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Topic 10: Inverted pendulum

Course : Control Theory I (VA1-A 18/19Z)
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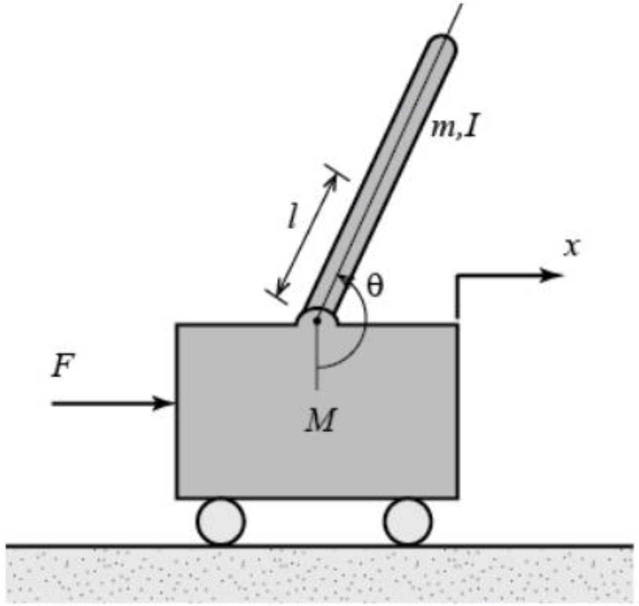
System modelling

The system in this example consists of an inverted pendulum mounted to a motorized cart. The inverted pendulum system is an example commonly found in control system textbooks and research literature. Its popularity derives in part from the fact that it is unstable without control, that is, the pendulum will simply fall over if the cart isn't moved to balance it. Additionally, the dynamics of the system are nonlinear. The objective of the control system is to balance the inverted pendulum by applying a force to the cart that the pendulum is attached to. A real-world example that relates directly to this inverted pendulum system is the attitude control of a booster rocket at take-off.

In this case we will consider a two-dimensional problem where the pendulum is constrained to move in the vertical plane shown in the figure below. For this system, the control input is the force F that moves the cart horizontally and the outputs are the angular position of the pendulum θ and the horizontal position of the cart x.



θ

Problem setup and design requirements

For this example, let's assume the following quantities:

M mass of the cart 0.5 kg
m mass of the pendulum 0.2 kg
b coefficient of friction for cart 0.1 N/m/sec
l length to pendulum centre of mass 0.3 m
I mass moment of inertia of the pendulum 0.006 kg.m^2
F force applied to the cart (x) cart position coordinate
θ pendulum angle from vertical (down)

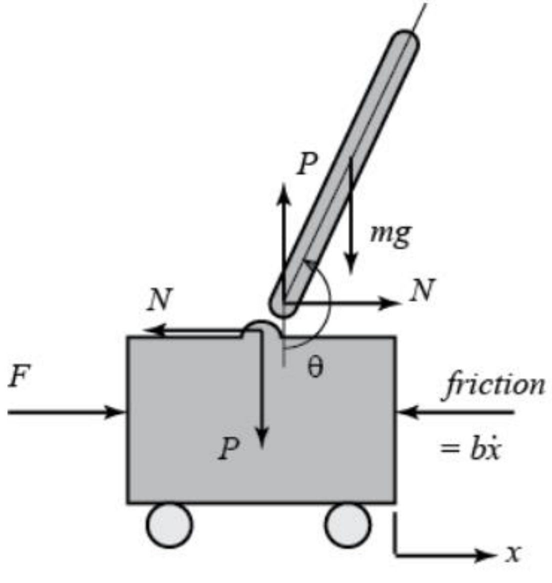
For the PID, root locus, and frequency response sections of this problem, we will be interested only in the control of the pendulum's position. This is because the techniques used in these sections are best-suited for single-input, single-output (SISO) systems. Therefore, none of the design criteria deal with the cart's position. We will, however, investigate the controller's effect on the cart's position after the controller has been designed. For these sections, we will design a controller to restore the pendulum to a vertically upward position after it has experienced an impulsive "bump" to the cart.

The pendulum will initially begin in the vertically upward equilibrium, $\theta = \pi$.

Pendulum angle θ pendulum angle not travel more than 20 degrees (0.35 radians) away from the vertically upward position.

Force analysis and system equations

Below are the free-body diagrams of the two elements of the inverted pendulum system.



Summing the forces in the free-body diagram of the cart in the horizontal direction, you get the following equation of motion.

$$M\ddot{x} + b\dot{x} + N = F$$

Force analysis and system equations

Note that you can also sum the forces in the vertical direction for the cart, but no useful information would be gained.

Summing the forces in the free-body diagram of the pendulum in the horizontal direction, you get the following expression for the reaction force N.

$$N = m\ddot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta$$

If you substitute this equation into the first equation, you get one of the two governing equations for this system.

$$(M + m)\ddot{x} + b\dot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta = F$$

Force analysis and system equations

To get the second equation of motion for this system, sum the forces perpendicular to the pendulum. Solving the system along this axis greatly simplifies the mathematics. You should get the following equation.

$$P sin\theta + N cos\theta - mg sin\theta = ml\ddot{\theta} + m\ddot{x} cos\theta$$

To get rid of the P and N terms in the equation above, sum the moments about the centroid of the pendulum to get the following equation.

$$-Pl sin\theta - Nl cos\theta = I\ddot{\theta}$$

Combining these last two expressions, you get the second governing equation.

$$(I + ml^2)\ddot{\theta} + mgl sin\theta = -ml\ddot{x} cos\theta$$

Force analysis and system equations

Since the analysis and control design techniques we will be employing in this example apply only to linear systems, this set of equations needs to be linearized. Specifically, we will linearize the equations about the vertically upward equilibrium position, $\theta = \pi$ and will assume that the system stays within a small neighbourhood of this equilibrium. This assumption should be reasonably valid since under control we desire that the pendulum not deviate more than 20 degrees from the vertically upward position. Let π represent the deviation of the pendulum's position from equilibrium, that is, $\theta = \pi + \varphi$ Again presuming a small deviation φ from equilibrium, we can use the following small angle approximations of the nonlinear functions in our system equations:

$$cos\theta = \cos(\pi + \varphi) \approx -1$$
$$sin\theta = sin(\pi + \varphi) \approx -\varphi$$

$$\dot{\theta}^2 = \varphi^2 \approx 0$$

Force analysis and system equations

Differential equations:

After substituting the above approximations into our nonlinear governing equations, we arrive at the two linearized equations of motion. Note u has been substituted for the input F.

$$(I + ml^2)\ddot{\varphi} - mgl\varphi = ml\ddot{x}$$

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\varphi} = u$$

Transfer function:

To obtain the transfer functions of the linearized system equations, we must first take the Laplace transform of the system equations assuming zero initial conditions. The resulting Laplace transforms are shown below.

$$(I + ml^2)\varphi(s)s^2 - mgl\varphi(s) = mlX(s)s^2$$

$$(M + m)X(s)s^2 + bX(s)s - ml\varphi(s)s^2 = U(s)$$

Force analysis and system equations

Recall that a transfer function represents the relationship between a single input and a single output at a time. To find our first transfer function for the output $\varphi(s)$ and an input of $U(s)$ we need to eliminate $X(s)$ from the above equations. Solve the first equation for $X(s)$.

$$X(s) = \left[\frac{(I + ml^2)}{ml} - \frac{g}{s^2}\right]\varphi(s)$$

Then substitute the above into the second equation.

$$(M + m)\left[\frac{(I + ml^2)}{ml} - \frac{g}{s^2}\right]\varphi(s)s^2 + b\left[\frac{(I + ml^2)}{ml} - \frac{g}{s^2}\right]\varphi(s)s - ml\varphi(s)s^2 = U(s)$$

Rearranging, the transfer function is then the following

$$\frac{\varphi(s)}{U(s)} = \frac{\frac{ml}{q}s^2}{s^4 + b\frac{(I + ml^2)}{q}s^3 - (M + m)\frac{mgl}{q}s^2 - \frac{bmgl}{q}s} \rightarrow \text{where}, q = [(M + m)(I + ml^2) - (ml^2)]$$

Force analysis and system equations

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From the transfer function above it can be seen that there is both a pole and a zero at the origin. These can be cancelled and the transfer function becomes the following.

$$P_{pend} = \frac{\varphi(s)}{U(s)} = \frac{\frac{ml}{q}s^2}{s^4 + b\frac{(I+ml^2)}{q}s^3 - (M+m)\frac{mgl}{q}s^2 - \frac{bmgl}{q}s} \left[\frac{rad}{N}\right]$$

Second, the transfer function with the cart position $X(s)$ as the output can be derived in a similar manner to arrive at the following.

$$P_{cart} = \frac{X(s)}{U(s)} = \frac{\frac{(I+ml^2)s^2 - gml}{q}}{s^4 + b\frac{(I+ml^2)}{q}s^3 - (M+m)\frac{mgl}{q}s^2 - \frac{bmgl}{q}s} \left[\frac{m}{N}\right]$$

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