Optimal Threshold Dividend Policy in Insurance

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Abstract

In this paper we discuss the optimal threshold dividend strategy under a maximum payout rate constraint and a minimum barrier height constraint. The optimal strategy is obtained by *HJB* functions, which also derives the optimized utility function in conditions of different constraints of minimum barrier height and maximum dividend payout per unit time. We also prove the convexity of the utility function and that the optimal dividend pay-out strategy with proportional reinsurance saves the insurance company from insolvency with probability 1.

1 Models, Notations and Preliminaries

We take into account an insurance company whose reserve (i.e. risk process) at time t is described by a stochastic process R_t. Motivated by the model in [3], with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the standard Brownian Motion $\{W_t\}_{t\geq 0}$ and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ induced by it. Following numerous literatures in insurance, We assume that, when there is no control, the risk process evolves as:

$$dR_t = \mu dt + \sigma dW_t \tag{1.1}$$

where $\mu, \sigma > 0$ and the initial R_0 is $\mathcal{F}_0$ measurable[1]. The model is valid when applied to describe large insurance companies where the claim size is small compared with its capital reserve. For a further discussion of this model, see, for example, [1] and [3].

A control strategy π is described by a two-dimensional stochastic process $\{a^\pi(t), L_t^\pi\}_{t\geq 0}$, where $0 \leq a_t^\pi \leq 1$ and $L_t^\pi \geq 0$. For a strategy $\pi = \{a^\pi(t), L_t^\pi\}$, $a^\pi(t)$ corresponds to the risk exposure at time t, or the reinsurance proportion at time t; $L_t^\pi \geq 0$ is a non-decreasing and $\mathcal{F}_t$ adapted process. As a result, we refer R_t^π to the controlled process as follows:

$$dR_t^\pi = a^\pi(t)\mu dt + a^\pi(t)\sigma dW_t - dL_t^\pi \tag{1.2}$$

$$R_0^\pi = x - L_0^\pi \tag{1.3}$$

where, as in (1.1), W_t is a standard Brownian Motions in $\mathbb{R}$. We define bankruptcy time as $\tau^\pi = \inf\{t \geq 0 : R_t^\pi < 0\}$. The collection of *admissible* strategies is denoted as $\Pi = \{\pi : \tau^\pi \leq T\}$, which is the collection of all strategies under which the company is free from insolvency before time T. For any given strategy π, we consider the discounted dividend payout until insolvency and let the defective dividend payout function be:

$$V_\pi(x) = \mathbb{E}\int_0^{\tau^\pi} e^{-ct} dL_t^\pi, \tag{1.4}$$

[1] We usually take it as a deterministic value x.

where c is the discount rate. The objective function we find is the optimal dividend payout function

$$V(x) = \sup_{\pi\in\Pi} V_\pi(x), \tag{1.5}$$

for any $x \geq 0$. In this paper, we set a minimum payout barrier $B \geq 0$ and the maximum payout rate $M < \infty$. To explicate, for a given strategy π at time t, if $R_t^\pi < B$, $L_t^\pi = 0$ *a.s.*, while $dL_t^\pi \leq M$ *a.s.*. And we denote Π_B^M as the set of all admissible strategies with a minimum payout barrier constraint B and a maximum payout rate constraint M. We first derive the *HJB* function that $V(x)$ satisfies. In our paper, we assume that $V(x)$ is twice continuously differentiable on $\mathbb{R}^+ = [0, +\infty)$ except for a finite number of points. For the convenience of our discussion, we define the differential operator $\mathcal{L}^{a,l}$ as follows

$$\mathcal{L}^{a,l} = \frac{1}{2}\sigma^2 a^2 \frac{d^2}{dx^2} + (\mu a - l)\frac{d}{dx} - c, \tag{1.6}$$

where $0 \leq a \leq 1$, denoting the proportion of reinsurance at time t, and $l \leq M$, referring to the dividend payout rate at t.

Theorem 1.1 Under the assumption that $V(x)$ defined by (1.5) is twice continuously differentiable on $(0, +\infty)$ except for a finite number of points, we have $V(x)$ satisfies the following *Hamilton-Jacobi-Bellman* equation

$$\max_{a\in[0,1], l\in[0,M]}[\frac{1}{2}\sigma^2 a^2 V''(x) + (\mu a - l)V'(x) - cV(x) + l] = 0 \tag{1.7}$$

with $V(0) = 0$.

Proof The proof, illuminated by a similar one in [3], is based on *dynamical programing principle*

$$V(x) = \sup_\pi \mathbb{E}[\int_0^{\tau^\pi\wedge\sigma} e^{-ct} l_\pi(s)ds + \mathbf{1}_{\{\sigma<\tau^\pi\}} e^{-c\sigma} V(R_\sigma^\pi)], \tag{1.8}$$

where σ is a stopping time. For any admissible $\pi \in \Pi_B^M$ and $h > 0$, let $\sigma_\pi^h = h \wedge \inf\{t : R_t^\pi \notin (x-h, x+h)\}$. Clearly, $\sigma_\pi^h < \infty$ *a.s.* and $\sigma_\pi^h \to 0$ *a.s.* as $h \to 0$. For any fixed a and l, and choose π such that $a^\pi(t) = a$ and $l^\pi(t) = l$. Additionally, let $h < x$, so we have $\sigma_\pi^h < \tau^\pi$ *a.s.*. We take $\sigma = \sigma_\pi^h$ in (1.8), thus

$$V(x) \geq \mathbb{E}[\int_0^{\sigma_\pi^h} e^{-ct} l dt + e^{-c\sigma_\pi^h} V(R_{\sigma_\pi^h}^\pi)]. \tag{1.9}$$

Applying Itô's formula to $g(t,x) = e^{-ct}V(R_t^\pi)$ in (1.9), we get

$$V(x) \geq \mathbb{E}[\int_0^{\sigma_\pi^h} e^{-ct} l dt] + V(x) + \mathbb{E}[\int_0^{\sigma_\pi^h} e^{-ct}\mathcal{L}^{a,l}V(R_t^\pi)dt].$$

So, we have:

$$\mathbb{E}\int_0^{\sigma_\pi^h} e^{-ct}(l + \mathcal{L}^{a,l}V(R_t^\pi))dt = \mathbb{E}[\int_0^{\sigma_\pi^h} e^{-ct} l dt] + \mathbb{E}[\int_0^{\sigma_\pi^h} e^{-ct}\mathcal{L}^{a,l}V(R_t^\pi)dt] \leq 0.$$

Dividing both sides by $\mathbb{E}[\sigma_\pi^h]$, we get:

$$0 \geq \frac{1}{\mathbb{E}[\sigma_\pi^h]}\mathbb{E}\int_0^{\sigma_\pi^h} e^{-ct}(l + \mathcal{L}^{a,l}V(R_t^\pi))dt \to l + \mathcal{L}^{a,l}V(x) \ a.s.,$$

as $h \to 0$. We take superiority in both sides, by (1.8) and Fatou's Lemma, so that (1.7) follows. □

Remark 1.2 From (1.7), we see that the objective function we optimize is quadratic with a_t and linear with l_t. As a result, l_t should be M or 0 depending on whether $V'(x) < 1$. As a maximizer of a quadratic function (the left-hand side of (1.7)), a_t should be 0, 1 or $-\frac{\mu V'(x)}{\sigma^2 V''(x)}$. It is also clear from the Markov property of R_t, that l_t and a_t is dependent on R_t, so the notions $a(x)$ and $l(x)$ are well-defined. Additionally, from (1.2) and (1.4), we derive that $V'(x) \geq 0$ for any x in the domain of $V'(\cdot)$.

2 Determining $V(x)$ without any Barrier Constraint

In this section, we try to solve (1.7) without any barrier constraint, i.e. $B = 0$. The forgoing argument is based on the one in [3], with a few corrections made. The proof of $V(x)$'s concavity in [3] is not rigorous since we can not make the path decomposition under stochastic strategy $\pi = \{a_t, l_t\}$. But we are going to prove that when x is in the right neighborhood of 0, $V''(x) < 0$.

Lemma 2.1 Suppose $V(x)$ satisfies (1.7), then $\exists \delta$ such that $V''(x) \leq 0$ when $x \in [0, \delta)$. ($V'(0)$ is taken as the right derivative at 0.)

Proof We will prove the assertion by pointing out contradictions under the condition that $V''(x+) > 0$. Let's first divide the problem into two cases: $V'(0+) \geq 1$ and $V'(0+) < 1$.

Case 1 $V'(0+) \geq 1$, so $l(x) = 0$ in the right neighborhood of 0.

In this case, we immediately have $-\frac{\mu V'(x)}{\sigma^2 V''(x)} < 0$, so $a(x) = 1$ when x is near 0. Then (1.7) is reduced to

$$\frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - cV(x) = 0 \tag{2.1}$$

With the initial value $V(0) = 0$, we have that, as long as x is close enough to 0,

$$V(x) = f_1(x) = C_1 \mathrm{e}^{d_+ x} - C_1 \mathrm{e}^{d_- x},$$

where C_1 is a positive constant and

$$d_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 2\sigma^2 c}}{\sigma^2}. \tag{2.2}$$

From (2.2), we have $d_+^2 < d_-^2$. However, $f''(x) = C_1(d_+^2 \mathrm{e}^{d_+ x} - d_-^2 \mathrm{e}^{d_- x}) < 0$ when x is close enough to 0, contradicting our assumption.

Case 2 $V'(0+) < 1$, so $l(x) = M$ in the right neighborhood of 0.

Similar argument as the one in the last case yields that $a(x) = 1$ when x is in a right neighborhood of 0. So the *HJB* function (1.7) is reduced to

$$\frac{1}{2}\sigma^2 V''(x) + (\mu - M)V'(x) - cV(x) + M = 0. \tag{2.3}$$

So we have, when x is close to 0,

$$V(x) = f_2(x) = \frac{M}{c} + \bar{C}_1 \mathrm{e}^{\bar{d}_+ x} - (\bar{C}_1 + \frac{M}{c})\mathrm{e}^{\bar{d}_- x},$$

where $\bar{C}_1$ is a constant and

$$\bar{d}_\pm = \frac{-(\mu - M) \pm \sqrt{(\mu - M)^2 + 2\sigma^2 c}}{\sigma^2}. \tag{2.4}$$

Clearly, $\bar{d}_+^2 \leq \bar{d}_-^2$, so we have $V''(0+) = \bar{C}_1 \bar{d}_+^2 - (\bar{C}_1 + \frac{M}{C})\bar{d}_+^2 \leq 0$. The assertion of the lemma follows from this contradiction. $\square$

Next, we prove that $V'(x) > 1$ near 0.

Lemma 2.2 Suppose $V(x)$ satisfies (1.7), then $\exists\delta$ such that $V'(x) > 1$ when $x \in [0, \delta)$. (At 0, the derivative is taken as the right derivative.)

Proof We still divide the problem into 2 cases:

Case 1 $-\frac{f'\mu}{\sigma^2 f''} \geq 1$, then $a = 1$ near $x = 0$

It follows from the same argument as in Case 2 of the proof of Lemma 2.1 that this case results in a contradiction.

Case 2 $-\frac{f'\mu}{\sigma^2 f''} < 1$, then $a = -\frac{f'\mu}{\sigma^2 f''}$ near $x = 0$

Plug a and l into (1.4), and solve the reduced equation, we have that

$$V(x) = f_4(x) = \frac{M}{c} + C_1^* \mathrm{e}^{\frac{1}{M}(c+\frac{\mu^2}{2\sigma^2})x},$$

contradicting the initial value $V(0) = 0$. The lemma follows. □

Lemma 2.1 and Lemma 2.2 tell us that $V(x)$ satisfies the conditions $V''(x) < 0$ and $V'(x) > 1$ when x is close to 0. Applying the method in [3], we derive the explicit form of $V(x)$ with some corrections to its argument and result. In the following paper, we let $u_1 = inf\{u : V'(u) = 1\}$, then $l(x) = 0$ when $x \leq u_1$. Hence, when $x < u_1$,

$$max_a[\frac{1}{2}\sigma^2 a^2 V''(x) + \mu a V'(x) - cV(x)] = 0. \tag{2.5}$$

As a maximizer of (2.5),

$$a(x) = -\frac{\mu V'(x)}{\sigma^2 V''(x)} \tag{2.6}$$

when $x < u_1$. Plug (2.6) into (2.5) and solve the equation. We have that

$$V(x) = g_1(x) = c_1 x^\gamma, \tag{2.7}$$

where $\gamma = \frac{c}{\frac{\mu^2}{2\sigma^2}+c}$. Hence, $a(x) = -\frac{\mu x}{\sigma^2(\gamma-1)}$. We assume $a(u_0) = 1$, and get that $u_0 = \frac{\sigma^2}{\mu}(1-\gamma)$. We first assume that $u_0 < u_1$ and , therefore, when $u_0 < x < u_1$

$$V(x) = g_2(x) = c_2 \mathrm{e}^{d_-(x-u_0)} + c_3 \mathrm{e}^{d_+(x-u_0)} \tag{2.8}$$

where $d_\pm$ is shown in (2.2). Clearly, when $x > u_1$, $l(x) = M$ and $V(x)$ satisfies (2.3). By the boundedness of V, we have that

$$V(x) = g_3(x) = \frac{M}{c} + c_4 \mathrm{e}^{\hat{d}(x-u_1)} \tag{2.9}$$

where $\hat{d} = \bar{d}_-$. From the discussion above, we have that

$$a(x) = \frac{\mu x}{\sigma^2(1-\gamma)} \tag{2.10}$$

when $x < u_0$ and

$$a(x) = 1 \tag{2.11}$$

when $x \geq u_0$. And $l(x)$ is given as follows

$$l(x) = 0 \tag{2.12}$$

when $x < u_1$ and

$$l(x) = M \tag{2.13}$$

when $x \geq u_1$. The constants in g_1, g_2 and g_3 are determined by the continuity assumption of V and V', and that $V'(u_1) = 1$. It follows that

$$c_1 \gamma u_0^{\gamma-1} = c_2 d_- + c_3 d_+ \tag{2.14}$$

and

$$c_1 u_0^{\gamma} = c_2 + c_3, \tag{2.15}$$

so we have

$$c_2 = c_1 \frac{u_0^{\gamma-1}(u_0 d_+ - \gamma)}{d_+ - d_-} \tag{2.16}$$

and

$$c_3 = c_1 \frac{u_0^{\gamma-1}(\gamma - d_- u_0)}{d_+ - d_-}. \tag{2.17}$$

Plug the previous value of c_2 and c_3 into g_2 and solve $g_2'(u_1) = g_3'(u_1) = 1$ $g_2(u_1) = g_3(u_1)$ and we get that

$$u_1 = u_0 + \frac{1}{d_+ - d_-} \ln \frac{(d_+ u_0 - \gamma)[d_-(\frac{M}{c} + \frac{1}{\hat{d}}) - 1]}{(\gamma - d_- u_0)[1 - d_+(\frac{M}{c} + \frac{1}{\hat{d}})]}, \tag{2.18}$$

$$c_4 = \frac{1}{\hat{d}} \tag{2.19}$$

and

$$\begin{aligned} c_1 = \frac{d_+ - d_-}{u_0^{\gamma-1}}(\frac{M}{c} + \frac{1}{\hat{d}})\{(d_+ u_0 - \gamma)[\frac{(d_+ u_0 - \gamma)[d_-(\frac{M}{c} + \frac{1}{\hat{d}}) - 1]}{(\gamma - d_- u_0)[1 - d_+(\frac{M}{c} + \frac{1}{\hat{d}})]}]^{\frac{d_-}{d_+ - d_-}} + \\ (\gamma - d_- u_0)[\frac{(d_+ u_0 - \gamma)[d_-(\frac{M}{c} + \frac{1}{\hat{d}}) - 1]}{(\gamma - d_- u_0)[1 - d_+(\frac{M}{c} + \frac{1}{\hat{d}})]}]^{\frac{1}{d_+ - d_-}}\} \end{aligned} \tag{2.20}$$

The remaining problem is that we have not verified that $u_1 \geq u_0$. If so, it follows that $u_1 \geq u_0$ if and only if

$$\frac{(d_+ u_0 - \gamma)[d_-(\frac{M}{c} + \frac{1}{\hat{d}}) - 1]}{(\gamma - d_- u_0)[1 - d_+(\frac{M}{c} + \frac{1}{\hat{d}})]} \geq 1 \tag{2.21}$$

Straightforward calculation implies that (2.21) is equivalent to

$$M \geq \frac{\mu}{2} + \frac{c\sigma^2}{\mu} \tag{2.22}$$

It is clear from its explicit form that $V(x)$ constructed above is concave and we have the following theorem.

Theorem 2.3 Assume $M \geq \mu/2 + c\sigma^2/\mu$ and $V(x)$ given by (2.7) (2.8) and (2.9), we have that $V(x)$ is a concave solution of (2.5).

Proof The proof is exactly the same as the one of Theorem 2.1 in [3]. The concavity is obvious so is the optimality of l. What remains to be shown is $a(x) = 1$ for $u_0 < x < u_1$. This is a direct and simple verification and we omit it here. For details, please refer to [3]. □

As is done in [3], for $M < \mu/2 + c\sigma^2/\mu$, $u_1 < u_0$, and the solution to (2.5) is given by

$$V(x) = h_1(x) = \frac{u_1}{\gamma}(\frac{x}{u_1})^{\gamma} \tag{2.23}$$

when $x \leq u_1$ and

$$V(x) = h_2(x) = \frac{M}{c}(1 - \gamma e^{-\frac{c}{M\gamma}(x - u_1)}) \tag{2.24}$$

when $x > u_1$. The maximizing function $a(x)$ is given by

$$a(x) = \frac{\mu x}{\sigma^2(1-\gamma)} \tag{2.25}$$

when $x < u_1$ and

$$a(x) = \frac{\mu u_1}{c\sigma^2(1-\gamma)} \tag{2.26}$$

when $x \geq u_1$. So, we have:

Theorem 2.4 Assume $M < \mu/2 + c\sigma^2/\mu$ and $V(x)$ given by (2.23) and (2.24), we have that $V(x)$ is a concave solution of (2.5).

Proof The proof is given in [3], Theorem 2.2. □

As usual, we need a verification theorem that indicates the solutions constructed above is optimal. That is:

Theorem 2.5 Suppose, $V(x)$ is given by (2.7) (2.8) and (2.9) for $M \geq \mu/2 + c\sigma^2/\mu$, and by (2.23) and (2.24) for $M < \mu/2 + c\sigma^2/\mu$. Then $V(x)$ satisfies (1.4), and $V(x) = V_{\pi^*}(x)$, where π^* is given by $a_{\pi^*}(t) = a(R_t^{\pi^*})$ and $l_{\pi^*}(t) = l(R_t^{\pi^*})$, for $t < \tau_{\pi^*}$, in which a and l are given by (2.10), (2.11) (2.25) (2.26) and (2.12), (2.13) respectively.

Proof The proof is the same as the one given in [3], Theorem 2.3. We need to apply Itô's formula to $\mathrm{e}^{-c(t\wedge\tau_\pi^\epsilon)}V(R_{t\wedge\tau_\pi^\epsilon})$, where $\tau_\pi^\epsilon = \inf\{t : R_t^\pi = \epsilon\}$ for a chosen $0 < \epsilon < x$, and the fact that $\int_0^{t\wedge\tau_\pi^\epsilon} \mathrm{e}^{-cs}\sigma a_\pi(s)V'(R_s^\pi)dW_s$ is a $0-$mean martingale. □

Remark 2.6 The structure of the optimal dividend policy is very clear. When the capital reserve is low, the major concern of an insurance company is to reduce the insolvency risk, so it is optimal for the firm to spread no dividend and have a greater reinsurance proportion. However, when the reserve capital is high, there is no immediate risk of insolvency, so the company can pay as much as possible and have no reinsurance at all. As we will show immediately, this policy not only optimizes the expected dividend payout but also completely reduces insolvency in finite time. Such property is a result of $a(x) = \frac{\mu x}{\sigma^2(1-\gamma)}$ when $x < u_0 \wedge u_1$. Plug it into (1.2) and we have that:

$$dR_t^{\pi^*} = \mu r R_t^{\pi^*} dt + \sigma r R_t^{\pi^*} dW_t \tag{2.27}$$

where $R_0 = x < u_0$ and $r = \frac{\mu}{\sigma^2(1-\gamma)}$. The solution to (2.27) is a geometric Brownian Motion, which is bigger than 0 with probability 1.

Theorem 2.7 Under policy π^*, in which $a_{\pi^*}(t) = a(R_t^{\pi^*})$ and $l_{\pi^*}(t) = l(R_t^{\pi^*})$, $\mathbb{P}(\tau^{\pi^*} = \infty) = 1$, independent of the initial capital x.

Proof By the Markov property of $R_t^{\pi^*}$, we only need to consider the case $x < u^* = u_0 \wedge u_1$. Let $\tau^* = \inf\{t : R_t^{\pi^*} > u^*\}$, and τ^* is also a stopping time, so we define $\sigma^* = \tau^* \wedge \tau^{\pi^*}$. Since $R_t^{\pi^*} > 0$ when $t < \sigma^*$, we can apply Itô's formula to $\ln(R_t^{\pi^*})$. That is

$$\begin{aligned}
\ln(R_t^{\pi^*}) - \ln(x) &= \int_0^t \frac{dR_s^{\pi^*}}{R_s^{\pi^*}} - \frac{1}{2}\int_0^t \frac{d < R^{\pi^*}, R^{\pi^*} >_t}{(R_s^{\pi^*})^2} \\
&= \int_0^t \mu r dt + \sigma r dW_t - \frac{1}{2}\int_0^t \frac{\sigma^2 r^2 (R_t^{\pi^*})^2}{(R_s^{\pi^*})^2} dt \\
&= \int_0^t (\mu r - \frac{1}{2}\sigma^2 r^2) dt + \int_0^t \sigma r dW_t \\
&= (\mu r - \frac{1}{2}\sigma^2 r^2)t + \sigma r W_t
\end{aligned} \tag{2.28}$$

for $t < \sigma^*$ and $x < u^*$. Take exponent in both sides of (2.28), we get that,

$$R_t^{\pi^*} = x\exp[(\mu r - \frac{1}{2}\sigma^2 r^2)t + \sigma r W_t] \tag{2.29}$$

for $t < \sigma^*$ and $x < u^*$. So, $R_t^{\pi^*}$ is a geometric Brownian Motion with a drift before τ^{π^*}, which can hit 0 with probability 0, thus concluding the proof. □

Remark 2.8 Theorem 2.7 can also be generalized to optimal barrier strategy under proportional reinsurance, which is discussed in [2]. In [2], the authors applied Girsarnov's Theorem to estimate the insolvency probability of company. However, Theorem 2.7 covers their results completely since the ruin probability is invariantly 0 regardless of the barrier. Consequently, the optimized barrier under ruin probability constraint is the same as the global optimal barrier, that is b^* in [2], not the one obtained therein according to the ruin probability estimation.

3 Determining $V(x)$ with a Barrier Constraint B

In this section, we deal with the condition in which there is a barrier constraint $B \neq 0$. As is done in the last section, we will solve (1.7) explicitly and determine the optimal strategy π_B^*. The *HJB* equation for the case with a barrier constraint can be reformulated as follows:

$$\max_{a\in[0,1]} \left[\frac{1}{2}\sigma^2 a^2 V''(x) + \mu a V'(x) - cV(x)\right] = 0 \quad 0 \leq x \leq B, \tag{3.1}$$

$$\max_{a\in[0,1], l\in[0,M]} \left[\frac{1}{2}\sigma^2 a^2 V''(x) + (\mu a - l)V'(x) - cV(x) + l\right] = 0 \quad x > B \tag{3.2}$$

The solution will be based on the results in the last section but with a higher degree of complexity.

Lemma 2.1 and Lemma 2.2 enables us to assume $V_B'(x) > 1$, $l(x) = 0$ and $a_B(x) = \frac{\mu x}{\sigma^2(1-\gamma)}$ when $x < \delta$. Since the solution to the *HJB* equation is different according to the value of M, we divide the problem into two cases: $M \leq \mu/2 + c\sigma^2/\mu$ and $M > \mu/2 + \sigma^2/\mu$.

3.1 $M \leq \mu/2 + c\sigma^2/\mu$

For different values of B, we have different solutions to the *HJB* equation. So, we consider the following 3 cases separately.

Case 1 $B \leq u_1$

The optimal strategies in this case are the same as the those without a barrier constraint. So, the solution in this case is the same as that without a barrier constraint given by equation (2.50) in [3].

Case 2 $u_1 < B < u_0$

For $x \leq B$, equation (3.1) is solved by

$$V_B(x) = v_1(x) = c_1' x^\gamma$$

with the maximizer $a(x) = -\frac{\mu V_B'(x)}{\sigma^2 V_B''(x)} = \frac{\mu x}{\sigma^2(1-\gamma)} < 1$, for $x \leq B < u_0$. Because $\Pi_B^* \subset \Pi^*$, the value function in this case is not bigger than the one without a barrier constraint, i.e. $c' \leq c_1$, where c_1 is defined by (2.50) in [3]. Now we get $v_1'(B) < 1$. The concavity of the value function implies that $V_B'(x) < 1$ for $x > B$. The equation (3.2) becomes

$$\max_{a\in[0,1]} \left[\frac{1}{2}\sigma^2 a^2 V^{''}(x) + (\mu a - M)V^{'}(x) - cV(x) + M\right] = 0 \qquad x > B$$

Inspired by [3], we can solve the above equation by the solution:

$$V_B(x) = v_2(x) = \frac{-M\eta}{1+c\eta} e^{\frac{1+c\eta}{-M\eta}(x-k_2)} + c_2' \qquad x > B,$$

where $\eta = 2\sigma^2/\mu^2$ and k_2, c_2' are unknown constants. To determine the unknown constants, we have the following equations by the continuity of $V_B(x)$, and its first derivative at B.

$$c_1' B^\gamma = \frac{-M\eta}{1+c\eta} e^{\frac{1+c\eta}{-M\eta}(B-k_2)} + c_2' \tag{3.3}$$

$$c_1' \gamma B^{\gamma-1} = e^{\frac{1+c\eta}{-M\eta}(B-k_2)} \tag{3.4}$$

Solve the above equations and we get the following value function:

$$V_B(x) = \begin{cases} \frac{M}{cB^\gamma + \gamma^2 B^{\gamma-1} M} x^\gamma & x \le B \\ \frac{M}{c} - \frac{\gamma^2 B^{\gamma-1} M^2}{c(cB^\gamma + \gamma^2 B^{\gamma-1} M)} e^{-\frac{c}{M\gamma}(x-B)} & x > B \end{cases}$$

And the maximizing function $a(x)$ is then given by

$$a(x) = \begin{cases} \frac{\mu x}{\sigma^2(1-\gamma)} & x \le B \\ \frac{M}{\frac{\mu}{2} + \frac{c\sigma^2}{\mu}} & x > B \end{cases}$$

Case 3 $B \ge u_0$

For $x < B$, let $a(x)$ be the maximizer of the left-hand side of (3.1). We still have that the interval for $0 < a(x) < 1$ is $(0, u_0)$. So, for $x \in (0, u_0)$, the solution of (3.1) is

$$v_1(x) = c_1' x^\gamma \tag{3.5}$$

For $x \in [u_0, B]$, the (3.1) becomes

$$\frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - cV(x) = 0$$

The solution of the above equation is

$$v_2(x) = c_2' e^{d_+ x} + c_3' e^{d_- x} \tag{3.6}$$

Again we get $v_2'(B) < 1$. The concavity of the value function implies that $V_B'(x) < 1$ for $x > B$. The equation (3.2) becomes

$$\max_{a\in[0,1]} \left[\frac{1}{2}\sigma^2 a^2 V''(x) + (\mu a - M)V'(x) - cV(x) + M \right] = 0 \qquad x > B$$

The solution of the above equation is

$$v_3(x) = \frac{M}{c} + c_4' e^{-\frac{c}{M\gamma}(x-B)} \tag{3.7}$$

To determine the constants c_1', c_2', c_3' and c_4', we apply the continuity and the continuous differentiability at u_0 and B. We get

$$c_1' = \frac{M}{c}[Ae^{d_B}(1 + \frac{M\gamma}{c}d_-) + Be^{d_+ B}(1 + \frac{M\gamma}{c}d_+)]^{-1}$$

$$c_2' = c_1' A$$

$$c_3' = c_2' B$$

$$c_4' = -c_1'(Ad_- e^{d_- B} + Bd_+ e^{d_+ B})\frac{M^2\gamma}{c^2}$$

where, $A = \frac{d_+ u_0 - \gamma u_0^{\gamma-1}}{(d_+ - d_-)e^{d_- u_0}}$, $B = \frac{\gamma u_0^{\gamma-1} - d_- u_0}{(d_+ - d_-)e^{d_+ u_0}}$. And the maximizing function $a(x)$ is given by

$$a(x) = \begin{cases} \frac{\mu x}{\sigma^2(1-\gamma)} & x < u_0 \\ 1 & u_0 \le x \le B \\ \frac{M}{\frac{\mu}{2} + \frac{c\sigma^2}{\mu}} & x > B \end{cases}$$

The solution in this case is given by plugging the values of $c_1',\ c_2',\ c_3',\ c_4',\ A$ and B into (3.5), (3.1) and (3.7). The verification of the concavity of V_B, $V_B'(B) < 1$ and $a(x) \le 0$ is straightforward but tedious, so we omit them here.

3.2 $M > \mu/2 + c\sigma^2/\mu$

In this case, we have that $u_0 < u_1$. Like in the case $M \le \mu/2 + c\sigma^2/\mu$, we consider the following two sub-cases.

Case 1 $B \le u_1$

By similar argument as that in the case of $M > \mu/2 + c\sigma^2/\mu$, we deduce that the solution of this case is the same as the one without barrier constraint, given by (2.31) in [3].

Case 2 $B > u_1$

Following the argument in Case 3 of $M \le \mu/2 + c\sigma^2/\mu$, we obtain

$$v_1(x) = c_1'' x^\gamma \qquad x < u_0 \tag{3.8}$$
$$v_2(x) = c_2'' e^{d_+ x} + c_3'' e^{d_- x} \qquad u_0 \le x \le B \tag{3.9}$$

For $x > B$

$$\max_{a\in[0,1]} \left[\frac{1}{2}\sigma^2 a^2 V''(x) + (\mu a - M)V'(x) - cV(x) + M\right] = 0 \qquad x > B$$

The solution for the above equation is

$$v_3(x) = \frac{M}{c} + c_4'' e^{\hat{d}(x-b)} \tag{3.10}$$

The constants $c_1,\ c_2,\ c_3$, and c_4 satisfy the following equations:

$$\begin{aligned} c_1'' &= \frac{M}{c}[Ae^{d_B}(1 + \frac{M\gamma}{c}d_-) + Be^{d_+ B}(1 + \frac{M\gamma}{c}d_+)]^{-1} \\ c_2'' &= c_1'' A \\ c_3'' &= c_2'' B \\ c_4'' &= c_1''(Ad_- e^{d_- B} + Bd_+ e^{d_+ B})\frac{M}{\hat{d}c} \end{aligned}$$

where $\hat{d} = \frac{-(\mu - M) - \sqrt{(\mu-M)^2 + 2c\sigma^2}}{\sigma^2}$. Solve the above equations and we have now explicitly solved (1.7) under barrier and dividend payout constraints.

References

[1] S. Asmussen and M. Taksar, Controlled Diffusion Models for Optimal Dividend Pay-Out, *Insurance: Math. & Economics* **20** (1997), 1-15

[2] L. He, P. Hou, Z.X. Liang, Optimal control of the insurance company with proportional reinsurance policy under solvency constraints, *Insurance: Math. & Economics* **43** (2008), 474-479

[3] B. Hojgaard and M. Taksar, Controlling Risk Exposure and Dividends Payout Schemes: Insurance Company Example, *Math. Finance.* **9** (1999), 153-182.