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Comparative Statics Analysis of An Inventory Management Model with Dynamic Pricing, Market Environment Fluctuation, and Delayed Differentiation

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We consider a general joint pricing and inventory management model with delayed differentiation, in which a firm serves a market with multiple products made from a generic one. The firm holds inventory for the generic product which is produced using multiple resources. Moreover, the market size, the attractiveness of each product, the firm's productivity, and the procurement cost of each resource all evolve over the planning horizon driven by an exogenous Markov process. Comparative statics analysis is essential for studying this model, offering insights on the impact of market environment fluctuation upon the firm's optimal pricing and inventory policy. Hence, we propose a new approach that combines the advantages of implicit function theorem (IFT) and monotone comparative statics (MCS) approaches to perform comparative statics analysis in our joint pricing and inventory management model under market environment fluctuation. The new approach applies to our model where the standard IFT and MCS methods are not easily amenable. Using our new comparative statics approach, we characterize the optimal pricing and production policy of the firm, and offer insights on how the firm should adaptively respond to market environment fluctuations.

Key words: joint pricing and inventory management; delayed differentiation; comparative statics analysis
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1. Introduction

It is a common practice that a firm has product lines of similar products. In this case, the firm usually first produces a single generic product, from which all the end products are made (see, e.g., Chapter 6.3 Snyder and Shen 2011). Such delayed differentiation strategy facilitates the firm to enjoy its risk-pooling effect under demand uncertainty. As an example, Swaminathan and Tayur (1998) report that IBM stores inventory of semi-finished products (vanilla boxes) and transforms the vanilla boxes into final individual products.

In the current unstable global market environment, firms are operated in the face of fluctuations in market size, product attractivenesses, firm productivity, and procurement costs of resources. For the demand side, as illustrated in Song and Zipkin (1993), customer demand fluctuations may be driven by changes in economic conditions and consumer tastes. For the supply side, volatilities of commodity prices are likely to lead to procurement cost fluctuations faced by the firm (see, e.g., Xiao et al. 2015). For example, Hewlett-Packard (HP) had a difficult time when the procurement cost of DRAM (dynamic random-access memory, an essential component of HP's printers) dropped by more than 90% in 2001, and then tripled in the next year (see Nagali et al. 2008). As empirically shown by Olley and Pakes (1996), technological changes can result in stochastic evolutions of firm-level productivities.

As a commonly adopted operations strategy, delayed differentiation has also received considerable attention in the supply chain management literature (e.g., Lee and Tang 1997, 1998, Swaminathan and Tayur 1998). However, most research in this literature does not take into account the pricing decision of the firm. In this study, we aim to study the optimal pricing and production policies in a general periodic-review joint pricing and inventory management model with delayed differentiation. The firm offers a portfolio of differentiated products made from a single generic product. The generic product is manufactured under the Cobb–Douglass production model using multiple resources procured from multiple supply channels with different costs. The firm holds

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inventory for the generic product, and the final products are offered in the market in a make-to-order fashion (see, e.g., Chapter 6.3 of Snyder and Shen 2011). In each period, the firm sets prices for all products and decides the production quantity of the generic product. On the demand side, the model is built upon the multi-nomial logit (MNL) choice model, whereas, on the supply side, our model incorporates the classical production function framework in the economics literature. The demand for each product is determined by the market size, the product price vector, and its attractiveness to customers. The market size, product attractivenesses, firm productivity, and procurement costs all evolve according to an underlying exogenous Markov process. Hence, our joint pricing and inventory management model captures both delayed differentiation and the high uncertainty and volatility of today's competitive and unstable marketplaces.

In this quite general dynamic pricing and inventory management model, comparative statics analysis plays an integral role in characterizing the impact of market environment fluctuation upon the optimal sales prices and order quantities. It delivers important insights regarding how the system should optimally respond to changes in the exogenous market condition and/or internal state over the planning horizon. For instance, a firm under an uncertain market environment often faces the conundrum that whether it should increase or decrease the sales price and procurement quantity under a higher procurement cost or a larger market size. Analogously, it is also important to modify the price and inventory policies in accordance to firm-level strategically changes like offering a new product or adopting a new production technology. As an essential tool in economics, engineering and operations management, comparative statics analysis offers a systematic method to study these challenges that are both common and essential in inventory management models under dynamic pricing.

In the economics and operations management literature, there are two standard methods to perform comparative statics analysis: (a) the implicit function theorem (IFT) approach, and (b) the monotone comparative statics (MCS) approach. For most of our analysis, we adopt the MCS approach to perform comparative statics analysis in our general joint pricing and inventory management model. Nevertheless, both the IFT and MCS methods are not amenable to characterize the impact of market size and firm productivity on the firm's optimal policy, due to the lack of second-order differentiability and supermodularity for the objective function in each period. Hence, we propose a new comparative statics analysis approach that combines the edges of both IFT and MCS approaches to perform comparative statics analysis in our model. This approach leverages the first-order optimality condition of the optimal policy in each period, and carefully analyzes how changes in parameter values impact the marginal value of each decision variable (i.e., the first-order partial derivative of the objective function).

To perform comparative statics analysis in each decision epoch of our general joint pricing and inventory management model, we integrate the standard MCS method and our new approach with the backward induction argument to iteratively link the comparison between optimizers and that between partial derivatives of the value functions and objective functions in each decision epoch. Furthermore, we simplify the optimization problem in each period by reducing the multiple decision variables into two, one on the demand side and the other on the supply side: the total purchasing probability and the normalized production quantity. We characterize the optimal joint pricing and ordering policy for an arbitrary number of end products and supply resources. In particular, the optimal production policy has a produce-up-to structure. We also show that the optimal sales prices are increasing in both market size and firm productivity. The optimal production quantity increases if market size is higher, but decreases with a higher firm productivity. When the product attractiveness increases, the firm increases the production quantity. On the other hand, when the procurement cost of some resource increases, the firm increases the sales price of each product. Offering new product options would prompt the firm to increase production quantity of the generic product. In anticipation of large future market size/high future attractiveness/high future productivity/high future costs, the firm should increase the prices of its products and the production quantity of the generic product.

To sum up, the contribution of this study can be summarized as follows. Whereas most papers in the joint pricing and inventory management literature focuses on single-item models, we are among the first to study dynamic pricing and inventory control in the presence of multiple products and delayed differentiation. We develop a new comprehensive joint pricing and inventory management model under delayed differentiation and uncertain market environment. Our model captures the important feature of delayed differentiation, multi-input production, and market environment fluctuation. We perform comparative statics analysis to unveil insights on how the firm should adaptively respond to the fluctuations of market environment. Characterizing the impact of market size and firm productivity with the standard methods is not feasible, so we propose a new approach that combines the advantages of MCS and IFT approaches. This new approach has the potential to be applicable

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in other models. Finally, our results offer insights on how the firm should adaptively respond to fluctuations in market environment under the delayed differentiation strategy.

The rest of the paper is organized as follows. We position this study in the related literature in section 2. Section 3 presents our general joint pricing and inventory management model with market environment fluctuation. In section 4, we analyze the joint pricing and inventory management problem under market environment fluctuation and propose a new comparative statics analysis approach. This study concludes with section 5. The proofs of the results are relegated to the Appendix.

2. Literature Review

This study is built upon three streams of literature: (a) dynamic pricing and inventory management; (b) comparative statics analysis in operations management; and (c) delayed differentiation in supply chain management.

This work is related to the growing literature on the dynamic pricing and inventory management problem under general stochastic demand. Federgruen and Heching (1999) provide a general treatment of this problem, and show the optimality of a base-stock/list-price policy. Feng et al. (2013) identify a set of new technical conditions under which a base-stock/list-price policy is optimal. Chen and Simchi-Levi (2004a, b, 2006) analyze the joint pricing and inventory control problem with fixed set-up cost, and show that (s, S, p) policy is optimal for finite horizon, infinite horizon, and continuous review models. Chen et al. (2006) and Huh and Janakiraman (2008) study the joint pricing and inventory control problem under lost sales. In the case of a single unreliable supplier with random yield, Li and Zheng (2006) show that supply uncertainty drives the firm to charge a higher price. A similar result is established by Feng (2010) in the context of uncertain capacity. Feng and Shi (2012) further extend this model to one with multiple suppliers and characterize the optimal joint pricing and sourcing decision therein. Gong et al. (2014) and Chao et al. (2016) characterize the joint dynamic pricing and dual-sourcing policy of an inventory system with the random yield risk and the disruption risk, respectively. Yang et al. (2014) characterize the optimal policy of a joint pricing and inventory management model in which the ordering quantity must be of integral multiples of a given specific batch size. When the replenishment leadtime is positive, the joint pricing and inventory control problem under periodic review is extremely difficult. For this problem, Pang et al. (2012) partially characterize the structure of the optimal policy. When adjusting the price is costly, Chen et al. (2015) demonstrate that inventory-based dynamic pricing can lead to significant profit improvement over static pricing, and limited number of price adjustments can yield a small profit loss relative to unlimited price adjustments. Xiao et al. (2015) study the dynamic pricing and inventory management problem under fluctuating procurement costs. We refer interested readers to Chen and Simchi-Levi (2012) for a comprehensive survey on joint pricing and inventory control models. We contribute to this stream of research by proposing a general joint pricing and inventory management model with delayed differentiation and market environment fluctuation. We also develop a new analytical approach for the comparative statics analysis in this model.

There is extensive application of comparative statics analysis in the operations management literature. See, for example Song (1994), Song et al. (2010), Huh et al. (2011), Federgruen and Wang (2013), Li and Yu (2014), Federgruen and Wang (2015). The majority of the papers in this stream of research apply the IFT and MCS approaches to establish comparative statics results and the structural properties of their models. Bish et al. (2012) is a notable exception that develops a novel analytical approach for the comparative statics analysis in multi-product, multi-resource newsvendor networks with responsive pricing. In their setting, the IFT approach is non-scalable and prohibitively difficult, whereas their new approach exploits the relationship between convex stochastic orders and dual variables, and is therefore scalable with respect to the numbers of products and resources. Our paper contributes to this line of research by developing a new approach for comparative statics analysis in a general joint pricing and inventory management model with delayed differentiation and market environment fluctuation. The proposed approach combines the advantages of both the IFT and MCS approaches and is easily amenable for comparative statics analysis in the model.

Finally, this study is also related to the delayed differentiation strategy in the supply chain literature. Lee and Tang (1997) develop a simple model capturing the costs and benefits of redesigning the product that achieves delayed differentiation. Lee and Tang (1998) examine the values of delayed differentiation to reduce demand variability for different sequences of differentiation. Swaminathan and Tayur (1998) study the optimal delayed differentiation configuration for multiple product lines and multiple semi-finished generic products. Our contribution toward this literature is that we endogenize the pricing decision of the firm under delayed differentiation and study how the firm should respond to market environment fluctuations with this strategy.

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3. Model

We consider a T-period joint pricing and inventory management model, in which a firm sells n differentiated products (say individualized computers) that are made from a generic one (the generic computer) in a market. The firm adopts the delayed differentiation strategy: It first procures input resources to manufacture the generic product, and then transforms the generic product into final ones in a make-to-order fashion. Hence, the firm holds inventory for the generic product, which is produced with an input of m resources. These resources can be capital, labor, raw materials, etc. We denote the set of n final products as $\mathcal{N} := \{1, 2, \cdots, n\}$, and the set of m resources as $\mathcal{M} := \{1, 2, \cdots, m\}$. The firm seeks to maximize its total discounted profit over the planning horizon by dynamically adjusting its joint pricing and inventory policy in each period. The periods are indexed backwards as $\{T, T-1, \cdots, 1\}$ and the discount factor is denoted as $\alpha \in (0,1)$.

In period t, the firm selects a vector of prices, $p_t = (p_t^1, p_t^2, \cdots, p_t^n)$, for different products. More specifically, for each $i \in \mathcal{N}$, p_t^i is the sales price for product i. Given the sales price vector p_t, the customer attractiveness vector $a_t = (a_t^1, a_t^2, \cdots, a_t^n)$, and the market size Λ_t, the demand of product i in period t is given as follows:

$$D_t^i(p_t, a_t, \Lambda_t) = \Lambda_t d^i(p_t|a_t) + \varepsilon_t^i, \tag{1}$$

where $\Lambda_t d^i(p_t|a_t)$ is the deterministic component of the demand function and ε_t^i is the random perturbation. Λ_t is the total number of potential customers in period t, and a_t^i measures the attractiveness of product i in period t. We assume that $d^i(p_t|a_t)$ is strictly decreasing in p_t^i and a_t^j ($j \neq i$), and strictly increasing in p_t^j ($j \neq i$) and a_t^i. The specific functional form of $d^i(p_t|a_t)$ is given by Equation (3) below. We also assume that the additive random perturbation term ε_t^i captures all other uncertainties not explicitly considered in this model. Specifically, $\{\varepsilon_t^i\}_{t=T}^1$ are continuous random variables independent of Λ_t and p_t with mean 0. Furthermore, $\{\varepsilon_t^i\}_{t=T}^1$ are *i.i.d.* across time t but may be dependent across different products. Hence, $D_t^i(p_t, a_t, \Lambda_t)$ follows a continuous distribution for any given (p_t, a_t, Λ_t) and $i \in \mathcal{N}$. We use $D_t(p_t, a_t, \Lambda_t) = (D_t^1(p_t, a_t, \Lambda_t), D_t^2(p_t, a_t, \Lambda_t), \cdots, D_t^n(p_t, a_t, \Lambda_t))$ to denote the demand vector for all products, with the sales price vector p_t, the customer preference vector a_t, and the market size Λ_t in period t. Given (p_t, a_t, Λ_t), the accumulative demand (for the common component) in period t is given by:

$$D_t^a(p_t, a_t, \Lambda_t) = \sum_{i \in \mathcal{N}} D_t^i(p_t, a_t, \Lambda_t) = \Lambda_t \left(\sum_{i=1}^n d^i(p_t|a_t) \right) + \varepsilon_t, \tag{2}$$

where $\varepsilon_t := \sum_{i=1}^n \varepsilon_t^i$ represents the accumulative demand perturbation in period t.

We further assume that the demand of each product follows the multi-nomial logit (MNL) choice model. Specifically, the demand function for product i is given by

$$d^i(p_t|a_t) = \frac{\exp(a_t^i - bp_t^i)}{1 + \sum_{j \in \mathcal{N}} \exp(a_t^j - bp_t^j)}, \tag{3}$$

where $b > 0$ is the price sensitivity of the customers. We remark that all our results remain valid under some different demand models, such as the completely segmented demand model (i.e., $d^i(\cdot|\cdot)$ is independent of p_t^j for all $j \neq i$) and the linear demand model (i.e., $d^i(p_t|a_t) = a_t^i - b_i p_t^i + \sum_{j \neq i} b_{i,j} p_t^j$, where $b_i, b_{i,j} > 0$).

The firm produces the generic product with an input of the resources in $\mathcal{M}$. In period t, the firm selects a vector of procurement/investment quantities for each resource, $q_t = (q_t^1, q_t^2, \cdots, q_t^m)$, under which the firm orders/invests $q_t^j \geq 0$ for resource j. The total cost of choosing the procurement vector q_t is therefore

$$\sum_{j \in \mathcal{M}} c_t^j q_t^j,$$

where c_t^j is the cost of resource j in period t. The output of the firm's production process Q_t follows the Cobb–Douglas production function:

$$Q_t = F(q_t|\Gamma_t) := \Gamma_t f(q_t) = \Gamma_t \prod_{j \in \mathcal{M}} (q_t^j)^{\gamma_j}, \tag{4}$$

where $\gamma_j > 0$ for each j and $\sum_{j \in \mathcal{M}} \gamma_j < 1$. The Cobb–Douglas production function is the standard way to model production in the economics literature (see, e.g., Mas-Colell et al. 1995). In particular, it is easy to verify that $f(q_t)$ is strictly increasing in q_t^j for any j, and it is strictly concave and supermodular in q_t on its domain. The productivity factor Γ_t of the firm in period t is called the total factor productivity in the economics literature. We refer interested to Olley and Pakes (1996) for the econometric method to estimate the evolution of firm-level productivity Γ_t.

The key feature of our model is market environment fluctuation, which is modeled as an exogenous Markov process $\{\theta_t : t = T, T-1, \cdots, 1\}$. Specifically, for each period t, the state of the market $\theta_t = (\Lambda_t, a_t, \Gamma_t, c_t)$ is an $(n+m+2)$–dimensional vector, where, as discussed above, Λ_t is the aggregate market size, $a_t = (a_t^1, a_t^2, \cdots, a_t^n)$ is the attractiveness vector of the n final products, Γ_t is the productivity factor of the firm, and $c_t = (c_t^1, c_t^2, \cdots, c_t^m)$ is the cost vector of the m resources. Note that the fluctuation of a_t captures the evolution of customers' preferences on the n products over the planning horizon. The productivity factor Γ_t measures how efficient the firm can produce the generic product from the input resources. The fluctuation of Γ_t captures the evolution of technology and organization changes that may affect the productivity of the firm.

For expositional ease, let $a_t^{-i} := (a_t^1, \cdots, a_t^{i-1}, a_t^{i+1}, \cdots, a_t^n)$ and $c_t^{-j} := (c_t^1, \cdots, c_t^{j-1}, c_t^{j+1}, \cdots, c_t^m)$. We assume that, for any $i \in \mathcal{N}$ (resp. $j \in \mathcal{M}$), conditioned on a_t^i (resp. c_t^j), a_{t-1}^i (resp. c_{t-1}^j) is independent of $(\Lambda_t, a_t^{-i}, \Gamma_t, c_t)$ (resp. $(\Lambda_t, a_t, \Gamma_t, c_t^{-j})$), that is, a_t^i (resp. c_t^j) is a sufficient statistic for a_{t-1}^i (resp. c_{t-1}^j). Analogously, we assume that Λ_t is a sufficient statistic of Λ_{t-1} and Γ_t is a sufficient statistic of Γ_{t-1}. Hence, the dynamics of θ_t can be represented as $\Lambda_{t-1} = \xi_t^{\Lambda}(\Lambda_t)$, $a_{t-1}^i = \xi_t^{a,i}(a_t^i)$, $\Gamma_{t-1} = \xi_t^{\Gamma}(\Gamma_t)$, and $c_{t-1}^j = \xi_t^{c,j}(c_t^j)$, where $\mathbb{E}\{\xi_t^{\Lambda}(\Lambda_t)|\theta_t\}$, $\mathbb{E}\{\xi_t^{a,i}(a_t^i)|\theta_t\}$, $\mathbb{E}\{\xi_t^{\Gamma}(\Gamma_t)|\theta_t\}$ and $\mathbb{E}\{\xi_t^{c,j}(c_t^j)|\theta_t\}$ are all finite. We further assume that, if $\hat{a}_t^i > a_t^i$ (resp. $\hat{c}_t^j > c_t^j$), $\xi_t^{a,i}(\hat{a}_t^i) \geq_{st} \xi_t^{a,i}(a_t^i)$ (resp. $\xi_t^{c,j}(\hat{c}_t^j) \geq_{st} \xi_t^{c,j}(c_t^j)$), where $\geq_{st}$ denotes the stochastic dominance in the usual stochastic order, that is, two random variables $X_1 \geq_{st} X_2$ if and only if $\mathbb{P}[X_1 \leq x] \leq \mathbb{P}[X_2 \leq x]$ for any $x \in \mathbb{R}$ (see Shaked and Shanthikumar 2007). Analogously, $\hat{\Lambda}_t > \Lambda_t$ (resp. $\hat{\Gamma}_t > \Gamma_t$) implies that $\xi_t^{\Lambda}(\hat{\Lambda}_t) \geq \xi_t^{\Lambda}(\Lambda_t)$ (resp. $\xi_t^{\Gamma}(\hat{\Gamma}_t) \geq_{st} \xi_t^{\Gamma}(\Gamma_t)$). This is an intuitive assumption, since a higher current market size is more likely to give rise to a higher market size in the next period, and the same is true for product attractiveness, productivity factor, and procurement cost.

Lipschitz continuity with the Lipschitz constant c_H, that is, for any $z_1, z_2 \in \mathbb{R}$, $|H(z_1) - H(z_2)| \leq c_H |z_1 - z_2|$. Without loss of generality and for the ease of exposition, we assume that the production cost of the generic product and that of the final products are normalized to zero. The former cost can be absorbed into the procurement costs of the resources, whereas the latter can be absorbed into the prices/margins of the final products. Finally, we remark that our model captures the important feature of high uncertainty and volatility in today's competitive and unstable marketplaces. Thus, it is widely applicable to studying the operational implications of market environment fluctuation on a firm's pricing and inventory strategies in various settings.

Note that although the demand and inventory penalty functions are assumed to be stationary for expositional convenience, all structural results in this study remain valid when they are time-dependent. It is also worth noting that, in our model, different products share the same backlogging cost (see, also, Li and Huh 2011). It would be an interesting extension to consider a joint pricing and inventory control model where different products have different backlogging costs.

To characterize the optimal joint pricing and inventory policy of the firm, we formulate the planning problem as a dynamic program. Let

$V_t(I_t|\theta_t)$ = the maximum expected discounted total profit in periods $t, t-1, \cdots, 0$, when the starting inventory level in period t is I_t and the realized market environment state is θ_t.

The sequence of events in each period unfolds as follows. At the beginning of period t, the firm reviews its inventory level I_t and the realized state of the market environment θ_t. The firm then simultaneously decides the sales price for each product and the order quantity for each resource, and pays the total procurement cost $\sum_{j \in \mathcal{M}} c_t^j q_t^j$. The orders are received immediately, after which the firm produces the generic product according to the production function (4). Then, the price-dependent stochastic demand vector for each final product $D_t(p_t, a_t, \Lambda_t)$ realizes. Depending on the realized demand vector, the firm transforms the generic product into final products in a make-to-order fashion, and collects revenue. Unmet demand is fully backlogged and excess inventory is fully carried over to the next period. Finally, the firm pays $H(z)$ for the inventory holding and backlogging cost for z units of ending net inventory of the generic product, where $H(\cdot)$ is a convex function with $H(0) = 0$ and $H(\cdot) > 0$ otherwise. Moreover, we assume that $H(\cdot)$ satisfies the

Without loss of generality, we assume that excess inventory at the end of the planning horizon is discarded without any salvage value, that is, $V_0(I_0|\theta_0) = 0$. The optimal value functions satisfy the following recursive scheme:

$$V_t(I_t|\theta_t) = \max_{(p_t, q_t) \in \mathcal{F}} \hat{J}_t(p_t, q_t, I_t|\theta_t), \tag{5}$$

where $\mathcal{F} := \{(p_t, q_t) : \forall i \in \mathcal{N}, p_t^i \geq 0, \forall j \in \mathcal{M}, q_t^j \geq 0\}$, and (6)

$$\begin{aligned}\hat{J}_t(p_t, q_t, I_t|\theta_t) := \; & \mathbb{E}\{\sum_{i \in \mathcal{N}} p_t^i D_t^i(p_t^i, a_t, \Lambda_t) - (\sum_{j \in \mathcal{M}} c_t^j q_t^j) \\ & -H(I_t + F(q_t|\Gamma_t) - D_t^a(p_t, a_t, \Lambda_t)) \\ & +\alpha V_{t-1}(I_t + F(q_t|\Gamma_t) \\ & -D_t^a(p_t, a_t, \Lambda_t)|\xi_t(\theta_t))|\theta_t\} \\ = \; & \Lambda_t(\sum_{i \in \mathcal{N}} p_t^i d^i(p_t|a_t^i)) - \sum_{j \in \mathcal{M}} c_t^j q_t^j \\ & +\Psi_t(I_t + \Gamma_t f(q_t) - \Lambda_t(\sum_{i \in \mathcal{N}} d^i(p_t|a_t^i))|\theta_t),\end{aligned} \tag{7}$$

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with $\Psi_t(z|\theta_t) := \mathbb{E}_{\theta_{t-1},\varepsilon_t}\{-H(z-\varepsilon_t)+\alpha V_{t-1}(z-\varepsilon_t|\theta_{t-1})|\theta_t\}$. (8)

Therefore, for each period t, the firm's profit-maximizing problem is to select a joint pricing and procurement policy $(p_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t)) \in \mathcal{F}$ to maximize $\hat{J}_t(p_t, q_t, I_t|\theta_t)$, with starting inventory level I_t and market environment state θ_t. If multiple maxima exist, the lexicographically smallest one is selected.

Throughout this study, we use ∂ to denote the derivative operator of a single variable function, and ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x. For any multivariate continuously differentiable function $f(x_1,x_2,\cdots,x_n)$ and $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ in $f(\cdot)$'s domain, we use $\partial_{x_i} f(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ to denote $\partial_{x_i} f(x_1, x_2, \cdots, x_n)|_{x=\tilde{x}}$ for any i. For any two n-dimensional vectors $v = (v_1,v_2,\cdots,v_n)$ and $\hat{v} = (\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_n)$, we use $\hat{v} > v$ to denote that $\hat{v}_i \geq v_i$ for each $1 \leq i \leq n$, and $\hat{v} \neq v$.

4. Model Analysis

4.1. Preliminary Analysis

To begin our analysis, we first transform the dynamic program so that the objective function in each period is concave. Following the standard technique in the joint pricing and inventory management literature, we use the purchasing probability vector $d_t = (d_t^1, d_t^2, \cdots, d_t^n)$ as the decision variable, where $d_t^i = \frac{\exp(a_t^i - bp_t^i)}{1+\sum_{j\in\mathcal{N}}\exp(a_t^j - bp_t^j)}$ ($b > 0$ is the sensitivity of price for customers). We first represent the price vector $p_t = (p_t^1, p_t^2, \cdots, p_t^n)$ in terms of the purchasing probability vector d_t:

$$p_t^i(d_t|a_t) = \frac{a_t^i + \log(1-\sum_{j\in\mathcal{N}} d_t^j) - \log(d_t^i)}{b},$$

where $d_t^i > 0$ for all i and $\sum_{j\in\mathcal{N}} d_t^j < 1$. Hence, we define the unit revenue function as a function of the purchasing probability vector d_t:

$$r(d_t|a_t) := \sum_{i\in\mathcal{N}} p_t^i(d_t|a_t) d_t^i = \sum_{i\in\mathcal{N}} \frac{a_t^i d_t^i}{b} + \left(\sum_{i\in\mathcal{N}} \frac{d_t^i}{b}\right) \log(1-\sum_{i\in\mathcal{N}} d_t^i) - \sum_{i\in\mathcal{N}} \frac{d_t^i \log(d_t^i)}{b}$$

Then, we define the optimal revenue given the total purchasing probability $N_t = \sum_{i\in\mathcal{N}} d_t^i$:

$$R(N_t|a_t) = \max_{\sum_{i\in\mathcal{N}} d_t^i = N_t} r(d_t|a_t), \text{ where } N_t \in (0, 1).$$

Let $d^*(N_t|a_t) := (d_1^*(N_t|a_t), d_2^*(N_t|a_t), \cdots, d_n^*(N_t|a_t))$ be the optimal (i.e., revenue-maximizing) purchasing probability vector associated with the total purchasing probability N_t and the attractiveness vector a_t. We also define $p_i^*(N_t|a_t) := p_t^i(d^*(N_t|a_t)|a_t)$ as the associated optimal price for product i ($i \in \mathcal{N}$). We have the following lemma on the properties of $R(\cdot|\cdot)$:

Lemma 1. *The following statements hold:*

(*a*) *$R(\cdot|a_t)$ is concave and continuously differentiable in N_t for any a_t Moreover, $R(N_t|a_t)$ is supermodular in (N_t, a_t^i) for any $i \in \mathcal{N}$.*
(*b*) *For each $i \in \mathcal{N}$, $d_i^*(N_t|a_t)$ is increasing in N_t, whereas $p_i^*(N_t|a_t)$ is decreasing in N_t*
(*c*) *For each $i \in \mathcal{N}$, $d_i^*(N_t|a_t)$ is increasing in a_t^i and decreasing in a_t^j $(j \neq i)$, whereas $p_i^*(N_t|a_t)$ is increasing in a_t^j for all $j \in \mathcal{N}$*

Lemma 1 suggests that the optimal purchasing probability of each final product is increasing in the total purchasing probability N_t. To induce this higher purchasing probability, the firm should decrease the price of this product. If a product is more attractive, the firm should adjust the prices of all products higher. As a consequence, the purchasing probability of each final product is increasing in its own attractiveness, but decreasing in the attractiveness of the other products. Furthermore, we have shown that the marginal revenue with respect to total purchasing probability is increasing in the attractiveness of any final product.

We now define the optimal cost function given the normalized production quantity M_t:

$$C(M_t|c_t) = \min_{f(q_t)=M_t} \sum_{i\in\mathcal{N}} c_t^i q_t^i$$

Hence, $\Gamma_t C(M_t|c_t)$ is the minimal cost to produce $\Gamma_t M_t$ generic products when the realized cost vector is c_t. Let $q_j^*(M_t|c_t)$ be the optimal order quantity for resource j with normalized production quantity M_t and procurement cost vector c_t. We have the following lemma on the properties of $C(\cdot|\cdot)$:

Lemma 2. *The following statements hold:*

(*a*) *$C(\cdot|c_t)$ is convexly increasing and continuously differentiable in M_t for any c_t Moreover, $C(M_t|c_t)$ is supermodular in (M_t, c_t^j) for any $j \in \mathcal{M}$.*
(*b*) *For each $j \in \mathcal{M}$, $q_j^*(M_t|c_t)$ is increasing in M_t item [(c)] For each $j \in \mathcal{M}$, $q_j^*(M_t|c_t)$ is decreasing in c_t^j and increasing in c_t^i $(i \neq j)$.*

Lemma 2 implies that the optimal procurement quantity of each resource is increasing in the normalized production quantity of the generic product M_t. The impact of the procurement cost for each resource is more subtle. The optimal order quantity for each resource is decreasing in the cost of this resource. To ensure that the production quantity of the generic product remains the same, the firm should increase the order quantity of all other resources. The

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supermodularity of the cost function suggests that the marginal total procurement cost is increasing in the cost of each resource.

With the help of Lemmas 1 and 2, we are now able to reformulate the dynamic program as follows:

$$V_t(I_t|\theta_t) = \max_{N_t \in (0,1),\, M_t \geq 0} J_t(N_t, M_t, I_t|\theta_t), \tag{9}$$

$$\text{where } J_t(N_t, M_t, I_t|\theta_t) := \Lambda_t R(N_t|a_t) - \Gamma_t C(M_t|c_t) + \Psi_t(I_t + \Gamma_t M_t - \Lambda_t N_t|\theta_t).. \tag{10}$$

It is clear by comparing the Bellman Equations (5) and (9) that the reformulation substantially simplifies our model. This will help us deliver insights on how the firm should dynamically adjust the pricing and inventory policy in response to market environment fluctuation. We use $(N_t^*(I_t, \theta_t), M_t^*(I_t, \theta_t))$ to denote the optimal decisions of the Bellman equation (9). Then, the optimal price of each product i can be represented by $p_t^{i*}(I_t, \theta_t) = p_i^*(N_t^*(I_t, \theta_t)|a_t)$, and the optimal order quantity of each resource j can be represented by $q_t^{j*}(I_t, \theta_t) = q_j^*(M_t^*(I_t, \theta)|c_t)$. We also define $Q_t^*(I_t, \theta_t) = \Gamma_t M_t^*(I_t, \theta_t)$. The following lemma characterizes the concavity and differentiability of the value and objective functions.

LEMMA 3. *For $t = T, T-1, \cdots, 1$ and any given market state θ_t, the following statements hold:*

(*a*) *$\Psi_t(\cdot|\theta_t)$ is concave and continuously differentiable in z.*
(*b*) *$V_t(\cdot|\theta_t)$ is concave and continuously differentiable in I_t.*
(*c*) *$J_t(\cdot, \cdot, \cdot|\theta_t)$ is jointly concave and continuously differentiable in (N_t, M_t, I_t).*

To conclude this subsection, we characterize the structure of the optimal pricing and inventory policy.

THEOREM 1. *For each period t, there exists a threshold, depending on the state of the environment θ_t, $\bar{I}_t(\theta_t)$, such that if $I_t < \bar{I}_t(\theta_t)$, $M_t^*(I_t, \theta_t) > 0$; otherwise, $I_t \geq \bar{I}_t(\theta_t)$, $M_t^*(I_t, \theta_t) = 0$.*

As shown in Theorem 1, the optimal policy of the firm bears a threshold structure. If the net inventory is below a state-dependent threshold $\bar{I}_t(\theta_t)$, the firm should produce some generic products ($M_t^*(I_t, \theta_t) > 0$). Otherwise, there is sufficient inventory, the firm should produce nothing. By the proof of Lemma 2, it is easy to verify that if $M_t^*(I_t, \theta_t) = 0$ (resp. $M_t^*(I_t, \theta_t) > 0$), we have $q_j^*(M_t^*(I_t, \theta_t)|c_t) = 0$ (resp. $q_j^*(M_t^*(I_t, \theta_t)|c_t) > 0$) for all $j \in \mathcal{M}$. Hence, the firm should order each resource if and only if its inventory is below the threshold $\bar{I}_t(\theta_t)$.

4.2. Comparative Statics Analysis

In this subsection, we investigate our joint pricing and inventory management model with market environment fluctuation using comparative statics analysis. Before running into the technical details, it is worthwhile briefly summarizing the key difficulties underlying the comparative statics analysis of our model:

1. Changes in current state variables and decisions will impact the states and value functions in the future. Thus, the comparative statics analysis of the model should highlight the intertemporal trade-off under market environment fluctuation.
2. The pricing and procurement decisions have intricate interactions with each other. Thus, the comparative statics analysis should summarize both the direct impact of the market environment evolution and its indirect impact via the interactions of different pricing and procurement decisions. Also note that the above two effects reinforce each other, thus making the comparative statics analysis of our model challenging.

To begin with, we study how the inventory level of the firm influences the optimal decisions $(p_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t))$ by performing comparative statics analysis with respect to I_t. We use $x_t := I_t + \Gamma_t M_t$ to denote the total produce-up-to level in period t, and $x_t^*(I_t, \theta_t) := I_t + \Gamma_t M_t^*(I_t, \theta_t)$ to denote the optimal total produce-up-to level. The following theorem characterizes how the inventory level influences the optimal policy of the firm.

THEOREM 2. *For $t = T, T-1, \cdots, 1$ and any given θ_t, the following statements hold:*

(*a*) *$N_t^*(I_t, \theta_t)$ is continuously increasing in I_t. Thus, $p_t^{i*}(I_t, \theta_t)$ is continuously decreasing in I_t for each $i \in \mathcal{N}$.*
(*b*) *$M_t^*(I_t, \theta_t)$ and $Q_t^*(I_t, \theta_t) = \Gamma_t M_t^*(I_t, \theta_t)$ are continuously decreasing in I_t. Thus, $q_t^{j*}(I_t, \theta_t)$ is continuously decreasing in I_t for each $j \in \mathcal{M}$.*
(*c*) *$x_t^*(I_t, \theta_t)$ is continuously increasing in I_t.*

Theorem 2 characterizes the impact of inventory level on the firm's optimal pricing and inventory policy. For the demand side, we show that the optimal total purchasing probability $N_t^*(I_t, \theta_t)$ is increasing in the inventory level of the firm. As a consequence, by Lemma 1, the optimal price of each end product is decreasing in the inventory level. With a high inventory level, the firm should decrease the price of each product to clear it. For the supply side, the optimal production quantity of the generic product $\Gamma_t M_t^*(I_t, \theta_t)$ is decreasing in the inventory level. Thus,

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by Lemma 2, the optimal order quantity of each resource is also decreasing in the inventory level. On-hand inventory and new production are substitutable supplies, so the firm should decrease the production quantity in the face of a higher inventory level. Although the production quantity of the generic product is decreasing in the inventory level, the total produce-up-to level is higher if the firm holds more inventory. Note that Theorem 2 generalizes Theorems 1 and 2 in Federgruen and Heching (1999) to our model with multiple products, multiple production resources, delayed differentiation, and market environment fluctuation.

Predicting with proxies: Transfer learning in high dimension. Next, we seek to study the impact of market environment fluctuation upon the firm's optimal pricing and inventory policy. To this end, we integrate comparative statics analysis with the standard backward induction argument in our model. To start with, we characterize how the firm responds to market size evolution. When the market size increases, the firm should increase its production quantity to match supply with demand. At the same time, the firm should increase the sales prices of all products to exploit the better market condition. Moreover, since the potential market size is more likely to become larger with a larger current market size, the overstocking risk decreases in this case. Hence, it is optimal for the firm to increase its total produce-up-to level in the face of a larger current market size. We formalize the above intuitions in the following theorem.

THEOREM 3. *For any given* t, *let* θ_t *and* $\hat{\theta}_t$ *be otherwise the same except that* $\hat{\Lambda}_t > \Lambda_t$. *For any* I_t, *the following statements hold:*

(a) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$.
(b) $N_t^*(I_t, \hat{\theta}_t) \leq N_t^*(I_t, \theta_t)$. *Hence,* $p_t^{i*}(I_t, \hat{\theta}_t) \geq p_t^{i*}(I_t, \theta_t)$ *for all* $i \in \mathcal{N}$.
(c) $M_t^*(I_t, \hat{\theta}_t) \geq M_t^*(I_t, \theta_t)$ *and* $Q_t^*(I_t, \hat{\theta}_t) \geq Q_t^*(I_t, \theta_t)$. *Hence,* $q_t^{j*}(I_t, \hat{\theta}_t) \geq q_t^{j*}(I_t, \theta_t)$ *for all* $j \in \mathcal{M}$ *and* $x_t^*(I_t, \hat{\theta}_t) \geq x_t^*(I_t, \theta_t)$.

The comparative statics analysis of the optimal pricing and inventory policy with respect to the market size evolution is challenging. We first discuss why the standard IFT and MCS approaches do not apply in this setting. Then we propose a new approach that combines the advantages of both approaches to obtain the comparative statics results. The biggest issue related to the IFT approach is that $J_t(\cdot, \cdot, \cdot|\cdot)$ is not necessarily twice continuously differentiable on its domain. For example, if θ_t is a Markov chain taking values from a discrete set, $J_t(\cdot, \cdot, \cdot|\cdot)$ is clearly not twice continuously differentiable. For the MCS approach, the objective function $J_t(\cdot, \cdot, \cdot|\theta_t)$ does not even have a clear component-wise supermodularity relationship between Λ_t and the decision variable N_t or M_t. Specifically, the marginal profit of the total purchasing probability $\partial_{N_t} J_t(N_t, M_t, I_t|\theta_t) = \Lambda_t \partial_{N_t} R(N_t|a_t) - \Lambda_t \partial_y \Psi_t(I_t + \Gamma_t M_t - \Lambda_t N_t|\theta_t)$ may be decreasing or decreasing in Λ_t depending on the signs of $\partial_{N_t} R(N_t|a_t)$ and $\partial_y \Psi_t(I_t + \Gamma_t M_t - \Lambda_t N_t|\theta_t)$. The same is true for the marginal profit of the normalized production quantity $\partial_{M_t} J_t(N_t, M_t, I_t|\theta_t) = -\Gamma_t \partial_{M_t} C(M_t|c_t) + \Gamma_t \partial_y \Psi_t(I_t + \Gamma_t M_t - \Lambda_t N_t|\theta_t)$. Hence, the well-established MCS theory is not readily applicable to understanding how the firm would adjust its price and inventory policy in response to market size evolution.

Therefore, to show Theorem 3, we propose a new approach for comparative statics analysis that combines the advantages of the classic IFT and MCS approaches. As the IFT approach, the proposed approach studies the first-order (KKT) condition which must be satisfied under the optimal price and inventory decisions. Analogous to the MCS approach, our new approach analyzes the impact of the market size upon the marginal value of the total purchasing probability and that of the production quantity for the generic product in detail. As a result, we are able to link the comparison between the optimal decision variables with that between the partial derivatives, and obtain a contradiction when the desired comparative statics prediction is reversed.

We now illustrate our new approach by sketching the proof of Theorem 3. The complete proof is relegated to the Appendix. We first introduce a lemma that plays a key role in our new approach:

LEMMA 4. *Let* $F_i(z, Z_i)$ *be a first-order differentiable function in* (z, Z_i) *for* $i = 1,2$, *where* $z \in [\underline{z}, \bar{z}]$ ($\underline{z}$ *and* $\bar{z}$ *might be infinite) and* $Z_i \in \mathcal{Z}_i$, *where* $\mathcal{Z}_i$ *is the feasible set of* Z_i. *For* $i = 1,2$, *let*

$$(z_i^*, Z_i^*) := \underset{(z, Z_i) \in [\underline{z}, \bar{z}] \times \mathcal{Z}_i}{\arg\max} F_i(z, Z_i),$$

be the optimizer of $F_i(\cdot,\cdot)$. *If* $z_1^* < z_2^*$, *we have:* $\partial_z F_1(z_1^*, Z_1^*) \leq \partial_z F_2(z_2^*, Z_2^*)$.

The proof of Theorem 3 is sketched as follows. We first assume (1) $\partial_y \Psi_t(y|\hat{\theta}_t) \geq \partial_y \Psi_t(y|\theta_t)$ if $\hat{\Lambda}_t > \Lambda_t$, and (2) to the contrary, $N_t^*(I_t, \hat{\theta}_t) > N_t^*(I_t, \theta_t)$. Then, we use Lemma 4 to establish that

$$\partial_y \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) > \partial_y \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t) \tag{11}$$

and $M_t^*(I_t, \hat{\theta}_t) < M_t^*(I_t, \theta_t)$. We then employ Lemma 4 again to show that

$$\partial_y \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) < \partial_y \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t),$$

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which forms a contradiction to inequality (11). Thus, the initial assumption is incorrect and we must have $N_t^*(I_t, \hat{\theta}_t) \le N_t^*(I_t, \theta_t)$. Using an analogous proof-by-contradiction argument, we can show that $M_t^*(I_t, \hat{\theta}_t) \ge M_t^*(I_t, \theta_t)$ if $\hat{\Lambda}_t > \Lambda_t$. Finally, invoking Lemma 4 again, we complete the backward induction by showing that $\partial_y \Psi_{t-1}(y|\hat{\theta}_{t-1}) \ge \partial_y \Psi_{t-1}(y|\theta_{t-1})$ for $\hat{\Lambda}_{t-1} > \Lambda_{t-1}$.

As shown in the proof of Theorem 3, our approach combines the advantages of the IFT and MCS approaches, and is able to perform comparative statics analysis for a model where IFT and MCS methods are not easily applicable. This approach can also be adapted to establish the impact of firm productivity on the optimal pricing and inventory decisions. Our next result shows that a more efficient firm (i.e., with a higher productivity factor Γ_t) will increase its production quantity of the generic product, but it will also decrease the procurement quantity of each resource as well. As a result, the firm will also decrease the price and increase the purchasing probability of each product to match demand with supply.

Theorem 4. *For any given t, let θ_t and $\hat{\theta}_t$ be otherwise the same except that $\hat{\Gamma}_t > \Gamma_t$. For any I_t, the following statements hold:*

(*a*) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \le \partial_{I_t} V_t(I_t|\theta_t)$.
(*b*) $N_t^*(I_t, \hat{\theta}_t) \ge N_t^*(I_t, \theta_t)$. *Hence,* $p_t^{i*}(I_t, \hat{\theta}_t) \le p_t^{i*}(I_t, \theta_t)$ *for all* $i \in \mathcal{N}$.
(*c*) $M_t^*(I_t, \hat{\theta}_t) \le M_t^*(I_t, \theta_t)$ *and* $Q_t^*(I_t, \hat{\theta}_t) \ge Q_t^*(I_t, \theta_t)$. *Hence,* $q_t^{j*}(I_t, \hat{\theta}_t) \le q_t^{j*}(I_t, \theta_t)$ *for all* $j \in \mathcal{M}$ *and* $x_t^*(I_t, \hat{\theta}_t) \ge x_t^*(I_t, \theta_t)$.

Similar to Theorem 3, Theorem 4 is difficult, if not impossible to prove with the standard IFT and MCS approaches. The objective function $J_t(\cdot, \cdot, \cdot|\cdot)$ is not necessarily twice continuously differentiable on its domain, and it does not even bear a clear componentwise supermodularity relationship between Γ_t and the decision variable N_t or M_t. Therefore, we resort to our new approach to establish the comparative statics results of the optimal price and inventory decisions with respect to firm productivity. Interested readers are referred to the Appendix for proof details.

Next, we examine the firm's optimal response towards the evolution of each product's attractiveness a_t^i.

Theorem 5. *For any given t, let θ_t and $\hat{\theta}_t$ be otherwise the same except that $\hat{a}_t^i > a_t^i$ for some i. For any I_t, the following statements hold:*

(*a*) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \ge \partial_{I_t} V_t(I_t|\theta_t)$
(*b*) $M_t^*(I_t, \hat{\theta}_t) \ge M_t^*(I_t, \theta_t)$ *and* $Q_t^*(I_t, \hat{\theta}_t) \ge Q_t^*(I_t, \theta_t)$ *Hence,* $q_t^{j*}(I_t, \hat{\theta}_t) \ge q_t^{j*}(I_t, \theta_t)$ *for all* $j \in \mathcal{M}$ *and* $x_t^*(I_t, \hat{\theta}_t) \ge x_t^*(I_t, \theta_t)$.

As an arbitrary product becomes more attractive to customers, the firm could earn a higher profit from the products and the marginal value of inventory is also higher. As a result, the firm should produce more and keep a higher total produce-up-to level. It is interesting to note that the optimal price of each product and the total purchasing probability of all products may not be monotone in the attractiveness of any product.

In Theorem 6 below, we show that, if the procurement cost of any resource is higher, the marginal value of inventory is also higher, and the firm charges a higher sales price for each product, which partially passes the cost fluctuation to customers.

Theorem 6. *For any given t, let θ_t and $\hat{\theta}_t$ be otherwise the same except that $\hat{c}_t^j > c_t^j$ for some j. For any I_t, the following statements hold:*

(*a*) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \ge \partial_{I_t} V_t(I_t|\theta_t)$.
(*b*) $N_t^*(I_t, \hat{\theta}_t) \le N_t^*(I_t, \theta_t)$. *Hence,* $p_t^{i*}(I_t, \hat{\theta}_t) \ge p_t^{i*}(I_t, \theta_t)$ *for all* $i \in \mathcal{N}$.

Analogous to Theorem 5, Theorem 6 is silent about the comparative statics prediction about the impact of procurement cost on the production quantity. Similar observation has also been established in the literature (see, e.g., Xiao et al. 2015).

In addition to the current market condition, the firm should also take into account the future market trend to achieve the long-run optimality. Comparative statics analysis also enables us to offer insights on the optimal responses of the firm to potential changes in the future market condition. We first study the impact of future market size trend on the firm's optimal decisions.

Theorem 7. *Let the two inventory systems be otherwise equivalent except that $\hat{\xi}_t^{\Lambda}(\Lambda_t) \ge_{st} \xi_t^{\Lambda}(\Lambda_t)$. For any t and (I_t, θ_t), the following statements hold:*

(*a*) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \ge \partial_{I_t} V_t(I_t|\theta_t)$.
(*b*) $\hat{N}_t^*(I_t, \theta_t) \le N_t^*(I_t, \theta_t)$. *Hence,* $\hat{p}_t^{i*}(I_t, \theta_t) \ge p_t^{i*}(I_t, \theta_t)$ *for all* $i \in \mathcal{N}$.
(*c*) $\hat{M}_t^*(I_t, \theta_t) \ge M_t^*(I_t, \theta_t)$ *and* $\hat{Q}_t^*(I_t, \theta_t) \ge Q_t^*(I_t, \theta_t)$. *Hence,* $\hat{q}_t^{j*}(I_t, \theta_t) \ge q_t^{j*}(I_t, \theta_t)$ *for all* $j \in \mathcal{M}$ *and* $\hat{x}_t^*(I_t, \theta_t) \ge x_t^*(I_t, \theta_t)$.

Theorem 7 shows that, under a higher market size trend, it is optimal to charge higher sales prices for all products to exploit the better (potential) market condition. On the other hand, a higher market size trend implies higher future demand, so the firm should

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order more for each of the resources and set a higher produce-up-to level, thus stocking more inventory of the generic product in the face of high potential market size. Similarly, we can show that a higher trend in product attractiveness or procurement cost also prompts the firm to charge higher sales prices and increase the production quantity of the generic product, as shown in Theorems 9, 10, and 11 in Appendix B.

In addition to the market environment fluctuations, we are also interested in examining how the optimal price and inventory policy would change if the firm offers an additional final product. When the firm introduces a new product into the market, the market coverage expands at the cost of cannibalized demands for the original end products. As a consequence, the marginal value of inventory increases and, therefore, the firm should also increase the production quantity of the generic product.

Theorem 8. *Let the two inventory systems be otherwise equivalent except for $\mathcal{N} \subset \hat{\mathcal{N}}$. For $t = T, T-1, \cdots, 1$, and any (I_t, θ_t), the following statements hold:*

(*a*) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$.
(*b*) $\hat{M}_t^*(I_t, \theta_t) \geq M_t^*(I_t, \theta_t)$ *and* $\hat{Q}_t^*(I_t, \theta_t) \geq Q_t^*(I_t, \theta_t)$. *Hence,* $\hat{q}_t^{j*}(I_t, \theta_t) \geq q_t^{j*}(I_t, \theta_t)$ *for all* $j \in \mathcal{M}$ *and* $\hat{x}_t^*(I_t, \theta_t) \geq x_t^*(I_t, \theta_t)$.

To sum up, comparative statics analysis is essential in our general joint pricing and inventory management model with market environment fluctuation. Although the standard IFT and MCS approaches may not be amenable for this complex model to characterize the impact of market size and firm productivity upon the optimal policy, we propose a new comparative statics approach that combines the advantages of IFT and MCS approaches. This new approach facilitates us to study the impact of market environment fluctuation upon the optimal policy. We believe this new approach has the potential to be applicable in other models as well.

5. Concluding Remarks

In this study, we study the optimal pricing and production policies in a general periodic-review joint pricing and inventory management model under delayed differentiation and market environment fluctuation. The firm offers multiple partially substitutable end products made from a single generic product. The firm holds inventory for this generic product, which is manufactured using multiple resources procured at different costs. The final products are offered in the market in a make-to-order fashion. A salient feature of our model is that the market size, product attractivenesses, firm productivity, and procurement costs all evolve according to an exogenous underlying Markov process. Thus, our model captures the high uncertainty and volatility of the market environment.

The key to analyzing the joint pricing and inventory management problem is comparative statics analysis. More specifically, we are interested in characterizing the structure of the optimal pricing and inventory policy. Moreover, we seek to investigate how market environment fluctuation would impact the optimal policy of the firm. Due to the lack of second-order differentiability and supermodularity, the standard IFT and MCS approaches are not readily applicable to characterizing the impact of market size and firm productivity on the optimal policy. Thus, we propose a new comparative statics analysis approach that combines the advantages of IFT and MCS approaches. More specifically, we analyze the first-order optimality condition of the optimal policy in each period, and carefully study how changes in market size and firm productivity impact the marginal value of each decision variable. We identify a simple yet powerful lemma which translates the monotonicity relationship between the optimizers into that between the partial derivatives of the objective function under different parameter values. Assuming to the contrary that the comparative statics prediction is reversed, our new approach employs this lemma and leverages some structural properties of our model to construct a contradiction by iteratively linking the monotone relationship between the optimizers and that between the partial derivatives of the objective function. In short, our approach paves the way to make component-wise comparisons between the optimizers under different parameter values by integrating the advantages of IFT and MCS approaches. Although our comparative statics approach is devised for our joint pricing and inventory management model with delayed differentiation and market environment fluctuation, we believe this new approach has the potential to be applicable in other settings.

We characterize the optimal joint pricing and ordering policy for arbitrary numbers of final products and resources as a produce-up-to policy. The optimal sales price of each product and the optimal order quantity are decreasing in the starting inventory level of the firm, and increasing in the market size and the product attractiveness of each product. When the firm productivity improves, the firm should decrease the price of each final product and the procurement quantity of each resource. Because of the higher productivity, the total production quantity of the generic product increases. When the costs of some resources increase, the firm increases the sales price of each product. Expanding the product line drives the firm

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to enlarge the market coverage, and, thus, to increase the production quantity of the generic product.

In summary, we propose a joint pricing and inventory management model with delayed differentiation and market environment fluctuation. Comparative statics analysis is crucial to analyze this general joint pricing and inventory management model. By combining the advantages of the standard IFT and MCS methods, we propose a new comparative statics analysis approach to characterize the impact of market environment fluctuation, to which IFT and MCS approaches are not applicable. We believe the new approach is promising for comparative statics analysis in other operations management models as well. Our work is silent about the general framework under which our proposed comparative statics approach can be applied while the stand IFT and MCS methods are not amenable. This would be an interesting direction for future research.

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References

Bish, E., X. Zeng, J. Liu, D. Bish. 2012. Comparative statics analysis of multiproduct newsvendor networks under responsive pricing. *Oper. Res.* **60**(5), 1111–1124.

Chao, X., X. Gong, S. Zheng. 2016. Optimal pricing and inventory policies with reliable and random-yield suppliers: characterization and comparison. *Ann. Oper. Res.* **241**(1-2), 35–51.

Chen, X., D. Simchi-Levi. 2004a. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Oper. Res.* **52**(6), 887–896.

Chen, X., D. Simchi-Levi. 2004b. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Math. Oper. Res.* **29**(3), 698–723.

Chen, X., D. Simchi-Levi. 2006. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The continuous review model. *Oper. Res. Lett.* **34**, 323–332.

Chen, X., D. Simchi-Levi. 2012 Pricing and inventory management. R. Philips, O. Ozalp, eds., *The Oxford Handbook of Pricing Management*. Oxford University Press, Oxford, 784–824.

Chen, Y., S. Ray, Y. Song. 2006. Optimal inventory and pricing control policy in periodic review system with fixed ordering cost and lost-sales. *Naval Res. Logist.* **53**, 117–136.

Chen, W., Q. Feng, S. Seshadri. 2015. Inventory-based dynamic pricing with costly price adjustment. *Prod. Oper. Manag.* **24**(5), 732–749.

Federgruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Oper. Res.* **47**(3), 454–475.

Federgruen, A., M. Wang. 2013. Monotonicity properties of a class of stochastic inventory systems. *Ann. Oper. Res.* **208**(1), 155–186.

Federgruen, A., M. Wang. 2015. A continuous review model with general shelf age and delay-dependent inventory costs. *Prob. Eng. Inform. Sci.* **29**(4), 507–525.

Feng, Q. 2010. Integrating dynamic pricing and replenishment decisions under supply capacity uncertainty. *Management Sci.* **56**(12), 2154–2172.

Feng, Q., R. Shi. 2012. Sourcing from multiple suppliers for price-dependent demands. *Prod. Oper. Manag.* **21**(3), 547–563.

Feng, Q., S. Luo, D. Zhang. 2013. Dynamic inventory-pricing control under backorder: Demand estimation and policy optimization. *Manuf. Serv. Oper. Manag.* **16**(1), 149–160.

Gong, X., X. Chao, S. Zheng. 2014. Dynamic pricing and inventory management with dual suppliers of different lead times and disruption risks. *Prod. Oper. Manag.* **23**(12), 2058–2074.

Huh, W., G. Janakiraman. 2008. (s;S), optimality in joint inventory-pricing control: An alternative approach. *Oper. Res.* **56**(3), 783–790.

Huh, W., C. Krishnamurthy, R. Weber. 2011. Concavity and monotonicity properties in a groundwater management model. *Nav. Res. Logist.* **58**(7), 670–675.

Lee, H. L., C. S. Tang. 1997. Modelling the costs and benefits of delayed product differentiation. *Management Sci.* **43**(1), 40–53.

Lee, H. L., C. S. Tang. 1998. Variability reduction through operations reversal. *Management Sci.* **44**(2), 162–172.

Li, H., W. Huh. 2011. Pricing multiple products with the multinomial logit and nested logit models: Concavity and implications. *Manuf. Serv. Oper. Manag.* **13**(4), 549–563.

Li, Q., P. Yu. 2014. Multimodularity and its application in three stochastic dynamic inventory problems. *Manuf. Serv. Oper. Manag.* **16**(3), 455–463.

Li, Q., S. Zheng. 2006. Joint inventory replenishment and pricing control for systems with uncertain yield and demand. *Oper. Res.* **54**(4), 696–705.

Mas-Colell, A., M. Whinston, J. Green. 1995. *Microeconomic Theory*. Oxford University Press, Oxford.

Nagali, V., J. Hwang, D. Sanghera, M. Gaskins, M. Pridgen, T. Thurston, P. Mackenroth, D. Branvold, P. Scholler, G. Shoemaker. 2008. Procurement risk management (prm), at Hewlett-Packard company. *Interfaces*. **38**(1), 51–60.

Olley, S, A. Pakes. 1996. The dynamics of productivity in the telecommunications equipment industry. *Econometrica*. **64**(6), 1263–1297.

Pang, Z., F. Chen, Y. Feng. 2012. A note on the structure of joint inventory-pricing control with leadtimes. *Oper. Res.* **60**(3), 581–587.

Shaked, M., J. G. Shanthikumar. 2007. *Stochastic Orders*. Springer Science & Business Media, Berlin.

Snyder, L. V., Z. M. Shen. 2011. *Fundamentals of Supply Chain Theory*. John Wiley & Sons, Hoboken.

Song, J. S. 1994. The effect of leadtime uncertainty in a simple stochastic inventory model. *Management Sci.* **40**(5), 603–613.

Song, J.S., P. Zipkin. 1993. Inventory control in a fluctuating demand environment. *Oper. Res.* **41**(2), 351–370.

Song, J.S., H. Zhang, Y. Hou, M. Wang. 2010. The effect of lead time and demand uncertainties in (r, q), inventory systems. *Oper. Res.* **58**(1), 68–80.

Swaminathan, J., S. Tayur. 1998. Managing broader product lines through delayed differentiation using vanilla boxes. *Management Sci.* **44**(12-part-2), S161–S172.

Xiao, G., N. Yang, R. Zhang. 2015. Dynamic pricing and inventory management under uctuating procurement costs. *Manuf. Serv. Oper. Manag.* **17**(3), 321–334.

Yang, Y., Y. Chen, Y. Zhou. 2014. Coordinating inventory control and pricing strategies under batch ordering. *Oper. Res.* **62**(1), 25–37.

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Appendix A: Proofs of Statements

Proof of Lemma 1. **Part(a).** By Theorem 1 of Li and Huh (2011), $r(\cdot|a_t)$ is jointly concave in d_t for any given a_t. The differentiability of $R(\cdot|a_t)$ (in N_t) then follows immediately from the envelope theorem. The concavity of $R(\cdot|a_t)$ (in N_t) follows from that concavity is preserved under maximization. To show that $R(\cdot|\cdot)$ is supermodular in (N_t, a_t^i), it suffices to show that $\partial_{N_t} R(N_t|a_t)$ is increasing in a_t^i. By the envelope theorem, $\partial_{N_t} R(N_t|a_t) = \xi^*$, where ξ^* is the Lagrangian multiplier with respect to the constraint $\sum_{i=1}^n d_t^i = N_t$. The first-order KKT condition can be written as

$$\frac{a_t^i}{b} + \frac{\log(1-N_t)}{b} - \frac{\sum_{j=1}^n d_t^j/b}{1-N_t} - \frac{\log(d_t^i)}{b} - \frac{1}{b} = \xi^*, \text{ for all } i$$

Hence,

$$\frac{a_t^i + \log(1-N_t) - 1 - \log(d_t^i)}{b} = C, \text{ for all } i,$$

where $C := \xi^* + \frac{N_t}{b(1-N_t)}$. Thus, $d_t^i = \exp(a_t^i - 1 + \log(1-N_t) - bC)$. Since $\sum_{i=1}^n d_t^i = N_t$,

$$\sum_{i=1}^n \exp(a_t^i - 1 - bC) = \frac{N_t}{1-N_t}.$$

For any i, assume that a_t^i increases, and then we have C increases as well. Thus, $\xi^* = C - \frac{N_t}{b(1-N_t)}$ also increases. Therefore, $\partial_{N_t} R(N_t|a_t) = \xi^*$ is increasing in a_t^i. This completes the proof of Part (a).

Part (b). As shown in the proof of Part (a), if N_t increases,

$$d_i^*(N_t|a_t) = \frac{N_t \exp(a_t^i - 1)}{\sum_{j=1}^n \exp(a_t^j - 1)} = \frac{N_t \exp(a_t^i)}{\sum_{j=1}^n \exp(a_t^j)}$$

will increase as well. Furthermore,

$$p_i^*(N_t|a_t) = \frac{\log(1-N_t) - \log(d_i^*(N_t|a_t))}{b}$$
$$= \frac{\log(1-N_t) - \log(N_t) + \log(\sum_{j=1}^n \exp(a_t^j))}{b}$$

is decreasing in N_t.

Part (c). As shown in the proof of Part (a),

$$d_i^*(N_t|a_t) = \frac{N_t \exp(a_t^i - 1)}{\sum_{j=1}^n \exp(a_t^j - 1)}$$

is increasing in a_t^i and decreasing in a_t^j for any $i \in \mathcal{N}$ and $j \neq i$. Furthermore,

$$p_i^*(N_t|a_t) = \frac{\log(1-N_t) - \log(N_t) + \log(\sum_{j=1}^n \exp(a_t^j))}{b}$$

is increasing in a_t^j for all $j \in \mathcal{N}$. This concludes the proof of Lemma 1.

Proof of Lemma 2. **Part (a)**. The production problem can be formulated as

$$\min \sum_{j=1}^m c_t^j q_t^j$$
$$s.t. \prod_{j \in \mathcal{M}} (q_t^j)^{\gamma_j} = M_t$$

Since $c_t^j q_t^j$ and $\prod_{j \in \mathcal{M}} (q_t^j)^{\gamma_j}$ are both increasing in q_t^j (for all j), the optimization can be relaxed as

$$\min \sum_{j=1}^m c_t^j q_t^j$$
$$s.t. \prod_{j \in \mathcal{M}} (q_t^j)^{\gamma_j} \geq M_t$$

The convexity of $C(\cdot|c_t)$ follows from that convexity is preserved under minimization. The continuous differentiability follows from the envelope theorem. To show that $C(\cdot|\cdot)$ is supermodular in (M_t, c_t^j), it suffices to show that $\partial_{M_t} C(M_t|c_t)$ is increasing in c_t^j for all j.

Note that $\partial_{M_t} C(M_t|c_t) = \mu^*$, where μ^* is the Lagrangian multiplier associated with the constraint $\prod_{j \in \mathcal{M}} (q_t^j)^{\gamma_j} \geq M_t$. The KKT condition with respect to q_t^j implies that

$$c_t^j = \mu^* \gamma_j \prod_{j \in \mathcal{M}} (q_t^j)^{\gamma_j} / q_t^j = \mu^* \gamma_j M_t / q_t^j.$$

Hence, $q_t^j = \mu^* \gamma_j M_t / c_t^j$. Therefore,

$$M_t = \prod_{j \in \mathcal{M}} (q_t^j)^{\gamma_j} = (\mu^*)^{\sum_{j \in \mathcal{M}} \gamma_j} \prod_{j \in \mathcal{M}} \left(\frac{\gamma_j}{c_t^j}\right)^{\gamma_j} (M_t)^{\sum_{j \in \mathcal{M}} \gamma_j}$$

Hence, it is clear that, if c_t^j increases, μ^* will increase as well. Hence, $\partial_{M_t} C(M_t|c_t) = \mu^*$ is increasing in c_t^j for any j.

Part (b). As shown in the proof of part (a),

$$q_j^*(M_t|c_t) = \mu^* \gamma_j M_t / c_t^j.$$

By the proof of part (a), $\mu^* M_t$ is proportional to $(M_t)^{\frac{1}{\sum_j \gamma_j}}$. Hence, $q_j^*(M_t|c_t)$ is increasing in M_t.

Part (c). By the proof of parts (a) and (b),

$$q_j^*(M_t|c_t) = \mu^* \gamma_j M_t / c_t^j = \frac{1}{c_t^j} \left(\prod_{j \in \mathcal{M}} \left(\frac{c_t^j}{\gamma_j}\right)^{\gamma_j} M_t \right)^{\frac{1}{\sum_j \gamma_j}}.$$

It is clear from the expression of $q_j^*(M_t|c_t)$ that $q_j^*(M_t|c_t)$ is decreasing in c_t^j and increasing in c_t^i for $i \neq j$. This concludes the proof of Lemma 2.

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Proof of Lemma 3. Since $V_0(\cdot|\theta_0) \equiv 0$ is concave and continuously differentiable in I_0 for any θ_0, it suffices to show that if $V_{t-1}(\cdot|\theta_{t-1})$ is concave and continuously differentiable in I_{t-1} for any θ_{t-1}, then, for any θ_t, (i) $\Psi_t(\cdot|\theta_t)$ is concave and continuously differentiable in z, (ii) $J_t(\cdot, \cdot, \cdot|\theta_t)$ is jointly concave and continuously differentiable in (N_t, M_t, I_t), and (iii) $V_t(\cdot|\theta_t)$ is concave and continuously differentiable in I_t.

Since $-H(\cdot)$ and $V_{t-1}(\cdot|\theta_{t-1})$ are concave and concavity is preserved under expectation, by Equation (8), $\Psi_t(z|\theta_t)$ is concave in z for any θ_t. Since ϵ_t follows a continuous distribution, $\Psi_t(z|\theta_t)$ is continuously differentiable in z.

Since $R(N_t|a_t)$ is continuously differentiable in N_t and $C(M_t|c_t)$ is continuously differentiable in M_t, $J_t(M_t, N_t, I_t|\theta_t) = \Lambda_t R(N_t|a_t) - \Gamma_t C(M_t|c_t) + \Psi_t(I_t + \Gamma_t M_t - \Lambda_t N_t|\theta_t)$ is continuously differentiable in (N_t, M_t, I_t) for any θ_t.

Since concavity is preserved under maximization, by Equation (5), $V_t(\cdot|\theta_t)$ is concave in I_t for any θ_t. The continuous differentiability of $V_t(\cdot|\theta_t)$ follows from the envelope theorem and its derivative is given by

$$\partial_{I_t} V_t(I_t|\theta_t) = \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t), \tag{A1}$$

where the first equality follows from the envelope theorem. This completes the induction and thus the proof of Lemma 3.

Proof of Theorem 1. Theorem 1 follows directly from part (b) of Theorem 2 that $M_t^*(I_t, \theta_t)$ is continuously decreasing in I_t.

Hence, $\bar{I}_t(\theta_t) = \max\{I_t : M_t^*(I_t, \theta_t) > 0\}$.

Proof of Theorem 2. **Part (a).** We define $x_t := I_t + \Gamma_t M_t$. Then, $J_t(\cdot, \cdot, \cdot|\theta_t)$ can be rewritten as

$$\Lambda_t R(N_t|a_t) - \Gamma_t C\left(\frac{x_t - I_t}{\Gamma_t}|c_t\right) + \Psi_t(x_t - \Lambda_t N_t|\theta_t),$$

which is supermodular in (N_t, x_t, I_t). Furthermore, the feasible set $\{(N_t, x_t, I_t) : N_t \in (0, 1), x_t - I_t \ge 0, I_t \in \mathbb{R}\}$ is a complete lattice. Therefore, by the Topkis' Theorem, $N_t^*(I_t, \theta_t)$ is increasing in I_t. By Lemma 1, $p_i^*(\cdot|a_t)$ is decreasing in N_t. Hence, $p_t^{i*}(I_t, \theta_t) = p_i^*(N_t^*(I_t, \theta_t)|a_t)$ is decreasing in I_t for each $i \in \mathcal{N}$.

Part (b). We define $y_t := I_t - \Lambda_t N_t$ and $z_t = -M_t$. Then, $J_t(\cdot, \cdot, \cdot|\theta_t)$ can be rewritten as

$$\Lambda_t R\left(\frac{I_t - y_t}{\Lambda_t}|a_t\right) - \Gamma_t C(-z_t|c_t) + \Psi_t(y_t - z_t|\theta_t),$$

which is supermodular in (y_t, z_t, I_t). Furthermore, the feasible set $\{(y_t, z_t, I_t) : I_t - y_t \in (0, \Lambda_t), z_t \le 0, I_t \in \mathbb{R}\}$ is a complete lattice. Therefore, by the Topkis' Theorem, the optimizer $z_t^*(I_t, \theta_t)$ is increasing in I_t, i.e., $M_t^*(I_t, \theta_t)$ is decreasing in I_t. By Lemma 2, $q_j^*(\cdot|c_t)$ is increasing in M_t. Hence, $q_t^{j*}(I_t, \theta_t) = q_j^*(M_t^*(I_t, \theta_t)|c_t)$ is decreasing in I_t for each $j \in \mathcal{M}$.

Part (c). By the proof of Part (a), the supermodularity of the transformed objective function implies that $x_t^*(I_t, \theta_t)$ is increasing in I_t. This concludes the proof of Theorem 2.

Proof of Lemma 4. $z_1^* < z_2^*$, so $\underline{z} \le z_1^* < z_2^* \le \bar{z}$. Hence,

$$\partial_z F_1(z_1^*, Z_1^*) \begin{cases} = 0 & \text{if } z_1^* > \underline{z}, \\ \le 0 & \text{if } z_1^* = \underline{z}; \end{cases} \quad \text{and}$$

$$\partial_z F_2(z_2^*, Z_2^*) \begin{cases} = 0 & \text{if } z_2^* < \bar{z}, \\ \ge 0 & \text{if } z_2^* = \bar{z}, \end{cases} \quad \text{i.e., } \partial_z F_1(z_1^*, Z_1^*) \le 0 \le$$

$$\partial_z F_2(z_2^*, Z_2^*).$$

Proof of Theorem 3. We show all parts together by backward induction. More specifically, we prove that if $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{\theta}_{t-1}) \ge \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ for all I_{t-1} and $\hat{\Lambda}_{t-1} > \Lambda_{t-1}$, then we have (i) $N_t^*(I_t, \hat{\theta}_t) \ge N_t^*(I_t, \theta_t)$, (ii) $M_t^*(I_t, \hat{\theta}_t) \ge M_t^*(I_t, \theta_t)$, and (iii) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \ge \partial_{I_t} V_t(I_t|\theta_t)$ for all I_t and $\hat{\Lambda}_t > \Lambda_t$. Since $\partial_{I_0} V_0(I_0|\hat{\theta}_0) = \partial_{I_0} V_0(I_0|\theta_0) = 0$ for all I_0 and $\hat{\Lambda}_0 > \Lambda_0$, the initial condition is satisfied. Since $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{\theta}_{t-1}) \ge \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ and $\xi_t^{\Lambda,i}(\hat{\Lambda}_t^i) \ge \xi_t^{\Lambda,i}(\Lambda_t^i)$ for any $i \in \mathcal{N}$, $\partial_z \Psi_t(z|\hat{\theta}_t) \ge \partial_z \Psi_t(z|\theta_t)$ for any z.

First, we show that $N_t^*(I_t, \hat{\theta}_t) \le N_t^*(I_t, \theta_t)$. Assume, to the contrary, that $N_t^*(I_t, \hat{\theta}_t) > N_t^*(I_t, \theta_t)$. Lemma 4 yields that $\partial_{N_t} J_t(N_t^*(I_t, \hat{\theta}_t), M_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \ge 0 \ge \partial_{N_t} J_t(N_t^*(I_t, \theta_t), M_t^*(I_t, \theta_t), I_t|\theta_t)$, i.e.,

$$\partial_{N_t} J_t(N_t^*(I_t, \hat{\theta}_t), M_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t)/\hat{\Lambda}_t \ge 0 \ge \partial_{N_t} J_t(N_t^*(I_t, \theta_t), M_t^*(I_t, \theta_t), I_t|\theta_t)/\Lambda_t.$$

Thus,

$$\begin{aligned} &\partial_{N_t} R(N_t^*(I_t, \hat{\theta}_t)|a_t) - \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) \\ &\ge 0 \ge \partial_{N_t} R(N_t^*(I_t, \theta_t)|a_t) - \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) \\ &\quad - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t) \end{aligned} \tag{A2}$$

Since $N_t^*(I_t, \hat{\theta}_t) > N_t^*(I_t, \theta_t)$, the concavity of $R(\cdot|a_t)$ implies that $\partial_{N_t} R(N_t^*(I_t, \hat{\theta}_t)|a_t) < \partial_{N_t} R(N_t^*(I_t, \hat{\theta}_t)|a_t)$. Hence, Equation (A2) yields that $\partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) < \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t)$. Since $\partial_z \Psi_t(z|\hat{\theta}_t) \ge \partial_z \Psi_t(z|\theta_t)$, the concavity of $\Psi_t(\cdot|\theta_t)$ implies that

$$\begin{aligned} I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t) > I_t + \Gamma_t M_t^*(I_t, \theta_t) \\ - \Lambda_t N_t^*(I_t, \theta_t). \end{aligned}$$

Since $\hat{\Lambda}_t > \Lambda_t$ and $N_t^*(I_t, \hat{\theta}_t) > N_t^*(I_t, \theta_t)$, $M_t^*(I_t, \hat{\theta}_t) > M_t^*(I_t, \theta_t)$. Lemma 4 suggests that $\partial_{M_t} J_t(N_t^*(I_t, \hat{\theta}_t), M_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \ge \partial_{M_t} J_t(N_t^*(I_t, \theta_t), M_t^*(I_t, \theta_t), I_t|\theta_t)$, i.e.,

$$
\begin{aligned}
&-\partial_{M_t} C(M_t^*(I_t, \hat{\theta}_t)|c_t) + \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) \\
&\geq 0 \geq -\partial_{M_t} C(M_t^*(I_t, \theta_t)|c_t) + \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) \\
&\quad -\Lambda_t N_t^*(I_t, \theta_t)|\theta_t) \qquad (A3)
\end{aligned}
$$

The convexity of $C(\cdot|c_t)$ implies that $\partial_{M_t} C(M_t^*(I_t, \hat{\theta}_t)|c_t) > \partial_{M_t} C(M_t^*(I_t, \theta_t)|c_t)$. Therefore, Equation (A3) suggests that $\partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) > \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t)$, which contradicts that $\partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) < \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t)$. Therefore, the previous assumption $N_t^*(I_t, \hat{\theta}_t) > N_t^*(I_t, \theta_t)$ is invalid and, hence, $N_t^*(I_t, \hat{\theta}_t) \leq N_t^*(I_t, \theta_t)$. Then, $p_t^{i*}(I_t, \hat{\theta}_t) = p_i^*(N_t^*(I_t, \hat{\theta}_t)|a_t) \geq p_i^*(N_t^*(I_t, \theta_t)|a_t) = p_t^{i*}(I_t, \theta_t)$.

Next, we show that $M_t^*(I_t, \hat{\theta}_t) \geq M_t^*(I_t, \theta_t)$. Assume, to the contrary, that $M_t^*(I_t, \hat{\theta}_t) < M_t^*(I_t, \theta_t)$. Lemma 4 yields that $\partial_{M_t} J_t(N_t^*(I_t, \hat{\theta}_t), M_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \leq \partial_{M_t} J_t(N_t^*(I_t, \theta_t), M_t^*(I_t, \theta_t), I_t|\theta_t)$, i.e.,

$$
\begin{aligned}
&-\partial_{M_t} C(M_t^*(I_t, \hat{\theta}_t)|c_t) + \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) \\
\leq\ &-\partial_{M_t} C(M_t^*(I_t, \theta_t)|c_t) + \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t) \qquad (A4)
\end{aligned}
$$

Since $M_t^*(I_t, \hat{\theta}_t) < M_t^*(I_t, \theta_t)$, the convexity of $C(\cdot|c_t)$ implies that $\partial_{M_t} C(M_t^*(I_t, \hat{\theta}_t)|c_t) < \partial_{M_t} C(N_t^*(I_t, \hat{\theta}_t)|c_t)$. Hence, Equation (A4) yields that $\partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) < \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t)$. Since $\partial_z \Psi_t(z|\hat{\theta}_t) \geq \partial_z \Psi_t(z|\theta_t)$, the concavity of $\Psi_t(\cdot|\theta_t)$ implies that

$$
I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t) > I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t).
$$

Since $\hat{\Lambda}_t > \Lambda_t$ and $M_t^*(I_t, \hat{\theta}_t) < M_t^*(I_t, \theta_t)$, $N_t^*(I_t, \hat{\theta}_t) < N_t^*(I_t, \theta_t)$. Lemma 4 suggests that $\partial_{N_t} J_t(N_t^*(I_t, \hat{\theta}_t), M_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \leq \partial_{N_t} J_t(N_t^*(I_t, \theta_t), M_t^*(I_t, \theta_t), I_t|\theta_t)$, i.e.,

$$
\begin{aligned}
&\partial_{N_t} R(N_t^*(I_t, \hat{\theta}_t)|a_t) - \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) \\
\leq\ & 0 \leq \partial_{N_t} R(N_t^*(I_t, \theta_t)|a_t) - \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t) \qquad (A5)
\end{aligned}
$$

The concavity of $R(\cdot|a_t)$ implies that $\partial_{N_t} R(N_t^*(I_t, \hat{\theta}_t)|a_t) > \partial_{N_t} R(N_t^*(I_t, \theta_t)|a_t)$. Therefore, Equation (A5) suggests that $\partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) > \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t)$, which contradicts that $\partial_z \Psi_t(I_t + \Gamma_t M_t(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) < \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t)$. Therefore, the previous assumption $M_t^*(I_t, \hat{\theta}_t) < M_t^*(I_t, \theta_t)$ is invalid and, hence, $M_t^*(I_t, \hat{\theta}_t) \geq M_t^*(I_t, \theta_t)$. Then, $q_t^{j*}(I_t, \hat{\theta}_t) = q_j^*(M_t^*(I_t, \hat{\theta}_t)|c_t) \geq q_j^*(M_t^*(I_t, \theta_t)|c_t) = q_t^{j*}(I_t, \theta_t)$, $Q_t^*(I_t, \hat{\theta}_t) = \Gamma_t M_t^*(I_t, \hat{\theta}_t) \geq \Gamma_t M_t^*(I_t, \theta_t) = Q_t^*(I_t, \theta_t)$, and $x_t^*(I_t, \hat{\theta}_t) = I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) \geq I_t + \Gamma_t M_t^*(I_t, \theta_t) = x_t^*(I_t, \theta_t)$.

Finally, to complete the induction, we show that $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$. Recall that $N_t^*(I_t, \hat{\theta}_t) \leq N_t^*(I_t, \theta_t)$ and $M_t^*(I_t, \hat{\theta}_t) \geq M_t^*(I_t, \theta_t)$. If $N_t^*(I_t, \hat{\theta}_t) = N_t^*(I_t, \theta_t)$ and $M_t^*(I_t, \hat{\theta}_t) = M_t^*(I_t, \theta_t)$, $I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t) \leq I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)$. By the inductive hypothesis $\partial_z \Psi_t(\cdot|\hat{\theta}_t) \geq \partial_z \Psi_t(\cdot|\theta_t)$, we have

$$
\begin{aligned}
&\partial_{I_t} V_t(I_t|\hat{\theta}_t) = \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) \\
&\geq \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).
\end{aligned}
$$

If $N_t^*(I_t, \hat{\theta}_t) < N_t^*(I_t, \theta_t)$, Lemma 4 implies that $\partial_{N_t} J_t(N_t^*(I_t, \hat{\theta}_t), M_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \leq \partial_{N_t} J_t(N_t^*(I_t, \theta_t), M_t^*(I_t, \theta_t), I_t|\theta_t)$, that is, Equation (A5) holds. The concavity of $R(\cdot|a_t)$ implies that $\partial_{N_t} R(N_t^*(I_t, \hat{\theta}_t)|a_t) > \partial_{N_t} R(N_t^*(I_t, \theta_t)|a_t)$. Hence, Equation (A5) suggests that $\partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) > \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t)$. This implies that

$$
\begin{aligned}
&\partial_{I_t} V_t(I_t|\hat{\theta}_t) = \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) > \\
&\partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).
\end{aligned}
$$

Finally, if If $M_t^*(I_t, \hat{\theta}_t) > M_t^*(I_t, \theta_t)$, Lemma 4 implies that $\partial_{M_t} J_t(N_t^*(I_t, \hat{\theta}_t), M_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \geq \partial_{M_t} J_t(N_t^*(I_t, \theta_t), M_t^*(I_t, \theta_t), I_t|\theta_t)$, i.e., Equation (A3) holds. The convexity of $C(\cdot|c_t)$ implies that $\partial_{M_t} C(M_t^*(I_t, \hat{\theta}_t)|c_t) > \partial_{M_t} C(M_t^*(I_t, \theta_t)|c_t)$. Hence, Equation (A3) suggests that $\partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) > \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t)$. This implies that

$$
\begin{aligned}
&\partial_{I_t} V_t(I_t|\hat{\theta}_t) = \partial_z \Psi_t(I_t + \Gamma_t M_t^*(I_t, \hat{\theta}_t) - \hat{\Lambda}_t N_t^*(I_t, \hat{\theta}_t) \\
&|\hat{\theta}_t) > \partial_z \Psi_t(I_t + \Gamma_t \\
&M_t^*(I_t, \theta_t) - \Lambda_t N_t^*(I_t, \theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).
\end{aligned}
$$

This concludes the induction and, thus, the proof 3.

Proof of Theorem 4. The proof of Theorem 4 follows from the similar argument as that of Theorem 3. So we omit the details for brevity.

Proof of Theorem 5. We show all parts together by backward induction. More specifically, we prove that if $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{\theta}_{t-1}) \geq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ for all I_{t-1} and $\hat{a}_{t-1}^i > a_{t-1}^i$, then we have (i) $M_t^*(I_t, \hat{\theta}_t) \geq M_t^*(I_t, \theta_t)$, and (ii) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$ for all I_t and $\hat{a}_t^i > a_t^i$. Since $\partial_{I_0} V_0(I_0|\hat{\theta}_0) = \partial_{I_0} V_0(I_0|\theta_0) = 0$ for all I_0 and $\hat{\Lambda}_0 > \Lambda_0$, the initial condition is satisfied. Since $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{\theta}_{t-1}) \geq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ and $\xi_t^{a,i}(\hat{a}_t^i) \geq_{st} \xi_t^{a,i}(a_t^i)$ for any $i \in \mathcal{N}$, $\partial_z \Psi_t(z|\hat{\theta}_t) \geq \partial_z \Psi_t(z|\theta_t)$ for any z.

First, we show that $M_t^*(I_t, \hat{\theta}_t) \geq M_t^*(I_t, \theta_t)$. Assume, to the contrary, that $M_t^*(I_t, \hat{\theta}_t) < M_t^*(I_t, \theta_t)$. Lemma 4 yields that $\partial_{M_t} J_t(N_t^*(I_t, \hat{\theta}_t), M_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \leq \partial_{M_t} J_t(N_t^*(I_t, \theta_t), M_t^*(I_t, \theta_t), I_t|\theta_t)$, Thus,

$$
\begin{aligned}
&-\partial_{M_t}C(M_t^*(I_t,\hat{\theta}_t)|c_t)+\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)\\
\leq\ &-\partial_{M_t}C(M_t^*(I_t,\theta_t)|c_t)+\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)
\end{aligned} \tag{A6}
$$

Since $M_t^*(I_t,\hat{\theta}_t)<M_t^*(I_t,\theta_t)$, the convexity of $C(\cdot|c_t)$ implies that $\partial_{M_t}C(M_t^*(I_t,\hat{\theta}_t)|c_t)<\partial_{M_t}C(M_t^*(I_t,\theta_t)|c_t)$. Hence, Equation (A6) yields that $\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)<\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. Since $\partial_z\Psi_t(z|\hat{\theta}_t)\geq\partial_z\Psi_t(z|\theta_t)$, the concavity of $\Psi_t(\cdot|\theta_t)$ implies that

$$
I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)>I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t).
$$

Since $M_t^*(I_t,\hat{\theta}_t)<M_t^*(I_t,\theta_t)$, we must have $N_t^*(I_t,\hat{\theta}_t)<N_t^*(I_t,\theta_t)$. Lemma 4 suggests that $\partial_{N_t}J_t(N_t^*(I_t,\hat{\theta}_t),M_t^*(I_t,\hat{\theta}_t),I_t|\hat{\theta}_t)\leq\partial_{M_t}J_t(N_t^*(I_t,\theta_t),M_t^*(I_t,\theta_t),I_t|\theta_t)$, i.e.,

$$
\begin{aligned}
&\partial_{N_t}R(N_t^*(I_t,\hat{\theta}_t)|\hat{a}_t)-\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)\\
\geq\ &\partial_{N_t}R(N_t^*(I_t,\theta_t)|a_t)-\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)
\end{aligned} \tag{A7}
$$

The concavity and supermodularity of $R(\cdot|a_t)$ imply that $\partial_{N_t}R(N_t^*(I_t,\hat{\theta}_t)|\hat{a}_t)>\partial_{N_t}R(M_t^*(I_t,\theta_t)|c_t)$. Therefore, Equation (A7) suggests that $\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)<\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$, which contradicts the previous statement that $\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta})>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. Therefore, the previous assumption $M_t^*(I_t,\hat{\theta}_t)<M_t^*(I_t,\theta_t)$ is invalid and, hence, $M_t^*(I_t,\hat{\theta}_t)\geq M_t^*(I_t,\theta_t)$. Then, $q_t^{j^*}(I_t,\hat{\theta}_t)=q_j^*(M_t^*(I_t,\hat{\theta}_t)|c_t)\geq q_j^*(M_t^*(I_t,\theta_t)|c_t)=q_t^{j^*}(I_t,\theta_t)$. Furthermore, $Q_t^*(I_t,\hat{\theta}_t)=\Gamma_tM_t^*(I_t,\hat{\theta}_t)\geq\Gamma_tM_t^*(I_t,\theta_t)=Q_t^*(I_t,\theta_t)$, and $x_t^*(I_t,\hat{\theta}_t)=I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)\geq I_t+\Gamma_tM_t^*(I_t,\theta_t)=x_t^*(I_t,\theta_t)$.

Finally, to complete the induction, we show that $\partial_{I_t}V_t(I_t|\hat{\theta}_t)\geq\partial_{I_t}V_t(I_t|\theta_t)$. If $M_t^*(I_t,\hat{\theta}_t)>M_t^*(I_t,\theta_t)$, Lemma 4 implies that $\partial_{M_t}J_t(N_t^*(I_t,\hat{\theta}_t),M_t^*(I_t,\hat{\theta}_t),I_t|\hat{\theta}_t)\geq\partial_{M_t}J_t(N_t^*(I_t,\theta_t),M_t^*(I_t,\theta_t),I_t|\theta_t)$,

$$
\begin{aligned}
&-\partial_{M_t}C(M_t^*(I_t,\hat{\theta}_t)|c_t)+\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)\\
\geq\ &-\partial_{M_t}C(M_t^*(I_t,\theta_t)|c_t)+\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)
\end{aligned} \tag{A8}
$$

The convexity of $C(\cdot|c_t)$ implies that $\partial_{M_t}C(M_t^*(I_t,\hat{\theta}_t)|c_t)>\partial_{M_t}C(M_t^*(I_t,\theta_t)|c_t)$. Hence, Equation (A8) suggests that $\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. This implies that

$$
\begin{aligned}
\partial_{I_t}V_t(I_t|\hat{\theta}_t)&=\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)\\
&>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)=\partial_{I_t}V_t(I_t|\theta_t).
\end{aligned}
$$

If $M_t^*(I_t,\hat{\theta}_t)=M_t^*(I_t,\theta_t)$ and $N_t^*(I_t,\hat{\theta}_t)\geq N_t^*(I_t,\theta_t)$, $I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)\leq I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)$. By the inductive hypothesis $\partial_z\Psi_t(\cdot|\hat{\theta}_t)\geq\partial_z\Psi_t(\cdot|\theta_t)$, we have

$$
\begin{aligned}
\partial_{I_t}V_t(I_t|\hat{\theta}_t)&=\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)\\
&\geq\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)=\partial_{I_t}V_t(I_t|\theta_t).
\end{aligned}
$$

If $M_t^*(I_t,\hat{\theta}_t)=M_t^*(I_t,\theta_t)$ and $N_t^*(I_t,\hat{\theta}_t)<N_t^*(I_t,\theta_t)$, Lemma 4 implies that $\partial_{N_t}J_t(N_t^*(I_t,\hat{\theta}_t),M_t^*(I_t,\hat{\theta}_t),I_t|\hat{\theta}_t)\leq\partial_{N_t}J_t(N_t^*(I_t,\theta_t),M_t^*(I_t,\theta_t),I_t|\theta_t)$, i.e.,

$$
\begin{aligned}
&\partial_{N_t}R(N_t^*(I_t,\hat{\theta}_t)|\hat{a}_t)-\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)\\
\leq\ &\partial_{N_t}R(N_t^*(I_t,\theta_t)|a_t)-\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)
\end{aligned} \tag{A9}
$$

The convexity and supermodularity of $R(\cdot|a_t)$ implies that $\partial_{N_t}R(N_t^*(I_t,\hat{\theta}_t)|a_t)>\partial_{N_t}R(N_t^*(I_t,\theta_t)|a_t)$. Hence, Equation (A9) suggests that $\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. This implies that

$$
\begin{aligned}
&\partial_{I_t}V_t(I_t|\hat{\theta}_t)=\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\hat{\theta}_t)-\Lambda_tN_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t)>\\
&\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)=\partial_{I_t}V_t(I_t|\theta_t).
\end{aligned}
$$

This concludes the induction and, thus, the proof of Theorem 5.

Proof of Theorem 6. The proof of Theorem 6 follows from the same argument as that of Theorem 5, so we omit it for brevity.

Proof of Theorem 7. We prove by backward induction. More specifically, we prove that if $\partial_{I_{t-1}}\hat{V}_{t-1}(I_{t-1}|\theta_{t-1})\geq\partial_{I_{t-1}}V_{t-1}(I_{t-1}|\theta_{t-1})$ for all I_{t-1}, then we have (i) $\hat{N}_t^*(I_t,\theta_t)\leq N_t^*(I_t,\theta_t)$, (ii) $\hat{M}_t^*(I_t,\theta_t)\geq M_t^*(I_t,\theta_t)$, and (iii) $\partial_{I_t}\hat{V}_t(I_t|\theta_t)\geq\partial_{I_t}V_t(I_t|\theta_t)$ for all I_t. Note that $\partial_{I_0}\hat{V}_0(I_0|\theta_0)=\partial_{I_0}V_0(I_0|\theta_0)$ for all I_0, so the initial condition is satisfied. Since $\partial_{I_{t-1}}\hat{V}_{t-1}(I_{t-1}|\theta_{t-1})\geq\partial_{I_{t-1}}V_{t-1}(I_{t-1}|\theta_{t-1})$, by Theorem 2(a), $\partial_z\hat{\Psi}_t(z|\theta_t)\geq\partial_z\Psi_t(z|\theta_t)$ for any z.

First, we show that $\hat{N}_t^*(I_t,\theta_t)\leq N_t^*(I_t,\theta_t)$. Assume, to the contrary, that $\hat{N}_t^*(I_t,\theta_t)>N_t^*(I_t,\theta_t)$. Lemma 4 yields that $\partial_{N_t}\hat{J}_t(\hat{N}_t^*(I_t,\theta_t),\hat{M}_t^*(I_t,\theta_t),I_t|\theta_t)\geq\partial_{N_t}J_t(N_t^*(I_t,\theta_t),M_t^*(I_t,\theta_t),I_t|\theta_t)$, Thus,

$$
\begin{aligned}
&\partial_{N_t}R(\hat{N}_t^*(I_t,\theta_t)|a_t)-\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)\\
\geq\ &\partial_{N_t}R(N_t^*(I_t,\theta_t)|a_t)-\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)
\end{aligned} \tag{A10}
$$

Since $\hat{N}_t^*(I_t,\theta_t)>N_t^*(I_t,\theta_t)$, the concavity of $R(\cdot|a_t)$ implies that $\partial_{N_t}R(\hat{N}_t^*(I_t,\theta_t)|a_t)<\partial_{N_t}R(\hat{N}_t^*(I_t,\theta_t)|a_t)$. Hence, Equation (A10) yields that $\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)<\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. Since $\partial_z\hat{\Psi}_t(z|\theta_t)\geq\partial_z\Psi_t(z|\theta_t)$, the concavity of $\Psi_t(\cdot|\theta_t)$ implies that

$$I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)>I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t).$$

Since $\hat{N}_t^*(I_t,\theta_t)>N_t^*(I_t,\theta_t)$, $\hat{M}_t^*(I_t,\theta_t)>M_t^*(I_t,\theta_t)$. Lemma 4 suggests that $\partial_{M_t}\hat{J}_t(\hat{N}_t^*(I_t,\theta_t),\hat{M}_t^*(I_t,\theta_t),I_t|\theta_t)\ge\partial_{M_t}J_t(N_t^*(I_t,\theta_t),M_t^*(I_t,\theta_t),I_t|\theta_t)$, i.e.,

$$\begin{aligned}&-\partial_{M_t}C(\hat{M}_t^*(I_t,\theta_t)|c_t)+\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)\\ \ge\;&-\partial_{M_t}C(M_t^*(I_t,\theta_t)|c_t)+\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)\end{aligned}\tag{A11}$$

The convexity of $C(\cdot|c_t)$ implies that $\partial\hat{C}(\hat{M}_t^*(I_t,\theta_t)|c_t)>\partial_{M_t}C(M_t^*(I_t,\theta_t)|c_t)$. Therefore, (A11) suggests that $\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$, which contradicts that $\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)<\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. Therefore, the previous assumption $\hat{N}_t^*(I_t,\theta_t)>N_t^*(I_t,\theta_t)$ is invalid and, hence, $\hat{N}_t^*(I_t,\theta_t)\le N_t^*(I_t,\theta_t)$. Then, $\hat{p}_t^{i*}(I_t,\theta_t)=p_i^*(\hat{N}_t^*(I_t,\theta_t)|a_t)\ge p_i^*(N_t^*(I_t,\theta_t)|a_t)=p_t^{i*}(I_t,\theta_t)$.

Next, we show that $\hat{M}_t^*(I_t,\theta_t)\ge M_t^*(I_t,\theta_t)$. Assume, to the contrary, that $\hat{M}_t^*(I_t,\theta_t)<M_t^*(I_t,\theta_t)$. Lemma 4 yields that $\partial_{M_t}\hat{J}_t(\hat{N}_t^*(I_t,\theta_t),\hat{M}_t^*(I_t,\theta_t),I_t|\theta_t)\le\partial_{M_t}J_t(N_t^*(I_t,\theta_t),M_t^*(I_t,\theta_t),I_t|\theta_t)$, i.e.,

$$\begin{aligned}&-\partial_{M_t}C(\hat{M}_t^*(I_t,\theta_t)|c_t)+\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)\\ \le\;&-\partial_{M_t}C(M_t^*(I_t,\theta_t)|c_t)+\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)\end{aligned}\tag{A12}$$

Since $\hat{M}_t^*(I_t,\theta_t)<M_t^*(I_t,\theta_t)$, the convexity of $C(\cdot|c_t)$ implies that $\partial_{M_t}C(\hat{M}_t^*(I_t,\theta_t)|c_t)<\partial_{M_t}C(M_t^*(I_t,\theta_t)|c_t)$. Hence, Equation (A12) yields that $\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)<\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. Since $\partial_z\hat{\Psi}_t(z|\theta_t)\ge\partial_z\Psi_t(z|\theta_t)$, the concavity of $\Psi_t(\cdot|\theta_t)$ implies that

$$I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)>I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t).$$

Since $\hat{M}_t^*(I_t,\theta_t)<M_t^*(I_t,\theta_t)$, $\hat{N}_t^*(I_t,\theta_t)<N_t^*(I_t,\theta_t)$. Lemma 4 suggests that $\partial_{N_t}\hat{J}_t(\hat{N}_t^*(I_t,\theta_t),\hat{M}_t^*(I_t,\theta_t),I_t|\theta_t)\le\partial_{N_t}J_t(N_t^*(I_t,\theta_t),M_t^*(I_t,\theta_t),I_t|\theta_t)$, i.e.,

$$\begin{aligned}&\partial_{N_t}R(\hat{N}_t^*(I_t,\theta_t)|a_t)-\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)\\ \le\;&\partial_{N_t}R(N_t^*(I_t,\theta_t)|a_t)-\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)\end{aligned}\tag{A13}$$

The concavity of $R(\cdot|a_t)$ implies that $\partial_{N_t}R(\hat{N}_t^*(I_t,\theta_t)|a_t)>\partial_{N_t}R(\hat{N}_t^*(I_t,\theta_t)|a_t)$. Therefore, Equation (A13) suggests that $\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$, which contradicts that $\partial_z\hat{\Psi}_t(I_t+\Gamma\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)<\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. Therefore, the previous assumption $\hat{M}_t^*(I_t,\theta_t)<M_t^*(I_t,\theta_t)$ is invalid and, hence, $\hat{M}_t^*(I_t,\theta_t)\ge M_t^*(I_t,\theta_t)$. Then, $\hat{q}_t^{j*}(I_t,\theta_t)=q_j^*(\hat{M}_t^*(I_t,\theta_t)|c_t)\ge q_j^*(M_t^*(I_t,\theta_t)|c_t)=q_t^{j*}(I_t,\theta_t)$, $\hat{Q}_t^*(I_t,\theta_t)=\Gamma_t\hat{M}_t^*(I_t,\theta_t)\ge\Gamma_tM_t^*(I_t,\theta_t)=Q_t^*(I_t,\theta_t)$, and $\hat{x}_t^*(I_t,\theta_t)=I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)\ge I_t+\Gamma_tM_t^*(I_t,\theta_t)=x_t^*(I_t,\theta_t)$.

Finally, to complete the induction, we show that $\partial_{I_t}V_t(I_t|\hat{\theta}_t)\ge\partial_{I_t}V_t(I_t|\theta_t)$. Recall that $\hat{N}_t^*(I_t,\theta_t)\le{}_N^t(I_t,\theta_t)$ and $\hat{M}_t^*(I_t,\theta_t)\ge M_t^*(I_t,\theta_t)$. If $\hat{N}_t^*(I_t,\theta_t)=N_t^*(I_t,\theta_t)$ and $\hat{M}_t^*(I_t,\theta_t)=M_t^*(I_t,\theta_t)$, $I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)\le I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)$. By the inductive hypothesis $\partial_z\hat{\Psi}_t(\cdot|\theta_t)\ge\partial_z\Psi_t(\cdot|\theta_t)$, we have

$$\begin{aligned}\partial_{I_t}\hat{V}_t(I_t|\theta_t)&=\partial_z\Psi_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)\\ &\ge\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)=\partial_{I_t}V_t(I_t|\theta_t).\end{aligned}$$

If $\hat{N}_t^*(I_t,\theta_t)<N_t^*(I_t,\theta_t)$, Lemma 4 implies that $\partial_{N_t}\hat{J}_t(\hat{N}_t^*(I_t,\theta_t),\hat{M}_t^*(I_t,\theta_t),I_t|\theta_t)\le\partial_{N_t}J_t(N_t^*(I_t,\theta_t),M_t^*(I_t,\theta_t),I_t|\theta_t)$, i.e., Equation (A13) holds. The concavity of $R(\cdot|a_t)$ implies that $\partial_{N_t}R(\hat{N}_t^*(I_t,\theta_t)|a_t)>\partial_{N_t}R(N_t^*(I_t,\theta_t)|a_t)$. Hence, Equation (A13) suggests that $\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. This implies that

$$\begin{aligned}\partial_{I_t}\hat{V}_t(I_t|\theta_t)&=\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t\\ &(I_t,\theta_t)|\theta_t)>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)\\ &=\partial_{I_t}V_t(I_t|\theta_t).\end{aligned}$$

Finally, if $\hat{M}_t^*(I_t,\theta_t)>M_t^*(I_t,\theta_t)$, Lemma 4 implies that $\partial_{M_t}\hat{J}_t(\hat{N}_t^*(I_t,\theta_t),\hat{M}_t^*(I_t,\theta_t),I_t|\theta_t)\ge\partial_{M_t}J_t(N_t^*(I_t,\theta_t),M_t^*(I_t,\theta_t),I_t|\theta_t)$, i.e., Equation (A11) holds. The convexity of $C(\cdot|c_t)$ implies that $\partial_{M_t}C(\hat{M}_t^*(I_t,\theta_t)|c_t)>\partial_{M_t}C(M_t^*(I_t,\theta_t)|c_t)$. Hence, Equation (A11) suggests that $\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)$. This implies that

$$\begin{aligned}\partial_{I_t}\hat{V}_t(I_t|\theta_t)&=\partial_z\hat{\Psi}_t(I_t+\Gamma_t\hat{M}_t^*(I_t,\theta_t)\\ &-\Lambda_t\hat{N}_t^*(I_t,\theta_t)|\theta_t)>\partial_z\Psi_t(I_t+\Gamma_tM_t^*(I_t,\theta_t)\\ &-\Lambda_tN_t^*(I_t,\theta_t)|\theta_t)=\partial_{I_t}V_t(I_t|\theta_t).\end{aligned}$$

This concludes the induction and, thus, the proof of Theorem 7.

Proof of Theorem 8. First, by the proof of Lemma 1, $\partial_{N_t}\hat{R}(N_t|a_t)>\partial_{N_t}R(N_t|a_t)$ for any (N_t,a_t). Therefore, the same argument as the proof of Theorem 5 yields that $\partial_{I_t}\hat{V}_t(I_t|\theta_t)\ge\partial_{I_t}V_t(I_t|\theta_t)$, $\hat{M}_t^*(I_t,\theta_t)\ge M_t^*(I_t,\theta_t)$, $\hat{Q}_t^*(I_t,\theta_t)\ge Q_t^*(I_t,\theta_t)$, $\hat{x}_t^*(I_t,\theta_t)\ge x_t^*(I_t,\theta_t)$, and $\hat{q}_t^{j*}(I_t,\theta_t)\ge q_t^{j*}(I_t,\theta_t)$ for all j.

Appendix B: Additional Results

This section presents some additional results on the impact of product attractiveness trend and procurement cost trend. Theorems 9 and 11 below show that higher trends in product attractiveness and procurement costs have similar impact as a higher trend in market size, under which the firm charges higher

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prices and places larger orders. The proofs of Theorem 9, 10, and 11 are analogous to that of Theorem 7, so we omit them for brevity.

THEOREM 9. *Let the two inventory systems be equivalent except that* $\hat{\xi}_t^{a,i_0}(a_t^{i_0}) \geq_{st} \xi_t^{a,i_0}(a_t^{i_0})$ *for some* $i_0 \in \mathcal{N}$. *For any* t *and* (I_t, θ_t), *the following statements hold:*

(*a*) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$.
(*b*) $\hat{N}_t^*(I_t, \theta_t) \leq N_t^*(I_t, \theta_t)$. *Hence,* $\hat{p}_t^{i*}(I_t, \theta_t) \geq p_t^{i*}(I_t, \theta_t)$ *for all* $i \in \mathcal{N}$.
(*c*) $\hat{M}_t^*(I_t, \theta_t) \geq M_t^*(I_t, \theta_t)$ *and* $\hat{Q}_t^*(I_t, \theta_t) \geq Q_t^*(I_t, \theta_t)$. *Hence,* $\hat{q}_t^{j*}(I_t, \theta_t) \geq q_t^{j*}(I_t, \theta_t)$ *for all* $j \in \mathcal{M}$ *and* $\hat{x}_t^*(I_t, \theta_t) \geq x_t^*(I_t, \theta_t)$.

THEOREM 10. *Let the two inventory systems be equivalent except that* $\hat{\xi}_t^{\Gamma}(\Gamma_t) \geq_{st} \xi_t^{\Gamma}(\Gamma_t)$. *For any* t *and* (I_t, θ_t), *the following statements hold:*

(*a*) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \leq \partial_{I_t} V_t(I_t|\theta_t)$.
(*b*) $\hat{N}_t^*(I_t, \theta_t) \geq N_t^*(I_t, \theta_t)$. *Hence,* $\hat{p}_t^{i*}(I_t, \theta_t) \leq p_t^{i*}(I_t, \theta_t)$ *for all* $i \in \mathcal{N}$.
(*b*) $\hat{M}_t^*(I_t, \theta_t) \leq M_t^*(I_t, \theta_t)$ *and* $\hat{Q}_t^*(I_t, \theta_t) \leq Q_t^*(I_t, \theta_t)$. *Hence,* $\hat{q}_t^{j*}(I_t, \theta_t) \leq q_t^{j*}(I_t, \theta_t)$ *for all* $j \in \mathcal{M}$ *and* $\hat{x}_t^*(I_t, \theta_t) \leq x_t^*(I_t, \theta_t)$.

THEOREM 11. *Let the two systems be equivalent except that* $\hat{\xi}_t^{c,j_0}(c_t^{j_0}) \geq_{st} \xi_t^{c,j_0}(c_t^{j_0})$ *for some* $j_0 \in \mathcal{M}$. *For any* t *and* (I_t, θ_t), *the following statements hold:*

(*a*) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$.
(*b*) $\hat{N}_t^*(I_t, \theta_t) \leq N_t^*(I_t, \theta_t)$. *Hence,* $\hat{p}_t^{i*}(I_t, \theta_t) \geq p_t^{i*}(I_t, \theta_t)$ *for all* $i \in \mathcal{N}$.
(*c*) $\hat{M}_t^*(I_t, \theta_t) \geq M_t^*(I_t, \theta_t)$ *and* $\hat{Q}_t^*(I_t, \theta_t) \geq Q_t^*(I_t, \theta_t)$. *Hence,* $\hat{q}_t^{j*}(I_t, \theta_t) \geq q_t^{j*}(I_t, \theta_t)$ *for all* $j \in \mathcal{M}$ *and* $\hat{x}_t^*(I_t, \theta_t) \geq x_t^*(I_t, \theta_t)$.

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