Robust Policy Optimization in Continuous-time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Stochastic Control

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***Abstract*—Following the recent resurgence in linear quadratic control for establishing theoretical benchmarks for reinforcement leaning-based policy optimization for complex dynamical systems with continuous state and action spaces, an optimal control problem for an infinite-dimensional linear stochastic system possessing additive Brownian motion is optimized on a cost that is an exponent of the quadratic form of the state, input, and disturbance terms. While recent efforts in deterministic LQ regulator and additive Gaussian noise settings abound, our analyses are distinguished by many natural systems characterized by additive Wiener process, amenable to Îto's stochastic differential calculus in dynamic game settings. Over arcs of trajectories that emanate from the phase space , the Hamilton-Jacobi Bellman equation parameterizes trajectories' costs so that with input and state data only, we circumvent the need for system dynamics knowledge. We thus arrive at a reinforced learning-based policy optimization algorithm (enforced via policy iteration) and we prescribe rigorous robustness analyses to the optimal policy via the input-to-state stability formalism. Our approach provides basic insight into principled machine learning based framework for recovering robustly optimal policies in stochastic policy optimization settings.**

***Index Terms*—Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control, Policy Optimization, Robust Reinforcement Learning, Policy Iteration.**

I. Introduction

Lately, various system-theoretic results analyzing the global convergence [1] and computational complexity [2] of nonconvex, constrained [3] gradient-based [4] and derivative-free [5], [6] policy optimization [7] in sampling-based reinforcement learning (RL) when the complete set of decision (or state feedback) variables are not previously known have appeared as control benchmarks [1], [8], [9], [10]. The most basic setting consists in optimizing over a decision variable K which must be determined from a (restricted) class of controllers $\mathcal{K}$ i.e. $\min_{K\in\mathcal{K}} J(K)$ where $J(K)$ is an objective (e.g. tracking error, safety assurance, goal-reaching measure of performance e.t.c.) required to be satisfied. In principle, K can be realized as a linear controller, a linear-in-the-parameters polynomial, or as a nonlinear kernel in the form of a radial basis function, or neural network.

These policy optimization (PO) schemes apply to a broad range of problems and have enjoyed wide success in complex systems where analytic models are difficult to derive [11], [12]. While they have become a popular tool for modern learning-based control [13], [14], the theoretical underpinning of their convergence, sample complexity, and robustness guarantees are little understood *in the large*. Only recently have rigorous analyses tools emerged [9], [15] for benchmarking RL with deterministic and additive Gaussian disturbance linear quadratic (LQ) controllers [16], [17], [18].

Tools for analyzing the convergence, sample complexity, or robustness of RL-based PO largely fall into one of infinite-horizon (i) discrete-time LQ regulator (LQR) settings [18]; (ii) discrete-time LQ problems under multiplicative noise [4]; or (iii) Zames' risk-sensitive [19] $\mathcal{H}_\infty$-control [20], [21] and discrete- and continuous-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design [22], [23], [3], [9] where the upper bound on the $\mathcal{H}_2$ cost is minimized subject to satisfying a set of risk-sensitive (often $\mathcal{H}_\infty$) constraints that *attenuate* [24], [25], [26] an unknown disturbance.

We focus on continuous-time linear systems in which disturbances enter additively as random stochastic Wiener processes following recent efforts on policy optimization for LQ regulator problems [18]; these systems may be modeled more accurately with uncertain additive Brownian noise. Here, diffusion processes modeled with Îto's stochastic calculus are the theoretical machinery for analysis. Prominent linear systems featuring additive Wiener processes occur in economics and finance [27], [28], stock options trading [29], protein kinetics, population growth models, dynamics of murmurations [30], and models involving computations with round-off error in floating point arithmetic calculations such as overparameterized neural network.

Our goal is to keep a controlled process, z, small in an infinite-horizon constrained optimization setting under a minimizing policy $\pi\in\Pi$ in spite of unforeseen additive vector-valued stochastic Brownian process $w(t)\in\mathbb{R}^q$ — which may be of large noise intensity. In terms of the L_2 norm, we can write $\|z\|_2 = \left(\int |z(t)|^2 dt\right)^{1/2}$. The associated performance criteria can be realized as minimizing the expected value of the risk-sensitive linear exponential functions of positive definite quadratic forms state and control variables

$$\begin{aligned}\min_{\pi\in\Pi} \mathcal{J}_{exp}(x_0,\pi) :=& \mathbb{E}\Big|_{x_0\in\mathcal{P}_0} \exp\left[\frac{\alpha}{2}\int_0^\infty z^\top(t)z(t)\mathrm{d}t\right],\ \alpha>0\\ \text{subject to } \mathrm{d}x(t) &= Ax(t)\mathrm{d}t + Bu(t)\mathrm{d}t + D\mathrm{d}w(t),\\ z(t) &= Cx(t)+Eu(t)\end{aligned} \tag{1}$$

with state process $x\in\mathbb{R}^n$, output process $z\in\mathbb{R}^p$ to be controlled, and control input $u\in\mathbb{R}^m$. The derivative of $w(t)\in\mathbb{R}^v$ *i.e.* dw/dt is a zero-mean Gaussian white noise with variance W, and $x(0)$ is a zero-mean Gaussian random vector independent of $w(t)$, $z(0)=0$, and $A\in\mathbb{R}^{n\times n}$,

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$B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{n\times q}$, and $E \in \mathbb{R}^{p\times m}$ are constant matrix functions. The random signal $x(0)$ and the process $w(t)$ are defined over a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Suppose that we carry out a Taylor series expansion about $\alpha = 0$ in (1), the variance term, $\alpha^2 \texttt{var}(\int_0^\infty z^\top z)$, will be small after minimization. Then α can be seen as a measure of *risk-aversion* if $\alpha > 0$. It is important to note that in this paper, we only consider state feedback. In particular when noise is present in the system, the value of α signifies the level of noise attenuation that penalizes the covariance matrix of the system's noise.

We adopt an adaptive policy iteration (PI) method in a continuous PO scheme. This can be seen as an instance of the actor-critic (AC) configuration in RL-based neural network *online* policy optimization schemess. Without explicit access to internal dynamics (system matrices), we must iterate between steps of policy evaluation and policy improvement. Mimicking the actor in RL AC settings, a parameterized controller must be evaluated relative to a parameterized cost function (the critic). The new policy is then used to improve the erstwhile (actor) policy by aiming to drive the cost to an extremal on the overall.
Contributions: We focus on the more sophisticated case of optimizing an *unknown stochastic linear policy* class $\mathcal{K}$ in an *infinite-horizon* LQ cost setting such that optimization iterates enjoy the *implicit regularization (IR) property* [10]—satisfying $\mathcal{H}_\infty$ robustness constraints. We place PO for *continuous-time linear stochastic controllers* on a rigorous *global convergence* and *robustness* footing. This is a distinguishing feature of our work. Our contributions are stated below.

- Reiterating the usual connection between risk-sensitive policy design and $\mathcal{H}_\infty$ zero-sum two-person dynamic differential games, we propose a *two-loop iterative alternating best-response procedure* for computing the optimal mixed-design policy – that accelerates the optimization scheme's convergence better than [4], [18], [8], [10] – in model- and sampling-based (i.e. model-free) cases;
- Rigorous local, global, and uniform convergence analyses follow for the loop updates in model-based settings;
- When the system model is not known, we provide a robust PO algorithm which can be interpreted as a hybrid of a system possessing nonlinear dynamics but with discrete-time policy iterates. Its robustness is analyzed in an input-to-state (ISS) framework, analyses to perturbations and uncertainties for the loop updates.
- Lastly, we benchmark our results against the natural policy gradient [31] (which enjoys the IR property [10]) in the spirit of recent system-theoretic analysis works [18], [8], [32], [3].

Within the limits of our knowledge, we are not aware of other rigorous convergence and robustness analyses for the PO algorithms we that present.
Notations: The set of all symmetric matrices with dimension n is $\mathbb{S}^n$ and $\mathbb{R}$ (resp. $\mathbb{N}_+$) is the set of real numbers (positive integers). The Kronecker product is denoted by $\otimes$. The Euclidean (Frobenius) norm of a vector or the spectral norm of a matrix is $\|\cdot\|$ ($\|\cdot\|_F$). Let $\|\cdot\|_\infty$ denote the supremum norm of a matrix-valued signal, i.e. $\|\Delta\|_\infty = \sup_{s\in\mathbb{Z}_+}\|\Delta_s\|_F$. The open ball of radius δ is $\mathcal{B}_\delta(X) = \{Y \in \mathbb{R}^{m\times n} | \|Y - X\|_F < \delta\}$. The maximum and minimum singular values (eigenvalues) of a matrix A are respectively denoted by $\bar{\sigma}(A)$ ($\bar{\lambda}(A)$) and $\sigma_{min}(A)$ ($\lambda_{min}(A)$). The eigenvalues of $A \in \mathbb{R}^{n\times n}$ are $\lambda_i(A)$ for $i = 1, 2, \cdots, n$. For the transfer function $G(s)$, its $\mathcal{H}_\infty$ norm is $\|G\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \bar{\sigma}(G(j\omega))$.

The n-dimensional identity matrix is I_n. Denote by x_{ij} the (ij)'th entry of $X \in \mathbb{R}^{m\times n}$ and by x_i the i'th element of $x \in \mathbb{R}^n$. The full vectorization of $X \in \mathbb{R}^{m\times n}$ is the $mn \times 1$ vector, $\texttt{vec}(X) = [x_{11}, x_{21}, \cdots, x_{m1}, x_{12}, \cdots, x_{m2}, \cdots, x_{mn}]^\top$. Let $P \in \mathbb{S}^n$, then the half-vectorization of P is the $n(n+1)/2$ column vector as a result of a vectorization of upper-triangular part of P i.e. $\texttt{svec}(P) = [p_{11}, \sqrt{2}p_{12}, \cdots, \sqrt{2}p_{1n}, p_{22}, \cdots, \sqrt{2}p_{n-1,n}, p_{nn}]^\top$. The vectorization of the dot product $\langle x, x^\top\rangle$, where $x \in \mathbb{R}^n$, is $\texttt{vecv}(x) = [x_1^2, \cdots, x_1x_n, x_2^2, x_2x_3, \cdots, x_n^2]^\top$. The inverse of $\texttt{vec}(x)$ and $\texttt{svec}(y)$ are respectively the full and symmetric matricizations: $\texttt{mat}_{m\times n}(x) = \left(\texttt{vec}(I_n)^\top \otimes I_m\right)(I_n \otimes x)$, and $\texttt{smat}_m(P)$ so that $\texttt{smat}(\texttt{svec}(p)) = P$. Here, $x \in \mathbb{R}^{mn}$ and $y \in \mathbb{R}^{m(m+1)/2}$ for $n, m \in \mathbb{R}_{\geq 0}$. Finally, we denote by $T_{\texttt{vec}}(A)$ the vectorization of $A^\top$ i.e. $\texttt{vec}(A^\top) = T_{\texttt{vec}}(\texttt{vec}(A))$.
Paper Structure: The rest of this article is structured as follows. LEQG connections to dynamic games are briefly established in Section II. In Section III, we present a double-loop PI procedure for robust policy recovery in a mixed $\mathcal{H}_2$/$\mathcal{H}_\infty$ PO settings (both in model-free and model-based optimization settings); this is followed by a rigorous analysis of their convergence and robustness properties. We demonstrate the efficacy of our proposed algorithm on numerical examples, discuss findings, and draw conclusions in Section IV.

II. PO: Dynamic Games Connection

In this section, we connect PO under linear controllers to the theory of two-person dynamic games. Let us first impose conditions to make the problem introduced in (1) amenable to our analysis.

Assumption 1 : *We take $C^\top C \triangleq Q \succ 0$, $E^T(C, E) = (0, R)$ for some matrix-valued function $R \succ 0$; and since in (1), we want $w(t)$ to be statistically independent, we take DD^T = 0. Seeing we are seeking a linear feedback controller for (1), we require that the pair (A, B) be stabilizable. We expect to compute solutions via an optimization process, therefore we require that unstable modes of A must be observable through Q. Whence $(\sqrt{Q}, A)$ must be detectable.*

Given Assumption 1, the LEQG cost functional now becomes

$$\mathcal{J}_{exp}(x_0, \pi) = \mathbb{E}\Big|_{x_0\in\mathcal{P}_0} \exp\left[\frac{\alpha}{2}\int_0^\infty \left(x^\top(t)Qx(t) + u^\top(t)Ru(t)\right)\mathrm{d}t\right], \tag{2}$$

for a fixed $\alpha > 0$ and the closed-loop transfer function is

$$T_{zw}(K) = (C - EK)(sI - A + BK)^{-1}D. \tag{3}$$

The set of all *suboptimal* control policies that robustly stabilizes the linear system against all (finite gain) stable perturbations Δ, interconnected to the system by $w = \Delta z$, such that $\|\Delta\|_\infty \leq 1/\gamma$ is

$$\mathcal{K} = \{\, K:\ \lambda_i(A - B_1 K) < 0,\ \|T_{zw}(K)\|_\infty < \gamma\}. \tag{4}$$

Observe: Cost (2), without the log term introduced in [6], mitigates against gradient bias in PG-based derivative-free optimization as we will show in subsequent sections — this objective provably converges to the optimal solution. We now establish the optimal control gain matrix for the LEQG problem.

Proposition 1 : *[33, Th. II.1] The optimal control to the LEQG optimization problem (1) and cost functional (2) under π in the infinite-horizon setting is of a linear-in-the-data form i.e. $u^\star(t) = -K^\star_{leqg}\hat{x}(t)$ where gain $K^\star_{leqg} = R^{-1}B^\top P_\tau$, and P_τ is the unique, symmetric, positive definite solution to the algebraic Riccati equation (ARE)*

$$A^\top P_\tau + P_\tau A - P_\tau(BR^{-1}B^\top - \alpha^{-2}DD^\top)P_\tau = -Q. \tag{5}$$

Corollary 1 : *In the infinite-horizon time-invariant case with constant system matrices and a stabilizable (A, B), by the theorem on "limit of monotonic operators" [34] and [20, Theorem 9.7], we find that $P^\star \triangleq P_\infty = \lim_{\tau\to\infty} P_\tau$, and $K^\star_{leqg} \triangleq K_\infty = \lim_{\tau\to\infty} K_\tau$.*

Remark 1 : *It is well-known by now that directly solving the linear exponential quadratic Gaussian problem (1) in policy-gradient frameworks incurs biased gradient estimates during iterations; this may affect the preservation of risk-sensitivity in infinite-horizon LTI settings (see [6], [10]). As such, we introduce a workaround with an equivalent dynamic game formulation to the stochastic LQ PO control problem in what follows.*

Lemma 1 (Closed-loop Two-Player Game Connection): *Consider the parameterized soft-constrained upper value, with a stochastically perturbed noise process $w(t)$, which enters the system dynamics as an additive bounded Gaussian with known statistics*[1]*,*

$$\begin{aligned} \min_{u\in\mathcal{U}}\max_{\xi\in W} \bar{\mathcal{J}}_\gamma(x_0, u, \xi) := \mathbb{E}\Big|_{x_0\sim\mathcal{P}_0,\,\xi(t)} \int_0^\infty \big[x^\top(t)Qx(t) + \\ u^\top(t)Ru(t) - \gamma^2\xi^\top(t)\xi(t)\big]\,\mathrm{d}t \\ \textit{subject to } \mathrm{d}x(t) = Ax(t)\mathrm{d}t + Bu(t)\mathrm{d}t + D\xi(t), \\ z(t) = Cx(t) + Eu(t) \end{aligned} \tag{6}$$

with $\xi(\equiv dw)$ as the zero-mean Gaussian noise with variance W(equivalent to dw/dt in (1)), scalar $\gamma > 0$ denoting the level of disturbance attenuation, and x_0 an arbitrary initial state. Suppose that there exists a non-negative definite (nnd) solution of (5) (with α replaced by γ), then its minimal realization, $P^\star$, exists. If $(A, Q^{\frac{1}{2}})$ is observable, then every nnd solution $P^\star$ of (5) is positive definite. For a nnd $P^\star$, there exists a common upper and lower value for the game and if $\bar{\mathcal{J}}_\gamma$ is finite for some $\gamma := \hat{\gamma} > 0$, then $\bar{\mathcal{J}}_\gamma$ is bounded (if and only if the pair (A, B) is stabilizable) and equivalent to the lower value $\underline{\mathcal{J}}_\gamma$[2]*. In addition, for a bounded $\bar{\mathcal{J}}_\gamma$ for some $\gamma = \hat{\gamma}$ and for optimal gain matrices, $K^\star = R^{-1}B^\top P_{K,L}$, $L^\star = \gamma^{-2}D^\top P_{K,L}$, $\bar{\mathcal{J}}_\gamma$ admits the following Hurwitz feedback matrices for all $\gamma > \hat{\gamma}$*

$$A^\star_K = A - BK^\star,\ A^\star_{K,L} = A^\star_K + DL^\star \tag{7}$$

where the nnd $P_{K,L}$ is the unique solution of (5) for $\gamma > \hat{\gamma}$ in the class of nnd matrices if it renders $A^\star_{K,L}$ Hurwitz. Whence, the saddle-point optimal controllers are

$$u^\star(x(t)) = -K^\star x(t),\ \xi^\star(x(t)) = L^\star x(t). \tag{8}$$

Proof. The proof follows that in [20, Th. 9.7] exactly if we preserve the γ^{-1} term in the ARE of equation 9.31 in [20] and replace it by γ^{-2} as we have here. $\square$

It follows that for any stabilizing control pair (K, L), if (8) is applied to the system in (6), the resulting cost from (6) is $\bar{J}_\gamma = (x_0^\top P_{K,L}x_0)$ [35]. Furthermore, let the state correlation matrix be defined as $\Sigma_{K,L} = \mathbb{E}_{x_0\sim\mathcal{D}}(x^\top(t)P_{K,L}x(t))$. In what follows, we present a double-loop iterative solver for the gains K and L in (8) for finding the saddle point (equivalently Nash Equilibrium) policies (8)

III. Policy Optimization via Policy Iteration

We now present a special case to Kleinman's policy iteration (PI) algorithm [35] via a nested double loop PI scheme when (i) exact models are known; this will provide a barometer for our later analysis when (ii) exact system models are unknown.

Let p and q be indices of nested iterations between updating the closed-loop minimizing player's controller K_p (in an outer loop) and the maximizing player's controller $L_q(K_p)$ (in an inner-loop) for $p = \{1, 2, \ldots, \bar{p}\}$ and $q = \{1, 2, \ldots, \bar{q}\}$ for $(\bar{p}, \bar{q}) \in \mathbb{N}_+$. Furthermore, define the identities

$$\begin{aligned} A^p_K &= A - BK_p, \quad A^{p,q}_{K,L} = A^p_K + DL_q(K_p), \\ Q^p_K &= Q + K_p^\top RK_p,\ A^\gamma_K = A^p_K + \gamma^{-2}DD^\top P^p_K. \end{aligned} \tag{9}$$

For the soft-constrained value functional (6) at the p'th iterate of the minimizing controller K we have the following value iteration form for (5),

$$A^{p\top}_K P^p_K + P^p_K A^p_K + Q^p_K + \gamma^{-2}P^p_K DD^\top P^p_K = 0, \tag{10a}$$

$$K_{p+1} = R^{-1}B^\top P^p_K \tag{10b}$$

where P^p_K is the p'th iterate's solution to (10). Similarly, for the maximizing controller, $L_q(K_p)$, the following closed-loop continuous-time ARE (CARE) iteration applies

$$A^{(p,q)^\top}_{K,L} P^{p,q}_{K,L} + P^{p,q}_{K,L}A^{p,q}_{K,L} + Q^p_K - \gamma^2 L_q^\top(K_p)L_q(K_p) = 0 \tag{11a}$$

$$K_{p+1} = R^{-1}B^\top P^{p,q}_K,\ L_{q+1}(K_p) = \gamma^{-2}D^\top P^{p,q}_{K,L} \tag{11b}$$

where $P^{p,q}_{K,L}$ is the solution to (11) for gains $[K_p, L_q(K_p)]$. Choosing a stabilizing minimizing player control gain, K_p

[1]Since the time derivative of a Brownian process $w(t)$ is $dw(t)$, we maximize over the Gaussian $dw(t)$, rather than the unbounded stochastic noise $w(t)$.

[2]The lower value is constructed by reversing the order of play in the value defined in (6).

Algorithm 1: (Model-Based) PO via Policy Iteration

Input: Max. outer iteration $\bar{p}$, $q = 0$, and an $\epsilon > 0$;
Input: Desired risk attenuation level $\gamma > 0$;
Input: Minimizing player's control matrix $R \succ 0$.
1 Compute $(K_0, L_0) \in \mathcal{K}$; ▷ From [36, Alg. 1];
2 Set $P^{0,0}_{K,L} = Q^0_K$; ▷ See equation (9);
3 **for** $p = 0, \ldots, \bar{p}$ **do**
4 Compute Q^p_K and A^p_K ▷ See equation (9);
5 Obtain P^p_K by evaluating K_p on (10);
6 **while** $\|P^p_K - P^{p,q}_{K,L}\|_F \le \epsilon$ **do**
7 Compute $L_{q+1}(K_p) := \gamma^{-2} D^\top P^{p,q}_{K,L}$;
8 Solve (11) until $\|P^p_K - P^{p,q+1}_{K,L}\|_F \le \epsilon$;
9 $\bar{q} \leftarrow q + 1$
10 **end**
11 Compute $K_{p+1} = R^{-1} B^\top P^{p,\bar{q}}_{K,L}$ ▷ See (11b) ;
12 **end**

we first evaluate u's performance by solving (10). *This is the policy evaluation step in PI*. The policy is then improved in a following iteration by solving for the cost matrix in (11b) – *this is the policy improvement step*. The process can thus be seen as *a policy iteration algorithm where the performance of an initial control gain K_p is first evaluated against a cost function. A newer evaluation of the cost matrix $P^{p,q}_{K,L}$ is then used to improve the controller gain K_{p+1} in an outer loop.*

Problem 1 (Model-Based Policy Iteration): *Given system matrices A, B, C, D, E, find the optimal controller gains K_p, $L_q(K_p)$ that robustly stabilizes (1) such that the controller gains do not leave the set of all suboptimal controllers denoted by*

$$\begin{aligned}\breve{\mathcal{K}} = \{(K_p, L_q(K_p)) : \lambda_i(A^p_K) < 0, \lambda_i(A^{p,q}_{K,L}) < 0, \\ \|T_{zw}(K_p, L_q(K_p))\|_\infty < \gamma \text{ for all } (p,q) \in \mathbb{N}\}. \quad (12)\end{aligned}$$

A. Double Loop (DL) Successive Substitution

The procedure for obtaining the optimal $P^\star$ in Problem 1 is described in Algorithm 1. It finds a global Nash Equilibrium (NE) (or equivalently a saddle-point equilibrium) [8] of the LQ zero-sum game (6) by solving the nonlinear ARE (11) in a nested two-loop policy iteration (PI) scheme.

An initial (K_0, L_0) control pair that guarantees the iterates' feasibility upon projection onto the set $\breve{\mathcal{K}}$ is first determined in order to enforce the condition (12). We refer readers to our recent conference paper [36] where this procedure is further elaborated[3]. Afterwards, the Riccati equation's (11) solution i.e. $P^{0,0}_{K,L} \triangleq Q^0_K$ must be computed and the gains $[K_p, L_q(K_p)]$ are updated as in (11b). *As an RL-based PO procedure, Line 11 in Alg. 1 can be seen as a reinforcement over the iterates interval p to $p+1$*. After the last update of the inner loop maximizing player gain $L_{\bar{q}}(K_p)$, the outer-loop update of the minimizing controller gain K_p robustly stabilizes the closed-loop transfer function from w to z i.e. $T_{zw}(K_p, L_q(K_p))$ under gains K_p and $L_{\bar{q}}(K_p)$ against all (finite gain) disturbances Δ interconnected to system (6) (by $w = \Delta z$) such that $\|\Delta\|_\infty \le \gamma^{-2}$.

[3]We remark that this can also be found via linear matrix inequality approaches [37], [38].

B. Outer Loop Stability, Optimality, and Convergence

We now discuss the convergence guarantees of the iterations under perfect dynamics. Let us introduce the following preliminary Lemma to guide the establishment of our results.

Lemma 2 : *Under Assumption 1 and for the ARE (10), if $K_0 \in \mathcal{K}$[4], then for any $p \in \mathbb{N}_+$, we must have the following conditions for the optimal $K^\star$ and $P^\star$,*
(1) $K_p \in \mathcal{K}$;
(2) $P^0_K \succeq P^1_K \succeq \cdots P^p_K \succeq \cdots \succeq P^\star$;
(3) $\lim_{p\to\infty}\|K_p - K^*\|_F = 0$, $\lim_{p\to\infty}\|P^p_K - P^*\|_F = 0$.

Proof of Lemma 2. When $p = 0$, $K_0 \in \mathcal{K}$, and it satisfies (1) (See [36, Alg. 1].) For $p > 0$, introduce the identities,

$$RK_{p+1} = B^\top P^p_K, \qquad K^\top_{p+1} R = P^p_K B, \tag{13a}$$

$$A^{p^\top}_K P^p_K = A^{(p+1)^\top}_K P^p_K + (K_{p+1} - K_p)^\top B^\top P^p_K, \tag{13b}$$

$$P^p_K A^p_K = P^p_K A^{(p+1)}_K + P^p_K B(K_{p+1} - K_p). \tag{13c}$$

Therefore, equation (10) becomes

$$\begin{aligned}&A^{(p+1)^\top}_K P^p_K + P^p_K A^{(p+1)}_K + \gamma^{-2} P^p_K D D^\top P^p_K + C^\top C \quad (14)\\ &+ K^\top_{p+1} R K_{p+1} + (K_{p+1} - K_p)^\top R (K_{p+1} - K_p) = 0.\end{aligned}$$

Thus, for a stabilizing $K_{p+1} (\neq K_p)$ we must have $(K_{p+1} - K_p)^\top R(K_{p+1} - K_p) \succ 0$ so that

$$A^{(p+1)^\top}_K P^p_K + P^p_K A^{(p+1)}_K + \gamma^{-2} P^p_K D D^\top P^p_K + Q^{p+1}_K \prec 0. \tag{15}$$

If (read: since) the inequality (15) holds, the bounded real Lemma [10, Lemma A.1, statement 3] stipulates that a $P^p_K \succ 0$ exists; by [10, Lemma A.1, statement 1], $\|T_{zw}(K_p)\|_\infty < \gamma$ given that $\lambda_i(A^{(p+1)}_K) < 0$ in (15). *A fortiori*, $K_p \in \mathcal{K}$ for $p > 0$ by the bounded real Lemma. This proves the first statement.

The proof for statement (2) now follows. At the $(p+1)$'th iteration, it can be verified that (10) admits the form

$$\begin{aligned}A^{(p+1)^\top}_K P^{p+1}_K + P^{p+1}_K A^{(p+1)}_K + C^\top C + K^\top_{p+1} R K_{p+1} \\ + \gamma^{-2} P^{p+1}_K D D^\top P^{p+1}_K = 0, \quad (16)\end{aligned}$$

so that subtracting (16) from (14) (at the p'th iteration) and using the statistical independence property of the noise term $w(t)$ (from Ass. 1) i.e. $DD^\top = 0$, we have

$$\begin{aligned}A^{(p+1)^\top}_K \left[P^p_K - P^{p+1}_K\right] + \left[P^p_K - P^{p+1}_K\right] A^{(p+1)}_K \\ (K_{p+1} - K_p)^\top R (K_{p+1} - K_p) = 0. \quad (17)\end{aligned}$$

Observe: If we let $\tilde{P}^{p+1}_K = (P^p_K - P^{p+1}_K)$, $\tilde{K}_{p+1} = (K_{p+1} - K_p)$, and $\tilde{Q}^{p+1}_K = \tilde{K}^\top_{p+1} R \tilde{K}_{p+1}$, then (17) is a Lyapunov equation of the form

$$A^{(p+1)^\top}_K \tilde{P}^{p+1}_K + \tilde{P}^{p+1}_K A^{(p+1)}_K + \tilde{Q}^{p+1}_K = 0. \tag{18}$$

Since $A^{(p+1)}_K$ is Hurwitz and satisfies the above Lyapunov equation, we must have $\tilde{P}^{p+1}_K \succeq 0$ because $\tilde{Q}^{p+1}_K \succeq 0$ by

[4]The $\mathcal{K}$ defined here refers to the one defined in (4).

Lemma 13. Whence, $\tilde{P}_K^{p+1} \succeq 0$ implies that $P_K^p \succeq P_K^{p+1}$ and $\tilde{Q}_K^{p+1} \succeq 0$ implies $K_{p+1} \geq K_p$. This proves the second statement. In this sentiment, the sequence $\{P_K^p\}_{p=1}^{\infty}$ is decreasing, bounded below by 0 and has a finite norm so that $\{P_K^p\}_{p=1}^{\infty}$ converges to P_K^{∞}. This satisfies (5). Observe from equation (10) that P_K^p is self-adjoint so that from the "limit of monotonic positive operators theorem" [34, p. 189], $\lim_{p\to\infty}\|P_K^p - P^*\|_F = 0$. By a similar argument for decreasing operators sequences [34, p. 190], the sequence $\{K_K^p\}_{p=0}^{\infty}$ is increasing and upper bounded by K_K^{∞}. Hence, $\lim_{p\to\infty}\|K_p - K^*\|_F = 0$. The third statement is thus proven. $\square$

In [10, Theorem A.7 and A.8], the authors showed that the controller update phase in the outer-loop has a global sub-linear and local quadratic convergence rates. We now demonstrate that the outer-loop iteration has a global linear convergence rate. Let us first establish a few preliminary results that we will need in the proof of our main result i.e. Theorem 1.

Lemma 3 : *Let $\Psi = \tilde{Q}_K^{p+1}$ so that $\Psi = \Psi^\top \succeq 0$. Furthermore, let $\Phi \in \mathbb{R}^{n\times n}$ be Hurwitz, $\Theta = \int_0^\infty e^{(\Phi^\top t)}\Psi e^{(\Phi t)}dt$, and $a(\Phi) = \log(5/4)\|\Phi\|^{-1}$. Then, $\|\Theta\| \geq \frac{1}{2}a(\Phi)\|\Psi\|$.*

Proof. Define $S(t) = \sum_{k=1}^{\infty}(\Phi t)^k/k!$ so that $e^{\Phi t} = I_n + \sum_{k=1}^{\infty}(\Phi t)^k/k! = I_n + S(t)$ after a Taylor series expansion. Whence $\|S(t)\| = \sum_{k=1}^{\infty}(\|\Phi\|t)^k/k!$ or $\|S(t)\| \geq e^{\|\Phi\|t} - 1$. For $x_0 \neq 0$ satisfying $x_0^\top \Psi x_0 = \|\Psi\|\|x_0\|^2$, we have

$$\begin{aligned}
x_0^\top \Theta x_0 &\geq \int_0^{a(\Phi)} x_0^\top e^{\Phi^\top}\Psi e^{\Phi}x_0 \mathrm{d}t, \\
&\geq \int_0^{a(\Phi)} x_0^\top \left(I_n + S(t)\right)^\top \Psi \left(I_n + S(t)\right) x_0 \mathrm{d}t, \\
&\geq \int_0^{a(\Phi)} \|\Psi\|\|x_0\|^2 - 2(e^{(\|\Phi\|a(\Phi))} - 1)\|\Psi\|\|x_0\|^2 \mathrm{d}t, \\
&\geq \int_0^{a(\Phi)} \frac{1}{2}\|\Psi\|\|x_0\|^2\mathrm{d}t \geq \frac{1}{2}a(\Phi)\|\Psi\|\|x_0\|^2. \qquad (19)
\end{aligned}$$

A fortiori, Lemma 3's proof follows from (19). $\square$

Remark 2 : *For $A_K = A - BK$, we know from the bounded real Lemma [10, Lemma A.1] that the Riccati equation*

$$A_K^\top P_K + P_K A_K + Q_K + \gamma^{-2}P_K DD^\top P_K = 0 \qquad (20)$$

admits a unique positive definite solution $P_K \succ 0$ with a Hurwitz $(A_K + \gamma^{-2}DD^\top P_K)$.

Lemma 4 (Optimality of the iteration): *Consider any $K \in \mathcal{K}$, let $K' = R^{-1}B^\top P_K$ (where P_K is the solution to (20), and $E_K = (K - K')^\top R(K - K')$. If $E_K = 0$, then $K = K^\star$.*

Lemma 5 (Bounds on Gain Difference Matrix): *For any $h > 0$, define $\mathcal{K}_h := \{K \in \mathcal{K} | Tr(P_K - P^\star) \leq h\}$. For any $K \in \mathcal{K}_h$, let $K' := R^{-1}B^\top P_K$, where P_K is the solution to (20), and $E_K := (K - K')^\top R(K - K')$. Then, there exists $b(h) > 0$, such that $\|P_K - P^\star\|_F \leq b(h)\|E_K\|_F$.*

Theorem 1 : *For any $h > 0$ and $K_0 \in \mathcal{K}_h$, there exists $\alpha(h) \in [0, 1)$, such that $Tr(P_K^{p+1} - P^\star) \leq \alpha(h)Tr(P_K^p - P^\star)$. That is, $P^\star$ is an exponentially stable equilibrium.*

Proof. For a Hurwitz $A_K^{(p+1)}$ and an $E_K^p = (K_p - K_{p+1})^\top R(K_p - K_{p+1})$, Lemma 13 and equation (17) imply that

$$P_K^p - P_K^{p+1} \succeq \int_0^\infty e^{A_K^{(p+1)^\top}t}E_K^p e^{A_K^{(p+1)}t}\mathrm{d}t =: H_K^p. \qquad (21)$$

From Lemma 2, we have for $p > 0$ that $P_K^0 \succeq P_K^p$ so that

$$\|A_K^{p+1}\| \leq \|A\| + (\|BR^{-1}B^\top\| + \gamma^{-2}\|DD^\top\|)h. \qquad (22)$$

Set $c(h) = \log(5/4)/\|A\| + (\|BR^{-1}B^\top\| + \gamma^{-2}\|DD^\top\|)h$ so that we have $\|H_K^p\| \geq \frac{1}{2}c(h)\|E_K^p\|$ from Lemma 3. Using Lemmas 15 and 5, and taking the trace of (21) we find that

$$\begin{aligned}
&Tr(P_K^{p+1} - P^\star) \leq Tr(P_K^p - P^\star) - Tr(H_K^p) \\
&\leq Tr(P_K^p - P^\star) - c(h)\|E_K^p\|/2 \\
&\leq Tr(P_K^p - P^\star) - \frac{c(h)}{2\sqrt{n}}\|E_K^p\|_F \\
&\leq Tr(P_K^p - P^\star) - \frac{c(h)}{2\sqrt{n}b(h)}\|P_K^p - P^\star\|_F \\
&\leq \left(1 - \frac{c(h)}{2nb(h)}\right)Tr(P_K^p - P^\star). \qquad (23)
\end{aligned}$$

The proof follows by setting $\alpha(h) = 1 - c(h)/2nb(h)$. $\square$

C. Inner Loop Stability, Optimality, and Convergence

We now analyze the monotonic convergence rate of the inner loop. Given arbitrary gains $K_p \in \mathcal{K}$ and $L_q(K_p)$, let $P_{K,L}^{p,q}$ be the positive definite solution of the associated Lyapunov equation (11). The following lemma shows that the cost matrix $P_{K,L}^{p,q}$ monotonically converges to (11)'s solution.

Lemma 6 : *Suppose that $L_0(K_0)$ is stabilizing, then for any $q \in \mathbb{N}_+$ (with $P_{K,L}^{p,\bar{q}}$ as the solution to (11)),*

1) *$A_{K,L}^{p,q}$ is Hurwitz;*
2) *$P_{K,L}^{p,\bar{q}} \succeq \cdots \succeq P_K^{(p,q+1)} \succeq P_K^{p,q} \succeq \cdots \succeq P_{K,L}^{p,0}$; and*
3) *$\lim_{q\to\infty}\|P_{K,L}^{p,q} - P_{K,L}^{p,\bar{q}}\|_F = 0$.*

Proof. To prove the first statement, we proceed by induction. For a $p \geq 0$ we have $K_p \in \breve{\mathcal{K}}$ by Theorem 1. Subtracting (11) from (10) yields

$$\begin{aligned}
0 =& A_K^{p\top}(P_K^p - P_{K,L}^{p,q}) + (P_K - P_{K,L}^{p,q})A_K^p + \\
&\gamma^2\left[L_{q+1}(K_p) - L_q(K_p)\right]^\top\left[L_{q+1}(K_p) - L_q(K_p)\right]. \quad (24)
\end{aligned}$$

In equation (24), we have that $\left[L_{q+1}(K_p) - L_q(K_p)\right]^\top\left[L_{q+1}(K_p) - L_q(K_p)\right] \succeq 0$ so that (24) admits a Lyapunov equation form. Following statement 2 of Lemma 13, we must have $(P_K^p - P_{K,L}^{p,q}) \succeq 0$. *A fortiori*, we must have $A_{K,L}^{(p,q)}$ Hurwitz in (24) following Lemma 13. This proves the first statement.

To prove the second statement, we proceed thus. Abusing notation by dropping the templated argument in $L_q(K_p)$, let us consider the identities,

$$\begin{aligned}
A_{K,L}^{(p,q)^\top}P_{K,L}^{p,q} &= A_{K,L}^{(p,q+1)^\top}P_{K,L}^{p,q} - \gamma^2\left[L_{q+1} - L_q\right]^\top L_{q+1} \\
P_{K,L}^{p,q}A_{K,L}^{(p,q)} &= P_{K,L}^{p,q}A_{K,L}^{(p,q+1)} - \gamma^2 L_{q+1}^\top\left[L_{q+1} - L_q\right]. \quad (25)
\end{aligned}$$

We now rewrite (11) in light of (25) as

$$A_{K,L}^{(p,q+1)^\top} P_{K,L}^{p,q} + P_{K,L}^{p,q} A_{K,L}^{(p,q+1)} - \gamma^2 \left[L_{q+1} - L_q\right]^\top L_{q+1} + Q_K - \gamma^2 L_{q+1}^\top \left[L_{q+1} - L_q\right] - \gamma^2 (L_q^\top L_q) = 0. \tag{26}$$

At the $(q+1)$'th iteration, we have (11) as

$$A_{K,L}^{(p,q+1)^\top} P_{K,L}^{p,q+1} + P_{K,L}^{p,q+1} A_{K,L}^{(p,q+1)} + Q_K - \gamma^2 L_{q+1}^\top (K_p) L_{q+1}(K_p) = 0. \tag{27}$$

Subtracting (26) from (27), we have

$$A_{K,L}^{(p,q+1)^\top} \left[P_{K,L}^{p,q+1} - P_{K,L}^{p,q}\right] + \left[P_{K,L}^{p,q+1} - P_{K,L}^{p,q}\right] A_{K,L}^{(p,q+1)} + + \gamma^2 \left[L_{q+1} - L_q\right]^\top \left[L_{q+1} - L_q\right] = 0. \tag{28}$$

Since $\left[L_{q+1} - L_q\right]^\top \left[L_{q+1} - L_q\right] \succeq 0$, (28) is indeed a Lyapunov equation so that $P_{K,L}^{p,q+1} \succeq P_{K,L}^{p,q}$ holds following Lemma 13. Whence, we must have $A_{K,L}^{(p,q+1)^\top}$ Hurwitz. Following the argument for all $(q, q') \in \bar{q}$ with $q \neq q'$, statement 2) holds.

Observe: $P_{K,L}^{p,q}$ is self-adjoint by reason of (10). By the theorem on the "limit of monotonically decreasing operators" [34, pp. 190], statement 2) implies that the sequence $\{P_{K,L}^{p,\bar{q}}, \cdots, P_{K,L}^{p,q=0}\}$ is monotonically decreasing and bounded from above by $P_{K,L}^{p,\bar{q}} \equiv P_{K,L}^{\star}$. That is, $P_{K,L}^{p,q}$ exists and is the solution of (10) and $P_{K,L}^{p,\bar{q}}$ is the unique positive definite solution to (11). *A fortiori*, we must have $\lim_{q\to\infty} P_{K,L}^{p,q} = P_{K,L}^{p,\infty}$. This establishes the third statement. □

1) Convergence Rate Analysis: We now analyze the monotonic convergence of the inner loop of the nested double loop algorithm. Let us first discuss a preliminary result.

Lemma 7 (Monotonic Convergence of the Inner-Loop): *For any $K \in \mathcal{K}$, let $L(K)$ be the control gain for the player w such that $A_K + DL(K)$ is Hurwitz. In addition, let P_K^L be the solution of*

$$\left(A_K + DL(K)\right)^\top P_K^L + P_K^L \left(A_K + DL(K)\right) + Q_K - \gamma^2 L(K)^\top L(K) = 0. \tag{29}$$

And define $L'(K) = \gamma^{-2} D^\top P_K^L$ and $E_K^L = \gamma^{-2}(L'(K) - L(K))^\top (L'(K) - L(K))$. Then, for a $c(K) = Tr\left(\int_0^\infty e^{(A_K+DL(K^\star))t} e^{(A_K+DL(K^\star))^\top t} \mathrm{d}t\right)$, the following inequality holds $Tr(P_K - P_K^L) \leq \|E_K^L\| c(K)/$

Theorem 2 : *For a $K \in \check{\mathcal{K}}$, and for any $q \in \mathbb{N}_+$, there exists $\beta(K) \in [0, 1)$, such that*

$$Tr(P_K^p - P_{K,L}^{p,q+1}) \leq \beta(K) Tr(P_K^p - P_{K,L}^{p,q}). \tag{30}$$

That is, the inner loop has a global linear convergence rate.

Proof. By Lemma 6, $A_{K,L}^{p,q}$ is Hurwitz. It follows from Lemma 13 and (14) that

$$P_{K,L}^{p,q+1} - P_{K,L}^{p,q} = \underbrace{\int_0^\infty e^{(A_{K,L}^{(p,q+1)^\top})t} E_K^q e^{A_{K,L}^{(p,q+1)t}} \mathrm{d}t}_{F_K^q}. \tag{31}$$

By Lemma 6, $P_K^p \succeq P_{K,L}^{p,q}$ so that

$$\|A_{K,L}^{(p,q+1)}\| \leq \|A - BK_p\| + \gamma^{-2}\|DD^\top\|\|P_K\|. \tag{32}$$

$$\text{Let } d(K) = \log(5/4)/\left(\|A_K\| + \gamma^{-2}\|DD^\top\|\|P_K^p\|\right), \tag{33}$$

so that from the trace of (32), we find that

$$\begin{aligned} Tr(P_K^p - P_{K,L}^{p,q+1}) &= Tr(P_K^p - P_{K,L}^{p,q}) - Tr(F_K^q), && \text{(34a)}\\ &\leq Tr(P_K^p - P_{K,L}^{p,q}) - \|F_K^q\|, && \text{(34b)}\\ &\leq Tr(P_K^p - P_{K,L}^{p,q}) - \frac{1}{2} d(K)\|E_K^j\|, && \text{(34c)} \end{aligned}$$

where we have used Lemma 15 to arrive at the inequality in (34b), and Lemma 3 for (34c). Furthermore, from Lemma 7

$$Tr(P_K^p - P_{K,L}^{p,q}) \leq \left(1 - \frac{1}{2}\frac{d(K)}{c(K)}\right) Tr(P_K^p - P_{K,L}^{p,q}). \tag{35}$$

The proof follows if we set $\beta(K) = 1 - d(K)/2c(K)$. □

Remark 3 : *As seen from Lemma 6, $P_K - P_K^j \succeq 0$. From Lemma 15 and the result of Theorem 2, we have $\|P_K - P_{K,L}^{p,q}\|_F \leq Tr(P_K - P_{K,L}^{p,q}) \leq \beta^j(K) Tr(P_K)$, i.e. $P_{K,L}^{p,q}$ exponentially converges to P_K in the Frobenius norm.*

The following lemma guarantees uniform convergence after an equal number of inner-loop iterations so that $P_{K,L}^{p,q}$ and $L_q(K)$ enter the vicinity of P_K and $L(K^\star)$ irrespective of the different values of K.

Lemma 8 (Uniform Convergence of Iterates): *For any $h > 0$, $K \in \mathcal{K}_h$, and $\epsilon > 0$, there exists $q'(h) \in \mathbb{N}_+$ independent of K, such that if $q \geq q'(h)$, $\|P_{K,L}^{p,q} - P_K\|_F \leq \epsilon$.*

Proof of Lemma 8. This Lemma is an immediate outcome of Theorems 1 and 2. □

D. Sampling-based PI on Hybrid Nonlinear System

In practice, the exact knowledge of the system matrices are unavailable so that the policy evaluation step will result in biased estimates. When errors are present from using I/O or state data for the PO procedure in Alg. 1, residuals from early termination of numerically solving Line 8 in Alg. 1, or using an approximate cost function owing to inexact values of Q and R, the algorithm may fail to converge. In a data sampling-based scheme, we must guarantee the stability of the closed-loop system and its robustness because there is the possibility for a divergence from the stability-robustness feasibility set $\check{\mathcal{K}}$ since the inner loop is computed in a finite number of steps. The problem is stated in Problem 2.

Problem 2 (Sampling-based Policy Iteration): *If A, B, C, D, E, P are all replaced by approximate matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{P}$, under what conditions will the sequences $\{\hat{P}_{K,L}^{p,q}\}_{(p,q)=1}^{(p,q)=\infty}$, $\{\hat{K}_p\}_{p=0}^{\infty}$, $\{\hat{L}_q\}_{q=0}^{\infty}$ converge to a small neighborhood of the optimal values $\{P_{K,L}^{\star}\}_{(p,q)=0}^{(p,q)=\infty}$, $\{K_p^\star\}_{p=0}^{\infty}$, and $\{L_q^\star\}_{q=0}^{\infty}$.*

1) Discrete-Time Nonlinear System Interpretation: From Assumption 1, a $P_K^0 \in \mathbb{S}^n$ exists such that when applied to find K_0 i.e. $K_0 = R^{-1}B^\top P_K^0$, such a K_0 will be stabilizing. Now, factoring in approximation errors between the policy evaluation and improvement structures, we end up with a hybrid system consisting of a continuous-time policy gain pair $(\hat{K}_p, \hat{L}_q(\hat{K}_p))$ and a learning algorithm that is essentially a discrete sampled data from a nonlinear system (owing to errors from various sources lumped together as disturbance input). We will leverage Lemmas 2 and 6 to show that under inexact loop updates, an online PI scheme converges to the optimal solution and closed-loop dynamic stability is guaranteed in an input-to-state stability framework (ISS) [39]. Hence the loops are discrete-time nonlinear systems.

2) Online (Model-Free) Nested Loop Reparameterization: Consider (10b) and suppose that $\hat{P}_K^0 \in \mathbb{S}^n$ is chosen following Assumption 1. It follows that a $\hat{K}_k^1 = R^{-1}B^\top \hat{P}_K^0$ will be stabilizing since $\tilde{K}_k^1 \equiv \hat{K}_k^1 - K_k^1 \triangleq 0$. The same argument applies for L_0. For $(p,q) > 0$, we must show that for $\tilde{K}_k^p \equiv \hat{K}_k^p - K_k^p \triangleq 0$ so that the sequence $\{P_{K,L}^{p,q}\}_{(p,q)=0}^\infty$ will converge to the locally exponentially stable $\hat{P}_{K,L}^\star$ going by Lemma 2 and 6.

Under inexact outer loop update, the iterate K_{p+1} becomes inaccurate so that the inexact outer-loop iteration involves the recursions

$$\hat{A}_K^{p\top}\hat{P}_K^p + \hat{P}_K^p\hat{A}_K^p + \hat{Q}_K^p + \gamma^{-2}\hat{P}_K^p DD^\top \hat{P}_K^p = 0, \tag{36a}$$
$$\hat{K}_{p+1} = R^{-1}B^\top \hat{P}_K^p \tag{36b}$$

where $\hat{A}_K^p = A - B\hat{K}_p$ and $\hat{Q}_K^p = Q + \hat{K}_p^\top R\hat{K}_p$. Similar argument applies to the inner loop updates so that the inexact inner loop update is

$$\hat{A}_{K,L}^{p,q\top}\hat{P}_{K,L}^{p,q} + \hat{P}_{K,L}^{p,q}\hat{A}_{K,L}^{p,q} + \hat{Q}_K^p - \gamma^2\hat{L}_q^\top\hat{L}_q(\hat{K}_p) = 0 \tag{37a}$$
$$\hat{K}_{p+1} = R^{-1}B^\top\hat{P}_K^{p,q},\ \hat{L}_{q+1}(\hat{K}_p) = \gamma^{-2}D^\top\hat{P}_{K,L}^{p,q} \tag{37b}$$

Consider the transformation of the infinite-dimensional stochastic differential equation (1) in light of the identities (9) under inexact updates for $(p,q) > 0$

$$\mathrm{d}x = [\hat{A}_{K,L}^{p,q}x + B(\hat{K}_p x - D\hat{L}_q(K_p) + u)]\mathrm{d}t + D\mathrm{d}w. \tag{38}$$

On a time interval $[s, s+\delta s]$, it follows from Itô's stochastic calculus and the Hamilton-Jacobi-Bellman equation that

$$\begin{aligned} d\left[x^\top(s+\delta s)\hat{P}_{K,L}^{p,q}(s+\delta s) - x^\top(s)\hat{P}_{K,L}^{p,q}x(s)\right] = \\ (\mathrm{d}x)^\top\hat{P}_{K,L}^{p,q}x + x^\top\hat{P}_{K,L}^{p,q}\mathrm{d}x + (\mathrm{d}x)^\top\hat{P}_{K,L}^{p,q}(\mathrm{d}x). \end{aligned} \tag{39}$$

Along the trajectories of equation (38) and using the gains in (11), the r.h.s. in the foregoing becomes

$$\begin{aligned} &x^\top\left[\hat{A}_{K,L}^{p,q\top}\hat{P}_{K,L}^{p,q} + \hat{P}_{K,L}^{p,q}\hat{A}_{K,L}^{p,q}\right]x\mathrm{d}t + 2x^\top\hat{P}_{K,L}^{p,q}D\mathrm{d}w \\ &+ 2x^\top\hat{P}_{K,L}^{p,q}B(K_p x - D\hat{L}_q(K_p) + u)dt + Tr(D^\top PD), \\ &= -x^\top\hat{Q}_K^p x\mathrm{d}t - \gamma^{-2}x^\top\hat{P}_{K,L}^{p,q}DD^\top\hat{P}_{K,L}^{p,q}x\mathrm{d}t + Tr(D^\top\hat{P}_{K,L}^{p,q}D) \\ &+ 2x^\top\hat{P}_{K,L}^{p,q}B\left[\hat{K}_p x - D\hat{L}_q(K_p) + u\right]\mathrm{d}t + 2x^\top\hat{P}_{K,L}^{p,q}D\mathrm{d}w, \end{aligned} \tag{40}$$

so that $x^\top(s+\delta s)\hat{P}_{K,L}^{p,q}(s+\delta s) - x^\top(s)\hat{P}_{K,L}^{p,q}x(s)$

$$\begin{aligned} &= \int_s^{s+\delta s}\left[(-x^\top\hat{Q}_K^p x - \gamma^2 w^\top w)\mathrm{d}t + 2\gamma^2 x^\top\hat{L}_{q+1}^\top(K_p)\mathrm{d}w\right] \\ &+ \int_s^{s+\delta s} 2x^\top\hat{K}_{p+1}^\top R\left[\hat{K}_p x - D\hat{L}_q(\hat{K}_p) + u\right]\mathrm{d}t \\ &\quad + \int_s^{s+\delta s} Tr(D^\top\hat{P}_{K,L}^{p,q}D)\mathrm{d}t. \end{aligned} \tag{41}$$

Observe: The system dependent matrices $\hat{A}_{K,L}^{p,q}, B, C, D$ from equation (40) are now replaced by input and state terms including $\hat{Q}_K^p$, $\hat{K}_{p+1}$, and $\hat{L}_{q+1}$ which are all retrievable via online measurements. *We essentially end up with an input-to-state system.* The price we pay is that the noise feedthrough matrix D must be known precisely. In this article, as is common in many linear stochastic system with Brownian motion, D is taken to be identity [40], [41].

3) Sampling-based PI Scheme: Our goal is to explore the system model until exact equality of $\hat{A}_{K,L}^{p,q}, P_{K,L}^{p,q}$ and $K_{p+1}, L_{q+1}(K_p)$ to the corresponding terms in (11). To this end, (41) allows us to explore with the controls $u = -K_0 x + \eta_i$ and $w = -L_0 x + \eta_i$ where η_i is drawn uniformly at random over matrices with a Frobenius norm r for a smoothing parameter r [4], [18]. Let us now introduce the following identities,

$$\begin{aligned} &x^\top\hat{Q}_K^p x = (x^\top\otimes x^\top)\,\mathtt{vec}(\hat{Q}_K^p), \\ &\gamma^2 w^\top w = \gamma^2(w^\top\otimes w^\top)\ \mathtt{vec}(I_v), \\ &2\gamma^2 x^\top\hat{L}_{q+1}^\top(\hat{K}_p)\mathrm{d}w = 2\gamma^2(I_n\otimes x^\top)\mathrm{d}w\ \mathtt{vec}(\hat{L}_{q+1}^\top(\hat{K}_p)), \\ &2x^\top\hat{K}_{p+1}^\top R\hat{K}_p x = 2(x^\top\otimes x^\top)(I_n\otimes\hat{K}_p^\top)\,\mathtt{vec}(\hat{K}_{p+1}^\top R), \\ &2x^\top\hat{K}_{p+1}^\top RD\hat{L}_q(\hat{K}_p) = 2(\hat{L}_q^\top(\hat{K}_p)D^\top\otimes x^\top)\,\mathtt{vec}(\hat{K}_{p+1}^\top R), \\ &2x^\top\hat{K}_{p+1}^\top Ru = 2(u^\top\otimes x^\top)\,\mathtt{vec}(\hat{K}_{p+1}^\top R), \\ &Tr(D^\top\hat{P}_{K,L}^{p,q}D) = \mathtt{vec}^\top(D)\,\mathtt{vec}(\hat{P}_{K,L}^{p,q}D). \end{aligned} \tag{42}$$

Furthermore, consider the matrices $\delta_{xx} \in \mathbb{R}^{\frac{n(n+1)}{2}l}$, $\delta_{ww} \in \mathbb{R}^{\frac{v(v+1)}{2}l}$, $I_{xx} \in \mathbb{R}^{l\times n^2}$, and $I_{ux} \in \mathbb{R}^{l\times nm}$ for $l \in \mathbb{N}_+$ so that

$$\begin{aligned} \Delta_{xx} &= \left[\mathtt{vecv}(x_1),\ldots,\mathtt{vecv}(x_l)\right]^\top,\ x_l = x_{l+1} - x_l, \\ \Delta_{ww} &= \left[\mathtt{vecv}(w_1),\ldots,\mathtt{vecv}(w_l)\right]^\top,\ w_l = w_{l+1} - w_l, \\ I_{xx} &= \left[\int_{s_0}^{s_1} x\otimes x\,\mathrm{d}t,\ldots,\int_{s_{l-1}}^{s_l} x\otimes x\,\mathrm{d}t\right]^\top, \\ I_{ww} &= \left[\int_{s_0}^{s_1} w\otimes w\,\mathrm{d}t,\ldots,\int_{s_{l-1}}^{s_l} w\otimes w\,\mathrm{d}t\right]^\top, \\ I_{xw} &= \left[\int_{s_0}^{s_1}(I_n\otimes x)\mathrm{d}w,\ldots,\int_{s_{l-1}}^{s_l}(I_n\otimes x)\mathrm{d}w\right]^\top, \\ I_{ux} &= \left[\int_{s_0}^{s_1} u\otimes x\,\mathrm{d}t,\ldots,\int_{s_{l-1}}^{s_l} u\otimes x\,\mathrm{d}t\right]^\top. \end{aligned} \tag{43}$$

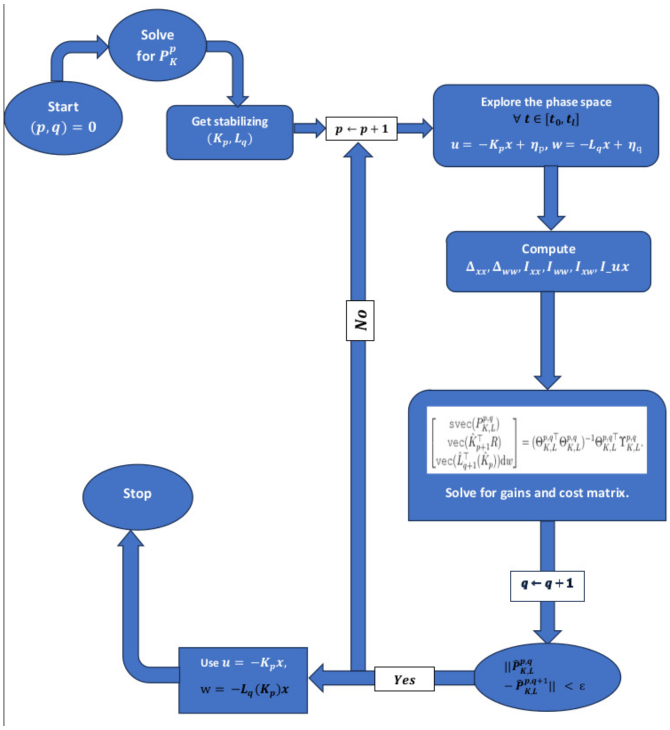

Fig. 1: Flowchart for Sampling-based PO in Continuous-time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Stochastic Control.

Next, set

$$\Theta^{p,q}_{K,L} = \Big[\Delta_{xx}, -2I_{xx}(I_n \otimes \hat{K}_p^\top) + 2(\hat{L}_q^\top(\hat{K}_p)D^\top \otimes x^\top) \\ -2I_{ux}, -2\gamma^2 I_{xw}, -\texttt{vec}^\top(D)\texttt{vec}(\hat{P}^{p,q}_{K,L}D)\Big], \tag{44a}$$

$$\Upsilon^{p,q}_{K,L} = \Big[-I_{xx}\texttt{vec}(\hat{Q}^p_K),\ -\gamma^2 I_{ww}\texttt{vec}(I_v)\Big]. \tag{44b}$$

Define $\mathbf{1}_{q^2}$ as one-vector with dimension q^2. Thus,

$$\Theta^{p,q}_{K,L}\begin{bmatrix} \texttt{svec}(P^{p,q}_{K,L}) \\ \texttt{vec}(\hat{K}^\top_{p+1}R) \\ \texttt{vec}(\hat{L}^\top_{q+1}(\hat{K}_p)) \\ \mathbf{1}_{q^2} \end{bmatrix} = \Upsilon^{p,q}_{K,L}. \tag{45}$$

Suppose that $\Theta^{p,q}_{K,L}$ is of full rank, then we can retrieve the unknown matrices via least squares estimation i.e.

$$\begin{bmatrix} \texttt{svec}(P^{p,q}_{K,L}) \\ \texttt{vec}(\hat{K}^\top_{p+1}R) \\ \texttt{vec}(\hat{L}^\top_{q+1}(\hat{K}_p))\text{d}w \\ \mathbf{1}_{q^2} \end{bmatrix} = (\Theta^{p,q\top}_{K,L}\Theta^{p,q}_{K,L})^{-1}\Theta^{p,q\top}_{K,L}\Upsilon^{p,q}_{K,L}. \tag{46}$$

We thus end up with a scheme for retrieving the system matrices provided that the algorithm is robust to perturbations upon iterating through (46) for each (p,q). The full scheme is summarized in the flowchart of Fig. 1. We next state the condition under which $\Theta^{p,q}_{K,L}$ is of full rank.

Lemma 9 : *[42, Lemma 6] If there exists an integer $l_0 > 0$ such that for all $l \geq l_0$, $rank(I_{xx}, I_{ux}, I_{xw}, \mathbf{1}_{q^2}) = n(n+1) + mn + nq + q^2$, then $\Theta^{p,q}_{K,L}$ has full rank for all $(p,q) \in (\bar{p}, \bar{q})$.*

Remark 4 : *Lemma 9 allows the convergence assurance of Fig. 1 under the condition that the rank condition be fulfilled.*

4) Robustness of Minimizing Controller to Perturbations: We now analyze the robustness of the sampling-based scheme as a hybrid nonlinear discrete time system gains with continuous-time dynamics. Let $\tilde{P} = P_K - \hat{P}_K$ and $\tilde{K} = K - \hat{K}$ denote errors arising from the inexact updates.

Lemma 10 (Outer-Loop Robustness to Perturbations): *For any $K \in \mathcal{K}$, there exists an $e(K) > 0$ such that for a perturbation $\tilde{K}$, $K + \tilde{K} \in \mathcal{K}$, as long as $\|\tilde{K}\| < e(K)$.*

Proof of Lemma 10. Let $F(\tilde{P}, \tilde{K})$ be

$$(A_K + \gamma^{-2}DD^\top P_K)^\top \tilde{P} + \tilde{P}(A_K + \gamma^{-2}DD^\top P_K) \\ - \tilde{K}^\top B^\top(P_K + \tilde{P}) - (P_K + \tilde{P})B\tilde{K} + \tilde{K}^\top RK + K^\top R\tilde{K} \\ + \tilde{K}^\top R\tilde{K} + \gamma^{-2}\tilde{P}DD^\top\tilde{P}. \tag{47}$$

Observe: the pair $(P_K + \tilde{P}, K + \tilde{K})$ satisfies (20) iff $F(\tilde{P}, \tilde{K}) = 0$ and that $F(\tilde{P}, \tilde{K}) = 0$ implies an implicit function of $\tilde{P}$ with respect to $\tilde{K}$ since if $\tilde{P} \in \mathbb{S}^n$ exists, $\tilde{K}$ must exist under controllability and observability assumptions. Let $\mathcal{F}(\tilde{P}, \tilde{K}) = \texttt{vec}(F(\tilde{P}, \tilde{K}))$ so that,

$$\mathcal{F}(\tilde{P}, \tilde{K}) = \big[I_n \otimes (A_K + \gamma^{-2}DD^\top P_K)^\top \\ + (A_K + \gamma^{-2}DD^\top P_K)^\top \otimes I_n\big]\,\texttt{vec}(\tilde{P}) \\ - (P_K B \otimes I_n)\texttt{vec}(\tilde{K}^\top) - (I_n \otimes P_K B)\texttt{vec}(\tilde{K}) \\ - (I_n \otimes \tilde{K}^\top B^\top + \tilde{K}^\top B^\top \otimes I_n)\texttt{vec}(\tilde{P}) \\ + (K^\top R \otimes I_n)\texttt{vec}(\tilde{K}^\top) + (I_n \otimes K^\top R)\texttt{vec}(\tilde{K}) \\ + \texttt{vec}(\tilde{K}^\top R\tilde{K}) + \gamma^{-2}\texttt{vec}(\tilde{P}DD^\top\tilde{P}). \tag{48}$$

Thus,

$$\frac{\partial\mathcal{F}(\tilde{P}, \tilde{K})}{\partial\texttt{vec}(\tilde{P})} = I_n \otimes [(A_K + \gamma^{-2}DD^\top P_K) - B\tilde{K}]^\top \\ + [(A_K + \gamma^{-2}DD^\top P_K) - B\Delta K]^\top \otimes I_n \\ + \tilde{P}DD^\top \otimes I_n + I_n \otimes \tilde{P}DD^\top \\ - (P_K B \otimes I_n)\texttt{vec}(\tilde{K}^\top) - (I_n \otimes P_K B)\texttt{vec}(\tilde{K}) \\ + (K^\top R \otimes I_n)\texttt{vec}(\tilde{K}^\top) + (I_n \otimes K^\top R)\texttt{vec}(\tilde{K}), \tag{49}$$

where we have used [43, Theorem 9], to obtain $\partial\texttt{vec}(\tilde{P}DD^\top\tilde{P})/\partial\texttt{vec}(\tilde{P}) = \tilde{P}DD^\top \otimes I_n + I_n \otimes \tilde{P}DD^\top$. Since $\mathcal{F}(0,0) = 0$, $(A_K + \gamma^{-2}DD^\top P_K)$ is Hurwitz, hence $\partial\mathcal{F}(\tilde{P}, \tilde{K})/\partial\texttt{vec}(\tilde{P})|_{\tilde{P}=0,\tilde{K}=0}$ is invertible. From the implicit function theorem, there must exist an $e_1(K) > 0$, such that $\tilde{P}$ is continuously differentiable with respect to $\tilde{K}$ for any $\tilde{K} \in \mathcal{B}_{e_1(K)}(0)$. Thus, $\|\tilde{P}\| \to 0$ as $\|\tilde{K}\| \to 0$. Since $K \in \mathcal{K}$ by [10, Lemma A.1], we must have $P_K \succ 0$. Therefore, there exists $e(K) > 0$, such that $\sigma_{\max}(\tilde{P}) < \sigma_{\min}(P_K)$, i.e. $P_K - \tilde{P} \succ 0$, as long as $\|\tilde{K}\| < e(K)$.

Since $\tilde{P}$ and $\tilde{K}$ satisfy $F(\tilde{P}, \tilde{K}) = 0$, we have

$$(A - BK - B\tilde{K})^T(P_K + \tilde{P}) + (P_K + \tilde{P}) + Q + \tag{50} \\ (K + \tilde{K})^T R(K + \tilde{K}) + \gamma^{-2}(P_K + \tilde{P})DD^T(P_K + \tilde{P}) = 0$$

Since $P_K + \tilde{P} \succ 0$ when $\|\tilde{K}\| < e(K)$, by [10, Lemma A.1], $K + \tilde{K} \in \mathcal{K}$. That is, as long as $\tilde{K}$ is small, if we start the PI with a robustly stabilizing $K \in \mathcal{K}$, we can guarantee the feasibility of the iterates. $\square$

Lemma 11 : *For any $h > 0$ and $K \in \mathcal{K}_h$, let $K' = R^{-1}B^\top P_K$, where P_K is the solution of (20), and $\hat{K}' = K' + \tilde{K}$. Then, there exists $f(h) > 0$, such that $\hat{K}' \in \mathcal{K}_h$ as long as $\|\tilde{K}\| < f(h)$.*

Proof. Since $\mathcal{K}_h$ is compact, it follows from Lemma 10 that $\underline{e}(h) := \inf_{K\in\mathcal{K}_h} e(K) > 0$. In addition, $\hat{K}' \in \mathcal{K}$ when $\|\tilde{K}\| < \underline{e}(h)$. By [10, Lemma A.1], $P_{\hat{K}'} = P_{\hat{K}'}^\top \succ 0$ is the solution of

$$A_{\hat{K}'}^\top P_{\hat{K}'} + P_{\hat{K}'}A_{\hat{K}'} + Q_{\hat{K}'} + \gamma^{-2}P_{\hat{K}'}DD^\top P_{\hat{K}'} = 0, \quad (51)$$

where $A_{\hat{K}'} = A - B\hat{K}'$ and $Q_{\hat{K}'} = Q + (\hat{K}')^\top R\hat{K}'$. Let $A^\star_{\hat{K}'} = A - B\hat{K}' + \gamma^{-2}DD^T P_{\hat{K}'}$. It follows from [10, Lemma A.1] that $A^\star_{\hat{K}'}$ is Hurwitz. Subtracting (51) from (20), using $K' = R^{-1}B^T P_K$, and completing the squares,

$$\begin{aligned} &A^{\star\top}_{\hat{K}'}(P_K - P_{\hat{K}'}) + (P_K - P_{\hat{K}'})A^\star_{\hat{K}'} \\ &+ (K'-K)^\top R(K'-K) - \tilde{K}^\top R\tilde{K} \\ &+ \gamma^{-2}(P_K - P_{\hat{K}'})DD^\top(P_K - P_{\hat{K}'}) = 0. \end{aligned} \quad (52)$$

From Lemma 13, we have $(P_K - P_{\hat{K}'}) \succeq$

$$\int_0^\infty e^{A^{\star\top}_{\hat{K}'}t}E_K e^{A^\star_{\hat{K}'}t}\mathrm{d}t - \int_0^\infty e^{A^{\star\top}_{\hat{K}'}t}\tilde{K}^\top R\tilde{K}e^{A^\star_{\hat{K}'}t}\mathrm{d}t, \quad (53)$$

so that taking the trace, using Lemma 3 and [44, Theorem 2],

$$\begin{aligned} Tr(P_K - P_{\hat{K}'}) \geq &\frac{\log(5/4)}{2\|A^\star_{\hat{K}'}\|}\|E_K\| \\ &- Tr\left(\int_0^\infty e^{A^{\star\top}_{\hat{K}'}t}e^{A^\star_{\hat{K}'}t}\mathrm{d}t\right)\|R\|\|\tilde{K}\|^2. \end{aligned} \quad (54)$$

It follows from Lemmas 5 and 15 that

$$\begin{aligned} Tr(P_{\hat{K}'} - P^\star) \leq &\left(1 - \underbrace{\frac{\log(5/4)b(h)}{2n\|A^\star_{\hat{K}'}\|}}_{f_1(\hat{K}')}\right)Tr(P_K - P^\star) \\ &+ \underbrace{Tr\left(\int_0^\infty e^{A^{\star\top}_{\hat{K}'}t}e^{A^\star_{\hat{K}'}t}\mathrm{d}t\right)}_{f_2(\hat{K}')}\|R\|\|\tilde{K}\|^2. \end{aligned} \quad (55)$$

Since $f_1(\hat{K}')$ and $f_2(\hat{K}')$ are continuous with respect to $\hat{K}'$,

$$\underline{f}_1(h) = \inf_{\hat{K}'\in\mathcal{K}_h} f_1(\hat{K}') > 0, \bar{f}_2(h) = \sup_{\hat{K}'\in\mathcal{K}_h} f_2(\hat{K}') < \infty. \quad (56)$$

It follows from (55) that if $\|\tilde{K}\| < \sqrt{\frac{\underline{f}_1(h)h}{\bar{f}_2(h)\|R\|}}$, then $Tr(P_{\hat{K}'} - P^\star) < h$. In summary, if

$$\|\tilde{K}\| < \min\left\{\underline{e}(h), \sqrt{\frac{\underline{f}_1(h)h}{\bar{f}_2(h)\|R\|}}\right\} =: f(h), \quad (57)$$

we have $\hat{K}' \in \mathcal{K}_h$. □

Theorem 3 : *The inexact outer loop is small-disturbance ISS. That is, for any $h > 0$ and $\hat{K}_0 \in \mathcal{K}_h$, if $\|\tilde{K}\| < f(h)$, there exist a $\mathcal{KL}$-function $\beta_1(\cdot,\cdot)$ and a $\mathcal{K}_\infty$-function $\gamma_1(\cdot)$ such that*

$$\|P^p_{\hat{K}} - P^\star\| \leq \beta_1(\|P^0_{\hat{K}} - P^*\|, p) + \gamma_1(\|\tilde{K}\|). \quad (58)$$

Proof. From Lemma 11, $\hat{K}^p_K \in \mathcal{K}_h$ for any $p \in \mathbb{N}_+$. From (55), at the p'th iteration, we have

$$\begin{aligned} &Tr(P^p_{\hat{K}} - P^\star) \leq (1 - \underline{f}_1(h))Tr(P^{p-1}_{\hat{K}} - P^\star) \\ &+ \bar{f}_2(h)\|R\|\|\tilde{K}^p_K\|^2. \end{aligned} \quad (59)$$

Repeating (59) for $p, p-1, \cdots, 1$,

$$Tr[P^p_{\hat{K}} - P^\star] \leq (1 - \underline{f}_1)^p Tr(P^1_{\hat{K}} - P^\star) + \frac{\bar{f}_2\|R\|\|\tilde{K}\|^2_\infty}{\underline{f}_1(h)}. \quad (60)$$

It follows from (60) and [44, Theorem 2] that

$$\|P^p_{\hat{K}} - P^\star\|_F \leq (1 - \underline{f}_1)^p\sqrt{n}\|P^1_{\hat{K}} - P^\star\|_F + \frac{\bar{f}_2\|R\|\|\tilde{K}\|^2_\infty}{\underline{f}_1}. \quad (61)$$

As $p \to \infty$, $P^p_{\hat{K}} \to P^\star$. The radius of the neighbor of $P^\star$ is proportional to $\|\tilde{K}\|^2_\infty$. Thus, the proof follows. □

5) Robustness of Maximizing Controller to Perturbations: The perturbed inner-loop iteration (37) has inexact matrix $\hat{A}^{p,q}_{K,L}$, and sequences $\{\hat{L}_{q+1}(K_p)\}^\infty_{q=0}$, and $\{\hat{P}^{p,q}_{K,L}\}^\infty_{q=0}$. We next analyze its robustness to perturbations when it differs from the exact loop matrices and sequences.

Lemma 12 (Inner-Loop Robustness to Perturbations): *Given $K \in \breve{\mathcal{K}}$, there exists a $g \in \mathbb{R}_+$, such that if $\|\tilde{L}_{q+1}(K_p)\|_F \leq g$, $\hat{A}^{p,q}_{K,L}$ is Hurwitz for all $q \in \mathbb{N}_+$.*

Proof. Define $\tilde{L}^\top_q(K_p) = L^\top_q(K_p) - \hat{L}^\top_q(K_p)$ and $\tilde{P}^{p,q}_{K,L} = P^{p,q}_{K,L} - \hat{P}^{p,q}_{K,L}$. Further, assume $\hat{A}^{p,q}_{K,L}$ is Hurwitz. From (10),

$$\begin{aligned} &\hat{A}^{p,q\top}_{K,L}P^{p,q}_{K,L} + P^{p,q}_{K,L}\hat{A}^{p,q+1}_{K,L} + Q_K - \gamma^{-2}\hat{P}^{p,q}_{K,L}DD^\top\hat{P}^{p,q}_{K,L} + \\ &\gamma^{-2}(P^{p,q}_{K,L} - \hat{P}^{p,q}_{K,L})DD^\top(P^{p,q}_{K,L} - \hat{P}^{p,q}_{K,L}) - \tilde{L}^\top_q(K_p)D^\top P^{p,q}_{K,L} \\ &- P^{p,q}_{K,L}D\tilde{L}_q(K_p) = 0. \end{aligned} \quad (62)$$

Set $\|\tilde{L}^\top_q(K_p)\| < \sigma_{\min}(Q_K - \gamma^{-2}P^{p,q}_{K,L}DD^\top P^{p,q}_{K,L})/2\|D^\top P^{p,q}_{K,L}\| \triangleq e$, it follows from (27) that $Q \succ \gamma^2 L^\top_{q+1}(K_p)L_{q+1}(K_p)$ by reason of it being admissible as a Lyapunov equation and the inequality $P^{p,q}_{K,L} \succeq \hat{P}^{p,q}_{K,L}$ that

$$\begin{aligned} &-\gamma^{-2}\hat{P}^{p,q}_{K,L}DD^\top\hat{P}^{p,q}_{K,L} + \gamma^{-2}(P^{p,q}_{K,L} - \hat{P}^{p,q}_{K,L})DD^\top(P^{p,q}_{K,L} \\ &- \hat{P}^{p,q}_{K,L}) - (\tilde{L}^j_K)^\top D^\top P^{p,q}_{K,L} - P^{p,q}_{K,L}D\tilde{L}_q(K_p) + Q_K \succ 0. \end{aligned}$$

Consequently, $\hat{A}^{p,q+1}_{K,L}$ is Hurwitz. Since $\hat{L}_q(K_0) = 0$ and $K \in \breve{\mathcal{K}}$, $\hat{A}^{p,0}_{K,L} = A - BK$ is Hurwitz. Hence, $\hat{A}^{p,q}_{K,L}$ is Hurwitz for all $q \in \mathbb{N}_+$ as long as $\|\tilde{L}_q(K_p)\|_F \leq e$. □

Theorem 4 : *Assume $\|\tilde{L}_q(K_p)\| < e$ for all $q \in \mathbb{N}_+$. There exists $\hat{\beta}(K) \in [0,1)$, and $\lambda(\cdot) \in \breve{\mathcal{K}}_\infty$, such that*

$$\|\hat{P}^{p,q}_{K,L} - P^{p,q}_{K,L}\|_F \leq \hat{\beta}^{j-1}(K)Tr(P^{p,q}_{K,L}) + \lambda(\|\tilde{L}\|_\infty). \quad (63)$$

Proof. When $\|\tilde{L}_q(K_p)\| < e$, we have an Hurwitz $\hat{A}^{p,q}_{K,L}$ going by 12. Rewriting (37) for the $(p+1)$th iteration and subtracting it from (37), we have

$$\begin{aligned} &\hat{A}^{(p,q+1)\top}_{K,L}(\hat{P}^{(p,q+1)}_{K,L} - \hat{P}^{p,q}_{K,L}) + (\hat{P}^{(p,q+1)}_{K,L} - \hat{P}^{p,q}_{K,L})\hat{A}^{(p,q+1)\top}_{K,L} \\ &+ \gamma^{-2}(\gamma^2\hat{L}_q(K_p) - D^\top\hat{P}^{p,q}_{K,L})^\top(\gamma^2\hat{L}_q(K_p) - D^\top\hat{P}^{p,q}_{K,L}) \\ &- \gamma^2\tilde{L}^\top_q(K_p)\tilde{L}_q(K_p) = 0. \end{aligned} \quad (64)$$

Suppose that $\hat{E}^{p,q}_{K,L} = \gamma^{-2}(\gamma^2 \hat{L}_q(K_p) - D^\top \hat{P}^{p,q}_{K,L})^\top(\gamma^2\hat{L}_q(K_p) - D^\top \hat{P}^j_K)$. It follows that since $\hat{A}^{p,q+1}_{K,L}$ is Hurwitz, $\hat{P}^{(p,q+1)}_{K,L} - \hat{P}^{p,q}_{K,L}$ becomes

$$\int_0^\infty e^{\hat{A}^{(p,q+1)^\top}_{K,L} t}\left[\hat{E}^{p,q}_{K,L} - \gamma^2 \tilde{L}_q^\top(K_p)\tilde{L}_q^\top(K_p)\right] e^{\hat{A}^{(p,q+1)}_{K,L} t} dt. \tag{65}$$

Now let $\hat{F}^q_K = \int_0^\infty e^{\hat{A}^{(p,q+1)^\top}_{K,L} t}\hat{E}^{p,q}_{K,L} e^{\hat{A}^{(p,q+1)}_{K,L} t} dt$ so that

$$\begin{aligned} &P^{p,q+1}_{K,L} - \hat{P}^{p,q+1}_{K,L} = P_{K,L} - \hat{P}^p_{K,L} - \hat{F}^q_K \\ &+ \int_0^\infty e^{\hat{A}^{(p,q+1)^\top}_{K,L} t}\left(\gamma^2 \tilde{L}_q^\top(K_p)\tilde{L}_q(K_p)\right) e^{\hat{A}^{p,q+1}_{K,L} t} dt. \end{aligned} \tag{66}$$

Let $f_K = \sup_{q\in\mathbb{N}_+}\|\hat{A}^{p,q+1}_{K,L}\|$. From Lemma 7, we can write $-\|\hat{F}^q_K\| \leq -\frac{\log(5/4)}{2f_K}\|\hat{E}^{p,q}_{K,L}\|$. Furthermore, by Lemma 7, we can write $-\|\hat{E}^{p,q}_{K,L}\| \leq -\frac{1}{c(K)}Tr(P^{p,q}_{K,L} - \hat{P}^{p,q}_{K,L})$, where $c(K) = Tr(\int_0^\infty e^{(A_K + DL_q(K_p))t} e^{(A_K+DL_q(K_p))^\top t} dt)$. Therefore, the trace of (66) becomes

$$\begin{aligned} &Tr(P^{p,q}_{K,L} - \hat{P}^{p,q+1}_{K,L}) \leq \left(1 - \frac{\log(5/4)}{2f_K c(K)}\right) Tr(P_K - \hat{P}^{p,q}_{K,L}) \\ &+ Tr\left(\int_0^\infty e^{(\hat{A}^{p,q+1}_{K,L})t} e^{(\hat{A}^{p,q+1}_{K,L})^\top t} dt\right)\gamma^2\|\tilde{L}_q(K_p)\|^2. \end{aligned} \tag{67}$$

$$\text{Let } g = \sup_{q\in\mathbb{N}_+} Tr\left(\int_0^\infty e^{(\hat{A}^{p,q+1}_{K,L})t} e^{(\hat{A}^{p,q+1}_{K,L})^\top t} dt\right), \tag{68}$$

and $\hat{\beta}(K) = 1 - \frac{\log(5/4)}{2f_K c(K)}$, so that

$$Tr(P^{p,q}_{K,L} - \hat{P}^{p,q}_{K,L}) \leq \hat{\beta}^{j-1}(K) Tr(P^{p,q}_{K,L}) + \lambda(\|\tilde{L}\|_\infty), \tag{69}$$

where $\lambda(\|\tilde{L}\|_\infty) := \frac{1}{1-\hat{\beta}(K)}\gamma^2 g \|\tilde{L}\|^2_\infty$. As $\|P_K - \hat{P}^{p,q}_{K,L}\|_F \leq Tr(P^{p,q}_{K,L} - \hat{P}^{p,q}_{K,L})$, we establish the theorem. □

From Theorem 4, as $j\to\infty$, $\hat{P}^{p,q}_{K,L}$ approaches the solution P_K and enters the ball centered by $P^{p,q}_{K,L}$. The radius of ball is proportional to $\|\tilde{L}\|_\infty$. Hence, the proposed inner-loop iterative algorithm approximates $P^{p,q}_{K,L}$ well despite the perturbation.

IV. Numerical Experiments

We consider a humanoid robot model [45], [46] as a three-link kinematic chain. The system is non-minimum phase, underactuated, and possesses badly damped poles. In addition, owing to the passive joint, there exists inherent (Wiener process) noise that additively perturbs the system dynamics.

This model has three states: two upper hinge (the hip and knee) *actuated joints* and a lower hinge (the ankle) passive joint. The state's velocity dynamics is $x = [\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3]^T$, where θ_1, θ_2, and θ_3 are the angles of the ankle, hip, and knee respectively. The linearized model of the triple inverted pendulum admits a form of the infinite dimensional linear PDE in (1), where $A \in \mathbb{R}^{6\times 6}$ and $B \in \mathbb{R}^{6\times 2}$(see [47, Section 3]), and $D = \left[0_{3\times 3}, I_3\right]^\top$. We impose an $\mathcal{H}_\infty$ norm bound of $\gamma = 5$ on the robot and set the initial state to $x(0) = [0, -5, 10, 10, -10, 10]^\top$ and set $C = \left[I_6, 0_{2\times 6}\right]^\top$, $E = \left[0_{6\times 2}, I_2\right]^\top$. Throughout, $w(t)$ is set to a Wiener process such that dw is drawn from $\mathcal{N}(0, I)$ and we chose a time step size, $dt = 0.0001$ for the numerical integration.

TABLE I: Computational Time of Alg. 1 vs. NPG.

Computational time (sec)					
Double Inverted Pendulum			Triple Inverted Pendulum		
Alg. 1	Fig. 1	NPG	Alg. 1	Fig. 1	NPG
0.0901	0.3061	2.1649	0.1455	0.7829	2.3209

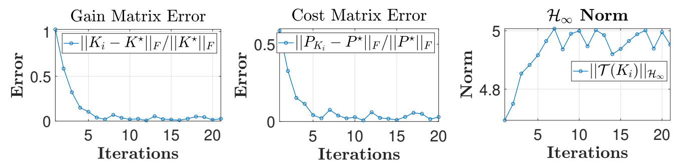

Fig. 2: Alg. 1 with $\|\tilde{K}\|_\infty = 0.15$.

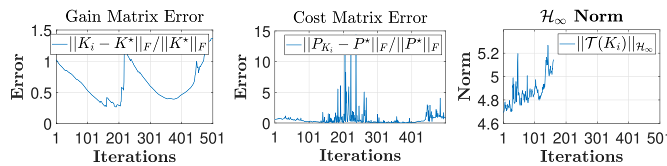

Fig. 3: Model-based Alg. 1 vs. NPG with $\|\tilde{K}\|_\infty = 0.1$.

In addition, we consider a double pendulum and compare the efficacy of the algorithms we have presented thus far against NPG. In what follows, we report various findings when running the model-based and model-free algorithm versus the natural policy gradient algorithm (NPG) [31], which shares similar character with our PI-based PO scheme. For other numerical experiment reports, we refer readers to our recent conference paper [36].

A. Model-based Mixed Design vs. NPG

Let us describe numerical experiments on the algorithms described so far. At each iteration, $\tilde{K}_p$ is sampled from a uniform Gaussian distribution whose Frobenius norm is 0.15. We chose

$$\hat{K}_0 = \begin{bmatrix} -203.3 & -74.2 & -31.4 & -67.7 & -28.4 & -16.5 \\ -529.5 & -198.8 & -77.8 & -175.5 & -78.7 & -39.0 \end{bmatrix}.$$

The results are shown in Figures 2 and 3. The robust mixed design PI scheme approaches the optimal solution after the 5'th iteration (See Fig. 2). At the last iteration, the deviation from the optimal cost matrix[5] is 2.9%, while the gain error[6] is 2.6%. In contrast, NPG exhibits cost matrix and controller gain errors that are unbounded as the iteration lengthens.

We compared the time it takes to compute the optimal policies in Alg. 1 against NPG in Table I. We see that for the double and triple inverted pendulums, the computational time of our algorithm is much less than that of NPG by around 90%. This is in fact a validation of our superior convergence rate (i.e. a global linear and local quadratic rate) compared to NPG's sublinear convergence rate.

[5]Calculated as $\|\hat{P}^{20}_K - P^\star\|/\|P^\star\|_F$.
[6]Calculated as $\|\hat{K}^{20}_K - K^\star\|_F/\|K^\star\|_F$.

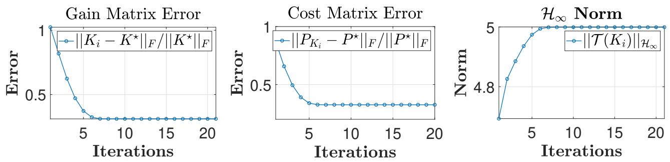

Fig. 4: Sampling-based Scheme Results.

B. Sampling-based Mixed Design vs. NPG

For the parameters of the algorithm in Fig. 1, we set $\bar{p} = 20$ and $\bar{q} = 30$, and found the maximum data collection time before attaining the full rank condition of Lemma 9 to be $t_l = 1500s$. The parameters of the A and B matrices are unknown but the initial controller $\hat{K}_1 \in \mathcal{K}$ is searched for following [36, Alg. 1].

We run the algorithm in Fig. 1 on (1). As seen in Fig. 4, the controller $\hat{K}_p$ found at each iteration converges after 5 iterations alongside $\hat{P}_{K_i}$ also converges. At 20th iteration, the relative error of $||\hat{K}_{20} - K_*||/||K_*|| = 31.5\%$ and $||\hat{P}_{K_{20}} - P_*||/||P_*|| = 31.6\%$. These demonstrate that the proposed algorithm can find an approximate optimal solution using the noisy data.

Appendix A: Proofs to Lemmas

In this appendix, we introduce a series of lemmas to guide our problem description and proposed solution.

Proof of Lemma 4. Since $R \succ 0$, $E_K = 0$ implies $K = K'$. Therefore at $E_K = 0$, we must have $K = K'$ which implies that $P_K = P'_K$. If $K = K'$ and $P_K = P'_K$, it suffices to conclude that $K' = K \equiv K^\star$ where $K^\star = R^{-1}B^\top P^\star$. Hence, $E_K = 0$ is tantamount to $P_K = P^\star$ and $K = K^\star$. □

Proof of Lemma 5. Define $A^\star = A - BR^{-1}B^TP^\star + \gamma^{-2}DD^TP^\star$ so that (20) becomes

$$\begin{aligned} &A^{\star\top}P_K + P_KA^\star + Q_K + (K^\star - K)^\top RK' \\ &+ K'^\top R(K^\star - K) - \gamma^{-2}P^\star DD^\top P_K - \gamma^{-2}P_KDD^\top P^\star \\ &+ \gamma^{-2}P_KDD^\top P_K = 0, \end{aligned} \tag{A.1}$$

In addition, (5) can be rewritten (replacing α with γ)as

$$\begin{aligned} &A^{\star\top}P^\star + P^\star A^\star + Q + K^{\star\top}RK^\star \\ &- \gamma^{-2}P^\star DD^\top P^\star = 0. \end{aligned} \tag{A.2}$$

Subtracting (A.2) from (A.1) and completing squares, we have

$$\begin{aligned} &A^{\star\top}(P_K - P^\star) + (P_K - P^\star)A^\star + E_K \\ &+ \gamma^{-2}(P_K - P^\star)DD^\top(P_K - P^\star) \\ &- (K' - K^\star)^\top R(K' - K^\star) = 0. \end{aligned} \tag{A.3}$$

Let $\Delta P_K := P_K - P^\star$. It follows from $K^\star = R^{-1}B^\top P^\star$ and (A.3) that

$$\begin{aligned} &A^{\star\top}\Delta P_K + \Delta P_K(A - BR^{-1}B^\top P_K + \gamma^{-2}DD^\top P_K) \\ &\quad + E_K = 0, \end{aligned} \tag{A.4}$$

whereupon $\mathcal{A}(K)\mathtt{vect}(\Delta P_K) = -\mathtt{vect}(E_K)$ with $\mathcal{A}(K)$ being

$$\begin{aligned} \mathcal{A}(K) := \{&(I_n \otimes A^{\star\top}) \\ &+ [(A - BR^{-1}B^\top P_K + \gamma^{-2}DD^\top P_K)^\top \otimes I_n]\}. \end{aligned} \tag{A.5}$$

From (20) and the implicit function theorem, P_K is a continuously differentiable function of $K \in \mathcal{K}$. Since $A^\star$ is Hurwitz, there exists a ball $\mathcal{B}(K^\star, \delta) := \{K \in \mathcal{K} | \|K - K^\star\|_F \leq \delta\}$, such that $\mathcal{A}(K)$ is invertible for any $K \in \mathcal{K}_h \cap \mathcal{B}(K^\star, \delta)$. Therefore, for any $K \in \mathcal{K}_h \cap \mathcal{B}(K^\star, \delta)$, it follows that

$$\|\Delta P_K\|_F \leq \underline{\sigma}^{-1}(\mathcal{A}(K))\|E_K\|_F. \tag{A.6}$$

On the other hand, for any $K \in \mathcal{K}_h \cap \mathcal{B}^c(K^\star, \delta)$, where $\mathcal{B}^c$ is the complement of $\mathcal{B}$, $E_K \neq 0$, and there exists a constant $b_1 > 0$, such that $\|E_K\|_F \geq b_1$. Thus, by Lemma 15, we have

$$\|\Delta P_K\|_F \leq Tr(P_K) \leq \frac{h + Tr(P^\star)}{b_1}\|E_K\|_F. \tag{A.7}$$

Suppose that $b_2 := \max_{K \in \mathcal{K}_h \cap \mathcal{B}(K^\star, \delta)} \underline{\sigma}^{-1}(\mathcal{A}(K))$ and $b(h) := \max\{b_2, \frac{h+Tr(P^\star)}{b_1}\}$, then the proof follows from (A.6) and the foregoing. □

Proof of Lemma 7. Subtracting (29) from (20), and using $L(K^\star) = \gamma^{-2}D^\top P_K$, we find that

$$\begin{aligned} &(A_K + DL(K^\star))^\top(P_K - P_K^L) + (P_K - P_K^L)(A_K + \\ &DL(K^\star)) + E_K^L - \gamma^{-2}(P_K^L - P_K)DD^\top(P_K^L - P_K) = 0. \end{aligned} \tag{A.8}$$

Since $A_K + DL(K^\star)$ is Hurwitz, it follows from [48, Lemma 3.18] that we must have

$$P_K - P_K^L \preceq \int_0^\infty e^{(A_K + DL(K^\star))^\top t}E_K^L e^{(A_K + DL(K^\star))t}\mathrm{d}t. \tag{A.9}$$

Taking the trace of the lhs, using [44, Theorem 2], and the cyclic property of the trace, the proof follows. □

Lemma 13 : *Assume $A \in \mathbb{R}^{n\times n}$ is Hurwitz and satisfies $A^TP + PA + Q = 0$. Then, the following properties hold*

1) *$P = \int_0^\infty e^{A^Tt}Qe^{At}\mathrm{d}t$;*
2) *$P \succ 0$ if $Q \succ 0$, and $P \succeq 0$ if $Q \succeq 0$;*
3) *If $Q \succeq 0$, then (Q, A) is observable iff $P \succ 0$;*
4) *For $P' \in \mathbb{S}^n$ satisfying $A^TP' + P'A + Q' = 0$, where $Q' \preceq Q$, we have $P' \preceq P$.*

Lemma 14 : *[48, Lemma 3.19] Suppose that P satisfies $A^TP + PA + Q = 0$, then the following statements hold:*

1) *A is Hurwitz if $P \succ 0$ and $Q \succ 0$.*
2) *A is Hurwitz if $P \succeq 0$, $Q \succeq 0$ and (Q, A) is detectable.*

Proof of Lemma 13. The first three statements are proven in [48, Lemma 3.18]. Consequently, P' can be expressed as

$$P' = \int_0^\infty e^{A^Tt}Q'e^{At}\mathrm{d}t. \tag{A.10}$$

Since $Q' \preceq Q$, $P' \preceq P$. □

Lemma 15 : *Norm of a Matrix Trace [49, Theorem 4.2.2] For any positive semi-definite matrix $P \in \mathbb{S}^n$, $\|P\|_F \leq Tr(P) \leq \sqrt{n}\|P\|_F$, and $\|P\| \leq Tr(P) \leq n\|P\|$. For any $x \in \mathbb{R}^n$, $x^TPx \geq \underline{\lambda}(P)\|x\|^2$.*

Lemma 16 : *For $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{n\times p}$, $\|XY\|_F \le \|X\|\|Y\|_F$.*

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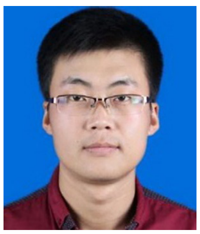

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