Random-Turn Hex and Other Selection Games

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1 INTRODUCTION.

Overview. The game of *Hex*, invented independently by Piet Hein in 1942 and John Nash in 1948 [9], has two players who take turns placing stones of their respective colors on the hexagons of a rhombus-shaped hexagonal grid (see Figure 1). A player wins by completing a path connecting the two opposite sides of his or her color. Although it is easy to show that player I has a winning strategy, it is far from obvious what that strategy is. Hex is usually played on an 11×11 board (e.g., the commercial version by Parker Brothers® and the Computer Olympiad Hex Tournament both use 11×11 boards), for which the optimal strategy is not yet known. (For a book on practical Hex strategy see [3], and for further information on Hex see [23], [7], [1], or [6].)

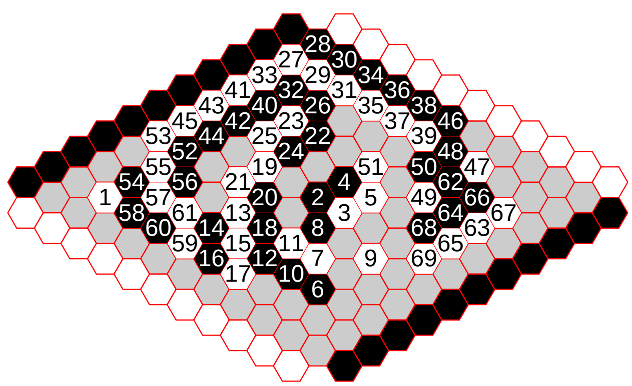

Figure 1: A game of Hex played at the 5th Computer Olympiad in London, August 24, 2000 [1]. Queenbee [18] (black) played against Hexy [2] (white). (There is a special rule regarding the first move whose purpose is to offset the advantage of the first player, see [1].)

Random-Turn Hex is the same as ordinary Hex, except that instead of alternating turns, players toss a coin before each turn to decide who gets to place the next stone. Although ordinary Hex is famously difficult to analyze, the optimal strategy for Random-Turn Hex turns out to be very simple. We introduce random-turn games in part because they are in many cases more tractable than their deterministic analogs; they also exhibit surprising structure and symmetry. In one general class of games called *selection games*, which includes Hex, the probability that player I wins when both players play *optimally* is precisely the probability that player I wins when both players play *randomly*. Combining this with Smirnov's recent result about percolation on the hexagonal lattice [21], we will see that in a certain "fine lattice limit" winning probabilities of Random-Turn Hex are a "conformal invariant" of the board shape. In this and other games, the set of moves played during an entire game (when both players play optimally) has an intriguing fractal structure.

On a philosophical note, random-turn games are natural models for real-world conflicts, where opposing agents (political parties, lobbyists, businesses, militaries, etc.) do not alternate turns. Instead, they continually seek to improve their positions incrementally. The extent to which they succeed is (at least partially) unpredictable and may be modeled using randomness. Random-turn games are also closely related to “Richman games” [11]. In a Richman game, each player offers money to the other player for the right to make the next move, and the player who offers more gets to move. (At the end of the Richman game, the money has no value.) Another class of random-turn games, where the moves of one player may be reversed by the other player, is studied in [11] and [16].

For simplicity, we limit the discussion in this article to two-player, zero-sum, random-turn games with perfect information.

Random-turn selection games. Now we describe a general class of games that includes the famous game of Hex. Let S be an n-element set, which will sometimes be called the *board*, and let f be a function from the 2^n subsets of S to $\mathbb{R}$. A *selection game* is played as follows: the first player selects an element of S, the second player selects one of the remaining $n-1$ elements, the first player selects one of the remaining $n-2$, and so forth, until all elements have been chosen. Let S_1 and S_2 signify the sets chosen by the first and second players, respectively. Then player I receives a payoff of $f(S_1)$ and player II a payoff of $-f(S_1)$ (selection games are zero-sum). The following are examples of selection games:

Hex. Here S is the set of hexagons on a rhombus-shaped $L \times L$ hexagonal grid, and $f(S_1)$ is 1 if S_1 contains a left-right crossing, -1 otherwise (see Figures 1 and 2). In this case, once S_1 contains a left-right crossing or S_2 contains an up-down crossing (which precludes the possibility of S_1 having a left-right crossing), the outcome is determined and there is no need to continue the game.

We also consider Hex played on other types of boards. In the general setting, some hexagons are given to the first or second players before the game has begun. One of the reasons for considering such games is that after a number of moves are played in ordinary Hex, the remaining game has this form.

Surround. The famous game of *Go* is not a selection game (for one, a player can remove an opponent's pieces), but the game of *Surround*, in which, as in Go, surrounding area is important, is a selection game. In this game S is the set of n hexagons in a hexagonal grid (of any shape). At the end of the game, each hexagon is recolored to be the color of the outermost cluster surrounding it (if there is such a cluster). The payoff $f(S_1)$ is the number of hexagons recolored black minus the number of hexagons recolored white. (Another natural payoff function is $f^*(S_1) = \text{sign}(f(S_1))$.)

Full-Board Tic-Tac-Toe. Here S is the set of spaces in a 3×3 grid, and $f(S_1)$ is the number of horizontal, vertical, or diagonal rows in S_1 minus the number of horizontal, vertical, or diagonal rows in $S \backslash S_1$. This is different from ordinary tic-tac-toe in that the game does not end after the first row is completed.

Team Captains. Two team captains are choosing baseball teams from a finite set S of n players for the purpose of playing a single game against each other. The payoff $f(S_1)$ for the first captain is the probability that the players in S_1 (together with the first captain) would beat the players in S_2 (together with the second captain). The payoff function may be very complicated (depending on which players are skilled at which positions, which players have played together before, which players get along well with which captain, etc.). Because we have not specified the payoff function, this game is as general as the class of selection games.

Every selection game has a random-turn variant in which at each turn a fair coin is tossed to decide who moves next.

Consider the following questions:

1. What can one say about the probability distribution of S_1 after a typical game of optimally played Random-Turn Surround?

2. More generally, in a generic random-turn selection game, how does the probability distribution of the final state depend on the payoff function f?

3. Less precise: Are the teams chosen in Random-Turn Team Captains "good teams" in any objective sense?

The answers are surprisingly simple.

2 OPTIMAL STRATEGY.

A (pure) *strategy* for a given player in a random-turn selection game is a map M from pairs of disjoint subsets (T_1, T_2) of S to elements of S. Here $M(T_1, T_2)$ indicates the element that the player will pick if given a turn at a time in the game at which player I has thus far picked the elements of T_1 and player II has picked the elements of T_2.

Denote by $E(T_1, T_2)$ the expected payoff for player I at this stage in the game, assuming that both players play optimally with the goal of maximizing expected payoff. As is true for all finite perfect-information, two-player games, E is well defined, and one can compute E and the set of possible optimal strategies inductively as follows. First, if $T_1 \cup T_2 = S$, then $E(T_1, T_2) = f(T_1)$. Next, suppose that we have computed $E(T_1, T_2)$ whenever $|S \backslash (T_1 \cup T_2)| \leq k$. Then if $|S \backslash (T_1 \cup T_2)| = k + 1$, and player I has the chance to move, player I will play optimally if and only if she chooses an s from $S \backslash (T_1 \cup T_2)$ for which $E(T_1 \cup \{s\}, T_2)$ is maximal. (If she chose any other s, this would reduce her expected payoff.) Similarly, player II plays optimally if and only if she minimizes $E(T_1, T_2 \cup \{s\})$ at each stage.

The foregoing analysis also demonstrates a well-known fundamental fact about finite, turn-based, perfect-information games: both players have optimal pure strategies (i.e., strategies that do not require flipping coins), and knowing the other player's strategy does not give a player any advantage when both players play optimally. (This contrasts with the situation in which the players play "simultaneously," as they do in Rock-Paper-Scissors.)

Theorem 2.1. *The value of a random-turn selection game is the expectation of $f(T)$ when a set T is selected randomly and uniformly among all subsets of S. Moreover, any optimal strategy for one of the players is also an optimal strategy for the other player.*

Proof. If player II plays any optimal strategy, player I can achieve the expected payoff $\mathbb{E}[f(T)]$ by playing exactly the same strategy (since, when both players play the same strategy, each element will belong to S_1 with probability $1/2$, independently). Thus, the value of the game is at least $\mathbb{E}[f(T)]$. However, a symmetric argument applied with the roles of the players interchanged implies that the value is no more than $\mathbb{E}[f(T)]$.

Suppose that M is an optimal strategy for the first player. We have seen that when both players use M, the expected payoff is $\mathbb{E}[f(T)] = E(\emptyset, \emptyset)$. Since M is optimal for player I, it follows that when both players use M player II always plays optimally (otherwise, player I would gain an advantage, since she is playing optimally). This means that $M(\emptyset, \emptyset)$ is an optimal first move for player II, and therefore every optimal first move for player I is an optimal first move for player II. Now note that the game started at any position is equivalent to a selection game. We conclude that every optimal move for one of the players is an optimal move for the other, which completes the proof. $\square$

If f is identically zero, then all strategies are optimal. However, if f is *generic* (meaning that all of the values $f(S_1)$ for different subsets S_1 of S are linearly independent over $\mathbb{Q}$), then the preceding argument shows that the optimal choice of s is always unique and that it is the same for both players. We thus have the following result:

Proposition 2.2. *If f is generic, then there is a unique optimal strategy and it is the same strategy for both players. Moreover, when both players play optimally, the final S_1 is equally likely to be any one of the 2^n subsets of S.*

Theorem 2.1 and Proposition 2.2 are in some ways quite surprising. In the baseball team selection, for example, one has to think very hard in order to play the game optimally, knowing that at each stage there is exactly one correct choice and that the adversary can capitalize on any miscalculation. Yet, despite all of that mental effort by the team captains, the final teams look no different than they would look if at each step both captains chose players uniformly at random.

Also, for purposes of illustration, suppose that there are only two players who know how to pitch and that a team without a pitcher always loses. In the alternating turn game, a captain can always wait to select a pitcher until just after the other captain selects a pitcher. In the random-turn game, the captains must try to select the pitchers in the opening moves, and there is an even chance the pitchers will end up on the same team.

Theorem 2.1 and Proposition 2.2 generalize to random-turn selection games in which the player to get the next turn is chosen using a biased coin. If player I gets each turn with probability p, independently, then the value of the game is $\mathbb{E}[f(T)]$, where T is a random subset of S for which each element of S is in T with probability p, independently. For the corresponding statement of the proposition to hold, the notion of "generic" needs to be modified. For example, it suffices to assume that the values of f are linearly independent over $\mathbb{Q}[p]$. The proofs are essentially the same. We leave it as an exercise to the reader to generalize the proofs further so as to include the following two games:

Restaurant Selection. Two parties (with opposite food preferences) want to select a dinner location. They begin with a map containing 2^n distinct points in $\mathbb{R}^2$, indicating restaurant locations. At each step, the player who wins a coin toss may draw a straight line that divides the set of remaining restaurants exactly in half and eliminate all the restaurants on one side of that line. Play continues until one restaurant z remains, at which time player I receives payoff $f(z)$ and player II receives $-f(z)$.

Balanced Team Captains. Suppose that the captains wish to have the final teams equal in size (i.e., there are $2n$ players and we want a guarantee that each team will have exactly n players in the end). Then instead of tossing coins, the captains may shuffle a deck of $2n$ cards (say, with n red cards and n black cards). At each step, a card is turned over and the captain whose color is shown on the card gets to choose the next player.

3 WIN-OR-LOSE SELECTION GAMES.

We say that a game is a *win-or-lose* game if $f(T)$ takes on precisely two values, which we may as well assume to be -1 and 1. If $S_1 \subset S$ and $s \in S$, we say that s is *pivotal* for S_1 if $f(S_1 \cup \{s\}) \neq f(S_1 \setminus \{s\})$. A selection game is *monotone* if f is monotone; that is, $f(S_1) \geq f(S_2)$ whenever $S_1 \supset S_2$. Hex is an example of a monotone, win-or-lose game. For such games, the optimal moves have the following simple description:

Lemma 3.1. *In a monotone, win-or-lose, random-turn selection game, a first move s is optimal if and only if s is an element of S that is most likely to be pivotal for a random-uniform subset T of S. When the position is (S_1, S_2), the move s in $S \setminus (S_1 \cup S_2)$ is optimal if and only if s is an element of $S \setminus (S_1 \cup S_2)$ that is most likely to be pivotal for $S_1 \cup T$, where T is a random-uniform subset of $S \setminus (S_1 \cup S_2)$.*

The proof of the lemma is straightforward at this point and is left to the reader.

For win-or-lose games, such as Hex, the players may stop making moves after the winner has been determined, and it is interesting to calculate how long a random-turn, win-or-lose, selection game will last when both players play optimally. Suppose that the game is a monotone game and that, when there is more than one optimal move, the players break ties in the same way. Then we may take the point of view that the playing of the game is a (possibly randomized) decision procedure for evaluating the payoff function f when the items are randomly allocated. Let $\vec{x}$ denote the allocation of the items, where $x_i = \pm 1$ according to whether the ith item goes to the first or second player. We may think of the x_i as input bits, and the playing of the game is one way to compute $f(\vec{x})$. The number of turns played is the number of bits of $\vec{x}$ examined before $f(\vec{x})$ is computed. We can use certain inequalities from the theory of Boolean functions to bound the average length of play.

Let $I_i(f)$ denote the influence of the ith bit on f (i.e., the probability that flipping x_i

will change the value of $f(\vec{x})$). The following inequality is from O'Donnell and Servedio [15]:

$$\sum_i I_i(f) = \mathbb{E}\left[\sum_i f(\vec{x})x_i\right] = \mathbb{E}\left[f(\vec{x})\sum_i x_i 1_{x_i \text{ examined}}\right] \leq \text{(by Cauchy-Schwarz)}$$
$$\sqrt{\mathbb{E}[f(\vec{x})^2]\,\mathbb{E}\left[\left(\sum_{i:\ x_i \text{ examined}} x_i\right)^2\right]} = \sqrt{\mathbb{E}\left[\left(\sum_{i:\ x_i \text{ examined}} x_i\right)^2\right]} = \sqrt{\mathbb{E}[\#\text{ bits examined}]}\,. \tag{1}$$

The last equality is justified by noting that $\mathbb{E}[x_i\, x_j\, 1_{x_i \text{ and } x_j \text{ both examined}}] = 0$ when $i \neq j$, which holds since conditioned on x_i being examined before x_j, conditioned on the value of x_i, and conditioned on x_j being examined, the expected value of x_j is zero. By (1) we have

$$\mathbb{E}[\#\text{ turns}] \geq \left[\sum_i I_i(f)\right]^2 .$$

We shall shortly apply this bound to the game of Random-Turn Hex.

We mention one other inequality from the theory of Boolean functions, this one due to O'Donnell, Saks, Schramm, and Servedio [14, Theorem 3.1]:

$$\mathrm{Var}[f] \leq \sum_i \Pr[x_i \text{ examined}] I_i(f). \tag{2}$$

For the random-turn game this implies that

$$\mathbb{E}[\#\text{ turns}] \geq \frac{\mathrm{Var}[f]}{\max_i I_i(f)}.$$

4 RANDOM-TURN HEX.

Odds of winning on large boards and under biased play. In the game of Hex, the propositions discussed earlier imply that the probability that player I wins is given by the probability that there is a left-right crossing in independent Bernoulli percolation on the sites (i.e., when the sites are independently and randomly colored black or white). One perhaps surprising consequence of the connection to Bernoulli percolation is that, if player I has a slight edge in the coin toss and wins the coin toss with probability $1/2 + \varepsilon$, then for any $\varepsilon > 0$, any $\delta > 0$, and any $r > 0$ there is a strategy for player I that wins with probability at least $1 - \delta$ on the $L \times rL$ board, provided that L is sufficiently large.

Random-Turn Hex on ordinary-size boards. Reisch proved in 1981 that determining which player has a winning strategy on an arbitrarily shaped Hex board (with some of the sites already colored in) is PSPACE-complete [17]. The online community has, however, made a good deal of progress (much of it unpublished) on solving the problem for smaller boards. Jing Yang [24] has announced the solution of Hex (and provided associated computer programs) on boards of size up to 9×9. Hex is usually played on an 11×11 board, for which the optimal strategy is not yet known.

What is the situation with Random-Turn Hex? We do not know if the correct move in Random-Turn Hex can be found in polynomial time. On the other hand, for any fixed ε a computer can sample $O(L^4\varepsilon^{-2}\log(L^4/\varepsilon))$ percolation configurations (filling in the empty hexagons at random) to estimate which empty site is most likely to be pivotal given the current board configuration. Except with probability $O(\varepsilon/L^2)$, the computer will pick a site that is within $O(\varepsilon/L^2)$ of being optimal. This simple randomized strategy provably beats an optimal opponent $50-\varepsilon$ percent of the time.

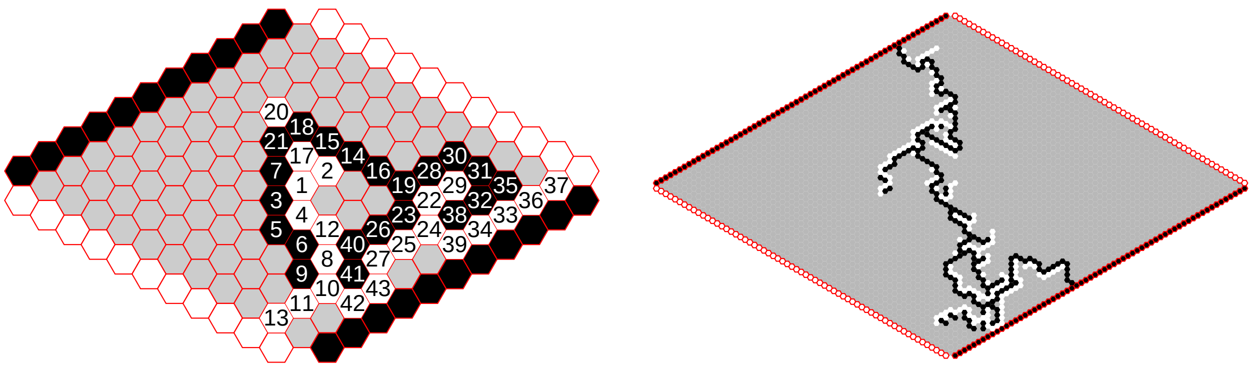

Figure 2: Random-Turn Hex on boards of size 11×11 and 63×63 under (near) optimal play.

Typical games under optimal play. What can we say about how long an average game of Random-Turn Hex will last, assuming that both players play optimally? (Here we assume that the game is stopped once a winner is determined.) If the side length of the board is L, we wish to know how the expected length of a game grows with L (see Figure 2 for games on a large board). Computer simulations on a variety of board sizes suggest that the exponent is in the range 1.5–1.6. As far as rigorous bounds go, a trivial upper bound is $O(L^2)$. Since the game does not end until a player has found a crossing, the length of the shortest crossing in percolation is a lower bound, and empirically this distance grows as $L^{1.1306\pm 0.0003}$ [8], where the exponent is known to be strictly larger than 1. We give a stronger lower bound:

Theorem 4.1. *Random-Turn Hex under optimal play on an order L board, when the two players break ties in the same manner, takes at least $L^{3/2+o(1)}$ time on average.*

Proof. To use the O'Donnell-Servedio bound (1), we need to know the influence that the sites have on whether or not there is a percolation crossing (a path of black hexagons connecting the two opposite black sides). The influence $I_i(f)$ is the probability that flipping site i changes whether there is a black crossing or a white crossing. The "4-arm exponent" for percolation is 5/4 [22] (as predicted earlier in [5]), so $I_i(f) = L^{-5/4+o(1)}$ for sites i "away from the boundary," say in the middle ninth of the region. Thus $\sum_i I_i(f) \geq L^{3/4+o(1)}$, so $\mathbb{E}[\#\text{ turns}] \geq L^{3/2+o(1)}$. $\square$

Remark. The function $\sum_i x_i 1_{x_i \text{ examined}}$ is the number of extra times that player I wins the coin toss at the time that the game terminates. Naturally this is correlated with the winner $f(\vec{x})$ of the game, and for large board sizes this correlation is noticeable. Since the only inequality used in (1) was the Cauchy-Schwarz inequality, if we know $\sum_i I_i(f)$ and the expected length of the game, we can determine how correlated the winner is with who won most of the coin tosses. Using more detailed knowledge of how $I_i(f)$ behaves near the boundary, one can show that for the standard lozenge-shaped Hex board, $\sum_i I_i(f) = L^{3/4+o(1)}$. If the game lasts for $L^{>1.5}$ steps on average, this would imply that the correlation between the winner of the game and the winner of the majority of the coin tosses before the game is won tends to 0 as $L \to \infty$.

Remark. The question of determining how many bits need to be examined before one can decide whether or not there is a percolation crossing arose in the context of dynamical percolation, where the sites flip according to independent Poisson clocks [19]. Roughly speaking, if few bits need to be examined, then it is easier for there to be exceptional times at which there is an infinite percolating cluster in the plane [19]. One possible algorithm for determining whether or not there is a crossing in an order L region would be to follow the black-white interfaces starting at the corners to see how they connect, which exposes $L^{7/4+o(1)}$ hexagons in expectation [22]. While this algorithm has the best currently provable bound on the number of exposed hexagons, the "play Random-Turn Hex" algorithm appears to do better (taking about $L^{1.5\text{–}1.6}$ time).

An optimally played game of Random-Turn Hex on a small board may occasionally have a move that is disconnected from the other played hexagons, as the game in Figure 3 shows. But this is very much the exception rather than the rule. For moderate- to large-sized boards it appears that in almost every optimally played game, the set of played hexagons remains a connected set throughout the game (which is in sharp contrast to the usual game of Hex). We do not have an explanation for this phenomenon, nor is it clear to us if it persists as the board size increases beyond the reach of simulations.

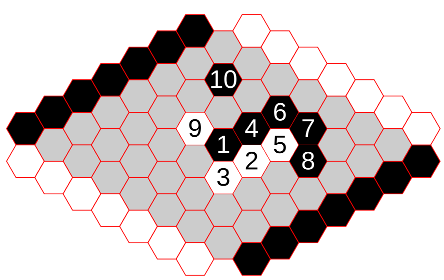

Figure 3: A rare occurrence—a game of Random-Turn Hex under (near) optimal play with a disconnected play.

Conformal (non)invariance of Random-Turn Hex. Combined with Smirnov's celebrated recent work in percolation [21], the connection between Random-Turn Hex and percolation enables us to use Cardy's formula [4] to approximate player I's probability of winning on very large boards of various shapes, such as the $L \times rL$ rectangle. The way this is done is as follows. Suppose that the players play the game on a very large board that, when suitably scaled, approximates some simply connected domain D in the complex plane, where the boundary of D alternates colors at the points b_1, b_2, b_3, b_4 with one black side between b_1 and b_2 and the other between b_3 and b_4. By the Riemann Mapping Theorem, there is an analytic function ϕ that bijectively maps the domain D to the upper half plane (i.e., a conformal bijection) such that $\phi(b_1) = 0$, $\phi(b_3) = 1$, and $\phi(b_4) = \infty$. Smirnov proved that in percolation on boards that are shaped like the domain D, the probability of a black percolation crossing is a function of $\phi(b_2)$ plus an error term that goes to zero as the board size (the length of the shortest path connecting opposite sides) goes to infinity. This function of $\phi(b_2)$ is an explicit formula (involving a hypergeometric function) that is known as *Cardy's formula*. These crossing probabilities are known as a "conformal invariant" of percolation because they remain invariant under conformal maps (up to error terms that go to zero as the board size goes to infinity). The connection between Random-Turn Hex and percolation tells us that when the two players play optimally, the probability of black winning is (up to an additive $o(1)$ error) conformally invariant, and in particular is given by Cardy's formula.

However, the actual game play is *not* conformally invariant, and indeed, even the location of the first move is not conformally invariant (see Figure 4). The reason for this is not difficult to understand. Recall that the first move is played at the site that is most likely to be pivotal. The probability that a site is pivotal is not scale invariant (for larger boards, the probability that a site is pivotal is smaller). Since a general conformal map scales different parts of the domain by different amounts, the location of the site most likely to be pivotal is not conformally invariant.

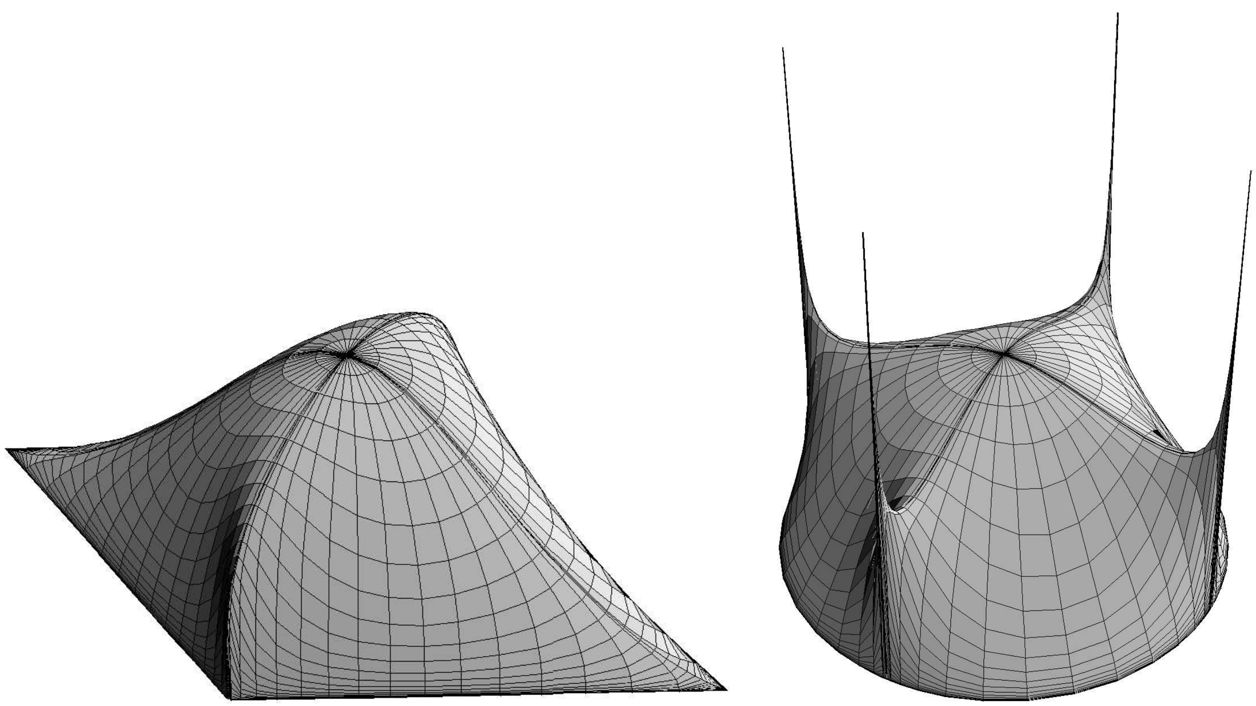

Figure 4: The location of the best first move in Random-Turn Hex is the site that is most likely to be pivotal for a percolation crossing. Shown here are estimates of the probability that a site is critical when the board is the standard Hex board (on the left) and when the board is a disk (on the right), calculated using SLE [20]. For the standard board the best first move is near the center, while for the disk-shaped board the best first move is near one of the four points where the black boundary and white boundary meet.

5 THREE EXAMPLES ON TREES.

In this section, we study two different random-turn games based on trees. Although these particular examples can be analyzed rather well, there are other natural games based on trees that we do not know how to analyze. We mention one such example at the end of the section (Recursive Three-Fold Majority).

AND-OR trees. We believe that the expected length of game play in Random-Turn Hex grows like a nonintegral power of the selection set size $|S|$ and that the set of hexagons played has a random fractal structure. The game of Random-Turn AND-OR (see Figure 5) is a selection game for which we can actually prove analogous statements.

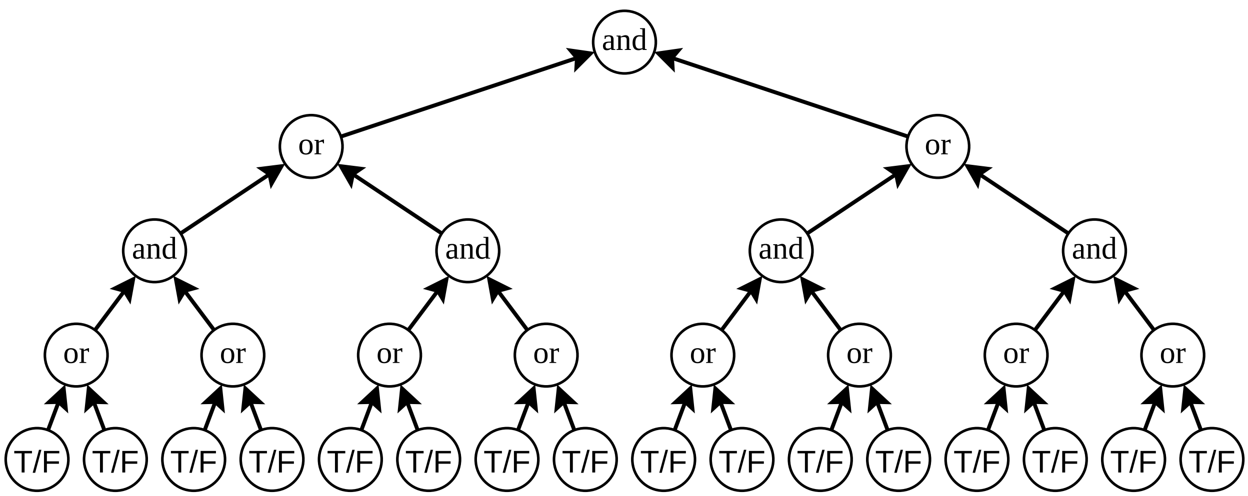

Figure 5: AND-OR tree of depth 4. In Random-Turn AND-OR, the player who wins the coin toss sets the bit at a leaf node. The value of the function determines the winner of the game.

The selection set S is the set of leaves in a depth h complete binary tree. We number the levels of the tree from 0 (the root) to h (the leaves). The leaves are treated as binary bits, with labels of T ("True") or F ("False") if they are chosen by player I or II, respectively. The internal nodes of the tree are also treated as binary bits; the players cannot select the internal nodes, instead the label of a bit on the kth level of the tree ($k < h$) is the AND of its two children if k is even and the OR if k is odd. Player I wins if the root label is T; player II wins if it is F.

When player I always wins the coin toss with probability p (which need not be 1/2), it is straightforward to estimate the probability that she wins the game. Let q_k be the probability that a label at the kth level is T. Then $q_k = q_{k+1}^2$ (if k is even) and $q_k = 2q_{k+1} - q_{k+1}^2$ (if k is odd). When k is even, $q_k = (2q_{k+2} - q_{k+2}^2)^2$. The map $q \mapsto (2q - q^2)^2$ has a fixed point when

$$q^4 - 4q^3 + 4q^2 - q = q(q-1)(q^2 - 3q + 1) = 0$$

(i.e., if $q \in \{0, 1, (3 \pm \sqrt{5})/2\}$); the fixed points in $[0,1]$ are 0, 1, and $(3-\sqrt{5})/2 \approx .382$. If p is fixed and h tends to ∞ along even integers, then the probability player I wins tends to 0 if $p < (3-\sqrt{5})/2$ and to 1 if $p > (3-\sqrt{5})/2$.

Theorem 5.1. *Consider an optimally played game of AND-OR on a level h tree with coin-toss probability p. If a move is played in some subtree T below some vertex v, then the succeeding moves are all played in T until the label of v is determined. Moreover, the labels of the level $h-1$ vertices are determined in an order that is an optimally played game on the tree truncated at level $h-1$.*

Proof. When a complete set of labels of the leaves (and hence all other vertices) is fixed, we say a vertex on level k is pivotal if it is a pivotal element of the game on the tree truncated at level k. A vertex on level k is pivotal if and only if its parent is pivotal *and* its sibling is T (if k is odd) or F (if k is even). Thus, a vertex is pivotal if and only if the path from that vertex to the root has the property that every sibling of a vertex along that path is T or F as the level is odd or even. Note that the event "v is pivotal" is independent of labels on the subtree below v.

We assume, inductively, that the statement of the theorem is true for a tree with fewer than h levels and for the first t steps on a level h tree. Before the $(t+1)$th step is played, there are either 0 or 1 partially determined vertices at level $h-1$ (that is, vertices that are undetermined but for which a leaf below them has been played). If there are none, then the leaves that are most likely to be pivotal are children of the level $h-1$ vertices that are most likely to be pivotal, so any optimal $(t+1)$th step also satisfies the desired condition. Suppose now that there is exactly one partially determined vertex v at level $h-1$, and call its labeled child x and its yet unlabeled child y. Let z be any unlabeled leaf different from y that might still affect the outcome of the game. Leaf x must have been labeled at step t. We need to show that playing y as the $(t+1)$th move is better than playing z.

Let $r_t(u)$ be the conditional probability that a vertex u is pivotal given the labels that have been set prior to move t. Let $\bar{p} = p$ if h is odd and $\bar{p} = 1-p$ if h is even ($\bar{p}$ is the probability that the value of a leaf node does not determine its parent). Since x was preferred over z, we have $r_t(x) \geq r_t(z)$. Now $r_t(z)$ is a nonnegative martingale, which is to say that $\mathbb{E}[r_{t+1}(z)|\text{labels at time } t] = r_t(z) \geq 0$. Since x got labeled the way it did with probability $\bar{p}$, we have $\bar{p}r_{t+1}(z) \leq r_t(z) \leq r_t(x) = \bar{p}r_t(v) = \bar{p}r_{t+1}(y)$. Thus, we see that at step $t+1$, playing y is at least as good as playing z. We need only rule out the case of equality, namely, $r_{t+1}(z) = r_{t+1}(y)$.

Let a be the least common ancestor of y and z, let b_1 be the child of a above y, and let b_2 be the child of a above z. Suppose that x was the first move played in the subtree below a. Regardless of which player got leaf x, the label of a would not be determined in turn t. Thus, z would still be pivotal with positive probability, so the martingale argument would actually give the strict inequality $\bar{p}r_{t+1}(z) < r_t(z)$ in place of the weak inequality we used earlier. Consequently, we get $r_{t+1}(z) < r_{t+1}(y)$, as required.

On the other hand, suppose that x was not the first move played below a. Then by induction, the $(t-1)$th move was played below b_1 and, again invoking the inductive hypothesis, we see that $r_t(z) < r_t(x)$. Thus, we get $r_{t+1}(z) < r_{t+1}(y)$ in this case as well. ∎

Note that the Theorem 5.1 completely characterizes the optimally played games, whether or not the two players break ties in the same way. In fact, for every optimally played game there is an embedding of the tree in the plane for which at each turn the played leaf is the leftmost leaf that at that point has a positive probability to be pivotal.

Theorem 5.2. *If h is even and $p = (3-\sqrt{5})/2$, then the expected length of the game (i.e., the expected number of labeled leaves) is precisely*

$$\left(\frac{1+\sqrt{5}}{2}\right)^h = (1.6180\ldots)^h.$$

Proof. When all leaves are randomly labeled, a level k vertex is 1 with probability p if k is even and $1-p$ if k is odd. Under optimal play, once a vertex's label is determined, with probability p it determines the label of its parent. Thus, given that a vertex v is labeled during an optimally played game, the expected number of labeled children of v is $1+1-p$. □

Since we know the precise expected length of the game, we take a moment to compare this with the lower bounds from the theory of Boolean functions. When the bits x_i are true with probability p, rather than taking $x_i = \pm 1$ according to whether the bit is true or false, it turns out to be more natural to take $\mathsf{T} = \sqrt{(1-p)/p}$ and $\mathsf{F} = -\sqrt{p/(1-p)}$. Then $\mathbb{E}[x_i] = 0$ and $\mathbb{E}[x_i^2] = 1$, so the O'Donnell-Servedio bound is still valid. For AND-OR trees with $p = (3-\sqrt{5})/2$, the influence of each bit is $(\frac{-1+\sqrt{5}}{2})^h$, and there are 2^h bits, whence the O'Donnell-Servedio bound gives

$$\mathbb{E}[\#\text{ turns}] \geq (-1+\sqrt{5})^{2h} = (1.5279\ldots)^h.$$

The O'Donnell-Saks-Schramm-Servedio bound (2) generalizes to the p-biased case [14, Theorem 3.1] and yields

$$\mathbb{E}[\#\text{ turns}] \geq \frac{1}{(\frac{-1+\sqrt{5}}{2})^h} = \left(\frac{1+\sqrt{5}}{2}\right)^h = (1.6180\ldots)^h.$$

This lower bound is exactly tight for AND-OR trees.

Shannon Switching Game. We now discuss a game for which we can prove that the set of moves is always connected under optimal play. In some ways, the analysis is similar to the analysis of AND-OR games.

The *Shannon Switching Game* is played by two players (named Cut and Short) on a graph with two distinguished vertices. When it is Cut's turn, he may delete any unplayed edge of the graph, while Short may render an edge immune to being cut. Cut wins if he manages to disconnect the two distinguished vertices, otherwise Short wins. When the graph is an $(L+1) \times L$ grid, with the vertices of the left side merged into one distinguished vertex and the vertices on the other side merged into another distinguished vertex, then the game is called Bridg-It (also known as Gale after its inventor David Gale). Oliver Gross showed that the first player in Bridg-It has a (simple) winning strategy, and then Lehman [12] (see also [13]) showed how to solve the general Shannon Switching Game. Here we consider the random-turn version of the Shannon Switching Game.

Just as Random-Turn Hex is connected to site percolation on the triangular lattice, where the vertices of the lattice (or, equivalently, faces of the hexaognal lattice) are independently colored black or white with probaiblity 1/2, Random-Turn Bridg-It is connected to bond percolation on the square lattice, where the edges of the square lattice are independently colored black or white with probability 1/2. We don't know the optimal strategy for Random-Turn Bridg-It, but as with Random-Turn Hex, we can make a randomized algorithm that plays near optimally. Less is known about bond percolation than site percolation, but it is believed that the crossing probabilities for these two processes are asymptotically the same on "nice" domains [10], so the probability that Cut wins in Random-Turn Bridg-It is well approximated by the probability that a player wins in Random-Turn Hex on a similarly shaped board.

Consider the Random-Turn Shannon Switching Game on a tree, where the root is one distinguished vertex and the leaves are (collectively) the other vertex. Thus, Short wins if at the end of the game there is a path from the root to one of the leaves. Under optimal play, the probability that Short wins is just the probability that there is a path from the root to some leaf when each edge is independently deleted with probability 1/2. For the complete ternary tree with h levels, the probability that Short wins under optimal play converges to $3-\sqrt{5} \approx 0.764$ for large h (see the proof of Theorem 5.5). For the complete binary tree with h levels Short wins with probability $\sim 4/h$, so we define the "enhanced binary tree" to be the tree where the root has $\lfloor h \log 2/2 \rfloor$ children each of whom fathers a complete binary tree with $h-1$ levels. Then the Random-Turn Shannon Switching Game on this enhanced binary tree is an approximately fair game when h is large, meaning that Short wins with probability $1/2 + o(1)$. But what are the optimal strategies, and how long does an average game last if both players play optimally? We will discuss these issues presently.

There is a natural coupling of a game with bond percolation on the tree. The edges played by Short are the open edges, and those played by Cut are the closed edges. The unplayed edges may be considered undecided. We assume that the game terminates when the outcome is decided.

In the following result, a subtree below a vertex v in a rooted tree consists of v and all the vertices and edges that are separated from the root by v.

Theorem 5.3. *Consider the Random-Turn Shannon Switching Game on a tree of depth h in which each internal node (except possibly the root) has b children and all the leaves are at level h. Under optimal play, the set of moves played by Short forms a connected set of edges containing the root, and the set of moves played by Cut are all adjacent to the edges played by Short. At any position, the set of optimal moves (for either player) consists of all the edges closest to the leaves among the undecided edges adjacent to shorted edges. Consequently, whenever an edge is played in some subtree T below some vertex, the subsequent moves are played in T until all the leaves of T are disconnected from the root by cut edges or until Short wins.*

Proof. After Short and Cut have made some moves, define the residual tree to be the graph formed from the original tree by deleting the subtrees disconnected from the root by cut edges and contracting edges that have been shorted. Consider some position of an optimally played game. By induction, we assume that in the residual tree corresponding to the current position the direct descendants of the root are themselves roots of regular b-ary subtrees of various depths whose leaves are leaves of the original tree. (The base of the induction is clear.) Suppose that $e = [x, y]$ is an edge in the residual tree, where x is closer to the root than y. The probability that e is pivotal for bond percolation from the root to the leaves (i.e., given the status of the other edges, there is a connection from the root to the leaves if e is uncut and there is no connection if e is cut) is precisely the probability that y is connected to the leaves (by uncut edges), x is connected to the root, and every open path from the root to the leaves passes through e. This is clearly maximized by edges in the residual tree adjacent to the root. Among the edges adjacent to the root, this is maximized by those closest to the leaves, because the probability of having an open path to the root in Bernoulli percolation on the regular b-ary tree is monotone decreasing in the depth of the tree. By Lemma 3.1, these correspond to the optimal moves. When one of these edges is shorted or cut, the stated structure of the residual tree is preserved. Thus, the induction step is established. The theorem follows. $\square$

Next, we consider how long an average optimally played game lasts. In the following two theorems we use the notation (common in computer science) $\Theta(g)$ to denote a quantity that is bounded between $c_1 g$ and $c_2 g$, where c_1 and c_2 are positive universal constants. (The related notation $O(g)$ denotes an expression that is upper bounded by $c_2 g$, but which may or may not be bounded below.)

Theorem 5.4. *Random-Turn Switching on the enhanced binary tree of depth h lasts on average $\Theta(h^2)$ turns under optimal play.*

Proof. For the complete binary tree of depth h the expected number of vertices in the percolation component of the root is $h+1$. Accordingly, the expected number of played edges of the enhanced binary tree is $O(h^2)$. For the lower bound, note that for the complete binary tree the expected number of vertices in the percolation component of the root conditional on there being no leaf in this component is $(1+o(1))h$. (This quantity is easily calculated inductively, for example.) For the enhanced binary tree, there is a good chance that $\Theta(h)$ subtrees of the root are explored, so that on average at least $\Theta(h^2)$ moves are played. $\square$

Theorem 5.5. *Random-Turn Switching on the complete ternary tree of order h lasts on average $\Theta(h)$ turns under optimal play.*

Proof. For the complete ternary tree of depth h let q_h be the probability that Cut wins, let μ_h be the expected number of explored (played) edges conditional on Cut winning, and let ν_h be the expected number of explored edges conditional on Short winning. We have $q_0 = 0$, $\mu_0 = 0$ (say), and $\nu_0 = 0$. When Cut wins, the top three edges are always explored. Conditional on Cut winning, these top edges are open with probability $q_{h-1}/(1+q_{h-1})$, independently. Consequently, the following recursions hold:

$$\begin{aligned}
q_{h+1} &= \left(\frac{1+q_h}{2}\right)^3, \\
\mu_{h+1} &= 3 + 3\frac{q_h}{1+q_h}\mu_h\,, \\
\nu_{h+1} &= \frac{1-q_h}{1-q_h^3}\left[1+\nu_h\right] + \frac{q_h(1-q_h)}{1-q_h^3}\left[2+\mu_h+\nu_h\right] + \frac{q_h^2(1-q_h)}{1-q_h^3}\left[3+2\mu_h+\nu_h\right] \\
&= \nu_h + \frac{1+q_h+q_h^2-3q_h^3}{1-q_h^3} + \frac{q_h+q_h^2-2q_h^3}{1-q_h^3}\mu_h\,.
\end{aligned}$$

We then have $q_h \to \sqrt{5}-2$ $(< 1/2)$, $\mu_h = \Theta(1)$, $\nu_h = \Theta(h)$, and $\mathbb{E}[\#\text{ turns}] = \Theta(h)$. $\square$

Recursive Majority. Let $S_h = S$ be a subset of the leaves of the complete ternary tree of depth h. Inductively, let S_j be the set of nodes at level j such that the majority of the nodes at level $j+1$ under them is in S_{j+1}. The payoff function $f(S)$ for *Recursive Three-Fold Majority* is -1 if $S_0 = \emptyset$ and $+1$ if $S_0 = \{\text{root}\}$. It seems that this random-turn game cannot be analyzed using the methods we have used for the Random-Turn AND-OR game or the Random-Turn Shannon Switching Game. In particular, we do not know how long the game takes (on average) when played optimally.

6 OPEN PROBLEMS.

We recall here some of the open problems raised earlier in the article. It would be interesting to know the expected length of a game of optimally played Random-Turn Hex and whether or not the true optimal move (not just near-optimal) can be found efficiently. The algorithm "play Random-Turn Hex" appears to be an efficient algorithm (in terms of expected number of input bits examined) for determining whether or not there is a percolation crossing, but is there an asymptotically more efficient algorithm? It would also be interesting to know, in optimally played Random-Turn Hex on large boards, whether or not the fraction of disconnected moves is asymptotically zero. Finally, we do not know how long Random-Turn Ternary Recursive Majority takes, or whether or not it is the most efficient algorithm (in terms of expected number of input bits read) for evaluating Recursive Ternary Majority.

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