Structure `fuzzy`.

Let $\boldsymbol{\varepsilon} = \{\varepsilon_x\}_x$ be i.i.d. random variables with $\mathsf{E}\,\varepsilon_x = 0$, and f be an unobservable function. What one gets to observe instead is $g(x) = f(x) + \varepsilon_x$.

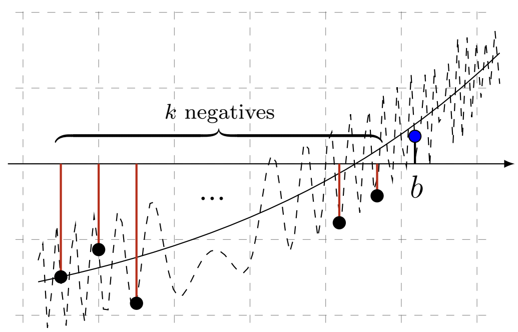
One possible question could then be to find the zero of f, based on observations g. Since said observations are noisy, the problem can be rephrased as follows: find $\underline{z} = \underline{z}(\boldsymbol{\varepsilon})$ and $\overline{z} = \overline{z}(\boldsymbol{\varepsilon})$, such that

$$\mathsf{P}(f(\underline{z}) < 0) \qquad \text{and} \qquad \mathsf{P}(f(\overline{z}) > 0)$$

are sufficiently large.

Monotone function. Suppose f is strictly increasing.

To get a lower bound on the zero of f, we want to find a sequence of points where the observed value g is negative, followed by a single positive observation.



Algorithm (Estimating zero of a noisy increasing function from below). This algorithm takes in five parameters: (i) the observed function g; (ii) x, the initial point where to evaluate the function; (iii) $h > 0$, the initial step size; (iv) integer $m \geqslant 1$, the number of times to refine the search grid; (v) integer $k \geqslant 1$, the number of points where the observations have to be of the same sign.

1. The aim of this step is to find a point where the observed value is positive.

   To this end, start at x with initial step $h > 0$. Then while $g(x) < 0$, update $x \leftarrow x + h$. Once g is non-negative, set $b = x$, and repeat step (2) $m \geqslant 1$ times.

2. The aim of this step is to have exactly k negative observations, followed by a non-negative $g(b)$.

   Halve the step size: $h \leftarrow h/2$.

   Evaluate g at points $\{b - jh\}$, $1 \leqslant j \leqslant k$. If at least one of the values of g is non-negative, push b to said point, and re-evalute the k observations to the left of b.

   Finally, we will have obtained the following:

$$\begin{aligned} &g(b) \geqslant 0, \\ &g(b - jh) < 0 \qquad \text{for } 1 \leqslant j \leqslant k. \end{aligned}$$

3. Use $(b - h)$ as the lower empirical bound for the zero of f.

Note that if f is decreasing, and we replace the step size $h > 0$ with $-h$, the algorithm above will yield a lower bound for the zero of f. This can be generalized in the following fasion. Let h denote the initaial step size. If $h > 0$ it means you go right from the initial point, and the generalized algorithm will yield a lower bound; otherwise you go left, and the algorithm will yield an upper bound.

Algorithm (Estimating zero of a noisy monotone function). This algorithm takes in six parameters: (i) the observed function g; (ii) a flag indicating whether f is increasing or decreasing; (iii) x, the initial point where to evaluate the function; (iv) h, the initial step size; (v) integer $m \geqslant 1$, the number of times to refine the search grid; (vi) integer $k \geqslant 1$, the number of points where the observations have to be of the same sign.

1. The aim of this step is to determine the sign of the "correct" side of the plane (corresponding to the circled sign in the table below).

$$s = \begin{cases} \mathbb{1}\{h > 0\} - \mathbb{1}\{h < 0\} & \text{if } f \text{ is increasing,} \\ \mathbb{1}\{h < 0\} - \mathbb{1}\{h > 0\} & \text{if } f \text{ is decreasing.} \end{cases}$$

f	increasing	decreasing	
$h > 0$	$---\oplus$	$+++\ominus$	Lower bound
$h < 0$	$\ominus+++$	$\oplus---$	Upper bound

2. The aim of this step is to find a point where the observed value is on the "correct" side of the plane.

   To this end, start at x with initial step h. Then while $s \cdot g(x) < 0$, update $x \leftarrow x + h$. Once $s \cdot g(x) \geqslant 0$, set $b = x$, and repeat step (3) $m \geqslant 1$ times.

3. The aim of this step is to have exactly k observations of sign $(-s)$, followed (or preceeded, depending on the sign of h) by $g(b)$ of sign s (or zero).

   Halve the step size: $h \leftarrow h/2$.

   Evaluate g at points $\{b - jh\}$, $1 \leqslant j \leqslant k$. If for at least one of them $s \cdot g(b - jh) \geqslant 0$, set $b \leftarrow (b - jh)$, and re-evalute the k adjacent observations.

   Finally, we will have obtained the following:

$$\begin{aligned} & s \cdot g(b) \geqslant 0, \\ & s \cdot g(b - jh) < 0 \qquad \text{for } 1 \leqslant j \leqslant k. \end{aligned}$$

4. If $h > 0$, then $(b - h)$ can be used as the lower empirical bound for the zero of f. If $h < 0$, then $(b - h)$ can be used as the upper empirical bound for the zero of f.