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BRNO UNIVERSITY OF TECHNOLOGY

FACULTY OF MECHANICAL ENGINEERING

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Topic 5: Transfer Function, PID controller

Course : Control Theory I (VA1-A 18/19Z)
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Location : A1/0636

Transfer function

The *transfer function* of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Consider the linear time-invariant system defined by the following differential equation:

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y$$
$$= b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

where y is the output of the system and x is the input. The transfer function of this system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero, or

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$
$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

By using the concept of transfer function, it is possible to represent system dynamics by algebraic equations in s. If the highest power of s in the denominator of the transfer function is equal to n, the system is called an *nth-order system*.

Example: Transfer function

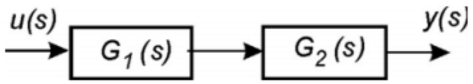
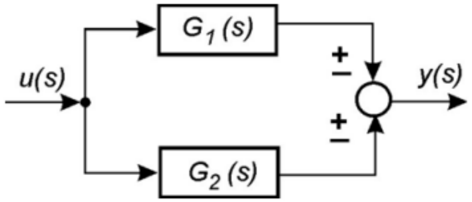
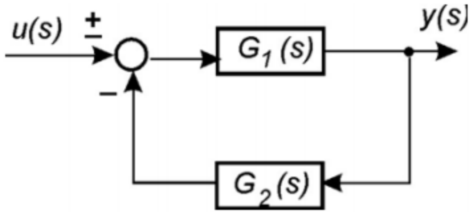
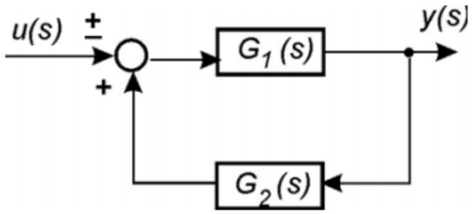
1. Transfer function $G(s)$ = ?

$$y''' + 4y'' + 0{,}5y' + 2y = 6u' + 3u$$

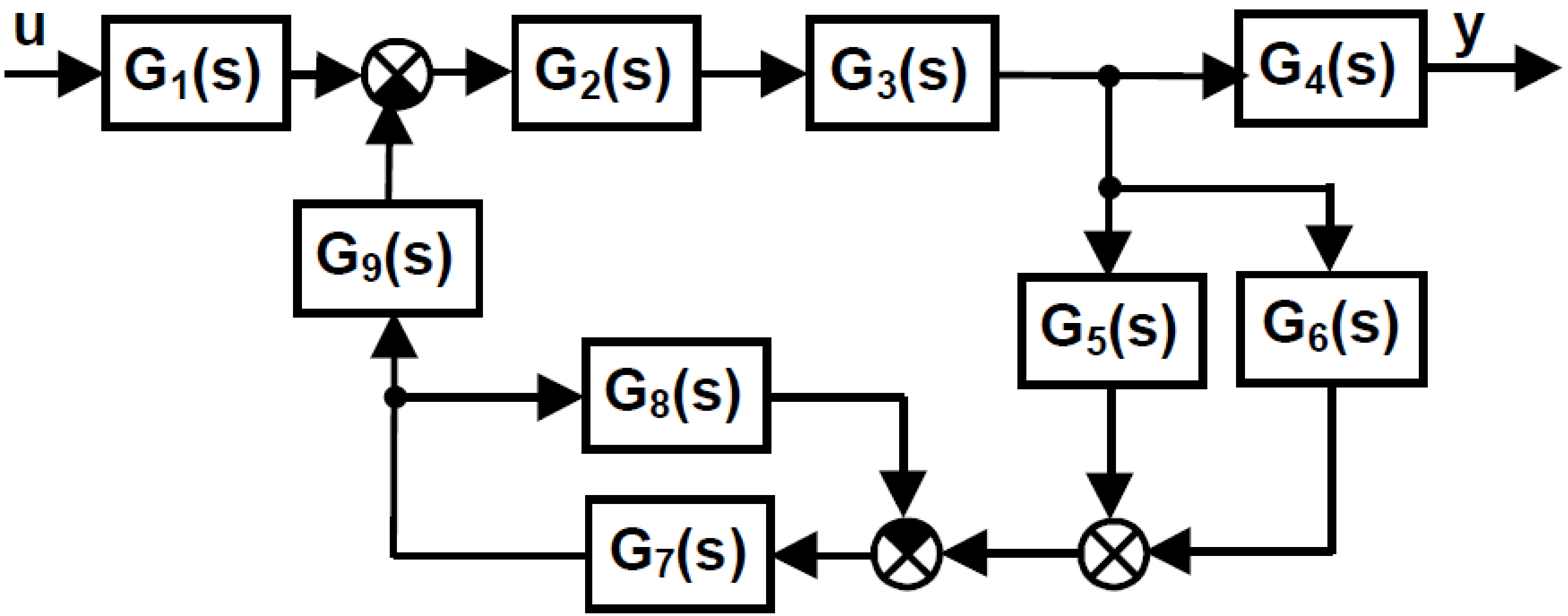
2. Differential equation ?

$$G(s) = \frac{7s^2 + 6s + 2}{s^3 + 5s^2 + 2s + 8}$$

Block diagram

Connection type	Transfer function	Block diagram
Serial connection (chain)	$G(s) = G_1(s)G_2(s)$	
Parallel connection	$G(s) = \pm G_1(s) \pm G_2(s)$	
Negative feedback loop	$G(s) = \dfrac{\pm G_1(s)}{1 + G_1(s)G_2(s)}$	
Positive feedback loop	$G(s) = \dfrac{\pm G_1(s)}{1 - G_1(s)G_2(s)}$	

Example: Block diagram

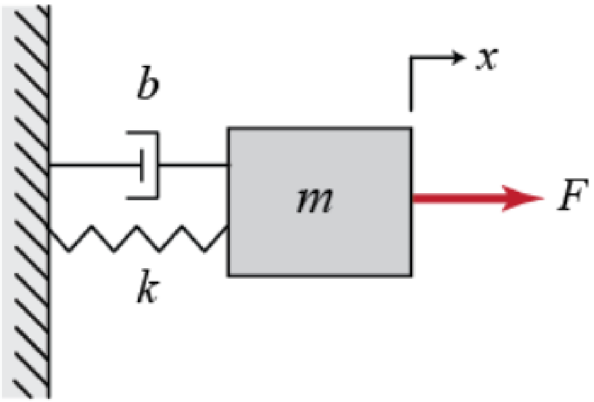


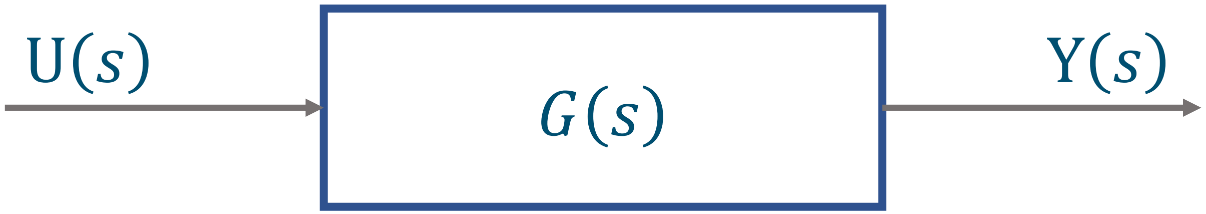
Example: Mass-Spring-Damper System

Parameters of the system:

```
m   = 1;    % mass                 [kg]
k   = 1;    % spring constant      [N/m]
b   = 0.2;  % damping constant     [Ns/m]
F   = 1;    % input force          [N]
```

1. Differential equation ?
2. Transfer function $G(s)$ = ?



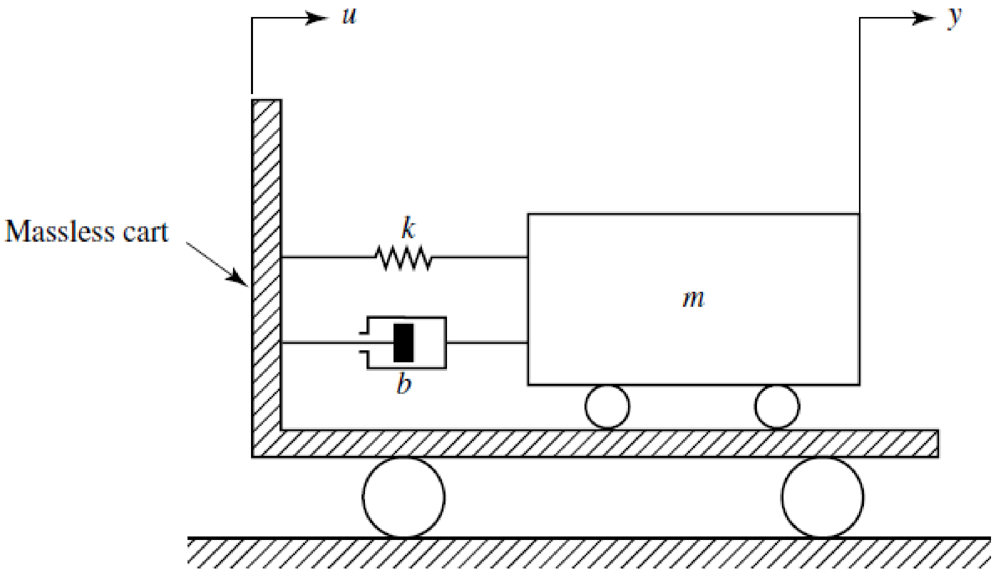


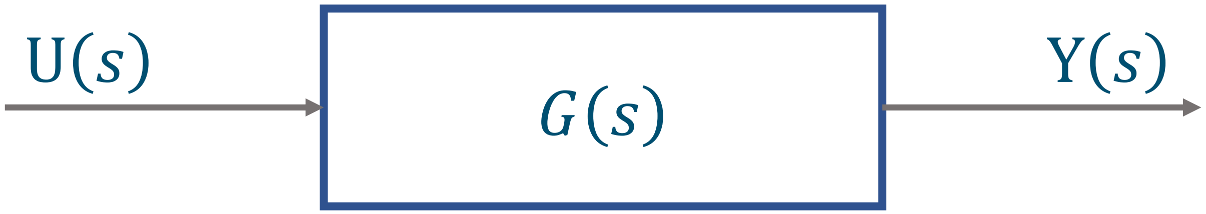
Example: MSD system mounted on a massless cart

Parameters of the system:

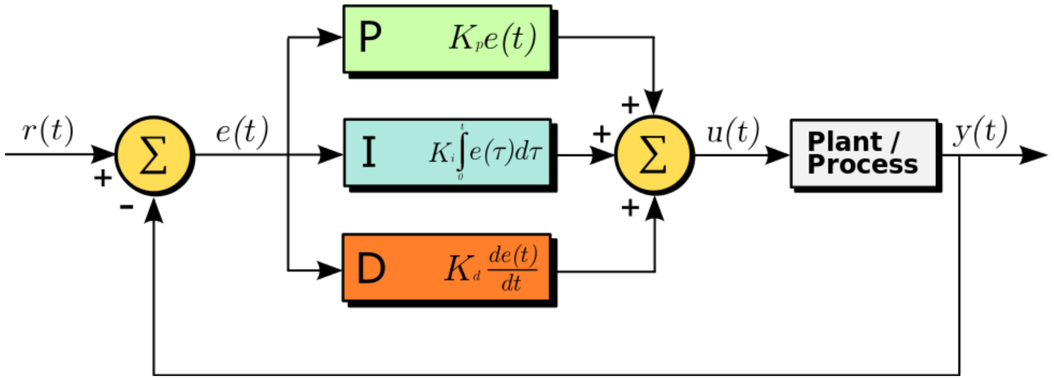
```
m   = 1;     % mass                 [kg]
k   = 1;     % spring constant      [N/m]
b   = 0.2;   % damping constant     [Ns/m]
F   = 1;     % input force          [N]
```

1. Differential equation ?
2. Transfer function $G(s)$ = ?





PID: Proportional-Integral-Derivative controller



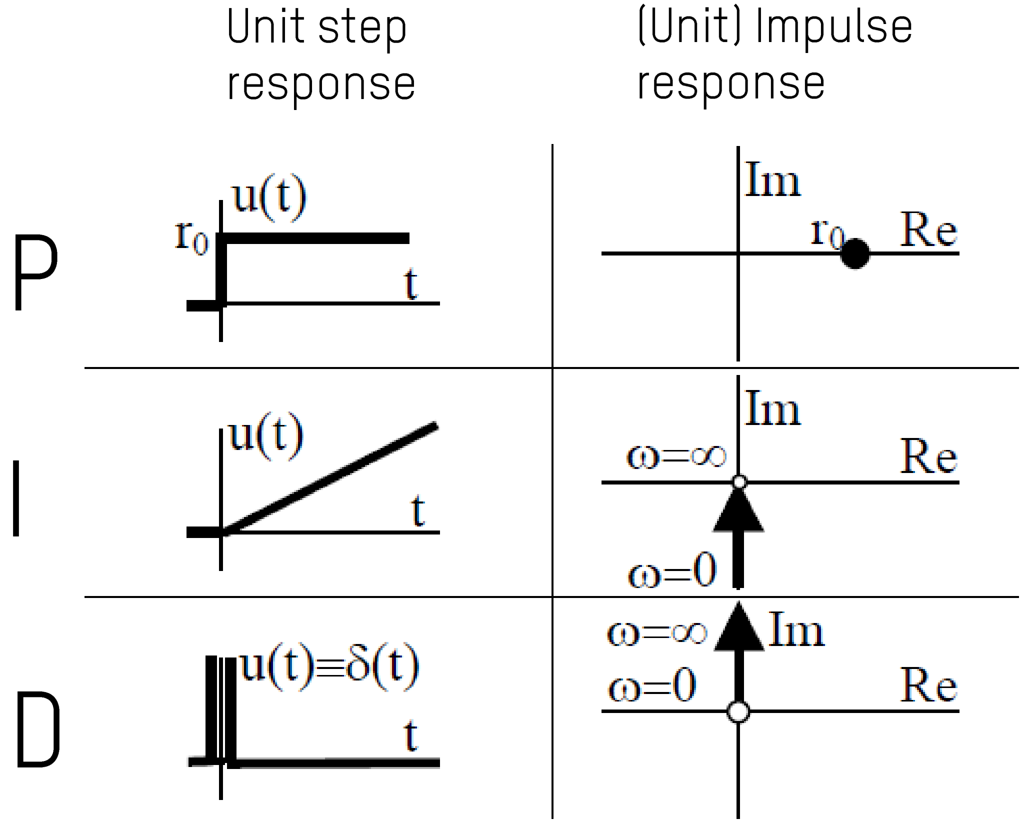
r(t): desired setpoint/ command variable
y(t): controlled variable
e(t): error value
u(t): manipulated variable

Mathematical form:

$$u(t) = K_p e(t) + K_i \int_0^t e(t')\,dt' + K_d \frac{de(t)}{dt},$$

Laplace transform:

$$G(s) = K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}$$



Example: PID

$$\mathrm{U}(s) \longrightarrow \boxed{G_P(s) = \frac{1}{(s+1)}} \longrightarrow \mathrm{Y}(s)$$

$$\mathrm{U}(s) \longrightarrow \boxed{G_I(s) = \frac{1}{s}} \longrightarrow \mathrm{Y}(s)$$

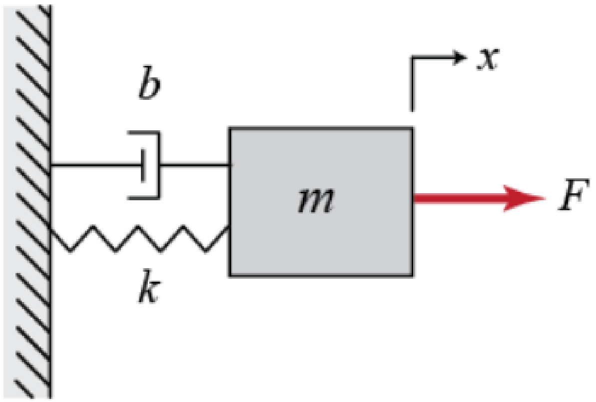
$$\mathrm{U}(s) \longrightarrow \boxed{G_D(s) = \frac{10s}{3s+1}} \longrightarrow \mathrm{Y}(s)$$

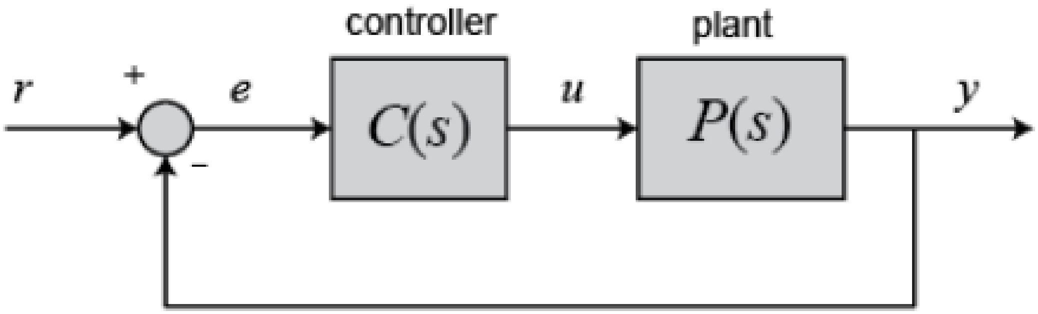
Example: MSD P, PI, PD, PID control

Parameters of the system:

```
m  = 1;   % mass              [kg]
k  = 20;  % spring constant   [N/m]
b  = 10;  % damping constant  [Ns/m]
F  = 1;   % input force       [N]
```

$$P(s) = \frac{1}{ms^2+bs+k}$$





1. Without controller
2. C(s) = P controller
3. C(s) = PD controller
4. C(s) = PI controller
5. C(s) = PID controller

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