Optimal Growth of a Two-Sided Platform with Heterogeneous Agents

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Problem Definition: We consider the dynamics of a two-sided platform, where the agent population on both sides experiences growth over time with heterogeneous growth rates. The compatibility between buyers and sellers is captured by a bipartite network. The platform sets commissions to optimize its total profit over T periods, considering the trade-off between short-term profit and growth as well as the spatial imbalances in supply and demand. **Methodology/Results:** We design an asymptotically optimal policy with the profit loss upper-bounded by a constant independent of T, in contrast with a myopic policy shown to be arbitrarily bad. To obtain this policy, we first develop a single-period benchmark problem that captures the optimal steady state of the platform, then delicately boost the growth of the agent types with the lowest relative population ratio compared with the benchmark in each period. We further examine the impact of the growth potential and the network structure on the platform's optimal profit, the agents' payment/income, and the optimal commissions at the optimal steady state. To achieve that, we introduce innovative metrics to quantify the long-run growth potential of each agent type. Using these metrics, we first show that a "balanced" network, where the relative long-run growth potential between sellers and buyers for all submarkets is the same as that for the entire market, allows the platform to achieve maximum profitability. For each agent type, higher ratios of their compatible counterparts' long-run growth potential to their own cause lower payments/higher income. Finally, the impact of the relative long-run growth potential on the optimal commissions in a submarket depends on the convexity/concavity of the value distribution function of agents. **Managerial Implications**: Our study provides insight into how the growth potential and network structure jointly influence the commission policy in the growth process and the optimal steady state.

Key words: two-sided market, platform growth, market structure.

1. Introduction

In recent years, the rapid growth of consumer-to-consumer (C2C) platforms such as Airbnb, eBay, and Upwork has transformed buyer-seller interactions. Their success relies on efficiently growing the agent base on both sides, which drives transaction volume and ultimately enhances platform profitability. Existing literature suggests that a pivotal strategy of the platform involves initially subsidizing agents to stimulate their growth and subsequently implementing charges to ensure long-term profitability (Lian and Van Ryzin 2021). Throughout this process, it is crucial to strike a balance between long-run growth and short-term profitability via a tailored commission structure. However, determining which agent segment to subsidize or charge higher fees becomes challenging, particularly considering the heterogeneity in their growth potentials and compatibility with other platform participants.

In general, the growth of an active agent base over periods hinges on two primary factors: retaining current agents and encouraging new adoptions. Word-of-mouth communication between potential adopters and current agents through online reviews or comments can encourage new adoption. Different agent types exhibit varying retention rates and word-of-mouth effects. For example, tourists seeking vacation homes on Airbnb may have lower retention rates than regular business travelers due to infrequent revisits (Hamilton et al. 2017). However, they rely more on transaction histories and online reviews from previous guests when selecting properties in unfamiliar destinations (Arndt 1967, Sundaram and Webster 1999). Platforms could tailor their commission structures based on distinct growth patterns across different agent segments. For example, Uber and Lyft usually send coupons to users based on their past behaviors to achieve churn management and keep users with low usage frequency (Yu and Zhu 2021). This targeted promotion campaign can be typically viewed as an indirect way to implement personalized pricing to alleviate backlash from customers.

Furthermore, based on previous works on the cross-side network effect of a two-sided market (Rochet and Tirole 2003, Eisenmann et al. 2006, Chu and Manchanda 2016), the growth on one side of the market has a positive impact on the growth of the other side. However, the value contributed to the opposite side of the network differs across various agent segments, as buyers and sellers are horizontally differentiated in terms of their "popularity" and preferences for agents on the other side of the market. This compatibility difference arises from varying tastes, geographical constraints, or skill mismatches (Birge et al. 2021). For instance, on Airbnb, listings located in popular tourist destinations or with a secure parking space tend to be more popular; on Upwork, freelancers who offer skills that match market demands and have flexible schedules tend to attract more companies. During the platform's growth phase, an increase in the number of "marquee users", typically prominent buyers or high-profile suppliers, can motivate the users on the other side to join the platform. Therefore, the platform must consider this compatibility difference of the agents when determining the commission in the growth process. For example, Airbnb charges different commissions based on the location of listings, room types, cancellation policies, and so on (Airbnb 2024, Thorn 2024).

With the intricate interplay of *intertemporal factors* marked by heterogeneous growth potentials and *spatial factors* characterized by the compatibility between agents, it becomes challenging for the platform to find an optimal commission policy to grow the agent base and maximize its long-term profits. Furthermore, gaining insights into how both the intertemporal and spatial factors affect the platform's optimal policy and profit is of utmost importance. These are the two primary focal points of our study.

Results and Contributions. We consider a two-sided platform that charges commissions to sellers and buyers for facilitating their transactions. The compatibility between the buyers and sellers is captured by a bipartite graph, and the transaction quantities and prices between the agents are determined endogenously in a general equilibrium setting. The mass of each agent type in each period depends on the mass and transaction volume in the previous period, capturing the effect of retention and word-of-mouth communication. The platform determines the commissions in each period to maximize the total profit in T periods, taking into account the trade-off between the immediate revenue and the potential for future expansion. Our main findings are summarized as follows.

First, we formulate the platform's problem as a multi-period pricing optimization model, which, however, is challenging to solve due to its high-dimensional state space (determined by the sizes of different agent types and planning time interval). To overcome this challenge, we first construct a single-period problem and show that the gap between T times its optimal objective value and that of the original problem is upper bounded by a constant (see Proposition 1). Therefore, the solution to this single-period problem captures the optimal steady state of the system, and we see it as a benchmark. We then develop a heuristic policy that is shown to be asymptotically optimal (see Theorem 1). The policy focuses only on the scarcest agent type relative to the benchmark problem in each period, and controls its payment/income at the benchmark level to boost its growth. The demand/supply quantities of other types are matched correspondingly to guarantee feasibility. We also provide some numerical examples to illustrate different growth trajectories of the platform and changes of commissions over time under different cases applying this policy (see Figure 1 and Figure 4 in Appendix B.3). Our result provides managerial insights for platform growth: the key is not to boost the growth of the agent type with the lowest mass in each period. Instead, the platform should first identify the target level at which it can maintain and maximize its long-run average profit, and then guarantee the service level (e.g., by offering subsidies or lowering commissions) for the agent types that lag behind this target level in each period. In comparison, we show that even if the platform serves as a monopoly intermediary in the market, the performance of a myopic policy without considering the growth dynamics in the marketplace could be arbitrarily bad (see Proposition 2). This highlights the significance of growth dynamics in the platform's profit optimization problem.

Second, we focus on the platform's optimal steady state characterized by the single-period benchmark problem. We analyze how the growth potential of agent types (intertemporal factor) and network structure (spacial factor) influence (1) the platform's profit, (2) the agents' payments/incomes, and (3) the optimal commissions. Regarding (1), previous literature (see Schrijver et al. 2003, Chou et al. 2011, Birge et al. 2021) considers static settings with exogenous agent bases and shows how the supply-demand imbalance across the network determines the system's performance. However, we find that the metric of "balances" in the literature fails in the dynamic setting (see Figure 2). To incorporate the intertemporal factor, we develop a novel metric to capture the long-run growth potential of each agent type. With a more specific growth function, we develop the intuition behind such a metric. We show that a "balanced" network, where the relative long-run growth potentials of sellers and buyers for all submarkets are the same as that for the entire market, leads to maximum platform profitability (see Theorem 2). In contrast, the extent of the "imbalance" of the network in terms of the relative long-run growth potentials between the two sides determines the lower bound of optimal profit the platform can achieve. Regarding (2), we show that the buyer (seller) type with a higher ratio of compatible sellers' (buyers') long-run growth potential to their own long-run growth potential experiences lower payments (higher income) at the optimal steady state (see Proposition 3). Based on this result, we conduct a sensitivity analysis to illustrate the impact of each agent type's long-run growth potential on its or others' income/payment (see Corollary 2). For (3), we show that the optimal commission charged from the submarket first decreases in the relative growth potential between sellers and buyers, and then increases (decreases) in it when the value distribution functions of both sides are convex (concave)(see Proposition 4). Our results suggest that the platform should strategically focus its marketing campaigns or loyalty programs on agents who exhibit relatively lower long-run growth potential compared with their compatible agents on the other side of the platform.

Organization of the Paper. The rest of the paper is organized as follows. After reviewing the relevant literature in Section 2, we introduce the model and discuss computational challenges in Section 3. In Section 4, we design a heuristic algorithm with provably good performance. In Section 5, we examine the impact of both the network structure and growth potential of agents on the platform's profit, agents' payments/incomes, and optimal commissions at the optimal steady state. The concluding remarks are drawn in Section 6.

Throughout the paper, we use "increasing" (and "decreasing") in a weak sense, i.e., meaning "non-decreasing" (and "non-increasing") unless otherwise specified. In addition, we use $\mathbb{R}_+$ to denote the non-negative real number set.

2. Literature Review

Pricing in two-sided platforms has been extensively studied in the field of Economics and Operations Management. Based on Caillaud and Jullien (2003), Rochet and Tirole (2003, 2006), Armstrong (2006), a growing literature has explored the pricing and matching problems in the context of online platforms (e.g., Hagiu 2009, Cachon et al. 2017, Taylor 2018, Bai et al. 2019, Benjaafar et al. 2019, Hu and Zhou 2020, Benjaafar et al. 2022, Cohen and Zhang 2022). Our work features network effects in a potentially incomplete two-sided market that evolves dynamically. Agents on one side of the market can only trade with a subset of agents on the other, and the platform's commissions influence the transactions and the growth of the agent base in the market. Therefore, our work is closely related to the following two streams of literature: (i) the growth of a marketplace and (ii) pricing in a networked market.

Early literature about the growth of a marketplace mainly focused on product diffusion, which provides a model to forecast the growth of the customer base for a new product, see e.g., Bass (1969), Kalish (1985), Norton and Bass (1987). Based on these papers, more recent literature studies how to leverage discounts or investment incentives to influence the growth of new products (e.g., Bass and Bultez 1982, Shen et al. 2014, Ajorlou et al. 2018) and that of two-sided platforms (e.g., Kabra et al. 2016, Lian and Van Ryzin 2021, He and Goh 2022). Specifically, Lian and Van Ryzin (2021) considered a two-sided market in which the platform can subsidize one or both sides to boost their growth. They show that the optimal policy is to employ a subsidy shock to rapidly steer the market towards its optimal long-term size. He and Goh (2022) studied the dynamics of a hybrid workforce comprising on-demand freelancers and traditional employees, both capable of fulfilling customer demands. They investigated how demand should be allocated between employees and freelancers, and under what conditions the system is sustainable in the long run. Our study differs from this stream of work in that agents have heterogeneous compatibility and growth potentials, which requires us to come up with a customized commission structure for different agent types; in addition, the transaction quantities and prices are both formed endogenously in a general equilibrium in each period.

Our study is also closely related to the literature on networked markets (e.g., Kranton and Minehart 2001, Bimpikis et al. 2019, Baron et al. 2022, Zheng et al. 2023, Chen and Wang 2023). In this line of work, the edges of the network capture the trading opportunities between agents, and the impacts of network effects on the market outcomes are analyzed. For example, Chen and Chen (2021) explored duopoly competition within a market involving network-connected buyers, and they showed that the existence of symmetric market share equilibrium for two identical sellers depends on the intensity of network effects and the quality of the product. More closely, some recent studies explore how to improve operational efficiency in a two-sided market using centralized price controls (e.g., Banerjee et al. 2015, Ma et al. 2022, Varma et al. 2023) or non-pricing controls (e.g., Kanoria and Saban 2021). For example, Hu and Zhou (2022) considered a platform that strategically matches buyers and sellers, who are categorized into distinct groups based on varying arrival rates and matching values. They provided sufficient conditions under which the optimal matching policy follows a priority hierarchy among matched pairs, determined by factors such as quality and distance. Our work adopts the framework proposed by Birge et al. (2021), in which a platform determines commission structure to maximize the total profit in a two-sided market, and the trades and prices are formed endogenously in a competitive equilibrium given the commissions. Differently, we delve into a dynamic setting and demonstrate that utilizing metrics for network imbalance from static settings in the prior studies to quantify the impact of network structure is inadequate. We introduce a novel metric that incorporates the intertemporal factor (i.e., the growth potentials of agents).

Some recent literature also explore the expansion of the platform's agent base in a network (e.g., Li et al. 2021, Alizamir et al. 2022). These studies assume a uniform retention and growth rate across agents from the same side or all agents in the network, with each agent's payoff determined by an exogenously specified function of the number of participants in the network. In contrast, we account for the heterogeneity of growth potentials among various agent types and introduce a novel metric that incorporates both spatial and intertemporal factors to assess the influence of the network structure on the platform's profitability.

3. Model

Consider a two-sided market in which a platform charges commissions to buyers and sellers for facilitating transactions. The compatibility between buyers and sellers is captured by a bipartite graph $(\mathcal{B} \cup \mathcal{S}, E)$, where $\mathcal{B} = \{1, 2, ..., N_b\}$ denotes the set of buyer type and $\mathcal{S} = \{1, 2, ..., N_s\}$ denotes the set of seller types; E is the set of edges that captures the potential trading opportunities between them. Specifically, $(i, j) \in E$ if and only if the service or product of type-i sellers can satisfy the demand of type-j buyers for $i \in \mathcal{S}$ and $j \in \mathcal{B}$. For example, on Upwork, this compatibility between supply and demand is mainly determined by the skills of freelancers and the task demands of the clients, which remains stable throughout the decision horizon (see e.g., Hu and Zhou 2022). These attributes are all recorded on the platform, or the platform can effectively infer them through user profiles or historical transaction data.

In each period $t \in \{1, ..., T\}$, the populations of type-i sellers and type-j buyers are respectively denoted by $s_i(t)$ for $i \in \mathcal{S}$ and $b_j(t)$ for $j \in \mathcal{B}$. Specifically, the initial population of each type is finite, i.e., $s_i(1) < \infty$ for $i \in \mathcal{S}$ and $b_j(1) < \infty$ for $j \in \mathcal{B}$. The buyers/sellers are infinitesimal, and each one of them supplies/demands at most one unit of product/service in one period if they trade in the market. For $t \in \{1, ..., T\}$, we use $q_i^s(t)$ and $q_j^b(t)$ respectively to denote the aggregate supply quantities of type-i sellers and the aggregate demand quantities of type-j buyers, where $q_i^s(t) \in [0, s_i(t)]$ for $i \in \mathcal{S}$ and $q_j^b(t) \in [0, b_j(t)]$ for $j \in \mathcal{B}$. Note that given the commission charged by the platform, the supply/demand vector $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ is endogenously determined in equilibrium, with mechanism details discussed later.

Population transition. A key feature of our model is that the mass of each agent type evolves dynamically at different rates (see the discussion in Section 1). For any $t \in \{1, ..., T-1\}$, we consider the following population transition equations:

$$s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t)), \qquad \forall i \in \mathcal{S} \tag{1a}$$

$$b_j(t+1) = \mathcal{G}_j^b(b_j(t), q_j^b(t)), \qquad \forall j \in \mathcal{B}. \tag{1b}$$

In (1), we assume that the mass of agents for the next period depends on the mass and transaction volume in the current period. A higher mass of agents in the current period contributes to a larger future agent base due to retention (see Lian and Van Ryzin 2021). A higher transaction quantity leads to a higher future agent base due to the word-of-mouth effect or the imitation effect (see Bass 1969, Mahajan and Peterson 1985), i.e., current agents who trade on the platform can share positive information about the platform with potential new adopters, attracting them to join the platform. $\mathcal{G}_i^s(\cdot,\cdot)$ and $\mathcal{G}_j^b(\cdot,\cdot)$ can have many possible forms, e.g., $\mathcal{G}_i^s(q,s) = sf(q/s)$, with concave $f(\cdot)$ capturing the agent type's average surplus (see Lian and Van Ryzin 2021).

For the rest of the paper, $\mathcal{G}_i^s(\cdot,\cdot)$ and $\mathcal{G}_j^b(\cdot,\cdot)$ in (1) will be referred to as the *growth functions*. Given these growth functions, the platform's commissions in each period will indirectly influence the growth of agents' bases by influencing the equilibrium transaction quantities $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$, as we will discuss later; the growth of each agent type is also indirectly affected by the number of agents in other categories for the same reason. We next introduce the assumptions for the growth functions. For simplicity of notation, we let $(\mathcal{G}_i^s)'_1(s,q), (\mathcal{G}_i^s)'_2(s,q)$ denote the partial derivatives of $\mathcal{G}_i^s(s,q)$ with respect to $s \geq 0$ and $q \geq 0$; similarly, $(\mathcal{G}_j^b)'_1(b,q), (\mathcal{G}_j^b)'_2(b,q)$ denote the partial derivatives of $\mathcal{G}_j^b(b,q)$ respectively in $b \geq 0$ and $q \geq 0$.

Assumption 1. **(growth functions)** *For any $i \in \mathcal{S}$ and any $j \in \mathcal{B}$,*

(i) $\mathcal{G}_i^s(0,0) = 0$ and $\mathcal{G}_j^b(0,0) = 0$;

(ii) $\mathcal{G}_i^s(s,q)$ is continuously differentiable, increasing and strictly concave in (s,q) for $0 \leq q \leq s$, and moreover, $\lim_{x\to\infty} \left[(\mathcal{G}_i^s)'_1(x,x) + (\mathcal{G}_i^s)'_2(x,x)\right] < 1$ for the seller side; the same properties hold for the buyer side with $\lim_{x\to\infty} \left[(\mathcal{G}_j^b)'_1(x,x) + (\mathcal{G}_j^b)'_2(x,x)\right] < 1$.

Assumption 1(i) implies that if the population mass is zero and there is no transaction from the previous period, then there is no retention or word-of-mouth effect. Assumption 1(ii) requires that the future agent base increases in the current population mass and transaction volume, but the marginal effects of these two factors decrease because the total mass of potential agents in a market is finite. It also requires that the total marginal effects of these two factors are lower than one when the transaction volume and the population mass approach infinity, which says that the number of agents in the system cannot grow infinitely large. We will delay the discussion about the class of examples under this assumption to Section 5.

We next discuss how the equilibrium supply/demand $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ is formed in a networked market given the commission by the platform in each period $t \in \{1, ..., T\}$.

Competitive equilibrium. In period $t \in \{1, ..., T\}$, the platform charges commission $r_i^s(t)$ to type-i sellers and $r_j^b(t)$ to type-j buyers if they trade. The commissions are homogeneous within the same agent type but may vary across different types. When $r_i^s(t) < 0$ or $r_j^b(t) < 0$, the platform subsidizes the sellers or buyers. In practice, platforms could implement heterogeneous prices to different user types through personalized coupon distribution (Park and Hwang 2020). Previous research also showed that revenue loss can be unbounded when using a uniform pricing strategy across types (Birge et al. 2021). Therefore, in our setting, we consider type-dependent, heterogeneous pricing (see e.g., Varma et al. 2023), which is also aligned with the practice of some platforms like Airbnb.

Given the commissions, type-i sellers offer their product/service at price $p_i(t)$ and receive $p_i(t) - r_i^s(t)$; type-j buyers pay $p_i(t) + r_j^b(t)$ if they trade with type-i sellers. The market prices $\boldsymbol{p}(t)$ are endogenously formed in equilibrium to match supply and demand, rather than controlled by the platform (see Definition 1 later). This is widely observed across various online platforms. For instance, hosts on Airbnb compete on their rental offers, and freelancers on Upwork compete on their hourly rates. We consider the case that a seller cannot charge different prices to different buyers, aligning with the standard practice of many online platforms, where seller prices are openly displayed on the web page. Finally, we assume for a type-j buyer, all compatible sellers (i.e., $i : (i, j) \in E$) provide perfectly substitutable products/services, and the type-j buyer does not have preference over the compatible sellers' products if their prices are the same. Similarly, it is indifferent for a seller to trade with any compatible buyers given that the market price is formed on the seller side. Note that vertical differentiation of sellers can be modeled by adding a quality term for each type of seller in the payoff function of buyers (see Birge et al. 2021), which does not fundamentally change our insights.

We use $F_{b_j} : [0, \overline{v}_{b_j}] \to [0, 1]$ and $F_{s_i} : [0, \overline{v}_{s_i}] \to [0, 1]$ to denote the cumulative distribution function of the (reservation) values respectively for type-j buyers and type-i sellers, in which $\overline{v}_{b_j}$ and $\overline{v}_{s_i}$ are finite for any $j \in \mathcal{B}$ and $i \in \mathcal{S}$. For simplicity, we refer to a seller by "he" and a buyer by "she". A type-i seller only engages in trades when the amount he receives from the transaction is weakly higher than his reservation value v, i.e., $p_i(t) - r_i^s(t) \geq v$; similarly, a type-j buyer trades when the total payment is weakly lower than her value v, i.e., $p_i(t) + r_j^b(t) \leq v$.

To simplify our analysis later, we extend the domains of the value distributions to $\mathbb{R}$: let $F_{b_j}(v) = 1$ for $v \geq \overline{v}_{b_j}$ and $F_{b_j}(v) = 0$ for $v \leq 0$ for any $j \in \mathcal{B}$; similarly, for the seller side, we let $F_{s_i}(v) = 1$ for $v \geq \overline{v}_{s_i}$ and $F_{s_i}(v) = 0$ for $v \leq 0$ for any $i \in \mathcal{S}$. In addition, define $f_{b_j}(v)$ and $f_{s_i}(v)$ respectively as the derivative of $F_{b_j}(v)$ for $v \in [0, \overline{v}_{b_j}]$ and $F_{s_i}(v)$ for $v \in [0, \overline{v}_{s_i}]$ (or the density function of the valuations). We impose the following assumption regarding the value distributions throughout the paper.

Assumption 2. **(value distribution)** *For any $j \in \mathcal{B}$ and $i \in \mathcal{S}$,*

(i) $F_{b_j}(v)$ and $F_{s_i}(v)$ are strictly increasing in $v \in [0, \overline{v}_{b_j}]$ and $v \in [0, \overline{v}_{s_i}]$;

(ii) $F_{b_j}(v)$ and $F_{s_i}(v)$ are continuously differentiable respectively in $v \in [0, \overline{v}_{b_j}]$ and $v \in [0, \overline{v}_{s_i}]$, and the density functions are lower bounded by a positive constant.

Under Assumption 2(i), we define the inverse function $F_{b_j}^{-1} : [0,1] \to [0, \overline{v}_{b_j}]$ and $F_{s_i}^{-1} : [0,1] \to [0, \overline{v}_{s_i}]$ such that $F_{b_j}^{-1}(F_{b_j}(v)) = v$ for $v \in [0, \overline{v}_{b_j}]$ and $F_{s_i}^{-1}(F_{s_i}(v)) = v$ for $v \in [0, \overline{v}_{s_i}]$. Under Assumption 2(ii), $F_{b_j}^{-1}(x)$ and $F_{s_i}^{-1}(x)$ are also continuous and differentiable in $x \in [0,1]$, and their density functions are also bounded. We further impose the following Assumption on $F_{b_j}^{-1}(x)$ and $F_{s_i}^{-1}(x)$.

Assumption 3. **(concavity)** *$F_{b_j}^{-1}(1 - a/b)a$ and $-F_{s_i}^{-1}(a/b)a$ are both strictly concave in (a, b) for $0 \leq a \leq b$.*

Assumptions 2 and 3 hold for many commonly used distributions such as uniform, truncated exponential, and truncated generalized Pareto distribution.

We can finally define the equilibrium in the network market in each period, given the commission vector $(r_i^s(t) : i \in \mathcal{S},\ r_j^b(t) : j \in \mathcal{B})$ by the platform and the population vector $(\boldsymbol{s}(t), \boldsymbol{b}(t))$. Denote by $x_{ij}(t)$ the amount type-j buyers purchase from type-i sellers, then the equilibrium should satisfy the following conditions:

Definition 1. **(competitive equilibrium)** In period $t \in \{1, \ldots, T\}$, given the platform's commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{N_s} \times \mathbb{R}^{N_b}$ and the population vector of sellers and buyers $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}_+^{N_s} \times \mathbb{R}_+^{N_b}$, a competitive equilibrium is defined as the price-flow vector $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ that satisfies the following conditions:

$$q_i^s(t) = s_i(t) F_{s_i}(p_i(t) - r_i^s(t)), \qquad \forall i \in \mathcal{S}, \tag{2a}$$

$$q_j^b(t) = b_j(t)\Big(1 - F_{b_j}\big(\min_{i' : (i', j) \in E} \{p_{i'}(t)\} + r_j^b(t)\big)\Big), \qquad \forall j \in \mathcal{B}, \tag{2b}$$

$$q_i^s(t) = \sum_{j':(i,j')\in E} x_{i,j'}(t), \qquad \forall i \in \mathcal{S}, \tag{2c}$$

$$q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B}, \tag{2d}$$

$$x_{ij}(t) \geq 0, \qquad \forall (i,j) \in E, \tag{2e}$$

$$x_{ij}(t) = 0, \qquad \forall i \notin \arg\min_{i':(i',j)\in E}\{p_{i'}\}, \;\; j \in \mathcal{B}. \tag{2f}$$

In Definition 1, Conditions (2a) and (2b) ensure that the total supply/demand quantities of type-i sellers and type-j buyers equal the mass of agents who can obtain nonnegative utilities from the transaction. Specifically, Condition (2b) assumes that type-j buyers only trade with compatible sellers with the lowest market price to maximize their utilities. Conditions (2c) and (2d) characterize the flow conservation conditions in the networked market. Finally, Condition (2e) requires that the transaction flow is non-negative, and Condition (2f) requires that the buyers only trade with their compatible sellers with the lowest prices.

Notice that equilibrium concepts similar to Definition 1 have also been adopted in the two-sided market literature by, e.g., Weyl (2010) and Birge et al. (2021). In our setting, the demand/supply quantities only depend on the prices and commissions in the current period, which is commonly seen in the literature about dynamic pricing, e.g., Chen and Gallego (2019), Birge et al. (2023). With Definition 1, given any commission profile and the total mass of agents in each period, we can show that the equilibrium always exists, and the equilibrium supply-demand vector $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ is always unique (see Proposition 5 in Appendix A.1).

Platform's profit optimization problem. Given the mass of different types of agents in the first period $(\boldsymbol{s}(1), \boldsymbol{b}(1))$, the platform aims to maximize its total T-period profit by determining the commission for each type in each period. For simplicity of notation, we let $(\boldsymbol{s}, \boldsymbol{b}) := (\boldsymbol{s}(t), \boldsymbol{b}(t))_{t=2}^T$, and $(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q^s}, \boldsymbol{q^b}) := (\boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))_{t=1}^T$, then the platform's T-period profit maximization problem can be expressed as:

$$\mathcal{R}^*(T) = \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{r},\boldsymbol{p},\boldsymbol{x},\boldsymbol{q^s},\boldsymbol{q^b}} \sum_{t=1}^T \left[\sum_{i\in\mathcal{S}} r_i^s(t) q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t) q_j^b(t)\right] \tag{3a}$$

$$\text{s.t.} \quad (\boldsymbol{s}(t), \boldsymbol{b}(t), \boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)) \text{ satisfies (2)}, \qquad \forall t \in \{1, ..., T\}, \tag{3b}$$

$$s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t)), \qquad \forall i \in \mathcal{S}, t \in \{1, ..., T-1\}, \tag{3c}$$

$$b_j(t+1) = \mathcal{G}_j^b(b_j(t), q_j^b(t)), \qquad \forall j \in \mathcal{B}, t \in \{1, ..., T-1\}. \tag{3d}$$

The platform's profit consists of the commissions from the sellers and buyers who trade in the market during the T periods. Constraint (3b) ensures that given the population vector $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ and commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ in period t, vector $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q}^{\boldsymbol{s}}(t), \boldsymbol{q}^{\boldsymbol{b}}(t))$ constitutes a competitive equilibrium; Constraints (3c)-(3d) indicate that the dynamics of populations follow the transition equations given in (1). Given that the equilibrium transaction quantities $(\boldsymbol{q}^{\boldsymbol{s}}(t), \boldsymbol{q}^{\boldsymbol{b}}(t))_{t=1}^T$ are unique under any commission $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))_{t=1}^T$ (see Proposition 5(ii) in Appendix A.1), the maximization problem in (3) is well-defined. In the rest of the paper, we refer to Problem (3) as OPT. Since OPT is non-convex (in $(\boldsymbol{r}, \boldsymbol{q})$), we will first reformulate it into a convex optimization problem and then discuss the challenges in solving it.

Reformulation and challenges. For any period $t \in \{1, \dots, T\}$, we can deduce from (2a) that type-i sellers' incomes per unit are bounded below by the highest reservation value among those who participate in trading, i.e., $p_i(t) - r_i^s(t) \geq F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right)$ for i with $q_i^s(t) > 0$. Similarly, type-j buyers' payments are bounded above by the lowest value among them, i.e., $p_i(t) + r_j^b(t) \leq F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right)$ for j with $q_j^b(t) > 0$. Therefore, the objective value of OPT is upper bounded by $\sum_{t=1}^T \left[\sum_{j \in \mathcal{B}} F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right) q_j^b(t) - \sum_{i \in \mathcal{S}} F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right) q_i^s(t)\right]$, which is concave in $(\boldsymbol{q}, \boldsymbol{s}, \boldsymbol{b})$ under Assumption 3. By further relaxing some constraints of OPT, we can obtain a convex optimization problem where the decision variables only consist of $(\boldsymbol{s}, \boldsymbol{b}, \boldsymbol{q}^s, \boldsymbol{q}^b, \boldsymbol{x})$ but not commission $(\boldsymbol{r}^s, \boldsymbol{r}^b)$. We present the formulation in Problem (13) and show that the relaxation is tight in Proposition 6 in Appendix A.1. After obtaining the optimal solution to this convex optimization problem, we can find a feasible commission profile $(\boldsymbol{r}^s, \boldsymbol{r}^b)$ that can induce this equilibrium by solving a system of linear inequalities in each period (see Lemma 2 in Appendix A.1). The feasible commissions always exist and are not necessarily unique, but the payments/incomes of agents with positive trades are uniquely determined in any equilibrium.

Even though the non-convexity challenge of OPT can be circumvented by the reformulation, solving Problem (13) is still computationally challenging when T is large (i.e., larger than $T \times (2N_s + 2N_b + |E|)$). Problem (13) can also be formulated as a deterministic dynamic program (DP) with high-dimensional state space, but the lack of structural properties for the DP formulation does not provide clear managerial implications for the growth strategy in the networked market. In Section 4 below, we propose a single-period convex problem, which returns the long-run average value of OPT; based on its optimal solution, we design a simple policy with provable performance guarantees. This policy provides clear guidance for the platform's growth strategy.

4. Asymptotically Optimal Policy

We define an admissible policy as a sequence of functions $\pi =: \{\pi_t : \mathcal{F}_t \to \mathcal{R}^{N_s+N_b}\}_{t=1}^{T}$ that outputs the commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ in each period t, where $\mathcal{F}_t$ is the history of population vectors $(\boldsymbol{s}(t'), \boldsymbol{b}(t') : t' = 1, \ldots, t)$ and transaction vectors $(\boldsymbol{x}(t'), \boldsymbol{q}^s(t'), \boldsymbol{q}^b(t') : t' = 1, \ldots, t-1)$ up to t. Let Π be the set of admissible policies and define $\mathcal{R}^\pi(T)$ as the platform's total profit in T periods for policy $\pi \in \Pi$. We evaluate the policy's performance by quantifying its profit loss relative to the optimal objective value $\mathcal{R}^*(T)$ in OPT, which is formally defined as

$$\mathcal{L}^\pi(T) \;=\; \mathcal{R}^*(T) - \mathcal{R}^\pi(T). \tag{4}$$

We focus on $\mathcal{L}^\pi(T)$ in the asymptotic setting as $T \to \infty$, and strive to devise an admissible policy with good performance.

Flynn (1978) studies heuristic policies for solving infinite-horizon deterministic dynamic programming problems. He provides the necessary and sufficient conditions for the existence and asymptotic optimality of "steady-state policy," which involves solving a static problem to identify the optimal steady state, moving the system to this state, and maintaining it there. Our algorithm shares a similar spirit of "steady-state policy." However, even though he provides examples of constructing feasible rules that move the system from the initial state to the target steady state, most of them involve implementing the action at the optimal steady state from the beginning or making straightforward modifications to it (Flynn 1975, 1981). We will see that those methods of constructing feasible rules cannot be applied to our setting due to the flow conservation constraints of the equilibrium in the networked market (i.e., Definition 1). To resolve this challenge, we propose a novel method called the Target-Ratio Policy (TRP) that only steers the growth of the scarcest agents in the network relative to the benchmark towards optimality in each period. Interestingly, we establish that such a policy can indeed achieve asymptotic optimality (see Theorem 1). On the other hand, we will show that the myopic policy, under which the platform completely neglects population growth, could perform arbitrarily badly in general (see Proposition 2).

Long-run Average Value Problem (AVG). Based on the reformulation of OPT, we first develop a corresponding steady-state problem. We will show that the optimal objective value for the steady-state problem could serve as a benchmark for the policy's profit loss in (4).

For convenience, define $\widetilde{F}_{b_j}(q_j^b,b_j) := F_{b_j}^{-1}\big(1-\frac{q_j^b}{b_j}\big)q_j^b$ for $b_j>0$ and $0\le q_j^b\le b_j$ and $\widetilde{F}_{b_j}(q_j^b,b_j) := 0$ for $q_j^b=b_j=0$. Similarly, define $\widetilde{F}_{s_i}(q_i^s,s_i) := F_{s_i}^{-1}\big(\frac{q_i^s}{s_i}\big)q_i^s$ for $s_i>0$ and $0\le q_i^s\le s_i$ and $\widetilde{F}_{s_i}(q_i^s,s_i) := 0$ for $q_i^s=s_i=0$. Then we consider the following optimization problem which we refer to as AVG:

$$
\begin{aligned}
\overline{\mathcal{R}} \;=\; \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{q^s},\boldsymbol{q^b},\boldsymbol{x}} \quad & \sum_{j\in\mathcal{B}} \widetilde{F}_{b_j}(q_j^b,b_j) - \sum_{i\in\mathcal{S}} \widetilde{F}_{s_i}(q_i^s,s_i), && && \text{(5a)}\\
\text{s.t.} \quad & q_i^s \;\le\; s_i, \sum_{j:(i,j)\in E} x_{ij} \;=\; q_i^s, && \forall i\in\mathcal{S}, && \text{(5b)}\\
& q_j^b \;\le\; b_j, \sum_{i:(i,j)\in E} x_{ij} \;=\; q_j^b, && \forall j\in\mathcal{B}, && \text{(5c)}\\
& x_{ij} \;\ge\; 0, && \forall (i,j)\in E. && \text{(5d)}\\
& s_i \;\le\; \mathcal{G}_i^s(s_i,q_i^s), && \forall i\in\mathcal{S}, && \text{(5e)}\\
& b_j \;\le\; \mathcal{G}_j^b(b_j,q_j^b), && \forall j\in\mathcal{B}. && \text{(5f)}
\end{aligned}
$$

We relax Constraint (3b) of OPT about equilibrium conditions to (5b)-(5d), and relax the population transition equations in Constraint (3c)-(3d) to inequalities in (5e)-(5f). Then the feasible region for Problem (5) is a convex set and the objective function is concave thanks to Assumption 1, which suggests that AVG is a tractable convex optimization problem. We next characterize the properties of the optimal solution to AVG:

Lemma 1. **(optimal solution to AVG)** *The optimal solution to Problem (5) exists, and*

(i) the optimal population $(\overline{\boldsymbol{s}},\overline{\boldsymbol{b}})$ *and the optimal supply-demand vector* $(\overline{\boldsymbol{q}}^s,\overline{\boldsymbol{q}}^b)$ *are unique;*

(ii) $\overline{s}_i=\mathcal{G}_i^s(\overline{s}_i,\overline{q}_i^s)$ *and* $\overline{b}_j=\mathcal{G}_j^b(\overline{b}_j,\overline{q}_j^b)$.

Lemma 1(ii) implies that the optimal mass of agents $(\overline{\boldsymbol{s}},\overline{\boldsymbol{b}})$ is at equilibrium by controlling the supply-demand vector at the level of $(\overline{\boldsymbol{q}}^s,\overline{\boldsymbol{q}}^b)$. We further show that the gap between T times the optimal objective value of AVG from (5) and that of OPT from (3) is upper bounded by a constant for any positive integer T.

Proposition 1. **(alternative benchmark)** *There exists a positive constant* C_1 *such that for any* $T=1,2,...,$

$$|\mathcal{R}^*(T)-T\overline{\mathcal{R}}| \;\le\; C_1.$$

Proposition 1 dictates that the difference between $\frac{1}{T}\mathcal{R}^*(T)$ and $\overline{\mathcal{R}}$ converges to zero as T approaches infinity. Therefore, the optimal solution to AVG $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b)$ captures a steady state where the long-run average profit is maximized, and so we call it the *optimal steady state* in the rest of the paper (see Flynn 1975, 1992). In addition, as we previously mentioned, in contrast to the high-dimensional problem OPT, AVG is a much more tractable static convex optimization problem. Therefore, we will consider $T\overline{\mathcal{R}}$, instead of $\mathcal{R}^*(T)$ as the benchmark for our algorithm design. We next propose the *Target Ratio Policy (TRP)* that admits fast convergence to the steady-state solutions to AVG and formally establish its asymptotic optimality.

Target Ratio Policy (TRP). For ease of illustration, we refer to $\frac{s_i(t)}{\bar{s}_i}$ for $i \in \mathcal{S}$ and $\frac{b_j(t)}{\bar{b}_j}$ for $j \in \mathcal{B}$ as the *population ratio* of type-i seller and type-j buyer, respectively. In addition, we notice $\frac{q_j^b(t)}{b_j(t)}$ $(\frac{q_i^s(t)}{s_i(t)})$ is the fraction of type-j buyers (type-i sellers) who trade in period t, and we refer to this fraction as the *service level* of the corresponding agent type. Recall that the service level also determines the payment/income of agents (i.e., $F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})$ and $F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})$).

Motivated by Proposition 1, we design our approximation algorithm to steer the mass of each agent type towards the optimal steady-state $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ in the network. A straightforward method is to control the service level of each type at the optimal service levels of AVG, i.e., $\frac{q_i^s(t)}{s_i(t)} \approx \frac{\bar{q}_i^s}{\bar{s}_i}$ for any $i \in \mathcal{S}$ and $\frac{q_j^b(t)}{b_j(t)} \approx \frac{\bar{q}_j^b}{\bar{b}_j}$ for any $j \in \mathcal{B}$ for $t \in \{1, ..., T\}$. This is also equivalent to controlling the income/payment of each agent type at the income/payment at the optimal steady state. However, the main challenge is that such a policy is not necessarily feasible in a network given the heterogeneous population ratios among different seller and buyer types. For example, consider a simple scenario of one buyer type and one seller type with a positive initial mass vector $(s(1), b(1))$. Given the flow conservation constraint $q^s(1) = q^b(1)$, if we control the supply quantity $q^s(1)$ such that the service level of the supply side is the same as that from AVG (i.e., $\frac{q^s(1)}{s(1)} = \frac{\bar{q}^s}{\bar{s}}$), the service level for buyers in the first period may be different from that of AVG in general (i.e., $\frac{q^b(1)}{b(1)} \neq \frac{\bar{q}^b}{\bar{b}}$ if $\frac{s(1)}{\bar{s}} \neq \frac{b(1)}{\bar{b}}$). In particular, the type with a lower population ratio will limit the transaction quantity of the other type with a higher ratio, which further restricts its growth. To circumvent this challenge, in each period, we focus on the type with the lowest population ratio and seek to boost its growth, while we match the transaction quantities of other types to guarantee the feasibility of the policy in the entire networked market. Towards this direction, we formally define the Target Ratio Policy in Algorithm 1.

One key advantage of TRP is its computational efficiency: it only requires solving the single-period optimization AVG once. It first identifies the agent types with strictly positive

Algorithm 1: Target Ratio Policy

Input: Optimal solution to AVG $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b, \overline{\boldsymbol{x}})$ and initial mass of agents $(\boldsymbol{s}(1), \boldsymbol{b}(1))$.
for $t = 1$ *to* T **do**
  $m(t) \leftarrow \min\left\{ \min_{i:\overline{s}_i>0} \frac{s_i(t)}{\overline{s}_i}, \min_{j:\overline{b}_j>0} \frac{b_j(t)}{\overline{b}_j} \right\}$;
  for $i = 1$ *to* N_s **do**
    $q_i^s(t) \leftarrow \overline{q}_i^s m(t)$;
  end
  for $j = 1$ *to* N_b **do**
    $q_j^b(t) \leftarrow \overline{q}_j^b m(t)$;
  end
  for $(i,j) \in E$ **do**
    $x_{ij}(t) \leftarrow \overline{x}_{ij} m(t)$;
  end
  Solve (12) in Appendix A.1 to obtain $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$;
  if there are multiple feasible solutions, select one arbitrarily;
  update population profile $(\boldsymbol{s}(t+1), \boldsymbol{b}(t+1))$ by the system dynamics in (1).
end
Output: $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))_{t=1}^T$.

populations in AVG solutions (for all j such that $\overline{b}_j > 0$ and i such that $\overline{s}_i > 0$). The types with zero population at the optimal steady state either have low growth potential or are located at less important positions in the network such that the platform should de-prioritize their growth from the very beginning. Among the agent types with positive population masses in AVG, TRP finds the one with the lowest population ratio $m(t)$ in each period t and matches its service level to the optimal one from AVG (i.e., $\frac{q_i^s(t)}{s_i(t)} = \frac{\overline{q}_i^s}{\overline{s}_i}$ or $\frac{q_j^b(t)}{b_j(t)} = \frac{\overline{q}_j^b}{\overline{b}_j}$). For other types with higher population ratios, their demand/supply quantities are matched correspondingly to guarantee feasibility in the networked market. As mentioned, we can then find the commissions to induce the desired transaction quantities in each period by solving a system of linear inequalities (see (12) in Lemma 2, Appendix A.1).

Note that for those agent types with higher initial population ratios, their service level will be lower than that of AVG (i.e., $\frac{q_i^s(t)}{s_i(t)} < \frac{\overline{q}_i^s}{\overline{s}_i}$ or $\frac{q_j^b(t)}{b_j(t)} < \frac{\overline{q}_j^b}{\overline{b}_j}$). Their populations may grow slowly, or even decline at the beginning. As a result, the agent type with the lowest population ratio may change over time in the network, and the platform may focus on boosting the growth of different types across the planning horizon. A main result of this section is that perhaps surprisingly, by guaranteeing the growth of the agent types with the *lowest* population ratio in the network in each period, the entire network could eventually converge to the optimal state of AVG. Let $\mathcal{L}^{TR}(T)$ denote the profit loss of TRP relative to the optimal objective value $\mathcal{R}^*(T)$, then the following result gives a theoretical performance guarantee for TRP:

THEOREM 1. **(performance of TRP)** *There exists a constant C_2 such that for $T = 1, 2..$,*

$$\mathcal{L}^{TR}(T) \;\leq\; C_2.$$

Theorem 1 shows that the profit loss of TRP relative to the optimal policy is uniformly bounded (with respect to T) by a constant, which further suggests that boosting the growth of the agent type with the lowest population ratio in each period is asymptotically optimal in the networked market. To prove this result, we first show that under TRP, even though the type with the lowest ratio may change over time, the lowest ratio $m(t)$ *monotonically* converges to one at an exponential rate, i.e., $|m(t+1)-1| \leq \gamma |m(t)-1|$ for some $\gamma \in (0,1)$. Therefore, for each type, the transaction quantity $q_i^s(t) = \overline{q}_i^s m(t)$ or $q_j^b(t) = \overline{q}_j^b m(t)$ converges to the optimal level $\overline{q}_i^s$ or $\overline{q}_j^b$ for any $i \in \mathcal{S}$ and $j \in \mathcal{B}$, which ensures that the population profile $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ also converges to the optimal solution $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to AVG. By establishing the fast convergence rate, we observe that there exists a constant C_1' such that $|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)| \leq C_1'$. Together with the result in Proposition 1, we conclude that there exists a constant C_2 such that $|\mathcal{R}^*(T) - \mathcal{R}^{TR}(T)| \leq C_2$. The detailed proof of Theorem 1 is relegated to Appendix B.

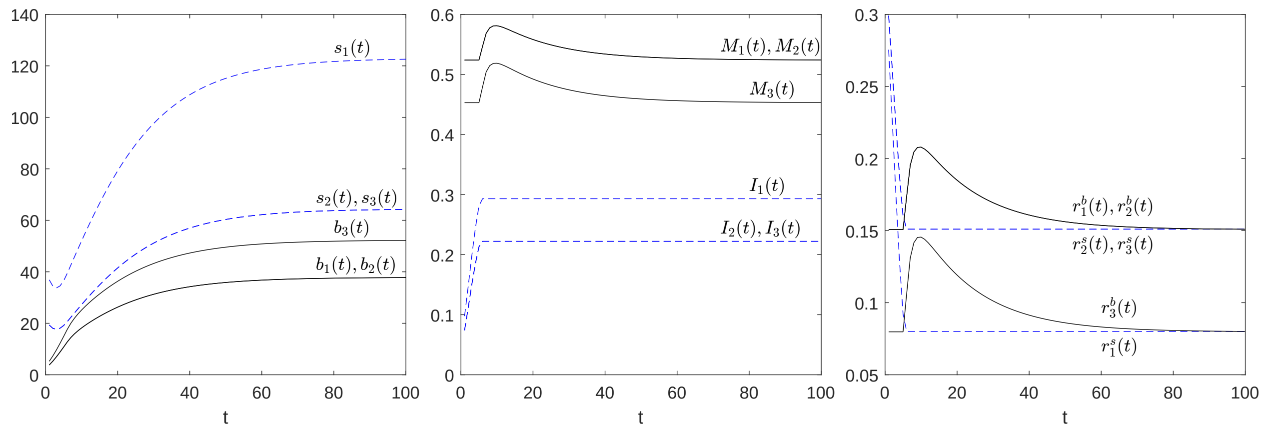
We next investigate the growth patterns of the agent types under TRP. We first establish that, under TRP, the transaction volume of every agent type in the market increases monotonically with time if the initial population of at least one agent type in the network is lower than that at the optimal steady state (which is usually the case for a start-up platform):

COROLLARY 1. **(monotonocity of agents' equilibrium transactions)** *Under TRP, if $m(1) = \min\left\{ \min_{i:\overline{s}_i>0} \frac{s_i(1)}{\overline{s}_i}, \min_{j:\overline{b}_j>0} \frac{b_j(1)}{\overline{b}_j} \right\} \leq 1$, then $q_i^s(t) \leq q_i^s(t+1)$ with $i \in \mathcal{S}$, $q_j^b(t) \leq q_j^b(t+1)$ with $j \in \mathcal{B}$ for for any $t \in \{1, ..., T-1\}$.*

This result immediately suggests that the total transaction volume in the platform $\sum_{i\in\mathcal{S}} q_i^s(t) + \sum_{j\in\mathcal{B}} q_j^b(t)$ grows over time under TRP if the lowest initial population ratio in the network is lower than one. A higher transaction volume not only promotes future growth but also serves as an important signal of the current platform's profitability. Therefore, the growth of transaction volume has been often highlighted as an important indicator in the platform's quarterly financial performance (Airbnb 2023). Our result emphasizes that the TRP ensures the growth of active trades under any network structure and growth function.

In comparison, the growth trajectory of agents' population, payment/income, and commission can be non-monotonic, as we show via the following numerical example.

Example 1. Consider an incomplete 3-by-3 network, in which type-1 sellers are compatible with all buyer types while type-2 and type-3 sellers can only trade with type-3 buyers (see Figure 3 for reference). The seller side has a higher initial population ratio (i.e., $30\% = \frac{s_i(1)}{\bar{s}_i} > \frac{b_j(1)}{\bar{b}_j} = 10\%$ for $i \in \mathcal{S}, j \in \mathcal{B}$). The retention rate of the seller side is assumed to be higher than the buyer side: $s_i(t+1) = 0.8 s_i(t) + 2(q_i^s(t))^{0.7}$ for $i \in \mathcal{S}$ and $b_j(t+1) = 0.7 b_j(t) + 2(q_j^b(t))^{0.7}$ for $j \in \mathcal{B}$. For expositional ease, we denote by $M_j(t) := \min_{i':(i',j)\in E} \left\{\overline{p}_{i'}(t) + \overline{r}_j^b(t)\right\}$ the payment of a type-j buyer at time t, and denote by $I_i(t) := \overline{p}_i(t) - \overline{r}_i^s(t)$ the income of a type-i seller at time t. The results are illustrated in Figure 1.



(a) Population (b) Payment/Income (c) Commission

Figure 1 Growth Trajectory for the platform with low initial population ratio from buyer sides under TRP.

Note: The blue dashed (black solid) lines denote the growth path of the seller (buyer) side.

In general, we will see that under TRP, the type with a higher (lower) population ratio should be charged with a higher (the same) commission compared with the value in AVG, but the relative population ratio could change over time. In Figure 1(a), the platform firstly promotes the growth of the buyer side due to their initially lower population ratio, while the number of sellers initially stagnates due to the limited transaction opportunity on the platform. Later, the buyers' population ratio surpasses the sellers', prompting the platform to grow the seller side. Eventually, the mass of all types stabilizes at an optimal steady state, at which the mass of type-1 seller (i.e., $\overline{s}_1$) is the highest among all types given its popularity and highest retention rate. To achieve that under TRP, in Figure 1(b), the platform firstly keeps the buyers' payments (i.e., $M_j(t)$) at the optimal steady-state level to stimulate their growth, and increases their payments once their population ratio becomes higher. Conversely, the sellers' incomes (i.e., $I_i(t)$) start lower than the steady-state value and gradually increase as the seller population ratio lags. Finally, at the optimal steady state, $\bar{I}_1$ ($\bar{M}_3$) are the highest (lowest) among the types of sellers (buyers) given their higher compatibility with the other side of the market. We construct a feasible (but not unique) commission in Figure 1(c). The commission charged from the seller side is higher initially, while that from the buyer side becomes higher later. Similar growth patterns could be observed in another case with different initial populations (see Figure 4 in Appendix B.3).

Myopic Policy (MP). Some prior studies have examined the effectiveness of the myopic pricing policy in the product diffusion process of a monopoly seller with mixed results under different diffusion functions. Robinson and Lakhani (1975) showed that myopic policy results in significant profit loss relative to the optimal policy if a lower current price could stimulate future demand. In contrast, Bass and Bultez (1982) considered the case that the diffusion process does not interact with price and showed by a numerical study that there is only a small difference in the discounted profits between the myopic and optimal policies. Here we examine how MP performs in our model.

Under MP, in each period t, the platform determines the commissions $(\boldsymbol{r}^b(t), \boldsymbol{r}^s(t))$ to maximize its profit in the current period (i.e., $\sum_{i\in\mathcal{S}} r_i^s(t)q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t)q_j^b(t)$) subject to the equilibrium constraints in (2), without considering the population dynamics in (1) and its impact on future profit. The formal definition of the myopic policy is given by Definition 2 in Appendix B. We let $\mathcal{R}^M(t)$ denote the platform's profit under MP in period t, and recall that $\overline{\mathcal{R}}$ is the optimal value of AVG and could be achieved under TRP. The following result shows that the performance of MP could be arbitrarily bad.

Proposition 2. **(performance of MP)** *Under MP, for any $\epsilon > 0$, there exists a problem instance such that $\lim_{t\to\infty} \mathcal{R}^M(t) := \overline{\mathcal{R}}^M < \infty$ and $\overline{\mathcal{R}}^M < \epsilon\overline{\mathcal{R}}$. Hence, there exists $C_3 > 0$ such that*

$$\mathcal{L}^{MP}(T) \geq C_3 T.$$

Proposition 2 suggests that ignoring the commissions' impact on the population growth could lead to significant profit loss even if the platform serves as a monopoly intermediary. In the proof of Proposition 2, we show that the commissions set by the platform under MP at the steady state are higher than those under TRP. One implication is that the platform must sacrifice some short-term margin to achieve long-term profitability.

Since the TRP requires controlling the service level (or equivalently, their payment/income) of different agent types at the level of the optimal steady state, we will see in the next section how the service level at the optimal steady state is determined by both the network structure $G(\mathcal{S}\cup\mathcal{B},E)$ and population dynamics from (1).

5. Impact of Population Dynamics and Network Structure

In this section, we investigate how the intertemporal factors marked by heterogeneous growth potentials and the spatial factors characterized by the compatibility influence the platform's profit (see Section 5.1) as well as the incomes/payments of agents and optimal commission (see Section 5.2) at the optimal steady state. Investigating the impacts of these spatial-temporal factors can provide insights into the platform's revenue management strategy.

The prior studies showed that in a static setting, a network that more efficiently matches supply with demand achieves a better performance from the platform's perspective, and the agent types connected to a larger population on the other side would gain higher surplus (see Schrijver et al. 2003, Chou et al. 2011, Birge et al. 2021). For example, Chou et al. (2011) showed that a bipartite network, in which every subset of nodes is linked to a sufficiently large number of neighboring nodes, is optimal for the system. Similarly, Birge et al. (2021) showed that supply-demand imbalance across the network, measured by the lowest seller-to-buyer population ratio among all submarkets, determines the lower bounds of the platform's achievable profit relative to that with a complete network. Interestingly, we see from the following numerical example that in a dynamic setting, the imbalance in terms of the equilibrium population ratio at the optimal steady state can no longer provide a profit guarantee for the platform.

Example 2. Consider a network shown in Figure 2. Suppose that buyers' and sellers' (reservation) values are uniformly distributed between $[0,1]$. Consider $s_i(t+1)=\alpha s_i(t)+\beta_i^s q_i^s(t)^\xi$ and $b_j(t+1)=\alpha b_j(t)+\beta_j^b q_j^b(t)^\xi$ for $i\in\{1,2\}, j\in\{1,2\}$ with parameters $\alpha=0.5$, $\xi=0.8, \beta_1^s=\beta_2^b=2, \beta_2^s=\beta_1^b=1$. The population ratio in AVG satisfies $\frac{\sum_{i\in N_E(\tilde{\mathcal{B}})}\bar{s}_i}{\sum_{j\in\tilde{\mathcal{B}}}\bar{b}_j}\geq 50\%\times\frac{\sum_{i\in\mathcal{S}}\bar{s}_i}{\sum_{j\in\mathcal{B}}\bar{b}_j}$ for any

$\widetilde{\mathcal{B}} \subseteq \mathcal{B}$, but the platform's optimal profit in AVG is only about 36 % of that in a complete market, i.e., $\overline{\mathcal{R}}(E, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b) = 36\% \times \overline{\mathcal{R}}(\overline{E}, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$

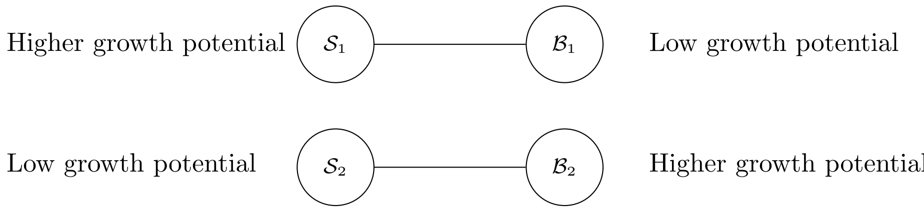


Figure 2 **Bias of Measuring Network Imbalance using Equilibrium Population Ratio**

Hence, it becomes crucial to incorporate temporal factors into the "imbalance" measure for the network, which requires us to first measure the growth potential of each agent type. As we mentioned in Section 1, the growth of an active agent base consists of retaining previous agents and encouraging word-of-mouth effect to attract new adoption. Therefore, to better quantify these two effects, we consider the following class of growth functions $\mathcal{G}^s(\cdot)$ and $\mathcal{G}^b(\cdot)$ in (1):

$$\mathcal{G}_i^s(s,q) \;=\; \alpha_i^s s + \beta_i^s g_s(s,q), \tag{6a}$$

$$\mathcal{G}_j^b(b,q) \;=\; \alpha_j^b b + \beta_j^b g_b(b,q). \tag{6b}$$

In (6), we have homogeneous-degree-ξ_s (and ξ_b) functions $g_s(\cdot,\cdot)$ (and $g_b(\cdot,\cdot)$) with $\xi_s \in (0,1)$ (and $\xi_b \in (0,1)$) for any $q, s, b \geq 0$. A function $g(\cdot,\cdot)$ is homogeneous of degree ξ means that $g(ns, nq) = n^{\xi} g(s,q)$ for any $s \geq q \geq 0$, $n > 0$.

In (6), $\alpha_i^s \in (0,1)$ and $\alpha_j^b \in (0,1)$ respectively denote the retention rate of type-i sellers and type-j buyers (Lian and Van Ryzin 2021, Alizamir et al. 2022, He and Goh 2022). The second term captures new agents' adoption, in which $\beta_i^s \in (0,1)$ and $\beta_j^b \in (0,1)$ measure the type-specific impact of the current user base and transactions on new adoption. Some previous studies about the growth of two-sided platforms assume that the new adoption depends on the transaction volume/price/surplus in the last period and the growth rates are homogeneous for agents from one side (see Lian and Van Ryzin 2021, He and Goh 2022). Different from them, we do not assume a specific expression for $g_s(s,q)$ and $g_b(s,q)$, but we impose the property of a homogeneous degree. This property means that the elasticity of the future user base with respect to the current transaction and user base is given by ξ. For example, $g_s(\cdot)$ can capture the average surplus of the agent (i.e., $\int_0^{\frac{q_i^s(t)}{s_i(t)}}(F_s^{-1})'(y)ydy$), which may contribute to the growth of new adoptions, as modeled by Lian and Van Ryzin (2021).

Furthermore, to isolate the impact of network structure and growth potential, we assume that different types of sellers/buyers have homogeneous value distributions.

ASSUMPTION 4. *$F_{s_i}(v)=F_s(v)$ for any $i\in\mathcal{S}$ and $F_{b_j}(v)=F_b(v)$ for any $j\in\mathcal{B}$.*

Agents' long-run growth potential. To obtain an intuitive expression of the long-run growth potential, we use, throughout this section, a simple polynomial term for g_s and g_b as an illustrative example. For $t\in\{1,...,T-1\}$,

$$s_i(t+1)=\alpha_i^s s_i(t)+\beta_i^s(q_i^s(t))^{\xi_s}, \qquad \text{for } i\in\mathcal{S}, \tag{7a}$$

$$b_j(t+1)=\alpha_j^b b_j(t)+\beta_j^b(q_j^b(t))^{\xi_b}, \qquad \text{for } j\in\mathcal{B}. \tag{7b}$$

Under this form, we can provide a closed-form expression for long-run growth potential, based on which we further deduce all the following results and managerial insights regarding the interplay of spatial-intertemporal factors. However, it is worth pointing out that our proofs do not rely on the exact expressions of (7). Moreover, the expressions for the long-run growth potential in (9) are the same for the more general class of functions in (6).

Based on (7), we proceed to develop the metric to measure the growth potential of each agent type. Given type-i sellers' service level $\frac{\overline{q}_i^s}{\overline{s}_i}$ induced by the platform's optimal commissions $(\overline{\boldsymbol{r}}^s,\overline{\boldsymbol{r}}^b)$ at the optimal steady state, the population of type-i seller converges to $\overline{s}_i$ that satisfies $\overline{s}_i=\alpha_i^s\overline{s}_i+\beta_i^s(\overline{q}_i^s)^{\xi_s}$. Algebraic manipulations suggest that

$$\overline{s}_i=\left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}}\left(\frac{\overline{q}_i^s}{\overline{s}_i}\right)^{\frac{\xi_s}{1-\xi_s}}, \qquad \overline{q}_i^s=\left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}}\left(\frac{\overline{q}_i^s}{\overline{s}_i}\right)^{\frac{1}{1-\xi_s}} \text{ where } i\in\mathcal{S}, \tag{8a}$$

Similarly, for the buyer side,

$$\overline{b}_j=\left(\frac{\beta_j^b}{1-\alpha_j^b}\right)^{\frac{1}{1-\xi_b}}\left(\frac{\overline{q}_j^b}{\overline{b}_j}\right)^{\frac{\xi_b}{1-\xi_b}}, \qquad \overline{q}_j^b=\left(\frac{\beta_j^b}{1-\alpha_j^b}\right)^{\frac{1}{1-\xi_b}}\left(\frac{\overline{q}_j^b}{\overline{b}_j}\right)^{\frac{1}{1-\xi_b}} \text{ where } j\in\mathcal{B}. \tag{8b}$$

Eqn. (8) reveals that given the service level $\frac{\overline{q}_i^s}{\overline{s}_i}$ for type-i sellers and $\frac{\overline{q}_j^b}{\overline{b}_j}$ for type-j buyers, the population of an agent type and the transaction quantities at the optimal steady state are proportional to the coefficients $\left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}}$ for type-i sellers and $\left(\frac{\beta_j^b}{1-\alpha_j^b}\right)^{\frac{1}{1-\xi_b}}$ for type-j buyers. Based on this, we formally define the long-run growth potential as follows:

$$\psi_i^s := \left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}}, \ i \in \mathcal{S}, \qquad \psi_j^b := \left(\frac{\beta_j^b}{1-\alpha_j^b}\right)^{\frac{1}{1-\xi_b}}, \ j \in \mathcal{B}. \tag{9}$$

We next provide some intuitive explanations for $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$. For simplicity, we omit the superscripts (s,b) and subscripts (i,j). In the population dynamics in (7), β captures the impact of transaction quantities on the population growth, and only a fraction $\alpha < 1$ of agents stays in the system after each period. As $\frac{\beta}{1-\alpha} = \sum_{t=0}^{\infty} \beta\alpha^t$, it captures the net present value for the long-run marginal impact of the transaction quantity q^{ξ}. Similarly, the impact of the population elasticity ξ after t periods can be captured by ξ^t. As $\frac{1}{1-\xi} = \sum_{t=0}^{\infty} \xi^t$, it represents the net present value of the long-term impact of the elasticity ξ. Therefore, we refer to ψ_i^s for $i \in \mathcal{S}$ and ψ_j^b for $j \in \mathcal{B}$ in (9) as the long-run growth potential of each agent type.

Rankings of relative growth potential in the network. Based on the long-run growth potential, we introduce a ranking of different types of agents. Let $N_E(X)$ denote the set of all neighbors of agent types $X \subseteq \mathcal{B} \cup \mathcal{S}$ in the graph $G(S \cup B, E)$ such that $N_E(X) = \{i \notin X : (i,j) \in E \text{ for } j \in X\}$. Given a network $G(\mathcal{S} \cup \mathcal{B}, E)$ and the long-run growth potential vector $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$, we first let $\mathcal{B}^0 = B$, $\mathcal{S}^0 = S$ and $E^0 = E$. For $\tau = 0, 1, ...$, we define $\mathcal{B}_\tau$ and $\mathcal{S}_\tau$ iteratively as follows:

$$\mathcal{B}_{\tau+1} = \underset{\widetilde{\mathcal{B}} \subseteq \mathcal{B}^\tau}{\arg\min} \frac{\sum_{i \in N_{E^\tau}(\widetilde{\mathcal{B}})} \psi_i^s}{\sum_{j \in \widetilde{\mathcal{B}}} \psi_j^b}, \tag{10a}$$

$$\mathcal{S}_{\tau+1} = N_{E^\tau}(\mathcal{B}_{\tau+1}). \tag{10b}$$

where $\mathcal{B}^{\tau+1} = \mathcal{B}^\tau \backslash \mathcal{B}_{\tau+1}$, $\mathcal{S}^{\tau+1} = \mathcal{S}^\tau \backslash \mathcal{S}_{\tau+1}$, $E^\tau = \{(i,j) \in E : i \in \mathcal{S}^\tau \text{ and } j \in \mathcal{B}^\tau\}$ and $N_{E^\tau}(B) = \{i \in \mathcal{S}^\tau : j \in \mathcal{B} \text{ and } (i,j) \in E^\tau\}$. If multiple sets achieve the minimum, the arg min operator returns the largest one.

In (10a), for each subset of buyer types $\widetilde{\mathcal{B}}$ of $\mathcal{B}^\tau$, $\frac{\sum_{i \in N_{E^\tau}(\widetilde{\mathcal{B}})} \psi_i^s}{\sum_{j \in \widetilde{\mathcal{B}}} \psi_j^b}$ is the ratio between the total long-run growth potential of its (remaining) compatible sellers and its own. We refer to the ratio as the *relative growth potential* between $N_{E^\tau}(\widetilde{\mathcal{B}})$ and $\widetilde{\mathcal{B}}$. This metric, similar to those used for comparing two economies, e.g., in Krugman (1989), captures the relative growth potential of sellers and buyers. In (10), we can iteratively identify a subgraph such that the relative growth potential of sellers is the lowest. Subsequently, we label it and remove this subgraph from the network, and then B^τ and S^τ are the remaining agent types and E^τ is the remaining graph after τ iterations. We repeat the procedure until the remaining subgraph is empty. As a result, the subnetwork with a higher index τ has a higher relative growth potential of sellers against buyers in the graph. This ranking incorporates both intertemporal factors captured by the long-run growth potential $\boldsymbol{\psi}$ and spatial factors captured by the graph structure $G(\mathcal{B} \cup \mathcal{S}, E)$.

We use the example below to illustrate the rankings of relative growth potential. This example illustrates the compatibility between freelance coders and clients in need of IT services on Upwork. Specifically, clients needing AI Services can only be served by coders with AI skills, and clients requiring immediate delivery of work can only choose coders with flexible working hours. By enumeration, we can obtain the index of each type, and the solid (dotted) line represents the lines between sets with the same (different) index. For a large-scale network, we can obtain the ranking by solving a convex optimization problem This procedure borrows the algorithmic idea to characterize the lexicographically optimal bases of polymatroids from Fujishige (1980).

Example 3. Consider a network as shown in Figure 3. Suppose that $\psi_i^s = \psi_j^b = 1$ for $i = \{1, 2, 3\}$ and $j = \{1, 2, 3\}$. Then by enumeration, we know $\{1, 2\} = \arg\min_{\widetilde{\mathcal{B}} \subseteq \mathcal{B}} \frac{\sum_{i \in N_E(\widetilde{\mathcal{B}})} \psi_i^s}{\sum_{j \in \widetilde{\mathcal{B}}} \psi_j^b}$, which means $\mathcal{B}_1 = \{1, 2\}$ and $\mathcal{S}_1 = \{1\}$ (blue nodes). After eliminating $\mathcal{B}_1$ and $\mathcal{S}_1$ from the network E, we have $\mathcal{B}^1 = \{3\}, \mathcal{S}^1 = \{2, 3\}, E^1 = \{(2, 3), (3, 3)\}$. Since there is only one buyer type left, we know $\mathcal{B}_2 = \{3\}$ and $\mathcal{S}_2 = \{2, 3\}$ (black nodes). Finally, all agent types are labeled with an index.

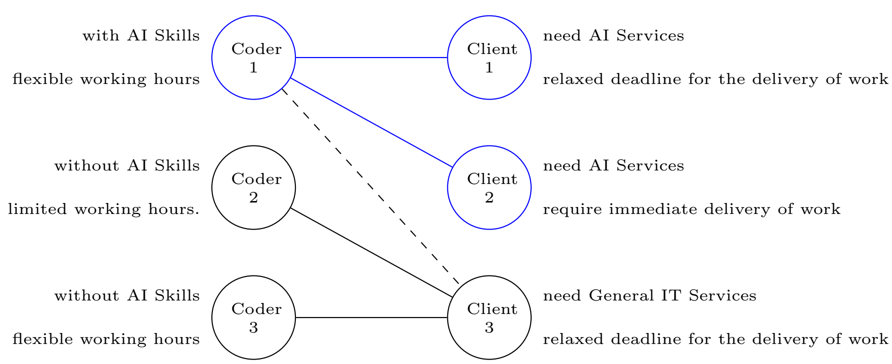


Figure 3 Compatibility between Freelance Coders and Clients in need of IT Services on Upwork.

We will next show the connection between the lowest relative growth potential in the network and the platform's profit at the optimal steady state.

5.1. Optimal Network for the Platform's Profit

To signify the dependence on the network structure $G(\mathcal{S}\cup\mathcal{B},E)$ and long-run growth potential $(\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$, we let $\overline{\mathcal{R}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$ denote the platform's optimal steady-state profit from AVG. Given that the feasible region for a complete graph is the largest in Problem (3), the platform can achieve the maximum optimal profit in a complete graph. Therefore, we let $\overline{E}$ denote the edge set for the complete graph with the set of seller types $\mathcal{S}$ and that of buyer types $\mathcal{B}$, and use $\overline{\mathcal{R}}(\overline{E},\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$ to benchmark the impact of network structure E on the platform's profit. The following theorem establishes a connection between the temporal-spatial factors and the platform's optimal profit in network $G(\mathcal{S}\cup\mathcal{B},E)$.

THEOREM 2. **($(1-\epsilon)$-optimal network structure)** *For any $\epsilon\in[0,1]$, if $G(\mathcal{B}\cup\mathcal{S},E)$ satisfies*

$$\frac{\sum_{i\in\mathcal{S}^1}\psi_i^s}{\sum_{j\in\mathcal{B}^1}\psi_j^b}\ \geq\ (1-\epsilon)\frac{\sum_{i\in\mathcal{S}}\psi_i^s}{\sum_{j\in\mathcal{B}}\psi_j^b}, \tag{11a}$$

then

$$\overline{\mathcal{R}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)\ \geq\ (1-\epsilon)\overline{\mathcal{R}}(\overline{E},\boldsymbol{\psi}^s,\boldsymbol{\psi}^b). \tag{11b}$$

In Condition (11a), the right-hand-side expression $\frac{\sum_{i\in\mathcal{S}}\psi_i^s}{\sum_{j\in\mathcal{B}}\psi_j^b}$ represents the relative long-run growth potential of all sellers to all buyers within the entire network $G(\mathcal{B}\cup\mathcal{S},E)$. Likewise, the left-hand-side is the relative growth potential of the compatible sellers to a subset of buyers $\mathcal{B}^1$, whose relative long-run growth potential is the lowest (see (10)). Therefore, ϵ quantifies the degree of imbalance: a positive value of ϵ indicates that there exists no submarket in which the relative growth potential is ϵ lower than that of the entire market. Then (11b) implies that the degree of imbalance ϵ in the network does not cause more than ϵ optimal profit loss for the platform. When $\epsilon=0$, the condition in (11a) ensures that the relative growth potential for all submarkets is equal to that for the entire market. In other words, the long-run growth potentials are "balanced" in the network. In this case, even though the market E may be incomplete, the lower bound in (11b) is tight, and the platform's optimal profit achieves the maximum possible optimal profit, i.e., $\overline{\mathcal{R}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)=\overline{\mathcal{R}}(\overline{E},\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$.

The managerial insight derived from Theorem 2 suggests that the platform should aim to enhance the balance of the network in terms of long-run growth potential to maximize its steady-state optimal profit. Specifically, the platform could target its marketing campaign on agent types with relatively low long-run growth potential to increase their retention and attract new users.

Remark. A related work by Alizamir et al. (2022) considers a monopoly firm providing service to a network of individual customers with externality. Differently, they find that a balanced network with symmetric mutual interactions among agents results in the lowest profit for the firm. In their setting, they assume a linear impact of agents' consumption on others, and the effects of network externalities go beyond immediate neighbors over time, causing increasing externalities in a network. In contrast, in our setting, increasing the population of one agent type leads to higher transaction quantities on the other side in equilibrium, and the marginal impact on the future population on the other side is decreasing. The diminishing marginal effect of agents' consumption and population on growth can be explained by the fact that the potential market size is usually finite in practice. $\Diamond$

5.2. Agent Payments/Incomes and Platform Commissions

In this subsection, we analyze the impact of agents' growth potential on the platform's commission decisions. Recall that the optimal commission $(\overline{\boldsymbol{r}}^{\boldsymbol{s}}, \overline{\boldsymbol{r}}^{\boldsymbol{b}})$ at the optimal steady state is not necessarily unique, but any optimal commission profile induces the same (net) payments and incomes for agent types engaged in transactions (see Proposition 5 and Lemma 2). Furthermore, the total commission generated from a transaction (i.e., $r_i^s + r_j^b$ for $(i,j) \in E$), which represents the difference between buyers' payments and sellers' incomes, is inherently unique. Therefore, in this subsection, we will first study the impact of network structure and growth potentials on (net) payments and incomes for agent types and then analyze their impact on the total optimal commission.

Buyers' payments and sellers' incomes. We next establish that the ranking of the relative growth potentials of sellers to buyers given in (10) determines the ranking of buyers' payments and sellers' incomes at the optimal steady state. We denote by $M_j = \min_{i':(i',j)\in E} \{\overline{p}_{i'} + \overline{r}_j^b\}$ the payment of any type-j buyers, and denote by $I_i = \overline{p}_i - \overline{r}_i^s$ the income of any type-i sellers at the optimal steady state.

Proposition 3. **(ranking of buyers' payments and sellers' incomes)** *In the network $G(\mathcal{S} \cup \mathcal{B}, E)$, under any platform's optimal commission profile $(\overline{\boldsymbol{r}}^{\boldsymbol{s}}, \overline{\boldsymbol{r}}^{\boldsymbol{b}})$ at the steady state,*

(1) for any $j_1 \in \mathcal{B}_{\tau_1}$ and $j_2 \in \mathcal{B}_{\tau_2}$ with $\tau_1 \leq \tau_2$, $M_{j_1} \geq M_{j_2}$ and $\frac{\overline{q}_{j_1}^b}{\overline{b}_{j_1}} \leq \frac{\overline{q}_{j_2}^b}{\overline{b}_{j_2}}$;

(2) for any $i_1 \in \mathcal{S}_{\tau_1}$ *and* $i_2 \in \mathcal{S}_{\tau_2}$ *with* $\tau_1 \le \tau_2$*,* $I_{i_1} \ge I_{i_2}$ *and* $\frac{\overline{q}^b_{i_1}}{\overline{b}_{i_1}} \ge \frac{\overline{q}^b_{i_2}}{\overline{b}_{i_2}}$.

Proposition 3 posits that under the platform's optimal commissions at the steady state, with a higher relative long-run growth potential of sellers to buyers (i.e., higher index τ indicates higher $\frac{\sum_{i\in\mathcal{S}_\tau}\psi^s_i}{\sum_{j\in\mathcal{B}_\tau}\psi^b_j}$ in (10)), the buyers pay less and experience a higher service level, while the sellers earn a lower income and experience a lower service level in equilibrium. By using the Example 3 to illustrate, the payments on the buyer (i.e., client) side satisfy $M_1 = M_2 > M_3$ given that $\mathcal{B}_1 = \{1,2\}$ and $\mathcal{B}_2 = \{3\}$; the incomes on the seller (i.e., coder) side satisfy that $I_1 > I_2 = I_3$ given that $\mathcal{S}_1 = \{1\}$ and $\mathcal{S}_2 = \{2,3\}$. The managerial implication from Proposition 3 is that while determining the service level, the platform needs to consider not only the retention rate and growth potentials of the focal agent types but also their trading partners on the other side of the market. Specifically, the platform should incentivize the agents with lower relative growth potential by offering them higher commissions and extract a higher surplus from those with higher relative growth potential.

Proposition 3 suggests that any change in the values of $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ induces changes in the service level of each agent type, ultimately affecting the equilibrium demand, supply, and population at the optimal steady state. Lastly, we examine the influence of the long-run growth potential $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ to offer guidance for the platform's commission decisions.

Corollary 2. **(impact of the long-run growth potential)** *Given any* $\xi_s \in (0,1)$ *and* $\xi_b \in (0,1)$*, at the optimal steady state,*

(1) for the service levels,

(i) given $j \in \mathcal{B}$*,* $\overline{q}^b_j/\overline{b}_j$ *decreases in* $\psi^b_{j'} \ge 0$ *for any* $j' \in \mathcal{B}$ *and increases in* $\psi^s_{i'} \ge 0$ *for any* $i' \in \mathcal{S}$*;*

(ii) given $i \in \mathcal{S}$*,* $\overline{q}^s_i/\overline{s}_i$ *decreases in* $\psi^s_{i'} \ge 0$ *for any* $i' \in \mathcal{S}$ *and increases in* $\psi^b_{j'} \ge 0$ *for any* $j' \in \mathcal{B}$*;*

(2) for the transaction quantities and populations,

(i) given $j \in \mathcal{B}$*,* $(\overline{q}^b_j, \overline{b}_j)$ *increases in* $\psi^b_j \ge 0$*, decreases in* $\psi^b_{j'} \ge 0$ *for any* $j' \in \mathcal{B}$ *with* $j' \ne j$*, and increases in* $\psi^s_{i'} \ge 0$ *for any* $i' \in \mathcal{S}$*;*

(ii) given $i \in \mathcal{S}$*,* $(\overline{q}^s_i, \overline{s}_i)$ *increases in* $\psi^s_i \ge 0$*, decreases in* $\psi^s_{i'} \ge 0$ *for any* $i' \in \mathcal{S}$ *with* $i' \ne i$ *and increases in* $\psi^b_{j'} \ge 0$ *for any* $j' \in \mathcal{B}$*.*

Note that for any $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, the vectors $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ are determined by the retention rates $(\boldsymbol{\alpha}^s, \boldsymbol{\alpha}^b)$ and the growth coefficients $(\boldsymbol{\beta}^s, \boldsymbol{\beta}^b)$. Corollary 2(1) suggests that the service level of any agent decreases in the growth potential of all types from the same side but increases in those on the other side of the market. Corollary 2(2) implies that the transaction volume and population of each type are increasing in their own growth potential and those on the other side of the network, but decreasing in those of other types on the same side.

We discuss the intuition using the buyer side as an example. Both a high long-run growth potential and a high service level contribute to an increase in the population of a buyer type at the optimal steady state. Consequently, when the long-run growth potential of a buyer type is high, the platform can maintain a high population by inducing a relatively lower service level. However, if other buyer types have higher long-run growth potential, their equilibrium demand will rise, resulting in increased prices for the sellers and a reduced service level for our focal buyer type. Conversely, if the corresponding sellers have higher long-run growth potential, their supply will increase, leading to lower prices and benefiting all buyers.

Platform's optimal commissions. We now focus on the total commission charged by the platform from one transaction, viz., the difference between the buyers' payments and the sellers' incomes. Note that under the optimal commission, type-i sellers with $i \in \mathcal{S}_\tau$ only trade with type-j buyers with $j \in \mathcal{B}_\tau$. Therefore, we will examine how the total commission charged from one transaction between sellers in $\mathcal{S}_\tau$ and buyers in $\mathcal{B}_\tau$ depends on the ranking of the relative growth potential of sellers to buyers τ given in (10). Here, we assume $\xi_s = \xi_b$ to isolate the impact of value distribution.

Proposition 4. **(ranking the platform's optimal commissions)** *Assume that F_s and F_b are twice differentiable in their domains and $\xi_s = \xi_b$. There exists $\widetilde{\tau}$ such that*

(1) $r_i^s + r_j^b$ for $i \in \mathcal{S}_\tau, j \in \mathcal{B}_\tau$ is decreasing in τ for $\tau < \widetilde{\tau}$;

(2) $r_i^s + r_j^b$ for $i \in \mathcal{S}_\tau, j \in \mathcal{B}_\tau$ is decreasing in τ for $\tau \geq \widetilde{\tau}$ if $F_s(v)$ and $F_b(v)$ are concave; whereas it is increasing in τ for $\tau \geq \widetilde{\tau}$ if $F_s(v)$ and $F_b(v)$ are convex.

In Proposition 4(1), when the relative growth potential of sellers to buyers falls below a threshold, the total commission charged from the transaction decreases with the relative growth potential between sellers and buyers. In Proposition 4(2), the concavity of $F_s(v)$ and $F_b(v)$ implies a higher density of agents with lower (reservation) value. In this case, when the relative growth potential of sellers to buyers is higher, the optimal total commission charged by the platform should be lower. Similarly, the convexity of $F_s(v)$ and $F_b(v)$ implies that the number of agents with higher (reservation) value is higher. In this scenario, the platform charges lower (higher) total commissions for transactions involving agents with moderate (high or low) relative growth potentials of sellers to buyers.

Intuitively, when the relative growth potential between sellers and buyers is below a threshold, the number of sellers is significantly smaller than that of buyers. In such cases, the platform uses its commission to keep the sellers' income at a sufficiently high level to ensure the participation of sellers. As the relative growth potential increases, the number of sellers rises, prompting the platform to gradually reduce buyer payments to stimulate demand. As a result, the total commission charged from the transaction, which is the difference between buyer payments and seller incomes, decreases with the relative growth potential between sellers and buyers.

When the relative growth potential between sellers and buyers exceeds the threshold, the number of sellers is already large in the market, and the platform no longer needs to provide high subsidies to ensure their participation. In this case, an increase in the relative growth potential between sellers and buyers suggests that the platform should reduce the service level for sellers and increase the service level for buyers, aimed at achieving a balance between supply and demand. When most agents have a low valuation of the product or service, the platform needs to offer buyers a large price cut to increase their demand, but a slight decrease in sellers' earnings can dampen the supply. As a result, the total commission from the transaction decreases with the relative growth potential. Conversely, when most agents highly value the product or service, providing buyers with a modest price reduction is sufficient to encourage their participation, and the platform can substantially reduce sellers' earnings without significantly impacting their supply. As a result, the total commission charged from the transaction increases with the relative growth potential between sellers and buyers.

6. Conclusion

In this study, we consider a two-sided platform that facilitates transactions between buyers and sellers with heterogeneous growth potentials. The compatibility between buyer and seller types is captured by a bipartite graph, which is not necessarily complete. The platform sets the commissions to maximize its T-period profit. To address the complexity of the platform's profit optimization problem, we consider the long-run average problem (AVG) as a benchmark and propose an algorithm called TRP with a provable performance guarantee: We show that boosting the growth of the agent type with the lowest population ratio compared with the long-run average benchmark each period leads to a profit loss bounded by a constant that is independent of T.

Furthermore, we delve into the optimal steady state obtained via AVG and explore how the growth potentials of agents and network structure influence the agents' income/payment in the market and the platform's profit. We begin by introducing a set of metrics designed to capture the growth potentials of agents. Based on it, we show that a balanced network, in which sellers with relatively high (low) growth potentials trade with buyers with relatively high (low) growth potentials, results in maximum profitability, while the degree of imbalance in the network establishes a lower bound for the platform's optimal profit (relative to that under the complete graph). We then show that buyer (seller) types compatible with higher sellers' (buyers') growth potentials experience lower payments (higher income). A sensitivity analysis demonstrates the impact of agent type's long-run growth potential on income/payment. Finally, the commission charged by the platform in a submarket depends on the relative growth potentials from the two sides of the market.

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Online Appendix

A. Additional Results and Proof in Section 3

We first present some additional results in Appendix A.1. We provide some Auxiliary Results used to prove the results in Section 3 in Appendix A.2 and we prove the results in Section 3 in Appendix A.3.

A.1. Additional Results in Section 3

Proposition 5. **(existence and uniqueness of equilibrium)** *For any $t \in \{1,...,T\}$, given a commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{N_s} \times \mathbb{R}^{N_b}$ and the total mass of agents $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}_+^{N_s} \times \mathbb{R}_+^{N_b}$,*

(i) a competitive equilibrium $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ always exists;

(ii) all competitive equilibria share the same supply-demand vector $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$, and they share the same prices $p_i(t)$ for $0 < q_i^s(t) < s_i(t)$.

Lemma 2. **(commissions for feasible transactions)** *For any $t \in \{1,...,T\}$, given any positive population vector $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ and non-negative trading vector $(\boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ that satisfy (i) the flow conservation conditions in (2c)-(2e) and (ii) $\boldsymbol{q}^s(t) \le \boldsymbol{s}(t)$ and $\boldsymbol{q}^b(t) \le \boldsymbol{b}(t)$, a commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ supports $(\boldsymbol{s}(t), \boldsymbol{b}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ in a competitive equilibrium if there exists a price vector $\boldsymbol{p}(t) \in \mathbb{R}^{N_s}$ that satisfies the following system of linear inequalities:*

$$p_i(t) - r_i^s(t) \; = \; F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big), \qquad \forall i : q_i^s(t) \; > \; 0, \tag{12a}$$

$$p_i(t) - r_i^s(t) \; \le \; F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big), \qquad \forall i : q_i^s(t) \; = \; 0, \tag{12b}$$

$$p_i(t) + r_j^b(t) \; = \; F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big), \qquad \forall (i,j) : x_{ij}(t) \; > \; 0, \tag{12c}$$

$$p_i(t) + r_j^b(t) \; \ge \; F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big), \qquad \forall (i,j) : x_{ij}(t) \; = \; 0. \tag{12d}$$

Consider the following convex optimization problem:

$$\mathcal{R}(T) \; = \; \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{x},\boldsymbol{q^s},\boldsymbol{q^b}} \quad \sum_{t=1}^{T} \left[\sum_{j \in \mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big) q_j^b(t) - \sum_{i \in \mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big) q_i^s(t) \right] \tag{13a}$$

$$\begin{aligned}
\text{s.t.} \quad & q_i^s(t) \le s_i(t), && \forall i \in \mathcal{S}, t \in \{1,...,T\}, && (13b)\\
& q_j^b(t) \le b_j(t), && \forall j \in \mathcal{B},\ t \in \{1,...,T\}, && (13c)\\
& \sum_{j' : (i,j') \in E} x_{i,j'}(t) = q_i^s(t), && \forall i \in \mathcal{S},\ t \in \{1,...,T\}, && (13d)\\
& q_j^b(t) = \sum_{i' : (i',j) \in E} x_{i',j}(t), && \forall j \in \mathcal{B},\ t \in \{1,...,T\}, && (13e)\\
& x_{ij}(t) \ge 0, && \forall (i,j) \in E,\ t \in \{1,...,T\}, && (13f)\\
& s_i(t+1) \le \mathcal{G}_i^s(s_i(t), q_i^s(t)), && \forall i \in \mathcal{S},\ t \in \{1,...,T-1\}, && (13g)\\
& b_j(t+1) \le \mathcal{G}_j^b(b_j(t), q_j^b(t)), && \forall j \in \mathcal{B},\ t \in \{1,...,T-1\}. && (13h)
\end{aligned}$$

From Problem (13), we can establish Proposition 6, which enables us to solve a concave maximization problem to obtain the optimal solution $(\boldsymbol{s}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{q^s}, \boldsymbol{q^b})$ to Problem (13), from which we can further establish the optimal commission profile $(\boldsymbol{r}^s, \boldsymbol{r}^b)$ by solving a set of linear inequalities in (12) of Lemma 2.

Proposition 6. **(tightness of relaxation)** *For any $T \geq 1$, Problem (13) is a tight relaxation of Problem (3): $\mathcal{R}^*(T) = \mathcal{R}(T)$ and any optimal solution $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}})$ to Problem (13) is also optimal to Problem (3).*

A.2. Auxiliary Results for Section 3

Lemmas 3 - 5 are needed to prove Proposition 5. In Lemma 5, we establish the connection between the equilibrium and the optimal solution to an optimization problem in (15). Before that, we establish some properties for the optimization problem in Lemma 3. We also establish the existence of the optimal solution to this optimization problem in Lemma 4, and show that it is essentially unique. These lemmas enable us to establish the existence and uniqueness of the competitive equilibrium in Definition 1. The proof of Auxiliary Results follows a similar argument as the proof of Proposition EC.1 and Proposition 9 in Birge et al. (2021). Therefore, we omit the detail of the proof of auxiliary results for simplicity.

For simplicity of notation, we first define that

$$W^t_{b_j}(q^b_j(t)) \;:=\; \int_0^{q^b_j(t)} F^{-1}_{b_j}\Big(1 - \frac{z}{b_j(t)}\Big) dz - r^b_j(t) q^b_j(t), \tag{14a}$$

$$W^t_{s_i}(q^s_i(t)) \;:=\; -\int_0^{q^s_i(t)} F^{-1}_{s_i}\Big(\frac{z}{s_i(t)}\Big) dz - r^s_i(t) q^s_i(t). \tag{14b}$$

Note that the sum of $W^t_{b_j}(q^b_j(t))$ and $W^t_{s_i}(q^s_i(t))$ can be viewed as the total surplus of buyers and sellers trading in the platform, and is the objective function in Problem (15). Let $W^{t'}_{b_j}(q)$ be the derivative of $W^t_{b_j}(q)$ at $q = q^b_j(t)$ for any $0 < q^b_j(t) < b_j(t)$, and abusing some notation, $W^{t'}_{b_j}(0) = \lim_{q^b_j(t)\downarrow 0} W^t_{b_j}(q^b_j(t))$ and $W^{t'}_{b_j}(b_j(t)) = \lim_{q^b_j(t)\uparrow b_j(t)} W^t_{b_j}(q^b_j(t))$ given Assumption 2(i). Similarly, we let $W^{t'}_{s_i}(q)$ be the derivative of $W^t_{s_i}(q)$ at $q = q^s_i(t)$ for any $0 < q^s_i(t) < s_i(t)$, and we let $W^{t'}_{s_i}(0) = \lim_{q^s_i(t)\downarrow 0} W^t_{s_i}(q^s_i(t))$ and $W^{t'}_{s_i}(s_i(t)) = \lim_{q^s_i(t)\uparrow s_i(t)} W^t_{s_i}(q^s_i(t))$ given Assumption 2(i). We consider the following properties of functions $W^t_{b_j}(q^b_j(t))$ and $W^t_{s_i}(q^s_i(t))$.

Lemma 3. *For any $j \in \mathcal{B}$, $i \in \mathcal{S}$ and $t \in \{1, \ldots, T\}$,*

(i) $W^t_{b_j}(q)$ is continuously differentiable and strictly concave in $q \in (0, b_j(t))$; moreover, both $W^t_{b_j}(q)$ and $W^{t'}_{b_j}(q)$ are right continuous at $q = 0$ and left continuous $q = b_j(t)$.

(ii) $W^t_{s_i}(q)$ is continuously differentiable and strictly concave in $q \in (0, s_i(t))$; moreover, both $W^t_{s_i}(q)$ and $W^{t'}_{s_i}(q)$ are right continuous at $q = 0$ and left continuous $q = s_i(t)$.

For any $t \in \{1, ..., T\}$, we proceed to consider the following optimization problem:

$$W(t) = \max_{\boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)} \sum_{i \in \mathcal{B}} \left(\int_0^{q^b_j(t)} F^{-1}_{b_j}\left(1 - \frac{z}{b_j(t)}\right) dz - r^b_j(t) q^b_j(t) \right)$$
$$- \sum_{i \in \mathcal{S}} \left(\int_0^{q^s_i(t)} F^{-1}_{s_i}\left(\frac{z}{s_i(t)}\right) dz + r^s_i(t) q^s_i(t) \right) \tag{15a}$$

$$\text{s.t. } q^b_j(t) = \sum_{i' : (i', j) \in E} x_{i', j}(t), \qquad \forall j \in \mathcal{B}, \tag{15b}$$

$$\sum_{j' : (i, j') \in E} x_{i, j'}(t) = q^s_i(t), \qquad \forall i \in \mathcal{S}, \tag{15c}$$

$$q^b_j(t) \leq b_j(t), \qquad \forall j \in \mathcal{B}, \tag{15d}$$

$$q_i^s(t) \le s_i(t), \qquad \forall i \in \mathcal{S}, \tag{15e}$$

$$x_{ij}(t) \ge 0, \qquad \forall (i,j) \in E. \tag{15f}$$

From Problem (15), we establish the result below. Before that, we define the notation "$a \le 0 \perp b \ge 0$" as $a \le 0, b \ge 0, ab = 0$.

LEMMA 4. *(i) There exists an optimal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ to Problem (15).*

(ii) Given any optimal primal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$, there exists a dual multiplier vector $(\boldsymbol{\theta^b}(t), \boldsymbol{\theta^s}(t), \boldsymbol{\eta}^b(t), \boldsymbol{\eta}^s(t), \boldsymbol{\pi}(t))$ associated with constraints (15b)-(15f) that satisfy the KKT conditions below:

$$F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right) - r_j^b(t) - \theta_j^b(t) - \eta_j^b(t) = 0, \qquad \forall j \in \mathcal{B}, \tag{16a}$$

$$F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right) + r_i^s(t) - \theta_i^s(t) + \eta_i^s(t) = 0, \qquad \forall i \in \mathcal{S}, \tag{16b}$$

$$\theta_j^b(t) - \theta_i^s(t) + \pi_{ij}(t) = 0, \qquad \forall (i,j) \in E, \tag{16c}$$

$$q_j^b(t) - b_j(t) \le 0 \perp \eta_j^b(t) \ge 0, \qquad \forall j \in \mathcal{B}, \tag{16d}$$

$$q_i^s(t) - s_i(t) \le 0 \perp \eta_i^s(t) \ge 0, \qquad \forall i \in \mathcal{S}, \tag{16e}$$

$$x_{ij}(t) \ge 0 \perp \pi_{ij}(t) \ge 0, \qquad \forall (i,j) \in E, \tag{16f}$$

$$q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B}, \tag{16g}$$

$$q_i^s(t) = \sum_{j':(i,j')\in E} x_{i,j'}(t), \qquad \forall i \in \mathcal{S}. \tag{16h}$$

In addition, these KKT conditions in (16) are necessary and sufficient conditions for the optimality of solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$.

(iii) All primal optimal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ share the same vector $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$;

(iv) The dual solution $\theta_i^s(t)$ for $i \in \{i' : 0 < q_{i'}^s < s_{i'}\}$ that satisfies (16) is unique.

The conditions in Lemma 5(i)-(ii) are sufficient and necessary conditions, while those in Lemma 5(iii) are only sufficient conditions for equilibrium, as the prices for type $i \in \{i' : q_{i'}^s(t) = 0$ or $q_{i'}^s(t) = s_{i'}(t)\}$ are not necessarily unique.

LEMMA 5. *In each period $t \in \{1, ..., T\}$, given any commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{B}|}$ and population vector $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{B}|}$,*

(i) $(\boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ satisfies the equilibrium conditions in Definition 1 if and only if it is an optimal solution to Problem (15);

(ii) for $i \in \{i' : 0 < q_{i'}^s(t) < s_{i'}(t)\}$, $p_i(t)$ satisfies the equilibrium conditions in Definition 1 if and only if

$$p_i(t) = \theta_i^s(t). \tag{17a}$$

(iii) for $i \in \{i' : q_{i'}^s(t) = 0$ or $q_{i'}^s(t) = s_{i'}(t)\}$, $p_i(t)$ satisfies the equilibrium conditions in Definition 1 if

$$p_i(t) = \theta_i^s(t). \tag{17b}$$

Before proceeding, note that functions $F_{s_i}^{-1}(\cdot)$ and $F_{b_j}^{-1}(\cdot)$ have the following properties in an equilibrium:

(1) On the seller side, if $p_i(t) - r_i^s(t) \le 0$, then $q_i^s(t) = 0$ and

$$F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big) \;\ge\; p_i(t) - r_i^s(t), \tag{18a}$$

if $0 < p_i(t) - r_i^s(t) < \overline{v}_{s_i}$, then $0 < q_i^s(t) < s_i(t)$ and

$$F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big) \;=\; p_i(t) - r_i^s(t), \tag{18b}$$

if $\overline{v}_{s_i} \le p_i(t) - r_i^s(t)$, then $q_i^s(t) = s_i(t)$ and

$$F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big) \;\le\; p_i(t) - r_i^s(t). \tag{18c}$$

(2) On the buyer side, if $\min_{i':(i',j)\in E}\{p_{i'}(t) + r_j^b(t)\} \le 0$, then $q_j^b(t) = b_j(t)$ and

$$F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big) \;\ge\; \min_{i':(i',j)\in E}\{p_{i'}(t) + r_j^b(t)\}, \tag{19a}$$

if $0 < \min_{i':(i',j)\in E}\{p_{i'}(t) + r_j^b(t)\} < \overline{v}_{b_j}$, then $0 < q_j^b(t) < b_j(t)$ and

$$F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big) \;=\; \min_{i':(i',j)\in E}\{p_{i'}(t) + r_j^b(t)\}, \tag{19b}$$

if $\min_{i':(i',j)\in E}\{p_{i'}(t) + r_j^b(t)\} \;\ge\; \overline{v}_{b_j}$, then $q_j^b(t) = 0$ and

$$F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big) \le \min_{i':(i',j)\in E}\{p_{i'}(t) + r_j^b(t)\}. \tag{19c}$$

A.3. Proof of Results for Section 3

Based on Lemmas 3 - 5, Proposition 5 is proved as below:

Proof of Proposition 5. We establish the following two claims of this result.

Claim (i). Lemma 4(i) implies that the optimal primal solution to (15) always exists, and Lemma 5(i) implies that the $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is the equilibrium if and only if it is the optimal primal solution to (15). Therefore, the equilibrium transaction vector $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ exists.

Lemma 4(ii) implies that the optimal dual solution to (15) always exists, and Lemma 5(ii) implies that $\boldsymbol{p}$ that satisfies the equality in (17) is the equilibrium price vector. Therefore, there exists a corresponding equilibrium price vector.

Claim (ii). Lemma 4(iii) implies that the optimal primal solution $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ to (15) is unique. Lemma 5(i) implies that the $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is the equilibrium if and only if it is the optimal primal solution to (15). Therefore, the equilibrium supply-demand vector $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is unique.

Lemma 4(iv) implies that the optimal dual solution $\boldsymbol{\theta}^s$ to Problem (15) is unique for $i \in \{i' : 0 < q_{i'}^s < s_{i'}\}$, and Lemma 5(ii) implies that $p_i(t) = \theta_i^s(t)$ for i that satisfies $0 < q_i^s(t) < s_i(t)$. Therefore, the equilibrium price is unique for i that satisfies $0 < q_i^s(t) < s_i(t)$. ■

Proof of Lemma 2. We establish the sufficiency of (12) in Step 1 and construct a feasible commission profile in Step 2 to show that the feasible commission profile always exists.

Step 1: Sufficiency. We show that for any $(\boldsymbol{q}^b(t), \boldsymbol{q}^s(t), \boldsymbol{x}(t))$ that satisfies (2c)-(2e), if vector $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ satisfies the conditions in (12), then it satisfies the conditions in Definition 1.

We first verify the conditions in Definition 1, in which (2c)-(2e) immediately follow from our conditions.

(2a) We consider the following two cases:

When $q_i^s(t)>0,\ s_i(t)F_{s_i}(p_i(t)-r_i^s(t)) \overset{(a)}{=} s_i(t)F_{s_i}(F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})) = q_i^s(t)$, (a) follows from (12a).

When $q_i^s(t)=0,\ 0\le s_i(t)F_{s_i}(p_i(t)-r_i^s(t)) \overset{(b)}{\le} s_i(t)F_{s_i}(F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})) = q_i^s(t)=0$, (b) follows from (12b).This implies that the inequalities are all tight, then $s_i(t)F_{s_i}(p_i(t)-r_i^s(t))=q_i^s(t)$.

(2b) We consider the following two cases:

When $q_j^b(t)=0$, then $x_{ij}(t)=0$ for $\forall i:(i,j)\in E$, then $0\le b_j(t)\Big(1-F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t))\Big) \overset{(c)}{\le} b_j(t)\Big(1-F_{b_j}(F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)}))\Big)=q_j^b(t)=0$, where (c) follows from (12d).This implies that the inequalities are all tight, then $b_j(t)\Big(1-F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t))\Big)=q_j^b(t)$.

When $q_j^b(t)>0$, pick a i_1 such that $x_{i_1 j}(t)>0$ we have $p_{i_1}(t)=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ based on (12c); if there exists any i_2 such that $x_{i_2 j}(t)=0$, we have $p_{i_2}(t)\ge F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ based on (12d); then $\min_{i':(i',j)\in E}\{p_{i'}(t)\}=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$, then $b_j(t)\Big(1-F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t))\Big)=b_j(t)\Big(1-F_{b_j}(F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)}))\Big)=q_j^b(t)$.

(2f) We consider two cases: When $q_j^b(t)=0$, then $x_{ij}(t)=0$ for $\forall i:(i,j)\in E$. When $q_j^b(t)>0$, we show in proof of (2b) that $p_i(t)\ge \min_{i':(i',j)\in E}\{p_{i'}\}=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ for $x_{ij}(t)=0$.

Step 2: construct an instance. In each period, given $(\boldsymbol{q}^b(t),\boldsymbol{q}^s(t),\boldsymbol{x}(t))$ that satisfies (2c)-(2e), consider the following one-period problem:

$$
\begin{aligned}
\widetilde{R}_t = \max_{\boldsymbol{q^s},\boldsymbol{q^b},\boldsymbol{x}} & \Big[\sum_{j\in\mathcal{B}} q_j^b + \sum_{i\in\mathcal{S}} q_i^s\Big] && \\
\text{s.t. } & q_j^b \le q_j^b(t), && \forall j\in\mathcal{B} && \text{(20a)}\\
& q_i^s \le q_i^s(t), && \forall i\in\mathcal{S} && \text{(20b)}\\
& \sum_{j':(i,j')\in E} x_{i,j'} = q_i^s, && \forall i\in\mathcal{S} && \text{(20c)}\\
& q_j^b = \sum_{i':(i',j)\in E} x_{i',j}, && \forall j\in\mathcal{B} && \text{(20d)}\\
& x_{ij}\ge 0, && \forall (i,j)\in E. && \text{(20e)}
\end{aligned}
$$

Note that the feasible solution set is not empty, as $q_j^b=q_j^b(t)$ for $\forall j\in\mathcal{B}$, $q_i^s=q_i^s(t)$ for $\forall i\in\mathcal{S}$ and $x_{ij}=x_{ij}(t)$ for $\forall(i,j)\in E$ is a feasible solution. Since the constraints are all linear, the KKT conditions are necessary for the optimal solution in (20). Let $(\omega_i^s(t),\omega_j^b(t),\pi_{ij}(t))$ be the Lagrange multipliers corresponding to the constraint in (20c)-(20e), then we can write down the KKT conditions corresponding to $\boldsymbol{x}$:

$$
\begin{aligned}
& \omega_i^s(t)-\omega_j^b(t)-\pi_{ij}(t)=0, && \forall (i,j)\in E, && \text{(21a)}\\
& x_{ij}(t)\ge 0 \perp \pi_{ij}(t)\ge 0, && \forall i\in\mathcal{S}, \forall (i,j)\in E. && \text{(21b)}
\end{aligned}
$$

Then we consider the commission and equilibrium price as follows:

$$
\begin{aligned}
& p_i(t)=\omega_i^s(t), && \forall i\in\mathcal{S}, && \text{(22a)}\\
& r_i^s(t)=\omega_i^s(t)-F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right), && \forall i\in\mathcal{S}, && \text{(22b)}
\end{aligned}
$$

$$r_j^b(t) = F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right) - \omega_j^b(t), \qquad \forall j \in \mathcal{B}. \tag{22c}$$

then the conditions (12a)-(12b) immediately follow. For (12c),

$$p_i(t) + r_j^b(t) = \omega_i^s(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(a)}{=} \omega_j^b(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) = F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}).$$

where (a) follows from (21a) and (21b) that $\pi_{ij}(t) = 0$ when $x_{ij}(t) \geq 0$.

For (12d),

$$p_i(t) + r_j^b(t) = \omega_i^s(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(b)}{=} \omega_j^b(t) + \pi_{ij}(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(c)}{\geq} F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}).$$

where (b) follows from (21a) and (c) follows from (21b). In summary, (12) holds for our construction in (22).■

Proof of Proposition 6 We need to prove that the optimal solutions to (3) exist and that they achieve an objective value of $\mathcal{R}^* = \mathcal{R}$. We first show that $\mathcal{R}^* \leq \mathcal{R}$ in step 1, and construct a solution to (3) whose value equals to $\mathcal{R}$ in step 2, which implies that $\mathcal{R}^* = \mathcal{R}$ and the solution is optimal.

Step 1: Establish that $\mathcal{R}^* \leq \mathcal{R}$. We show that any feasible solution to (3) is feasible in Problem (13) in Step 1.1, and we further show that it leads to a higher objective value in Problem (13) in Step 1.2.

Step 1.1: Any feasible solution in (3) is feasible in (13). To prove the claim, it is sufficient to verify the constraints (13b)-(13c), as other constraints immediately follow from the constraints in (3).

Based on (2a) and (2b), we have $q_i^s(t) = s_i(t)F_{s_i}(p_i(t) - r_i^s(t)) \leq s_i(t)$ as $F_{s_i}(p_i(t) - r_i^s(t)) \in [0,1]$ and $q_j^b(t) = b_j(t)[1 - F_{b_j}(\min_{i:(i,j)\in E}\{p_i(t)\} + r_j^b(t))] \leq b_j(t)$ as $F_{b_j}(\min_{i:(i,j)\in E}\{p_i(t)\} + r_j^b(t)) \in [0,1]$. Therefore, the constraints (13b)-(13c) are satisfied.

Step 1.2: Any feasible solution in (3) results in a higher objective value in (13). We first show that the optimal solution to Problem (3) satisfies the following:

$$\left(F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right)\right)q_i^s(t) \leq \left(p_i(t) - r_i^s(t)\right)q_i^s(t), \qquad \forall i \in \mathcal{S}, t \in \{1,...,T\}, \tag{23a}$$

$$\left(F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right)\right)q_j^b(t) \geq \left(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t)\right)q_j^b(t), \qquad \forall j \in \mathcal{B}, t \in \{1,...,T\}. \tag{23b}$$

For (23a), when $q_i^s(t) = 0$, (23a) immediately holds; when $q_i^s(t) > 0$, (23a) follows from (18b) and (18c) in the proof of Lemma 5. For (23b), when $q_j^b(t) = 0$, (23b) immediately holds; when $q_j^b(t) > 0$, (23b) follows from (19a) and (19b) in the proof of Lemma 5.

Given (23), the objective function in (3a) satisfies the following:

$$\begin{aligned}
\mathcal{R}^* &= \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} r_j^b(t)q_j^b(t) + \sum_{i\in\mathcal{S}} r_i^s(t)q_i^s(t)\Big] \\
&\overset{(a)}{=} \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} r_j^b(t) \sum_{i':(i',j)\in E} x_{i'j}(t) + \sum_{i\in\mathcal{S}} r_i^s(t) \sum_{j':(i,j')\in E} x_{ij'}(t)\Big] \\
&= \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} \sum_{i':(i',j)\in E} \Big(p_{i'}(t) + r_j^b(t)\Big)x_{i'j}(t) - \sum_{i\in\mathcal{S}} \Big(p_i(t) - r_i^s(t)\Big) \sum_{j':(i,j')\in E} x_{ij'}(t)\Big]
\end{aligned}$$

$$\overset{(b)}{=}\sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t)\Big)\sum_{i':(i',j)\in E}x_{i'j}(t)-\sum_{i\in\mathcal{S}}\Big(p_i(t)-r_i^s(t)\Big)\sum_{j':(i,j')\in E}x_{ij'}(t)\Big]$$

$$=\sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t)\Big)q_j^b(t)-\sum_{i\in\mathcal{S}}\Big(p_i(t)-r_i^s(t)\Big)q_i^s(t)\Big]$$

$$\overset{(c)}{\leq}\sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}}F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t)-\sum_{i\in\mathcal{S}}F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\Big]=\mathcal{R},$$

where (a) follows from (2c)-(2d); (b) follows from (2f) that $x_{ij}=0$ for $i\notin \underset{i':(i',j)\in E}{argmin}\{p_i+r_i^s\}$; (c) follows from (23).

Step 2: Establish that $\mathcal{R}^*=\mathcal{R}$. Given any feasible solution to (13), we construct a feasible solution for (3) in Step 2.1, and we further obtain an objective value that equals $\mathcal{R}$ in Step 2.2.

Step 2.1: Construct a feasible solution for Problem (3).

In each period, given the solution for Problem (13), we consider the construction from (22) as in the proof of Lemma 2. We need to verify that all the constraints in (3) hold. Notice that we only need to verify that (2a) (2b) (2f) (3c) and (3d) hold, as other constraints exist in (13) and automatically hold.

(2a) from the construction of $p_i(t)$ and $r_i^s(t)$, we can establish that

$$s_i(t)F_{s_i}(p_i(t)-r_i^s(t))=s_i(t)F_{s_i}\Big(F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)\Big)=q_i^s(t).$$

(2b) We consider the following two cases:

(i) if $q_j^b>0$, we pick a i' such that $(i',j)\in E$, then there are two further cases: (1) $x_{i'j}>0$, then $p_{i'}(t)\overset{(a)}{=}\omega_{i'}^s(t)\overset{(b)}{=}\omega_j^b(t)+\pi_{i'j}(t)\overset{(c)}{=}\omega_j^b(t)$, where (a) follows form the construction of $p_{i'}(t)$; (b) follows from (21a); (c) follows from (21b) for $x_{i'j}>0$; (2) $x_{i'j}=0$, then $p_{i'}(t)=\omega_{i'}^s(t)=\omega_j^b(t)+\pi_{i'j}(t)\overset{(d)}{\geq}\omega_j^b(t)$, where (d) follows from (21b) for $x_{i'j}=0$. In summary, $\min_{i':(i',j)\in E}\{p_{i'}(t)\}=\omega_j^b(t)$, then

$$b_j(t)[1-F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t))]=b_j(t)[1-F_{b_j}(\omega_j^b(t)+r_j^b(t))]\overset{(e)}{=}b_j(t)[1-F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})]=q_j^b(t),$$

where (e) follows from the construction of $r_j^b(t)$;

(ii) if $q_j^b=0$, we have $p_{i'}(t)=\omega_{i'}^s(t)=\omega_j^b(t)+\pi_{i'j}(t)\geq\omega_j^b(t)$, then $0\overset{(f)}{\leq}b_j(t)[1-F_{b_j}(\min\{p_i(t)+r_j^b(t)\})]\leq b_j(t)[1-F_{b_j}(\omega_j^b(t)+r_j^b(t))]\overset{(g)}{=}b_j(t)[1-F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})]=q_j^b(t)=0,$, where (f) follows from $F_{b_j}(\cdot)\leq1$, (g) follows from the construction of $r_j^b(t)$. This implies that inequality must be tight. Therefore, (2b) holds.

(2f) We have verified in the proof of (2b) that for any $(i,j)\in E$, we have $p_i=\omega_j^b$ for $x_{ij}>0$ and $p_i\geq\omega_j^b$ for $x_{ij}=0$. Therefore, $x_{ij}=0$ for $i\notin\underset{i':(i',j)\in E}{\arg\min}\,p_{i'}$.

(3c) We first prove (13g) holds as equality by contradiction. Suppose that $s_i(t+1)<\mathcal{G}_i^s(s_i(t),q_i^s(t))$ in the optimal solution to (13), then let $s_i'(t+1)=\mathcal{G}_i^s(s_i(t),q_i^s(t))$, we can obtain higher objective value by replacing the $s_i(t+1)$ in the optimal solution with $s_i'(t+1)$ as (13a) increases in $s_i(t+1)$; in addition, $s_i(t+2)\leq\mathcal{G}_i^s(s_i(t+1),q_i^s(t+1))<\mathcal{G}_i^s(s_i'(t+1),q_i^s(t+1))$, which implies that the constraint in (13g) still hold. This contradicts to our assumption that $s_i(t+1)<\mathcal{G}_i^s(s_i(t),q_i^s(t))$ is the optimal solution to (13). Therefore, $s_i(t+1)=\mathcal{G}_i^s(s_i(t),q_i^s(t))$ in the optimal solution to (13), and (3c) immediately holds.

(3d) follows the same argument in (3c).

Step 2.2: Obtain a value that equals $\mathcal{R}$. We can deduce that

$$\begin{aligned}
\mathcal{R}^* &= \sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}} r_i^s(t)q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t)q_j^b(t)\right] \\
&\overset{(a)}{=} \sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}}(\omega_i^s(t) - F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)}))q_i^s(t) + \sum_{j\in\mathcal{B}}(F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t))q_j^b(t)\right] \\
&\overset{(b)}{=} \sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right] \\
&\quad + \sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}}\omega_i^s(t)\sum_{j':(i,j')\in E} x_{ij'}(t) - \sum_{j\in\mathcal{B}}\omega_j^b(t)\sum_{i':(i',j)\in E} x_{i'j}(t)\right] \\
&= \sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right] + \sum_{t=1}^{T}\left[\sum_{(i,j)\in E}\Big(\omega_i^s(t) - \omega_j^b(t)\Big)x_{ij}(t)\right] \\
&\overset{(c)}{=} \sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right] = \mathcal{R},
\end{aligned}$$

where (a) follows from the construction of $r_i^s(t)$ and $r_j^b(t)$, (b) follows from (13d) and (13e), (c) follows from (21a) and (21b) that when $x_{ij} > 0$, $\omega_i^s = \omega_j^b$, while when $x_{ij} = 0$, $\omega_i^s \geq \omega_j^b$. ∎

B. Proof of Results in Section 4

We provide and prove some auxiliary results in Appendix B.1 and prove the result in Section 4 in Appendix B.2.

B.1. Auxiliary Results for Section 4

Given the definitions of the value functions $\widetilde{F}_{b_j}$ for any $j\in\mathcal{B}$ and $\widetilde{F}_{s_i}$ for any $i\in\mathcal{S}$ from Problem (5), we have the following lemma.

LEMMA 6. *$\widetilde{F}_{b_j}(q,b)$ is continuous at $(0,0)$ for $i\in\mathcal{S}$ and $\widetilde{F}_{s_i}(q,s)$ is continuous at $(0,0)$ for $j\in\mathcal{B}$.*

Proof of Lemma 6. We need to show that $\lim_{(q,b)\downarrow(0,0)}\widetilde{F}_{b_j}(q,b) = \widetilde{F}_{b_j}(0,0) = 0$ and $\lim_{(q,s)\downarrow(0,0)}\widetilde{F}_{s_i}(q,s) = \widetilde{F}_{s_i}(0,0) = 0$, which holds because

$$\begin{aligned}
0 \;\leq\; \lim_{(q,b)\downarrow(0,0)}\widetilde{F}_{b_j}(q,b) \;&=\; \lim_{(q,b)\downarrow(0,0)} F_{b_j}^{-1}\Big(1-\frac{q}{b}\Big)q \;\leq\; \overline{v}_{b_j}\times 0 \;=\; 0, \\
0 \;\leq\; \lim_{(q,s)\downarrow(0,0)}\widetilde{F}_{s_i}(q,s) \;&=\; \lim_{(q,s)\downarrow(0,0)} F_{s_i}^{-1}\Big(\frac{q}{s}\Big)q \;\leq\; \overline{v}_{s_i}\times 0 \;=\; 0,
\end{aligned}$$

where given Assumption 2, all of the inequalities above follow from $F_{b_j}^{-1}(x)\in[0,\overline{v}_{b_j}]$ for $x\in[0,1]$ where $\overline{v}_{b_j} < \infty$ and $F_{s_i}^{-1}(x)\in[0,\overline{v}_{s_i}]$ for $x\in[0,1]$ where $\overline{v}_{s_i} < \infty$. ∎

We next develop an auxiliary result about the growth of populations. To simplify the notation, we let $\mathcal{N} := \{1,...,|\mathcal{S}|,|\mathcal{S}|+1,...,|\mathcal{S}|+|\mathcal{B}|\}$, where the first $|\mathcal{S}|$ nodes represent the types from the seller side and the last $|\mathcal{B}|$ nodes represent the types from the buyer side. In addition, we use $n_i(t)$ and $q_i(t)$ to respectively denote the population and transaction quantity of type $i\in\mathcal{N}$ at time $t\in\{1,\dots,T\}$. We define $\mathcal{G}_i(\cdot,\cdot) := \mathcal{G}_i^s(\cdot,\cdot)$ for $i\in\{1,...,|\mathcal{S}|\}$ and $\mathcal{G}_i(\cdot,\cdot) := \mathcal{G}_{i-|\mathcal{S}|}^b(\cdot,\cdot)$ for $i\in\{|\mathcal{S}|+1,...,|\mathcal{S}|+|\mathcal{B}|\}$. In addition, we define $\mathcal{N}^+ := \{i\in\mathcal{N} : \overline{n}_i > 0\}$.

Recall that

$$m(t) = \min_{i\in\mathcal{N}^+} \frac{n_i(t)}{\overline{n}_i}. \tag{24}$$

Given the minimum population ratio $m(t)$ in (24), we let $l(t)$ be the agent type with the lowest population ratio at time t or "the lowest node at time t" for short:

$$l(t) := \arg\min_{i\in\mathcal{N}^+} \frac{n_i(t)}{\overline{n}_i}. \tag{25}$$

If there is more than one i such that $\frac{n_i(t)}{\overline{n}_i} = m(t)$, we can set $l(t)$ as any node with the minimum population ratio. After the population evolves in period t, it is worth noting that the lowest node can change. Let $\tau_0 := 0$ and $m(\tau_0)$ be a dummy agent type with the minimum ratio in period 0 with $m(\tau_0) \notin \mathcal{S}\cup\mathcal{B}$. Moreover, we let X be the total number of times that the lowest node changes in Algorithm 1 for some $X \in \{1,\dots,T\}$. we let $\tau_x := \min\{t : t > \tau_{x-1}, l(t) \neq l(\tau_{x-1})\}$ for $t \in \{1,\dots,T\}$, in which τ_x is the x^{th} time that the lowest node changes for $x \in \{1,\dots,X\}$. For example, for $x \in \{0,1,...,X\}$, if node i has the lowest ratio at time $\tau_x - 1$, then $n_{l(\tau_x-1)}(\tau_x)$ denotes the population ratio of the node i at time τ_x.

Given the lowest node $l(t) \in \mathcal{S}\cup\mathcal{B}$ we let

$$g_t(n) := \mathcal{G}_{l(t)}\left(n, n\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}}\right), \tag{26}$$

where $n \geq 0$. Then $g_t(n)$ is the transition equation for the lowest node in period t, as by the population transition specified in Algorithm 1 and the definition of $g_t(\cdot)$, we have that

$$n_{l(t)}(t+1) = \mathcal{G}_{l(t)}\left(n_{l(t)}(t), n_{l(t)}(t)\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}}\right) = g_t(n_{l(t)}(t)). \tag{27}$$

We have the following observation about function $g_t(\cdot)$.

Lemma 7. *$g_t(n)$ is differentiable, increasing and strictly concave in $n \geq 0$. Moreover, its derivative satisfies $g_t'(\overline{n}_{l(t)}) < 1$ for all $t \in \{1,...,T\}$. Moreover, $g_t(n) - n < 0$ for $n > \overline{n}_{l(t)}$ and $g_t(n) - n > 0$ for $0 < n < \overline{n}_{l(t)}$.*

Proof of Lemma 7. We divide the proof arguments into the following components.

Differentiability and monotonicity. From Assumption 1, we have that function $\mathcal{G}_i(n,q)$ is continuously differentiable, increasing and strictly concave in $n \geq 0$, which directly implies that $g_t(n)$ is differentiable, increasing and strictly concave in $n \geq 0$.

$g_t'(\overline{n}_{l(t)}) < 1$ for all $t \in \{1,...,T\}$. By Algorithm 1, we have that $\overline{n}_{l(t)} > 0$. Since $g_t(n)$ is continuous in $n \in [0, \overline{n}_{l(t)}]$ and differentiable $(0, \overline{n}_{l(t)})$, by the mean value theorem, there exists a $\widetilde{n}_{l(t)} \in (0, \overline{n}_{l(t)})$ such that $g_t'(\widetilde{n}_{l(t)}) = \frac{g_t(\overline{n}_{l(t)}) - g_t(0)}{\overline{n}_{l(t)} - 0} \overset{(a)}{=} \frac{\overline{n}_{l(t)} - g_t(0)}{\overline{n}_{l(t)} - 0} \overset{(b)}{=} \frac{\overline{n}_{l(t)} - 0}{\overline{n}_{l(t)} - 0} = 1$, where (a) follows from Lemma 1(ii) and (b) follows from Assumption 1(i). Since $g_t(n)$ is strictly concave in $n \geq 0$, its derivative strictly decreases in $n \geq 0$, which implies that $g_t'(\overline{n}_{l(t)}) < 1$ given that $\widetilde{n}_{l(t)} \in (0, \overline{n}_{l(t)})$.

$g_t(n) - n < 0$ for $n > \overline{n}_{l(t)}$. we define that $y_t(n) := g_t(n) - n$, and it remains to show that $y_t(n) < 0$ for $n > \overline{n}_{l(t)}$. Since $y_t'(n_{l(t)}) = g_t'(n_{l(t)}) - 1 < 0$ for $n_{l(t)} > \overline{n}_{l(t)}$ and $y_t(\overline{n}_{l(t)}) = 0$ based on Lemma 1(ii), $y_t(n_{l(t)}) < 0$ for $n_{l(t)} > \overline{n}_{l(t)}$.

$g_t(n) - n > 0$ for $0 < n < \overline{n}_{l(t)}$. It remains to show that $y_t(n) > 0$ for $0 < n < \overline{n}_{l(t)}$. Note that $y_t(n)$ is concave in n. Since $y_t(0) = g_t(0) - 0 = 0$ and $y_t(\overline{n}_{l(t)}) = g_t(\overline{n}_{l(t)}) - \overline{n}_{l(t)} = 0$, we know $y_t((1-a)\times\overline{n}_{l(t)}) > a y_t(0) + (1-a) y_t(\overline{n}_{l(t)}) = 0 + 0 = 0$ for $a \in (0,1)$, therefore $y_t(n) > 0$ for $0 < n < \overline{n}_{l(t)}$. ■

Lastly, we formally define the myopic policy and establish its tractability as a supporting result for our proof arguments for Section 4.

Definition 2. **(myopic policy)** For $t \in \{1,...,T\}$, given the current population $(\boldsymbol{s}^M(t), \boldsymbol{b}^M(t))$, the myopic policy solves the following optimization problem:

$$\mathcal{R}^{M*}(t) \;=\; \max_{\boldsymbol{r}(t),\boldsymbol{p}(t),\boldsymbol{x}(t),\boldsymbol{q^s}(t),\boldsymbol{q^b}(t)} \sum_{i\in\mathcal{S}} r_i^s(t)q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t)q_j^b(t) \tag{28a}$$

$$\text{s.t.} \quad (\boldsymbol{s^M}(t), \boldsymbol{b^M}(t), \boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)) \text{ satisfies (2)}, \forall t \in \{1,...,T\}. \tag{28b}$$

To solve Problem (28), we consider the following optimization problem:

$$\mathcal{R}^{M}(t) \;=\; \max_{\boldsymbol{q}^s(t),\boldsymbol{q}^b(t),\boldsymbol{x}(t)} \sum_{j\in\mathcal{B}} F_{b_j}^{-1}\Big(1-\frac{q_j^b(t)}{b_j^M(t)}\Big)q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i^M(t)}\Big)q_i^s(t) \tag{29a}$$

$$\text{s.t.} \quad q_i^s(t) \le s_i^M(t), \sum_{j':(i,j')\in E} x_{i,j'}(t) = q_i^s(t), \qquad \forall i\in\mathcal{S}, t\in\{1,...,T\}, \tag{29b}$$

$$q_j^b(t) \le b_j^M(t),\; q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j\in\mathcal{B},\; t\in\{1,...,T\}, \tag{29c}$$

$$x_{ij}(t) \ge 0, \qquad \forall (i,j)\in E,\; t\in\{1,...,T\}. \tag{29d}$$

Recalling the observations about Problem (13), we can apply exactly the same arguments as in the proof of Proposition 6 to establish the following result about Problem (29), whose proof will be neglected for avoiding repetition:

Corollary 3. *For any $t\in\{1,...,T\}$, Problem (29) is a tight relaxation of Problem (28), i.e., $\mathcal{R}^{M*}(t) = \mathcal{R}^M(t)$ and any optimal solution $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t), \boldsymbol{x}(t))$ to Problem (29) is also optimal to Problem (28).*

B.2. Proof of Results in Section 4

Proof of Lemma 1.

Show that AVG's optimal solution and objective value are finite. On the seller side, for any $i\in\mathcal{S}$, we first show that the optimal solution $(\overline{q}_i^s, \overline{s}_i)$ is finite for all $i\in\mathcal{S}$. We first show that $\overline{s}_i$ is finite. The constraint of AVG requires that $s_i \le \mathcal{G}_i^s(s_i, q_i^s) \le \mathcal{G}_i^s(s_i, s_i)$, which requires that $\mathcal{G}_i^s(s_i,s_i) - s_i \ge 0$. Given that $\lim_{x\to\infty}((\mathcal{G}_i^s)_1'(x,x) + (\mathcal{G}_i^s)_2'(x,x)) < 1$ and $\mathcal{G}_i^s(x,x)$ is continuously differentiable in $x\ge 0$ by Assumption 1, there exists a constant $a<1$ and $\hat{s}_i > 0$ such that $(\mathcal{G}_i^s)_1'(\hat{s}_i,\hat{s}_i) + (\mathcal{G}_i^s)_2'(\hat{s}_i,\hat{s}_i) = a < 1$. Therefore, for any $s_i > \hat{s}_i$, the constraint requires that

$$\begin{aligned}\mathcal{G}_i^s(s_i,s_i) - s_i \le& \mathcal{G}_i^s(\hat{s}_i,\hat{s}_i) + (\mathcal{G}_i^s)_1'(\hat{s}_i,\hat{s}_i)(s_i-\hat{s}_i) + (\mathcal{G}_i^s)_2'(\hat{s}_i,\hat{s}_i)(s_i-\hat{s}_i) - s_i\\ =&\mathcal{G}_i^s(\hat{s}_i,\hat{s}_i) + a(s_i-\hat{s}_i) - s_i\end{aligned}$$

which indicates that for any $s_i > \max\{\hat{s}_i, \frac{\mathcal{G}_i^s(\hat{s}_i,\hat{s}_i)-a\hat{s}_i}{1-a}\}$, we have $\mathcal{G}_i^s(s_i,s_i) - s_i < 0$ and therefore is not feasible. Therefore, it is without loss of optimality to focus on the compact set $[0,\hat{s}_i]$ for the optimal solution $\overline{s}_i$. Since $q_i \le s_i$, this suggests that the optimal solution $\overline{q}_i^s \in [0,\hat{s}_i]$, which is also finite. The same arguments hold for the buyer side.

Show that optimal solution $(\overline{\boldsymbol{q}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to AVG exists. For any $u \in [0,1]$, we have that $F_{s_i}^{-1}(u) \leq \overline{v}_{s_i} < \infty$ for any $i \in \mathcal{S}$ and $F_{b_j}^{-1}(u) \leq \overline{v}_{b_j} < \infty$ for all $j \in \mathcal{B}$. Therefore, the objective value of AVG is also finite. We have already shown that the feasible set of $(\boldsymbol{q}, \boldsymbol{s}, \boldsymbol{b})$ is closed and bounded. The constraints in (5b)-(5c) also ensure that the feasible set of $\boldsymbol{x}$ is closed and bounded. In summary, the feasible set characterized by constraint (5b)-(5f) is compact. In addition, the feasible set is not empty, as solution $\mathbf{0}$ is feasible. Furthermore, the objective function in (5a) is continuous in this compact set based on Assumption 2(i). By the extreme value theorem, an optimal solution $(\overline{\boldsymbol{q}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to AVG exists.

We proceed to prove the lemma.

(i). By the extreme value theorem, the optimal solution to (5) exists. Since the objective function is strictly concave and the feasible region is a convex set, the optimal solution to (5) is unique.

(ii). If there exists a $i \in \mathcal{S}$ such that $\mathcal{G}_i^s(\overline{s}_i, \overline{q}_i^s) - \overline{s}_i > 0$, then given that $\mathcal{G}_i^s(s_i, q_i^s)$ is continuous on s_i, we can always find a $\epsilon > 0$ small enough such that $\mathcal{G}_i^s(\overline{s}_i + \epsilon, \overline{q}_i^s) - (\overline{s}_i + \epsilon) > 0$. In addition, $\overline{s}_i + \epsilon > \overline{s}_i \geq \overline{q}_i^s$. By replacing $\overline{s}_i$ with $\overline{s}_i + \epsilon$, we obtain a higher objective value since the objective function strictly increases in s_i. Therefore, the assumption $\mathcal{G}_i^s(\overline{s}_i, \overline{q}_i^s) - \overline{s}_i > 0$ contradicts the optimality of $(\overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to Problem (5). The same proof arguments can be applied to the buyer side. ■

Proof of Proposition 1. By Proposition 6, $\mathcal{R}(T) = \mathcal{R}^*(T)$. So it suffices to show that there exists a constant C_1 such that $|\mathcal{R}(T) - T\overline{\mathcal{R}}| \leq C_1$. To prove the result, we establish the following two claims.

Claim 1: $\mathcal{R}(T) - T\overline{\mathcal{R}} \geq -C_1'$. We delay the proof to Step 3 in the proof of Theorem 1 that there exists a constant C_1' and a policy π such that $\mathcal{R}^\pi(T) - T\overline{\mathcal{R}} \geq -C_1'$, which further implies that $\mathcal{R}(T) - T\overline{\mathcal{R}} \geq \mathcal{R}^\pi(T) - T\overline{\mathcal{R}} \geq -C_1'$ given that $\mathcal{R}(T) \geq \mathcal{R}^\pi(T)$.

Claim 2: $\mathcal{R}(T) - T\overline{\mathcal{R}} \leq C_1''$. Before proving the claim, we first consider the following optimization problem for any $T > 0$:

$$
\begin{aligned}
\widetilde{\mathcal{R}} \; = \; \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{q^s},\boldsymbol{q^b},\boldsymbol{x}} \quad & \sum_{j\in\mathcal{B}} \widetilde{F}_{b_j}(q_j^b, b_j) - \sum_{i\in\mathcal{S}} \widetilde{F}_{s_i}(q_i^s, s_i) && && \text{(30a)}\\
\text{s.t.} \quad & q_i^s \leq s_i, && \forall i \in \mathcal{S}, && \text{(30b)}\\
& q_j^b \leq b_j, && \forall j \in \mathcal{B}, && \text{(30c)}\\
& \sum_{j:(i,j)\in E} x_{ij} = q_i^s, && \forall i \in \mathcal{S}, && \text{(30d)}\\
& q_j^b = \sum_{i:(i,j)\in E} x_{ij}, && \forall j \in \mathcal{B}, && \text{(30e)}\\
& x_{ij} \geq 0, && \forall (i,j) \in E, && \text{(30f)}\\
& s_i \leq \mathcal{G}_i^s(s_i, q_i^s) + \frac{s_i(1)}{T}, && \forall i \in \mathcal{S}, && \text{(30g)}\\
& b_j \leq \mathcal{G}_j^b(b_j, q_j^b) + \frac{b_j(1)}{T}, && \forall j \in \mathcal{B}. && \text{(30h)}
\end{aligned}
$$

Note that the only difference between Problem (30) and Problem (5) is the right-hand side of the constraints (30g)-(30h). Given that $s_i(1) > 0$ for all $i \in \mathcal{S}$ and $b_j(1) > 0$ for all $j \in \mathcal{B}$, Problem (30) could be viewed as a relaxation of Problem (5). We first show that $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$ and then show that there exists a positive constant C_1'' such that $T\widetilde{\mathcal{R}} - T\overline{\mathcal{R}} \leq C_1''$ for any $T > 0$. Consequently, we can have $\mathcal{R}(T) - T\overline{\mathcal{R}} \leq C_1''$ for any $T > 0$.

Step 2.1: Show that $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$. For any optimal solution $(\boldsymbol{s}(t), \boldsymbol{b}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t), \boldsymbol{x}(t) : t = 1, \ldots, T)$ to Problem (13), we construct the following alternative solution vector $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}}, \overline{\boldsymbol{x}})$ for Problem (30):

$$\begin{aligned}
&\overline{s}_i = \frac{1}{T}\sum_{t=1}^{T} s_i(t) \text{ and } \overline{q}_i^s = \frac{1}{T}\sum_{t=1}^{T} q_i^s(t), && \forall i \in \mathcal{S},\\
&\overline{b}_j = \frac{1}{T}\sum_{t=1}^{T} b_j(t) \text{ and } \overline{q}_j^b = \frac{1}{T}\sum_{t=1}^{T} q_j^b(t), && \forall j \in \mathcal{B},\\
&\overline{x}_{ij} = \frac{1}{T}\sum_{t=1}^{T} x_{ij}(t), && \forall (i,j) \in E
\end{aligned}$$

We establish the feasibility of $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}}, \overline{\boldsymbol{x}})$ for Problem (30) in Step 2.1.1 and then show that $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$ in Step 2.1.2.

Step 2.1.1: Feasibility. First, from the constraints in Problem (13), we can easily show (30b) - (30f) hold. In particular, $\overline{q}_i^s = \frac{1}{T}\sum_{t=1}^{T} q_i^s(t) \overset{(a)}{\leq} \frac{1}{T}\sum_{t=1}^{T} s_i(t) = \overline{s}_i$. The same argument applies for $\overline{q}_j^b$ and $\overline{b}_j$ on the buyer side. For (30d)-(30e), $\overline{q}_i^s = \frac{1}{T}\sum_{t=1}^{T} q_i^s(t) \overset{(b)}{=} \frac{1}{T}\sum_{j':(i,j')\in E}\sum_{t=1}^{T} x_{ij'}(t) = \sum_{j':(i,j')\in E}\overline{x}_{ij}$. and $\overline{q}_j^b = \frac{1}{T}\sum_{t=1}^{T} q_j^b(t) \overset{(c)}{=} \frac{1}{T}\sum_{i':(i',j)\in E}\sum_{t=1}^{T} x_{i'j}(t) = \sum_{i':(i',j)\in E}\overline{x}_{ij}$. For (30f), $\overline{x}_{ij} = \frac{1}{T}\sum_{t=1}^{T} x_{ij}(t) \overset{(e)}{\geq} 0$.

For constraints in (30g)-(30h), we show that

$$\begin{aligned}
\overline{s}_i - \mathcal{G}_i^s(\overline{s}_i, \overline{q}_i^s) - \frac{s_i(1)}{T} &\overset{(a)}{=} \frac{1}{T}\sum_{t=1}^{T} s_i(t) - \mathcal{G}_i^s\Big(\frac{1}{T}\sum_{t=1}^{T} s_i(t), \frac{1}{T}\sum_{t=1}^{T} q_i^s(t)\Big) - \frac{s_i(1)}{T}\\
&\overset{(b)}{\leq} \frac{1}{T}\sum_{t=1}^{T}\Big[s_i(t) - \mathcal{G}_i^s(s_i(t), q_i^s(t))\Big] - \frac{s_i(1)}{T}\\
&= \frac{1}{T}\sum_{t=1}^{T-1}\Big[s_i(t+1) - \mathcal{G}_i^s(s_i(t), q_i^s(t))\Big] + \frac{1}{T}(s_i(1) - \mathcal{G}_i^s(s_i(T), q_i^s(T))) - \frac{s_i(1)}{T}\\
&\leq 0 + \frac{1}{T}\Big(-\mathcal{G}_i^s(s_i(T), q_i^s(T))\Big) \ \leq\ 0,
\end{aligned}$$

where (a) follows from the construction of $\overline{s}_i$ and $\overline{q}_i^s$ at the beginning of Step 2.1; (b) follows the Assumption 1(ii) that $\mathcal{G}_i^s(\cdot)$ is concave. This proves that Constraint (30g) holds. Following the same argument, we can show that Constraint (30h) holds.

Step 2.1.2: $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$. Given the construction of $\overline{s}_i$ and $\overline{b}_j$, we obtain that $\overline{s}_i > 0$ and $\overline{b}_j > 0$. Given the definitions of $\widetilde{F}_b(\overline{q}_j^b, \overline{b}_j)$ and $\widetilde{F}_s(\overline{q}_i^s, \overline{s}_i)$ in Problem (5), the objective value in (5a) is given by $\sum_{j\in\mathcal{B}} F_{b_j}^{-1}\big(1 - \frac{\overline{q}_j^b}{\overline{b}_j}\big)\overline{q}_j^b - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\big(\frac{\overline{q}_i^s}{\overline{s}_i}\big)\overline{q}_i^s$. This allows us to establish that

$$\begin{aligned}
T\widetilde{\mathcal{R}} &\overset{(a)}{=} T\Big[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{\frac{1}{T}\sum_{t=1}^{T} q_j^b(t)}{\frac{1}{T}\sum_{t=1}^{T} b_j(t)}\Big)\frac{1}{T}\sum_{t=1}^{T} q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\Big(\frac{\frac{1}{T}\sum_{t=1}^{T} q_i^s(t)}{\frac{1}{T}\sum_{t=1}^{T} s_i(t)}\Big)\frac{1}{T}\sum_{t=1}^{T} q_i^s(t)\Big]\\
&\overset{(b)}{\geq} T \times \frac{1}{T}\sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right) q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right) q_i^s(t)\Big] = \mathcal{R}(T).
\end{aligned}$$

where (a) follows from the construction of $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}}, \overline{\boldsymbol{x}})$ in Step 2-1; (b) follows from the concavity of $F_{b_j}^{-1}(1 - \frac{a}{b})a$ and $-F_{s_i}^{-1}(\frac{a}{b})a$ by Assumption 3.

Summarizing the arguments in these two steps, we have $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$.

Step 2.2: Show that $T\widetilde{\mathcal{R}} - T\overline{\mathcal{R}} \leq C_1'' $ for some $C_1'' > 0$. Let $(\boldsymbol{\mu}^s, \boldsymbol{\mu}^b)$ be the dual optimal solution corresponding to the constraint $s_i \leq \mathcal{G}_i^s(s_i, q_i^s)$ and $b_j \leq \mathcal{G}_j^b(b_j, q_j^b)$ in Problem (5), then $\mu_i^s \geq 0$ for $\forall i \in \mathcal{S}$ and $\mu_j^b \geq 0$ for $\forall j \in \mathcal{B}$ according to duality theory. Note that the only difference between Problem (5) and Problem (30) is the right-hand side of the constraints in (30g)-(30h). Therefore, based on (5.57) in Boyd et al. (2004), we can establish that

$$\widetilde{\mathcal{R}} \leq \overline{\mathcal{R}} + \sum_{i \in \mathcal{S}} \mu_i^s \times \frac{1}{T} s_i(1) + \sum_{j \in \mathcal{B}} \mu_j^b \times \frac{1}{T} b_j(1),$$

which further implies that

$$T(\widetilde{\mathcal{R}} - \overline{\mathcal{R}}) \leq T\Big(\sum_{i \in \mathcal{S}} \mu_i^s \times \frac{1}{T} s_i(1) + \sum_{j \in \mathcal{B}} \mu_j^b \times \frac{1}{T} b_j(1)\Big) = \sum_{i \in \mathcal{S}} \mu_i^s s_i(1) + \sum_{j \in \mathcal{B}} \mu_j^b b_j(1).$$

We let $C_1'' := \sum_{i \in \mathcal{S}} \mu_i^s s_i(1) + \sum_{j \in \mathcal{B}} \mu_j^b b_j(1)$, and obtain the desired result.

In summary, $|\mathcal{R}(T) - T\overline{\mathcal{R}}| \leq C_1$, where $C_1 = \max\{|C_1'|, |C_1''|\}$. ■

Proof of Theorem 1. We divide the proof arguments into the following steps: in Step 1, we show that the solution generated by the TRP is feasible to Problem (13); in Step 2, we show under the TRP, there exists a constant $\gamma \in (0,1)$ such that $|m(t+1) - 1| \leq \gamma |m(t) - 1|$ for $\forall t \in \{1, ..., T-1\}$; in Step 3, we show that there exists a constant C_1' such that $T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T) \leq C_1'$. Then, together with Proposition 1, we conclude that there exists a constant $C_2 := C_1 + C_1'$ such that $\mathcal{L}^{TR}(T) = \mathcal{R}^*(T) - \mathcal{R}^{TR}(T) = \big(\mathcal{R}^*(T) - T\overline{\mathcal{R}}\big) + \big(T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\big) \leq C_2$.

Step 1: Show that the solution generated by the TRP is feasible to Problem (13).

We let $(\overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b, \overline{\boldsymbol{x}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ be the optimal solution to the AVG in Problem (5). Recall the definition of $m(t)$ in (24), we have that the TRP uses the commissions $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ in Algorithm 1 to induce the populations and transaction quantities that satisfy $q_i^s(t) = \overline{q}_i^s m(t)$ and $s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t))$ for $i \in \mathcal{S}$. Similarly, for the buyer side, $q_j^b(t) = \overline{q}_j^b m(t)$ and $b(t+1) = \mathcal{G}_j^b(b_j(t), q_j^b(t))$ for $j \in \mathcal{B}$.

We first verify the feasibility of the transaction vector $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t), \boldsymbol{x}(t))$ to Constraints (13b) - (13h).

(13b)-(13c). $q_i^s(t) \overset{(a)}{=} \overline{q}_i^s m(t) \overset{(b)}{\leq} s_i(t) \frac{\overline{q}_i^s}{\overline{s}_i} \overset{(c)}{\leq} s_i(t)$, where (a) follows from Algorithm 1; (b) follows directly from the definition of $m(t)$ in (24); (c) follows from Constraint (5b) that $\overline{q}_i^s \leq \overline{s}_i$. Similarly, $q_j^b(t) = \overline{q}_j^b m(t) \leq b_j(t) \frac{\overline{q}_j^b}{\overline{b}_j} \leq b_j(t)$.

(13d)-(13e). $q_i^s(t) = \overline{q}_i^s m(t) \overset{(a)}{=} \sum_{j':(i,j') \in E} \overline{x}_{i,j'} m(t) \overset{(b)}{=} \sum_{j':(i,j') \in E} x_{i,j'}(t)$, where (a) follows from (5b); (b) follows from Algorithm 1. Similarly, $q_j^b(t) = \overline{q}_j^b m(t) = \sum_{i':(i',j) \in E} \overline{x}_{i',j} m(t) = \sum_{i':(i',j) \in E} x_{i',j}(t)$.

(13f) . $x_{i,j} = \overline{x}_{i,j} m(t) \geq 0$ follows from (5d).

(13g)-(13h). Given $s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t))$, the inequality is a relaxation, which directly follows. A similar argument holds for the buyer side.

Summarizing the arguments above, the solution generated by the TRP is feasible to Problem (13).

Step 2: Show that there exists a constant $\gamma \in (0,1)$ such that $|m(t+1) - 1| \leq \gamma |m(t) - 1|$ for $t \in \{1, ..., T-1\}$.

Recall the definition of $l(t)$ and $g_t(n)$ in (25) and (26), respectively. We discuss three cases: (1) $m(1) > 1$, (2) $m(1) < 1$ and (3) $m(1) = 1$. In each case, we will first show that $m(t)$ gets closer to 1 as t increases, and then we show that the convergence rate can be upper bounded by $\gamma < 1$.

Step 2 - Case 1: $m(1)>1$.

Step 2 - Case 1 - Step 2.1: Show that $m(1)>m(2)>...>m(T-1)>m(T)>1$. To prove the claim of this case, we show that for any $t\in\{1,\dots,T-1\}$, if $m(t)>1$, then $m(t)>m(t+1)>1$. Let $X>0$ denote the number of times the agent type with the lowest ratio changes. We consider the following two cases for $\forall t\in\{1,\dots,T\}$: (1) the lowest node does not change in the next period, i.e., $\tau_x\le t\le\tau_{x+1}-2$ for $x\in\{0,...,X-1\}$; (2) the lowest node changes in next step, i.e., $t=\tau_{x+1}-1$ for $x\in\{0,...,X-1\}$.

(1) For any $\tau_x\le t\le\tau_{x+1}-2$ with $x\in\{0,...,X-1\}$, we show that if $m(t)>1$, then $m(t)>m(t+1)>1$.

Recall that $m(t)=\frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}$ and $m(t+1)=\frac{n_{l(t+1)}(t+1)}{\overline{n}_{l(t+1)}}\overset{(a)}{=}\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}$, where (a) holds given that $l(t)=l(t+1)$ for $\tau_x\le t\le\tau_{x+1}-2$ and $x\in\{0,\dots,X-1\}$. Then, to show that $m(t)>m(t+1)>1$, it is equivalent to establish that $n_{l(t)}(t)>n_{l(t)}(t+1)>\overline{n}_{l(t)}$. First, we have

$$n_{l(t)}(t+1)-n_{l(t)}(t)\ \overset{(b)}{=}\ g_t(n_{l(t)}(t))-n_{l(t)}(t)\ \overset{(c)}{<}\ 0,$$

where (b) follows from (27); (c) follows directly from Lemma 7. Second, we deduce that

$$n_{l(t)}(t+1)-\overline{n}_{l(t)}\overset{(d)}{=}g_t(n_{l(t)}(t))-\overline{n}_{l(t)}\overset{(e)}{=}g_t(n_{l(t)}(t))-g_t(\overline{n}_{l(t)})\overset{(f)}{>}0,$$

where (d) follows from (27); (e) follows from Lemma 1(ii); (f) follows from $n_{l(t)}(t)>\overline{n}_{l(t)}$ given that $m(t)=\frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}>1$ and that $g_t(n)$ increases in $n\ge 0$ from Lemma 7.

In summary, for $\tau_x\le t\le\tau_{x+1}-2$, if $m(t)>1$, then $m(t)>m(t+1)>1$.

(2) For $t=\tau_x-1$ with $x\in\{1,\dots,X\}$, we want to show that if $m(\tau_x-1)>1$, then $m(\tau_x-1)>m(\tau_x)>1$. To prove this, we can deduce that

$$m(\tau_x)\ =\ \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}}\ \overset{(a)}{\le}\ \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}\ \overset{(b)}{<}\ \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}\ =\ m(\tau_x-1),$$

where (a) follows directly from the definition that $l(\tau_x)$ in (25); (b) follows from $n_{l(\tau_x-1)}(\tau_x)=g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x-1))<n_{l(\tau_x-1)}(\tau_x-1)$, where the second inequality follows from $n_{l(\tau_x-1)}(\tau_x-1)>\overline{n}_{l(\tau_x-1)}$ given that $m(\tau_x-1)=\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}>1$ and Lemma 7. Therefore, $m(\tau_x)<m(\tau_x-1)$.

Next, we show that $m(\tau_x)>1$. Since

$$\begin{aligned}m(\tau_x)\ =\ \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}}\ &\overset{(c)}{=}\ \frac{\mathcal{G}_{l(\tau_x)}\Big(n_{l(\tau_x)}(\tau_x-1),\overline{q}_{l(\tau_x)}\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}\Big)}{\overline{n}_{l(\tau_x)}}\\ &\overset{(d)}{\ge}\ \frac{\mathcal{G}_{l(\tau_x)}(n_{l(\tau_x)}(\tau_x-1),\overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}}\ \overset{(e)}{>}\ \frac{\mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)},\overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}}\ =\ 1,\end{aligned}$$

where (c) follows from Algorithm 1; (d) follows from the condition that $\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}=m(\tau_x-1)>1$ and $\mathcal{G}_{l(\tau_x)}(n,q)$ increases in $q\ge 0$; (e) follows from $\frac{n_{l(\tau_x)}(\tau_x-1))}{\overline{n}_{l(\tau_x)}}\ge m(\tau_x-1)>1$. Therefore, $m(\tau_x)>1$.

Based on the arguments above, if $m(t)>1$, then $m(t)>m(t+1)>1$, which holds for any $t\in\{1,...,T-1\}$. Thus, we can conclude that if $m(1)>1$, then $m(1)>m(2)>...>m(T-1)>m(T)>1$.

Step 2 - Case 1 - Step 2.2: Show that there exists a constant $\gamma_1\in(0,1)$ such that $|m(t+1)-1|\le\gamma_1|m(t)-1|$ for any $t\in\{1,\dots,T\}$. Again, we consider the following two cases: (1) the lowest node does not change in the next step, i.e., $\tau_x\le t\le\tau_{x+1}-2$ for any $x\in\{0,...,X-1\}$; (2) the lowest node changes in next step, i.e., $t=\tau_{x+1}-1$ for any $x\in\{0,...,X-1\}$. For both cases, we first show that $\big|m(t+1)-1\big|\le g_t'(\overline{n}_{l(t)})\big|m(t)-1\big|$. Then we show that there exists a $\gamma_1\in(0,1)$ independent from T such that for any positive integer T, $\max\limits_{t=1,\dots,T}g_t'(\overline{n}_{l(t)})\le\gamma_1<1$.

(1) For $\tau_x \leq t \leq \tau_{x+1}-2$, we observe that

$$\begin{aligned}\left|n_{l(t)}(t+1)-\overline{n}_{l(t)}\right| &\overset{(a)}{=} n_{l(t)}(t+1)-\overline{n}_{l(t)} \overset{(b)}{=} g_t(n_{l(t)}(t))-g_t(\overline{n}_{l(t)}) \\ &\overset{(c)}{<} (n_{l(t)}(t)-\overline{n}_{l(t)})g_t'(\overline{n}_{l(t)}) \overset{(d)}{=} \left|n_{l(t)}(t)-\overline{n}_{l(t)}\right|g_t'(\overline{n}_{l(t)}),\end{aligned}$$

where (a) follows from $\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}} \geq m(t+1) > 1$ for any $t \in \{1,...,T-1\}$; (b) follows from (27) and Lemma 1(ii); (c) follows from Lemma 7 given that $g_t(n)$ is strictly concave in $n \geq 0$; (d) follows from $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}} > 1$ for any $t \in \{1,...,T\}$. Therefore, $\left|m(t+1)-1\right| = \left|\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}-1\right| < g_t'(\overline{n}_{l(t)})\left|\frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}-1\right| = g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$.

(2) For $t = \tau_x - 1$,

$$\begin{aligned}\left|m(\tau_x)-1\right| &\overset{(a)}{=} m(\tau_x)-1 = \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}}-1 \overset{(b)}{\leq} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}-1 \\ &\overset{(c)}{=} \frac{g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x-1))-g_{\tau_x-1}(\overline{n}_{l(\tau_x-1)})}{\overline{n}_{l(\tau_x-1)}} \overset{(d)}{<} \left(\frac{n_{l(\tau_x-1)}(\tau_x-1)-\overline{n}_{l(\tau_x-1)}}{\overline{n}_{l(\tau_x-1)}}\right)g_{\tau_x-1}'(\overline{n}_{l(\tau_x-1)}) \\ &= (m(\tau_x-1)-1)g_{\tau_x-1}'(\overline{n}_{l(\tau_x-1)}) \overset{(e)}{=} \left|m(\tau_x-1)-1\right|g_{\tau_x-1}'(\overline{n}_{l(\tau_x-1)}),\end{aligned}$$

where (a) follows from $m(t) \geq 1$ for any $t \in \{1,...,T\}$; (b) follows from $\frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = m(\tau_x) \leq \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}$; (c) follows from $g_t(\cdot)$ in (26) and Lemma 1(ii); (d) follows from the strict concavity of $g_t(\cdot)$ in Lemma 7; (e) follows from $m(\tau_x-1) = \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} > 1$.

In summary, $\left|m(t+1)-1\right| \leq g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$ for any $t \in \{1,\ldots,T\}$. Define $\gamma_1 := \max_{i\in\mathcal{N}^+} \frac{\partial \mathcal{G}_i}{\partial n}(n, n\frac{\overline{q}_i}{\overline{n}_i})$, then

$$\max_{t=1,\ldots,T} g_t'(\overline{n}_{l(t)}) \overset{(a)}{=} \max_{t=1,\ldots,T} \frac{\partial \mathcal{G}_{l(t)}}{\partial n}(n, n\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}})|_{n=\overline{n}_{l(t)}} \leq \max_{i\in\mathcal{N}^+} \frac{\partial \mathcal{G}_i}{\partial n}(n, n\frac{\overline{q}_i}{\overline{n}_i})|_{n=\overline{n}_i} = \gamma_1 \overset{(b)}{<} 1,$$

where (a) follows from the definition of $g_t(\cdot)$ in (26) and (b) follows from the finite network $G(\mathcal{S}\cup\mathcal{B}, E)$ and discussion in Lemma 7. This allows us to conclude the contraction arguments for the case of $m(1) > 1$.

Step 2 - Case 2: $m(1) < 1$.

Step 2 - Case 2 - Step 2.1: Show that $m(1) < m(2) < ... < m(T-1) < m(T) < 1$. Similar to the discussions in Step 2 - Case 1, we consider the following two cases: (1) the lowest node does not change in the next step, i.e., $\tau_x \leq t \leq \tau_{x+1}-2$ for any $x \in \{0,...,X-1\}$; (2) the lowest node changes in next step, i.e., $t = \tau_{x+1}-1$ for any $x \in \{0,...,X-1\}$.

(1) For $\tau_x \leq t \leq \tau_{x+1}-2$, we want to show that if $m(t) < 1$, then $m(t) < m(t+1) < 1$.

Recall that $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}$ and $m(t+1) = \frac{n_{l(t+1)}(t+1)}{\overline{n}_{l(t+1)}} \overset{(a)}{=} \frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}$, where (a) holds as $l(t) = l(t+1)$ for $\tau_x \leq t \leq \tau_{x+1}-2$. Therefore, $m(t) < 1$ implies that $n_{l(t)}(t) < \overline{n}_{l(t)}$. We observe that $m(t) < m(t+1) < 1$ is then equivalent to $n_{l(t)}(t) < n_{l(t)}(t+1) < \overline{n}_{l(t)}$, which holds because

$$n_{l(t)}(t+1) - n_{l(t)}(t) \;=\; g_t(n_{l(t)}(t)) - n_{l(t)}(t) \;>\; 0,$$

where the equality follows from (27) and the inequality follows from the condition that $0 < n_{l(t)}(t) < \overline{n}_{l(t)}$ and Lemma 7. In addition,

$$n_{l(t)}(t+1) - \overline{n}_{l(t)} = g_t(n_{l(t)}(t)) - g_t(\overline{n}_{l(t)}) < 0,$$

given that $n_{l(t)}(t) < \overline{n}_{l(t)}$ and that $g_t(n)$ increases in $n \geq 0$ based on Lemma 7. The derivations above allow us to establish that $n_{l(t)}(t) < n_{l(t)}(t+1) < \overline{n}_{l(t)}$.

(2) For $t=\tau_x-1$, we show that $m(\tau_x-1)<m(\tau_x)<1$ if $m(\tau_x-1)<1$, then

$$\begin{aligned} m(\tau_x) &\overset{(a)}{=} \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = \frac{\mathcal{G}_{l(\tau_x)}(n_{l(\tau_x)}(\tau_x-1), \overline{q}_{l(\tau_x)}\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}})}{\overline{n}_{l(\tau_x)}} \overset{(b)}{\geq} \frac{\mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)}\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}, \overline{q}_{l(\tau_x)}\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}})}{\overline{n}_{l(\tau_x)}} \\ &\overset{(c)}{>} \frac{\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}\mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)}, \overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} \overset{(d)}{=} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} = m(\tau_x-1), \end{aligned}$$

where (a) follows the definition of $m(\tau_x)$ in (24) and $l(\tau_x)$ in (25); (b) follows from $\frac{n_{l(\tau_x)}(\tau_x-1)}{\overline{n}_{l(\tau_x)}} \geq m(\tau_x-1) = \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}$ given the definition of $m(\tau_x-1)$ in (24); (c) follows from $\mathcal{G}_i(a\overline{n}_i, a\overline{q}_i) = \mathcal{G}_i(a\overline{n}_i + (1-a)0, a\overline{q}_i + (1-a)0) > a\mathcal{G}_i(\overline{n}_i, \overline{q}_i) + (1-a)\mathcal{G}_i(0,0) = a\mathcal{G}_i(\overline{n}_i, \overline{q}_i)$ for $0<a<1$ given that $\mathcal{G}_i(0,0)=0$ and $\mathcal{G}_i(n_i,q_i)$ is strictly concave in (n_i,q_i); in addition, (d) follows from $\mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)}, \overline{q}_{l(\tau_x)}) = \overline{n}_{l(\tau_x)}$. In summary, we have $m(\tau_x) > m(\tau_x-1)$.

To proceed, we further observe that

$$m(\tau_x) \;=\; \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} \overset{(d)}{\leq} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}} \overset{(e)}{<} 1,$$

where (d) follows from $\frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = m(\tau_x) \leq \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}$ given the definition of $m(\tau_x)$ in (24); (e) follows from Lemma 7 that $n_{l(\tau_x-1)}(\tau_x) = g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x-1)) < \overline{n}_{l(\tau_x-1)}$ for $n_{l(\tau_x-1)}(\tau_x-1) < \overline{n}_{l(\tau_x-1)}$. Thus, we have that $m(\tau_x)<1$.

In summary, $m(t)<m(t+1)<1$ if $m(t)<1$ for $\forall t\in\{1,...,T-1\}$. Since $m(t)<1$, we obtain that $m(1)<m(2)<...<m(T-1)<m(T)<1$.

Step 2 - Case 2 - Step 2.2: Show that there exists a constant $\gamma_2\in(0,1)$ such that $|m(t+1)-1| \leq \gamma_2|m(t)-1|$ for any $t\in\{1,\dots,T\}$. Following a similar argument in the previous step, we can obtain the desired results.

Step 2 - Case 3: $m(1)=1$. When $m(1)=1$, we want to show that $m(t)=1$ for any $t\in\{1,...,T\}$. To establish the claim, we show that inductively, if $m(t)=1$ then $m(t+1)=1$ for any $t\in\{1,...,T-1\}$. We observe that

$$n_{l(t)}(t+1) \overset{(a)}{=} \mathcal{G}_{l(t)}(n_{l(t)}(t), \overline{q}_{l(t)}m(t)) \overset{(b)}{=} \mathcal{G}_{l(t)}(\overline{n}_{l(t)}, \overline{q}_{l(t)}) \overset{(c)}{=} \overline{n}_{l(t)},$$

where (a) follows from the population transition induced by Algorithm 1; (b) holds given that $m(t)=1$, which further implies that $n_{l(t)}(t) = \overline{n}_{l(t)}$; (c) follows from Lemma 1(ii). Thus, $\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}} = 1$.

In addition, for $i\in\mathcal{N}^+$ with $i\neq l(t)$, we can deduce that

$$n_i(t+1) \;=\; \mathcal{G}_i(n_i(t), \overline{q}_i m(t)) \overset{(d)}{\geq} \mathcal{G}_i(\overline{n}_i, \overline{q}_i) \;=\; \overline{n}_i,$$

where (d) follows from $\frac{n_i(t)}{\overline{n}_i} \geq m(t) = 1$ given the definition of $m(t)$ in (24) and the condition that $i\neq l(t)$. The observation above implies that $\frac{n_i(t+1)}{\overline{n}_i} \geq 1$ for $i\in\mathcal{N}^+$ with $i\neq l(t)$. Therefore, we can establish that

$$m(t+1) \;=\; \min\left\{\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}, \min_{\substack{i\in\mathcal{N}^+,\\ i\neq l(t)}}\left\{\frac{n_i(t+1)}{\overline{n}_i}\right\}\right\} \;=\; 1.$$

Given that $m(1)=1$, by inductively establishing that $m(t+1)=1$ for any $t\in\{1,...,T-1\}$, we have that $m(t)=1$ for any $t\in\{1,...,T\}$. Thus, we obtain that $\big|m(t+1)-1\big| = 0 \leq \gamma_3\big|m(t)-1\big| = 0$ for any $\gamma_3\in(0,1)$.

In summary of the three cases above for $m(t)<1$, $m(t)>1$ and $m(t)=1$, by letting $\gamma=\max\{\gamma_1,\gamma_2,\gamma_3\}$, We have that for some $\gamma\in(0,1)$,

$$\big|m(t+1)-1\big|\le\gamma\big|m(t)-1\big|,$$

for any $t=\{1,...,T-1\}$.

Step 3: Show that there exists a constant C_1' such that $|T\overline{\mathcal{R}}-\mathcal{R}^{TR}(T)|\le C_1'$. We prove this by the following steps. Given $\boldsymbol{q}(t)$ and $\boldsymbol{n}(t)$ induced by TRP, we show in Step 3.1 that there exists a positive constant C_{q_i} such that $\lim_{T\to\infty}\sum_{t=1}^{T}|q_i(t)-\overline{q}_i|\le C_{q_i}$; In Step 3.2, we show that the previous two steps induce a positive constant $C_{\frac{q_i}{n_i}}$ that satisfies $\lim_{T\to\infty}\sum_{t=1}^{T}|\frac{\overline{q}_i}{\overline{n}_i}-\frac{q_i(t)}{n_i(t)}|\le C_{\frac{q_i}{n_i}}$ for any $i\in\mathcal{N}^+$; In Step 3.3, based on Steps 3.1 - 3.2, we conclude that there exists a constant C_1' such that $\big|T\overline{\mathcal{R}}-\mathcal{R}^{TR}(T)\big|\le C_1'$.

Step 3.1: Show that there exists constants C_{q_i} such that $\lim_{T\to\infty}\sum_{t=1}^{T}|q_i(t)-\overline{q}_i|<C_{q_i}$ for any $i\in\mathcal{N}^+$. Notice that

$$\begin{aligned}\lim_{T\to\infty}\sum_{t=1}^{T}\Big|q_i(t)-\overline{q}_i\Big| &\overset{(a)}{=}\lim_{T\to\infty}\sum_{t=1}^{T}\overline{q}_i\Big|m(t)-1\Big|\overset{(b)}{\le}\lim_{T\to\infty}\sum_{t=1}^{T}\overline{q}_i\Big|m(1)-1\Big|\gamma^{t-1}\\ &=\lim_{T\to\infty}\overline{q}_i\Big|m(1)-1\Big|\frac{1-\gamma^T}{1-\gamma}\overset{(c)}{=}\frac{1}{1-\gamma}\overline{q}_i\Big|m(1)-1\Big|,\end{aligned}$$

where (a) follows from $q_i(t)=\overline{q}_i m(t)$ in Algorithm 1; (b) follows from the contraction arguments in Step 2; (c) follows from $\gamma<1$ in Step 2. Let $C_{q_i}=\frac{\overline{q}_i|m(1)-1|}{1-\gamma}$, and then the result follows.

Before proceeding, we provide some supporting results whose proofs will be provided towards the end of this section:

Lemma 8. *For any $i\in\mathcal{N}^+$ with $n_i(1)\ge\overline{n}_i$, there exists a positive constant C_{n_i} such that $\lim_{T\to\infty}\sum_{t=1}^{T}|n_i(t)-\overline{n}_i|<C_{n_i}$. Moreover, for any $i\in\mathcal{N}^+$ with $n_i(1)<\overline{n}_i$, if $m(1)<1$, then $n_i(t)<\overline{n}_i$ for $t\in\{1,...,T\}$.*

Step 3.2: Show that there exists positive constants $C_{\frac{q_i}{n_i}}$ such that $\lim_{T\to\infty}\sum_{t=1}^{T}|\frac{\overline{q}_i}{\overline{n}_i}-\frac{q_i(t)}{n_i(t)}|\le C_{\frac{q_i}{n_i}}$ for any $i\in\mathcal{N}^+$. To show the claim for this step, we notice that for any $i\in\mathcal{N}_+$,

$$\Big|\frac{\overline{q}_i}{\overline{n}_i}-\frac{q_i(t)}{n_i(t)}\Big|\overset{(a)}{=}\Big|\frac{\overline{q}_i}{\overline{n}_i}-\frac{\overline{q}_i m(t)}{n_i(t)}\Big|=\frac{\overline{q}_i}{\overline{n}_i}\Big|1-\frac{\overline{n}_i m(t)}{n_i(t)}\Big|\overset{(b)}{\le}\frac{\overline{q}_i}{\overline{n}_i}\left(\Big|1-\frac{\overline{n}_i}{n_i(t)}\Big|+\frac{\overline{n}_i}{n_i(t)}\Big|1-m(t)\Big|\right),$$

where (a) follows from the population transition induced by Algorithm 1, and (b) follows directly from the triangle inequality. Therefore,

$$\begin{aligned}\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\frac{\overline{q}_i}{\overline{n}_i}-\frac{q_i(t)}{n_i(t)}\Big| &\le\lim_{T\to\infty}\sum_{t=1}^{T}\frac{\overline{q}_i}{\overline{n}_i}\left(\Big|1-\frac{\overline{n}_i}{n_i(t)}\Big|+\frac{\overline{n}_i}{n_i(t)}\Big|1-m(t)\Big|\right)\\ &=\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\frac{\overline{n}_i}{n_i(t)}\Big|1-\frac{n_i(t)}{\overline{n}_i}\Big|+\sum_{t=1}^{T}\frac{\overline{n}_i}{n_i(t)}\Big|1-m(t)\Big|\right)\\ &\overset{(c)}{\le}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\frac{1}{m(t)}\Big|1-\frac{n_i(t)}{\overline{n}_i}\Big|+\sum_{t=1}^{T}\frac{1}{m(t)}\Big|1-m(t)\Big|\right),\quad(*)\end{aligned}\tag{31}$$

where (c) follow from the definition of $m(t)$ in (24).

Notice that if $m(1)=\min_{i\in\mathcal{N}^+}\frac{n_i(1)}{\overline{n}_i}\ge1$, then $n(1)\ge\overline{n}_i$ for any $i\in\mathcal{N}^+$. Thus, it is without loss of generality to consider the following three cases for any $i\in\mathcal{N}^+$ to further relax the term in the RHS of (31), which we denote by “(*)”.

(1) When $n_i(1) \geq \overline{n}_i$ and $m(1) \geq 1$, we show that

$$(*) \overset{(d)}{\leq} \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\left|1-\frac{n_i(t)}{\overline{n}_i}\right| + \sum_{t=1}^{T}\left|1-m(t)\right|\right) \overset{(e)}{\leq} \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{C_{n_i}}{\overline{n}_i} + \sum_{t=1}^{T}\left|1-m(1)\right|\gamma^{t-1}\right)$$
$$= \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{C_{n_i}}{\overline{n}_i} + \left|1-m(1)\right|\frac{1}{1-\gamma}\right),$$

where (d) follows from the result in Step 2 - Case 1- Step 2.1 and Step 2 - Case 3 that if $m(1) > 1$, then $m(1) \geq m(2) \geq ... \geq m(T) \geq 1$; (e) follows from Lemma 8 that $\lim_{T\to\infty} \sum_{t=1}^{T} |n_i(t) - \overline{n}_i| \leq C_{n_i}$ given that $n_i(1) \geq \overline{n}_i$, and we also have $|m(t)-1| \leq \gamma|m(t-1)-1|$ for $\gamma < 1$ and $t \in \{2,...,T\}$ by Step 2. Therefore, by letting $C_{\frac{q_i}{n_i}} := \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{C_{n_i}}{\overline{n}_i} + |1-m(1)|\frac{1}{1-\gamma}\right)$, we obtain the desired result.

(2) When $n_i(1) < \overline{n}_i$ and $m(1) < 1$, we show that

$$(*) \overset{(f)}{\leq} \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\frac{1}{m(t)}\left|1-m(t)\right| + \sum_{t=1}^{T}\frac{1}{m(t)}\left|1-m(t)\right|\right)$$
$$\overset{(g)}{\leq} \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\sum_{t=1}^{T}\left|1-m(1)\right|\gamma^{t-1} + \frac{1}{m(1)}\sum_{t=1}^{T}\left|1-m(1)\right|\gamma^{t-1}\right) \leq \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{2|1-m(1)|}{m(1)(1-\gamma)}\right),$$

where (f) follows from the observation that $m(t) \leq \frac{n_i(t)}{\overline{n}_i} < 1$, where the first inequality follows from the definition of $m(t)$ in (24) and the second inequality follows from Lemma 8 that if $n_i(1) < \overline{n}_i$ and $m(1) < 1$, then $n_i(t) < \overline{n}_i$ for $t \in \{1,...,T\}$; (g) follows from the observation that $|m(t)-1| \leq \gamma|m(t-1)-1|$ for $\gamma < 1$ and $t \in \{2,...,T\}$ by Step 2, and therefore $|m(t)-1| \leq \gamma^{t-1}|m(1)-1|$; in addition, we show in Step 2 - Case 2- Step 2.1 that when $m(1) < 1$, we have $m(1) \leq m(t)$ for any $t \in \{1,...,T\}$. Therefore, we can let $C_{\frac{q_i}{n_i}} := \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{2|1-m(1)|}{m(1)(1-\gamma)}\right)$, and then obtain the desired result.

(3) When $n_i(1) \geq \overline{n}_i$ and $m(1) < 1$, we show that

$$(*) \overset{(h)}{<} \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i} + \sum_{t=1}^{T}\frac{1}{m(1)}\left|1-m(t)\right|\right)$$
$$\overset{(i)}{\leq} \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i} + \sum_{t=1}^{T}\frac{1}{m(1)}\left|1-m(1)\right|\gamma^{t-1}\right) \overset{(j)}{=} \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i} + \left|\frac{1}{m(1)}-1\right|\frac{1}{1-\gamma}\right),$$

where (h) follows from the observation in Step 2 -Case 2- Step 2.1 that $m(1) < m(2) < ... < m(T) < 1$ when $m(1) < 1$ and the result in Lemma 8 that $\lim_{T\to\infty} \sum_{t=1}^{T} |n_i(t) - \overline{n}_i| \leq C_{n_i}$ when $n_i(1) \geq \overline{n}_i$; (i) follows from the results in Step 2 that $|m(t+1)-1| \leq \gamma|m(t)-1|$; (j) follows from the observation in Step 2 that $\gamma < 1$. Therefore, by letting $C_{\frac{q_i}{n_i}} := \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i} + |\frac{1}{m(1)}-1|\frac{1}{1-\gamma}\right)$, we can establish the desired result.

In summary, we have that for any $i \in \mathcal{N}^+$, there exists a positive constant $C_{\frac{q_i}{n_i}}$ such that

$$\lim_{T\to\infty} \sum_{t=1}^{T}\left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}\right| \leq C_{\frac{q_i}{n_i}}.$$

<u>Step 3.3: Show that there exists a constant C_1' such that $|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)| \leq C_1'$.</u> Note that for $j \in \mathcal{B}$ with $\overline{b}_j = 0$, we have $\widetilde{F}_{b_j}(\overline{q}_j^b, \overline{b}_j) = 0$ based on the definition of $\widetilde{F}_{b_j}$ before the formulation of (5). Since $\overline{q}_j^b \leq \overline{b}_j = 0$, we have $q_j^b(t) = \overline{q}_j^b m(t) = 0$ induced by Algorithm 1, which further implies that $F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t) = 0$. Therefore,

$$\lim_{T\to\infty} \sum_{t=1}^{T} \sum_{j\in\mathcal{B}:\,\overline{b}_j=0}\left(\widetilde{F}_{b_j}(\overline{q}_j^b, \overline{b}_j) - F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t)\right) = 0.$$

Similarly, we can establish that for any $i\in\mathcal{S}$ with $\overline{s}_i=0$, we have that $\widetilde{F}_{s_i}(\overline{q}^s_i,\overline{s}_i)=0$, which further implies that $q^s_i(t)=\overline{q}^s_i m(t)=0$. Thus, we have that

$$\lim_{T\to\infty}\sum_{t=1}^{T}\sum_{i\in\mathcal{S}:\overline{s}_i=0}\left(\widetilde{F}_{s_i}(\overline{q}^s_i,\overline{s}_i)-F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})q^s_i(t)\right) \;=\; 0.$$

Based on the two observations above, with $(\boldsymbol{q}^s(t),\boldsymbol{q}^b(t),\boldsymbol{s}(t),\boldsymbol{b}(t):t=1,\ldots T)$ induced by the TRP, we can deduce that

$$\begin{aligned}
&\lim_{T\to\infty}\sum_{t=1}^{T}\left|T\overline{\mathcal{R}}-\mathcal{R}^{TR}(T)\right| \\
&=\lim_{T\to\infty}\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}}\left(\widetilde{F}_{b_j}(\overline{q}^b_j,\overline{b}_j)-F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)q^b_j(t)\right)-\sum_{i\in\mathcal{S}}\left(\widetilde{F}_{s_i}(\overline{q}^s_i,\overline{s}_i)-F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big)q^s_i(t)\right)\right] \\
&=\lim_{T\to\infty}\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(F^{-1}_{b_j}\Big(1-\frac{\overline{q}^b_j}{\overline{b}_j}\Big)\overline{q}^b_j-F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)q^b_j(t)\right)-\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(F^{-1}_{s_i}\Big(\frac{\overline{q}^s_i}{\overline{s}_i}\Big)\overline{q}^s_i-F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})q^s_i(t)\right)\right] \\
&\overset{(a)}{\le}\lim_{T\to\infty}\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\left|F^{-1}_{b_j}\Big(1-\frac{\overline{q}^b_j}{\overline{b}_j}\Big)\overline{q}^b_j-F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)\overline{q}^b_j\right|+\left|F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)\overline{q}^b_j-F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)q^b_j(t)\right|\right)\right. \\
&\quad\left.+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\left|F^{-1}_{s_i}\Big(\frac{\overline{q}^s_i}{\overline{s}_i}\Big)\overline{q}^s_i-F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big)\overline{q}^s_i\right|+\left|F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big)\overline{q}^s_i-F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big)q^s_i(t)\right|\right)\right] \\
&\overset{(b)}{\le}\lim_{T\to\infty}\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}^b_j\frac{1}{d^b_j}\left|\frac{\overline{q}^b_j}{\overline{b}_j}-\frac{q^b_j(t)}{b_j(t)}\right|+F^{-1}_{b_j}(1-\frac{q^b_j(t)}{b_j(t)})\left|\overline{q}^b_j-q^b_j(t)\right|\right)\right. \\
&\quad\left.+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}^s_i\frac{1}{d^s_i}\left|\frac{\overline{q}^s_i}{\overline{s}_i}-\frac{q^s_i(t)}{s_i(t)}\right|+F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})\left|\overline{q}^s_i-q^s_i(t)\right|\right)\right] \\
&\le\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}^b_j\frac{1}{d^b_j}\lim_{T\to\infty}\sum_{t=1}^{T}\left|\frac{\overline{q}^b_j}{\overline{b}_j}-\frac{q^b_j(t)}{b_j(t)}\right|+\max_t F^{-1}_{b_j}(1-\frac{q^b_j(t)}{b_j(t)})\lim_{T\to\infty}\sum_{t=1}^{T}\left|\overline{q}^b_j-q^b_j(t)\right|\right) \\
&\quad+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}^s_i\frac{1}{d^s_i}\lim_{T\to\infty}\sum_{t=1}^{T}\left|\frac{\overline{q}^s_i}{\overline{s}_i}-\frac{q^s_i(t)}{s_i(t)}\right|+\max_t F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})\lim_{T\to\infty}\sum_{t=1}^{T}\left|\overline{q}^s_i-q^s_i(t)\right|\right) \\
&\overset{(c)}{\le}\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}^b_j\frac{1}{d^b_j}C_{q^b_j/b_j}+\overline{v}^b_jC_{q^b_j}\right)+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}^s_i\frac{1}{d^s_i}C_{q^s_i/s_i}+\overline{v}^s_iC_{q^s_i}\right)\;:=\;C'_1.
\end{aligned}$$

where (a) follows from the triangle inequality; (b) follows from Assumption 2(ii) that the derivative of F_{b_j} (F_{s_i}) is lower bounded by a positive constant d^b_j (d^s_i), and therefore the derivative of $F^{-1}_{b_j}$ ($F^{-1}_{s_i}$) is upper bounded by $\frac{1}{d^b_j}$ ($\frac{1}{d^s_i}$), then $|F^{-1}_{b_j}(x_1)-F^{-1}_{b_j}(x_2)|\le\frac{1}{d^b_j}|x_1-x_2|$ for $\forall x_1,x_2$ in the domain, otherwise $\frac{|F^{-1}_{b_j}(x_1)-F^{-1}_{b_j}(x_2)|}{|x_1-x_2|}>\frac{1}{d^b_j}$ implies that there exists a $x_3\in(x_1,x_2)$ such that $f'(x_3)=\frac{|F^{-1}_{b_j}(x_1)-F^{-1}_{b_j}(x_2)|}{|x_1-x_2|}>\frac{1}{d^b_j}$ by mean value theorem, which contradicts to the fact that the derivative of $F^{-1}_{b_j}$ is upper bounded by $\frac{1}{d^b_j}$; following the same argument, $|F^{-1}_{s_i}(x_1)-F^{-1}_{s_i}(x_2)|\le\frac{1}{d^s_i}|x_1-x_2|$ for $\forall x_1,x_2$ in the domain. (c) follows from the results in Step 3.1- Step 3.2 that $\lim_{T\to\infty}\sum_{t=1}^{T}|q_i(t)-\overline{q}_i|<C_{q_i}$ and $\lim_{T\to\infty}\sum_{t=1}^{T}|\frac{\overline{q}_i}{\overline{n}_i}-\frac{q_i(t)}{n_i(t)}|\le C_{\frac{q_i}{n_i}}$ for any $i\in\mathcal{N}^+$; in addition, $F^{-1}_{b_j}\le\overline{v}_{b_j}$ and $F^{-1}_{s_i}\le\overline{v}_{s_i}$. Note that we have $\overline{v}_{b_j}<\infty$ for $j\in\mathcal{B}$ and $\overline{v}_{s_i}<\infty$ for $i\in\mathcal{S}$ and $\frac{1}{d^b_j}<\infty$ for $j\in\mathcal{B}$ and $\frac{1}{d^s_i}<\infty$ for $i\in\mathcal{S}$ given Assumption 2(ii).

Together with the observation from Proposition 1, we can conclude that there exists a constant $C_2 := C_1 + C_1'$ such that $\mathcal{L}^\pi(T) = \mathcal{R}^*(T) - \mathcal{R}^\pi(T) = \big(\mathcal{R}^*(T) - T\overline{\mathcal{R}}\big) + \big(T\overline{\mathcal{R}} - T\mathcal{R}^\pi(T)\big) \le C_1 + C_1' = C_2$. ■

Proof of Corollary 1 Follows from Step 2 - Case 2 - Step 2.1 of Proof of Theorem 1. ■

Proof of Proposition 2. We denote by $(\boldsymbol{r^M}(t), \boldsymbol{p^M}(t), \boldsymbol{q}^{s,M}(t), \boldsymbol{q}^{b,M}(t), \boldsymbol{x}^M(t))$ the optimal solution to the optimization problem for the MP in Definition 2. We consider the following problem instance: Consider a simple network in which there is only one buyer type and one seller type with initial population $s(1) = b(1) > 0$. Given the commissions $\boldsymbol{r^M}(t)$ induced by the MP, we let the populations for the next period be $(\boldsymbol{s}^M(t+1), \boldsymbol{b}^M(t+1))$ is updated by $s^M(t+1) = \alpha s^M(t) + \beta(q^{s,M}(t))^\xi$ and $b^M(t+1) = \alpha b^M(t) + \beta(q^{b,M}(t))^\xi$, where we assume $\beta > 0$ and $0 < \xi < 1$ so that the Assumption 1 holds. In addition, we let $F_s(\cdot)$ and $F_b(\cdot)$ be the distribution functions over $[0,1]$ from the uniform distribution.

We establish two claims to complete the proof.

Claim 1: $\lim_{t\to\infty} R^M(t)$ exists . We divide the proof arguments into the following steps. In Step 1.1, we show that if a steady state induced by the MP exists, we characterize the properties of the steady state. In Step 1.2, we show that the populations converge to the steady state under the platform's MP. For simplicity of notations, we let $\mathcal{R}^M(t)$ denote the profit in period t under the MP.

Step 1.1. Characterize the quantity $\overline{q}^M$ and the profit $\overline{R}^M$ in a steady state. We first define a steady state as such that the populations and transaction quantities remain unchanged after the population transition in each period. Given the definition of a steady state, under the platform's myopic policy, the steady-state population vector $(\overline{s}^M, \overline{b}^M, \overline{q}^M)$ should satisfy the following three conditions:

$$\overline{q}^M = \underset{0 \le q \le \min\{\overline{s}^M, \overline{b}^M\}}{\arg\max} \Big[\Big(1 - \frac{q}{\overline{s}^M} - \frac{q}{\overline{b}^M}\Big) q\Big], \tag{32a}$$

$$\overline{s}^M = \alpha \overline{s}^M + \beta(\overline{q}^M)^\xi, \tag{32b}$$

$$\overline{b}^M = \alpha \overline{b}^M + \beta(\overline{q}^M)^\xi. \tag{32c}$$

Condition (32a) ensures that given the population in each period $(\overline{s}^M, \overline{b}^M)$, the platform's commissions r could induce the equilibrium quantity $\overline{q}^M$ to maximize its profit in the current period (see Corollary 3 for the formulation of optimization problem); (32b) and (32c) ensure that the population vector $(\overline{s}^M, \overline{b}^M)$ remains unchanged after the update in each period.

For Problem (32a), from the first-order-condition $\frac{\partial}{\partial q}\big[\big(1 - \frac{q}{\overline{s}^M} - \frac{q}{\overline{b}^M}\big)q\big] = 0$, we can obtain that $\overline{q}^M = \frac{\overline{s}^M \overline{b}^M}{2\overline{s}^M + 2\overline{b}^M}$, which falls in the region $(0, \min\{\overline{s}^M, \overline{b}^M\})$. Thus, the optimal solution to (32a) is an interior point. Together with the equations in (32b)-(32c), we obtain that

$$\overline{q}^M = \Big(\frac{k}{4}\Big)^{\frac{1}{1-\xi}}, \overline{b}^M = k\Big(\frac{k}{4}\Big)^{\frac{\xi}{1-\xi}}, \overline{s}^M = k\Big(\frac{k}{4}\Big)^{\frac{\xi}{1-\xi}}.$$

where we let $k = \frac{\beta}{1-\alpha}$ for simplicity of notations. This allows us to show that the profit induced by the platform's MP satisfies that

$$\overline{\mathcal{R}}^M = \Big(1 - \frac{\overline{q}^M}{\overline{s}^M} - \frac{\overline{q}^M}{\overline{b}^M}\Big)\overline{q}^M = \frac{1}{2}\Big(\frac{k}{4}\Big)^{\frac{1}{1-\xi}}.$$

Step 1.2: For the seller side, show that there exists a $\gamma \in (0,1)$ such that $|\overline{s}^M - s^M(t+1)| \leq \gamma|(\overline{s}^M - s^M(t))|$.

Next, we establish the convergence of the platform's MP. Without loss of generality, we prove the convergence on the seller side, and notice that the same argument would hold for the buyer side as well.

Since we have $s^M(1) = b^M(1)$ in the problem instance, and in each iteration we have $s^M(t+1) = \alpha s^M(t) + \beta(q^M(t))^\xi$ and $b^M(t+1) = \alpha b^M(t) + \beta(q^M(t))^\xi$, we obtain that $s^M(t) = b^M(t)$ for any $t \in \{1, ..., T\}$. Based on this observation, we can obtain that

$$\begin{aligned} q^M(t) &= \underset{0<q<\min\{s^M(t), b^M(t)\}}{\arg\max} \left\{\left(1 - \frac{q}{s^M(t)} - \frac{q}{b^M(t)}\right) q\right\} \\ &= \underset{0<q<s^M(t)}{\arg\max} \left\{\left(1 - \frac{q}{s^M(t)} - \frac{q}{s^M(t)}\right) q\right\} = \frac{s^M(t)}{4}. \end{aligned}$$

From the optimal solution $q^M(t)$ above, we obtain that

$$s^M(t+1) = \alpha s^M(t) + \beta(q^M(t))^\xi = \alpha s^M(t) + \beta\left(\frac{s^M(t)}{4}\right)^\xi.$$

Abusing some notations, we let $g_s(s) := \alpha s + \beta(\frac{s}{4})^\xi$ for any $s \geq 0$ such that $g_s(\overline{s}^M) = \overline{s}^M$ based on the condition in (32b). To proceed, we consider the following two cases that $s^M(1) \geq \overline{s}^M$ and $s^M(1) < \overline{s}^M$:

(1) When $s^M(1) \geq \overline{s}^M$, we want to show that $s^M(t) \geq \overline{s}^M$ for $t \in \{1, .., T\}$. By induction, if $s^M(t) \geq \overline{s}^M$, we have $s^M(t+1) = g_s(s^M(t)) \geq g_s(\overline{s}^M) = \overline{s}^M$, where the inequality follows from the fact that $g_s(\cdot)$ is an increasing function. Since $s^M(1) \geq \overline{s}^M$, we obtain that $s^M(t) \geq \overline{s}^M$ for $t \in \{1, .., T\}$.

Based on the observation above, we can establish that

$$\left|s^M(t+1) - \overline{s}^M\right| = \left|g_s(s^M(t)) - \overline{s}^M\right| \overset{(a)}{=} g_s(s^M(t)) - g_s(\overline{s}^M) \overset{(b)}{\leq} \left|s^M(t) - \overline{s}^M\right| g_s'(\overline{s}^M), \tag{33}$$

where (a) follows from the observation that $s^M(t) \geq \overline{s}^M$ for $t \in \{1, .., T\}$ in this case; (b) follows from the condion that g_s is concave given that $g_s(s) = \alpha s + \beta(\frac{s}{4})^\xi$ with $a \in (0,1)$. Moreover, we have $g_s'(\overline{s}^M) < 1$ given that $g_s(0) = 0$ and $g_s(\overline{s}^M) = \overline{s}^M$, and so by the mean value theorem, there exists a $\widetilde{s} \in (0, \overline{s}^M)$ such that $g_s'(\widetilde{s}) = \frac{g_s(\overline{s}^M) - g_s(0)}{\overline{s}^M - 0} = 1$. Since $g_s(\cdot)$ is concave, we have that $g_s'(\overline{s}^M) < g_s'(\widetilde{s}) = 1$ given that $\overline{s}^M > \widetilde{s}$. By letting $\gamma_1 := g_s'(\overline{s}^M)$, we establish that there exists $\gamma_1 \in (0,1)$ such that $|\overline{s}^M - s^M(t+1)| \leq \gamma_1|(\overline{s}^M - s^M(t))|$ for $t \in \{1, ..., T-1\}$ if $s^M(1) \geq \overline{s}^M$. From the definition of $g_s(\cdot)$ and $\overline{s}^M$, we see that γ_1 is independent of T.

(2) When $s^M(1) < \overline{s}^M$, we want to show that $s^M(t) < \overline{s}^M$ for $t \in \{1, .., T\}$. If $s^M(t) < \overline{s}^M$, we have $s^M(t+1) = g_s(s^M(t)) < g_s(\overline{s}^M) = \overline{s}^M$, where the inequality follows from that $g_s(\cdot)$ is an increasing function given that $s^M(t) < \overline{s}^M$. Since $s^M(1) < \overline{s}^M$, by induction we obtain that $s^M(t) < \overline{s}^M$ for any $t \in \{1, .., T\}$.

Then, we can establish that

$$\frac{\overline{s}^M - g_s(s^M(t))}{\overline{s}^M - s^M(t)} \overset{(c)}{<} \frac{\overline{s}^M - g_s(s^M(1))}{\overline{s}^M - s^M(1)} \overset{(d)}{<} 1,$$

where in Step (c), we establish the following set of observations: (c-i) we first establish that $\frac{\overline{s}^M - g_s(s)}{\overline{s}^M - s}$ decreases in $s \geq 0$ by showing that $\frac{\partial}{\partial s}\left(\frac{\overline{s}^M - g_s(s)}{\overline{s}^M - s}\right) = \frac{(s - \overline{s}^M)g_s'(s) - g_s(s) + \overline{s}^M}{(s - \overline{s}^M)^2} < 0$, with the inequality following as $g_s(s)$ is strictly concave in $s \geq 0$ such that $\overline{s}^M = g_s(\overline{s}^M) < g_s(s) + (\overline{s}^M - s)g_s'(s)$; (c-ii) we then show that $s^M(t) > s^M(1)$ for $t \in \{2, .., T\}$. Note that $g_s(0) = 0$ and $g_s(\overline{s}^M) = \overline{s}^M$. Since $g_s(s) - s$ is strictly concave in $s \geq 0$, by the Jensen's inequality, we obtain that $g_s(a\overline{s}^M) - a\overline{s}^M > a(g_s(\overline{s}^M) - \overline{s}^M) + (1-a)\big(g_s(0) - 0\big) = 0$ for $0 < a < 1$.

Therefore, we have $g_s(a\overline{s}^M) > a\overline{s}^M$ for $0<a<1$, which further implies that $s^M(t+1) = g_s(s^M(t)) > s^M(t)$ given that $0 < s^M(t) < \overline{s}^M$. Thus, we can obtain that $s^M(t) < s^M(t+1) < \overline{s}^M$ for $t \in \{1,..,T-1\}$. Combining the observations in (c-i) and (c-ii), since $\frac{\overline{s}^M - g_s(s^M(t))}{\overline{s}^M - s^M(t)}$ decreases in $s^M(t)$ and $s^M(t+1) > s^M(t) > s^M(1)$ for $t \in \{2,..,T-1\}$, we have that Step (c) holds. For Step (d), we have $s^M(1) < s^M(2) = g_s(s^M(1)) < g_s(\overline{s}^M) = \overline{s}^M$, where the first inequality follows from $s^M(t+1) = g_s(s^M(t)) > s^M(t)$ for $0 < s^M(t) < \overline{s}^M$ based on previous discussion; the second inequality follows from the condition that $s^M(1) < \overline{s}^M$ in this case and $g_s(.)$ is a increasing function; the last equation follows directly from the observation in (32b). Therefore, we have that $\frac{\overline{s}^M - g_s(s^M(1))}{\overline{s}^M - s^M(1)} < 1$.

By letting $\gamma_2 = \frac{\overline{s}^M - g_s(s^M(1))}{\overline{s}^M - s^M(1)}$, we obtain that $\frac{\overline{s}^M - g_s(s^M(t))}{\overline{s}^M - s^M(t)} \le \gamma_2$, which implies that

$$\left|\overline{s}^M - g_s(s^M(t))\right| \overset{(e)}{=} \overline{s}^M - g_s(s^M(t)) \le \gamma_2\left(\overline{s}^M - s^M(t)\right) \overset{(f)}{=} \gamma_2\left|\overline{s}^M - s^M(t)\right|$$

where (e) and (f) follow from the observations that $s^M(t) < \overline{s}^M$ for $t \in \{1,..,T\}$. In summary, there exists a $\gamma_2 \in (0,1)$ such that $|\overline{s}^M - s^M(t+1)| \le \gamma_2 |(\overline{s}^M - s^M(t))|$ for $t \in \{1,...,T-1\}$ if $s^M(1) < \overline{s}^M$. Again, from the definition of $g_s(\cdot)$, we see that γ_2 is independent of T.

In summary of the two cases above, we let $\gamma := \max\{\gamma_1, \gamma_2\}$, which allows us to obtain the desired result.

Claim 2: For any $\epsilon > 0$, there exists $a \in (0,1)$ for the population transition in this problem instance such that $\overline{\mathcal{R}}^M < \epsilon \overline{\mathcal{R}}$. For the AVG in (5) given the problem instance before Step 1, we have that

$$\begin{aligned} \overline{\mathcal{R}} \;=\; & \max_{s,b,q} \quad \Big(1 - \frac{q}{s} - \frac{q}{b}\Big) q \\ & \text{s.t. } 0 \le q \le s, \quad 0 \le q \le b, \quad s \le \alpha s + \beta q^{\xi}, \quad b \le \alpha b + \beta q^{\xi}. \end{aligned}$$

In addition, based on Lemma 1(ii), the inequalities in the last two constraints are both tight. Note that $s = \alpha s + \beta q^{\xi}$ and $b = \alpha b + \beta q^{\xi}$ are equivalent to $s = b = kq^{\xi}$, where $k = \frac{\beta}{1-\alpha}$. By plugging $s = b = kq^{\xi}$ into the objective function we obtain $\overline{\mathcal{R}} = \max_{0 \le q \le kq^{\xi}} (1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}})q$. Since $(1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}})q$ is concave in $q \ge 0$ for $0 < \xi < 1$, from the first-order condition, we have $\overline{q} = \big(\frac{k}{2(2-\xi)}\big)^{\frac{1}{1-\xi}}$, which satisfy $0 < \overline{q} < kq^{\xi}$. Thus, the optimal commission $\overline{r}$ and the optimal profit $\overline{R}$ for the instance of the AVG in (5) satisfies that

$$\begin{aligned} \overline{r} \;&=\; 1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}} = \frac{1-\xi}{2-\xi}, \\ \overline{\mathcal{R}} \;&=\; \Big(1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}}\Big) q = \frac{1-\xi}{2-\xi}\Big(\frac{k}{2(2-\xi)}\Big)^{\frac{1}{1-\xi}}, \end{aligned}$$

which further implies that $\frac{\overline{\mathcal{R}}^M}{\overline{\mathcal{R}}} = (\frac{2-\xi}{2})^{\frac{1}{1-\xi}} \frac{2-\xi}{2(1-\xi)}$. Therefore, we can obtain that

$$\lim_{\xi \to 1} \frac{\overline{\mathcal{R}}^M}{\overline{\mathcal{R}}} \;=\; \lim_{\xi \to 1} \Big(\frac{2-\xi}{2}\Big)^{\frac{1}{1-\xi}} \frac{2-\xi}{2(1-\xi)} \;=\; 0.$$

■

Proof of Lemma 8. We prove the two claims of this result separately. Given that the supporting lemma is located in Step 3 in the proof of Theorem 1, we would borrow some observations from Step 2 in the proof of Theorem 1 in the proof arguments below.

Claim 1. For $i \in \mathcal{N}^+$, when $n_i(1) \ge \overline{n}_i$, we further consider the following two cases: (1) $m(1) \ge 1$; (2) $m(1) < 1$.

(1) When $n_i(1) \geq \overline{n}_i$ and $m(1) \geq 1$, we first show that $n_i(t) \geq \overline{n}_i$ for any $t \in \{1, ..., T\}$. Given that $n_i(1) \geq \overline{n}_i$ for any $i \in \mathcal{N}^+$, we assume for induction purpose that $n_i(t) \geq \overline{n}_i$, and then we can establish that

$$n_i(t+1) \overset{(a)}{=} \mathcal{G}_i(n(t), \overline{q}_i m(t)) \overset{(b)}{\geq} \mathcal{G}_i(n(t), \overline{q}_i) \geq \mathcal{G}_i(\overline{n}_i, \overline{q}_i) \overset{(c)}{=} \overline{n}_i,$$

where (a) follows from Algorithm 1; (b) follows from our observations in Step 2 Case 1 in the proof of Theorem 1 that if $m(1) > 1$, then we have $m(1) > m(2) > ... > m(T) > 1$, and in Step 2 Case 3 that if $m(1) = 1$, then we have $m(1) = m(2) = ... = m(T) = 1$; (c) follows directly from Lemma 1(ii). By induction, with $n_i(1) \geq \overline{n}_i$ and $m(1) \geq 1$, we obtain that $n_i(t) \geq \overline{n}_i$ for any $t \in \{1, ..., T\}$.

To proceed, we further notice that for any $t \in \{1, ..., T\}$,

$$\begin{aligned} n_i(t) - \overline{n}_i \overset{(d)}{=} & \mathcal{G}_i(n_i(t-1), \overline{q}_i m(t-1)) - \mathcal{G}_i(\overline{n}_i, \overline{q}_i) \\ = & \mathcal{G}_i(n_i(t-1), \overline{q}_i m(t-1)) - \mathcal{G}_i(n_i(t-1), \overline{q}_i) + \mathcal{G}_i(n_i(t-1), \overline{q}_i) - \mathcal{G}_i(\overline{n}_i, \overline{q}_i) \\ \overset{(e)}{\leq} & \overline{q}_i(m(t-1)-1)(\mathcal{G}_i)'_2(n_i(t-1), \overline{q}_i) + (\overline{n}_i(t-1) - \overline{n}_i)(\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i), \end{aligned}$$

where (d) follows from Algorithm 1 and Lemma 1(ii); (e) follows from the concavity of $\mathcal{G}_i(\cdot, \cdot)$ by Assumption 1. Since $n_i(t) \geq \overline{n}_i$, the LHS of the inequality for (e) is nonnegative, and we can take the absolute values and obtain the following inequality:

$$\begin{aligned} \sum_{t=2}^{T} \Big| n_i(t) - \overline{n}_i \Big| & \leq \sum_{t=2}^{T} \Big[\Big| \overline{q}_i(m(t-1)-1)(\mathcal{G}_i)'_2(n_i(t-1), \overline{q}_i) \Big| + \Big| (\overline{n}_i(t-1) - \overline{n}_i)(\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i) \Big| \Big] \\ & \overset{(f)}{\leq} \sum_{t=2}^{T} \Big| \overline{q}_i(m(t-1)-1) \Big| + \sum_{t=2}^{T} \Big| (\overline{n}_i(t-1) - \overline{n}_i)(\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i) \Big| \\ & \leq \overline{q}_i \sum_{t=2}^{T} \gamma^{t-2} \Big| (m(1)-1) \Big| + \sum_{t=2}^{T} \Big| (\overline{n}_i(t-1) - \overline{n}_i)(\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i) \Big| \end{aligned}$$

For (f), we show that $(\mathcal{G}_i)'_2(n_i(t-1), \overline{q}_i) < 1$. Define $y(n) := \mathcal{G}(n, n\frac{\overline{q}_i}{n_i(t-1)})$, by the mean value theorem, there must exist a $\hat{n} \in (0, n_i(t-1))$ such that $y'(\hat{n}) = \frac{y(n_i(t-1)) - y(0)}{n_i(t-1) - 0} = \frac{\mathcal{G}(n_i(t-1), \overline{q}_i)}{n_i(t-1)} < 1$ for $n_i(t-1) > \overline{n}_i$. Therefore, given the concavity of $y(n)$, $y'(n_i(t-1)) < 1$, which suggest that $(\mathcal{G}_i)'_1(n_i(t-1), \overline{q}_i) + (\mathcal{G}_i)'_2(n_i(t-1), \overline{q}_i)\frac{\overline{q}_i}{n_i(t-1)} < 1$, which suggest that $(\mathcal{G}_i)'_2(n_i(t-1), \overline{q}_i) < 1$. Then

$$\begin{aligned} \sum_{t=1}^{T} \Big| n_i(t) - \overline{n}_i \Big| & \leq \frac{\overline{q}_i \sum_{t=2}^{T} \gamma^{t-2} \Big| (m(1)-1) \Big|}{1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i)} - \frac{(\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i)}{1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i)} \times \Big| n_i(T) - \overline{n}_i \Big| + \frac{\Big| n_i(1) - \overline{n}_i \Big|}{1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i)} \\ & \leq \frac{\overline{q}_i \sum_{t=2}^{T} \gamma^{t-2} \Big| (m(1)-1) \Big|}{1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i)} + \frac{\Big| n_i(1) - \overline{n}_i \Big|}{1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i)} \end{aligned}$$

Therefore, $\lim_{T \to \infty} \sum_{t=1}^{T} \Big| n_i(t) - \overline{n}_i \Big| \leq \frac{\overline{q}_i \Big| (m(1)-1) \Big|}{(1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\Big| n_i(1) - \overline{n}_i \Big|}{1 - (\mathcal{G}_i)'_1}$. In the end, we define the positive constant

$$C_{n_i} := \frac{\overline{q}_i \Big| (m(1)-1) \Big|}{(1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\Big| n_i(1) - \overline{n}_i \Big|}{1 - (\mathcal{G}_i)'_1},$$

which allows us to obtain the desired result.

(2) Given that $m(1) < 1$ and that $n_i(1) \geq \overline{n}_i$, we consider two cases. In the first case, we consider the scenario where there exists a $\widetilde{t} \in \{2, ..., T\}$ such that $n_i(\widetilde{t}) \geq \overline{n}_i$. In the second case, we consider the scenario where $n_i(t) \geq \overline{n}_i$ for all $t \in \{1, \ldots, T\}$.

In the first case, given $\widetilde{t} \in \{2, \ldots, T\}$ such that $n_i(\widetilde{t}) < \overline{n}_i$, we want to show that $n_i(t) < \overline{n}_i$ for $t \geq \widetilde{t}$. We prove the claim by induction. Given that $n_i(\widetilde{t}) < \overline{n}_i$, for any $t \geq \widetilde{t}$, suppose towards an induction purpose that $n_i(t) < \overline{n}_i$, and we can establish that

$$n_i(t+1) \overset{(a)}{=} \mathcal{G}_i(n(t), \overline{q}_i m(t)) \overset{(b)}{<} \mathcal{G}_i(n(t), \overline{q}_i) < \mathcal{G}_i(\overline{n}_i, \overline{q}_i) \overset{(c)}{=} \overline{n}_i, \tag{34}$$

where (a) follows from Algorithm 1; (b) follows from the condition that $\mathcal{G}_i(q)$ strictly increases in $q \geq 0$ and from the observation in Step 2.1 from the proofs of Theorem 1 that if $m(1) < 1$, then $m(1) < m(2) < ... < m(T) < 1$; (c) follows directly from Lemma 1(ii). Therefore, we obtain that if there exists a $\widetilde{t} \in \{2, \ldots, T\}$ such that $n_i(\widetilde{t}) < \overline{n}_i$, we have $n_i(t) < \overline{n}_i$ for $t \geq \widetilde{t}$. We then show that $\widetilde{t}$ is independent of T. Given the definition of $\widetilde{t}$ as the first time that $n_i(t) < \overline{n}_i$, it is equivalent to show that the value of $n_i(t)$ for $0 \leq t \leq \widetilde{t}$ is independent of T. This is true as given $n_i(1)$ and $m(1)$, for $t \in \{1, ..., \widetilde{t} - 1\}$, $n_i(t+1) = \mathcal{G}_i(n(t), \overline{q}_i m(t))$, where $m(t) = \min_{i' \in \mathcal{N}^+} \{\frac{n_{i'}(t)}{\overline{n}_{i'}}\}$ is independent of T for $1 \leq t \leq \widetilde{t} - 1$.

The observations above allow us to deduce that in the first case,

$$\begin{aligned}
\lim_{T\to\infty} \sum_{t=1}^{T} \Big| n_i(t) - \overline{n}_i \Big| &= \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \lim_{T\to\infty} \sum_{t=\widetilde{t}}^{T} \Big| n_i(t) - \overline{n}_i \Big| \\
&\overset{(d)}{=} \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \lim_{T\to\infty} \sum_{t=\widetilde{t}}^{T} \Big| m(t) - 1 \Big| \overset{(e)}{\leq} \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \lim_{T\to\infty} \sum_{t=\widetilde{t}}^{T} \Big| m(\widetilde{t}) - 1 \Big| \gamma^{t-\widetilde{t}} \\
&= \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \Big| m(\widetilde{t}) - 1 \Big| \frac{1}{1-\gamma} \overset{(f)}{\leq} \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \Big| m(1) - 1 \Big| \frac{1}{1-\gamma} \\
&\overset{(g)}{\leq} \frac{\overline{q}_i \Big| (m(1) - 1) \Big|}{(1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\Big| n_i(1) - \overline{n}_i \Big|}{1 - (\mathcal{G}_i)'_1} + \overline{n}_i \Big| m(1) - 1 \Big| \frac{1}{1-\gamma},
\end{aligned}$$

where (d) follows from the definition of $m(t)$, (e) follows from Step 2, and (f) follows from $m(1) < m(2) < ... < m(T) < 1$ if $m(1) < 1$ in Step 2.1; (g) follows from the Case (1). Then let $C_{n_i} = \frac{\overline{q}_i \left| (m(1)-1) \right|}{(1-(\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\left| n_i(1) - \overline{n}_i \right|}{1-(\mathcal{G}_i)'_1} + \overline{n}_i \Big| m(1) - 1 \Big| \frac{1}{1-\gamma}$, we obtain the desired result.

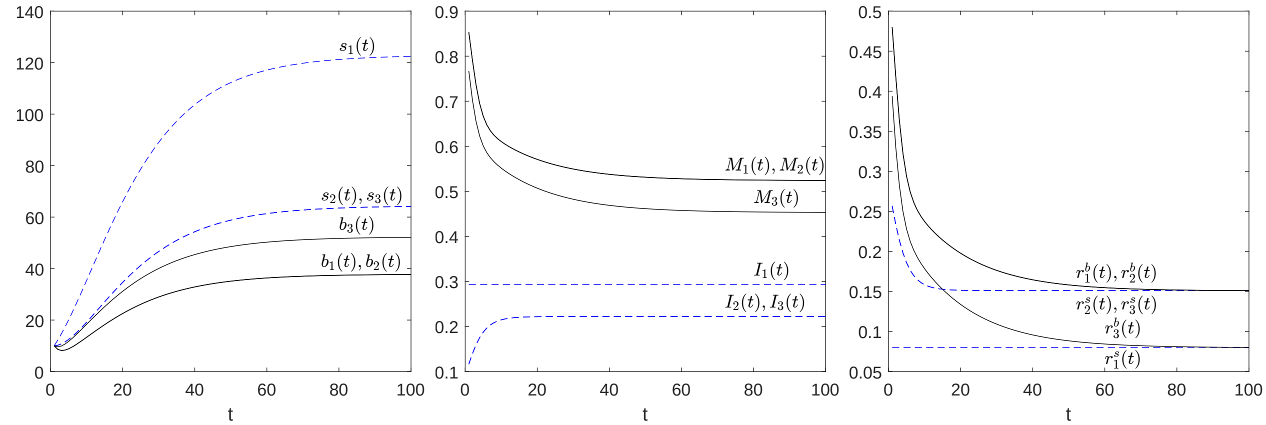
In the second case, if $n_i(t) \geq \overline{n}_i$ for all $t \in \{1, ..., T\}$, we can apply the same upper bound as in Case (1) above under Claim 1.

Claim 2. To establish the second claim of this result, when $n_i(1) \leq \overline{n}_i$ and $m(1) < 1$, by applying the same induction arguments as in (34) from the previous claim, we can establish that $n_i(t) \leq \overline{n}_i$ for any $t \in \{1, \ldots, T\}$.

Summarizing the arguments above, we complete the proofs of the two claims in this result. ■

B.3. Additional Numerical Results

In Figure 4, we consider the case with the same mass of agent for all types, i.e., $s_i(1) = b_j(1) = 10$ for $i \in \mathcal{S}, j \in \mathcal{B}$. Given that the mass of type-1 sellers is the highest among all types at the optimal steady state, its initial



(a) Population (b) Payment/Income (c) Commission

Figure 4 Growth Trajectory for the platform with same initial populations for all types under TRP.

population ratio is the lowest in the network (see Figure 4(a)). Throughout the growth process, the platform focuses on promoting the growth of type-1 sellers. A feasible commission strategy is to consistently charge low commissions to type-1 sellers to ensure their income (i.e., $I_1(t)$) remains at the optimal steady state. For other types, higher commissions are charged initially because they are relatively sufficient, but later the commissions are lower to increase their service level (see Figure 4(b)-4(c)). From Figures 1 and 4, it can be seen that although the different initial populations result in different growth processes, the mass of the agent base and the service level of all types stabilize at the same level at the end.

C. Proof of Results in Section 5

In this section, we develop some auxiliary results that are needed for the proofs of results in Section 5 in C.1. We then respectively prove the results from Section 5.2 in C.2 and those from 5.1 in C.3.

C.1. Auxiliary Results for Section 5.

In this section, we first develop a simper formulation for Problem (5) in (40). To do that, we first characterize the properties of Problem (5) in Lemma 9 and Lemma 10. Next, we reformulate it in Lemma 11, and will further simplify its formulation into (40) in Lemma 12. We then show the connection between the optimal solution to (40) $\boldsymbol{w}^*$ and $(\mathcal{S}_\tau, \mathcal{B}_\tau)$ constructed in (10) in Lemma 13. The proof of the auxiliary results follows a similar argument to the proof of Lemma 1, Lemma 2 and Proposition 10 in Birge et al. (2021). Therefore, we omit the detail of the proof of auxiliary results for simplicity.

To develop an equivalent reformulation in $(\boldsymbol{q}, \boldsymbol{x})$ for **AVG**, recall from Lemma 1(ii) that the relaxed population dynamics constraints $s_i \le \alpha_i^s s_i + \mathcal{G}_i^s(q_i^s)$ and $b_j \le \alpha_j^b b_j + \mathcal{G}_j^b(q_j^b)$ with the optimal solutions to **AVG** are tight. Together with (7), on the seller side, we have $s_i = \frac{\beta_i^s (q_i^s)^{\xi_s}}{1-\alpha_i^s}$ for any $i \in \mathcal{S}$. We further let $k_i^s := \frac{\beta_i^s}{1-\alpha_i^s}$, which allows us to obtain that $s_i = k_i^s (q_i^s)^{\xi_s}$ for any $i \in \mathcal{S}$. Similarly, on the buyer side, we have $b_j = k_j^b (q_j^b)^{\xi_b}$ for any $j \in \mathcal{B}$, where $k_j^b = \frac{\beta_j^b}{1-\alpha_j^b}$. Plugging the expressions of $s_i = k_i^s (q_i^s)^{\xi_s}$ and $b_j = k_j^b (q_j^b)^{\xi_b}$ into **AVG**, we obtain the following reformulation of **AVG**:

$$\overline{\mathcal{R}} = \max_{\boldsymbol{q^s}, \boldsymbol{q^b}, \boldsymbol{x}} \Big[\sum_{j \in \mathcal{B}} \widetilde{F}_b(q_j^b, k_j^b (q_j^b)^{\xi_b}) - \sum_{i \in \mathcal{S}} \widetilde{F}_s(q_i^s, k_i^s (q_i^s)^{\xi_s}) \Big] \tag{35a}$$

$$\text{s.t. } q_i^s \le k_i^s (q_i^s)^{\xi_s}, \qquad \forall i \in \mathcal{S}, \tag{35b}$$

$$q_j^b \le k_j^b (q_j^b)^{\xi_b}, \qquad \forall j \in \mathcal{B}, \tag{35c}$$

$$\sum_{j:(i,j)\in E} x_{ij} = q_i^s, \qquad \forall i \in \mathcal{S}, \tag{35d}$$

$$q_j^b = \sum_{i:(i,j)\in E} x_{ij}, \qquad \forall j \in \mathcal{B}, \tag{35e}$$

$$x_{ij} \ge 0, \qquad \forall (i,j) \in E. \tag{35f}$$

where $\widetilde{F}_b(\cdot)$ and $\widetilde{F}_s(\cdot)$ are defined before Problem (5).

For $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, define $y_b(q) := F_b^{-1}\left(1-(q)^{1-\xi_b}\right) q$ for $0 \le q \le 1$. Define $y_s(q,u) := -F_s^{-1}\left(\frac{(q)^{1-\xi_s}}{u^{1-\xi_s}}\right) q$ for $0 \le q \le u$ and $u > 0$, $y_s(0,0) := \lim_{(q,u)\to(0,0)} y_s(q,u)$. For simplicity of notations, we let $y_b'(q) := \frac{dy_b(q)}{dq}$ for $0 < q < 1$ and $(y_s)_1'(q,u) := \frac{\partial y_s(q,u)}{\partial q}$ for $0 < q < u$. Furthermore, we let $y_b'(0) := \lim_{q\downarrow 0} y_b'(q)$, $y_b'(1) := \lim_{q\uparrow 1} y_b'(q)$; for $u > 0$, we let $(y_s)_1'(0,u) := \lim_{q\to 0} (y_s)_1'(q,u)$, $(y_s)_1'(u,u) := \lim_{q\to u} (y_s)_1'(q,u)$; for $q > 0$, we let $(y_s)_2'(q,q) := \lim_{u\to q} (y_s)_2'(q,u)$. We show in the following lemma that all of the limiting values are finite.

Lemma 9. *(i) $y_b(q)$ is continuously differentiable and strictly concave in $q \in [0,1]$;*

(ii) $y_s(q,u)$ is continuous and strictly concave in $(q,u) \in \{(q',u') : 0 \le q' \le u'\}$; moreover, $y_s(q,u)$ is continuously differentiable in $(q,u) \in \{(q',u') : 0 \le q' \le u', u' > 0\}$;

(iii) for any $0 < \xi_s < 1$, $-(1-\xi_s)\left[F_s^{-1}\right]'(x)x - F_s^{-1}(x)$ strictly decreases in $x \in [0,1]$.

Before the next auxiliary result, we define

$$\rho(u) := \operatorname*{arg\,max}_{0 \le q \le \min\{1,u\}} \Big(y_b(q) + y_s(q,u)\Big), \qquad \text{for } u \ge 0, \tag{36}$$

$$h(u) = \max_{0 \le q \le \min\{1,u\}} \Big(y_b(q) + y_s(q,u)\Big), \qquad \text{for } u \ge 0. \tag{37}$$

Given the definition of $\rho(u)$ and $h(u)$ above, we proceed to consider the following auxiliary result about $(\rho(u), h(u))$ for $u \ge 0$. Notice that $-(y_s)_1'(u,u) = (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s > 0$, which is a constant. To support our proof arguments below, when $u > 0$, if $y_b'(0) > (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s$, we let $\widetilde{u} := (y_b')^{-1}((1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s)$; if $y_b'(0) \le (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s$, we let $\widetilde{u} := 0$.

Lemma 10. *(i) $\rho(u)$ is a well-defined and strictly increasing in $u \ge 0$; moreover, given $\widetilde{u} \ge 0$ defined before the lemma statement, $\frac{\rho(u)}{u} = 1$ for $u \in (0, \widetilde{u}]$ and $\frac{\rho(u)}{u}$ strictly decreases in $u \ge \widetilde{u}$;*

(ii) $h(u)$ is continuous, strictly increasing and strictly concave in $u \ge 0$.

We next develop an alternative optimization for Problem (35). Consider the following optimization problem:

$$\overline{\mathcal{V}} = \max_{\boldsymbol{w},\boldsymbol{z}} \sum_{j\in\mathcal{B}} \left[(k_j^b)^{\frac{1}{1-\xi_b}} h\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big)\right] \tag{38a}$$

$$\text{s.t.} \quad (w_j)^{\frac{1}{1-\xi_b}} = \sum_{i:(i,j)\in E} z_{ij}, \qquad j \in \mathcal{B} \tag{38b}$$

$$\sum_{j:(i,j)\in E} z_{ij} = (k_i^s)^{\frac{1}{1-\xi_s}}, \qquad i \in \mathcal{S}, \tag{38c}$$

$$z_{ij} \geq 0, \qquad \forall (i,j) \in E. \tag{38d}$$

where

$$h(u) = \max_{0 \leq \widetilde{q}_j \leq \min\{1,u\}} F_b^{-1}(1-(\widetilde{q}_j)^{1-\xi_b})\widetilde{q}_j - F_s^{-1}\Big(\frac{(\widetilde{q}_j)^{1-\xi_s}}{u^{1-\xi_s}}\Big)\widetilde{q}_j \text{ for any } u > 0 \tag{39}$$

and $h(0)=0$. We consider the following result:

Lemma 11. *We have the following equivalence properties between Problem (38) and Problem (39):*

(i) let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be the optimal solution to Problem (35), and construct $(\boldsymbol{w}, \boldsymbol{z})$ such that $w_j = (\frac{q_j^b}{q_i^s}(k_i^s)^{\frac{1}{1-\xi_s}})^{1-\xi_b}$ for any $i : x_{ij} > 0$ and $z_{ij} = \frac{x_{ij}}{q_i^s}(k_i^s)^{\frac{1}{1-\xi_s}}$, $\widetilde{q}_j = \frac{q_j^b}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, then $(\boldsymbol{w}, \boldsymbol{z})$ is the optimal solution to Problem (38) and $\widetilde{q}_j$ is the optimal solution to Problem (39) with $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$;

(ii) let $(\boldsymbol{w}, \boldsymbol{z})$ be the optimal solution to Problem (38) and $\widetilde{q}_j$ is the optimal solution to Problem (39) with $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, then construct $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ such that $x_{ij} = \frac{z_{ij}(k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j}{(w_j)^{\frac{1}{1-\xi_b}}}$ and $q_i^s = \frac{(k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j(k_i^s)^{\frac{1}{1-\xi_s}}}{w_j^{\frac{1}{1-\xi_b}}}$ for $j : z_{ij} > 0$, $q_j^b = (k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j$, then $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is the optimal solution to (35);

(iii) Problem (35) and Problem (38) share the same optimal objective value, i.e., $\overline{\mathcal{R}} = \overline{\mathcal{V}}$.

We can further simplify the formulation in (38) in the following Lemma 12.

Lemma 12. *Problem (38) and the following problem share the same optimal solution vector $\boldsymbol{w}$,*

$$\overline{\mathcal{Y}} = \max_{\boldsymbol{w}} \sum_{j \in \mathcal{B}} \Big[(k_j^b)^{\frac{1}{1-\xi_b}} h\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big)\Big] \tag{40a}$$

$$s.t. \quad \sum_{j \in \widetilde{\mathcal{B}}} (w_j)^{\frac{1}{1-\xi_b}} \leq \sum_{i \in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}}, \qquad \forall \widetilde{\mathcal{B}} \subseteq \mathcal{B}, \tag{40b}$$

$$w_j \geq 0, \qquad \forall j \in \mathcal{B}, \tag{40c}$$

and moreover, $\overline{\mathcal{Y}} = \overline{\mathcal{V}}$ where $\overline{\mathcal{V}}$ is the optimal objective value for Problem (38).

The next lemma establishes the connection between the optimal solution $\boldsymbol{w}^*$ to Problem (40) and the network components $G(\mathcal{S}_\tau \cup \mathcal{B}_\tau, E_\tau)$ constructed in (10). Given the finiteness of the network $G(\mathcal{S} \cup \mathcal{B}, E)$, the iteration in (10) yields a maximum index $\overline{\tau}$.

Lemma 13. *For any $\tau \in \{1, ..., \overline{\tau}\}$ and any $j' \in \mathcal{B}_\tau$, we have $\frac{(w_{j'}^*)^{\frac{1}{1-\xi_b}}}{(k_{j'}^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i \in \mathcal{S}_\tau}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j \in \mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$.*

C.2. Proof of Results in Section 5.2.

Proof of Proposition 3. Recall that we have established the connection for the optimal solution and the optimal objective value of Problem (35) with those of Problem (38) and Problem (40) in Lemma 11 and Lemma 12. Therefore, we focus on characterizing the properties of optimization problems in (38) and (40) instead of (35) in this proof. We have already shown that (38) and (40) share the same optimal solution $\boldsymbol{w}^*$ in Lemma 12. To prove the claim, we consider the buyer side in Step 1 and the seller side in Step 2.

Step 1: Establish the ranking of buyers' service levels and payments. Based on Lemma 11(ii), we let $(\boldsymbol{w}, \boldsymbol{z})$ be the optimal solution to Problem (38) and $\widetilde{q}_j$ is the optimal solution to Problem (39) with the parameter $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$. We know the optimal solution to Problem (35) satisfies

$$\frac{q_j^b}{b_j} \overset{(a)}{=} \frac{(q_j^b)^{1-\xi_b}}{k_j^b} \overset{(b)}{=} (\widetilde{q}_j)^{1-\xi_b} \overset{(c)}{=} \rho^{1-\xi_b}\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big),$$

where Step (a) follows from the observation that $b_j = k_j^b (q_j^b)^{\xi_b}$ in Problem (35); Step (b) follows from the solution property of $\widetilde{q}_j$ in Problem (39) by Lemma 11(ii); Step (c) follows from the definition of the optimal solution ρ to Problem (36). Therefore, the ranking of service levels $(\frac{q_j^b}{b_j})_{j\in\mathcal{B}}$ is the same as that of $\Big(\rho\big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\big)\Big)_{j\in\mathcal{B}}$.

For buyers' payments, we know that

$$\min_{i':(i',j)\in E}\{p_{i'}^s\} + r_j^b \;=\; F_b^{-1}\Big(1 - \frac{q_j^b}{b_j}\Big) \;=\; F_b^{-1}\Big(1 - \rho\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big)\Big).$$

Therefore, the ranking of buyers' payments $(\min_{i':(i',j)\in E}\{p_{i'}^s\} + r_j^b)_{j\in\mathcal{B}}$ is the opposite of $\Big(\rho\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big)\Big)_{j\in\mathcal{B}}$.

By Lemma 10(i), we have that $\rho(u)$ strictly increases in $u > 0$. From Lemma 13, we know that $\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for $j\in\mathcal{B}_\tau$ and $\tau = 1,\dots\overline{\tau}$. Furthermore, the definition in (10) implies that $\frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ strictly increases in $\tau = 1,\dots,\overline{\tau}$. Therefore, we have

$$\begin{aligned}
&\frac{q_{j_1}^b}{b_{j_1}} = \frac{q_{j_2}^b}{b_{j_2}}, && \text{for } j_1, j_2 \in \mathcal{B}_\tau, \tau\in\{1,\dots,\overline{\tau}\},\\
&\frac{q_{j_1}^b}{b_{j_1}} < \frac{q_{j_2}^b}{b_{j_2}}, && \text{for } j_1 \in \mathcal{B}_{\tau_1}, j_2\in\mathcal{B}_{\tau_2}, \tau_1,\tau_2\in\{1,\dots,\overline{\tau}\} \text{ and } \tau_1<\tau_2.
\end{aligned}$$

and

$$\begin{aligned}
&\min_{i':(i',j_1)\in E}\{p_{i'}^s\} + r_{j_1}^b = \min_{i':(i',j_2)\in E}\{p_{i'}^s\} + r_{j_2}^b, && \text{for } j_1, j_2 \in \mathcal{B}_\tau, \tau\in\{1,\dots,\overline{\tau}\},\\
&\min_{i':(i',j_1)\in E}\{p_{i'}^s\} + r_{j_1}^b > \min_{i':(i',j_2)\in E}\{p_{i'}^s\} + r_{j_2}^b, && \text{for } j_1 \in \mathcal{B}_{\tau_1}, j_2\in\mathcal{B}_{\tau_2}, \tau_1,\tau_2\in\{1,\dots,\overline{\tau}\} \text{ and } \tau_1<\tau_2.
\end{aligned}$$

Step 2: Establish the ranking of sellers' service levels and incomes. To establish the ranking of sellers' service levels, given the optimal solution $\boldsymbol{w}$ to Problem (40) and the optimal solution $\widetilde{q}_j$ to Problem (39) with parameter $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, we have that for any $i\in\mathcal{S}$ and $j : x_{ij} > 0$,

$$\frac{q_i^s}{s_i} \overset{(a)}{=} \frac{(q_i^s)^{1-\xi_s}}{k_i^s} \overset{(b)}{=} \left(\frac{\rho\big((w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}\big)}{(w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}}\right)^{1-\xi_s}, \tag{41}$$

where (a) follows from our discussion before Problem (35) that $s_i = k_i^s (q_i^s)^{\xi_s}$; (b) follows from Lemma 11(ii) for $j : x_{ij} > 0$.

We next show that for any $\tau_1\neq\tau_2$, we have $x_{ij} = 0$ with $i\in\mathcal{S}_{\tau_1}$ and $j\in B_{\tau_2}$. Based on Lemma 11(ii), it is equivalent to show the optimal solution to Problem (38) satisfies that for any $\tau_1\neq\tau_2$, $z_{ij} = 0$ for $i\in\mathcal{S}_{\tau_1}$ and $j\in B_{\tau_2}$. We show it by induction. Again, to simplify the notation in Problem (38), we let $W_j := (w_j)^{\frac{1}{1-\xi_b}}$ and

$\psi^b_j := (k^b_j)^{\frac{1}{1-\xi_b}}$ for any $j \in \mathcal{B}$ and let $\psi^s_i := (k^s_i)^{\frac{1}{1-\xi_s}}$ for any $i \in \mathcal{S}$. We first consider $\tau = 1$. The buyers in $\mathcal{B}_1$ can only trade with the sellers in $\mathcal{S}_1$ given that they are not connected to any other seller types. It remains to show that the sellers in $\mathcal{S}_1$ only trade with the buyers in $\mathcal{B}_1$ at the platform's optimal commissions. Suppose towards a contradiction that there exist $\tau_1 \neq 1$ such that $z_{ij} > 0$ for some $i \in \mathcal{S}_1$ and $j \in B_{\tau_1}$, then

$$\begin{aligned}\sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E} z_{ij} &= \sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E, j\in\mathcal{B}_1} z_{ij} + \sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E, j\notin\mathcal{B}_1} z_{ij} \\ &\overset{(a)}{>} \sum_{j\in\mathcal{B}_1}\sum_{i:(i,j)\in E, i\in\mathcal{S}_1} z_{ij} \overset{(b)}{=} \sum_{j\in\mathcal{B}_1} W_j \overset{(c)}{=} \sum_{j\in\mathcal{B}_1}\psi^b_j \frac{\sum_{i\in\mathcal{S}_1}\psi^s_i}{\sum_{j\in\mathcal{B}_1}\psi^b_j} = \sum_{i\in\mathcal{S}_1}\psi^s_i \end{aligned} \tag{42}$$

where (a) follows from the assumption that $z_{ij} > 0$ for some $i \in \mathcal{S}_1$ and some $j \in B_{\tau_1}$ with $\tau_1 \neq 1$; (b) follows from (38b); (c) follows from the observation in Lemma 13. In summary, $\sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E} z_{ij} > \sum_{i\in\mathcal{S}_1}\psi^s_i$, which violate Constraint (38c). In summary, we have that $z_{ij} = 0$ for all $i \in \mathcal{S}_1$ and $j \in B_{\tau_1}$ if $\tau_1 \neq 1$. Assuming that $\mathcal{B}_\tau$ only trade with $\mathcal{S}_\tau$ and vice versa, we proceed to show that $\mathcal{B}_{\tau+1}$ only trade with $\mathcal{S}_{\tau+1}$ and vice versa. First, the buyers in $\mathcal{B}_{\tau+1}$ only trade with the sellers in $\mathcal{S}_{\tau+1}$, because they are not adjacent to the seller types from $\mathcal{S}_{\tau'}$ for any $\tau' \geq \tau + 1$; and the seller types with an index lower than $\tau + 1$ does not trade with them based on our previous discussion. Second, $\mathcal{S}_{\tau+1}$ only trade with $\mathcal{B}_{\tau+1}$, otherwise we can also obtain $\sum_{i\in\mathcal{S}_{\tau+1}}\sum_{j:(i,j)\in E} z_{ij} > \sum_{i\in\mathcal{S}_{\tau+1}}\psi^s_i$ following the same argument in (42), which violate Constraint (38c) to Problem (38) given that Problem (40) is a reformulation without loss of optimality. In summary, for any $\tau_1 \neq \tau_2$, $x_{ij} = 0$ for $i \in \mathcal{S}_{\tau_1}$ and $j \in B_{\tau_2}$. This allows us to show that for any $i \in \mathcal{S}_\tau$ with $\tau = 1, \ldots, \overline{\tau}$, we have that if $j : x_{ij} > 0$, then we obtain that $j \in \mathcal{B}_\tau$.

Thus, regarding the sellers' incomes, for any $i \in \mathcal{S}_\tau$ with $\tau = 1, \ldots, \overline{\tau}$ and any $j : x_{ij} > 0$, we have that

$$p^s_i - r^s_i = F_s^{-1}\Big(\frac{q^s_i}{s_i}\Big) = F_s^{-1}\Bigg(\frac{\rho\Big((w_j)^{\frac{1}{1-\xi_b}}/(k^b_j)^{\frac{1}{1-\xi_b}}\Big)}{(w_j)^{\frac{1}{1-\xi_b}}/(k^b_j)^{\frac{1}{1-\xi_b}}}\Bigg).$$

Since $\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k^b_j)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k^b_j)^{\frac{1}{1-\xi_b}}}$ for $j \in \mathcal{B}_\tau$ with $\tau = 1, \ldots, \overline{\tau}$ in Lemma 13, we can next focus on the ranking of $\frac{\rho\Big(\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k^s_i)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k^b_j)^{\frac{1}{1-\xi_b}}\Big)}{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k^s_i)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k^b_j)^{\frac{1}{1-\xi_b}}}$ for $\tau = 1, \ldots, \overline{\tau}$. Recall from Step 1 that $\frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k^b_j)^{\frac{1}{1-\xi_b}}}$ strictly increases in $\tau = 1, \ldots \overline{\tau}$. Based on Lemma 10, for some constant $\widetilde{u} \geq 0$, we have that $\frac{\rho(u)}{u} = 1$ for $0 < u \leq \widetilde{u}$ and $\frac{\rho(u)}{u}$ strictly decreases in u for $u > \widetilde{u}$. Define $\widetilde{\tau} := \max\{\tau | u_j < \widetilde{u} \text{ for } j \in \mathcal{B}_\tau\}$. We observe that (i) for any $\tau \leq \widetilde{\tau}$, we have $\frac{q^s_i}{s_i} = \frac{\rho(u)}{u} = 1$ and $p^s_i - r^s_i = F_s^{-1}\big(\frac{\rho(u)}{u}\big) = F_s^{-1}(1) = \overline{v}_{s_i}$ for $i \in S_\tau$; (ii) for any $\tau > \widetilde{\tau}$, we have $\frac{\rho(\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k^s_i)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k^b_j)^{\frac{1}{1-\xi_b}})}{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k^s_i)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k^b_j)^{\frac{1}{1-\xi_b}}}$ strictly decreases in τ. Therefore, we can summarize that

$$\begin{aligned} &\frac{q^s_{i_1}}{s_{i_1}} = \frac{q^s_{i_2}}{s_{i_2}}, && \text{for } i_1, i_2 \in \mathcal{S}_\tau, \tau \in \{1, ..., \overline{\tau}\}, \\ &\frac{q^s_i}{s_i} = 1, && \text{for } i \in \mathcal{S}_\tau, \tau \leq \widetilde{\tau}, \\ &\frac{q^s_{i_1}}{s_{i_1}} > \frac{q^s_{i_2}}{s_{i_2}}, && \text{for } i_1 \in \mathcal{S}_{\tau_1}, i_2 \in \mathcal{S}_{\tau_2}, \tau_1, \tau_2 \in \{\widetilde{\tau}+1, ..., \overline{\tau}\} \text{ and } \tau_1 < \tau_2. \end{aligned}$$

and

$$p^s_{i_1} - r^s_{i_1} = p^s_{i_1} - r^s_{i_1}, \qquad \text{for } i_1, i_2 \in \mathcal{S}_\tau, \tau \in \{1, ..., \overline{\tau}\},$$

$$p_i^s - r_i^s = \overline{v}_{s_i}, \qquad \text{for } i \in \mathcal{S}_\tau, \tau \le \widetilde{\tau},$$

$$p_{i_1}^s - r_{i_1}^s > p_{i_2}^s - r_{i_2}^s, \qquad \text{for } i_1 \in \mathcal{S}_{\tau_1}, i_2 \in \mathcal{S}_{\tau_2}, \tau_1, \tau_2 \in \{\widetilde{\tau}+1, ..., \overline{\tau}\} \text{ and } \tau_1 < \tau_2.$$

Summarizing the two steps above, we conclude the claims in this result. ■

Proof of Corollary 2. Given the definition of $(\boldsymbol{k}^s,\boldsymbol{k}^b)$ at the beginning of Appendix C.1, for any $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, we first let $\psi_i^s = (k_i^s)^{\frac{1}{1-\xi_s}}$ and $\psi_j^b = (k_j^b)^{\frac{1}{1-\xi_b}}$ for simplicity of notations. We consider the equivalent reformulation in Problem (38) with decision variables $(\boldsymbol{w}, \boldsymbol{z})$ by Lemma 11 and Problem (40) with the decision variable vector $\boldsymbol{w}$ and Lemma 12. We let $W_j = (w_j)^{\frac{1}{1-\xi_b}}$ for all $j \in \mathcal{B}$.

Notice that it is without loss of generality to consider a connected graph $G(\mathcal{S} \cup \mathcal{B}, E)$ for the proof arguments. We prove the impact of $\boldsymbol{\psi}^s$ and $\boldsymbol{\psi}^b$ on the service levels in Step 1, and then the impacts on supply/demand and population in Step 2.

Proof of Claim (1): Establish the impact of $\boldsymbol{\psi}^s$ and $\boldsymbol{\psi}^b$ on the service levels. Recall from Step 1 in the proof arguments of Proposition 3 that for any $j \in \mathcal{B}$, when $\frac{W_j}{\psi_j^b}$ becomes larger under the optimal solution $\boldsymbol{W}$ to Problem (40), $\frac{q_j^b}{b_j}$ becomes larger at the optimal solution as well. As a result, we can focus on the impact of $\boldsymbol{\psi}^s$ and $\boldsymbol{\psi}^b$ on $\frac{W_j}{\psi_j^b}$ for $j \in \mathcal{B}_\tau$.

Step (1-i): Establish the impact of $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ on the service levels of the buyer side. Let $(\boldsymbol{W}, \boldsymbol{z})$ be the optimal solution to (38) given parameters $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ and let $\{(\mathcal{S}_\tau, \mathcal{B}_\tau) : \tau = 1, \dots \overline{\tau}\}$ be the network components obtained from (10) given this parameter set. We define the index set $\tau_i := \{\tau | i \in S_\tau\}$ and $\tau_j := \{\tau | j \in B_\tau\}$. We consider an alternative vector $(\hat{\boldsymbol{\psi}}^s, \hat{\boldsymbol{\psi}}^b)$ in which we pick any $\widetilde{i} \in \mathcal{S}$, and let $\hat{\psi}_{\widetilde{i}}^s > \psi_{\widetilde{i}}^s$; we also let $\hat{\psi}_i^s := \psi_i^s$ for all $i \neq \widetilde{i}$ and let $\hat{\psi}_j^b := \psi_j^b$ for all $j \in \mathcal{B}$. Then we obtain that the parameter vector $(\hat{\psi}^s, \hat{\psi}^b)$ has only one entry on the seller side that is higher than in $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$. Let $(\hat{W}, \hat{z})$ be the optimal solution to (38) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$, and let $\{(\hat{\mathcal{S}}_\tau, \hat{\mathcal{B}}_\tau) : \tau = 1, \dots \widetilde{\tau}\}$ be the network components obtained from (10) given this parameter set for some positive integer $\widetilde{\tau}$.

To prove the claim of this step, we want to show that $W_j \le \hat{W}_j$ for all $j \in \mathcal{B}$. This leads to the observation that $\frac{W_j}{\psi_j^b} \le \frac{\hat{W}_j}{\hat{\psi}_j^b}$ given our construction that $\hat{\psi}_j^b := \psi_j^b$ for all $j \in \mathcal{B}$. In this way, we can claim that a higher ψ_i^s leads to weakly higher $\frac{W_j}{\psi_j^b}$ for all $j \in \mathcal{B}$.

Suppose towards a contradiction that there exists a $j_1 \in \mathcal{B}$ such that $W_{j_1} > \hat{W}_{j_1}$ at the optimal solution. Based on Constraint (38b), we have that $\sum_{i \in N_E(j_1)} z_{ij_1} = W_{j_1} > \hat{W}_{j_1} = \sum_{i \in N_E(j_1)} \hat{z}_{ij_1}$, which implies that there exists a $i_1 \in N_E(j_1)$ such that $z_{i_1 j_1} > \hat{z}_{i_1 j_1} \ge 0$. Similarly, given $i_1 \in N_E(j_1)$, based on Constraint (38c), we have that $\sum_{j \in N_E(i_1)} z_{i_1 j} = \psi_{i_1}^s \le \hat{\psi}_{i_1}^s = \sum_{j \in N_E(i_1)} \hat{z}_{i_1 j}$ where the inequality follows from the construction of $\hat{\boldsymbol{\psi}}$ above. This implies that there exists $j_2 \in N_E(i_1)$ such that $0 \le z_{i_1 j_2} < \hat{z}_{i_1 j_2}$. Using the same argument as above, there must exist a $i_2 \in N_E(j_2), i_2 \neq i_1$ such that $z_{i_2 j_2} > \hat{z}_{i_2 j_2} \ge 0$ and there exists some $j_3 \in N_E(i_2)$ such that $0 \le z_{i_2 j_3} < \hat{z}_{i_2 j_3}$. In this iteration, given the finiteness of the graph, we have that there exists a finite list $(j_1, i_1, j_2, i_2 ..., j_n)$ such that $W_{j_1} > \hat{W}_{j_1}$ and $W_{j_n} \le \hat{W}_{j_n}$. We let $\mathbb{B}_1 = \{j_1\}$, and $\mathbb{S}_1 = \{i | i \in N_E(j_1), z_{i_1 j_1} > \hat{z}_{i_1 j_1} \ge 0\}$. For $t \in \{2, 3 \dots\}$, we further let $\mathbb{B}_t = \{j | j \in N_E(i), 0 \le z_{ij} < \hat{z}_{ij}, \forall i \in \mathbb{S}_{t-1}\}$, and $\mathbb{S}_t = \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \ge 0, \forall j \in \mathbb{B}_{t-1}\}$. We have that $\mathscr{B}_t := \cup_{l \in \{1,...,t\}} \mathbb{B}_l$ and $\mathscr{S}_t := \cup_{l \in \{1,...,t\}} \mathbb{S}_l$ are the sets of all possible buyer types and seller types accessed within the first $2t$ steps in this iteration. Since $\mathscr{B}_{t-1} \subset \mathscr{B}_t \subset \mathcal{B}$ and $|\mathcal{B}|$ is finite, there exists a finite $\overline{t}$ such that

$\mathscr{B}_{\bar{t}} = \mathscr{B}_{\bar{t}-1}$, i.e., the set $\mathscr{B}_t$ stops expanding. Under the assumption that $W_{j_1} > \hat{W}_{j_1}$ at the optimal solution for $j_1 \in \mathbb{B}_1$, we next show that there exists $j \in \mathscr{B}_{\bar{t}}$ such that $W_j < \hat{W}_j$. We further suppose towards a contradiction that $W_j > \hat{W}_j$ for any $j \in \mathscr{B}_{\bar{t}}$. Consider the set of seller types $\widetilde{S} := \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \geq 0,\ \forall j \in \mathscr{B}_{\bar{t}}\}$. We can show that $\widetilde{S} \subseteq \mathscr{S}_{\bar{t}}$ by definition. Moreover, we would obtain that

$$\begin{aligned}
\sum_{i\in\widetilde{S}} \hat{\psi}_i^s &= \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}<\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} \hat{z}_{ij} \\
&\overset{(a)}{=} \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} \hat{z}_{ij} \\
&< \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} z_{ij} \\
&\leq \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} z_{ij} \ = \ \sum_{i\in\widetilde{S}} \psi_i^s
\end{aligned}$$

where in Step (a), with $\widetilde{S} \subseteq \mathscr{S}_{\bar{t}} = \cup_{l\in\{1,\ldots,\bar{t}\}} \mathbb{S}_l$, in the iterative construction above, given that $\mathbb{B}_t = \{j | j \in N_E(i), 0 \leq z_{ij} < \hat{z}_{ij},\ \forall i \in \mathbb{S}_{t-1}\}$ and that $\mathscr{B}_{\bar{t}} = \cup_{l\in\{1,\ldots,\bar{t}\}} \mathbb{B}_l$, the subset of buyer types $\{j : z_{ij} < \hat{z}_{ij}$ for some $i \in \widetilde{S}\}$ should be a subset of $\mathscr{B}_{\bar{t}}$; based on the definition $\widetilde{S} = \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \geq 0,\ \forall j \in \mathscr{B}_{\bar{t}}\}$, we have that $z_{ij} > \hat{z}_{ij}$ for any $i \in \widetilde{S}$ and $j \in \mathscr{B}_{\bar{t}}$, which further implies that $\{j : z_{ij} < \hat{z}_{ij},\ \forall i \in \widetilde{S}\} = \emptyset$ and that $\sum_{i\in\widetilde{S}} \sum_{j:z_{ij}<\hat{z}_{ij}} \hat{z}_{ij} = 0$. However, the observation that $\sum_{i\in\widetilde{S}} \hat{\psi}_i^s < \sum_{i\in\widetilde{S}} \psi_i^s$ contradicts with the fact that $\sum_{i\in\widetilde{S}} \hat{\psi}_i^s \geq \sum_{i\in\widetilde{S}} \psi_i^s$ by construction of $(\hat{\boldsymbol{\psi}}^s, \hat{\boldsymbol{\psi}}^b)$ above. Therefore, such a contradiction implies that there exists a $j_l \in \mathbb{B}_l \subset \mathscr{B}_{\bar{t}}$ for some $l \in \mathbb{N}_+$ such that $W_{j_l} \leq \hat{W}_{j_l}$. Thus, there must exist a finite path $(j_1, i_1, j_2, i_2 ..., j_l)$ for $j_t \in \mathbb{B}_t$ and $i_t \in \mathbb{S}_t$ such that $z_{i_t j_t} > 0$ for $t \in \{1, .., l\}$ and $\hat{z}_{i_{t-1} j_t} > 0$ for $t \in \{2, \ldots, l\}$ under the assumption that $W_{j_l} \leq \hat{W}_{j_l}$. For any $t \in \{1, \ldots, l-1\}$, we let τ_{i_t} and τ_{j_t} be the corresponding index for the seller subgroup for $\mathcal{S}_\tau$ and the buyer subgroup $\mathcal{B}_\tau$ by the iterative construction in (10). Since $z_{i_t j_t} > 0$, we know that $\tau_{i_t} = \tau_{j_t}$. With the iterative construction, we have $j_{t+1} \in N_E(i_t)$, which satisfies that $\tau_{i_t} \leq \tau_{j_{t+1}}$ given that S_{i_t} is not adjacent to $\mathcal{B}_l$ with $l < \tau_{i_t}$ with the iterative construction in (10). In summary, $\tau_{j_1} = \tau_{i_1} \leq \tau_{j_2} = ... \leq \tau_{j_l}$, which implies that $\frac{W_{j_n}}{\psi_{j_n}^b} \geq \frac{W_{j_1}}{\psi_{j_1}^b}$ based on Lemma 13. Therefore, $\frac{\hat{W}_{j_n}}{\hat{\psi}_{j_n}^b} \geq \frac{W_{j_n}}{\psi_{j_n}^b} \geq \frac{W_{j_1}}{\psi_{j_1}^b} > \frac{\hat{W}_{j_1}}{\hat{\psi}_{j_1}^b}$.

We proceed to show that the constructed solution $(\hat{W}, \hat{z})$ cannot be the optimal solution to Problem (38) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$. We first send a flow ϵ along $j_n \to i_{n-1} \to j_{n-1} \to ... \to i_1 \to j_1$ to construct a new feasible solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$: since $\frac{\hat{W}_{j_n}}{\hat{\psi}_{j_n}^b} > \frac{\hat{W}_{j_1}}{\hat{\psi}_{j_1}^b}$ and $\hat{z}_{i_t, j_{t+1}} > 0$ for all $t \in \{1, ..., n-1\}$, we can pick any $\epsilon \in (0, \min\{(\hat{W}_{j_n} \hat{\psi}_{j_1}^b - \hat{W}_{j_1} \hat{\psi}_{j_n}^b)/(\hat{\psi}_{j_1}^b + \hat{\psi}_{j_n}^b), \min_{t\in\{1,\ldots,n-1\}} \{\hat{z}_{i_t, j_{t+1}}\}\})$; for $t \in \{1, ..., n-1\}$, let $\widetilde{z}_{i_t j_t} := \hat{z}_{i_t j_t} + \epsilon$, $\widetilde{z}_{i_t j_{t+1}} := \hat{z}_{i_t j_{t+1}} - \epsilon$, $\widetilde{z}_{ij} := \hat{z}_{ij}$ for all $(i, j) \neq (i_t j_{t+1}), (i, j) \neq (i_t j_t)$. Let $\widetilde{W}_{j_1} := \hat{W}_{j_1} + \epsilon$ and $\widetilde{W}_{j_n} := \hat{W}_{j_n} - \epsilon$, $\widetilde{W}_{j'} := \hat{W}_{j'}$ for all $j' \neq j_1, j' \neq j_n$. We next verify the feasibility of this new solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$ in Problem (38). Since $\epsilon \leq \min_{t\in\{1,\ldots,n-1\}} \{\hat{z}_{i_t, j_{t+1}}\}$, we can obtain that $\widetilde{z}_{i_t j_{t+1}} \geq 0$ such that Constraint (38d) is satisfied. In addition, in our construction of the new feasible solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$, since we only send a flow ϵ along $j_n \to i_{n-1} \to j_{n-1} \to ... \to i_1 \to j_1$, Constraints (38b) - (38c) are preserved. Thus, $(\widetilde{W}, \widetilde{z})$ is feasible in Problem (38). We define the super-gradient of $h(u)$ as $\partial h(u) = \{z \in \mathbb{R} | h(t) \leq h(u) + z(t-u), \forall t \geq 0\}$. In addition, we define $\partial_- h(u) := \inf\{\partial h(u)\}$ and $\partial_+ h(u) := \sup\{\partial h(u)\}$. Given the strict concavity of $h(u)$ for $u \geq 0$, we have that if $u_2 > u_1 > 0$, then $\partial_+ h(u_2) < \partial_- h(u_1)$, which implies that

$$\hat{\psi}_{j_1}^b h\Big(\frac{\widetilde{W}_{j_1}}{\hat{\psi}_{j_1}^b}\Big) + \hat{\psi}_{j_n}^b h\Big(\frac{\widetilde{W}_{j_n}}{\hat{\psi}_{j_n}^b}\Big) \ = \hat{\psi}_{j_1}^b h\Big(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}_{j_1}^b}\Big) + \hat{\psi}_{j_n}^b h\Big(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}_{j_n}^b}\Big)$$

$$> \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \epsilon\partial h_-\Big(\frac{\hat{W}_{j_1}+\epsilon}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big) - \epsilon\partial h_+\Big(\frac{\hat{W}_{j_n}-\epsilon}{\hat{\psi}^b_{j_n}}\Big)$$
$$\geq \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big)$$

where the first inequality follows from the concavity of $h(\cdot)$ in $\mathbb{R}_+$; for the second inequality, since $\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}$ and $\epsilon < \frac{\hat{W}_{j_n}\hat{\psi}_{j_1} + \hat{W}_{j_1}\hat{\psi}_{j_n}}{\hat{\psi}_{j_1}+\hat{\psi}_{j_n}}$, we have $\frac{\hat{W}_{j_n}-\epsilon}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1}+\epsilon}{\hat{\psi}^b_{j_1}}$, and therefore, $\partial_+ h(\frac{\hat{W}_{j_n}-\epsilon}{\hat{\psi}^b_{j_n}}) < \partial h_-(\frac{\hat{W}_{j_1}+\epsilon}{\hat{\psi}^b_{j_1}})$. Since other terms in the objective function remain unchanged, $(\widetilde{W}, \widetilde{z})$ leads to a strictly higher objective value than $(\hat{W}, \hat{z})$, which contradicts with the fact that $(\hat{W}, \hat{z})$ be the optimal solution to (38) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$.

In conclusion, we have that $\frac{W_j}{\psi^b_j} \leq \frac{\hat{W}_j}{\hat{\psi}^b_j}$ for all $j \in \mathcal{B}$. This concludes the claim about the impact of ψ^s_i.

For the impact of ψ^b_j, we can apply exactly the same proof-by-contradiction arguments as above to establish that when $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \in \mathcal{B}$, then we have that the optimal solution $\frac{W_j}{\psi^b_j}$ decreases for any $j \in \mathcal{B}$.

Step (1-ii): Establish the impact of $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ on the service levels of the seller side. For the impact of $\boldsymbol{\psi}^s$ on the service levels of the seller side, we first recall the construction of $(\hat{\psi}^s, \hat{\psi}^b)$ based on $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ in Step (1-i), which satisfies that $\hat{\psi}^s_{\widetilde{i}} > \psi^s_{\widetilde{i}}$, $\hat{\psi}^s_i := \psi^s_i$ for all $i \neq \widetilde{i}$ and $\hat{\psi}^b_j := \psi^b_j$ for all $j \in \mathcal{B}$. Without loss of generality, we suppose that a type-i seller trades with type-j_1 buyer where $i \in \mathcal{S}_{l_1}$ and $j_1 \in \mathcal{B}_{l_1}$ given the parameter set $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$; and given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$, we suppose that the type-i seller trades with type-j_2 buyer for some $j_2 \in \mathcal{B}_{l_2}$. The index satisfies that $l_2 \geq l_1$ given that $\mathcal{S}_{l_1}$ is not connected with $\mathcal{B}_t$ for any $t < l_1$ by the iterative construction of network components in (10). Therefore, we have that $\frac{W_{j_1}}{\psi^b_{j_1}} \leq \frac{W_{j_2}}{\psi^b_{j_2}} \leq \frac{\hat{W}_{j_2}}{\hat{\psi}^b_{j_2}}$, where the first inequality follows from Lemma 13 given that $l_2 \geq l_1$, and the second inequality follows from the same arguments in Step (1-i). Since type-i sellers have positive trades with type-j_1 buyers in the optimal solutions given the parameters $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$, and with type-$j_2$ buyers in the optimal solutions given the parameters $(\hat{\boldsymbol{\psi}}^s, \hat{\boldsymbol{\psi}}^b)$, based on the observation that $\frac{W_{j_1}}{\psi^b_{j_1}} \leq \frac{\hat{W}_{j_2}}{\hat{\psi}^b_{j_2}}$, we can establish that

$$\frac{q^s_i}{s_i} \overset{(a)}{=} \Big(\frac{\rho(W_{j_1}/\psi^b_{j_1})}{W_{j_1}/\psi_{j_1}}\Big)^{1-\xi_s} \overset{(b)}{\geq} \Big(\frac{\rho(\hat{W}_{j_2}/\hat{\psi}^b_{j_2})}{\hat{W}_{j_2}/\hat{\psi}_{j_2}}\Big)^{1-\xi_s} \overset{(c)}{=} \frac{\hat{q}^s_i}{\hat{s}_i}, \tag{43}$$

where Step (a) and Step (c) follow from the optimality equation in (41) from the proof arguments in Proposition 3; Step (b) follows from the fact that $\frac{\rho(x)}{x}$ monotonically decreases in $x \geq 0$ (see Lemma 10). In summary, when $\psi^s_{\widetilde{i}}$ increases for any $\widetilde{i} \in \mathcal{S}$, we have that $\frac{q^s_i}{s_i}$ becomes weakly lower for all $i \in \mathcal{S}$.

Using the same arguments above, we could establish the impact of $\boldsymbol{\psi}^b$ on the seller side: when $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \in \mathcal{B}$, we have that $\frac{q^s_i}{s_i}$ becomes weakly higher for all $i \in \mathcal{S}$.

Proof of Claim (2): Establish the impact of $\boldsymbol{\psi}^s$ and $\boldsymbol{\psi}^b$ on transaction quantities and populations. Recall from (8) that we have $q^b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$ and $b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{\xi_b}{1-\xi_b}}$ for any $j \in \mathcal{B}$ at the optimal solution to Problem (5) given (7). We establish this claim in the following two substeps.

Step (2-i): Establish the impact of $\boldsymbol{\psi}^b$ on the transaction quantities and populations. For any $j \in \mathcal{B}$, recall from Step (1-i) above that if $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \neq j$, or if $\psi^s_{\widetilde{i}}$ increases for any $\widetilde{i} \in \mathcal{S}$, then $\frac{q^b_j}{b_j}$ weakly decreases at the optimal solution. Given that $q^b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$, we can establish that as $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \neq j$, then q^b_j weakly decreases at the optimal solution for any $j \in \mathcal{B}$. From $q^b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$, we have that $b_j = \psi^b_j(q^b_j)^{\xi_b}$ for any $j \in \mathcal{B}$, which further suggests that b_j weakly decreases at the optimal solution for any $j \in \mathcal{B}$.

For any $j \in \mathcal{B}$, it remains to consider the impact of ψ_j^b on (q_j^b, b_j) at the optimal solution for $j \in \mathcal{B}$. We first show that q_j^b increases in $\psi_j^b \geq 0$ for any $j \in \mathcal{B}$. Recall from Constraints (35d)-(35e) that $\sum_{i\in\mathcal{S}} q_i^s = \sum_{i\in\mathcal{S}}\sum_{j:(i,j)\in E} x_{ij} = \sum_{j\in\mathcal{B}}\sum_{i:(i,j)\in E} x_{ij} = \sum_{j\in\mathcal{B}} q_j^b$, which means that $q_j^b = \sum_{i\in\mathcal{S}} q_i^s - \sum_{j'\neq j, j'\in\mathcal{B}} q_{j'}^b$. Since higher ψ_j^b leads to weakly higher q_i^s for any $i \in \mathcal{S}$ and weakly lower $q_{j'}^b$ for any $j' \in \mathcal{B}$ with $j' \neq j$, we conclude that higher ψ_j^b leads to weakly higher q_j^b. Similarly, higher ψ_i^s leads to weakly higher q_i^s.

Step (2-ii): Establish the impact of $\boldsymbol{\psi}^s$ on the transaction quantities and populations. By applying the same arguments as in Step (2-i), we can establish that (q_i, s_i) weakly increases in ψ_i^s for all $i \in \mathcal{S}$, and q_i^s and s_i weakly decreases in $\psi_{i'}^s$ for any $i' \neq i$ and weakly increases in ψ_j^b for all $j \in \mathcal{B}$. ■

Proof of Proposition 4. Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be the optimal solution to Problem (35); we let $u_j := (w_j)^{\frac{1}{1-\xi_b}} / (k_j^b)^{\frac{1}{1-\xi_b}}$ for any $j \in \mathcal{B}$ where $(\boldsymbol{w}, \boldsymbol{z})$ is the optimal solution to the reformulation into Problem (38) (see Lemma 11). Recall that for given $\tau = 1, \ldots, \overline{\tau}$ from (10), type-i sellers for $i \in \mathcal{S}_\tau$ trade with type-j buyers for $j \in \mathcal{B}_\tau$. Moreover, for any $i \in \mathcal{S}_\tau$ and $j \in \mathcal{B}_\tau$,

$$r_i^s + r_j^b = F_b^{-1}\Big(1 - \frac{q_j^b}{k_j^b (q_j^b)^{\xi_b}}\Big) - F_s^{-1}\Big(\frac{q_i^s}{k_i^s (q_i^s)^{\xi_s}}\Big) \;=\; F_b^{-1}\Big(1 - \rho^{1-\xi_b}(u_j)\Big) - F_s^{-1}\Big(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\Big),$$

where the first equation follows from the conditions in (12a) and (12c) where the expressions of s_i and b_j are given before Problem (35); the second equation follows from the observations in Lemma 11(ii) and the definition of $\rho(u)$ in (36). In addition, at the optimal solution, the value of u_j for any $j \in \mathcal{B}_\tau$ increases in $\tau = 1, \ldots, \overline{\tau}$ (see Lemma 13 and the definition in (10)). For simplicity of notations, we let $r(u) := F_b^{-1}(1 - \rho^{1-\xi_b}(u)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u)}{u^{1-\xi_s}})$ for any $u > 0$. Recall the definition $\widetilde{u} := (y_b')^{-1}((1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s)$ before Lemma 10.

We prove the two claims of this result.

Claim (1). If $u_j \leq \widetilde{u}$, we have $\rho(u_j) = u_j$ (see Lemma 10(i)). This implies that $F_b^{-1}(1 - \rho^{1-\xi_b}(u_j)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}) = F_b^{-1}(1 - u_j^{1-\xi_b}) - F_s^{-1}(1)$, which is decreasing in $u_j \in [0,1]$ given that $F_b(\cdot)$ is a strictly increasing function in $[0, \overline{v}^b]$ (see Assumption 2). We let $\widetilde{\tau} := \max\{\tau | u_j < \widetilde{u} \text{ for } j \in \mathcal{B}_\tau\}$. Together with the fact that at the optimal solution, the value of u_j for $j \in \mathcal{B}_\tau$ increases in $\tau = 1, \ldots, \overline{\tau}$, we obtain that the value $r(u_j)$ increases in $\tau < \widetilde{\tau}$.

Claim (2). If $u_j \geq \widetilde{u}$, we know that $y_b'(\rho(u_j)) + (y_s)_1'(\rho(u_j), u_j) = 0$. Define $Y(\widetilde{q}_j, u_j) := y_b'(\widetilde{q}_j) + (y_s)_1'(\widetilde{q}_j, u_j)$ given the definitions of y_s and y_b before Lemma 9: for any $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, $y_b(q) = F_b^{-1}\left(1 - (q)^{1-\xi_b}\right) q$ for $0 \leq q \leq 1$ and $y_s(q,u) = -F_s^{-1}\left(\frac{(q)^{1-\xi_s}}{u^{1-\xi_s}}\right) q$ for $0 \leq q \leq u$ and $u > 0$, $y_s(0,0) := \lim_{(q,u)\to(0,0)} y_s(q,u)$. We have that

$$\begin{aligned} Y(\widetilde{q}_j, u_j) &= y_b'(\widetilde{q}_j) + (y_s)_1'(\widetilde{q}_j, u_j) \\ &= \left((\xi_b - 1)\widetilde{q}_j^{1-\xi_b}(F_b^{-1})'\left(1 - \widetilde{q}_j^{1-\xi_b}\right) + F_b^{-1}\left(1 - \widetilde{q}_j^{1-\xi_b}\right)\right) + \left((\xi_s - 1)\frac{\widetilde{q}_j}{u_j^{1-\xi_s}}(F_s^{-1})'\left(\frac{\widetilde{q}_j}{u_j^{1-\xi_s}}\right) - F_s^{-1}\left(\frac{\widetilde{q}_j}{u_j^{1-\xi_s}}\right)\right) \end{aligned}$$

Since F_s and F_b are twice differentiable, we know that F_s^{-1} and F_b^{-1} are continuously differentiable, and therefore $Y(\widetilde{q}_j, u_j)$ is continuously differentiable at $(\widetilde{q}_j, u_j)$ for $0 \leq \widetilde{q}_j \leq \min\{u_j, 1\}$. By the implicit function theorem, there exists a continuously differentiable function $\rho(u_j)$ such that $\widetilde{q}_j = \rho(u_j)$ given $Y(\widetilde{q}_j, u_j) = 0$. By differentiating $Y(\widetilde{\rho}(u_j), u_j) = 0$ with respect to u_j, we obtain

$$\rho'(u_j) = \frac{(\xi_s - 1)u_j^{\xi_s - 3}\rho(u_j)^{1-2\xi_s}\left((\xi_s - 1)\rho(u_j)u_j^{\xi_s}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + (\xi_s - 2)u_j\rho(u_j)^{\xi_s}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\right)}{(\xi_b - 1)\rho(u_j)^{-2\xi_b} f_b + (\xi_s - 1)u_j^{\xi_s - 2}\rho(u_j)^{-2\xi_s} f_s}$$

where

$$f_b := (\xi_b - 2)\rho(u_j)^{\xi_b}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) - (\xi_b-1)\rho(u_j)(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right),$$
$$f_s := (\xi_s - 1)\rho(u_j)u_j^{\xi_s}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + (\xi_s-2)u_j\rho(u_j)^{\xi_s}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right).$$

We proceed to show that $f_s < 0$ and $f_b < 0$ for later use:

$$\begin{aligned}
f_b &:= (1-\xi_b)\rho(u_j)^{\xi_b}\Big(\frac{(2-\xi_b)}{(\xi_b-1)}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) + \rho^{1-\xi_b}(u_j)(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)\Big)\\
&\overset{(a)}{<} (1-\xi_b)\rho(u_j)^{\xi_b}\Big(-2(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) + \rho^{1-\xi_b}(u_j)(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)\Big) \overset{(b)}{<} 0,\\
f_s &:= (\xi_s-1)u_j\rho^{\xi_s}(u_j)\Big(\rho^{1-\xi_s}(u_j)u_j^{\xi_s-1}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + \frac{\xi_s-2}{\xi_s-1}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\Big)\\
&\overset{(c)}{<} (\xi_s-1)u_j\rho^{\xi_s}(u_j)\Big(\rho^{1-\xi_s}(u_j)u_j^{\xi_s-1}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + 2(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\Big) \overset{(d)}{<} 0,
\end{aligned}$$

where (a) and (c) follow from the facts that $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, which imply that $\frac{2-\xi_b}{\xi_b-1} < -2$ and $\frac{\xi_s-2}{\xi_s-1} > 2$ given that $(F_b^{-1})' > 0$ and $(F_s^{-1})' > 0$ on the domains; (b) and (d) follow from the conditions that $-F_s^{-1}(a/b)a$ and $F_b^{-1}(1-a/b)a$ are concave in (a,b) for $0 \le a \le b$ and $b > 0$ by Assumption 3, and therefore $\frac{a}{b}(F_s^{-1})''\left(\frac{a}{b}\right) + 2(F_s^{-1})'\left(\frac{a}{b}\right) > 0$ and $\frac{a}{b}(F_b^{-1})''\left(1-\frac{a}{b}\right) - 2(F_b^{-1})'\left(1-\frac{a}{b}\right) < 0$. In summary, we have $f_s < 0$ and $f_b < 0$.

Finally, we want to establish how $r(u_j) = F_b^{-1}(1-\rho^{1-\xi_b}(u_j)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}})$ changes in $u_j > 0$. Again, given the continuity of $r(u)$, we define the sup-derivative

$$\partial r(u) = \{z \in \mathbb{R} \mid r(t) \le r(u) + z(t-u),\ \forall t \ge 0\},$$

which implies that

$$\partial r(u) \ = \ (\xi_b-1)\rho(u_j)^{-\xi_b}\rho'(u_j)(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) + (\xi_s-1)u_j^{\xi_s-2}\rho(u_j)^{-\xi_s}\left(u_j\rho'(u_j)-\rho(u_j)\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right).$$

Plugging in the expression of $\rho'(u_j)$, we obtain that

$$\partial r(u) = \frac{(\xi_b-1)(\xi_s-1)\rho(u_j)\left(f_1+f_2+f_3\right)}{u_j\left((\xi_b-1)s^{2-\xi_s}\rho(u_j)^{2\xi_s}f_b + (\xi_s-1)\rho(u_j)^{2\xi_b}f_s\right)},$$

where

$$f_1 = (\xi_b-1)u_j\rho(u_j)^{\xi_s+1}(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)$$
$$f_2 = (\xi_s-1)u_j^{\xi_s}\rho(u_j)^{\xi_b+1}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)$$
$$f_3 = -u_j(\xi_b-\xi_s)\rho(u_j)^{\xi_b+\xi_s}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right).$$

Based on the observation above, we discuss the two cases of this claim:

(i) if $F_s(v)$ and $F_b(v)$ are convex in $v \in [0,\overline{v}_s]$ and $v \in [0,\overline{v}_b]$, we have $(F_b^{-1})''(v) < 0$ and $(F_s^{-1})''(v) < 0$ in their domains. Given $(F_b^{-1})'(v) > 0$ and $(F_s^{-1})'(v) > 0$, $\rho(u_j) < 1$ (see (36)) and $\xi_s, \xi_b \in (0,1)$, we know that $f_1 > 0$ and $f_2 > 0$. Since $\xi_s = \xi_b$, $f_3 = 0$. Therefore, the numerator of $\frac{\partial r(u_j)}{\partial u_j}$ is positive. Since $f_s < 0$ and $f_b < 0$, the denominator of $\frac{\partial r(u_j)}{\partial u_j}$ is positive. In summary, $\frac{\partial r(u_j)}{\partial u_j} > 0$ for $u_j \ge \widetilde{u}$;

(ii) if $F_s(v)$ and $F_b(v)$ are concave in $v \in [0, \overline{v}_s]$ and $v \in [0, \overline{v}_b]$ respectively, we have $(F_b^{-1})''(v) > 0$ and $(F_s^{-1})''(v) > 0$, then $f_1 < 0$ and $f_2 < 0$. Therefore, $\frac{\partial r(u_j)}{\partial u_j} < 0$ for $u_j \geq \widetilde{u}$.

■

C.3. Proof of Results in Section 5.1.

Proof of Theorem 2. Recall that $\overline{\mathcal{R}}(E, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b), \overline{\mathcal{V}}(E, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b), \overline{\mathcal{Y}}(E, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ are respectively the optimal objective value to (35), (38) and (40). To simplify the notations, we use $\overline{\mathcal{R}}(E)$, $\overline{\mathcal{V}}(E)$, $\overline{\mathcal{Y}}(E)$ to denote $\overline{\mathcal{R}}(E, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$, $\overline{\mathcal{V}}(E, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$, $\overline{\mathcal{Y}}(E, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$. From Lemma 11 and 12, we have that $\overline{\mathcal{R}}(E) = \overline{\mathcal{V}}(E) = \overline{\mathcal{Y}}(E)$. Therefore, to prove the claim in this result, it is equivalent to focus on Problem (40) and show that $\overline{\mathcal{Y}}(E) \geq (1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$.

We next consider Problem (44) below with an additional constraint $F_b^{-1}(1 - q_j^{1-\xi_b}) - F_s^{-1}\big(\frac{q_j^{1-\xi_s}}{u_j^{1-\xi_s}}\big) \geq r$ for some $r \in \mathbb{R}$ in comparison with Problem (40). We then show that even the problem with this constraint can obtain the objective value weakly higher than $(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$, from which we can conclude that $\overline{\mathcal{Y}}(E) \geq \mathcal{Y}^h(E) \geq (1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. Given the edge set $\overline{E}$ of the complete graph, for any edge set $E \subset \overline{E}$, we define this auxiliary problem below

$$
\begin{aligned}
\mathcal{Y}^h(E) \;=\; & \max_{\boldsymbol{w}, r} \sum_{j \in \mathcal{B}} \Big[(k_j^b)^{\frac{1}{1-\xi_b}} h\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}, r \Big) \Big] && \text{(44a)} \\
\text{s.t.} \; & \sum_{j \in \widetilde{\mathcal{B}}} (w_j)^{\frac{1}{1-\xi_b}} \leq \sum_{i \in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}}, \qquad \forall \widetilde{\mathcal{B}} \subseteq \mathcal{B}, && \text{(44b)} \\
& w_j \geq 0, \qquad \forall j \in \mathcal{B} && \text{(44c)} \\
& r \leq \overline{v}_b, && \text{(44d)}
\end{aligned}
$$

where for any $u > 0$,

$$
h(u, r) = \max_{\substack{0 \leq \widetilde{q} \leq \min\{1, u\}, \\ F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u^{1-\xi_s}}\right) \geq r}} \Big(F_b^{-1}(1 - \widetilde{q}^{1-\xi_b}) - F_s^{-1}\Big(\frac{\widetilde{q}^{1-\xi_s}}{u^{1-\xi_s}}\Big) \Big) \widetilde{q}. \tag{44e}
$$

<u>Step 1: Show that $\overline{\mathcal{Y}}(E) \geq \mathcal{Y}^h(E)$.</u> Note that the only difference between (44) and (40) is that one more constraint $F_b^{-1}\big(1 - (\widetilde{q}_j)^{1-\xi_b}\big) - F_s^{-1}\big(\frac{\widetilde{q}_j^{1-\xi_s}}{u^{1-\xi_s}}\big) \geq r$ for any $(i,j) \in E$ is added to Problem (44). With $r \leq \overline{v}_b$, we have that the constraint for the maximization problem in (h, r) is non-empty given that solution $\widetilde{q} = 0$ is feasible. Therefore, the solution to Problem (44) is also feasible in Problem (40), and two problems share the same objective functions. Thus, we have that

$$
\overline{\mathcal{Y}}(E) \geq \mathcal{Y}^h(E).
$$

<u>Step 2: Show that $\mathcal{Y}^h(E) \geq (1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$.</u> To establish the claim, we first reformulate the optimization problems for $\mathcal{Y}^h(E)$ and $\overline{\mathcal{Y}}(\overline{E})$.

<u>Step 2.1: Reformulate the problem for $\mathcal{Y}^h(E)$.</u> With $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$ for any $j \in \mathcal{B}$, we define

$$
\hat{q}_j(r, u_j) := \max \Big\{ \widetilde{q} : \; r \leq F_b^{-1}(1 - (\widetilde{q})^{1-\xi_b}) - F_s^{-1}\Big(\frac{\widetilde{q}^{1-\xi_s}}{u_j^{1-\xi_s}} \Big), 0 \leq \widetilde{q} \leq \min\{1, u_j\} \Big\}. \tag{45}
$$

Note that since $F_b^{-1}(1 - \widetilde{q}^{1-\xi_b}) - F_s^{-1}\Big(\frac{\widetilde{q}^{1-\xi_s}}{u_j^{1-\xi_s}}\Big)$ strictly decreases in $\widetilde{q} \in [0, \min\{1, u_j\}]$, we know $\hat{q}_j(r, u_j)$ is unique given (r, u_j). Given that r is a lower bound of $F_b^{-1}(1 - (\widetilde{q})^{1-\xi_b}) - F_s^{-1}\Big(\frac{(\widetilde{q})^{1-\xi_s}}{(u_j)^{1-\xi_s}}\Big)$ and $\hat{q}_j(r, u_j)$ is suboptimal to Problem (44e), the optimal objective value $\mathcal{Y}^h(E)$ from Problem (44e) is weakly higher than the optimal objective value of following optimization problem

$$\begin{aligned}
\max_{\boldsymbol{w},r} \quad & \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r\hat{q}_j\Big(r, \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big) \\
\text{s.t.} \quad & \sum_{j\in\widetilde{\mathcal{B}}} w_j^{\frac{1}{1-\xi_b}} \le \sum_{i\in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}}, && \forall \widetilde{\mathcal{B}}\subseteq\mathcal{B}, \\
& w_j \ge 0, && \forall j\in\mathcal{B}, \\
& r \le \overline{v}_b.
\end{aligned}$$

For any $r\in(-\infty,\overline{v}^b]$ and $\epsilon\in(0,1)$, we observe that $(w_j)^{\frac{1}{1-\xi_b}} = (k_j^b)^{\frac{1}{1-\xi_b}}(1-\epsilon)\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}$ is feasible in the optimization problem above given that $w_j\ge 0$ for any $j\in\mathcal{B}$ and for any $\widetilde{\mathcal{B}}\subseteq\mathcal{B}$,

$$\sum_{\in\widetilde{\mathcal{B}}} w_j^{\frac{1}{1-\xi_b}} = \sum_{j\in\widetilde{\mathcal{B}}} (k_j^b)^{\frac{1}{1-\xi_b}}(1-\epsilon)\frac{\sum_{i'\in\mathcal{S}}(k_{i'}^s)^{\frac{1}{1-\xi_s}}}{\sum_{j'\in\mathcal{B}}(k_{j'}^b)^{\frac{1}{1-\xi_b}}} \le \sum_{i\in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}},$$

where the inequality follows directly from the condition in the theorem statement. By letting $\overline{u} := \frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}$, we have that

$$\mathcal{Y}^h(E) \ \ge\ \max_{r\le\overline{v}_b} \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r\hat{q}_j\Big(r, (1-\epsilon)\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}\Big) \ =\ \max_{r\le\overline{v}_b} \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r\hat{q}_j\big(r,(1-\epsilon)\overline{u}\big).$$

Step 2.2: Reformulate the problem for $\overline{\mathcal{Y}}(\overline{E})$. We first show that given the graph set to the complete graph $G(\mathcal{S}\cup\mathcal{B},\overline{E})$, the optimal solution to Problem (40) satisfies $(w_{j'}^*)^{\frac{1}{1-\xi_b}} = (k_{j'}^b)^{\frac{1}{1-\xi_b}}\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for any $j'\in\mathcal{B}$. Given the definition of $(\mathcal{S}_\tau,\mathcal{B}_\tau)$ in (10), in a complete graph, we have that $\mathcal{B}_1=\mathcal{B}$, as for any $\widetilde{\mathcal{B}}\subseteq\mathcal{B}$, we have that

$$\frac{\sum_{i\in N_E(\widetilde{\mathcal{B}})}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\widetilde{\mathcal{B}}}(k_j^b)^{\frac{1}{1-\xi_b}}} \overset{(a)}{=} \frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\widetilde{\mathcal{B}}}(k_j^b)^{\frac{1}{1-\xi_b}}} \overset{(b)}{\ge} \frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_E(\mathcal{B})}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}},$$

where Step (a) follows from the fact that network $G(\mathcal{S}\cup\mathcal{B},\overline{E})$ is complete; Step (b) follows from the condition that $\widetilde{\mathcal{B}}\subseteq\mathcal{B}$. By Lemma 13, we have $\frac{(w_{j'}^*)^{\frac{1}{1-\xi_b}}}{(k_{j'}^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for any $j'\in\mathcal{B}$. Therefore, we can obtain that

$$\overline{\mathcal{Y}}(\overline{E}) \ =\ \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} h\Bigg(\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}\Bigg).$$

Similar to Step 2.1, given definition of $h(.)$ in (39), we could reformulate $h(.)$ by defining that

$$\overline{q} := \operatorname*{arg\,max}_{\widetilde{q}\in[0,\min\{1,\overline{u}\}]} \Bigg(F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\Big(\frac{\widetilde{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}}\Big)\Bigg)\widetilde{q}, \tag{46}$$

where we recall that we have set $\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}} = \overline{u}$ in Step 2.1 above. By letting $\overline{r} := F_b^{-1}(1-(\overline{q})^{1-\xi_b}) - F_s^{-1}(\frac{(\overline{q})^{1-\xi_s}}{(\overline{u})^{1-\xi_s}})$, given definition of $h(.)$ in (39), we have that

$$\overline{\mathcal{Y}}(\overline{E}) \ =\ \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} h\Bigg(\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}\Bigg) \ =\ \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}}\,\overline{r}\overline{q}.$$

Step 2.3: Establish that $\mathcal{Y}^h(E) \geq (1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. To establish the claim, for any $j \in \mathcal{B}$, we want to show that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}) \geq (1-\epsilon)\overline{q}$.

By the definition of $\hat{q}_j(r,u)$ in (45), we have that for any $j \in \mathcal{B}$,

$$\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}) := \max\left\{\widetilde{q} : \overline{r} \leq F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\Big(\frac{\widetilde{q}^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}\Big), 0 \leq \widetilde{q} \leq \min\left\{1,(1-\epsilon)\overline{u}\right\}\right\}.$$

For simplicity of notations, we use $\hat{q}_j$ to denote $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})$. Since $F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}(\frac{(\widetilde{q})^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$ decreases in $\widetilde{q} \in [0,\min\{1,(1-\epsilon)\overline{u}\}]$, we have that either $\overline{r} = F_b^{-1}(1-(\hat{q}_j)^{1-\xi_b}) - F_s^{-1}(\frac{(\hat{q}_j)^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$ or $\hat{q}_j = \min\{1,(1-\epsilon)\overline{u}\}$.

For any $j \in \mathcal{B}$, to show that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}) \geq (1-\epsilon)\overline{q}$, we consider the following two cases:

(1) if $\hat{q}_j = \min\{1,(1-\epsilon)\overline{u}\}$, then $\hat{q}_j = \min\{1,(1-\epsilon)\overline{u}\} \geq (1-\epsilon)\min\{1,\overline{u}\} = (1-\epsilon)\overline{q}$, where the last equality follows from the constraint in Problem (46);

(2) if $\overline{r} = F_b^{-1}(1-\hat{q}_j^{1-\xi_b}) - F_s^{-1}(\frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$, then based on the definition that $\overline{r} = F_b^{-1}(1-\overline{q}^{1-\xi_b}) - F_s^{-1}(\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}})$ in Step 2.2, we have that

$$F_b^{-1}(1-\overline{q}^{1-\xi_b}) - F_s^{-1}\left(\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}}\right) = F_b^{-1}(1-\hat{q}_j^{1-\xi_b}) - F_s^{-1}\Big(\frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}\Big).$$

Note that $F_b^{-1}(1-q^{1-\xi_b}) - F_s^{-1}(\frac{q^{1-\xi_s}}{u^{1-\xi_s}})$ strictly increases in $u \geq q \geq 0$ and strictly decreases in $q \in [0,\min\{1,u\}]$. With the equation above, given that $0 < (1-\epsilon)\overline{u} \leq \overline{u}$, we have that $\overline{q} \geq \hat{q}_j$, which further implies that $\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}} \leq \frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}$. This allows us to establish that $\hat{q}_j^{1-\xi_s} \geq ((1-\epsilon)\overline{u})^{1-\xi_s}\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}} = (\overline{q})^{1-\xi_s}(1-\epsilon)^{1-\xi_s}$. Therefore, we have $\hat{q}_j \geq (1-\epsilon)\overline{q}$.

Summarizing the two cases above, we can establish that

$$\mathcal{Y}^h(E) \overset{(a)}{\geq} \sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}\overline{r}\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}) \overset{(b)}{\geq} \sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}\overline{r}(1-\epsilon)\overline{q} \overset{(c)}{=} (1-\epsilon)\overline{\mathcal{Y}}(\overline{E}),$$

where (a) follows from Step 2.1 and $\overline{r} = F_b^{-1}(1-(\overline{q})^{1-\xi_b}) - F_s^{-1}(\frac{(\overline{q})^{1-\xi_s}}{(\overline{u})^{1-\xi_s}}) \leq F_b^{-1}(1) = \overline{v}_b$; (b) follows from the observation that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}) \geq (1-\epsilon)\overline{q}$ for any $j \in \mathcal{B}$; (c) follows directly from the reformulation in Step 2.2. ■