Robust Policy Optimization in Continuous-time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Stochastic Control

Leilei Cui and Lekan Molu (*Member, IEEE*)

Abstract—Following the recent resurgence in establishing linear control theoretic benchmarks for reinforcement leaning (RL)-based policy optimization (PO) for complex dynamical systems with continuous state and action spaces, an optimal control problem for a continuous-time infinite-dimensional linear stochastic system possessing additive Brownian motion is optimized on a cost that is an exponent of the quadratic form of the state, input, and disturbance terms. We lay out a model-based and model-free algorithm for RL-based stochastic PO. For the model-based algorithm, we establish rigorous convergence guarantees. For the sampling-based algorithm, over trajectory arcs that emanate from the phase space, we find that the Hamilton-Jacobi Bellman equation parameterizes trajectory costs — resulting in a discrete-time (input and state-based) sampling scheme accompanied by unknown nonlinear dynamics with continuous-time policy iterates. The need for known dynamics operators is circumvented and we arrive at a reinforced PO algorithm (via policy iteration) where an upper bound on the $\mathcal{H}_2$ norm is minimized (to guarantee stability) and a robustness metric is enforced by maximizing the cost with respect to a controller that includes the level of noise attenuation specified by the system's H_∞ norm. Rigorous robustness analyses is prescribed in an input-to-state stability formalism. Our analyses and contributions are distinguished by many natural systems characterized by additive Wiener process, amenable to Îto's stochastic differential calculus in dynamic game settings.

Index Terms—Optimal control, Robust control, $\mathcal{H}_\infty$ control, Iterative learning control, Machine learning.

I. INTRODUCTION

Lately, various system-theoretic results analyzing the global convergence [1] and computational complexity [2] of non-convex, constrained [3] gradient-based [4] and derivative-free [5] policy optimization in sampling-based reinforcement learning (RL) when the complete set of decision (or state feedback) variables are not previously known have appeared as control benchmarks [6], [7]. The most basic setting consists in optimizing over a decision variable K which must be determined from a (restricted) class of controllers $\mathcal{K}$ i.e. $\min_{K \in \mathcal{K}} J(K)$ where $J(K)$ is an objective (e.g. tracking error, safety assurance, goal-reaching measure of performance e.t.c.) required to be satisfied. In principle, K can be realized as a linear controller, a linear-in-the-parameters polynomial, or as a nonlinear kernel in the form of a radial basis function, or neural network.

These policy optimization (PO) schemes apply to a broad range of problems and have enjoyed wide success in complex systems where analytic models are difficult to derive [8]. While they have become a popular tool for modern learning-based control [9], the theoretical underpinning of their convergence, sample complexity, and robustness guarantees are little understood *in the large*. Only recently have rigorous analyses tools emerged [6], [10] for benchmarking RL with deterministic and additive Gaussian disturbance linear quadratic (LQ) controllers [11], [1].

Tools for analyzing the convergence, sample complexity, or robustness of RL-based PO largely fall into one of infinite-horizon (i) discrete-time LQ regulator (LQR) settings i.e.

$$\min_{K \in \mathcal{K}} \mathbb{E} \sum_{t=0}^{\infty} (x_t^\top Q x_t + u_t^\top R u_t) \text{ s.t. } x_{t+1} = A x_t + B u_t, x_0 \sim \mathcal{P}_0$$

where $A, B, Q,$ and R are standard LQR matrices for state x_t, control input u_t and x_0 is drawn from a random distribution P_0 [1]; (ii) discrete-time LQ problems under multiplicative noise i.e. $\min_{\pi \in \Pi} \mathbb{E}_{x_0, \{\delta_i\}, \{\gamma_i\}} \sum_{t=0}^{\infty} (x_t^\top Q x_t + u_t^\top R u_t)$ subject to $x_{t+1} = (A + \sum_{i=1}^{p} \delta_{ti} A_i) x_t + (B + \sum_{i=1}^{q} \gamma_{ti} B_i) u_t$ with covariance $\mathbb{E}_{x_0}[x_0 x_0^T]$ and A, B, Q, R are the standard LQR matrices with δ_{ti} and γ_{tj} serving as the i.i.d zero-mean and mutually independent multiplicative noise terms [4]; or (iii) Risk-sensitive $\mathcal{H}_\infty$-control [12] and discrete- and continuous-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design [13], [3] where the upper bound on the $\mathcal{H}_2$ cost is minimized subject to satisfying a set of risk-sensitive (often $\mathcal{H}_\infty$) constraints that *attenuate* [14] an unknown disturbance. i.e. $\min_{K \in \mathcal{K}} J(K) := Tr(P_K DD^\top)$ subject to $\mathcal{K} := \{K | \rho(A - BK) < 1, \|T_{zw}(K)\|_\infty < \gamma\}$ where P_K is the solution to the generalized algebraic Riccati equation (GARE), A, B, D, K are standard closed-loop system matrices, $\|T_{zw}(K)\|_\infty$ denotes the $\mathcal{H}_\infty$-norm of the closed-loop transfer function from a disturbance input w to its output z, and $\gamma > 0$, Here, $\gamma > 0$, upper-bounded by $\gamma^\star$, a scalar measure of system risk-sensitivity [15].

We focus on continuous-time linear systems in which disturbances enter additively as random stochastic Wiener processes following recent efforts on policy optimization for LQ regulator problems [1]; these systems may be modeled more accurately with uncertain additive Brownian noise where diffusion processes modeled with Îto's stochastic calculus are the theoretical machinery for analysis. Prominent systems featuring such additive Wiener processes occur in economics and finance [16], stock options trading [17], protein kinetics, population growth models, and models involving computations with round-off error in floating point arithmetic calculations such as overparameterized neural network dynamics.

Our goal is to keep a controlled process, z, small in an infinite-horizon constrained optimization setting under a

L. Cui is with the Control and Networks Lab, Department of Electrical and Computer Engineering, Tandon School of Engineering, New York University, Brooklyn, NY 11201, USA. (email: l.cui@nyu.edu).

L. Molu is with the Reinforcement Learning group at Microsoft Research, 300 Lafayette Street, New York, NY 10012, USA. (email: lekanmolu@microsoft.com).

minimizing policy $u \in \mathcal{U}$ in spite of unforeseen additive vector-valued stochastic Brownian process $w(t) \in \mathbb{R}^q$ — which may be of large noise intensity. In terms of the L_2 norm, we can write $\|z\|_2 = \left(\int |z(t)|^2 dt \right)^{1/2}$. The associated performance criteria can be realized as minimizing the expected value of the risk-sensitive linear exponential functions of positive definite quadratic forms state and control variables

$$\min_{u \in \mathcal{U}} \mathcal{J}_{exp}(x_0, u, w) := \mathbb{E}\Big|_{x_0 \in \mathcal{P}_0} \exp\left[\frac{\alpha}{2} \int_0^\infty z^\top(t) z(t) dt \right],$$
$$\text{subject to } dx(t) = Ax(t) dt + Bu(t) dt + D dw(t),$$
$$z(t) = Cx(t) + Eu(t), \ \alpha > 0 \quad (1)$$

with state process $x \in \mathbb{R}^n$, output process $z \in \mathbb{R}^p$ to be controlled, and control input $u \in \mathbb{R}^m$. The derivative of $w(t) \in \mathbb{R}^v$ *i.e.* dw/dt is a zero-mean Gaussian white noise with variance W, and $x(0)$ is a zero-mean Gaussian random vector independent of $w(t)$, $z(0) = 0$, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{n \times q}$, and $E \in \mathbb{R}^{p \times m}$ are constant matrix functions. The random signal $x(0)$ and the process $w(t)$ are defined over a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that we carry out a Taylor series expansion about $\alpha = 0$ in (1), the variance term, $\alpha^2 \text{var}(\int_0^\infty z^\top z)$, will be small after minimization. Thus, α can be seen as a measure of *risk-aversion* if $\alpha > 0$. It is important to note that in this paper, we only consider state feedback when $\alpha > 0$. In particular when noise is present in the system, the value of α signifies the level of noise attenuation that penalizes the covariance matrix of the system's noise.

We adopt an adaptive policy optimzation policy iteration (PI) method in a continuous PO scheme. This can be seen as an instance of the actor-critic (AC) configuration in RL-based *online* policy optimization schemes. Without explicit access to internal dynamics (system matrices), we iterate between steps of policy evaluation and policy improvement. Mimicking the actor in an RL AC setting, a parameterized controller must be evaluated relative to a parameterized cost function (the critic). The new policy is then used to improve the erstwhile (actor) policy by aiming to drive the cost to an extremal on the overall.

Contributions: We focus on the more sophisticated case of optimizing an *unknown stochastic linear policy* class $\mathcal{K}$ in an *infinite-horizon* LQ cost setting such that optimization iterates enjoy the *implicit regularization (IR) property* [7]—satisfying $\mathcal{H}_\infty$ robustness constraints. We place PO for *continuous-time linear stochastic controllers* on a rigorous *global convergence* and *robustness* footing. This is a distinguishing feature of our work. Our contributions are stated below.

- We propose a *two-loop iterative alternating best-response procedure* for computing the optimal *mixed-design policy* parameterized by *continuous time linear quadratic stochastic control*– that accelerates the optimization scheme's convergence – in model- and sampling-based cases;
- Rigorous convergence analyses follow for the model-based loop updates;
- In the absence of exact system models, we provide a robust PO scheme as a hybrid system with discrete-time samples from a nonlinear dynamical system. Its robust-

ness is analyzed in an input-to-state (ISS) framework for robustness to perturbations and uncertainties for loop updates.

- Lastly, we benchmark our results against the natural policy gradient [18] in the spirit of recent system-theoretic analysis works [1], [19], [20], [3].

Notations: The set of all symmetric matrices with dimension n is $\mathbb{S}^n$ and $\mathbb{R}$ (respectively $\mathbb{N}_+$) is the set of real numbers (resp. positive integers). The Kronecker product is denoted by $\otimes$. The Euclidean (Frobenius) norm of a vector or the spectral norm of a matrix is $\|\cdot\|$ ($\|\cdot\|_F$). Let $\|\cdot\|_\infty$ denote the supremum norm of a matrix-valued signals, i.e. $\|\Delta\|_\infty = \sup_{s \in \mathbb{Z}_+} \|\Delta_s\|_F$. The open ball of radius δ is $\mathcal{B}_\delta(X) = \{ Y \in \mathbb{R}^{m \times n} | \|Y - X\|_F < \delta \}$. The maximum and minimum singular values (eigenvalues) of a matrix A are respectively denoted by $\bar{\sigma}(A)$ ($\bar{\lambda}(A)$) and $\sigma_{min}(A)$ ($\lambda_{min}(A)$). The eigenvalues of $A \in \mathbb{R}^{n \times n}$ are $\lambda_i(A)$ for $i = 1, 2, \cdots, n$. For the transfer function $G(s)$, its $\mathcal{H}_\infty$ norm is $\|G\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega))$.

The n-dimensional identity matrix is I_n. The full vectorization of $X \in \mathbb{R}^{m \times n}$ is $\text{vec}(X) = [x_{11}, x_{21}, \cdots, x_{m1}, x_{12}, \cdots, x_{m2}, \cdots, x_{mn}]^\top$. Let $P \in \mathbb{S}^n$; the half-vectorization of P is the $n(n+1)/2$ column vectorization of the upper-triangular part of P: $\text{svec}(P) = [p_{11}, \sqrt{2}p_{12}, \cdots, \sqrt{2}p_{1n}, p_{22}, \cdots, \sqrt{2}p_{n-1,n}, p_{nn}]^\top$. The vectorization of the dot product $\langle x, x^\top \rangle$, where $x \in \mathbb{R}^n$, is $\text{vecv}(x) = [x_1^2, \cdots, x_1 x_n, x_2^2, x_2 x_3, \cdots, x_n^2]^\top$.

Paper Structure: A linear exponential quadratic Gaussian (LEQG) stochastic optimal control connection to dynamic games is first established in Section II. In §III, we present a nested double-loop procedure for robust policy recovery in a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ PO (in model-free and model-based) settings; this is followed by a rigorous analysis of their convergence and robustness properties. We demonstrate the efficacy of our proposed algorithm on numerical examples, and discuss findings in §IV. To maintain coherency and adhere to paper length constraints, detailed proofs are deferred to the appendix.

II. PO: DYNAMIC GAMES CONNECTION

In this section, we connect PO under linear controllers to the theory of two-person dynamic games.

Assumption 1 : *We take $C^\top C \triangleq Q \succ 0$, $E^T (C, E) = (0, R)$ for some matrix-valued function $R \succ 0$; and since in (1), we want $w(t)$ to be statistically independent, we take $DD^T = 0$. Seeking a linear feedback controller for (1), we require that the pair (A, B) be stabilizable. We expect to compute solutions via an optimization process, therefore we require that unstable modes of A must be observable through Q. Whence $(\sqrt{Q}, A)$ must be detectable.*

Given Assumption 1, the LEQG cost functional now becomes

$$\mathcal{J}_{exp}(x_0, u) = \mathbb{E}\Big|_{x_0 \in \mathcal{P}_0} \exp\left\{ \frac{\alpha}{2} \int_0^\infty \big[x^\top(t) Q x(t) + u^\top(t) R u(t) \big] dt \right\}, \quad (2)$$

for a fixed $\alpha > 0$ and the closed-loop transfer function is

$$T_{zw}(K) = (C - EK)(sI - A + BK)^{-1}D. \qquad (3)$$

The set of all *suboptimal* controllers that robustly stabilizes the linear system against all (finite gain) stable perturbations Σ, interconnected to the system by $w = \Sigma z$, such that $\|\Sigma\|_\infty \leq 1/\gamma$ is

$$\mathcal{K} = \{K : \lambda_i(A - B_1 K) < 0, \ \|T_{zw}(K)\|_\infty < \gamma\}. \qquad (4)$$

Proposition 1 : *[21, Th. II.1] The optimal control to the LEQG optimization problem (1) and cost functional (2) under u in the infinite-horizon setting is of a linear-in-the-data form i.e. $u^\star(t) = -K_{leqg}^\star \hat{x}(t)$ where gain $K_{leqg}^\star = R^{-1}B^\top P_\tau$, and P_τ is the unique, symmetric, positive definite solution to the algebraic Riccati equation (ARE)*

$$A^\top P_\tau + P_\tau A - P_\tau (BR^{-1}B^\top - \alpha^{-2}DD^\top)P_\tau = -Q. \qquad (5)$$

Corollary 1 : *In the infinite-horizon time-invariant case with constant system matrices and a stabilizable (A, B), by the theorem on "limit of monotonic operators" [22] and [15, Theorem 9.7], we find that $P^\star \triangleq P_\infty = \lim_{\tau \to \infty} P_\tau$, and $K_{leqg}^\star \triangleq K_\infty = \lim_{\tau \to \infty} K_\tau$.*

Remark 1 : *It is well-known that directly solving the LEQG problem (1) in policy-gradient frameworks incurs biased gradient estimates during iterations; this may affect the preservation of risk-sensitivity in infinite-horizon LTI settings (see [5], [7]). As such, we introduce a workaround with an equivalent dynamic game formulation to the stochastic LQ PO control problem.*

Lemma 1 (Closed-loop Two-Player Game Connection): *Consider the parameterized soft-constrained upper value, with a stochastically perturbed noise process $w(t)$, which enters the system dynamics as an additive bounded Gaussian with known statistics[1],*

$$\min_{u \in \mathcal{U}} \max_{\xi \in W} \bar{\mathcal{J}}_\gamma(x_0, u, \xi) := \mathbb{E}\Big|_{x_0 \sim \mathcal{P}_0, \xi(t)} \int_0^\infty \big[x^\top(t)Qx(t) +$$
$$u^\top(t)Ru(t) - \gamma^2 \xi^\top(t)\xi(t)\big] \, \mathrm{d}t$$
$$\text{subject to } \mathrm{d}x(t) = Ax(t)\mathrm{d}t + Bu(t)\mathrm{d}t + D\xi(t),$$
$$z(t) = Cx(t) + Eu(t) \qquad (6)$$

with $\xi(\equiv dw)$ as the zero-mean Gaussian noise with variance W(equivalent to dw/dt in (1)), scalar $\gamma > 0$ denoting the level of disturbance attenuation, and x_0 an arbitrary initial state. Suppose that there exists a non-negative definite (nnd) solution of (5) (with α replaced by γ), then its minimal realization, $P^\star$, exists. If $(A, Q^{\frac{1}{2}})$ is observable, then every nnd solution $P^\star$ of (5) is positive definite. For a nnd $P^\star$, there exists a common upper and lower value for the game and if $\bar{\mathcal{J}}_\gamma$ is finite for some $\gamma := \hat{\gamma} > 0$, then $\bar{\mathcal{J}}_\gamma$ is bounded (if and only if the pair (A, B) is stabilizable) and equivalent to the lower value $\underline{\mathcal{J}}_\gamma$[2]. In addition, for a bounded $\bar{\mathcal{J}}_\gamma$ for some $\gamma = \hat{\gamma}$ and for optimal

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[1] Since the time derivative of a Brownian process $w(t)$ is $dw(t)$, we maximize over the Gaussian $dw(t)$, rather than the unbounded stochastic noise $w(t)$.

[2] The lower value is constructed by reversing the order of play in the value defined in (6).

gain matrices, $K^\star = R^{-1}B^\top P_{K,L}$, $L^\star = \gamma^{-2}D^\top P_{K,L}$, $\bar{\mathcal{J}}_\gamma$ admits the following Hurwitz feedback matrices for all $\gamma > \hat{\gamma}$

$$A_K^\star = A - BK^\star, \ A_{K,L}^\star = A_K^\star + DL^\star \qquad (7)$$

where the nnd $P_{K,L}$ is the unique solution of (5) for $\gamma > \hat{\gamma}$ in the class of nnd matrices if it renders $A_{K,L}^\star$ Hurwitz. Whence, the saddle-point optimal controllers are

$$u^\star(x(t)) = -K^\star x(t), \ \xi^\star(x(t)) = L^\star x(t). \qquad (8)$$

Proof. The proof follows that in [15, Th. 9.7] exactly if we preserve the γ^{-1} term in the ARE of equation 9.31 in [15] and replace it by γ^{-2} as we have here. $\qquad \square$

For any stabilizing control pair (K, L), if (8) is applied to the system in (6), the resulting cost from (6) is $\bar{J}_\gamma = (x_0^\top P_{K,L} x_0)$ [23]. In what follows, we present a double-loop iterative solver for the gains K and L in (8) – in model-based and model-free settings.

III. POLICY OPTIMIZATION VIA POLICY ITERATION

We now present a special case to Kleinman's policy iteration (PI) algorithm [23] in a PO setting via a nested double loop PI scheme when (i) exact models are known; this will provide a barometer for our later analysis when (ii) exact system models are unknown.

Let p and q be indices of nested iterations between updating the closed-loop minimizing player's controller K_p (in an outer loop) and the maximizing player's controller $L_q(K_p)$ (in an inner-loop) for $p = \{1, 2, \ldots, \bar{p}\}$ and $q = \{1, 2, \ldots, \bar{q}\}$ for $(\bar{p}, \bar{q}) \in \mathbb{N}_+$. Furthermore, define the identities

$$A_K^p = A - BK_p, \quad A_{K,L}^{p,q} = A_K^p + DL_q(K_p),$$
$$Q_K^p = Q + K_p^\top RK_p, \ A_K^\gamma = A_K^p + \gamma^{-2}DD^\top P_K^p. \qquad (9)$$

For the soft-constrained value functional (6) at the p'th iterate of the minimizing controller K we have the following value iteration form for (5),

$$A_K^{p\top} P_K^p + P_K^p A_K^p + Q_K^p + \gamma^{-2} P_K^p DD^\top P_K^p = 0, \quad (10a)$$
$$K_{p+1} = R^{-1}B^\top P_K^p \qquad (10b)$$

where P_K^p is the p'th iterate's solution to (10). Similarly, for the maximizing controller, $L_q(K_p)$, the following closed-loop continuous-time ARE (CARE) iteration applies

$$A_{K,L}^{(p,q)\top} P_{K,L}^{p,q} + P_{K,L}^{p,q} A_{K,L}^{p,q} + Q_K^p - \gamma^2 L_q^\top(K_p)L_q(K_p) = 0 \qquad (11a)$$

$$K_{p+1} = R^{-1}B^\top P_K^{p,q}, \ L_{q+1}(K_p) = \gamma^{-2}D^\top P_{K,L}^{p,q} \qquad (11b)$$

where $P_{K,L}^{p,q}$ is the solution to (11) for gains $[K_p, L_q(K_p)]$. Choosing a stabilizing minimizing player control gain, K_p we first evaluate u's performance by solving (10). *This is the policy evaluation step in PI.* The policy is then improved in a following iteration by solving for the cost matrix in (11b) – *this is the policy improvement step.* The process can thus be seen as *a policy iteration algorithm where the performance of an initial control gain K_p is first evaluated against a cost*

Algorithm 1: (Model-Based) PO via Policy Iteration

Input: Max. outer iteration $\bar{p}$, $q = 0$, and an $\epsilon > 0$;

Input: Desired risk attenuation level $\gamma > 0$;

Input: Minimizing player's control matrix $R \succ 0$.

1   Compute $(K_0, L_0) \in \mathcal{K}$;   ▷ From [24, Alg. 1];

2   Set $P_{K,L}^{0,0} = Q_K^0$;   ▷ See equation (9);

3   **for** $p = 0, \ldots, \bar{p}$ **do**

4   Compute Q_K^p and A_K^p ▷ See equation (9);

5   Obtain P_K^p by evaluating K_p on (10);

6   **while** $\|P_K^p - P_{K,L}^{p,q}\|_F \leq \epsilon$ **do**

7   Compute $L_{q+1}(K_p) := \gamma^{-2} D^\top P_{K,L}^{p,q}$;

8   Solve (11) until $\|P_K^p - P_{K,L}^{p,q}\|_F \leq \epsilon$;

9       $\bar{q} \leftarrow q + 1$

10   **end**

11   Compute $K_{p+1} = R^{-1} B^\top P_{K,L}^{p,\bar{q}}$ ▷ See (11b) ;

12 **end**

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function. A newer evaluation of the cost matrix $P_{K,L}^{p,q}$ is then used to improve the controller gain K_{p+1} in the outer loop.

Problem 1 (Model-Based Policy Iteration): *Given system matrices A, B, C, D, E, find the optimal controller gains K_p, $L_q(K_p)$ that robustly stabilizes (1) such that the controller gains do not leave the set of all suboptimal controllers denoted by*

$$\breve{\mathcal{K}} = \{(K_p, L_q(K_p)) : \lambda_i(A_K^p) < 0, \lambda_i(A_{K,L}^{p,q}) < 0,$$
$$\|T_{zw}(K_p, L_q(K_p))\|_\infty < \gamma \text{ for all } (p,q) \in \mathbb{N}\}. \quad (12)$$

A. Double Loop (DL) Successive Substitution

The procedure for obtaining the optimal $P^\star$ in Problem 1 is described in Algorithm 1. It finds a global Nash Equilibrium (NE) (or equivalently a saddle-point equilibrium) [19] of the LQ zero-sum game (6) by solving the nonlinear ARE (11) in a nested two-loop policy iteration (PI) scheme.

An initial (K_0, L_0) control pair that guarantees the iterates' feasibility upon projection onto the set $\breve{\mathcal{K}}$ is first determined in order to enforce the condition (12). We refer readers to our recent conference paper [24] where this procedure is further elaborated[3]. Afterwards, the Riccati equation's (11) solution i.e. $P_{K,L}^{0,0} \triangleq Q_K^0$ must be computed and the gains $[K_p, L_q(K_p)]$ are updated as in (11b). *As an RL-based PO procedure, Line 11 in Alg. 1 can be seen as a reinforcement over the iterates $\{p, p+1\}$. After the last update of the inner loop maximizing player gain $L_{\bar{q}}(K_p)$, the outer-loop update of the minimizing controller gain K_p robustly stabilizes the closed-loop transfer function from w to z i.e. $T_{zw}(K_p, L_q(K_p))$ under gains K_p and $L_{\bar{q}}(K_p)$ against all (finite gain) disturbances Σ interconnected to system (6) (by $w = \Sigma z$) such that $\|\Sigma\|_\infty \leq \gamma^{-2}$.*

B. Outer Loop Stability, Optimality, and Convergence

We now discuss the convergence guarantees of the iterations under perfect dynamics. Let us first introduce the preliminary results.

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[3]We remark that this can also be found by solving a convex optimization problem if the Riccati equation is formulated as a linear matrix inequality [25].

Lemma 2 : *Under Assumption 1 and for the ARE (10), if $K_0 \in \mathcal{K}$[4], then for any $p \in \mathbb{N}_+$, we must have the following conditions for the optimal $K^\star$ and $P^\star$,*

(1) $K_p \in \mathcal{K}$;

(2) $P_K^0 \succeq P_K^1 \succeq \cdots P_K^p \succeq \cdots \succeq P^\star$;

(3) $\lim_{p \to \infty} \|K_p - K^*\|_F = 0$, $\lim_{p \to \infty} \|P_K^p - P^*\|_F = 0$.

We provide a straightforward proof (with the robustness operator γ of (10)) in the appendix. An alternate proof is given in [23]. In [7, Theorem A.7 and A.8], the authors showed that the controller update phase in the outer-loop has a global sub-linear and local quadratic convergence rates. We now demonstrate that the outer-loop iteration has a global linear convergence rate. Let us first establish a few preliminary results that we will need in the proof of our main result.

Lemma 3 : *Let $\Psi = (K_{p+1} - K_p)^\top R(K_{p+1} - K_p)$; and $\Psi = \Psi^\top \succeq 0$. Furthermore, let $\Phi \in \mathbb{R}^{n \times n}$ be Hurwitz so that $\Theta = \int_0^\infty e^{(\Phi^\top t)} \Psi e^{(\Phi t)} dt$ and define $c(\Phi) = \log(5/4)\|\Phi\|^{-1}$. Then, $\|\Theta\| \geq \frac{1}{2} c(\Phi)\|\Psi\|$.*

Proof. Define $S(t) = \sum_{k=1}^\infty (\Phi t)^k / k!$ so that $e^{\Phi t} = I_n + \sum_{k=1}^\infty (\Phi t)^k / k! \triangleq I_n + S(t)$ after a Taylor series expansion. Whence $\|S(t)\| = \sum_{k=1}^\infty (\|\Phi\| t)^k / k!$ or $\|S(t)\| \geq e^{\|\Phi\| t} - 1$. For $x_0 \neq 0$ satisfying $x_0^\top \Psi x_0 = \|\Psi\| \|x_0\|^2$, it can be verified that

$$x_0^\top \Theta x_0 = \int_0^\infty x_0^\top e^{\Phi^\top} \Psi e^\Phi x_0 dt \geq \int_0^{c(\Phi)} x_0^\top e^{\Phi^\top} \Psi e^\Phi x_0 dt,$$
$$\geq \int_0^{c(\Phi)} \frac{1}{2} \|\Psi\| \|x_0\|^2 dt \geq \frac{1}{2} c(\Phi) \|\Psi\| \|x_0\|^2. \quad (13)$$

A fortiori, Lemma 3's proof follows from (13). $\square$

Remark 2 : *For $A_K = A - BK$, we know from the bounded real Lemma [7, Lemma A.1] that the Riccati equation*

$$A_K^\top P_K + P_K A_K + Q_K + \gamma^{-2} P_K D D^\top P_K = 0 \quad (14)$$

admits a unique positive definite solution $P_K \succ 0$ with a Hurwitz $(A_K + \gamma^{-2} D D^\top P_K)$.

Lemma 4 (Optimality of the iteration): *Consider any $K \in \mathcal{K}$, let $K' = R^{-1} B^\top P_K$ (where P_K is the solution to (14), and $\Psi_K = (K - K')^\top R(K - K')$. If $\Psi_K = 0$, then $K = K^\star$.*

Lemma 5 (Bound on Cost Difference Matrix): *For any $h > 0$, define $\mathcal{K}_h := \{K \in \mathcal{K} | Tr(P_K^p - P^\star) \leq h\}$. For any $K \in \mathcal{K}_h$, let $K' := R^{-1} B^\top P_K^p$, where P_K^p is the p'th iterate solution to (14), and $\Psi_{K_p} = (K_p - K_p')^\top R(K_p - K_p')$. Then, there exists $b(h) > 0$, such that $\|P_K^p - P^\star\|_F \leq b(h)\|\Psi_{K_p}\|_F$.*

Theorem 1 : *For any $h > 0$ and $K_0 \in \mathcal{K}_h$, there exists $\alpha(h) \in \mathbb{R}$ such that $Tr(P_K^{p+1} - P^\star) \leq \alpha(h) Tr(P_K^p - P^\star)$. That is, $P^\star$ is an exponentially stable equilibrium.*

Proof. Since $[P_K^p - P_K^{p+1}]$ satisfies the Lyapunov equation (A.5), $(K_{p+1} - K_p)^\top R(K_{p+1} - K_p) \geq 0$ implies that $P_K^p -$

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[4]The $\mathcal{K}$ defined here refers to the one defined in (4).

$P_K^{p+1} \succeq 0$ by Lemma 14. Hence $A_K^{(p+1)}$ must be Hurwitz going by Lemma 15. Define $H_K^p = \int_0^\infty e^{A_K^{(p+1)^\top} t} \Psi_p e^{A_K^{(p+1)} t} dt$ where Ψ_p is as defined in Lemma 3. Thus, we have by the second statement of Lemma 15 that the cost matrix admits the form (see statement 1 of Lemma 14)

$$P_K^p - P_K^{p+1} \succeq H_K^p. \tag{15}$$

From Lemma 2, we have for a $p > 0$ that $P_K^0 \succeq P_K^p$ so that for an $h \geq 0$, we find that

$$\|A_K^{p+1}\| \leq \|A\| + (\|BR^{-1}B^\top\| + \gamma^{-2}\|DD^\top\|)h. \tag{16}$$

Set $c(h) = \log(5/4)/\|A\| + (\|BR^{-1}B^\top\| + \gamma^{-2}\|DD^\top\|)h$ so that we have $\|H_K^p\| \geq \frac{1}{2}c(h)\|\Psi_p\|$ from Lemma 3. Using Lemmas 5 and 13, and taking the trace of (15) we find that

$$\begin{aligned} Tr(P_K^{p+1} - P^\star) &\leq Tr(P_K^p - P^\star) - Tr(H_K^p), \\ &\leq Tr(P_K^p - P^\star) - c(h)\|\Psi_p\|/2, \\ &\leq Tr(P_K^p - P^\star) - \frac{\sqrt{n}c(h)}{2n}\|\Psi_p\|_F, \\ &\leq Tr(P_K^p - P^\star) - \frac{c(h)}{2\sqrt{n}b(h)}\|P_K^p - P^\star\|_F, \\ &\leq \left(1 - \frac{c(h)}{2nb(h)}\right)Tr(P_K^p - P^\star). \end{aligned} \tag{17}$$

The proof follows if we set $\alpha(h) = 1 - c(h)/2nb(h)$. $\qquad\square$

C. Inner Loop Stability, Optimality, and Convergence

We now analyze the monotonic convergence rate of the inner loop. Given arbitrary gains $K_p \in \mathcal{K}$ and $L_q(K_p)$, let $P_{K,L}^{p,q}$ be the positive definite solution of the associated Lyapunov equation (11). The following lemma shows that the cost matrix $P_{K,L}^{p,q}$ monotonically converges to (11)'s solution.

Lemma 6 : *Suppose that $L_0(K_0)$ is stabilizing, then for any $q \in \mathbb{N}_+$ (with $P_{K,L}^{p,\bar{q}}$ as the solution to (11)),*

1) *$A_{K,L}^{p,q}$ is Hurwitz;*
2) *$P_{K,L}^{p,\bar{q}} \succeq \cdots \succeq P_K^{(p,q+1)} \succeq P_K^{p,q} \succeq \cdots \succeq P_{K,L}^{p,0}$; and*
3) *$\lim_{q\to\infty}\|P_{K,L}^{p,q} - P_{K,L}^{p,\bar{q}}\|_F = 0$.*

A proof is provided in the Appendix. We next analyze the monotonic convergence of the inner loop of the nested double loop algorithm. Let us first discuss a preliminary result.

Lemma 7 (Monotonic Convergence of the Inner-Loop): *For any $K \in \mathcal{K}$, let $L(K)$ be the control gain for the player w such that $A_K + DL(K)$ is Hurwitz. Let P_K^L be the solution of*

$$\begin{aligned} (A_K + DL(K))^\top P_K^L + P_K^L (A_K + DL(K)) + Q_K \\ - \gamma^2 L(K)^\top L(K) = 0. \end{aligned} \tag{18}$$

Let $L'(K) = \gamma^{-2}D^\top P_K^L$ and $\Psi_K^L = \gamma^{-2}(L'(K) - L(K))^\top(L'(K) - L(K))$. Then, for a $c(K) = Tr\left(\int_0^\infty e^{(A_K+DL(K^\star))t}e^{(A_K+DL(K^\star))^\top t}dt\right)$, the following inequality holds $Tr(P_K - P_K^L) \leq \|\Psi_K^L\|c(K)$.

Theorem 2 : *For a $K \in \check{\mathcal{K}}$, and for any $(p,q) \in \mathbb{N}_+$, there exists $\beta(K) \in \mathbb{R}$ such that*

$$Tr(P_K^p - P_{K,L}^{p,q+1}) \leq \beta(K)Tr(P_K^p - P_{K,L}^{p,q}). \tag{19}$$

Proof. Define $\Psi_K^q = \gamma^2 [L_{q+1} - L_q]^\top [L_{q+1} - L_q]$ in (A.18) and $F_K^q = \int_0^\infty e^{(A_{K,L}^{(p,q+1)})^\top t}\Psi_K^q e^{A_{K,L}^{(p,q+1)}t} dt$. By Lemma 6, $A_{K,L}^{p,q+1}$ is Hurwitz in (A.18) so that from Lemma 14 we have

$$P_{K,L}^{p,q+1} - P_{K,L}^{p,q} = F_K^q. \tag{20}$$

By Lemma 6, $P_{K,L}^{p,q+1} \succeq P_{K,L}^{p,q}$ so that

$$\|A_{K,L}^{(p,q+1)}\| \leq \|A - BK_p\| + \gamma^{-2}\|DD^\top\|\|P_K^p\|. \tag{21}$$

Let $d(K) = \log(5/4)/\left(\|A_K\| + \gamma^{-2}\|DD^\top\|\|P_K^p\|\right),\tag{22}$

so that subtracting both sides of (20) from P_K^p and taking the trace of the resulting expression, we find that

$$\begin{aligned} Tr(P_K^p - P_{K,L}^{p,q+1}) &= Tr(P_K^p - P_{K,L}^{p,q}) - Tr(F_K^q), &\text{(23a)} \\ &\leq Tr(P_K^p - P_{K,L}^{p,q}) - \|F_K^q\|, &\text{(23b)} \\ &\leq Tr(P_K^p - P_{K,L}^{p,q}) - \frac{1}{2}d(K)\|\Psi_K^q\|, &\text{(23c)} \end{aligned}$$

where we have used Lemma 13 to arrive at the inequality in (23b), and extended Lemma 3 to arrive at (23c) since $\Psi_K^q = \Psi_K^{q^\top}$. Furthermore, from Lemma 7

$$Tr(P_K^p - P_{K,L}^{p,q}) \leq \left(1 - \frac{d(K)}{2c(K)}\right)Tr(P_K^p - P_{K,L}^{p,q}). \tag{24}$$

The proof follows if we set $\beta(K) = 1 - d(K)/2c(K)$. $\qquad\square$

Remark 3 : *As seen from Lemma 6, $P_K^p - P_{K,L}^{p,q} \succeq 0$. From Lemma 13 and the result of Theorem 2, we have $\|P_K - P_{K,L}^{p,q}\|_F \leq Tr(P_K - P_{K,L}^{p,q}) \leq \beta(K)Tr(P_K)$, i.e. $P_{K,L}^{p,q}$ exponentially converges to P_K in the Frobenius norm.*

Lemma 8 (Uniform Convergence of Iterates): *For any $h > 0$, $K \in \mathcal{K}_h$, and $\epsilon > 0$, there exists $q'(h) \in \mathbb{N}_+$ independent of K, such that if $q \geq q'(h)$, $\|P_{K,L}^{p,q} - P_K\|_F \leq \epsilon$.*

Proof of Lemma 8. This Lemma is an immediate outcome of Theorems 1 and 2. $\qquad\square$

D. Sampling-based PO on Hybrid Discrete-Time Nonlinear System

The exact knowledge of the system matrices A, B, C, D, E are often unavailable so that the policy evaluation step will result in biased estimates. When errors are present from using I/O or state data for the PO procedure in Alg. 1, residuals from early termination of numerically solving Line 8 in Alg. 1, or using an approximate cost function owing to inexact values of Q and R, the algorithm may fail to converge.

Problem 2 (Sampling-based Policy Optimization): *If A, B, C, D, E, P are all replaced by approximate matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{P}$, under what conditions will the sequences $\{\hat{P}_{K,L}^{p,q}\}_{(p,q)=1}^{(p,q)=\infty}$, $\{\hat{K}_p\}_{p=0}^\infty$, $\{\hat{L}_q\}_{q=0}^\infty$ converge to a small neighborhood of the optimal values $\{P_{K,L}^\star\}_{(p,q)=0}^{(p,q)=\infty}$, $\{K_p^\star\}_{p=0}^\infty$, and $\{L_q^\star\}_{q=0}^\infty$.*

1) Discrete-Time Nonlinear System Interpretation

From Assumption 1, a $P_K^0 \in \mathbb{S}^n$ exists such that when applied to find K_0 i.e. $K_0 = R^{-1}B^\top P_K^0$, such a K_0 will be stabilizing. Now, factoring in approximation errors between the policy evaluation and improvement steps, we end up with a hybrid system consisting of a continuous-time policy gain pair $(\hat{K}_p, \hat{L}_q(\hat{K}_p))$ and a learning algorithm that is essentially a discrete sampled data from a nonlinear system (owing to errors from various sources). We will leverage Lemmas 2 and 6 to show that under inexact loop updates and lumping gain iterate estimate errors as system inputs to the online PO scheme, it converges to the optimal solution and closed-loop dynamic stability is guaranteed in an input-to-state stability framework (ISS) [26]. Hence the loops are discrete-time nonlinear systems.

2) Online (Model-Free) Nested Loop Reparameterization

Consider (10b) and suppose that $\hat{P}_K^0 \in \mathbb{S}^n$ is chosen following Assumption 1. It follows that a $\hat{K}_k^1 = R^{-1}B^\top \hat{P}_K^0$ will be stabilizing since $\tilde{K}_k^1 \equiv \hat{K}_k^1 - K_k^1 \triangleq 0$. The same argument applies for L_0. For $(p,q) > 0$, we must show that for $\tilde{K}_k^p \equiv \hat{K}_k^p - K_k^p \triangleq 0$ so that the sequence $\{P_{K,L}^{p,q}\}_{(p,q)=0}^\infty$ will converge to the locally exponentially stable $\hat{P}_{K,L}^\star$ going by Lemmas 2 and 6. We proceed by lumping the estimate errors as an input into the gain terms to be computed in the PO algorithm. Under inexact outer loop update, the iterate K_{p+1} becomes inaccurate so that the inexact outer-loop iteration involves the recursions

$$\hat{A}_K^{p\top}\hat{P}_K^p + \hat{P}_K^p \hat{A}_K^p + \hat{Q}_K^p + \gamma^{-2}\hat{P}_K^p DD^\top \hat{P}_K^p = 0, \quad (25a)$$

$$\hat{K}_{p+1} = R^{-1}B^\top \hat{P}_K^p + \tilde{K}_{p+1} \triangleq \bar{K}_{p+1} + \tilde{K}_{p+1}, \quad (25b)$$

where $\hat{A}_K^p = A - B\hat{K}_p$ and $\hat{Q}_K^p = Q + \hat{K}_p^\top R\hat{K}_p$. Similar argument applies to the inner loop updates so that the inexact inner loop update is

$$\hat{A}_{K,L}^{p,q\top}\hat{P}_{K,L}^{p,q} + \hat{P}_{K,L}^{p,q}\hat{A}_{K,L}^{p,q} + \hat{Q}_K^p - \gamma^2 \hat{L}_q^\top \hat{L}_q(\hat{K}_p) = 0 \quad (26a)$$

$$\hat{K}_{p+1} = R^{-1}B^\top \hat{P}_K^{p,q} + \tilde{K}_p, \quad (26b)$$

$$\hat{L}_{q+1}(\hat{K}_p) = \gamma^{-2}D^\top \hat{P}_{K,L}^{p,q} + \tilde{L}_{q+1}(\tilde{K}_p) \quad (26c)$$

$$\triangleq \bar{L}_{q+1}(\bar{K}_p) + \tilde{L}_{q+1}(\tilde{K}_p). \quad (26d)$$

Consider the transformation of the infinite-dimensional stochastic differential equation (1) in light of the identities (9) under inexact updates for $(p,q) > 0$

$$dx = [\hat{A}_{K,L}^{p,q}x + B(\hat{K}_p x - D\hat{L}_q(K_p) + u)]dt + Ddw. \quad (27)$$

On a time interval $[s, s+\delta s]$, it follows from Itô's stochastic calculus and the Hamilton-Jacobi-Bellman equation that

$$d\left[x^\top(s+\delta s)\hat{P}_{K,L}^{p,q}x(s+\delta s) - x^\top(s)\hat{P}_{K,L}^{p,q}x(s)\right] =$$
$$(dx)^\top \hat{P}_{K,L}^{p,q}x + x^\top \hat{P}_{K,L}^{p,q}dx + (dx)^\top \hat{P}_{K,L}^{p,q}(dx). \quad (28)$$

Along the trajectories of equation (27) and using the gains in (11), the r.h.s. in the foregoing becomes

$$x^\top\left[\hat{A}_{K,L}^{p,q\top}\hat{P}_{K,L}^{p,q} + \hat{P}_{K,L}^{p,q}\hat{A}_{K,L}^{p,q}\right]xdt + 2x^\top \hat{P}_{K,L}^{p,q}Ddw \quad (29)$$

$$+ 2x^\top \hat{P}_{K,L}^{p,q}B(K_p x - D\hat{L}_q(K_p) + u)dt + Tr(D^\top PD),$$

so that $x^\top(s+\delta s)\hat{P}_{K,L}^{p,q}(s+\delta s) - x^\top(s)\hat{P}_{K,L}^{p,q}x(s)$

$$= \int_s^{s+\delta s}\left[(-x^\top \hat{Q}_K^p x - \gamma^2 w^\top w)dt + 2\gamma^2 x^\top \hat{L}_{q+1}^\top(K_p)dw\right]$$

$$+ \int_s^{s+\delta s} 2x^\top \hat{K}_{p+1}^\top R\left[\hat{K}_p x - D\hat{L}_q(\hat{K}_p) + u\right]dt$$

$$+ \int_s^{s+\delta s} Tr(D^\top \hat{P}_{K,L}^{p,q}D)dt. \quad (30)$$

Observe: The system dependent matrices $\hat{A}_{K,L}^{p,q}, B, C, D$ from equation (29) are now replaced by input and state terms including $\hat{Q}_K^p$, $\hat{K}_{p+1}$, and $\hat{L}_{q+1}$ which are all retrievable via online measurements. *We essentially end up with an input-to-state system.* The price we pay is that the noise feedthrough matrix D must be known precisely. In this article, as is common in many linear stochastic system with Brownian motion, D is taken to be identity [27], [28].

3) Sampling-based PO Scheme

Our goal is to explore the system model until exact equality of $\hat{A}_{K,L}^{p,q}, \hat{P}_{K,L}^{p,q}$ and $\hat{K}_{p+1}, \hat{L}_{q+1}(K_p)$ to the corresponding terms in (11) occur. To this end, (30) allows us to explore with the controls $u = -K_0 x + \eta_p$ and $w = -L_0 x + \eta_q$ where (η_p, η_q) is drawn uniformly at random over matrices with a Frobenium norm r similar to [4], [1]. Let us now introduce the following identities,

$$x^\top \hat{Q}_K^p x = (x^\top \otimes x^\top)\,\mathrm{vec}(\hat{Q}_K^p),$$
$$\gamma^2 w^\top w = \gamma^2(w^\top \otimes w^\top)\,\mathrm{vec}(I_v),$$
$$2\gamma^2 x^\top \hat{L}_{q+1}^\top(\hat{K}_p)dw = 2\gamma^2(I_n \otimes x^\top)dw\,\mathrm{vec}(\hat{L}_{q+1}^\top(\hat{K}_p)),$$
$$2x^\top \hat{K}_{p+1}^\top R\hat{K}_p x = 2(x^\top \otimes x^\top)(I_n \otimes \hat{K}_p^\top)\,\mathrm{vec}(\hat{K}_{p+1}^\top R),$$
$$2x^\top \hat{K}_{p+1}^\top RD\hat{L}_q(\hat{K}_p) = 2(\hat{L}_q^\top(\hat{K}_p)D^\top \otimes x^\top)\,\mathrm{vec}(\hat{K}_{p+1}^\top R),$$
$$2x^\top \hat{K}_{p+1}^\top Ru = 2(u^\top \otimes x^\top)\,\mathrm{vec}(\hat{K}_{p+1}^\top R),$$
$$Tr(D^\top \hat{P}_{K,L}^{p,q}D) = \mathrm{vec}^\top(D)\,\mathrm{vec}(\hat{P}_{K,L}^{p,q}D). \quad (31)$$

Furthermore, consider the matrices $\Delta_{xx} \in \mathbb{R}^{\frac{n(n+1)}{2}l}$, $\Delta_{ww} \in \mathbb{R}^{\frac{v(v+1)}{2}l}$, $I_{xx} \in \mathbb{R}^{l \times n^2}$, and $I_{ux} \in \mathbb{R}^{l \times mn}$ for $l \in \mathbb{N}_+$ so that

$$\Delta_{xx} = [\mathrm{vecv}(x_1), \ldots, \mathrm{vecv}(x_l)]^\top, \ x_l = x_{l+1} - x_l,$$

$$\Delta_{ww} = [\mathrm{vecv}(w_1), \ldots, \mathrm{vecv}(w_l)]^\top, \ w_l = w_{l+1} - w_l,$$

$$I_{xx} = \left[\int_{s_0}^{s_1} x \otimes x\,dt, \ldots, \int_{s_{l-1}}^{s_l} x \otimes x\,dt\right]^\top,$$

$$I_{ww} = \left[\int_{s_0}^{s_1} w \otimes w\,dt, \ldots, \int_{s_{l-1}}^{s_l} w \otimes w\,dt\right]^\top,$$

$$I_{xw} = \left[\int_{s_0}^{s_1} (I_n \otimes x)dw, \ldots, \int_{s_{l-1}}^{s_l} (I_n \otimes x)dw\right]^\top,$$

$$I_{ux} = \left[\int_{s_0}^{s_1} u \otimes x\,dt, \ldots, \int_{s_{l-1}}^{s_l} u \otimes x\,dt\right]^\top. \quad (32)$$

Fig. 1: Flowchart for Sampling-based PO in Continuous-time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Stochastic Control.

Next, set

$$\Theta_{K,L}^{p,q} = \Big[\Delta_{xx}, -2I_{xx}(I_n \otimes \hat{K}_p^\top) + 2(\hat{L}_q^\top(\hat{K}_p)D^\top \otimes x^\top)$$
$$-2I_{ux}, -2\gamma^2 I_{xw}, -\text{vec}^\top(D)\text{vec}(\hat{P}_{K,L}^{p,q}D) \Big], \quad (33a)$$

$$\Upsilon_{K,L}^{p,q} = \Big[-I_{xx}\text{vec}(\hat{Q}_K^p), \; -\gamma^2 I_{ww}\text{vec}(I_v) \Big]. \quad (33b)$$

Define $\mathbf{1}_{q^2}$ as one-vector with dimension q^2. Thus,

$$\Theta_{K,L}^{p,q} \Big[\text{svec}(P_{K,L}^{p,q}) \quad \text{vec}(\hat{K}_{p+1}^\top R) \quad \text{vec}(\hat{L}_{q+1}^\top(\hat{K}_p)) \quad \mathbf{1}_{q^2} \Big]^\top$$
$$= \Upsilon_{K,L}^{p,q}. \quad (34)$$

Suppose that $\Theta_{K,L}^{p,q}$ is of full rank, then we can retrieve the unknown matrices via least squares estimation i.e.

$$\begin{bmatrix} \text{svec}(P_{K,L}^{p,q}) \\ \text{vec}(\hat{K}_{p+1}^\top R) \\ \text{vec}(\hat{L}_{q+1}^\top(\hat{K}_p))\text{d}w \\ \mathbf{1}_{q^2} \end{bmatrix} = (\Theta_{K,L}^{p,q\top}\Theta_{K,L}^{p,q})^{-1}\Theta_{K,L}^{p,q\top}\Upsilon_{K,L}^{p,q}. \quad (35)$$

We thus end up with a scheme for retrieving the system matrices provided that the algorithm is robust to perturbations upon iterating through (35) for each (p,q). The full scheme is summarized in the flowchart of Fig. 1. We next state the condition under which $\Theta_{K,L}^{p,q}$ is of full rank.

Lemma 9 : *[29, Lemma 6] If there exists an integer $l_0 > 0$ such that for all $l \geq l_0$, $\text{rank}(I_{xx}, I_{ux}, I_{xw}, \mathbf{1}_{q^2}) = n(n+1) + mn + nq + q^2$, then $\Theta_{K,L}^{p,q}$ has full rank for all $(p,q) \in (\bar{p}, \bar{q})$.*

Remark 4 : *Lemma 9 allows the convergence assurance of Fig. 1 under the condition that the rank condition be fulfilled.*

4) Robustness of Minimizing Controller to Perturbations

We now analyze the robustness of the sampling-based scheme as a hybrid nonlinear discrete time system gains with continuous-time dynamics. Let $\tilde{P} = P_K - \hat{P}_K$ and $\tilde{K} = K - \hat{K}$ denote errors arising from the inexact updates.

Lemma 10 (Outer-Loop Robustness to Perturbations): *For any $K \in \mathcal{K}$, there exists an $e(K) > 0$ such that for a perturbation $\tilde{K}$, $K + \tilde{K} \in \mathcal{K}$, as long as $\|\tilde{K}\| < e(K)$.*

As long as $\tilde{K}$ is small, if we start with a robustly stabilizing $K \in \mathcal{K}$, we can guarantee the feasibility of the iterates.

Theorem 3 : *The inexact outer loop is small-disturbance ISS. That is, for any $h > 0$ and $\hat{K}_0 \in \mathcal{K}_h$, if $\|\tilde{K}\| < f(h)$, there exist a $\mathcal{KL}$-function $\beta_1(\cdot, \cdot)$ and a $\mathcal{K}_\infty$-function $\gamma_1(\cdot)$ such that*

$$\|P_{\hat{K}}^p - P^\star\| \leq \beta_1(\|P_{\hat{K}}^0 - P^*\|, p) + \gamma_1(\|\tilde{K}\|). \quad (36)$$

Proof. From Lemma 12, $\hat{K}_K^p \in \mathcal{K}_h$ for any $p \in \mathbb{N}_+$. From (A.30), at the p'th iteration, we have

$$Tr(P_{\hat{K}}^p - P^\star) \leq (1 - \underline{f}_1(h))Tr(P_{\hat{K}}^{p-1} - P^\star)$$
$$+ \bar{f}_2(h)\|R\|\|\tilde{K}_K^p\|^2. \quad (37)$$

Repeating (37) for $p, p-1, \cdots, 1$,

$$Tr[P_{\hat{K}}^p - P^\star] \leq (1 - \underline{f}_1)^p Tr(P_{\hat{K}}^1 - P^\star) + \frac{\bar{f}_2\|R\|\|\tilde{K}\|_\infty^2}{\underline{f}_1(h)}. \quad (38)$$

It follows from (38) and [30, Theorem 2] that

$$\|P_{\hat{K}}^p - P^\star\|_F \leq (1 - \underline{f}_1)^p\sqrt{n}\|P_{\hat{K}}^1 - P^\star\|_F + \frac{\bar{f}_2\|R\|\|\tilde{K}\|_\infty^2}{\underline{f}_1}. \quad (39)$$

As $p \to \infty$, $P_{\hat{K}}^p \to P^\star$. The radius of the neighbor of $P^\star$ is proportional to $\|\tilde{K}\|_\infty^2$. Thus, the proof follows. $\square$

5) Robustness of Maximizing Controller to Perturbations

The perturbed inner-loop iteration (26) has inexact matrix $\hat{A}_{K,L}^{p,q}$, and sequences $\{\hat{L}_{q+1}(K_p)\}_{q=0}^\infty$, and $\{\hat{P}_{K,L}^{p,q}\}_{q=0}^\infty$. We next analyze its robustness to perturbations when it differs from the exact loop matrices and sequences.

Lemma 11 (Stability of the Inner-Loop's System Matrix): *Given $K \in \check{\mathcal{K}}$, there exists a $g \in \mathbb{R}_+$, such that if $\|\tilde{L}_{q+1}(K_p)\|_F \leq g$, $\hat{A}_{K,L}^{p,q}$ is Hurwitz for all $q \in \mathbb{N}_+$.*

Lemma 12 : *For any $h > 0$ and $K \in \mathcal{K}_h$, let $K' = R^{-1}B^\top P_K$, where P_K is the solution of (14), and $\hat{K}' = K' + \tilde{K}$. Then, there exists $f(h) > 0$, such that $\hat{K}' \in \mathcal{K}_h$ as long as $\|\tilde{K}\| < f(h)$.*

Theorem 4 : *Assume $\|\tilde{L}_q(K_p)\| < e$ for all $q \in \mathbb{N}_+$. There exists $\hat{\beta}(K) \in [0, 1)$, and $\lambda(\cdot) \in \check{\mathcal{K}}_\infty$, such that*

$$\|\hat{P}_{K,L}^{p,q} - P_{K,L}^{p,q}\|_F \leq \hat{\beta}^{q-1}(K)Tr(P_{K,L}^{p,q}) + \lambda(\|\tilde{L}\|_\infty). \quad (40)$$

From Theorem 4, as $q \to \infty$, $\hat{P}_{K,L}^{p,q}$ approaches the solution P_K and enters the ball centered at $P_{K,L}^{p,q}$ with radius proportional to $\|\tilde{L}\|_\infty$. Hence, the proposed inner-loop iterative algorithm well approximates $P_{K,L}^{p,q}$.

TABLE I: Computational Time: Model-based PO vs. NPG.

Policy Optimization Computational time (secs)					
Double Inverted Pendulum			Triple Inverted Pendulum		
Model-based	Model-free	NPG	Model-based	Model-free	NPG
0.0901	0.3061	2.1649	0.1455	0.7829	2.3209

IV. NUMERICAL EXPERIMENTS

We consider a humanoid robot model [31], [32] in the form of a three-link kinematic chain; and a standard double pendulum. The humanoid is non-minimum phase, underactuated, and possesses badly damped poles. Its passive joint can be modeled as a Wiener process noise that additively perturbs its dynamics.

This model has three states: two upper hinge (the hip and knee) *actuated joints* and a lower hinge (the ankle) *passive joint*. The dynamics is $x = [\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3]^\top$, where θ_1, θ_2, and θ_3 are the angles of the ankle, hip, and knee respectively. The linearized model of the triple inverted pendulum admits a form of the infinite dimensional linear PDE in (1), where $A \in \mathbb{R}^{6\times6}$ and $B \in \mathbb{R}^{6\times2}$ (see [33, Section 3]), and $D = \begin{bmatrix} 0_{3\times3}, I_3 \end{bmatrix}^\top$. We impose an $\mathcal{H}_\infty$ norm bound of $\gamma = 5$ on the robot, set the initial state to $x(0) = [0, -5, 10, 10, -10, 10]^\top$ and set $C = \begin{bmatrix} I_6, 0_{2\times6} \end{bmatrix}^\top$, $E = \begin{bmatrix} 0_{6\times2}, I_2 \end{bmatrix}^\top$. Throughout, $w(t)$ is set to a Wiener process such that its time derivative dw is drawn from a zero-mean Gaussian distribution with variance $\mathcal{X}^2$. We chose a step size, $dt = 0.0001$. We next report our findings for the model-based, model-free algorithm, and the natural policy gradient algorithm (NPG) [18]. For other numerical experiment reports, we refer readers to our recent conference paper [24].

A. Model-based Mixed Design vs. NPG

Let us describe numerical experiments on the algorithms described so far. At each iteration, $\tilde{K}_p$ is sampled from a uniform Gaussian distribution whose Frobenius norm is 0.15. We found

$$\hat{K}_0 = \begin{bmatrix} -203.3 & -74.2 & -31.4 & -67.7 & -28.4 & -16.5 \\ -529.5 & -198.8 & -77.8 & -175.5 & -78.7 & -39.0 \end{bmatrix}.$$

The results for running the model-based and NPD algorithms are shown in Figures 2 and 4. The robust mixed design PO scheme approaches the optimal solution after the 5'th iteration (See Fig. 2). At the last iteration, the deviation from the optimal cost matrix[5] is 2.9%, while the gain error[6] is 2.6%. In contrast, NPG exhibits cost matrix and controller gain errors that are unbounded as the iteration lengthens.

We compared the time it takes to compute the optimal policies in model-based nested algorithm against NPG in Table I. For the double and triple inverted pendulums, the computational time of our algorithm is much less than that of NPG by around 90%. This is a validation of our superior convergence rate compared to NPG's sublinear convergence rate.

B. Sampling-based Mixed Design vs. NPG

For the model-based algorithm, we set $\bar{p} = 20$ and found the maximum data collection time before attaining the full

[5]Calculated as $\|\hat{P}_K^{20} - P^\star\|/\|P^\star\|_F$.
[6]Calculated as $\|\hat{K}_K^{20} - K^\star\|_F/\|K^\star\|_F$.
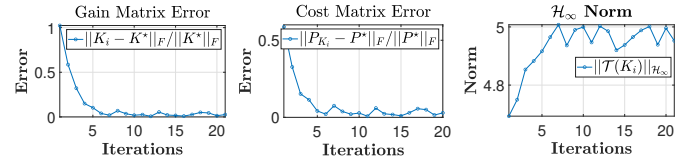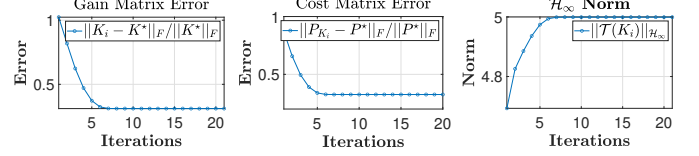

Fig. 2: Model-based design: $\|\tilde{K}\|_\infty = 0.15$.

Fig. 3: Sampling-based scheme results.

Fig. 4: Model-based design vs. NPG with $\|\tilde{K}\|_\infty = 0.1$.

rank condition of Lemma 9 to be $t_l = 1500s$. The system parameters A and B are unknown but the initial controller $\hat{K}_0 \in \mathcal{K}$ is searched for following [24, Alg. 1].

We run the sampling-based algorithm on (1). From the charts of Fig. 3, the controller $\hat{K}_p$ found at each iteration converges after 5 iterations alongside $\hat{P}_K^p$. At the 20'th iteration, the relative error $\|\hat{K}_{20} - K_*\|/\|K_*\| = 31.5\%$ and $\|\hat{P}_{K_{20}} - P_*\|/\|P_*\| = 31.6\%$. These demonstrate that the proposed algorithm does find an approximate optimal solution from the noisy data.

Comparing our algorithm under an additive Wiener process noise against the natural policy gradient, we find that the relative errors in gain and cost matrix errors are not well-behaved. The same disruption applies to the $\mathcal{H}_\infty$-norm plot. These further demonstrates the need for a robust PO scheme such as the one we have presented in this work for problems that fall into the class of systems (1).

V. CONCLUSIONS

We have proposed an iterative nested-loop reinforced PO scheme for the robust optimal control of linear plants with additive Wiener process. Rigorous convergence results is first established in model-based settings. For unknown model parameters, system trajectories' costs are expanded to obtain Riccati solution estimates for loops gains' computations. These gains may be biased owing to inexact knowledge of system dynamics; thus, we robustly bound deviations from known stable gain iterates in a small-disturbance input-to-state stability formalism. The learning pipeline's efficacy has been numerically tested with benchmarks against the natural policy gradient [18]. Our evaluations suggest recovery of stabilizing policies in the two problems we had set out to solve. We hope that our proposal guides researchers and practitioners the robust recovery of optimal policies for the stochastic infinite-dimensional linear PDEs with additive Wiener processes.

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APPENDIX A: LEMMAS AND PROOFS

In this appendix, we introduce a series of lemmas to guide our problem description and proposed solution.

Proof of Lemma 2. When $p = 0$, $K_0 \in \mathcal{K}$, and it satisfies (1) (See [24, Alg. 1].) For $p > 0$, introduce the identities,

$$RK_{p+1} = B^\top P_K^p, \qquad K_{p+1}^\top R = P_K^p B, \tag{A.1a}$$

$$A_K^{p\top} P_K^p = A_K^{(p+1)\top} P_K^p + (K_{p+1} - K_p)^\top B^\top P_K^p, \tag{A.1b}$$

$$P_K^p A_K^p = P_K^p A_K^{(p+1)} + P_K^p B(K_{p+1} - K_p). \tag{A.1c}$$

Therefore, equation (10) becomes

$$A_K^{(p+1)\top} P_K^p + P_K^p A_K^{(p+1)} + \gamma^{-2} P_K^p DD^\top P_K^p + C^\top C \tag{A.2}$$
$$+ K_{p+1}^\top RK_{p+1} + (K_{p+1} - K_p)^\top R(K_{p+1} - K_p) = 0.$$

Thus, for a stabilizing $K_{p+1}(\neq K_p)$ we must have $(K_{p+1} - K_p)^\top R(K_{p+1} - K_p) \succ 0$ so that

$$A_K^{(p+1)\top} P_K^p + P_K^p A_K^{(p+1)} + \gamma^{-2} P_K^p DD^\top P_K^p + Q_K^{p+1} \prec 0. \tag{A.3}$$

If (read: since) the inequality (A.3) holds, the bounded real Lemma [7, Lemma A.1, statement 3] stipulates that a $P_K^p \succ 0$ exists; by [7, Lemma A.1, statement 1], $\|T_{zw}(K_p)\|_\infty < \gamma$ given that $\lambda_i(A_K^{(p+1)}) < 0$ in (A.3). *A fortiori*, $K_p \in \mathcal{K}$ for $p > 0$ by the bounded real Lemma. This proves the first statement.

The proof for statement (2) now follows. At the $(p+1)$'th iteration, it can be verified that (10) admits the form

$$A_K^{(p+1)\top} P_K^{p+1} + P_K^{p+1} A_K^{(p+1)} + C^\top C + K_{p+1}^\top RK_{p+1}$$
$$+ \gamma^{-2} P_K^{p+1} DD^\top P_K^{p+1} = 0, \tag{A.4}$$

so that subtracting (A.4) from (A.2) (at the p'th iteration) and using the statistical independence property of the noise term $w(t)$ (from Ass. 1) i.e. $DD^\top = 0$, we have

$$A_K^{(p+1)\top}\left[P_K^p - P_K^{p+1}\right] + \left[P_K^p - P_K^{p+1}\right]A_K^{(p+1)}$$
$$(K_{p+1} - K_p)^\top R(K_{p+1} - K_p) = 0. \quad (A.5)$$

Observe: Equation (A.5) is a Lyapunov equation of the form

$$A_K^{(p+1)\top}\left[P_K^p - P_K^{p+1}\right] + \left[P_K^p - P_K^{p+1}\right]A_K^{(p+1)}$$
$$= -(K_{p+1} - K_p)^\top R(K_{p+1} - K_p) \triangleq -\Psi_p \quad (A.6)$$

Statement 1 of Lemma 15 implies that $A_K^{(p+1)}$ is Hurwitz since Ψ_p above is Hurwitz. Hence, we must have $P_K^p - P_K^{p+1} \succeq 0$ because $(K_{p+1} - K_p)^\top R(K_{p+1} - K) \succeq 0$ by statement (3) of Lemma 14. Whence, $P_K^p \succeq P_K^{p+1}$ and $K_{p+1} \geq K_p$. This proves the second statement. In this sentiment, the sequence $\{P_K^p\}_{p=0}^\infty$ is decreasing, bounded below by 0 and has a finite norm so that $\{P_K^p\}_{p=0}^\infty$ converges to P_K^∞. This satisfies (5). Observe from equation (10) that P_K^p is self-adjoint so that from the "limit of monotonic positive operators theorem" [22], p. 189], $\lim_{p\to\infty}\|P_K^p - P^*\|_F = 0$. By a similar argument for decreasing operators sequences [22, p. 190], the sequence $\{K_K^p\}_{p=0}^\infty$ is increasing and upper bounded by K_K^∞. Hence, $\lim_{p\to\infty}\|K_p - K^*\|_F = 0$. The third statement is thus proven. $\square$

Proof of Lemma 4. Since $R \succ 0$, $\Psi_K = 0$ implies $K = K'$. Therefore at $\Psi_K = 0$, we must have $K = K'$ which implies that $P_K = P_K'$. If $K = K'$ and $P_K = P_K'$, it suffices to conclude that $K' = K \triangleq K^\star$ where $K^\star = R^{-1}B^\top P^\star$. Hence, $\Psi_K = 0$ is tantamount to $P_K = P^\star$ and $K = K^\star$. $\square$

Proof of Lemma 5. Define $A^\star = A - BR^{-1}B^\top P^\star + \gamma^{-2}DD^\top P^\star$ so that (14) becomes

$$A^{\star\top}P_K^p + P_K^p A^\star + Q_{K_p} + (K^\star - K_p)^\top RK_p'$$
$$+ K_p'^\top R(K^\star - K_p) - \gamma^{-2}P^\star DD^\top P_K^p - \gamma^{-2}P_K^p DD^\top P^\star$$
$$+ \gamma^{-2}P_K^p DD^\top P_K^p = 0. \quad (A.7)$$

In addition, (5) can be rewritten (replacing α with γ) as

$$A^{\star\top}P^\star + P^\star A^\star + Q + K^{\star\top}RK^\star - \gamma^{-2}P^\star DD^\top P^\star = 0. \quad (A.8)$$

Subtracting (A.8) from (A.7) and completing squares, we have

$$A^{\star\top}(P_K^p - P^\star) + (P_K^p - P^\star)A^\star + \Psi_{K_p}$$
$$+ \gamma^{-2}(P_K^p - P^\star)DD^\top(P_K^p - P^\star) \quad (A.9)$$
$$- (K_p' - K^\star)^\top R(K_p' - K^\star) = 0.$$

Let $\tilde{P}_K^p := P_K^p - P^\star$. It follows from $K_p^\star = R^{-1}B^\top P^\star$ and (A.9) that

$$A^{\star\top}\tilde{P}_K^p + \tilde{P}_K^p(A - BR^{-1}B^\top P_K^p + \gamma^{-2}DD^\top P_K^p)$$
$$+ \Psi_{K_p} = 0, \quad (A.10)$$

whereupon $\mathcal{A}(K_p)\text{vec}(\tilde{P}_K^p) = -\text{vec}(\Psi_K)$ with $\mathcal{A}(K_p)$ being

$$I_n \otimes A^{\star\top} + (A - BR^{-1}B^\top P_K^p + \gamma^{-2}DD^\top P_K^p)^\top \otimes I_n. \quad (A.11)$$

From (14) and the implicit function theorem, P_K^p is a continuously differentiable function of $K_p \in \mathcal{K}$. Since $A^\star$ is Hurwitz, there exists a ball $\mathcal{B}_\delta(K^\star) := \{K \in \mathcal{K}|\|K - K^\star\|_F \leq \delta\}$, such that $\mathcal{A}(K)$ is invertible for any $K \in \mathcal{K}_h \cap \mathcal{B}_\delta(K^\star)$. Therefore, for any $K \in \mathcal{K}_h \cap \mathcal{B}_\delta(K^\star)$, it follows that

$$\|\tilde{P}_K^p\|_F \leq \underline{\sigma}^{-1}(\mathcal{A}(K_p))\|\Psi_{K_p}\|_F. \quad (A.12)$$

On the other hand, for any $K \in \mathcal{K}_h \cap \mathcal{B}_\delta^c(K^\star)$, where $\mathcal{B}^c$ is a complement of $\mathcal{B}$, $\Psi_{K_p} \neq 0$ and there exists a constant $b_1 > 0$ such that $\|\Psi_{K_p}\| \geq b_1$. Thus, by Lemma 13, we have

$$\|\tilde{P}_K^p\|_F \leq Tr(P_K^p) \leq \frac{h + Tr(P^\star)}{b_1}\|\Psi_{K_p}\|_F. \quad (A.13)$$

Suppose that $b_2 = \max_{K \in \mathcal{K}_h \cap \mathcal{B}_\delta(K^\star)}\underline{\sigma}^{-1}(\mathcal{A}(K))$ and $b(h) = \max\{b_2, \frac{h+Tr(P^\star)}{b_1}\}$, then the proof follows from (A.12) and the foregoing. $\square$

Proof of Lemma 6. To prove the first statement, we proceed by induction. For a $p \geq 0$ we have $K_p \in \check{\mathcal{K}}$ by Theorem 1. Subtracting (11) from (10) yields

$$0 = A_K^{p\top}(P_K^p - P_{K,L}^{p,q}) + (P_K^p - P_{K,L}^{p,q})A_K^p +$$
$$\gamma^2\left[L_{q+1}(K_p) - L_q(K_p)\right]^\top[L_{q+1}(K_p) - L_q(K_p)]. \quad (A.14)$$

In equation (A.14), we have that $[L_{q+1}(K_p) - L_q(K_p)]^\top[L_{q+1}(K_p) - L_q(K_p)] \succeq 0$ so that (A.14) admits a Lyapunov equation form. Following statement 2 of Lemma 14, we must have $(P_K^p - P_{K,L}^{p,q}) \succeq 0$. *A fortiori*, we must have $A_{K,L}^{(p,q)}$ as Hurwitz in (A.14) following statement 2 of Lemma 14. This proves the first statement.

To prove the second statement, we abuse notation by dropping the templated argument in $L_q(K_p)$. Let us consider the identities,

$$A_{K,L}^{(p,q)\top}P_{K,L}^{p,q} = A_{K,L}^{(p,q+1)\top}P_{K,L}^{p,q} - \gamma^2[L_{q+1} - L_q]^\top L_{q+1}$$
$$P_{K,L}^{p,q}A_{K,L}^{(p,q)} = P_{K,L}^{p,q}A_{K,L}^{(p,q+1)} - \gamma^2 L_{q+1}^\top[L_{q+1} - L_q]. \quad (A.15)$$

We now rewrite (11) in light of (A.15) as

$$A_{K,L}^{(p,q+1)\top}P_{K,L}^{p,q} + P_{K,L}^{p,q}A_{K,L}^{(p,q+1)} - \gamma^2[L_{q+1} - L_q]^\top L_{q+1}$$
$$+ Q_K - \gamma^2 L_{q+1}^\top[L_{q+1} - L_q] - \gamma^2(L_q^\top L_q) = 0. \quad (A.16)$$

At the $(q+1)$'st iteration, we have (11) as

$$A_{K,L}^{(p,q+1)\top}P_{K,L}^{p,q+1} + P_{K,L}^{p,q+1}A_{K,L}^{(p,q+1)} + Q_K$$
$$- \gamma^2 L_{q+1}^\top(K_p)L_{q+1}(K_p) = 0. \quad (A.17)$$

Subtracting (A.16) from (A.17), we have

$$A_{K,L}^{(p,q+1)\top}\left[P_{K,L}^{p,q+1} - P_{K,L}^{p,q}\right] + \left[P_{K,L}^{p,q+1} - P_{K,L}^{p,q}\right]A_{K,L}^{(p,q+1)} +$$
$$+ \gamma^2[L_{q+1} - L_q]^\top[L_{q+1} - L_q] = 0. \quad (A.18)$$

Since $[L_{q+1} - L_q]^\top[L_{q+1} - L_q] \succeq 0$, (A.18) is indeed a Lyapunov equation so that $P_{K,L}^{p,q+1} \succeq P_{K,L}^{p,q}$ holds following Lemma 14. Whence, we must have $A_{K,L}^{(p,q+1)\top}$ Hurwitz. Following the argument for all $(q, q') \in \bar{q}$ with $q \neq q'$, statement 2) holds.

Observe: $P_{K,L}^{p,q}$ is self-adjoint by reason of (10). By the theorem on the "limit of monotonically decreasing operators" [22, pp. 190], statement 2) implies that the sequence $\{P_{K,L}^{p,\bar{q}}, \cdots, P_{K,L}^{p,q=0}\}$ is monotonically decreasing and bounded from above by $P_{K,L}^{p,\bar{q}} \equiv P_{K,L}^\star$. That is, $P_{K,L}^{p,q}$ exists and is the solution of (10) and $P_{K,L}^{p,\bar{q}}$ is the unique positive definite solution to (11). *A fortiori*, we must have $\lim_{q\to\infty} P_{K,L}^{p,q} = P_{K,L}^{p,\infty}$. This establishes the third statement. $\qquad\square$

Proof of Lemma 7. Subtracting (18) from (14), and using $L(K^\star) = \gamma^{-2} D^\top P_K$, we find that

$$(A_K + DL(K^\star))^\top (P_K - P_K^L) + (P_K - P_K^L)(A_K + \quad \text{(A.19)}$$
$$DL(K^\star)) + \Psi_K^L - \gamma^{-2}(P_K^L - P_K)DD^\top(P_K^L - P_K) = 0.$$

Since $A_K + DL(K^\star)$ is Hurwitz, it follows from statements (1) and (3) of Lemma 14 that we must have

$$P_K - P_K^L \preceq \int_0^\infty e^{(A_K + DL(K^\star))^\top t} \Psi_K^L e^{(A_K + DL(K^\star))t} \mathrm{d}t.$$
$$\text{(A.20)}$$

Taking the trace of the lhs, using [30, Theorem 2], and employing the cyclic property of the trace, the proof follows. $\quad\square$

Proof of Lemma 10. Let $F(\tilde{P}, \tilde{K})$ be

$$(A_K + \gamma^{-2}DD^\top P_K)^\top \tilde{P} + \tilde{P}(A_K + \gamma^{-2}DD^\top P_K)$$
$$- \tilde{K}^\top B^\top (P_K + \tilde{P}) - (P_K + \tilde{P})B\tilde{K} + \tilde{K}^\top RK + K^\top R\tilde{K}$$
$$+ \tilde{K}^\top R\tilde{K} + \gamma^{-2}\tilde{P}DD^\top \tilde{P}. \quad \text{(A.21)}$$

Observe: the pair $(P_K + \tilde{P}, K + \tilde{K})$ satisfies (14) iff $F(\tilde{P}, \tilde{K}) = 0$ and that $F(\tilde{P}, \tilde{K}) = 0$ implies an implicit function of $\tilde{P}$ with respect to $\tilde{K}$ since if $\tilde{P} \in \mathbb{S}^n$ exists, $\tilde{K}$ must exist under the controllability and observability assumptions of Ass 1. Let $\mathcal{F}(\tilde{P}, \tilde{K}) = \mathrm{vec}(F(\tilde{P}, \tilde{K}))$ so that,

$$\mathcal{F}(\tilde{P}, \tilde{K}) = \big[I_n \otimes (A_K + \gamma^{-2}DD^\top P_K)^\top$$
$$+ (A_K + \gamma^{-2}DD^\top P_K)^\top \otimes I_n \big] \mathrm{vec}(\tilde{P})$$
$$- (P_K B \otimes I_n)\mathrm{vec}(\tilde{K}^\top) - (I_n \otimes P_K B)\mathrm{vec}(\tilde{K})$$
$$- (I_n \otimes \tilde{K}^\top B^\top + \tilde{K}^\top B^\top \otimes I_n)\mathrm{vec}(\tilde{P}) \quad \text{(A.22)}$$
$$+ (K^\top R \otimes I_n)\mathrm{vec}(\tilde{K}^\top) + (I_n \otimes K^\top R)\mathrm{vec}(\tilde{K})$$
$$+ \mathrm{vec}(\tilde{K}^\top R\tilde{K}) + \gamma^{-2}\mathrm{vec}(\tilde{P}DD^\top \tilde{P}).$$

Thus,

$$\frac{\partial \mathcal{F}(\tilde{P}, \tilde{K})}{\partial \mathrm{vec}(\tilde{P})} = I_n \otimes [(A_K + \gamma^{-2}DD^\top P_K) - B\tilde{K}]^\top \quad \text{(A.23)}$$
$$+ [(A_K + \gamma^{-2}DD^\top P_K) - B\tilde{K}]^\top \otimes I_n + \tilde{P}DD^\top \otimes I_n$$
$$+ I_n \otimes \tilde{P}DD^\top - (P_K B \otimes I_n)\mathrm{vec}(\tilde{K}^\top) - (I_n \otimes P_K B)$$
$$\mathrm{vec}(\tilde{K}) + (K^\top R \otimes I_n)\mathrm{vec}(\tilde{K}^\top) + (I_n \otimes K^\top R)\mathrm{vec}(\tilde{K}),$$

where we have used [34, Theorem 9], to obtain $\partial \mathrm{vec}(\tilde{P}DD^\top \tilde{P})/\partial \mathrm{vec}(\tilde{P}) = \tilde{P}DD^\top \otimes I_n + I_n \otimes \tilde{P}DD^\top$. Since $\mathcal{F}(0, 0) = 0$, $(A_K + \gamma^{-2}DD^\top P_K)$ is Hurwitz, hence $\partial \mathcal{F}(\tilde{P}, \tilde{K})/\partial \mathrm{vec}(\tilde{P})|_{\tilde{P}=0, \tilde{K}=0}$ is invertible. From the implicit function theorem, there must exist an $e_1(K) > 0$, such that $\tilde{P}$ is continuously differentiable with respect to $\tilde{K}$ for any $\tilde{K} \in \mathcal{B}_{e_1(K)}(0)$. Thus, $\|\tilde{P}\| \to 0$ as $\|\tilde{K}\| \to 0$. Since $K \in \mathcal{K}$ by [7, Lemma A.1], we must have $P_K \succ 0$. Therefore,

there exists $e(K) > 0$, such that $\sigma_{\max}(\tilde{P}) < \sigma_{\min}(P_K)$, i.e. $P_K - \tilde{P} \succ 0$, as long as $\|\tilde{K}\| < e(K)$.

Since $\tilde{P}$ and $\tilde{K}$ satisfy $F(\tilde{P}, \tilde{K}) = 0$, we have

$$(A - BK - B\tilde{K})^T (P_K + \tilde{P}) + (P_K + \tilde{P}) + Q + \quad \text{(A.24)}$$
$$(K + \tilde{K})^T R(K + \tilde{K}) + \gamma^{-2}(P_K + \tilde{P})DD^T(P_K + \tilde{P}) = 0$$

Since $P_K + \tilde{P} \succ 0$ when $\|\tilde{K}\| < e(K)$, by [7, Lemma A.1], $K + \tilde{K} \in \mathcal{K}$. That is, if $\tilde{K}$ is small, if we start the PI with a robustly stabilizing $K \in \mathcal{K}$, we can guarantee the feasibility of the iterates. $\qquad\square$

Proof of Lemma 11. Define $\tilde{L}_q^\top(K_p) = L_q^\top(K_p) - \hat{L}_q^\top(K_p)$ and $\tilde{P}_{K,L}^{p,q} = P_{K,L}^{p,q} - \hat{P}_{K,L}^{p,q}$. Further, assume $\hat{A}_{K,L}^{p,q}$ is Hurwitz. From (11),

$$\hat{A}_{K,L}^{p,q\top} P_{K,L}^{p,q} + P_{K,L}^{p,q} \hat{A}_{K,L}^{p,q+1} + Q_K - \gamma^{-2} \hat{P}_{K,L}^{p,q} DD^\top \hat{P}_{K,L}^{p,q} +$$
$$\gamma^{-2}(P_{K,L}^{p,q} - \hat{P}_{K,L}^{p,q})DD^\top(P_{K,L}^{p,q} - \hat{P}_{K,L}^{p,q}) - \tilde{L}_q^\top(K_p)D^\top P_{K,L}^{p,q}$$
$$- P_{K,L}^{p,q} D\tilde{L}_q(K_p) = 0. \quad \text{(A.25)}$$

Set $\|\tilde{L}_q^\top(K_p)\| < \sigma_{\min}(Q_K - \gamma^{-2} P_{K,L}^{p,q} DD^\top P_{K,L}^{p,q})/ 2\|D^\top P_{K,L}^{p,q}\| \triangleq e$. It follows from (A.17) that $Q \succeq \gamma^2 L_{q+1}^\top(K_p)L_{q+1}(K_p)$ by reason of it being admissible as a Lyapunov equation. The inequality $P_{K,L}^{p,q} \succeq \hat{P}_{K,L}^{p,q}$ so that

$$- \gamma^{-2}\hat{P}_{K,L}^{p,q}DD^\top \hat{P}_{K,L}^{p,q} + \gamma^{-2}(P_{K,L}^{p,q} - \hat{P}_{K,L}^{p,q})DD^\top(P_{K,L}^{p,q}$$
$$- \hat{P}_{K,L}^{p,q}) - (\tilde{L}_K^j)^\top D^\top P_{K,L}^{p,q} - P_{K,L}^{p,q} D\tilde{L}_q(K_p) + Q_K \succeq 0.$$

Consequently, $\hat{A}_{K,L}^{p,q+1}$ is Hurwitz. Since $\hat{L}_q(K_0) = 0$ and $K \in \check{\mathcal{K}}$, $\hat{A}_{K,L}^{p,0} = A - BK$ is Hurwitz. Hence, $\hat{A}_{K,L}^{p,q}$ is Hurwitz for all $q \in \mathbb{N}_+$ as long as $\|\tilde{L}_q(K_p)\|_F \le e$. $\qquad\square$

Proof of Lemma 12. Since $\mathcal{K}_h$ is compact, it follows from Lemma 10 that $\underline{e}(h) := \inf_{K \in \mathcal{K}_h} e(K) > 0$. In addition, $\hat{K}' \in \mathcal{K}$ when $\|\tilde{K}\| < \underline{e}(h)$. By [7, Lemma A.1], $P_{\hat{K}'} = P_{\hat{K}'}^\top \succ 0$ is the solution of

$$A_{\hat{K}'}^\top P_{\hat{K}'} + P_{\hat{K}'} A_{\hat{K}'} + Q_{\hat{K}'} + \gamma^{-2} P_{\hat{K}'} DD^\top P_{\hat{K}'} = 0,$$
$$\text{(A.26)}$$

where $A_{\hat{K}'} = A - B\hat{K}'$ and $Q_{\hat{K}'} = Q + (\hat{K}')^\top R\hat{K}'$. Let $A_{\hat{K}'}^\star = A - B\hat{K}' + \gamma^{-2}DD^T P_{\hat{K}'}$. It follows from [7, Lemma A.1] that $A_{\hat{K}'}^\star$ is Hurwitz. Subtracting (A.26) from (14), using $K' = R^{-1}B^T P_K$, and completing the squares,

$$A_{\hat{K}'}^{\star\top}(P_K - P_{\hat{K}'}) + (P_K - P_{\hat{K}'})A_{\hat{K}'}^\star$$
$$+ (K' - K)^\top R(K' - K) - \tilde{K}^\top R\tilde{K} \quad \text{(A.27)}$$
$$+ \gamma^{-2}(P_K - P_{\hat{K}'})DD^\top(P_K - P_{\hat{K}'}) = 0.$$

From Lemma 14, we have $(P_K - P_{\hat{K}'}) \succeq$

$$\int_0^\infty e^{A_{\hat{K}'}^{\star\top} t} E_K e^{A_{\hat{K}'}^\star t} \mathrm{d}t - \int_0^\infty e^{A_{\hat{K}'}^{\star\top} t} \tilde{K}^T R\tilde{K} e^{A_{\hat{K}'}^\star t} \mathrm{d}t,$$
$$\text{(A.28)}$$

so that taking the trace, using Lemma 3 and [30, Theorem 2],

$$Tr(P_K - P_{\hat{K}'}) \ge \frac{\log(5/4)}{2\|A_{\hat{K}'}^\star\|}\|E_K\|$$
$$- Tr\left(\int_0^\infty e^{A_{\hat{K}'}^{\star\top} t} e^{A_{\hat{K}'}^\star t} \mathrm{d}t\right)\|R\|\|\tilde{K}\|^2. \quad \text{(A.29)}$$

It follows from Lemmas 5 and 13 that

$$Tr(P_{\hat{K}'} - P^\star) \leq \left(1 - \frac{\log(5/4)b(h)}{2n\|A_{\hat{K}'}^\star\|}\right) Tr(P_K - P^\star)$$
$$+ Tr\left(\int_0^\infty e^{A_{\hat{K}'}^{\star\top}t} e^{A_{\hat{K}'}^\star t} dt\right) \|R\|\|\tilde{K}\|^2. \quad (A.30)$$

Let

$$f_1(\hat{K}') = \frac{\log(5/4)b(h)}{2n\|A_{\hat{K}'}^\star\|}, f_2(\hat{K}') = Tr\left(\int_0^\infty e^{A_{\hat{K}'}^{\star\top}t} e^{A_{\hat{K}'}^\star t} dt\right).$$

Since $f_1(\hat{K}')$ and $f_2(\hat{K}')$ are continuous with respect to $\hat{K}'$,

$$\underline{f}_1(h) = \inf_{\hat{K}' \in \mathcal{K}_h} f_1(\hat{K}') > 0, \bar{f}_2(h) = \sup_{\hat{K}' \in \mathcal{K}_h} f_2(\hat{K}') < \infty. \quad (A.31)$$

It follows from (A.30) that if $\|\tilde{K}\| < \sqrt{\frac{\underline{f}_1(h)h}{\bar{f}_2(h)\|R\|}}$, then $Tr(P_{\hat{K}'} - P^\star) < h$. In summary, if

$$\|\tilde{K}\| < \min\left\{\underline{e}(h), \sqrt{\frac{\underline{f}_1(h)h}{\bar{f}_2(h)\|R\|}}\right\} =: f(h), \quad (A.32)$$

we have $\hat{K}' \in \mathcal{K}_h$. $\quad\square$

Lemma 13 : *Norm of a Matrix Trace [35, Theorem 4.2.2] For any positive semi-definite matrix $P \in \mathbb{S}^n$, $\|P\|_F \leq Tr(P) \leq \sqrt{n}\|P\|_F$, and $\|P\| \leq Tr(P) \leq n\|P\|$. For any $x \in \mathbb{R}^n$, $x^\top Px \geq \underline{\lambda}(P)\|x\|^2$.*

Lemma 14 : *Assume $A \in \mathbb{R}^{n\times n}$ is Hurwitz and satisfies $A^\top P + PA + Q = 0$. Then, the following properties hold*
(1) $P = \int_0^\infty e^{A^\top t}Qe^{At}dt$;
(2) $P \succ 0$ if $Q \succ 0$, and $P \succeq 0$ if $Q \succeq 0$;
(3) If $Q \succeq 0$, then (Q, A) is observable iff $P \succ 0$;
(4) For a $P' \in \mathbb{S}^n$ satisfying $A^\top P' + P'A + Q' = 0$, where $Q' \preceq Q$, we have $P' \preceq P$.

Proof of Lemma 14. The first three statements are proven in [36, Lemma 3.18]. From the expression in the fourth statement above, P' can be expressed as

$$P' = \int_0^\infty e^{A^\top t}Q'e^{At}dt \quad (A.33)$$

owing to statement (1) above. If $Q' \preceq Q$, then comparing statement (1) with (A.33), we must have $P' \preceq P$. $\quad\square$

Lemma 15 : *[36, Lemma 3.19] Suppose that P satisfies $A^\top P + PA + Q = 0$, then the following statements hold:*
1) A is Hurwitz if $P \succ 0$ and $Q \succ 0$.
2) A is Hurwitz if $P \succeq 0$, $Q \succeq 0$ and (Q, A) is detectable.

Proof of Theorem 4. When $\|\tilde{L}_q(K_p)\| < e$, we have an Hurwitz $\hat{A}_{K,L}^{p,q}$ going by Lemma 11. Rewriting (26) for the $(p+1)$'th iteration and subtracting it from (26), we have

$$\hat{A}_{K,L}^{(p,q+1)\top}(\hat{P}_{K,L}^{(p,q+1)} - \hat{P}_{K,L}^{p,q}) + (\hat{P}_{K,L}^{(p,q+1)} - \hat{P}_{K,L}^{p,q})\hat{A}_{K,L}^{(p,q+1)\top}$$
$$+ \gamma^{-2}(\gamma^2\hat{L}_q(K_p) - D^\top\hat{P}_{K,L}^{p,q})^\top(\gamma^2\hat{L}_q(K_p) - D^\top\hat{P}_{K,L}^{p,q})$$
$$- \gamma^2\tilde{L}_q^\top(K_p)\tilde{L}_q(K_p) = 0. \quad (A.34)$$

Suppose that $\hat{\Psi}_{K,L}^{p,q} = \gamma^{-2}(\gamma^2\hat{L}_q(K_p) - D^\top\hat{P}_{K,L}^{p,q})^\top(\gamma^2\hat{L}_q(K_p) - D^\top\hat{P}_K^j)$. It follows that since $\hat{A}_{K,L}^{p,q+1}$ is Hurwitz, $\hat{P}_{K,L}^{(p,q+1)} - \hat{P}_{K,L}^{p,q}$ becomes

$$\int_0^\infty e^{\hat{A}_{K,L}^{(p,q+1)\top}t}\left[\hat{\Psi}_{K,L}^{p,q} - \gamma^2\tilde{L}_q^\top(K_p)\tilde{L}_q^\top(K_p)\right]e^{\hat{A}_{K,L}^{(p,q+1)}t}dt. \quad (A.35)$$

Now let $\hat{F}_K^q = \int_0^\infty e^{\hat{A}_{K,L}^{(p,q+1)\top}t}\hat{\Psi}_{K,L}^{p,q}e^{\hat{A}_{K,L}^{(p,q+1)}t}dt$ so that

$$P_{K,L}^{p,q+1} - \hat{P}_{K,L}^{p,q+1} = P_{K,L} - \hat{P}_{K,L}^p - \hat{F}_K^q$$
$$+ \int_0^\infty e^{\hat{A}_{K,L}^{(p,q+1)\top}t}\left(\gamma^2\tilde{L}_q^\top(K_p)\tilde{L}_q(K_p)\right)e^{\hat{A}_{K,L}^{(p,q+1)}t}dt. \quad (A.36)$$

Let $f_K = \sup_{q\in\mathbb{N}_+}\|\hat{A}_{K,L}^{p,q+1}\|$. From Lemma 7, we can write $-\|\hat{F}_K^q\| \leq -\frac{\log(5/4)}{2f_K}\|\hat{\Psi}_{K,L}^{p,q}\|$. Furthermore, by Lemma 7, we can write $-\|\hat{\Psi}_{K,L}^{p,q}\| \leq -\frac{1}{c(K)}Tr(P_{K,L}^{p,q} - \hat{P}_{K,L}^{p,q})$, where $c(K) = Tr(\int_0^\infty e^{(A_K+DL_q(K_p))t}e^{(A_K+DL_q(K_p))^\top t}dt)$. Therefore, the trace of (A.36) becomes

$$Tr(P_{K,L}^{p,q} - \hat{P}_{K,L}^{p,q+1}) \leq \left(1 - \frac{\log(5/4)}{2f_Kc(K)}\right)Tr(P_K - \hat{P}_{K,L}^{p,q})$$
$$+ Tr\left(\int_0^\infty e^{(\hat{A}_{K,L}^{p,q+1})t}e^{(\hat{A}_{K,L}^{p,q+1})^\top t}dt\right)\gamma^2\|\tilde{L}_q(K_p)\|^2. \quad (A.37)$$

Let $g = \sup_{q\in\mathbb{N}_+} Tr\left(\int_0^\infty e^{(\hat{A}_{K,L}^{p,q+1})t}e^{(\hat{A}_{K,L}^{p,q+1})^\top t}dt\right)$, $\quad (A.38)$

and $\hat{\beta}(K) = 1 - \frac{\log(5/4)}{2f_Kc(K)}$, so that

$$Tr(P_{K,L}^{p,q} - \hat{P}_{K,L}^{p,q}) \leq \hat{\beta}^{q-1}(K)Tr(P_{K,L}^{p,q}) + \lambda(\|\tilde{L}\|_\infty), \quad (A.39)$$

where $\lambda(\|\tilde{L}\|_\infty) := \frac{1}{1-\hat{\beta}(K)}\gamma^2g\|\tilde{L}\|_\infty^2$. As $\|P_K - \hat{P}_{K,L}^{p,q}\|_F \leq Tr(P_{K,L}^{p,q} - \hat{P}_{K,L}^{p,q})$, we establish the theorem. $\quad\square$