1 Normal (Gaussian)

Usually written as $\mathcal{N}\left(\mu, \sigma^2\right)$, it is parameterized by μ and σ^2. Its density is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

If $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$,

$$\mathsf{E}(X) = \mu, \qquad \mathsf{Var}(X) = \sigma^2.$$

2 Lognormal

Since Λ is log-normal, it has known partial moments: if $\Lambda \sim \ln\mathcal{N}\left(\mu_\Lambda, \sigma_\Lambda\right)$,

$$\begin{aligned}\int_a^b t^n \, d\mathbf{P}\left(\Lambda \leq t\right) &= e^{n\mu_\Lambda + \frac{1}{2}n^2\sigma_\Lambda^2}\left(\Phi\left[\frac{\mu_\Lambda + n\sigma_\Lambda^2 - \log a}{\sigma_\Lambda}\right] - \Phi\left[\frac{\mu_\Lambda + n\sigma_\Lambda^2 - \log b}{\sigma_\Lambda}\right]\right) \\ &= \frac{1}{2}e^{n\mu_\Lambda + \frac{1}{2}n^2\sigma_\Lambda^2}\left(\mathrm{Erf}\left[\frac{\mu_\Lambda + n\sigma_\Lambda^2 - \log a}{\sigma_\Lambda\sqrt{2}}\right] - \mathrm{Erf}\left[\frac{\mu_\Lambda + n\sigma_\Lambda^2 - \log b}{\sigma_\Lambda\sqrt{2}}\right]\right).\end{aligned}$$

3 Exponential

Consider a hypothesis for shift in mean: $\mathrm{Exp}\left(1/\mu\right) \longrightarrow \mathrm{Exp}\left(1/\theta\right)$, i.e.

$$\begin{aligned} f_0(x) &= \frac{1}{\mu} e^{-\frac{1}{\mu}x}, \\ f_1(x) &= \frac{1}{\theta} e^{-\frac{1}{\theta}x}. \end{aligned}$$

$$\begin{aligned} \Lambda(X) &= \frac{f_1(X)}{f_0(X)} = \frac{\mu}{\theta} e^{\left(\frac{1}{\mu}-\frac{1}{\theta}\right)X}, \qquad \Lambda^{-1}(t) = \frac{\log(t) - \log\left(\frac{\mu}{\theta}\right)}{\frac{1}{\mu}-\frac{1}{\theta}}, \\ \mathcal{L}(X) &= \log \Lambda(X) = \log\left(\frac{\mu}{\theta}\right) + \left(\frac{1}{\mu}-\frac{1}{\theta}\right) X, \\ \mathrm{I} &= \mathsf{E}_\theta\, \mathcal{L}(X) = \log\left(\frac{\mu}{\theta}\right) - 1 + \frac{\theta}{\mu}. \end{aligned}$$

$$\mathbf{P}\left(\Lambda \le t\right) = \mathbf{P}\left(\mathcal{L} \le \log t\right) = \mathbf{P}\left(\left(\frac{1}{\mu}-\frac{1}{\theta}\right) X \le \log t - \log\left(\frac{\mu}{\theta}\right)\right).$$

Here we have two cases: $\theta > \mu$ and $\theta < \mu$. In the former case

$$\begin{aligned} \mathbf{P}_\eta\left(\Lambda \le t\right) &= \mathbf{P}_\eta\left(X \le \frac{\log t - \log\left(\frac{\mu}{\theta}\right)}{\frac{1}{\mu}-\frac{1}{\theta}}\right) \\ &= \mathbf{P}_\eta\left(\frac{1}{\eta} X \le \frac{\log t - \log\left(\frac{\mu}{\theta}\right)}{\eta\left(\frac{1}{\mu}-\frac{1}{\theta}\right)}\right) \\ &= \begin{cases} 1 - \left(\frac{\mu/\theta}{t}\right)^{1/\left(\frac{\eta}{\mu}-\frac{\eta}{\theta}\right)} & \text{if} \quad t \ge \mu/\theta, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

i.e. $\Lambda \sim \mathrm{Pareto}\left(1/\left(\frac{\eta}{\mu}-\frac{\eta}{\theta}\right), \frac{\mu}{\theta}\right)$. In the latter case,

$$\begin{aligned} \mathbf{P}_\eta\left(\Lambda \le t\right) &= 1 - \mathbf{P}_\eta\left(\frac{1}{\eta} X \le \frac{-\log t + \log\left(\frac{\mu}{\theta}\right)}{\eta\left(\frac{1}{\theta}-\frac{1}{\mu}\right)}\right) \\ &= \begin{cases} \left(\frac{t}{\mu/\theta}\right)^{1/\left(\frac{\eta}{\theta}-\frac{\eta}{\mu}\right)} & \text{if} \quad 0 < t \le \mu/\theta, \\ 1 & \text{if} \quad t > \mu/\theta, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

i.e. Λ is a scaled version of Beta distribution with $\beta = 1$: $\Lambda \sim \mathrm{Be}_{(0,\mu/\theta)}\left(1/\left(\frac{\eta}{\theta}-\frac{\eta}{\mu}\right), 1\right)$.

In both cases, it has known partial moments:

$$\begin{aligned} \int_a^b t^n \, d\mathbf{P}\left(\Lambda \le t\right) &= \frac{\alpha_\Lambda k_\Lambda^{\alpha_\Lambda}}{n-\alpha_\Lambda}\left(b^{n-\alpha_\Lambda} - a^{n-\alpha_\Lambda}\right), && \text{if } \Lambda \sim \mathrm{Pareto}\left(\alpha_\Lambda, k_\Lambda\right), \\ \int_a^b t^n \, d\mathbf{P}\left(\Lambda \le t\right) &= \frac{\alpha_\Lambda k_\Lambda^{-\alpha_\Lambda}}{n+\alpha_\Lambda}\left(b^{n+\alpha_\Lambda} - a^{n+\alpha_\Lambda}\right), && \text{if } \Lambda \sim \mathrm{Be}_{(0,k_\Lambda)}\left(\alpha_\Lambda, 1\right). \end{aligned}$$

4 Chi-squared

Consider a hypothesis for shift in mean: $\chi^2(m) \longrightarrow \chi^2(n)$, i.e.

$$f_0(x) = \frac{1}{2^{m/2}\Gamma(m/2)} x^{\frac{m}{2}-1} e^{-\frac{x}{2}},$$
$$f_1(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}.$$

$$\Lambda(X) = \frac{f_1(X)}{f_0(X)} = \frac{\Gamma(m/2)}{\Gamma(n/2)} \left(\frac{X}{2}\right)^{\frac{n-m}{2}}, \qquad \Lambda^{-1}(t) = 2\left(\frac{\Gamma(n/2)}{\Gamma(m/2)}\, t\right)^{\frac{2}{n-m}},$$
$$\mathcal{L}(X) = \log\Lambda(X) = \log\left(\frac{\Gamma(m/2)}{\Gamma(n/2)}\right) + \frac{n-m}{2}\log\left(\frac{X}{2}\right),$$
$$\mathrm{I} = \mathsf{E}_n\,\mathcal{L}(X) = \log\left(\frac{\Gamma(m/2)}{\Gamma(n/2)}\right) + \frac{n-m}{2}\,\psi\left(\frac{n}{2}\right), \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

$$\mathbf{P}\left(\Lambda \le t\right) = \mathbf{P}\left(\left(\frac{X}{2}\right)^{\frac{n-m}{2}} \le \frac{\Gamma(n/2)}{\Gamma(m/2)} t\right).$$

Here we have two cases: $n > m$ and $n < m$. In the former case

$$\mathbf{P}_k\left(\Lambda \le t\right) = \mathbf{P}_k\left(X \le 2\left(\frac{\Gamma(n/2)}{\Gamma(m/2)}\, t\right)^{\frac{2}{n-m}}\right)$$

In the latter case,

$$\mathbf{P}_k\left(\Lambda \le t\right) = 1 - \mathbf{P}_k\left(X \le 2\left(\frac{\Gamma(n/2)}{\Gamma(m/2)}\, t\right)^{\frac{2}{n-m}}\right)$$