zero net force on the head of the patient. Each  $\tilde{F}_{c_i}$  in (67) is of the form (39).

### 6.2 End-effector Velocities

We define the velocity constraint dual of (66) as the constraint between the relative velocity of the head and that of the twist velocities of the contact point

$$\begin{pmatrix} \tilde{v}_{c_i} \\ \tilde{\omega}_{c_i} \end{pmatrix} = \begin{bmatrix} \mathbf{I} & \hat{\mathbf{w}}(r_{c_i}) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{pmatrix} v_{c_h} \\ \omega_{c_h} \end{pmatrix}. \quad (68)$$

<sup>5</sup>An hyperelastic material is one where the strain-energy function exists.

For a conjugate twist vector  $(v_c^T, \omega_c^T)^T$  to the forces exerted by the IABs,  $f_c$ , we have the following

$$\begin{pmatrix} v_c \\ \omega_c \end{pmatrix} = G^T \begin{pmatrix} v_{c_h} \\ \omega_{c_h} \end{pmatrix}. \quad (69)$$

Given a *selection matrix*  $B_i^T(\xi_h, \xi_{iab_i}) \in \mathbb{R}_i^m$  for a particular IAB, where  $m_i$  is the range of all the forces and moments for the chosen contact primitive (or union of contact primitives), the *manipulation map* for the  $i^{th}$  IAB can be written as,

$$G_i^T(\xi_h, \xi_{iab_i})\xi_h = B_i^T(\xi_h, \xi_{iab_i})\mathbf{J}_{c_i}(\xi_h, \mathbf{r}_{r_i})\dot{\xi}_{iab_i} \quad (70)$$

where  $\mathbf{J}_{c_i}$  is the contact Jacobian for the  $i^{th}$  actuator, and  $\xi_h$  denotes the velocity of the head. For the 8 soft actuators, the manipulation constraint of the system can be written as

$$\begin{bmatrix} G_1^T \\ G_2^T \\ \vdots \\ G_8^T \end{bmatrix} \begin{pmatrix} v_h \\ w_h \end{pmatrix} = \text{diag} \begin{pmatrix} B_1^T \mathbf{J}_{c_1} \\ B_2^T \mathbf{J}_{c_2} \\ \vdots \\ B_8^T \mathbf{J}_{c_8} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{r}}_{iab_1} \\ \dot{\mathbf{r}}_{iab_2} \\ \vdots \\ \dot{\mathbf{r}}_{iab_8} \end{pmatrix}. \quad (71)$$

## 7 Newton-Euler System of Equations

The dynamics of the head is a form of (59) but without the actuator torques. In local coordinates, it has the form

$$\mathbf{M}_h(\zeta)\ddot{\zeta} + \mathbf{C}_h(\zeta, \dot{\zeta})\dot{\zeta} + \mathbf{N}_h(\zeta, \dot{\zeta}) = 0 \quad (72)$$

with  $\zeta$  being a local parameterization of the position and orientation of the head in the Lie Group  $SE(3)$ , and  $\mathbf{N}_h$  being the gravitational and frictional forces exerted by/on the head. The head and the multi-DOF IAB system are connected via the manipulation constraint *i.e.*

$$G^T(\zeta, \mathbf{r})\dot{\zeta} = \mathbf{J}(\zeta, \mathbf{r})\dot{\mathbf{r}}. \quad (73)$$

Suppose that the velocity constraint produces a virtual displacement constraint in  $\delta\zeta$  and  $\delta\mathbf{r}$  such that for  $q = (\zeta, \mathbf{r})$ , we have

$$\delta\mathbf{r} = \mathbf{J}^{-1}(q)G^T(q)\delta\zeta$$

the Lagrange equations become

$$\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - (\boldsymbol{\tau}, 0) \right) \delta q = 0 \quad (74a)$$

$$\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{\partial L}{\partial \mathbf{r}} - \boldsymbol{\tau} \right)^T \begin{pmatrix} \delta\mathbf{r} \\ \delta\zeta \end{pmatrix} = 0 \quad (74b)$$

$$\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{\partial L}{\partial \mathbf{r}} - \boldsymbol{\tau} \right) \delta\mathbf{r} + \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\zeta}} - \frac{\partial L}{\partial \zeta} \right) \delta\zeta = 0 \quad (74c)$$

$$GJ^{-T} \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{\partial L}{\partial \mathbf{r}} - \boldsymbol{\tau} \right) \delta\zeta + \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\zeta}} - \frac{\partial L}{\partial \zeta} \right) \delta\zeta = 0 \quad (74d)$$

wherefore,

$$\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\zeta}} - \frac{\partial L}{\partial \zeta} \right) \delta\zeta + GJ^{-T} \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{\partial L}{\partial \mathbf{r}} \right) = GJ^{-T} \boldsymbol{\tau} \quad (75)$$

given the arbitrariness of  $\delta\zeta$ . Equations (75) alongside (73) completely describe the system dynamics. Putting (58) into (75), we have

$$\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\zeta}} - \frac{\partial L}{\partial \zeta} \right) \delta\zeta = GJ^{-T} \left( 1 - \frac{\rho}{2\|\mathbf{r}\|^2} \right) \boldsymbol{\tau}. \quad (76)$$

## 8 Conclusions

We have presented the constitutive model of these new class of soft actuators and we have presented the kinematic and dynamic equations that govern the behavior of the soft mechanism under mechanical loads. Future work will integrate the hardware for the tasks so described herein. Furthermore, in a follow-up paper (to be released shortly), we prescribe a trajectory optimization algorithm that uses the presented model on several manipulation and control tasks.

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## A Deformation Gradient Derivation

It can be verified that the orthonormal basis vectors for (9) are

$$\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} / |\frac{\partial \mathbf{r}}{\partial r}| = \begin{bmatrix} \cos \theta \sin \phi \\ \sin \theta \sin \phi \\ \cos \phi \end{bmatrix}$$

$$\mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} / |\frac{\partial \mathbf{r}}{\partial \phi}| = \begin{bmatrix} \cos \theta \cos \phi \\ \sin \theta \cos \phi \\ -\sin \phi \end{bmatrix}, \mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} / |\frac{\partial \mathbf{r}}{\partial \theta}| = \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix}.$$

From (12), we can write

$$\begin{aligned} \mathbf{F} = & \frac{\partial(\mathbf{dx}_r)}{\partial R} \mathbf{e}_r \otimes \mathbf{e}_R + \frac{1}{R} \frac{\partial}{\partial \Phi} (\mathbf{dx}_r \mathbf{e}_r) \otimes \mathbf{e}_\Phi \\ & + \frac{1}{R \sin \Phi} \frac{\partial(\mathbf{dx}_r \mathbf{e}_r)}{\partial \Theta} \otimes \mathbf{e}_\Theta + \frac{\partial}{\partial R} (\mathbf{dx}_\phi \mathbf{e}_\phi) \otimes \mathbf{e}_R \\ & + \frac{1}{R} \frac{\partial}{\partial \Phi} (\mathbf{dx}_\phi \mathbf{e}_\phi) \otimes \mathbf{e}_\Phi + \frac{1}{R \sin \Phi} \frac{\partial}{\partial \Theta} (\mathbf{dx}_\phi \mathbf{e}_\phi) \otimes \mathbf{e}_\Theta \\ & + \frac{\partial}{\partial R} (\mathbf{dx}_\theta \mathbf{e}_\theta) \otimes \mathbf{e}_R + \frac{1}{R \sin \Phi} \frac{\partial}{\partial \Theta} (\mathbf{dx}_\theta \mathbf{e}_\theta) \otimes \mathbf{e}_\Theta \end{aligned} \quad (77)$$

whereupon,

$$\begin{aligned} \mathbf{F} = & \frac{\partial(\mathbf{dx}_r)}{\partial R} \mathbf{e}_r \otimes \mathbf{e}_R + \frac{1}{R \sin \Phi} \frac{\partial(\mathbf{dx}_r)}{\partial \Theta} \mathbf{e}_r \otimes \mathbf{e}_\Theta \\ & + \mathbf{dx}_r \frac{\sin \phi}{R \sin \Phi} \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \frac{\mathbf{dx}_r}{R} \mathbf{e}_\phi \otimes \mathbf{e}_\Phi + \frac{1}{R} \frac{\partial(\mathbf{dx}_r)}{\partial \Phi} \mathbf{e}_r \otimes \mathbf{e}_\Phi \\ & + \frac{\partial(\mathbf{dx}_\phi)}{\partial R} \mathbf{e}_\phi \otimes \mathbf{e}_R - \frac{\mathbf{dx}_\phi}{R} \mathbf{e}_r \otimes \mathbf{e}_\Phi + \frac{1}{R} \frac{\partial(\mathbf{dx}_\phi)}{\partial \Phi} \mathbf{e}_\phi \otimes \mathbf{e}_\Phi \\ & + \cos \phi \frac{\mathbf{dx}_\phi}{R \sin \Phi} \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \frac{1}{R \sin \Phi} \frac{\partial(\mathbf{dx}_\phi)}{\partial \Theta} \mathbf{e}_\phi \otimes \mathbf{e}_\Theta \\ & + \frac{\partial(\mathbf{dx}_\theta)}{\partial R} \mathbf{e}_\theta \otimes \mathbf{e}_R + \frac{1}{R} \frac{\partial(\mathbf{dx}_\theta)}{\partial \Phi} \mathbf{e}_\theta \otimes \mathbf{e}_\Phi + \frac{1}{R \sin \Phi} \times \dots \\ & \frac{\partial(\mathbf{dx}_\theta)}{\partial \Theta} \mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \frac{\mathbf{dx}_\theta}{R \sin \Phi} (-\mathbf{e}_r \sin \phi - \mathbf{e}_\phi \cos \phi) \otimes \mathbf{e}_\Theta \end{aligned} \quad (78)$$

where we have made use of the chain rule, (10), together with the identities,

$$\frac{\partial \mathbf{e}_r}{\partial r} = \frac{\partial \mathbf{e}_\phi}{\partial r} = \frac{\partial \mathbf{e}_\theta}{\partial r} = \frac{\partial \mathbf{e}_\theta}{\partial \phi} = 0, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_r, \quad \frac{\partial \mathbf{e}_r}{\partial \phi} = \mathbf{e}_\phi, \quad (79)$$

$$\begin{aligned} \frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta \sin \phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \theta} = \mathbf{e}_\theta \cos \phi, \\ \frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -(\mathbf{e}_r \sin \phi + \mathbf{e}_\phi \cos \phi). \end{aligned} \quad (80)$$

in carrying out the partial derivatives of (78). In matrix form, we rewrite (78) as

$$\mathbf{F} = \begin{pmatrix} \frac{\partial(\mathbf{dx}_r)}{\partial R} & \frac{\partial(\mathbf{dx}_r)}{\partial \Phi} - \frac{\mathbf{dx}_\phi}{R} \\ \frac{\partial(\mathbf{dx}_\phi)}{\partial R} & \frac{\mathbf{dx}_r}{R} + \frac{1}{R} \frac{\partial(\mathbf{dx}_\phi)}{\partial \Phi} \\ \frac{\partial(\mathbf{dx}_\theta)}{\partial R} & \frac{1}{r} \frac{\partial \mathbf{dx}_\theta}{\partial \phi} \\ & \frac{1}{R \sin \Phi} \frac{\partial(\mathbf{dx}_r)}{\partial \Theta} - \frac{\sin \phi}{R \sin \Phi} \mathbf{dx}_\theta \\ & \frac{1}{R \sin \Phi} \frac{\partial(\mathbf{dx}_\phi)}{\partial \Theta} - \frac{\cos \phi}{R \sin \Phi} \mathbf{dx}_\theta \\ & \frac{1}{R \sin \Phi} \left( \frac{\partial(\mathbf{dx}_\theta)}{\partial \Theta} + \cos \phi \mathbf{dx}_\phi + \sin \phi \mathbf{dx}_r \right) \end{pmatrix} \quad (81)$$

which as a result of (10), becomes

$$\mathbf{F} = \begin{pmatrix} \frac{\partial(\mathbf{dx}_r)}{\partial R} & \frac{\partial(\mathbf{dx}_r)}{\partial \Phi} - \frac{\mathbf{dx}_\phi}{R} \\ \frac{\partial(\mathbf{dx}_\phi)}{\partial R} & \frac{\mathbf{dx}_r}{R} + \frac{1}{R} \frac{\partial(\mathbf{dx}_\phi)}{\partial \Phi} \\ \frac{\partial(\mathbf{dx}_\theta)}{\partial R} & \frac{1}{r} \frac{\partial \mathbf{dx}_\theta}{\partial \phi} \\ & \frac{1}{R \sin \Phi} \frac{\partial(\mathbf{dx}_r)}{\partial \Theta} - \frac{\mathbf{dx}_\theta}{R} \\ & \frac{1}{R \sin \Phi} \frac{\partial(\mathbf{dx}_\phi)}{\partial \Theta} - \cot \phi \frac{\mathbf{dx}_\theta}{R} \\ & \frac{\mathbf{dx}_\phi}{R} \cot \phi + \frac{\mathbf{dx}_r}{R} + \frac{1}{R \sin \Phi} \frac{\partial(\mathbf{dx}_\theta)}{\partial \Theta} \end{pmatrix} \quad (82)$$

or

$$\mathbf{F} = \begin{pmatrix} \frac{R^2}{r^2} & -\frac{\phi}{R} & -\frac{\theta}{R} \\ 0 & \frac{r}{R} + \frac{1}{R} & -\frac{\theta}{R} \cot \phi \\ 0 & 0 & \frac{r}{R} + \frac{\phi}{R} \cot \phi + \frac{1}{R \sin \Phi} \end{pmatrix}. \quad (83)$$

## B Robot-Load Kinetics

We now derive the overall dynamics for the elastic IAB in Eulerian form. Following (22), a point on the surface of the IAB has the following description

$$\mathbf{r} = \begin{bmatrix} \lambda \\ \phi \end{bmatrix} = \begin{bmatrix} r/R \\ \beta - \alpha \end{bmatrix} \quad (84)$$

so that the Eulerian time differentiation of  $\mathbf{r}$  yields

$$\dot{\mathbf{r}} = [\dot{\lambda}, \dot{\phi}]^T = [\dot{r}/R, -\dot{\alpha}]^T \quad (85)$$

which follows since  $\dot{R} = \dot{\beta} = 0$  in the reference configuration. Similarly, we find that  $\ddot{\mathbf{r}}$  is

$$\ddot{\mathbf{r}} = [\ddot{\lambda}, \ddot{\phi}]^T = [\ddot{r}/R, -\ddot{\alpha}]^T \quad (86)$$

Recall the kinetic energy form of a continuum body (§ 5)

$$T = \frac{1}{2} \rho \mathbf{v}(\mathbf{r}, t) \cdot \mathbf{v}(\mathbf{r}, t) = \frac{1}{2} \rho \|\dot{\mathbf{r}}\|^2. \quad (87)$$

Given the incompressibility of the IAB material body, the material mass density is uniform throughout the body at a configuration so that the rate of change of the body mass,  $\rho$ , vanishes.

### B.1 Case I: Cauchy Elastic IAB Material Skins

Suppose we choose a Cauchy Elastic material so that the constitutive equation that governs the Cauchy stress tensor,  $\boldsymbol{\sigma}$ , is independent of the path of the deformation from the reference configuration but is solely a function of the state of deformation. Then, it follows that  $V = 0$ . We have

$$T = \frac{1}{2} \rho \|\dot{\mathbf{r}}\|^2 = \frac{1}{2} \rho (\dot{r}^2/R^2 + \dot{\alpha}^2), \quad V = 0. \quad (88)$$

It follows that the Lagrangian is

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} \rho \|\dot{\mathbf{r}}\|^2 = \frac{1}{2} \rho (\dot{r}^2/R^2 + \dot{\alpha}^2) \quad (89)$$

and the derivatives of the canonical momenta are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{d}{dt} \left( \frac{\rho \dot{r}}{R^2} \right) = \rho \left( \frac{\ddot{r}}{R^2} - 2 \frac{\dot{r} \dot{R}}{R^3} \right) \equiv \frac{\rho}{R^2} \ddot{r} \quad (90a)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt} (\rho \ddot{\alpha}) = \rho \ddot{\alpha}, \quad (90b)$$

where (90a) follows from the fact that the radius is constant in the reference configuration. We therefore have the following associated generalized forces

$$\frac{\partial L}{\partial r} = 0, \frac{\partial L}{\partial R} = -\frac{\rho \dot{r}^2}{R^3} \text{ and } \frac{\partial L}{\partial \phi} = 0. \quad (91a)$$

Recalling the Euler-Lagrange equation from (51), we may write the torque that governs the  $j'th$  IAB as (we have dropped the  $j'th$  index)

$$\boldsymbol{\tau} = \frac{\rho \ddot{r}}{R^2} + \rho \ddot{\alpha} + \frac{\rho \dot{r}^2}{R^3} \quad (92)$$

and in matrix form for all the system of IABs, we have

$$\boldsymbol{\tau} = \underbrace{\begin{bmatrix} \rho/R^2 & 0 \\ 0 & \rho \end{bmatrix}}_{M_{iab}} \underbrace{\begin{bmatrix} \ddot{r} \\ \ddot{\alpha} \end{bmatrix}}_{\dot{\mathbf{r}}} + \underbrace{\begin{bmatrix} \dot{r} \\ \dot{\alpha} \end{bmatrix}}_T \underbrace{\begin{bmatrix} \rho/R^3 & 0 \\ 0 & 0 \end{bmatrix}}_{C_{iab}} \underbrace{\begin{bmatrix} \dot{r} \\ \dot{\alpha} \end{bmatrix}}_{\dot{\mathbf{r}}} \quad (93)$$

or rewritten compactly as,

$$\boldsymbol{\tau} = M_{iab}(\mathbf{r}) \dot{\mathbf{r}} + \dot{\mathbf{r}}^T C_{iab}(\mathbf{r}, \dot{\mathbf{r}}) \dot{\mathbf{r}} \quad (94)$$

## B.2 Case II: Green Elastic IAB Material Skins

When the stress tensor depends on the strain, we have from (62), that

$$V(\mathbf{r}) = \text{tr}(\boldsymbol{\sigma} \Sigma). \quad (95)$$

The associated force on the head is now a function of the kinetic and potential energies so that we have

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} \rho \|\dot{\mathbf{r}}\|^2 + \text{tr}(\boldsymbol{\sigma} \dot{\mathbf{F}}) \quad (96a)$$

$$= \frac{1}{2} \rho \left( \dot{r}^2 / R^2 + \dot{\alpha}^2 \right) + \text{tr}(\boldsymbol{\sigma} \dot{\mathbf{F}}). \quad (96b)$$

We thus have

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} \rho \left( \dot{r}^2 / R^2 + \dot{\alpha}^2 \right) + \text{tr}(\boldsymbol{\sigma} \dot{\mathbf{F}}) \quad (97)$$

Solving for the derivatives of the kinetic and potential energies as before, we have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\rho}{R^2} \ddot{r}, \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \rho \ddot{\alpha} \quad (98a)$$

with the following associated generalized forces

$$\frac{\partial L}{\partial r} = -4C_2 \left( \frac{2r^3}{R^5} - \frac{R}{r^3} \right) - 4C_1 \left( \frac{2R^3}{r^5} - \frac{r}{R^3} \right), \quad (99a)$$

$$\text{and } \frac{\partial L}{\partial \phi} = 0. \quad (99b)$$

We can now write the torque as

$$\begin{aligned} \boldsymbol{\tau} = & \frac{\rho \ddot{r}}{R^2} + \rho \ddot{\alpha} + \frac{\rho \dot{r}^2}{R^3} + 4C_1 \left( \frac{2R^3}{r^5} + \frac{r}{R^3} \right) \\ & + 4C_2 \left( \frac{2r^3}{R^5} + \frac{R}{r^3} \right) \end{aligned} \quad (100)$$