Dynamic Competition under Market Size Dynamics: Balancing the Exploitation-Induction Tradeoff

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Abstract

We study a dynamic competition model, in which retail firms periodically compete on promotional effort, sales price, and service level over a finite planning horizon. The key feature of our model is that the current decisions influence the future market sizes through the service effect and the network effect, i.e., the firm with a higher current service level and a higher current demand is more likely to have larger future market sizes and vice versa. Hence, the competing firms face the tradeoff between generating current profits and inducing future demands (i.e., the *exploitation-induction tradeoff*). Using the linear separability approach, we characterize the pure strategy Markov perfect equilibrium in both the simultaneous competition and the promotion-first competition. The exploitation-induction tradeoff has several important managerial implications under both competitions. First, to balance the exploitation-induction tradeoff, the competing firms should increase promotional efforts, offer price discounts, and improve service levels under the service effect and the network effect. Second, the exploitation-induction tradeoff is more intensive at an earlier stage of the sales season than at later stages, so the equilibrium sales prices are increasing, whereas the equilibrium promotional efforts and service levels are decreasing, over the planning horizon. Third, the competing firms need to balance the exploitation-induction tradeoff inter-temporally under the simultaneous competition, whereas they need to balance this tradeoff both inter-temporally and intra-temporally under the promotion-first competition. Finally, we show that, in the dynamic game with market size dynamics, the exploitation-induction tradeoff could be a new driving force for the "fat-cat" effect (i.e., the equilibrium promotional efforts are higher under the promotion-first competition than those under the simultaneous competition).

Key words: dynamic game; Markov perfect equilibrium; market size dynamics; exploitation-induction tradeoff

1 Introduction

In today's competitive and unstable market environment, it is prevalent that modern firms compete not only on generating current profits, but also on winning future market shares (see, e.g., Klemperer, 1995). The current decisions of all competing firms in the market not only determine their respective current profits, but also significantly influence their future demands. We refer to such inter-temporal dependence of future demands on the current decisions as market size dynamics. Under market size dynamics, myopically optimizing the current profit may lead to significant loss of future demands, and hurt the firm's profit in the long run. Therefore, the competing firms face an important tradeoff between generating current profits and inducing future demands, which we refer to as the *exploitation-induction tradeoff*.

Among others, we focus on two main drivers of the aforementioned exploitation-induction tradeoff: (a) The future demand is positively correlated with the current service level, which we refer to as the *service effect*; and (b) the future demand is positively correlated with the current demand, which we refer to as the *network effect*.

The service effect is driven by the well-recognized phenomenon that the past service experience of a customer significantly impacts his/her future purchasing decisions (see, e.g., Bolton et al., 2006; Aflaki and Popescu, 2014). A poor service (e.g., a low fill rate of a customer's orders) generally diminishes the goodwill of a customer, thus leading to lower future orders from this customer (Adelman and Mersereau, 2013). Moreover, it is widely observed in practice that stockouts can adversely impact future demands (see, e.g., Anderson et al., 2006; Gaur and Park, 2007). In the face of a stockout experience, a natural reaction of a customer is to order fewer items and/or switch the seller in a subsequent purchasing execution (see, e.g., Fitzsimons, 2000; Olsen and Parker, 2008). Therefore, good [poor] past services of a firm are likely to induce high [low] demands in the future.

The network effect, also known as network externalities, refers to the general phenomenon that a customer's utility of purchasing a product is increasing in the number of other customers buying the same product (see, e.g., Economides, 1996). Under the network effect, a higher current demand of a firm leads to more adoptions of its product, thus increasing the utility of purchasing its product for future customers and boosting future demands. There are three major mechanisms that give rise to the network effect: (a) the direct effect, under which an increase in the adoption of a product leads to a direct increase in the value of this product for other users (see, e.g., Katz and Shapiro, 1985); (b) the indirect effect, under which an increase in the adoption of a product enhances the value of its complementary products or services, which in turn increases the value of the original product (see, e.g., Cabral, 2011); and (c) the social effect, under which the value of a product is influenced by the social interactions of its customers with their peers (see, e.g., Bloch and Quérou, 2013).

In the highly inter-correlated and competitive market of the current era, the service effect and the network effect reinforce each other. This is because the fast development of information technology enables customers to easily learn the information (on, e.g., quality, service, popularity, etc.) of any product through communications with their friends and/or the customer reviews on online reviewing platforms and social media. Thus, the higher the current demand of a firm, the more information about its service quality will be released to the public, and, hence, the higher impact its service quality will have upon future demands. Moreover, the current service level of a firm impacts the future demands of itself as well as its competitors, because customers are likely to patronage the firms with good past service and abandon those with poor past service based on either their own purchasing experience or the social learning process.

The primary goal of this paper is to develop a model that can provide insights on how the exploitation-induction tradeoff impacts the equilibrium market behavior under both the service effect and network effect. To this end, we study a periodic-review dynamic competition model, in which firms in a retail market compete under a Markov game over a finite planning horizon. The random demand of each firm in each period is determined by its market size and the current sales prices and promotional efforts of all competing firms. The promotional effort (e.g., advertising, product innovation, and/or after sales service) of a firm boosts the current demand of itself and diminishes that of its competitors. The key feature of our model is that the market sizes of the competing firms are stochastically evolving throughout the planning horizon, and their evolutions are driven by the service effect and the network effect. More specifically, to capture the market size dynamics, we assume that the future market size of each firm is stochastically increasing in its current service level and demand, and stochastically decreasing in the current service levels of its competitors. Taking the market size dynamics into consideration, each firm chooses its promotional effort, sales price, and inventory stocking quantity in each decision period, with an attempt to balance generating current profits and inducing future demands in the dynamic and competitive market. We study two competitions: (a) the simultaneous competition, under which the firms simultaneously make their promotion, price, and inventory decisions in each period; and (b) the promotion-first competition, under which the firms first make their promotional efforts and, after observing the promotion decisions in the market, choose their sales prices and inventory levels in each period.

Conducting a dynamic game analysis, we make two main contributions in this paper: (a) We study a dynamic competition model with the inter-temporal influences of *current* decisions over *future* demands, and characterize the pure strategy Markov perfect equilibrium under both the simultaneous competition and the promotion-first competition; (b) we identify several important managerial implications of the exploitation-induction tradeoff upon the equilibrium market behavior of the dynamic competition under the service effect and the network effect.

We use the Markov perfect equilibrium paradigm to analyze our dynamic competition model, because the competing firms need to adaptively adjust their strategies based on their inventory levels and market sizes in each period. The analytical characterization of Markov perfect equilibria in a dynamic oligopoly with planning horizon length greater than two is, in general, prohibitively difficult (see, e.g., Olsen and Parker, 2014). To characterize the equilibrium market outcome in our model, we employ the linear separability approach (see, e.g., Olsen and Parker, 2008) and show that, under both the simultaneous competition and the promotion-first competition, the equilibrium profit of each firm in each period is linearly separable in its own inventory level and market size. Such linear separability greatly facilitates the analysis and enables us to characterize the pure strategy Markov perfect equilibrium under both competitions. Moreover, under both competitions, the pure strategy Markov perfect equilibrium has the nice feature that the equilibrium strategy of each firm only depends on the private information (i.e., inventory level and market size) of itself, but not on that of its competitors. Under the simultaneous competition, the subgame played by the competing firms in each period can be decomposed into a two-stage competition, in which the firms compete jointly on promotional effort and sales price in the first stage, and on service level in the second. Under the promotion-first competition, the subgame in each period can be decomposed into a three-stage competition, in which the firms compete on promotional effort in the first stage, on sales price in the second, and on service level in the third. Under both competitions, each stage of the subgame in each period has a pure strategy Nash equilibrium, thus ensuring the existence of a pure strategy Markov perfect equilibrium in the Markov game. We also provide mild sufficient conditions under which the Markov perfect equilibrium is unique under each competition.

Under both the simultaneous and the promotion-first competitions, the market size dynamics significantly impact the equilibrium behaviors of the competing firms via the exploitation-induction tradeoff. This tradeoff is quantified by the linear coefficient of market size for each firm in each period. The higher the market size coefficient, the more intensive the exploitation-induction tradeoff for the respective firm in the previous period. We identify three effective strategies under the service effect and the network effect: (a) improving promotional efforts, (b) offering price discounts, and (c) elevating service levels. These strategies are grounded on the uniform idea that, to balance the exploitation-induction tradeoff, the competing firms can induce higher future demands at the cost of reduced current margins. Our analysis demonstrates how the strength of the service effect and network effect impacts the equilibrium market outcome. Under stronger service and network effects, the exploitation-induction tradeoff is more intensive, so the competing firms make more promotional efforts, offer heavier price discounts, and maintain higher service levels. When the market is stationary, the intensity of the exploitation-induction tradeoff decreases over the sales season under both competitions. Hence, the equilibrium sales prices are increasing, whereas the equilibrium promotional efforts and service levels are decreasing, over the planning horizon.

Our analysis reveals two interesting differences between the simultaneous competition and the promotion-first competition under market size dynamics. First, under the simultaneous competition, the competing firms need to balance the exploitation-induction tradeoff inter-temporally, whereas, under the promotion-first competition, they have to balance this tradeoff both inter-temporally and intra-temporally. Second, we identify a new driving force for the "fat-cat" effect (i.e., in each period, the equilibrium promotional efforts may be higher under the promotion-first competition than those under the simultaneous competition): The exploitation-induction tradeoff is more intensive in the promotion-first competition than in the simultaneous competition, thus prompting the firms to make more promotional efforts under the promotion-first competition.

The rest of this paper is organized as follows. We position this paper in the related literature in Section 2. Section 3 introduces the model setup. We analyze the simultaneous competition model in Section 4, and the promotion-first competition model in Section 5. We compare the equilibrium outcomes in these two competitions in Section 6. Section 7 concludes this paper. All proofs are relegated to the Appendix.

2 Literature Review

Our work is related to several streams of research in the literature. The literature on the phenomenon that the current service level impacts future demands is rich. For example, Schwartz (1966, 1970) first studies the inventory management model, in which future demands are adversely affected by current poor service levels. Adelman and Mersereau (2013) consider the dynamic capacity allocation problem of a supplier, whose customers remember past service. Aflaki and Popescu (2014) propose a dynamic behavioral model to study the retention and service relationship management with the effect of past service experiences on future service quality expectations. The impact of current service on future demands has also been analyzed in a competitive environment. Hall and Porteus (2000) investigate a dynamic customer service competition, in which the duopoly firms compete by investing in capacity with a fixed total number of customers. Liu et al. (2007) study a dynamic inventory duopoly model, in which inventory is perishable and customers may defect to a competitor. Olsen and Parker (2008) generalize this model to the setting with non-perishable inventory and the setting in which the firms may attract dissatisfied customers from the competition. Gans (2002) investigates the supplier competition model, in which each customer switches among suppliers based on her past service quality experience. Gaur and Park (2007) study an inventory competition, in which each customer learns about a firm's service level from her previous shopping experience, and makes her potential patronage decision among different firms accordingly. The contribution of our paper to this literature is that we characterize the equilibrium market behavior in the joint promotional effort, sales price, and service level competition under the service effect.

The optimal pricing strategy under network externalities has received considerable attention in the economics and marketing literature. Dhebar and Oren (1986) characterize the optimal nonlinear pricing strategy for a network product with heterogenous customers. Xie and Sirbu (1995) examine the equilibrium dynamic pricing strategies of an incumbent and a later entrant under network externalities. Bensaid and Lesne (1996) consider the optimal dynamic monopoly pricing under network externalities and show that the equilibrium prices increase as time passes. Bloch and Quérou (2013) study the optimal pricing strategy in a network with a given network structure and characterize the relationship between optimal prices and consumers' centrality. We contribute to this stream of literature by analyzing the impact of network externalities upon the competing firms' operations decisions (i.e., the inventory policies) in a dynamic competition.

Our paper is also related to the extensive literature on dynamic pricing and inventory management. This literature diverges into two lines of research: (i) the monopoly model, in which a single firm maximizes its total expected profit over a finite or infinite planning horizon, and (ii) the competition model, in which multiple firms play a noncooperative game to maximize their respective expected per-period profits over an infinite planning horizon. The literature on the monopoly model of joint pricing and inventory management is very rich. Federgruen and Heching (1999) give a general treatment of this problem and show the optimality of the base-stock list-price policy. Chen and Simchi-Levi (2004a,b, 2006) study the joint pricing and inventory management problem with fixed ordering costs for the finite horizon, infinite horizon, and continuous review models. Chen et al. (2006) characterize the optimal policy in the joint pricing and inventory control model with fixed ordering costs and lost sales. Huh and Janakiraman (2008) identify a general condition under which (s, S)-type policies are optimal for a stationary joint pricing and inventory control model with fixed ordering costs. Li and Zheng (2006) study the joint pricing and inventory management problem with the random yield risk, and show that such risk drives the firm to charge a higher price in each period. The joint pricing and inventory control problem with periodic review and positive leadtime is extremely difficult. For this problem, Pang et al. (2012) and Chen et al. (2014) characterize the monotonicity properties of the optimal price and inventory policy for nonperishable and perishable products, respectively. We refer interested readers to Chen and Simchi-Levi (2012) for a comprehensive review on the monopoly models of joint pricing and inventory management.

The research on the competition model of dynamic pricing and inventory management is also abundant. Under deterministic demands, Bernstein and Federgruen (2003) study the EOQ model of a two-echelon distribution system, characterize the equilibrium pricing and replenishment strategies of the competing retailers under both Bertrand and Cournot competitions, and identify the perfect coordination mechanisms therein. Bernstein and Federgruen (2004a) address infinite-horizon models for oligopolies with competing retailers under price-sensitive uncertain demand. Bernstein and Federgruen (2004b) develop a stochastic general equilibrium inventory model, in which retailers compete on both sales price and service level throughout an infinite horizon. Bernstein and Federgruen (2007) generalize this model to a decentralized supply chain setting, and characterize the perfect coordinating mechanisms under price and service competition. Our work differs from this line of literature in that we study the exploitation-induction tradeoff with the service effect and the network effect in a dynamic and competitive market.

To this end, we adopt the Markov perfect equilibrium (i.e., the closed-loop equilibrium) in a finite-horizon model as opposed to the commonly used stationary strategy equilibrium (i.e., the open-loop equilibrium) in an infinite-horizon model.

Finally, from the methodological perspective, our work is related to the literature on the analysis of Markov perfect equilibrium in dynamic competition models. Markov perfect equilibrium is prevalent in the economics literature on dynamic oligopoly models (see, e.g., Maskin and Tirole, 1988; Ericson and Pakes, 1995; Curtat, 1996). In the operations management literature, this equilibrium concept has been widely adopted to study the equilibrium behaviors in dynamic games. Employing the linear separability approach, Hall and Porteus (2000); Liu et al. (2007); Olsen and Parker (2008) characterize the Markov perfect equilibrium in dynamic duopoly models with market size dynamics, and Ahn and Olsen (2007) analyze the structure of the pure strategy Markov perfect equilibria in a dynamic inventory competition with subscriptions. A similar approach based on the separability of player decisions and probability transition functions has been used by Albright and Winston (1979) to study a joint pricing and advertising competition, and by Nagarajan and Rajagopalan (2009) to study a multi-period inventory competition. Due to limited technical tractability, the analysis of Markov perfect equilibrium in nonlinear and nonseparable dynamic games is scarce. Martínez-de-Albéniz and Talluri (2011) characterize the Markov perfect equilibrium price strategy in a finite-horizon dynamic Bertrand competition with fixed capacities. Lu and Lariviere (2012) numerically compute the Markov perfect equilibrium in an infinite-horizon model, in which a supplier allocates its limited capacity to competing retailers. Olsen and Parker (2014) give conditions under which the stationary infinite-horizon equilibrium is also a Markov perfect equilibrium in the context of inventory duopolies. Our paper adopts the linear separability approach to characterize the pure strategy Markov perfect equilibrium of a dynamic joint promotion, price, and inventory competition under both the service effect and the network effect, and analyze the exploitation-induction tradeoff therein.

3 Model

Consider an industry with N competing retail firms, which serve the market with partially substitutable products over a $T-$period planning horizon, labeled backwards as $\{T, T-1, \cdots, 1\}$. In each period t, each firm i selects a promotional effort $\gamma_{i,t} \in [0, \bar{\gamma}_{i,t}]$, which represents the effort the firm makes in advertising, product innovation, and/or after-sales service to promote the demand of its product in the current period. We assume that, in any period t, the total promotional investment cost of each firm i is proportional to its realized demand in period t, $D_{i,t}$, and given by $\nu_{i,t}(\gamma_{i,t})D_{i,t}$. The per-unit demand cost rate, $\nu_{i,t}(\cdot)$, is a non-negative, convexly increasing, and twice continuously differentiable function of the promotional effort $\gamma_{i,t}$, with $\nu_{i,t}(0) = 0$. Before the demand is realized in period t, each firm i selects a sales price

$p_{i,t} \in [\underline{p}_{i,t}, \bar{p}_{i,t}]$ and adjusts its inventory level to $x_{i,t}$. We assume that the excess demand of each firm is fully backlogged. In summary, each firm i makes three decisions at the beginning of any period t: (i) the promotional effort $\gamma_{i,t}$, (ii) the sales price $p_{i,t}$, and (iii) the inventory level $x_{i,t}$.

The demand of each firm i in any period t depends on the entire vector of promotional efforts $\gamma_t := (\gamma_{1,t}, \gamma_{2,t}, \cdots, \gamma_{N,t})$ and the entire vector of sales prices $p_t := (p_{1,t}, p_{2,t}, \cdots, p_{N,t})$ in period t. We denote the demand of firm i as $D_{i,t}(\gamma_t, p_t)$. More specifically, we base our analysis on the following multiplicative form of $D_{i,t}(\cdot,\cdot)$:

$$D_{i,t}(\gamma_t, p_t) = \Lambda_{i,t} d_{i,t}(\gamma_t, p_t)\xi_{i,t}, \tag{1}$$

where $\Lambda_{i,t} > 0$ is the market size of firm i in period t, $d_{i,t}(\gamma_t, p_t) > 0$ captures the impact of γ_t and p_t on firm i's demand in period t, and $\xi_{i,t}$ is a positive continuous random variable with a connected support. Let $F_{i,t}(\cdot)$ be the *c.d.f.* and $\bar{F}_{i,t}(\cdot)$ be the *c.c.d.f.* of $\xi_{i,t}$. The market size $\Lambda_{i,t}$ is observable by firm i at the beginning of period t through the pre-order sign-ups and/or subscriptions before the release of its product in period t. The random perturbation term $\xi_{i,t}$ is independent of the market size vector $\Lambda_t := (\Lambda_{1,t}, \Lambda_{2,t}, \cdots, \Lambda_{N,t})$, the sales price vector p_t, and the promotional effort vector γ_t. Moreover, $\{\xi_{i,t} : t = T, T-1, \cdots, 1\}$ are independently distributed for each i. Without loss of generality, we normalize $\mathbb{E}[\xi_{i,t}] = 1$ for each i and any t, i.e., $\mathbb{E}[D_{i,t}(\gamma_t, p_t)] = \Lambda_{i,t} d_{i,t}(\gamma_t, p_t)$. Therefore, $d_{i,t}(\gamma_t, p_t)$ can be viewed as the normalized expected demand of firm i in period t.

We assume that $d_{i,t}(\cdot,\cdot)$ is twice continuously differentiable on $[0, \bar{\gamma}_{1,t}] \times [0, \bar{\gamma}_{2,t}] \times \cdots \times [0, \bar{\gamma}_{N,t}] \times [\underline{p}_{1,t}, \bar{p}_{1,t}] \times [\underline{p}_{2,t}, \bar{p}_{2,t}] \times \cdots \times [\underline{p}_{N,t}, \bar{p}_{N,t}]$, and satisfies the following monotonicity properties:

$$\frac{\partial d_{i,t}(\gamma_t, p_t)}{\partial \gamma_{i,t}} > 0, \ \frac{\partial d_{i,t}(\gamma_t, p_t)}{\partial \gamma_{j,t}} < 0, \ \frac{\partial d_{i,t}(\gamma_t, p_t)}{\partial p_{i,t}} < 0, \text{ and } \frac{\partial d_{i,t}(\gamma_t, p_t)}{\partial p_{j,t}} > 0, \text{ for all } j \neq i. \tag{2}$$

In other words, an increase in a firm's promotional effort increases the current-period demand of itself, and decreases the demands of its competitors. On the other hand, an increase in a firm's sales price decreases the demand of itself, and increases the demands of its competitors. Moreover, we assume that $d_{i,t}(\cdot,\cdot)$ is log-separable, i.e., $d_{i,t}(\gamma_t, p_t) = \psi_{i,t}(\gamma_t)\rho_{i,t}(p_t)$, where $\psi_{i,t}(\cdot)$ and $\rho_{i,t}(\cdot)$ are positive and twice-continuously differentiable. Inequalities (2) imply that

$$\frac{\partial \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t}} > 0, \ \frac{\partial \psi_{i,t}(\gamma_t)}{\partial \gamma_{j,t}} < 0, \ \frac{\partial \rho_{i,t}(p_t)}{\partial p_{i,t}} < 0, \text{ and } \frac{\partial \rho_{i,t}(p_t)}{\partial p_{j,t}} > 0, \text{ for all } j \neq i.$$

For technical tractability, we assume that $\psi_{i,t}(\cdot)$ and $\rho_{i,t}(\cdot)$ satisfy the log increasing differences and the diagonal dominance conditions for each i and any t, i.e.,

$$\frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t}^2} < 0, \ \frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t}\partial \gamma_{j,t}} \geq 0 \text{ for all } j \neq i, \text{ and } \left|\frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t}^2}\right| > \sum_{j\neq i} \frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t}\partial \gamma_{j,t}}; \tag{3}$$

$$\frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t}^2} < 0, \ \frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t}\partial p_{j,t}} \geq 0 \text{ for all } j \neq i, \text{ and } \left|\frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t}^2}\right| > \sum_{j\neq i} \frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t}\partial p_{j,t}}. \tag{4}$$

The log increasing differences and the diagonal dominance assumptions are not restrictive, and can be satisfied by a large set of commonly used demand models in the economics and operations management literature, such as the linear, logit, Cobb-Douglas, and CES demand functions (see, e.g., Milgrom and Roberts, 1990; Bernstein and Federgruen, 2004a,b).

The expected fill rate of firm i in period t, $z_{i,t}$, is given by

$$z_{i,t} = \frac{\mathbb{E}[x_{i,t}^+ \wedge D_{i,t}(\gamma_t, p_t)]}{\mathbb{E}[D_{i,t}(\gamma_t, p_t)]} = \frac{\mathbb{E}[(\Lambda_{i,t} d_{i,t}(\gamma_t, p_t) y_{i,t})^+ \wedge (\Lambda_{i,t} d_{i,t}(\gamma_t, p_t)\xi_{i,t})]}{\Lambda_{i,t} d_{i,t}(\gamma_t, p_t)} = \mathbb{E}(y_{i,t}^+ \wedge \xi_{i,t}),$$

where $y_{i,t} := \frac{x_{i,t}}{\Lambda_{i,t} d_{i,t}(\gamma_t, p_t)}$ and $a \wedge b := \min\{a, b\}$ for any $a, b \in \mathbb{R}$. Thus, $z_{i,t}$ is concavely increasing in $y_{i,t}$ for all $y_{i,t} \geq 0$. Moreover, $z_{i,t} = 0$ if $y_{i,t} \leq 0$, and $z_{i,t} \uparrow 1$, if $y_{i,t} \to +\infty$.

The key feature of our model is that current promotion, pricing, and inventory decisions impact upon future demands via the service effect and the network effect. To model these two effects, we assume that the market size of each firm in the next period is given by the following functional form:

$$\Lambda_{i,t-1} = \eta_{i,t}(z_t, D_{i,t}, \Lambda_{i,t}, \Xi_{i,t}) = \Lambda_{i,t}\Xi_{i,t}^1 + \alpha_{i,t}(z_t) D_{i,t}\Xi_{i,t}^2, \tag{5}$$

where $\Xi_{i,t}^1$ is a positive random variable representing the market size changes driven by exogenous factors such as economic environment. Let $\mu_{i,t} := \mathbb{E}[\Xi_{i,t}^1] > 0$. The term $\alpha_{i,t}(z_t) D_{i,t}\Xi_{i,t}^2$ summarizes the service effect and the network effect. Specifically, $\alpha_{i,t}(\cdot) \geq 0$ is a continuously differentiable function with

$$\frac{\partial \alpha_{i,t}(z_t)}{\partial z_{i,t}} \geq 0, \text{ and } \frac{\partial \alpha_{i,t}(z_t)}{\partial z_{j,t}} \leq 0, \text{ for all } j \neq i,$$

and $\Xi_{i,t}^2$ is a nonnegative random variable with $\mathbb{E}[\Xi_{i,t}^2] = 1$. $\Xi_{i,t}^2$ captures the random perturbations in the market size changes driven by the service effect and the network effect. We refer to $\{\alpha_{i,t}(\cdot) : 1 \leq i \leq N, T \geq t \geq 1\}$ as the market size evolution functions. Moreover, for technical tractability, we assume that $\alpha_{i,t}(\cdot)$ is additively separable, i.e.,

$$\alpha_{i,t}(z_t) = \kappa_{ii,t}(z_{i,t}) - \sum_{j \neq i} \kappa_{ij,t}(z_{j,t}),$$

where $\kappa_{ii,t}(\cdot) > 0$ is concave, increasing and continuously differentiable in $z_{i,t}$, and $\kappa_{ij,t}(\cdot) \geq 0$ is continuously increasing in $z_{j,t}$ for all $j \neq i$. Since $\alpha_{i,t}(\cdot) \geq 0$ for all z_t, $\kappa_{ii,t}(0) - \sum_{j \neq i} \kappa_{ij,t}(1) \geq 0$. Let $\eta_t(\cdot,\cdot,\cdot,\cdot) := (\eta_{1,t}(\cdot,\cdot,\cdot,\cdot), \eta_{2,t}(\cdot,\cdot,\cdot,\cdot), \cdots, \eta_{N,t}(\cdot,\cdot,\cdot,\cdot))$ denote the market size vector in the next period.

The evolution of the market sizes, (5), has several important implications. First, the future market size of each firm depends on its current market size in a Markovian fashion. Thus, the dynamic competition model in this paper falls into the regime of Markov games. Second, although the service level of each firm does not influence the current demand of any firm due to the unobservability of the firms' inventory information to customers, it will impact the firms' future demands. This phenomenon is driven by the service effect. The higher the service level of a firm, the better service experience the customers have with this firm in the current period, and the more customers will patronage this firm in the future. Analogously, if the service levels of a firm's competitors increase, customers will be more likely to purchase from its competitors in the future. Therefore, the future demand of each firm is stochastically increasing in the current service level of this firm and stochastically decreasing in the current service level of any of its competitors. Hence, the inventory decision of each firm has the demand-inducing value driven by the service effect. Third, the future demand of each firm is positively correlated with the current demand of this firm. This phenomenon is driven by the network effect. If the realized current demand of a firm is higher, potential customers can get higher utilities if purchasing from this firm, thus giving rise to higher future demand. Because of the network effect, the sales price and promotional effort not only affect the current demand, but also influence future demands. Fourth, the service effect and the network effect reinforce each other. More specifically, the impact of current service levels upon future market sizes is higher with higher realized current demands. With the explosive growth of online social media, customers could easily learn the service qualities of all firms through social learning. As a consequence, higher current demands lead to more intensive social interactions among customers, and, hence, magnify the impact of current service levels on future demands.

We introduce the following model primitives:

$$\begin{aligned}
\delta_i &= \text{discount factor of firm } i \text{ for revenues and costs in future periods, } 0 < \delta_i \leq 1,\\
w_{i,t} &= \text{per-unit wholesales price paid by firm } i \text{ in period } t,\\
b_{i,t} &= \text{per-unit backlogging cost paid by firm } i \text{ in period } t,\\
h_{i,t} &= \text{per-unit holding cost paid by firm } i \text{ in period } t.
\end{aligned}$$

Without loss of generality, we assume the following inequalities hold for each i and t:

$b_{i,t} > w_{i,t} - \delta_i w_{i,t-1}$:the backlogging penalty is higher than the saving from delaying an order to the next period for each firm in any period, so that no firm will backlog all of its demand,

$h_{i,t} > \delta_i w_{i,t-1} - w_{i,t}$:the holding cost is sufficiently high so that no firm will place a speculative order.

$\bar{p}_{i,t} > \delta_i w_{i,t-1} + b_{i,t} + \nu_{i,t}(\bar{\gamma}_{i,t})$:positive margin for backlogged demand with highest price and promotional effort.

We define the normalized expected holding and backlogging cost function for firm i in period t:

$$L_{i,t}(y_{i,t}) := \mathbb{E}\{h_{i,t}(y_{i,t} - \xi_{i,t})^+ + b_{i,t}(y_{i,t} - \xi_{i,t})^-\}, \text{ where } y_{i,t} \in \mathbb{R}. \tag{6}$$

The state of the Markov game is given by:

$$\begin{aligned}
I_t &= (I_{1,t}, I_{2,t}, \cdots, I_{N,t}) = \text{the vector for the starting inventories of all firms in period } t,\\
\Lambda_t &= (\Lambda_{1,t}, \Lambda_{2,t}, \cdots, \Lambda_{N,t}) = \text{the vector for the market sizes of all firms in period } t.
\end{aligned}$$

We use $\mathcal{S} := \mathbb{R}^N \times \mathbb{R}_+^N$ to denote the state space of each firm i in the dynamic competition.

To characterize how the market size dynamics (i.e., the service effect and the network effect) impact the equilibrium market outcome, we consider the Markov perfect equilibrium (MPE) in our dynamic competition model. An MPE satisfies two conditions: (a) in each period t, each firm i's promotion, price, and inventory strategy depends on the history of the game only through the current period state variables (I_t, Λ_t), and (b) in each period t, the strategy profile generates a Nash equilibrium in the associated proper subgame. In other words, MPE is a closed-loop equilibrium that satisfies subgame perfection in each period. Because of its simplicity and consistency with rationality, MPE is widely used in dynamic competition models in the economics (e.g., Maskin and Tirole, 1988) and operations management (e.g., Olsen and Parker, 2008) literature.

A major technical challenge to characterize the MPE in a dynamic inventory competition model is that when the starting inventories are higher than the equilibrium order-up-to levels, the model becomes illy behaved and analytically intractable (see, e.g., Olsen and Parker, 2014). This issue is worsened under endogenous pricing decisions (see, e.g., Bernstein and Federgruen, 2007). To overcome this technical challenge, we make the following assumption throughout our analysis.

Assumption **1** At the beginning of each period t, each firm i is allowed to sell (potentially part of) its onhand inventory to its supplier at the current-period per-unit wholesale price $w_{i,t}$.

Assumption 1 is imposed to circumvent the aforementioned technical challenge. As will be clear by our subsequent analysis, with this assumption, the equilibrium profit of each firm i in each period t is linearly separable in its starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$. Assumption 1 enables us to eliminate the influence of current inventory decision of any firm upon the future equilibrium behavior of the market, so as to single out and highlight the exploitation-induction tradeoff with the service effect and the network effect. Assumption 1 applies when the retail firms have such great market power that they can reach an agreement with their respective suppliers on the return policy with full price refund. Bernstein and Federgruen (2007), among others, also make this assumption to characterize the MPE in an infinite-horizon joint price and service level competition model. With Assumption 1, we can define the action space of each firm i in each period t: $\mathcal{A}_{i,t}(I_{i,t}) := [0, \bar{\gamma}_{i,t}] \times [\underline{p}_{i,t}, \bar{p}_{i,t}] \times [\min\{0, I_{i,t}\}, +\infty)$.

4 Simultaneous Competition

In this section, we study the simultaneous competition (SC) model where each firm i simultaneously chooses a combined promotion, price, and inventory strategy in any period t. This model applies to the scenarios where the market expanding efforts (e.g., advertising, trade-in programs, etc.) take effect instantaneously, so, in essence, the promotional effort and sales price decisions are made simultaneously in each period. Our analysis in this section focuses on characterizing the pure strategy MPE and providing insights on the impact of the exploitation-induction tradeoff in the SC model.

4.1 Equilibrium Analysis

In this subsection, we show that the simultaneous competition model has a pure strategy MPE. Moreover, we characterize a sufficient condition on the per-unit demand cost rate of promotional effort, $\nu_{i,t}(\cdot)$, under which the MPE is unique. Without loss of generality, we assume that, at the end of the planning horizon, each firm i salvages all the on-hand inventory and fulfills all the backlogged demand at unit wholesale price $w_{i,0} \geq 0$. The payoff function of each firm i is given by:

$$\begin{aligned} & \mathbb{E}\{\sum_{t=1}^{T}\delta_i^{T-t}[p_{i,t}D_{i,t}(\gamma_t,p_t) - w_{i,t}(x_{i,t}-I_{i,t}) - h_{i,t}(x_{i,t}-D_{i,t}(\gamma_t,p_t))^+ - b_{i,t}(x_{i,t}-D_{i,t}(\gamma_t,p_t))^- \\ & -\nu_{i,t}(\gamma_{i,t})D_{i,t}(\gamma_t,p_t)] + \delta_i^T w_{i,0}I_{i,0}|I_T,\Lambda_T\}, \qquad (7) \\ \text{s.t.} \quad & I_{i,t-1} = x_{i,t} - D_{i,t}(\gamma_t,p_t) \text{ for each } t, \\ \text{and} \quad & \Lambda_{i,t-1} = \Lambda_{i,t}\Xi^1_{i,t} + \alpha_{i,t}(z_t)D_{i,t}(\gamma_t,p_t)\Xi^2_{i,t} \text{ for each } t. \end{aligned}$$

Under an MPE, each firm i should try to maximize its expected payoff in each subgame (i.e., in each period t) conditioned on the realized inventory levels and market sizes in period t, (I_t, Λ_t):

$$\begin{aligned} & \mathbb{E}\{\sum_{\tau=1}^{t}\delta_i^{t-\tau}[p_{i,\tau}D_{i,\tau}(\gamma_\tau,p_\tau) - w_{i,\tau}(x_{i,\tau}-I_{i,\tau}) - h_{i,\tau}(x_{i,\tau}-D_{i,\tau}(\gamma_\tau,p_\tau))^+ - b_{i,\tau}(x_{i,\tau}-D_{i,\tau}(\gamma_\tau,p_\tau))^- \\ & -\nu_{i,\tau}(\gamma_{i,\tau})D_{i,\tau}(\gamma_\tau,p_\tau)] + \delta_i^t w_{i,0}I_{i,0}|I_t,\Lambda_t\}, \qquad (8) \\ \text{s.t.} \quad & I_{i,\tau-1} = x_{i,\tau} - D_{i,\tau}(\gamma_\tau,p_\tau) \text{ for each } \tau,\ t\geq\tau\geq 1, \\ \text{and} \quad & \Lambda_{i,\tau-1} = \Lambda_{i,\tau}\Xi^1_{i,\tau} + \alpha_{i,\tau}(z_\tau)D_{i,\tau}(\gamma_\tau,p_\tau)\Xi^2_{i,\tau} \text{ for each } \tau,\ t\geq\tau\geq 1. \end{aligned}$$

A (pure) Markov strategy profile in the SC model $\sigma^{sc} := \{\sigma^{sc}_{i,t}(\cdot,\cdot) : 1 \leq i \leq N, T \geq t \geq 1\}$ prescribes each firm i's combined promotion, price, and inventory strategy in each period t, where $\sigma^{sc}_{i,t}(\cdot,\cdot) := (\gamma^{sc}_{i,t}(\cdot,\cdot), p^{sc}_{i,t}(\cdot,\cdot), x^{sc}_{i,t}(\cdot,\cdot))$ is a Borel measurable mapping from $\mathcal{S}$ to $\mathcal{A}_{i,t}(I_{i,t})$. We use $\sigma^{sc}_t := \{\sigma^{sc}_{i,t}(\cdot,\cdot) : 1 \leq i \leq N, T \geq t \geq 1\}$ to denote the pure strategy profile in the induced subgame in period t, which prescribes each firm i's (pure) strategy from period t till the end of the planning horizon.

To evaluate the expected payoff of each firm i in each period t for any given Markov strategy profile σ^{sc} in the simultaneous competition, let

$V_{i,t}(I_t,\Lambda_t|\sigma^{sc}_t) =$ the total expected discounted profit of firm i in periods $t, t-1, \cdots, 1, 0$, when starting period t with the state variable (I_t, Λ_t) and the firms play strategy σ^{sc}_t in periods $t, t-1, \cdots, 1$.

Thus, by backward induction, $V_{i,t}(\cdot,\cdot|\sigma^{sc}_t)$ satisfies the following recursive scheme for each firm i in each period t:

$$V_{i,t}(I_t,\Lambda_t|\sigma^{sc}_t) = J_{i,t}(\gamma^{sc}_t(I_t,\Lambda_t), p^{sc}_t(I_t,\Lambda_t), x^{sc}_t(I_t,\Lambda_t), I_t, \Lambda_t|\sigma^{sc}_{t-1}),$$

where $\gamma_t^{sc}(\cdot,\cdot) = (\gamma_{1,t}^{sc}(\cdot,\cdot), \gamma_{2,t}^{sc}(\cdot,\cdot), \cdots, \gamma_{N,t}^{sc}(\cdot,\cdot))$ is the period t promotional effort vector prescribed by σ^{sc}, $p_t^{sc}(\cdot,\cdot) = (p_{1,t}^{sc}(\cdot,\cdot), p_{2,t}^{sc}(\cdot,\cdot), \cdots, p_{N,t}^{sc}(\cdot,\cdot))$ is the period t sales price vector prescribed by σ^{sc}, $x_t^{sc}(\cdot,\cdot) = (x_{1,t}^{sc}(\cdot,\cdot), x_{2,t}^{sc}(\cdot,\cdot), \cdots, x_{N,t}^{sc}(\cdot,\cdot))$ is the period t post-delivery inventory vector prescribed by σ^{sc},

$$\begin{aligned} J_{i,t}(\gamma_t, p_t, x_t, I_t, \Lambda_t|\sigma_{t-1}^{sc}) \quad = \quad & \mathbb{E}\{p_{i,t}D_{i,t}(\gamma_t, p_t) - w_{i,t}(x_{i,t} - I_{i,t}) - h_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^+ \\ & -b_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^- - \nu_{i,t}(\gamma_{i,t})D_{i,t}(\gamma_t, p_t) \\ & +\delta_i V_{i,t-1}(x_t - D_t(\gamma_t, p_t), \eta_t(z_t, D_t(\gamma_t, p_t), \Lambda_t, \Xi_t)|\sigma_{t-1}^{sc})|I_t, \Lambda_t\} \end{aligned} \quad (9)$$

and $V_{i,0}(I_t, \Lambda_t) = w_{i,0}I_{i,0}$. We now formally define the pure strategy MPE in the SC model.

Definition 1 A (pure) Markov strategy $\sigma^{sc*} = \{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \le i \le N, T \ge t \ge 1\}$ is a pure strategy MPE in the SC model if and only if, for each firm i, each period t, and each state variable (I_t, Λ_t),

$$\begin{aligned} & (\gamma_{i,t}^{sc*}(I_t, \Lambda_t), p_{i,t}^{sc*}(I_t, \Lambda_t), x_{i,t}^{sc*}(I_t, \Lambda_t)) \\ = & \text{argmax}_{(\gamma_{i,t}, p_{i,t}, x_{i,t}) \in \mathcal{A}_{i,t}(I_{i,t})}\{J_{i,t}([\gamma_{i,t}, \gamma_{-i,t}^{sc*}(I_t, \Lambda_t)], [p_{i,t}, p_{-i,t}^{sc*}(I_t, \Lambda_t)], [x_{i,t}, x_{-i,t}^{sc*}(I_t, \Lambda_t)], I_t, \Lambda_t|\sigma_{t-1}^{sc*})\}. \end{aligned} \quad (10)$$

By Definition 1, a (pure) Markov strategy profile in the SC model is a pure strategy MPE if it satisfies subgame perfection in each period t. Definition 1 does not guarantee the existence of an MPE, σ^{sc*}, in the SC model. In Theorem 1, below, we will show a pure strategy MPE always exists in the SC model. Moreover, under a mild additional assumption on $\nu_{i,t}(\cdot)$, the SC model has a unique pure strategy MPE. By Definition 1, the equilibrium strategy for firm i in period t, $(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot))$, may depend on the state vector of its competitors $(I_{-i,t}, \Lambda_{-i,t})$. In practice, however, each firm i's starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$ are generally its private information that is not accessible by its competitors in the market. We will show that the equilibrium strategy profile of each firm i in each period t is only contingent on its own realized state variables $(I_{i,t}, \Lambda_{i,t})$, but independent of its competitors' private information $(I_{-i,t}, \Lambda_{-i,t})$. The following theorem characterizes the existence and the uniqueness of MPE in the SC model.

Theorem 1 The following statements hold for the SC model:

(a) There exists a pure strategy MPE $\sigma^{sc*} = \{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \le i \le N, T \ge t \ge 1\}$.

(b) For each pure strategy MPE, σ^{sc*}, there exists a series of vectors $\{\beta_t^{sc} : T \ge t \ge 1\}$, where $\beta_t^{sc} = (\beta_{1,t}^{sc}, \beta_{2,t}^{sc}, \cdots, \beta_{N,t}^{sc})$ with $\beta_{i,t}^{sc} > 0$ for each i and t, such that

$$V_{i,t}(I_t, \Lambda_t|\sigma_t^{sc*}) = w_{i,t}I_{i,t} + \beta_{i,t}^{sc}\Lambda_{i,t}, \text{ for each firm } i \text{ and each period } t. \quad (11)$$

(c) If the following two conditions simultaneously hold for each i and t:

(i) $\nu'_{i,t}(\cdot) \leq 1$ for all $\gamma_{i,t} \in [0, \bar{\gamma}_{i,t}]$; and

(ii) $\nu''_{i,t}(\gamma_{i,t})(p_{i,t} - \delta w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \underline{c}_{i,t}) + [\nu'_{i,t}(\gamma_{i,t})]^2 \geq \nu'_{i,t}(\gamma_{i,t})$ for all $p_{i,t} \in [\underline{p}_{i,t}, \bar{p}_{i,t}]$ and $\gamma_{i,t} \in [0, \bar{\gamma}_{i,t}]$, where

$$\underline{c}_{i,t} := \max\{(\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) : y_{i,t} \geq 0\},$$

σ^{sc*} is the unique MPE in the SC model. In particular, if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, conditions (i) and (ii) are satisfied.

Theorem 1(a) demonstrates the existence of a pure strategy MPE in the simultaneous competition model. Moreover, in Theorem 1(b), we show that, for each pure strategy MPE σ^{sc*}, the corresponding profit function of each firm i in each period t is linearly separable in its starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$. We refer to the constant $\beta^{sc}_{i,t}$ as the SC market size coefficient of firm i in period t. As we will show later, the SC market size coefficient measures the intensity of the exploitation-induction tradeoff. The larger the $\beta^{sc}_{i,t}$, the more intensive the exploitation-induction tradeoff for firm i in the previous period $t+1$. Theorem 1(b) also implies that the equilibrium profit of each firm i in each period t only depends on the state variables of itself $(I_{i,t}, \Lambda_{i,t})$, but not on those of its competitors $(I_{-i,t}, \Lambda_{-i,t})$. Theorem 1(c) characterizes a sufficient condition for the uniqueness of an MPE in the SC model. In particular, if the promotional effort $\gamma_{i,t}$ refers to the actual monetary payment of promotional investment per-unit demand for each firm i in each period t (i.e., $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for each i and t), there exists a unique MPE in the SC model. For the rest of this paper, we assume that conditions (i) and (ii) are satisfied for each i and t and, hence, the SC model has a unique pure strategy MPE σ^{sc*}.

The linear separability of $V_{i,t}(\cdot, \cdot|\sigma^{sc*}_t)$ (i.e., Theorem 1(b)) enables us to characterize the MPE in the SC model. Plugging (11) into the objective function of firm i in period t, by $x_{i,t} = \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)y_{i,t}$ and $z_{i,t} = \mathbb{E}(y^+_{i,t} \wedge \xi_{i,t})$, we have:

$$\begin{aligned}
J_{i,t}(\gamma_t, p_t, x_t, I_t, \Lambda_t|\sigma^{sc*}_{t-1}) =& \mathbb{E}\{p_{i,t}D_{i,t}(\gamma_t, p_t) - w_{i,t}(x_{i,t} - I_{i,t}) - h_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^+ \\
&- b_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^- - \nu_{i,t}(\gamma_{i,t})D_{i,t}(\gamma_t, p_t) \\
&+ \delta_i V_{i,t-1}(x_t - D_t(\gamma_t, p_t), \eta_t(z_t, D_t(\gamma_t, p_t), \Lambda_t, \Xi_t)|\sigma^{sc*}_{t-1})|I_t, \Lambda_t\} \\
=& \mathbb{E}\{p_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t} - w_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - I_{i,t}) \\
&- h_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t})^+ \\
&- b_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t})^- \\
&- \nu_{i,t}(\gamma_{i,t})\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t} + \delta_i w_{i,t-1}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t}) \\
&+ \delta_i\beta^{sc}_{i,t-1}(\Lambda_{i,t}\Xi^1_{i,t} + \alpha_{i,t}(z_t)\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t}\Xi^2_{i,t})|I_t, \Lambda_t\} \\
=& w_{i,t}I_{i,t} + \Lambda_{i,t}\{\delta_i\beta^{sc}_{i,t-1}\mu_{i,t} + \psi_{i,t}(\gamma_t)\rho_{i,t}(p_t)[p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc}_{i,t}(y_t)]\},
\end{aligned} \tag{12}$$

$$\text{where } \pi^{sc}_{i,t}(y_t) = (\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i \beta^{sc}_{i,t-1}(\kappa_{ii,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}]) - \sum_{j\neq i} \kappa_{ij,t}(\mathbb{E}[y^+_{j,t} \wedge \xi_{j,t}])),$$
$$\text{and } \beta^{sc}_{i,0} := 0 \text{ for each } i.$$

We observe from (12) that the payoff function of each firm i in the subgame of period t has a nested structure. Hence, the subgame of period t can be decomposed into two stages, where the firms compete jointly on promotion and price in the first stage, and on inventory in the second stage. Since the service level of each firm i, as measured by the expected fill rate $z_{i,t}$, is increasing in the inventory decision $y_{i,t}$, we refer to the second-stage competition as the service level competition hereafter. By backward induction, we first study the second-stage service level competition. Let $\mathcal{G}^{sc,2}_t$ be the $N-$player noncooperative game that represents the second-stage service level competition in period t, where player i has payoff function $\pi^{sc}_{i,t}(\cdot)$ and feasible action set $\mathbb{R}$. The following proposition characterizes the Nash equilibrium of the game $\mathcal{G}^{sc,2}_t$.

Proposition **1** *For each period t, the second-stage service level competition $\mathcal{G}^{sc,2}_t$ has a unique pure strategy Nash equilibrium y^{sc*}_t. Moreover, for each i, $y^{sc*}_{i,t} > 0$ is the unique solution to the following equation:*

$$(\delta_i w_{i,t-1} - w_{i,t}) - L'_{i,t}(y^{sc*}_{i,t}) + \delta_i \beta^{sc}_{i,t-1} \bar{F}_{i,t}(y^{sc*}_{i,t}) \kappa'_{ii,t}(\mathbb{E}(y^{sc*}_{i,t} \wedge \xi_{i,t})) = 0. \tag{13}$$

Proposition 1 demonstrates the existence and uniqueness of a pure strategy Nash equilibrium of the second-stage service level competition. Moreover, $y^{sc*}_{i,t}$ can be obtained by solving the first-order condition $\partial_{y_{i,t}} \pi^{sc}_{i,t}(y^{sc*}_t) = 0$. Let $\pi^{sc*}_t := (\pi^{sc*}_{1,t}, \pi^{sc*}_{2,t}, \cdots, \pi^{sc*}_{N,t})$ be the equilibrium payoff vector of the second-stage service level competition in period t, where $\pi^{sc*}_{i,t} = \pi^{sc}_{i,t}(y^{sc*}_t)$. For each i and t, let

$$\Pi^{sc}_{i,t}(\gamma_t, p_t) := \psi_{i,t}(\gamma_t)\rho_{i,t}(p_t)[p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}]. \tag{14}$$

We define an $N-$player noncooperative game $\mathcal{G}^{sc,1}_t$ to represent the first-stage joint promotion and price competition in period t, where player i has payoff function $\Pi^{sc}_{i,t}(\cdot,\cdot)$ and feasible action set $[0, \bar{\gamma}_{i,t}] \times [\underline{p}_{i,t}, \bar{p}_{i,t}]$. We characterize the Nash equilibrium of the game $\mathcal{G}^{sc,1}_t$ in the following proposition.

Proposition **2** *For each period t, following statements hold:*

(a) The first-stage joint promotion and price competition, $\mathcal{G}^{sc,1}_t$, is a log-supermodular game.

(b) The game $\mathcal{G}^{sc,1}_t$ has a unique pure strategy Nash equilibrium $(\gamma^{sc*}_t, p^{sc*}_t)$, which is the unique serially undominated strategy of $\mathcal{G}^{sc,1}_t$.

(c) *The Nash equilibrium of* $\mathcal{G}_t^{sc,1}$ *is the unique solution to the following system of equations:*

$$\text{For each } i,\ \frac{\partial_{\gamma_{i,t}}\psi_{i,t}(\gamma_t^{sc*})}{\psi_{i,t}(\gamma_t^{sc*})} - \frac{\nu'_{i,t}(\gamma_{i,t}^{sc*})}{p_{i,t}^{sc*} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}^{sc*}) + \pi_{i,t}^{sc*}} \begin{cases} \le 0, & \text{if } \gamma_{i,t}^{sc*} = 0, \\ = 0, & \text{if } \gamma_{i,t}^{sc*} \in (0, \bar{\gamma}_{i,t}), \\ \ge 0 & \text{if } \gamma_{i,t}^{sc*} = \bar{\gamma}_{i,t}; \end{cases} \text{ and,}$$

$$\text{for each } i,\ \frac{\partial_{p_{i,t}}\rho_{i,t}(p_t^{sc*})}{\rho_{i,t}(p_t^{sc*})} + \frac{1}{p_{i,t}^{sc*} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}^{sc*}) + \pi_{i,t}^{sc*}} \begin{cases} \le 0, & \text{if } p_{i,t}^{sc*} = \underline{p}_{i,t}, \\ = 0, & \text{if } p_{i,t}^{sc*} \in (\underline{p}_{i,t}, \bar{p}_{i,t}), \\ \ge 0 & \text{if } p_{i,t}^{sc*} = \bar{p}_{i,t}. \end{cases} \tag{15}$$

(d) *Let* $\Pi_t^{sc*} := (\Pi_{1,t}^{sc*}, \Pi_{2,t}^{sc*}, \cdots, \Pi_{N,t}^{sc*})$ *be the equilibrium payoff vector of the first-stage joint promotion and price competition in period* t*, where* $\Pi_{i,t}^{sc*} = \Pi_{i,t}^{sc}(\gamma_t^{sc*}, p_t^{sc*})$*. We have* $\Pi_{i,t}^{sc*} > 0$ *for all* i*.*

Proposition 2 shows that the first-stage joint promotion and price competition $\mathcal{G}_t^{sc,1}$ is a log-supermodular game, and has a unique pure strategy Nash equilibrium $(\gamma_t^{sc*}, p_t^{sc*})$. The unique Nash equilibrium, $(\gamma_t^{sc*}, p_t^{sc*})$, is determined by (i) the serial elimination of strictly dominated strategies, or (ii) the system of first-order conditions (15). Under equilibrium, by Proposition 2(d) and the objective function of period t, (12), each firm i earns a positive normalized expected total discounted profit, $\Lambda_{i,t}(\delta_i\beta_{i,t-1}^{sc}\mu_{i,t} + \Pi_{i,t}^{sc*})$, in the subgame of period t. Summarizing Theorem 1, Proposition 1 and Proposition 2, we have the following theorem that sharpens the characterization of the MPE in the SC model.

THEOREM **2** For each period t, the following statements hold:

(a) For each i, $\beta_{i,t}^{sc} = \delta_i\beta_{i,t-1}^{sc}\mu_{i,t} + \Pi_{i,t}^{sc*}$.

(b) Under the unique (pure strategy) MPE σ^{sc*}, the policy of firm i is given by

$$(\gamma_{i,t}^{sc*}(I_t, \Lambda_t), p_{i,t}^{sc*}(I_t, \Lambda_t), x_{i,t}^{sc*}(I_t, \Lambda_t)) = (\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}, \Lambda_{i,t}y_{i,t}^{sc*}\rho_{i,t}(p_t^{sc*})\psi_{i,t}(\gamma_t^{sc*})). \tag{16}$$

Theorem 2(a) recursively computes the SC market size coefficient vectors $\{\beta_t^{sc} : T \ge t \ge 1\}$. Theorem 2(b) demonstrates that, under the MPE σ^{sc*}, each firm i's joint promotion, price, and inventory policy in each period t only depends on its own state variables $(I_{i,t}, \Lambda_{i,t})$, but not on those of its competitors $(I_{-i,t}, \Lambda_{-i,t})$, which are not accessible to firm i in general. Thus, for each firm i in each period t, its equilibrium strategy has the attractive feature that the strategy depends on its accessible information only.

In some of our analysis below, we will consider a special case of the SC model, where the market is symmetric, i.e., all competing firms have identical characteristics. We use the subscript "s" to denote the case of symmetric market. In this case, for all i, j, and t, let $\rho_{s,t}(\cdot) := \rho_{i,t}(\cdot)$, $\psi_{s,t}(\cdot) := \psi_{i,t}(\cdot)$, $\nu_{s,t}(\cdot) := \nu_{i,t}(\cdot)$, $\alpha_{s,t}(\cdot) := \alpha_{i,t}(\cdot)$, $\kappa_{sa,t}(\cdot) := \kappa_{ii,t}(\cdot)$, $\kappa_{sb,t}(\cdot) := \kappa_{ij,t}(\cdot)$, $w_{s,t} :=$

$w_{i,t}$, $h_{s,t} := h_{i,t}$, $b_{s,t} := b_{i,t}$, $\mu_{s,t} := \mu_{i,t}$, and $\delta_s := \delta_i$. Thus, let $L_{s,t}(\cdot) := L_{i,t}(\cdot)$ for each i. As shown in Theorem 1, there exists a unique pure strategy MPE in the symmetric SC model, which we denote as σ_s^{sc*}. The following proposition is a corollary of Theorems 1-2.

PROPOSITION **3** *The following statements hold for the symmetric SC model:*

(a) For each $t = T, T-1, \cdots, 1$, there exists a constant $\beta_{s,t}^{sc} > 0$, such that

$$V_{i,t}(I_t, \Lambda_t | \sigma_{s,t}^{sc*}) = w_{s,t} I_{i,t} + \beta_{s,t}^{sc} \Lambda_{i,t}, \text{ for all } i.$$

(b) In each period t, the second-stage service level competition $\mathcal{G}_{s,t}^{sc,2}$ is symmetric, with the payoff function for each firm i given by

$$\pi_{i,t}^{sc}(y_t) = (\delta_s w_{s,t-1} - w_{s,t}) y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta_{s,t-1}^{sc} (\kappa_{sa,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]) - \sum_{j \neq i} \kappa_{sb,t}(\mathbb{E}[y_{j,t}^+ \wedge \xi_{j,t}])).$$

Moreover, $\mathcal{G}_{s,t}^{sc,2}$ has a unique pure strategy Nash equilibrium which is symmetric, so we use $y_{s,t}^{sc*}$ [$\pi_{s,t}^{sc*}$] to denote the equilibrium strategy [payoff] of each firm in $\mathcal{G}_{s,t}^{sc,2}$.

(c) In each period t, the first-stage joint promotion and price competition $\mathcal{G}_{s,t}^{sc,1}$ is symmetric, with the payoff function for each firm i given by

$$\Pi_{i,t}^{sc}(\gamma_t, p_t) = \psi_{s,t}(\gamma_t) \rho_{s,t}(p_t) [p_{i,t} - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{i,t}) + \pi_{s,t}^{sc*}].$$

Moreover, $\mathcal{G}_{s,t}^{sc,1}$ has a unique pure strategy Nash equilibrium $(\gamma_{ss,t}^{sc*}, p_{ss,t}^{sc*})$ which is symmetric (i.e., $\gamma_{ss,t}^{sc*} = (\gamma_{s,t}^{sc*}, \gamma_{s,t}^{sc*}, \cdots, \gamma_{s,t}^{sc*})$ for some $\gamma_{s,t}^{sc*}$ and $p_{ss,t}^{sc*} = (p_{s,t}^{sc*}, p_{s,t}^{sc*}, \cdots, p_{s,t}^{sc*})$ for some $p_{s,t}^{sc*}$).

(d) Under the unique pure strategy MPE, σ_s^{sc*}, the policy of firm i in period t is

$$(\gamma_{i,t}^{sc*}(I_t, \Lambda_t), p_{i,t}^{sc*}(I_t, \Lambda_t), x_{i,t}^{sc*}(I_t, \Lambda_t)) = (\gamma_{s,t}^{sc*}, p_{s,t}^{sc*}, \Lambda_{i,t} y_{s,t}^{sc*} \rho_{s,t}(p_{ss,t}^{sc*}) \psi_{s,t}(\gamma_{ss,t}^{sc*})), \text{ for each } (I_t, \Lambda_t).$$

Proposition 3 characterizes the MPE, σ_s^{sc*}, and the market size coefficients, $\{\beta_{s,t}^{sc} : T \geq t \geq 1\}$, in the symmetric SC model. Proposition 3 shows that, in the symmetric SC model, all competing firms set the same promotional effort, sales price, and service level in each period under equilibrium, whereas the equilibrium market outcome may vary in different periods.

4.2 Exploitation-Induction Tradeoff

In this subsection, we study how the market size dynamics (i.e., the service effect and the network effect) influence the equilibrium market outcome in the SC model. We focus on the managerial implications of the exploitation-induction tradeoff in a dynamic and competitive market.

To begin with, we characterize the impact of the market size coefficient vectors $\{\beta_t^{sc} : T \geq t \geq 1\}$ upon the equilibrium market outcome. The following theorem serves as the building block of our subsequent analysis of the exploitation-induction tradeoff in the SC model.

THEOREM **3** For each period t, the following statements hold:

(a) For each i and $j \neq i$, $y_{i,t}^{sc*}$ is continuously increasing in $\beta_{i,t-1}^{sc}$ and independent of $\beta_{j,t-1}^{sc}$.

(b) For each i and $j \neq i$, $\pi_{i,t}^{sc*}$ is continuously increasing in $\beta_{i,t-1}^{sc}$ and continuously decreasing in $\beta_{j,t-1}^{sc}$.

(c) If the SC model is symmetric, $\gamma_{s,t}^{sc*}$ is continuously increasing in $\pi_{s,t}^{sc*}$, whereas $p_{s,t}^{sc*}$ is continuously decreasing in $\pi_{s,t}^{sc*}$.

(d) If the SC model is symmetric and $\psi_{s,t}(\cdot)$ and $\rho_{s,t}(\cdot)$ satisfy the following monotonicity condition

$$\sum_{i=1}^{N} \frac{\partial \psi_{s,t}(\gamma_t)}{\partial \gamma_{i,t}} > 0, \text{ for all } \gamma_t, \text{ and } \sum_{i=1}^{N} \frac{\partial \rho_{s,t}(p_t)}{\partial p_{i,t}} < 0, \text{ for all } p_t, \tag{17}$$

$\beta_{s,t}^{sc}$ is continuously increasing in $\pi_{s,t}^{sc*}$.

(e) If the SC model is symmetric and $\pi_{s,t}^{sc*}$ is increasing in $\beta_{s,t-1}^{sc}$, $\gamma_{s,t}^{sc*}$ is continuously increasing in $\beta_{s,t-1}^{sc}$, whereas $p_{s,t}^{sc*}$ is continuously decreasing in $\beta_{s,t-1}^{sc}$.

(f) In the symmetric SC model, if the monotonicity condition (17) holds and $\pi_{s,t}^{sc*}$ is increasing in $\beta_{s,t-1}^{sc}$, $\beta_{s,t}^{sc}$ is continuously increasing in $\beta_{s,t-1}^{sc}$.

Theorem 3 shows that the market size coefficients $\{\beta_{i,t}^{sc} : 1 \leq i \leq N, T \geq t \geq 1\}$ quantify the intensity of the exploitation-induction tradeoff in the SC model. More specifically, if $\beta_{i,t-1}^{sc}$ is larger, firm i faces stronger exploitation-induction tradeoff in period t. Therefore, to balance this strengthened tradeoff and to induce high future demands, each firm should improve service quality, decrease sales price, and increase promotional effort, as shown in parts (a) and (e) of Theorem 3. Moreover, Theorem 3(f) characterizes the relationship between the exploitation-induction tradeoffs in different periods, demonstrating that if the exploitation-induction tradeoff is more intensive in the next period, it is also stronger in the current period under a mild condition. The monotonicity condition (17) implies that a uniform increase of all N firms' promotional efforts leads to an increase in the demand of each firm, and a uniform price increase by all N firms gives rise to a decrease in the demand of each firm. This condition is commonly used in the literature (see, e.g., Bernstein and Federgruen, 2004b; Allon and Federgruen, 2007), and often referred to as the "dominant diagonal" condition for linear demand models. The assumption that $\pi_{s,t}^{sc*}$ is increasing in $\beta_{s,t-1}^{sc}$ is not restrictive either. In Lemma 4 in the Appendix, we give some sufficient conditions for this assumption. More specifically, Lemma 4 implies that $\pi_{s,t}^{sc*}$ is increasing in $\beta_{s,t-1}^{sc}$ if one of the following conditions holds: (i) The adverse effect of a firm's competitors' service upon its future market size is not strong; (ii) the network effect is sufficiently strong; or (iii) both the service effect and the network effect are sufficiently strong.

Now we consider a benchmark case without the service effect and the network effect. We use "~" to denote this model. Thus, in the benchmark model, the market size evolution function

$\tilde{\alpha}_{i,t}(\cdot) \equiv 0$ for each firm i and each period t. Without the service effect and the network effect, the current promotion, price, and service level decisions of any firm will not influence the future demands. Therefore, the competing firms can focus on generating current profits in each period without considering inducing future demands, i.e., the exploitation-induction tradeoff is absent in this benchmark case. To characterize the impact of the service effect and the network effect upon the equilibrium outcome, the following theorem compares the Nash equilibria in $\mathcal{G}_t^{sc,2}$ and $\tilde{\mathcal{G}}_t^{sc,2}$, and the Nash equilibria in $\mathcal{G}_t^{sc,1}$ and $\tilde{\mathcal{G}}_t^{sc,1}$.

THEOREM 4 (a) For each firm i and each period t , $y_{i,t}^{sc*} \geq \tilde{y}_{i,t}^{sc*}$, $z_{i,t}^{sc*} \geq \tilde{z}_{i,t}^{sc*}$, and $\pi_{i,t}^{sc*} \geq \tilde{\pi}_{i,t}^{sc*}$.

(b) Consider the symmetric SC model. For each period t, the following statements hold:

(i) $\gamma_{s,t}^{sc*} \geq \tilde{\gamma}_{s,t}^{sc*}$ and, thus, $\gamma_{i,t}^{sc*}(I_t, \Lambda_t) \geq \tilde{\gamma}_{i,t}^{sc*}(I_t, \Lambda_t)$ for all i and all (I_t, Λ_t).

(ii) $p_{s,t}^{sc*} \leq \tilde{p}_{s,t}^{sc*}$ and, thus, $p_{i,t}^{sc*}(I_t, \Lambda_t) \leq \tilde{p}_{i,t}^{sc*}(I_t, \Lambda_t)$ for all i and all (I_t, Λ_t).

(iii) If the monotonicity condition (17) holds, we have $x_{i,t}^{sc*}(I_t, \Lambda_t) \geq \tilde{x}_{i,t}^{sc*}(I_t, \Lambda_t)$ for all i and all (I_t, Λ_t).

Theorem 4 highlights the impact of market size dynamics upon the equilibrium market outcome. Specifically, Theorem 4(a) shows that, under the service effect and the network effect, each firm i should set a higher service level in each period t. In the symmetric SC model, Theorem 4(b-i) shows that each firm should increase its promotional effort in each period under the service effect and the network effect, in order to induce higher future demands. Analogously, Theorem 4(b-ii) shows that the service effect and the network effect give rise to lower equilibrium sales price of each firm in each period. Under the monotonicity condition (17), Theorem 4(b-i,ii) implies that the equilibrium expected demand of each firm in each period is higher under the service effect and the network effect. As a consequence, to match supply with the current demand and to induce future demands with the service effect, each firm should increase its base stock level in each period under the service effect and the network effect, as shown in Theorem 4(b-iii).

Theorem 4 identifies effective strategies for firms to balance the exploitation-induction trade-off under both the service effect and the network effect. In this case, the competing firms have to tradeoff generating current profits and inducing future demands. To balance the exploitation-induction trade-off, the firms can employ three strategies to exploit the service effect and the network effect: (a) elevating service levels, (b) offering price discounts, and (c) improving promotional efforts. Elevating service levels does not lead to a higher current demand, but helps the firm induce higher future demands via the service effect. Offering price discounts and improving promotional efforts do not increase the current profits but give rise to higher current demands and, thus, induce higher future demands via the network effect. In a nutshell, the uniform idea of all three strategies is that, to balance the exploitation-induction tradeoff under the service effect and the network effect, the competing firms should induce higher future demands at the cost of reduced current margins.

To deliver sharper insights on the managerial implications of the exploitation-induction tradeoff, we confine ourselves to the symmetric SC model for the rest of this section. The following theorem characterizes how the intensities of the service effect and the network effect influence the equilibrium market outcome in the symmetric SC model.

THEOREM **5** *Let two symmetric SC models be identical except that one with market size evolution functions $\{\hat{\alpha}_{s,t}(\cdot)\}_{T\geq t\geq 1}$ and the other with $\{\alpha_{s,t}(\cdot)\}_{T\geq t\geq 1}$. Assume that, for each period t, (i) the monotonicity condition (17) holds, and (ii) $\kappa_{sb,t}(\cdot) \equiv \kappa^0_{sb,t}$ for some constant $\kappa^0_{sb,t}$.*

(a) If $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for each period t and each z_t, we have, for each period t, $\hat{\beta}^{cs}_{s,t} \geq \beta^{cs}_{s,t}$, $\hat{\gamma}^{cs*}_{s,t} \geq \gamma^{cs*}_{s,t}$, and $\hat{p}^{cs*}_{s,t} \leq p^{cs*}_{s,t}$. Thus, for each period t, $\hat{\gamma}^{cs*}_{i,t}(I_t, \Lambda_t) \geq \gamma^{cs*}_{i,t}(I_t, \Lambda_t)$ and $\hat{p}^{cs*}_{i,t}(I_t, \Lambda_t) \leq p^{cs*}_{i,t}(I_t, \Lambda_t)$ for all i and all $(I_t, \Lambda_t) \in \mathcal{S}$.

(b) If, for each period t, $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t and $\hat{\kappa}'_{sa,t}(z_{i,t}) \geq \kappa'_{sa,t}(z_{i,t}) \geq 0$ for all $z_{i,t}$, we have, for each period t, $\hat{\beta}^{cs}_{s,t} \geq \beta^{cs}_{s,t}$, $\hat{\gamma}^{cs*}_{s,t} \geq \gamma^{cs*}_{s,t}$, $\hat{p}^{cs*}_{s,t} \leq p^{cs*}_{s,t}$, and $\hat{y}^{cs*}_{s,t} \geq y^{cs*}_{s,t}$. Thus, for each period t, $\hat{\gamma}^{cs*}_{i,t}(I_t, \Lambda_t) \geq \gamma^{cs*}_{i,t}(I_t, \Lambda_t)$, $\hat{p}^{cs*}_{i,t}(I_t, \Lambda_t) \leq p^{cs*}_{i,t}(I_t, \Lambda_t)$, and $\hat{x}^{cs*}_{i,t}(I_t, \Lambda_t) \geq x^{cs*}_{i,t}(I_t, \Lambda_t)$ for all i and all $(I_t, \Lambda_t) \in \mathcal{S}$.

Theorem 5 sharpens Theorem 4 by showing that if the intensities of the network effect and the service effect (captured by the magnitudes of $\alpha_{s,t}(\cdot)$ and $\kappa'_{sa,t}(\cdot)$, respectively) are higher, the exploitation-induction tradeoff becomes stronger. To balance the strengthened exploitation-induction tradeoff, each firm should increase its promotional effort, decrease its sales price, and improve its service level in each period. More specifically, Theorem 5(a) shows that a higher intensity of the network effect (i.e., larger $\alpha_{s,t}(\cdot)$) drives all the firms to make more promotional efforts and charge lower sales prices. Theorem 5(b) further suggests that higher intensities of both the network effect and the service effect (i.e., larger $\alpha_{s,t}(\cdot)$ and $\kappa'_{sa,t}(\cdot)$) prompt all the firms to make more promotional efforts, charge lower sales prices, and maintain higher service levels. Stronger service effect and network effect intensify the exploitation-induction tradeoff, thus driving the firms to put more weight on inducing future demands than on exploiting the current market. Therefore, to effectively balance the exploitation-induction tradeoff, all the firms should carefully examine the intensities of the service effect and the network effect.

Next, we analyze the exploitation-induction tradeoff from an inter-temporal perspective. Under the service effect and the network effect, how should the competing firms adjust their promotion, price, and service strategies throughout the sales season to balance the exploitation-induction tradeoff? To address this question, we characterize the evolution of the equilibrium market outcome in the stationary and symmetric SC model. In this model, the model primitives, demand functions, and market size evolution functions are identical for all firms throughout the planning horizon. In addition, the random perturbations in market demands and market size evolution are *i.i.d.* throughout the planning horizon. The following theorem characterizes the evolution of the equilibrium promotion, price, and service strategy in the stationary and symmetric SC model.

THEOREM **6** *Consider the stationary and symmetric SC model. Assume that, for each period t, (i) the monotonicity condition* (17) *holds, and (ii) $\pi^{sc*}_{s,t}$ is increasing in $\beta^{sc}_{s,t-1}$. For each period t, the following statements hold:*

(a) $\beta^{cs}_{s,t} \geq \beta^{cs}_{s,t-1}$, $\gamma^{cs*}_{s,t} \geq \gamma^{cs*}_{s,t-1}$, $p^{cs*}_{s,t} \leq p^{cs*}_{s,t-1}$, *and* $y^{cs*}_{s,t} \geq y^{cs*}_{s,t-1}$.

(b) $\gamma^{cs*}_{i,t}(I,\Lambda) \geq \gamma^{cs*}_{i,t-1}(I,\Lambda)$, $p^{cs*}_{i,t}(I,\Lambda) \leq p^{cs*}_{i,t-1}(I,\Lambda)$, *and* $x^{cs*}_{i,t}(I,\Lambda) \geq x^{cs*}_{i,t-1}(I,\Lambda)$ *for each i and each* $(I,\Lambda) \in \mathcal{S}$.

Theorem 6 sheds light on how to balance the exploitation-induction tradeoff from an intertemporal perspective. More specifically, we show that, if the market is symmetric and stationary, the exploitation-induction tradeoff is more intensive (i.e., $\beta^{sc}_{s,t}$ is larger) at the early stage of the sales season. Moreover, the equilibrium sales price is increasing, whereas the equilibrium promotional effort and service level are decreasing, over the planning horizon. The service effect and the network effect have greater impacts upon future demands (and, hence, future profits) when the remaining planning horizon is longer. Therefore, to adaptively balance the exploitation-induction tradeoff throughout the sales season, all the firms increase their sales prices and decrease their promotional efforts and service levels towards the end of the sales season. Our analysis justifies the widely used introductory price and promotion strategy with which firms offer discounts and launch promotional campaigns at the beginning of a sales season to attract more early purchases (see, e.g., Cabral et al., 1999; Parker and Van Alstyne, 2005; Eisenmann et al., 2006).

To summarize, under the service effect and the network effect, the competing firms have to trade off between generating current profits and inducing future demands. To effectively balance the exploitation-induction tradeoff, the firms should (a) increase promotional efforts, (b) offer price discounts, and (c) improve service levels. Moreover, the exploitation-induction tradeoff is more intensive (a) with stronger service effect and network effect, or (b) at the early stage of the sales season.

5 Promotion-First Competition

In this section, we consider the promotion-first competition (PF) model, i.e., in each period t, each firm i first selects its promotional effort and then, after observing the current-period promotional efforts of all firms, chooses a combined sales price and service level strategy. This model is suitable for the scenario in which the stickiness of market expanding choices is much higher than that of sales price and service level choices. For example, due to the long leadtime for technology development, decisions on research and development effort are usually made well in advance of sales price and service level decisions.

Employing the linear separability approach, we will show that, in the PF model, the firms engage in a three-stage competition in each period, the first stage on promotional effort, the second on sales price, and the last on service level. We will also demonstrate that the exploitation-induction tradeoff has more involved managerial implications in the PF model than its implications in the SC model. In the SC model, the competing firms balance the exploitation-induction tradeoff inter-temporally, whereas the firms in the PF model balance this tradeoff both inter-temporally and intra-temporally.

For tractability, we make the following additional assumption throughout this section:

$$\rho_{i,t}(p_t) = \phi_{i,t} - \theta_{ii,t}p_{i,t} + \sum_{j\neq i}\theta_{ij,t}p_{j,t}, \text{ for each } i \text{ and } t, \tag{18}$$

where $\phi_{i,t}, \theta_{ii,t} > 0$ and $\theta_{ij,t} \geq 0$ for each i, j, and t. Moreover, we assume that the diagonal dominance conditions hold for each $\rho_{i,t}(\cdot)$, i.e., for each i and t, $\theta_{ii,t} > \sum_{j\neq i}\theta_{ij,t}$ and $\theta_{ii,t} > \sum_{j\neq i}\theta_{ji,t}$. In addition, we make the same assumption as Allon and Federgruen (2007) as follows:

Assumption 2 For each i and t, the minimum [maximum] allowable price $\underline{p}_{i,t}$ [$\bar{p}_{i,t}$] is sufficiently low [high] so that it will have no impact on the equilibrium market behavior.

We will give a sufficient condition for Assumption 2 in the discussion after Proposition 5.

5.1 Equilibrium Analysis

In this subsection, we use the linear separability approach to characterize the pure strategy MPE in the PF model. In this model, a (pure) Markov strategy profile of firm i in period t is given by $\sigma^{pf}_{i,t} = (\gamma^{pf}_{i,t}(\cdot,\cdot), p^{pf}_{i,t}(\cdot,\cdot,\cdot), x^{pf}_{i,t}(\cdot,\cdot,\cdot))$, where $\gamma^{pf}_{i,t}(I_t,\Lambda_t)$ prescribes the promotional effort given the state variable (I_t,Λ_t), and $(p^{pf}_{i,t}(I_t,\Lambda_t,\gamma_t), x^{pf}_{i,t}(I_t,\Lambda_t,\gamma_t))$ prescribes the sales price and the post-delivery inventory level, given the state variable (I_t,Λ_t) and the current period promotional effort vector γ_t. Let $\gamma^{pf}_t(\cdot,\cdot) := (\gamma^{pf}_{1,t}(\cdot,\cdot), \gamma^{pf}_{2,t}(\cdot,\cdot), \cdots, \gamma^{pf}_{N,t}(\cdot,\cdot))$, $p^{pf}_t(\cdot,\cdot,\cdot) := (p^{pf}_{1,t}(\cdot,\cdot,\cdot), p^{pf}_{2,t}(\cdot,\cdot,\cdot), \cdots, p^{pf}_{N,t}(\cdot,\cdot,\cdot))$, and $x^{pf}_t(\cdot,\cdot,\cdot) := (x^{pf}_{1,t}(\cdot,\cdot,\cdot), x^{pf}_{2,t}(\cdot,\cdot,\cdot), \cdots, x^{pf}_{N,t}(\cdot,\cdot,\cdot))$. We use σ^{pf}_t to denote the (pure) strategy profile of all firms in the subgame of period t, which prescribes their (pure) strategies from period t to the end of the planning horizon.

To evaluate the expected payoff of each firm i in each period t for any given Markov strategy profile σ^{pf} in the PF model, let

$V_{i,t}(I_t,\Lambda_t|\sigma^{pf}_t) =$ the total expected discounted profit of firm i in periods $t, t-1, \cdots, 1, 0$, when starting period t with the state variable (I_t,Λ_t) and the firms play strategy σ^{pf}_t in periods $t, t-1, \cdots, 1$.

Thus, by backward induction, $V_{i,t}(\cdot,\cdot|\sigma^{pf}_t)$ satisfies the following recursive scheme for each firm i and each period t:

$$V_{i,t}(I_t,\Lambda_t|\sigma^{pf}_t) = J_{i,t}(\gamma^{pf}_t(I_t,\Lambda_t), p^{pf}_t(I_t,\Lambda_t,\gamma^{pf}_t(I_t,\Lambda_t)), x^{pf}_t(I_t,\Lambda_t,\gamma^{pf}_t(I_t,\Lambda_t)), I_t,\Lambda_t|\sigma^{pf}_{t-1}),$$

where

$$
\begin{aligned}
J_{i,t}(\gamma_t, p_t, x_t, I_t, \Lambda_t|\sigma_{t-1}^{pf}) \;=\; & \mathbb{E}\{p_{i,t}D_{i,t}(\gamma_t, p_t) - w_{i,t}(x_{i,t} - I_{i,t}) - h_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^+ \\
& - b_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^- - \nu_{i,t}(\gamma_{i,t})D_{i,t}(\gamma_t, p_t) \\
& + \delta_i V_{i,t-1}(x_t - D_t(\gamma_t, p_t), \eta_t(z_t, D_t(\gamma_t, p_t), \Lambda_t, \Xi_t)|\sigma_{t-1}^{pf})|I_t, \Lambda_t\}, \qquad (19)
\end{aligned}
$$

and $V_{i,0}(I_t, \Lambda_t) = w_{i,0}I_{i,0}$. We now define the pure strategy MPE in the PF model.

Definition 2 A (pure) Markov strategy $\sigma^{pf*} = \{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \leq i \leq N, T \geq t \geq 1\}$ is a pure strategy MPE in the PF model if and only if, for each firm i, period t, and state variable $(I_t, \Lambda_t) \in \mathcal{S}$,

$$
\begin{aligned}
& (p_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t), x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)) \\
&= \mathrm{argmax}_{p_{i,t} \in [\underline{p}_{i,t}, \bar{p}_{i,t}], x_{i,t} \geq \min\{0, I_{i,t}\}} [J_{i,t}(\gamma_t, [p_{i,t}, p_{-i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)], [x_{i,t}, x_{-i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)], I_t, \Lambda_t|\sigma_{t-1}^{pf*})], \text{ for all } \gamma_t;
\end{aligned}
$$

and $\gamma_{i,t}^{pf*}(I_t, \Lambda_t)$

$$
= \mathrm{argmax}_{\gamma_{i,t} \in [0, \bar{\gamma}_{i,t}]} [J_{i,t}([\gamma_{i,t}, \gamma_{-i,t}^{pf*}(I_t, \Lambda_t)], p_t^{pf*}(I_t, \Lambda_t, [\gamma_{i,t}, \gamma_{-i,t}^{pf*}(I_t, \Lambda_t)]), x_t^{pf*}(I_t, \Lambda_t, [\gamma_{i,t}, \gamma_{-i,t}^{pf*}(I_t, \Lambda_t)]), I_t, \Lambda_t|\sigma \qquad (20)
$$

Definition 2 suggests that a pure strategy MPE in the PF model is a (pure) Markov strategy profile that satisfies subgame perfection in each stage of the competition in each period t. The following theorem shows that there exists a pure strategy MPE in the PF model.

Theorem 7 The following statements hold for the PF model:

(a) There exists a pure strategy MPE $\sigma^{pf*} = \{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \leq i \leq N, T \geq t \geq 1\}$.

(b) For each pure strategy MPE σ^{pf*}, there exists a series of vectors $\{\beta_t^{pf} : T \geq t \geq 1\}$, where $\beta_t^{pf} = (\beta_{1,t}^{pf}, \beta_{2,t}^{pf}, \cdots, \beta_{N,t}^{pf})$ with $\beta_{i,t}^{pf} > 0$ for each i and t, such that

$$
V_{i,t}(I_t, \Lambda_t|\sigma_t^{pf*}) = w_{i,t}I_{i,t} + \beta_{i,t}^{pf}\Lambda_{i,t}, \text{ for each } i, t, \text{ and } (I_t, \Lambda_t) \in \mathcal{S}. \qquad (21)
$$

(c) If $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for each i and t, σ^{pf*} is the unique MPE in the PF model.

Theorem 7 demonstrates the existence of a pure strategy MPE in the PF model. As in the SC model, in Theorem 7(b), we show that, for each pure strategy MPE σ^{pf*}, the associated profit function of each firm i in each period t is linearly separable in its own starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$. We refer to the constant $\beta_{i,t}^{pf}$ as the PF market size coefficient of firm i in period t, which measures the exploitation-induction tradeoff intensity in the PF model. Theorem 7(c) shows that the MPE in the PF model is unique if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, i.e., the promotional effort $\gamma_{i,t}$ is the the actual per-unit demand market expanding expenditure of firm i in period t. For the rest of this section, we assume that $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for each i and t, and, hence, σ^{pf*} is the unique pure strategy MPE in the PF model. We use $\{\beta_t^{pf} : T \geq t \geq 1\}$ to denote the PF market size coefficient associated with σ^{pf*} hereafter.

The linear separability of $V_{i,t}(\cdot,\cdot|\sigma_t^{pf*})$ enables us to have a sharper characterization of MPE in the PF model. As in the SC model, we can rewrite the objective function of firm i in period t as follows.

$$
\begin{aligned}
J_{i,t}(\gamma_t, p_t, x_t, I_t, \Lambda_t|\sigma_{t-1}^{pf*}) =& \mathbb{E}\{p_{i,t}D_{i,t}(\gamma_t, p_t) - w_{i,t}(x_{i,t} - I_{i,t}) - h_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^+ \\
& - b_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^- - \nu_{i,t}(\gamma_{i,t})D_{i,t}(\gamma_t, p_t) \\
& + \delta_i V_{i,t-1}(x_t - D_t(\gamma_t, p_t), \eta_t(z_t, D_t(\gamma_t, p_t), \Lambda_t, \Xi_t)|\sigma_{t-1}^{pf*})|I_t, \Lambda_t\} \\
=& \mathbb{E}\{p_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t} - w_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - I_{i,t}) \\
& - h_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t})^+ \\
& - b_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t})^- \\
& - \nu_{i,t}(\gamma_{i,t})\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t} + \delta_i w_{i,t-1}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t}) \\
& + \delta_i\beta_{i,t-1}^{pf}(\Lambda_{i,t}\Xi_{i,t}^1 + \alpha_{i,t}(z_t)\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t}\Xi_{i,t}^2)|I_t, \Lambda_t\} \\
=& w_{i,t}I_{i,t} + \Lambda_{i,t}\{\delta_i\beta_{i,t-1}^{pf}\mu_{i,t} + \psi_{i,t}(\gamma_t)\rho_{i,t}(p_t)[p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf}(y_t)]\},
\end{aligned}
\tag{22}
$$

where $\pi_{i,t}^{pf}(y_t) = (\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i\beta_{i,t-1}^{pf}(\kappa_{ii,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]) - \sum_{j\neq i}\kappa_{ij,t}(\mathbb{E}[y_{j,t}^+ \wedge \xi_{j,t}]))$,

and $\beta_{i,0}^{pf} := 0$ for each i.

We observe from (22) that, in the PF model, the payoff function of each firm i in each period t has a nested structure. Hence, the competition in each period t can be decomposed into three stages: In the first stage, the firms compete on promotional effort; in the second stage, they compete on sales price; in the third stage, they compete on service level. By backward induction, we start the equilibrium analysis with the third-stage service level competition. Let $\mathcal{G}_t^{pf,3}$ be the $N-$player noncooperative game that represents the third-stage service level competition in period t, where player i has the payoff function $\pi_{i,t}^{pf}(\cdot)$ and the feasible action set $\mathbb{R}$. The following proposition characterizes the Nash equilibrium of the game $\mathcal{G}_t^{pf,3}$.

Proposition 4 *For each period t, the third-stage service level competition $\mathcal{G}_t^{pf,3}$ has a unique pure strategy Nash equilibrium y_t^{pf*}. Moreover, for each i, $y_{i,t}^{pf*} > 0$ is the unique solution to the following equation:*

$$(\delta_i w_{i,t-1} - w_{i,t}) - L'_{i,t}(y_{i,t}^{pf*}) + \delta_i\beta_{i,t-1}^{pf}\bar{F}_{i,t}(y_{i,t}^{pf*})\kappa'_{ii,t}(\mathbb{E}(y_{i,t}^{pf*} \wedge \xi_{i,t})) = 0. \tag{23}$$

Proposition 4 characterizes the unique pure strategy Nash equilibrium of the third-stage service level competition. Moreover, $y_{i,t}^{pf*}$ is the solution to the first-order condition $\partial_{y_{i,t}}\pi_{i,t}^{pf}(y_t^{pf*}) = 0$. Let $\pi_t^{pf*} := (\pi_{1,t}^{pf*}, \pi_{2,t}^{pf*}, \cdots, \pi_{N,t}^{pf*})$ be the equilibrium payoff vector of the third-stage service level competition in period t, where $\pi_{i,t}^{pf*} = \pi_{i,t}^{pf}(y_t^{pf*})$. For each i and t, let

$$\Pi_{i,t}^{pf,2}(p_t|\gamma_t) := \rho_{i,t}(p_t)(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}). \tag{24}$$

Therefore, given the outcome of the first-stage promotion competition, γ_t, we can define an $N-$player noncooperative game $\mathcal{G}_t^{pf,2}(\gamma_t)$ to represent the second-stage price competition in period t, where player i has the payoff function $\Pi_{i,t}^{pf,2}(\cdot|\gamma_t)$ and the feasible action set $[\underline{p}_{i,t}, \bar{p}_{i,t}]$. We define A_t as an $N \times N$ matrix with entries defined by $A_{ii,t} := 2\theta_{ii,t}$ and $A_{ij,t} := -\theta_{ij,t}$ where $i \neq j$. By Lemma 2(a) in the Appendix, A_t is invertible. Let $f_t(\gamma_t)$ be an $N-$dimensional vector with $f_{i,t}(\gamma_t) := \phi_{i,t} + \theta_{ii,t}(\delta_i w_{i,t-1} + \nu_{i,t}(\gamma_{i,t}) - \pi_{i,t}^{pf*})$. We characterize the Nash equilibrium of the game $\mathcal{G}_t^{pf,2}(\gamma_t)$ in the following proposition.

Proposition **5** *For each period t and any given γ_t, the following statements hold:*

(a) The second-stage price competition $\mathcal{G}_t^{pf,2}(\gamma_t)$ has a unique pure strategy Nash equilibrium $p_t^{pf*}(\gamma_t)$.

(b) $p_t^{pf*}(\gamma_t) = A_t^{-1} f_t(\gamma_t)$. Moreover, $p_{i,t}^{pf*}(\gamma_t)$ is continuously increasing in $\gamma_{j,t}$ for each i and j.

(c) Let $\Pi_t^{pf*,2}(\gamma_t) := (\Pi_{1,t}^{pf*,2}(\gamma_t), \Pi_{2,t}^{pf*,2}(\gamma_t), \cdots, \Pi_{N,t}^{pf*,2}(\gamma_t))$ be the equilibrium payoff vector of the second-stage price competition in period t, where $\Pi_{i,t}^{pf*,2}(\gamma_t) = \Pi_{i,t}^{pf,2}(p_t^{pf*}(\gamma_t)|\gamma_t)$. We have $\Pi_{i,t}^{pf*,2}(\gamma_t) = \theta_{ii,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})^2 > 0$ for all i.

Proposition 5 shows that, for any given promotional effort vector γ_t, the second-stage price competition $\mathcal{G}_t^{pf,2}(\gamma_t)$ has a unique pure strategy Nash equilibrium $p_t^{pf*}(\gamma_t) = A_t^{-1} f_t(\gamma_t)$. By Proposition 5(b), we have $p_{i,t}^{pf*}(\mathbf{0}) \leq p_{i,t}^{pf*}(\gamma_t) \leq p_{i,t}^{pf*}(\bar{\gamma}_t)$ for each i and γ_t, where $\mathbf{0}$ is an N-dimensional vector with each entry equal to 0 and $\bar{\gamma}_t := (\bar{\gamma}_{1,t}, \bar{\gamma}_{2,t}, \cdots, \bar{\gamma}_{N,t})$. Thus, a sufficient condition for Assumption 2 is that $\underline{p}_{i,t} \leq p_{i,t}^{pf*}(\mathbf{0})$ and $\bar{p}_{i,t} \geq p_{i,t}^{pf*}(\bar{\gamma}_t)$ for all i and t.

Now we study the first-stage promotion competition in period t. Let

$$\Pi_{i,t}^{pf,1}(\gamma_t) := \Pi_{i,t}^{pf*,2}(\gamma_t)\psi_{i,t}(\gamma_t) = \theta_{ii,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})^2 \psi_{i,t}(\gamma_t). \tag{25}$$

Thus, we can define an $N-$player noncooperative game $\mathcal{G}_t^{pf,1}$ to represent the first-stage promotion competition in period t, where player i has the payoff function $\Pi_{i,t}^{pf,1}(\cdot)$ and the feasible action set $[0, \bar{\gamma}_{i,t}]$. We characterize the Nash equilibrium of the game $\mathcal{G}_t^{pf,1}$ in the following proposition.

Proposition **6** *For each period t, the following statements hold:*

(a) The first-stage promotion competition $\mathcal{G}_t^{pf,1}$ is a log-supermodular game.

(b) There exists a unique pure strategy Nash equilibrium in the game $\mathcal{G}_t^{pf,1}$, which is the unique serially undominated strategy of $\mathcal{G}_t^{pf,1}$.

(c) The unique Nash equilibrium of $\mathcal{G}_t^{pf,1}$, γ_t^{pf*}, is the solution to the following system of

equations:

$$\text{for each } i,\ \frac{\partial_{\gamma_{i,t}}\psi_{i,t}(\gamma_t^{pf*})}{\psi_{i,t}(\gamma_t^{pf*})}-\frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})\nu'_{i,t}(\gamma_{i,t}^{pf*})}{p_{i,t}^{pf*}(\gamma_t^{pf*})-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t}^{pf*})+\pi_{i,t}^{pf*}}\begin{cases}\leq 0, & \text{if } \gamma_{i,t}^{pf*}=0,\\ =0, & \text{if } \gamma_{i,t}^{pf*}\in(0,\bar{\gamma}_{i,t}),\\ \geq 0 & \text{if } \gamma_{i,t}^{pf*}=\bar{\gamma}_{i,t}.\end{cases} \tag{26}$$

(d) *Let* $\Pi_t^{pf*,1} := (\Pi_{1,t}^{pf*,1}, \Pi_{2,t}^{pf*,1}, \cdots, \Pi_{N,t}^{pf*,1})$ *be the equilibrium payoff vector associated with* γ_t^{pf*}*, i.e.,* $\Pi_{i,t}^{pf*,1} = \Pi_{i,t}^{pf,1}(\gamma_t^{pf*})$ *for each* i*. We have* $\Pi_{i,t}^{pf*,1} > 0$ *for all* i*.*

As shown in Proposition 6, in the PF model, the first-stage promotion competition in period t is a log-supermodular game and has a unique pure strategy Nash equilibrium. Moreover, the unique Nash equilibrium promotional effort vector γ_t^{pf*} can be determined by (i) the serial elimination of strictly dominated strategies, or (ii) the system of first-order conditions (26).

The following theorem summarizes Theorem 7 and Propositions 4-6, and characterizes the MPE in the PF model.

Theorem **8** For each period t, the following statements hold:

(a) For each i, $\beta_{i,t}^{pf} = \delta_i \beta_{i,t-1}^{pf}\mu_{i,t} + \Pi_{i,t}^{pf*,1}$.

(b) Under the unique pure strategy MPE σ^{pf*}, the policy of firm i in period t is given by

$$(\gamma_{i,t}^{pf*}(I_t,\Lambda_t), p_{i,t}^{pf*}(I_t,\Lambda_t,\gamma_t), x_{i,t}^{pf*}(I_t,\Lambda_t,\gamma_t)) = (\gamma_{i,t}^{pf*}, p_{i,t}^{pf*}(\gamma_t), \Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t))\psi_{i,t}(\gamma_t)). \tag{27}$$

In particular, for any (I_t, Λ_t), the associated (pure strategy) equilibrium price and inventory decisions of firm i are $p_{i,t}^{pf*}(\gamma_t^{pf*})$ and $\Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t^{pf*}))\psi_{i,t}(\gamma_t^{pf*})$, respectively.

Theorem 8(a) recursively determines the PF market size coefficient vectors, $\{\beta_t^{pf} : T \geq t \geq 1\}$, associated with the unique pure strategy MPE σ^{pf*}. Theorem 8(b) demonstrates that, under the unique pure strategy MPE σ^{pf*}, each firm i's promotion, price, and inventory decisions in each period t depend on its private information (i.e., $(I_{i,t}, \Lambda_{i,t})$) only, but not on that of its competitors (i.e., $(I_{-i,t}, \Lambda_{-i,t})$). Hence, the unique pure strategy MPE in the PF model has the attractive feature that the strategy of each firm is contingent on its accessible information only.

As in the SC model, we will perform some of our analysis below with the symmetric PF model, where all firms have identical characteristics. We use the subscript "s" to denote the case of symmetric market in the PF model. In this case, $\rho_{s,t}(p_t) = \phi_{s,t} - \theta_{sa,t}p_{i,t} + \sum_{j\neq i}\theta_{sb,t}p_{j,t}$ for some nonnegative constants $\phi_{s,t}$, $\theta_{sa,t}$, and $\theta_{sb,t}$, where $\theta_{sa,t} > (N-1)\theta_{sb,t}$. We use σ_s^{pf*} to denote the unique pure strategy MPE in the symmetric PF model. The following proposition characterizes σ_s^{pf*} in the PF model.

Proposition **7** *The following statements hold for the symmetric PF model.*

(a) *For each* $t = T, T-1, \cdots, 1$, *there exists a constant* $\beta^{pf}_{s,t} > 0$, *such that*

$$V_{i,t}(I_t, \Lambda_t | \sigma^{pf*}_{s,t}) = w_{s,t} I_{i,t} + \beta^{pf}_{s,t} \Lambda_{i,t}, \text{ for all } i.$$

(b) *In each period* t, *the third-stage service level competition* $\mathcal{G}^{pf,3}_{s,t}$ *is symmetric, with the payoff function for each firm* i *given by*

$$\pi^{pf}_{i,t}(y_t) = (\delta_s w_{s,t-1} - w_{s,t}) y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta^{pf}_{s,t-1}(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}]) - \sum_{j \neq i} \kappa_{sb,t}(\mathbb{E}[y^+_{j,t} \wedge \xi_{j,t}])).$$

Moreover, $\mathcal{G}^{pf,3}_{s,t}$ *has a unique pure strategy Nash equilibrium, which is symmetric, so we use* $y^{pf*}_{s,t}$ [$\pi^{pf*}_{s,t}$] *to denote the equilibrium strategy [payoff] of each firm in* $\mathcal{G}^{pf,3}_{s,t}$.

(c) *In each period* t, *the second-stage price competition* $\mathcal{G}^{pf,2}_{s,t}(\gamma_t)$ *is symmetric if* $\gamma_{i,t} = \gamma_{j,t}$ *for all* $1 \leq i, j \leq N$. *In this case,* $\mathcal{G}^{pf,2}_{s,t}(\gamma_t)$ *has a unique pure strategy Nash equilibrium* $p^{pf*}_{ss,t}(\gamma_t)$, *which is symmetric (i.e.,* $p^{pf*}_{ss,t}(\gamma_t) = (p^{pf*}_{s,t}(\gamma_t), p^{pf*}_{s,t}(\gamma_t), \cdots, p^{pf*}_{s,t}(\gamma_t))$ *for some* $p^{pf*}_{s,t}(\gamma_t) \in [\underline{p}_{s,t}, \bar{p}_{s,t}]$*).*

(d) *In each period* t, *the first-stage promotion competition* $\mathcal{G}^{pf,1}_{s,t}$ *is symmetric. Moreover,* $\mathcal{G}^{pf,1}_{s,t}$ *has a unique pure strategy Nash equilibrium* $\gamma^{pf*}_{ss,t}$, *which is symmetric (i.e.,* $\gamma^{pf*}_{ss,t} = (\gamma^{pf*}_{s,t}, \gamma^{pf*}_{s,t}, \cdots, \gamma^{pf*}_{s,t})$ *for some* $\gamma^{pf*}_{s,t} \in [0, \bar{\gamma}_{s,t}]$*).*

(e) *Under the unique pure strategy MPE* σ^{pf*}_s, *the policy of firm* i *in period* t *is*

$$(\gamma^{pf*}_{i,t}(I_t, \Lambda_t), p^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t), x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t)) = (\gamma^{sc*}_{s,t}, p^{pf*}_{i,t}(\gamma_t), \Lambda_{i,t} y^{pf*}_{s,t} \rho_{s,t}(p^{pf*}_t(\gamma_t)) \psi_{s,t}(\gamma_t)),$$

for all (I_t, Λ_t) *and* γ_t. *In particular, for each firm* i *and any* (I_t, Λ_t), *the equilibrium price is* $p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t})$, *and the equilibrium post-delivery inventory level is* $\Lambda_{i,t} y^{pf*}_{s,t} \rho_{s,t}(p^{pf*}_{ss,t}(\gamma^{pf*}_{ss,t})) \psi_{s,t}(\gamma^{pf*}_{ss,t})$.

Proposition 7 shows that, in the symmetric PF model, all competing firms make the same promotional effort, charge the same sales price, and maintain the same service level in each period. The PF market size coefficient is also identical for all firms in each period.

5.2 Exploitation-Induction Tradeoff

In this subsection, we study how the exploitation-induction tradeoff impacts the equilibrium market outcome in the PF model. As in the SC model, we first characterize the impact of the PF market size coefficient vectors, $\{\beta^{pf}_t : T \geq t \geq 1\}$.

THEOREM **9** For each period t, the following statements hold:

(a) For each i and $j \neq i$, $y^{pf*}_{i,t}$ is continuously increasing in $\beta^{pf}_{i,t-1}$ and independent of $\beta^{pf}_{j,t-1}$.

(b) For each i and $j \neq i$, $\pi^{pf*}_{i,t}$ is continuously increasing in $\beta^{pf}_{i,t-1}$ and continuously decreasing in $\beta^{pf}_{j,t-1}$.

(c) For each i, j, and γ_t , $p^{pf*}_{i,t}(\gamma_t)$ is continuously decreasing in $\pi^{pf*}_{j,t}$.

(d) If the PF model is symmetric, $\gamma^{pf*}_{s,t}$ is continuously increasing in $\pi^{pf*}_{s,t}$. If, in addition, the monotonicity condition (17) holds, $\beta^{pf}_{s,t}$ is continuously increasing in $\pi^{pf*}_{s,t}$ as well.

(e) If the PF model is symmetric and $\pi^{pf*}_{s,t}$ is increasing in $\beta^{pf}_{s,t-1}$, $\gamma^{pf*}_{s,t}$ is continuously increasing in $\beta^{pf}_{s,t-1}$, whereas $p^{pf*}_{i,t}(\gamma_t)$ is continuously decreasing in $\beta^{pf}_{s,t-1}$. If, in addition, the monotonicity condition (17) holds, $\beta^{pf}_{s,t}$ is continuously increasing in $\beta^{pf}_{s,t-1}$ as well.

Theorem 9 demonstrates that the market size coefficients $\{\beta^{pf}_{i,t} : 1 \leq i \leq N, T \geq t \geq 1\}$ quantify the intensity of the exploitation-induction tradeoff in the PF model. More specifically, a larger $\beta^{pf}_{i,t-1}$ implies more intensive exploitation-induction tradeoff of firm i in period t.

As in the SC model, we use "$\tilde{}$" to denote the benchmark case without the service effect and the network effect, where the market size evolution function $\tilde{\alpha}_{i,t}(\cdot) \equiv 0$ for each firm i and each period t. Thus, the exploitation-induction tradeoff is absent in this benchmark model, and it suffices for the firms to myopically maximize their current-period profits. The following theorem characterizes the impact of the service effect and the network effect in the PF model.

Theorem **10** (a) For each firm i and each period t, $y^{pf*}_{i,t} \geq \tilde{y}^{pf*}_{i,t}$, $z^{pf*}_{i,t} \geq \tilde{z}^{pf*}_{i,t}$, and $\pi^{pf*}_{i,t} \geq \tilde{\pi}^{pf*}_{i,t}$.

(b) For each firm i and each period t, $p^{pf*}_{i,t}(\gamma_t) \leq \tilde{p}^{pf*}_{i,t}(\gamma_t)$ for all γ_t. Moreover, if the PF model is symmetric and (17) holds, $x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t) \geq \tilde{x}^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t)$ for all i, t, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$.

(c) Consider the symmetric PF model. For each period t, $\gamma^{pf*}_{s,t} \geq \tilde{\gamma}^{pf*}_{s,t}$. Thus, $\gamma^{pf*}_{i,t}(I_t, \Lambda_t) \geq \tilde{\gamma}^{pf*}_{i,t}(I_t, \Lambda_t)$ for all i and all $(I_t, \Lambda_t) \in \mathcal{S}$.

Consistent with Theorem 4(a), Theorem 10(a) shows that, the service effect and the network effect drive the competing firms to maintain higher service levels in the PF model. Theorem 10(b) reveals the impact of the exploitation-induction tradeoff upon the competing firms' price and inventory strategy in the PF model. Specifically, given any outcome of the first-stage promotion competition γ_t, in the second-stage price competition, each firm i should charge a lower sales price under the service effect and the network effect, so as to exploit the network effect and induce higher future demands. Moreover, in each period t, the equilibrium post-delivery inventory levels contingent on any realized promotional effort vector γ_t are also higher in the PF model under the service effect and the network effect. Theorem 10(c) sheds light on how the exploitation-induction tradeoff influences the equilibrium promotion strategies under the service effect and the network effect. In the symmetric PF model, the equilibrium promotional effort of each firm i in each period t is higher under the service effect and the network effect.

Note that, in the PF model, the equilibrium price and inventory outcomes under the service effect and the network effect, $p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t})$ and $x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma^{pf*}_{ss,t})$, may be either higher or lower than those without market size dynamics, $\tilde{p}^{pf*}_{ss,t}(\tilde{\gamma}^{pf*}_{s,t})$ and $\tilde{x}^{pf*}_{i,t}(I_t, \Lambda_t, \tilde{\gamma}^{pf*}_{ss,t})$. This phenomenon contrasts with the equilibrium market outcomes in the SC model, where the equilibrium sales price [post-delivery inventory level] of each firm in each period is lower [higher] under the service effect and the network effect (i.e., Theorem 4(b-i,iii)). This discrepancy is driven by the fact that, in the PF model, each firm observes the promotion decisions of its competitors before making its pricing decision. Hence, under the service effect and the network effect, the competing firms may either decrease the sales prices to induce future demands or increase the sales prices to exploit the better market condition from the increased promotional efforts (recall that $\gamma^{pf*}_{s,t} \geq \tilde{\gamma}^{pf*}_{s,t}$). In general, either effect may dominate, so we do not have a general monotonicity relationship between either the equilibrium price outcomes (i.e., $p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t})$ and $\tilde{p}^{pf*}_{s,t}(\tilde{\gamma}^{pf*}_{ss,t})$) or the equilibrium inventory outcomes (i.e., $x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma^{pf*}_{ss,t})$ and $\tilde{x}^{pf*}_{i,t}(I_t, \Lambda_t, \tilde{\gamma}^{pf*}_{ss,t})$). Therefore, the exploitation-induction tradeoff in the PF model is more involved than that in the SC model. The competing firms only need to trade off between generating current profits and inducing future demands intertemporally in the SC model, whereas they need to balance this tradeoff both inter-temporally and intra-temporally in the PF model.

To deliver sharper insights on the managerial implications of the exploitation-induction tradeoff, we confine ourselves to the symmetric PF model for the rest of this section.

Theorem **11** *Let two symmetric PF models be identical except that one with market size evolution functions $\{\hat{\alpha}_{s,t}(\cdot)\}_{T\geq t\geq 1}$ and the other with $\{\alpha_{s,t}(\cdot)\}_{T\geq t\geq 1}$. Assume that, for each period t, (i) the monotonicity condition* (17) *holds, and (ii) $\kappa_{sb,t}(\cdot) \equiv \kappa^0_{sb,t}$ for some constant $\kappa^0_{sb,t}$.*

(a) If $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for each period t and all z_t, we have, for each period t, $\hat{\beta}^{pf}_{s,t} \geq \beta^{pf}_{s,t}$, $\hat{p}^{pf*}_{i,t}(\gamma_t) \leq p^{pf*}_{i,t}(\gamma_t)$ for all i and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$, and $\hat{\gamma}^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t}$. Thus, for each period t, $\hat{p}^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t) \leq p^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t)$ and $\hat{\gamma}^{pf*}_{i,t}(I_t, \Lambda_t) \geq \gamma^{pf*}_{i,t}(I_t, \Lambda_t)$ for all i, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$.

(b) If, for each period t, $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t and $\hat{\kappa}'_{sa,t}(z_{i,t}) \geq \kappa'_{sa,t}(z_{i,t}) \geq 0$ for all $z_{i,t}$, we have, for each period t, $\hat{\beta}^{pf}_{s,t} \geq \beta^{pf}_{s,t}$, $\hat{y}^{pf*}_{s,t} \geq y^{pf*}_{s,t}$, $\hat{p}^{pf*}_{i,t}(\gamma_t) \leq p^{pf*}_{i,t}(\gamma_t)$, and $\hat{\gamma}^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t}$. Thus, for each period t, $\hat{p}^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t) \leq p^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t)$, $\hat{x}^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t) \geq x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t)$, $\hat{\gamma}^{pf*}_{i,t}(I_t, \Lambda_t) \geq \gamma^{pf*}_{i,t}(I_t, \Lambda_t)$ for all i, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$.

Theorem 11(a) shows that, in the symmetric PF model, higher intensity of the network effect (i.e., larger $\alpha_{s,t}(\cdot)$) drives all the competing firms to make more promotional efforts and charge lower sales prices for each observed promotion vector. Moreover, if the intensities of both the network effect and the service effect (i.e., the magnitudes of $\alpha_{s,t}(\cdot)$ and $\kappa'_{sa,t}(\cdot)$) are higher, Theorem 11(b) demonstrates that all the competing firms are prompted to maintain higher service levels as well. Therefore, in the PF model, the exploitation-induction tradeoff is stronger with more intensive service effect and network effect.

Theorem **12** *Consider the stationary symmetric PF model. Assume that, for each period t, (i) the monotonicity condition* (17) *holds, and (ii) $\pi^{pf*}_{s,t}$ is increasing in $\beta^{pf}_{s,t-1}$. For each period t, the following statements hold:*

(a) $\beta^{pf}_{s,t} \geq \beta^{pf}_{s,t-1}$, $y^{pf*}_{s,t} \geq y^{pf*}_{s,t-1}$, $p^{pf*}_{s,t}(\gamma) \leq p^{pf*}_{s,t-1}(\gamma)$ for each $\gamma \in [0, \bar{\gamma}_s]^N$, and $\gamma^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t-1}$.

(b) $p^{pf*}_{i,t}(I, \Lambda, \gamma) \leq p^{pf*}_{i,t-1}(I, \Lambda, \gamma)$, $x^{pf*}_{i,t}(I, \Lambda, \gamma) \geq x^{pf*}_{i,t-1}(I, \Lambda, \gamma)$, and $\gamma^{pf*}_{i,t}(I, \Lambda) \geq \gamma^{pf*}_{i,t-1}(I, \Lambda)$ for each i, $(I, \Lambda) \in \mathcal{S}$, and $\gamma \in [0, \bar{\gamma}_{s,t}]^N$.

Analogous to Theorem 6, Theorem 12 justifies the widely used introductory price and promotion strategy. More specifically, this result shows that if the market is stationary and symmetric in the PF model, the competing firms should decrease the promotional efforts (i.e., $\gamma^{pf*}_{s,t}$) and service levels (i.e., $y^{pf*}_{s,t}$), and increase the sales prices contingent on any realized promotional efforts (i.e., $p^{pf*}_{s,t}(\gamma_t)$), over the planning horizon. Hence, Theorem 12 suggests that, in the PF model, the exploitation-induction tradeoff is more intensive at the early stage of the sales season than at later stages.

To conclude this section, we remark that, because of the aforementioned intra-temporal exploitation-induction tradeoff under the promotion-first competition, Theorems 11-12 cannot give the monotone relationships on the equilibrium outcomes of each firm i's sales price (i.e., $p^{pf*}_{i,t}(I_t, \Lambda_t, \gamma^{pf*}_{ss,t})$) and post-deliver inventory level (i.e., $x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma^{pf*}_{ss,t})$).

6 Comparison of the Two Competition Models

As demonstrated above, the exploitation-induction tradeoff is more involved in the PF model than that in the SC model. In this section, we compare the unique MPE in the SC model and that in the PF model, and discuss how the exploitation-induction tradeoff impacts the equilibrium market outcomes under different competitions.

Theorem **13** Consider the symmetric SC and PF models. Assume that, for each period t, (i) the demand function $\rho_{i,t}(\cdot)$ is linear and given by (18), (ii) $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, (iii) the monotonicity condition (17) holds, (iv) Assumption 2 holds, (v) $\pi^{sc*}_{s,t}$ is increasing in $\beta^{sc}_{s,t-1}$, and (vi) $\pi^{pf*}_{s,t}$ is increasing in $\beta^{pf}_{s,t-1}$. The following statements hold:

(a) If $\beta^{pf}_{s,t-1} \geq \beta^{sc}_{s,t-1}$, $y^{pf*}_{s,t} \geq y^{sc*}_{s,t}$ and $\gamma^{pf*}_{s,t} \geq \gamma^{sc*}_{s,t}$.

(b) For each period t, there exists an $\epsilon_t \in [0, \frac{1}{N-1}]$, such that, if $\theta_{sb,t} \leq \epsilon_t \theta_{sa,t}$, we have

  (i) $\beta^{pf}_{s,t} \geq \beta^{sc}_{s,t}$ and, thus, $V_{i,t}(I_t, \Lambda_t | \sigma^{pf*}_t) \geq V_{i,t}(I_t, \Lambda_t | \sigma^{sc*}_t)$ for each firm i and all $(I_t, \Lambda_t) \in \mathcal{S}$;

  (ii) $y^{pf*}_{s,t} \geq y^{sc*}_{s,t}$;

  (iii) $\gamma^{pf*}_{s,t} \geq \gamma^{sc*}_{s,t}$.

Theorem 13 shows that, if the product differentiation is sufficiently high (as captured by the condition that $\theta_{sb,t} \leq \epsilon_t \theta_{sa,t}$), the PF competition leads to stronger exploitation-induction tradeoff (i.e., $\beta^{pf}_{s,t} \geq \beta^{sc}_{s,t}$). As a consequence, the competing firms should set higher service levels and promotional efforts in the PF model. Compared with the simultaneous competition, the promotion-first competition enables the firm to responsively adjust their sales prices in accordance to the market condition and their competitors' promotion strategies. If the product differentiation is sufficiently high, such pricing flexibility gives rise to higher expected profits of all firms and more intensive exploitation-induction tradeoff in the PF model.

Theorem 13 also reveals the "fat-cat" effect in our dynamic competition model: When the price decisions are made after observing the promotional efforts in each period, the firms tend to "overinvest" in promotional efforts. As shown in the literature (e.g., Fudenberg and Tirole, 1984; Allon and Federgruen, 2007), one driving force for this phenomenon is that, under the PF competition, the firms can charge higher prices in the subsequent price competition with increased promotional efforts in each period. Theorem 13 identifies a new driving force for the "fat-cat" effect: The firms under the PF competition make more promotional efforts to balance the more intensive exploitation-induction tradeoff therein. Therefore, our analysis delivers a new insight to the literature that the exploitation-induction tradeoff may give rise to the "fat-cat" effect in dynamic competition.

7 Conclusion

This paper studies a dynamic joint promotion, price, and service competition model, in which current decisions influence future demands through the service effect and the network effect. Our model highlights an important tradeoff in a dynamic and competitive market: the tradeoff between generating current profits and inducing future demands (i.e., the exploitation-induction tradeoff). We characterize the impact of the exploitation-induction tradeoff upon the equilibrium market outcome under the service effect and the network effect, and identify the effective strategies to balance this tradeoff under dynamic competition.

We employ the linear separability approach to characterize the pure strategy MPE both in the SC model and in the PF model. An important feature of the MPE in both models is that the equilibrium strategy of each firm in each period only depends on the private inventory and market size information of itself, but not on that of its competitors. Moreover, the exploitation-induction tradeoff is more intensive if the service effect and the network effect are stronger; and this trade-off decreases over the planning horizon. The exploitation-induction tradeoff is more involved in the PF model than in the SC model. This is because the competing firms need to balance this tradeoff both inter-temporally and intra-temporally in the PF model, whereas they only need to balance it inter-temporally in the SC model. More specifically, in the SC model, to effectively balance the exploitation-induction tradeoff, the firms should (a) increase promotional efforts, (b) offer price discounts, and (c) improve service levels. In the PF model, the firms should increase promotional efforts under the service effect and the network effect. Given the same promotional effort in the first stage competition, the firms need to decrease their sales prices under the network effect. However, with an increased promotional effort in the first stage competition, the equilibrium sales prices in the second stage competition may either decrease to increase. Analogously, the equilibrium post-delivery inventory levels may either decrease or increase in the PF model under the service effect and the network effect. Finally, we identify the "fat-cat" effect in our dynamic competition model: If the product differentiation is sufficiently high, under the MPE, the firms make more promotional efforts in the PF model than in the SC model. The driving force of this phenomenon is that the exploitation-induction tradeoff is more intensive under the promotion-first competition than under the simultaneous competition.

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Appendix A: Proofs of Statements

We use ∂ to denote the derivative operator of a single variable function, and ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x. For any multivariate continuously differentiable function $f(x_1, x_2, \cdots, x_n)$ and $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ in $f(\cdot)$'s domain, $\forall i$, we use $\partial_{x_i} f(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ to denote $\partial_{x_i} f(x_1, x_2, \cdots, x_n)|_{x=\tilde{x}}$. The following lemma is used throughout our proof.

LEMMA 1 Let $G_i(z, Z)$ be a continuously differentiable function in (z, Z), where $z \in [\underline{z}, \bar{z}]$ ($\underline{z}$ and $\bar{z}$ might be infinite) and $Z \in \mathbb{R}^{n_i}$ for $i = 1, 2$. For $i = 1, 2$, let $(z_i, Z_i) := \text{argmax}_{(z,Z)} G_i(z, Z)$ be the optimizers of $G_i(\cdot, \cdot)$. If $z_1 < z_2$, we have: $\partial_z G_1(z_1, Z_1) \leq \partial_z G_2(z_2, Z_2)$.

Proof: $z_1 < z_2$, so $\underline{z} \leq z_1 < z_2 \leq \bar{z}$. Hence, $\partial_z G_1(z_1, Z_1) \begin{cases} = 0 & \text{if } z_1 > \underline{z}, \\ \leq 0 & \text{if } z_1 = \underline{z}; \end{cases}$ and $\partial_z G_2(z_2, Z_2) \begin{cases} = 0 & \text{if } z_2 < \bar{z}, \\ \geq 0 & \text{if } z_2 = \bar{z}, \end{cases}$ i.e., $\partial_z G_1(z_1, Z_1) \leq 0 \leq \partial_z G_2(z_2, Z_2)$. $\square$

Proof of Theorems 1-2 and Propositions 1-2: We show Theorem 1, Proposition 1, Proposition 2, and Theorem 2 together by backward induction. More specifically, we show that, if $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{sc*}) = w_{i,t-1}I_{i,t-1} + \beta_{i,t-1}^{sc}\Lambda_{i,t-1}$ for all i, (a) Proposition 1 holds for period t, (b) Proposition 2 holds for period t, (c) there exists a Markov strategy profile $\{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \leq i \leq N\}$ which forms a Nash equilibrium in the subgame of period t, (d) under conditions (i) and (ii) in Theorem 1(c), the Nash equilibrium in the subgame of period t, $\{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \leq i \leq N\}$, is unique, and (e) there exists a positive vector β_t^{sc}, such that $V_{i,t}(I_t, \Lambda_t|\sigma_t^{sc*}) = w_{i,t}I_{i,t} + \beta_{i,t}^{sc}\Lambda_{i,t}$ for all i. Because $V_{i,0}(I_0, \Lambda_0) = w_{i,0}I_{i,0}$ for all i, the initial condition is satisfied.

Since $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{sc*}) = w_{i,t-1}I_{i,t-1} + \beta_{i,t-1}^{sc}\Lambda_{i,t-1}$ for all i, Equation (12) implies that the objective function of player i in $\mathcal{G}_t^{sc,2}$ is

$$\pi_{i,t}^{sc}(y_t) = (\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i\beta_{i,t-1}^{sc}(\kappa_{ii,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]) - \sum_{j \neq i} \kappa_{ij,t}(\mathbb{E}[y_{j,t}^+ \wedge \xi_{j,t}])).$$

Thus, for any given strategy of other players $y_{-i,t}$, player i maximizes the following univariate function:

$$\zeta_{i,t}^{sc}(y_{i,t}) := (\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i\beta_{i,t-1}^{sc}\kappa_{ii,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]).$$

If $y_{i,t} < 0$, $(y_{i,t} - \xi_{i,t})^+ = 0$, $(y_{i,t} - \xi_{i,t})^- = \xi_{i,t} - y_{i,t}$, and, thus, $-L_{i,t}(y_{i,t}) = -b_{i,t}\mathbb{E}(\xi_{i,t} - y_{i,t}) = -b_{i,t} + b_{i,t}y_{i,t}$. Moreover, $y_{i,t} < 0$ implies that $\delta_i\beta_{i,t-1}^{sc}\kappa_{ii,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]) \equiv \delta_i\beta_{i,t-1}^{sc}\kappa_{ii,t}(0)$. Hence, if $y_{i,t} < 0$,

$$\zeta_{i,t}^{sc}(y_{i,t}) = -b_{i,t} + (\delta_i w_{i,t-1} - w_{i,t} + b_{i,t})y_{i,t} + \delta_i\beta_{i,t-1}^{sc}\kappa_{ii,t}(0).$$

Because $b_{i,t} > w_{i,t} - \delta_i w_{i,t-1}$, $\zeta_{i,t}^{sc}(\cdot)$ is strictly increasing in $y_{i,t}$ for $y_{i,t} \leq 0$.

Observe that $-L_{i,t}(\cdot)$ is concave and continuously differentiable in $y_{i,t}$. Since $\mathbb{E}(y_{i,t}^+ \wedge \xi_{i,t})$ is concavely increasing and continuously differentiable in $y_{i,t}$ for $y_{i,t} \geq 0$, and $\kappa_{ii,t}(\cdot)$ is concavely increasing and continuously differentiable, $\kappa_{ii,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}])$ is concavely increasing and continuously differentiable in $y_{i,t}$ for $y_{i,t} \geq 0$. Hence, $\zeta_{i,t}^{sc}(\cdot)$ is concave and continuously differentiable in $y_{i,t}$ for $y_{i,t} \geq 0$. Observe that

$$\partial_{y_{i,t}}\zeta_{i,t}^{sc}(0+) = \delta_i w_{i,t-1} - w_{i,t} + b_{i,t} + \delta_i\beta_{i,t-1}^{sc}\bar{F}_{i,t}(0)\kappa'_{ii,t}(\mathbb{E}(0 \wedge \xi_{i,t})) = \delta_i w_{i,t-1} - w_{i,t} + b_{i,t} + \delta_i\beta_{i,t-1}^{sc}\kappa'_{ii,t}(0) > 0,$$

where the inequality follows from $\delta_i w_{i,t-1} - w_{i,t} + b_{i,t} > 0$ and $\kappa'_{ii,t}(0) \geq 0$. Therefore, the optimizer of $\zeta_{i,t}^{sc}(\cdot)$, $y_{i,t}^{sc*}$, is the solution to the first-order condition: $\partial_{y_{i,t}}\zeta_{i,t}^{sc}(y_{i,t}^{sc*}) = 0$, or, equivalently,

$$(\delta_i w_{i,t-1} - w_{i,t}) - L'_{i,t}(y_{i,t}^{sc*}) + \delta_i\beta_{i,t-1}^{sc}\bar{F}_{i,t}(y_{i,t}^{sc*})\kappa'_{ii,t}(\mathbb{E}(y_{i,t}^{sc*} \wedge \xi_{i,t})) = 0.$$

Because $\xi_{i,t}$ is continuously distributed, $y_{i,t}^{sc*}$ is unique for each i. Moreover, $y_{i,t}^{sc*} > 0$ and $\zeta_{i,t}^{sc}(y_{i,t}^{sc*}) > \zeta_{i,t}^{sc}(0) = -b_{i,t} + \delta_i \beta_{i,t-1}^{sc} \kappa_{ii,t}(0)$ for each i.

We now show that Proposition 2 holds for period t. Since $\zeta_{i,t}^{sc}(y_{i,t}^{sc*}) > \zeta_{i,t}^{sc}(0) = -b_{i,t} + \delta_i \beta_{i,t-1}^{sc} \kappa_{ii,t}(0)$ and $\alpha_{i,t}(z_t) \geq \kappa_{ii,t}(0) - \sum_{j \neq i} \kappa_{ij,t}(1) \geq 0$, we have $\pi_{i,t}^{sc*} > \zeta_{i,t}^{sc}(0) - \delta_i \beta_{i,t-1}^{sc} \sum_{j \neq i} \kappa_{ij,t}(1) \geq -b_{i,t}$. Observe that

$$\bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*} > \bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\bar{\gamma}_{i,t}) - b_{i,t} > 0.$$

Thus, if $p_{i,t} = \bar{p}_{i,t}$, $p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*} > 0$. Therefore, each firm i could at least earn a positive payoff of $(\bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\bar{\gamma}_{i,t}) - b_{i,t}) \underline{\epsilon}_{i,t}$ by charging the maximum allowable price $\bar{p}_{i,t}$, where

$$\underline{\epsilon}_{i,t} := \min\{\psi_{i,t}(\gamma_t)\rho_{i,t}(p_t) : \gamma_t \in [0, \bar{\gamma}_{1,t}] \times [0, \bar{\gamma}_{2,t}] \times \cdots \times [0, \bar{\gamma}_{N,t}] \times [\underline{p}_{1,t}, \bar{p}_{1,t}] \times [\underline{p}_{2,t}, \bar{p}_{2,t}] \times \cdots \times [\underline{p}_{N,t}, \bar{p}_{N,t}]\} > 0.$$

Let

$$\bar{\epsilon}_{i,t} := \max\{\psi_{i,t}(\gamma_t)\rho_{i,t}(p_t) : \gamma_t \in [0, \bar{\gamma}_{1,t}] \times [0, \bar{\gamma}_{2,t}] \times \cdots \times [0, \bar{\gamma}_{N,t}] \times [\underline{p}_{1,t}, \bar{p}_{1,t}] \times [\underline{p}_{2,t}, \bar{p}_{2,t}] \times \cdots \times [\underline{p}_{N,t}, \bar{p}_{N,t}]\} \geq \underline{\epsilon}_{i,t}.$$

Hence, we can restrict the feasible action set of firm i in $\mathcal{G}_t^{sc,1}$ to

$$\mathcal{A}_{i,t}^{sc,1} := \{(\gamma_{i,t}, p_{i,t}) \in [0, \bar{\gamma}_{i,t}] \times [\underline{p}_{i,t}, \bar{p}_{i,t}] : p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*} \geq \frac{(\bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\bar{\gamma}_{i,t}) - b_{i,t})\underline{\epsilon}_{i,t}}{\bar{\epsilon}_{i,t}} > 0\},$$

which is a nonempty and complete sublattice of $\mathbb{R}^2$. Thus, $\Pi_{i,t}^{sc}(\gamma_t, p_t) > 0$ and

$$\log(\Pi_{i,t}^{sc}(\gamma_t, p_t)) = \log(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*}) + \log(\psi_{i,t}(\gamma_t)) + \log(\rho_{i,t}(p_t)) \tag{28}$$

is well-defined on $\mathcal{A}_{i,t}^{sc,1}$. Because $\rho_{i,t}(\cdot)$ and $\psi_{i,t}(\cdot)$ satisfy (3) and (4), for each i and $j \neq i$, we have

$$\frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial \gamma_{i,t} \partial p_{i,t}} = \frac{\partial^2 \log(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*})}{\partial \gamma_{i,t} \partial p_{i,t}} = \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*})^2} \geq 0,$$

$$\frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial \gamma_{i,t} \partial p_{j,t}} = 0, \quad \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} = \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} \geq 0,$$

$$\frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial p_{i,t} \partial \gamma_{j,t}} = 0, \text{ and } \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial p_{i,t} \partial p_{j,t}} = \frac{\partial^2 \log(\rho_{i,t}(p_t))}{\partial p_{i,t} \partial p_{j,t}} \geq 0.$$

Hence, $\mathcal{G}_t^{sc,1}$ is a log-supermodular game and, thus, has pure strategy Nash equilibria which are the smallest and largest undominated strategies (see Theorem 5 in Milgrom and Roberts, 1990).

Next, we show that if conditions (i) and (ii) in Theorem 1(c) hold, the Nash equilibrium of $\mathcal{G}_t^{sc,1}$ is unique. First, we show that under conditions (i) and (ii) in Theorem 1(c),

$$\frac{\partial^2 \log \Pi_{i,t}^{sc}(\gamma_t, p_t)}{\partial p_{i,t}^2} < 0, \ |\frac{\partial^2 \log \Pi_{i,t}^{sc}(\gamma_t, p_t)}{\partial p_{i,t}^2}| > \sum_{j \neq i} \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial p_{i,t} \partial p_{j,t}} + \sum_{j=1}^{N} \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial p_{i,t} \partial \gamma_{j,t}}, \tag{29}$$

$$\frac{\partial^2 \log \Pi_{i,t}^{sc}(\gamma_t, p_t)}{\partial \gamma_{i,t}^2} < 0, \text{ and } |\frac{\partial^2 \log \Pi_{i,t}^{sc}(\gamma_t, p_t)}{\partial \gamma_{i,t}^2}| > \sum_{j \neq i} \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} + \sum_{j=1}^{N} \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial \gamma_{i,t} \partial p_{j,t}}. \tag{30}$$

Note that, by (28) and (4),

$$\frac{\partial^2 \log \Pi_{i,t}^{sc}(\gamma_t, p_t)}{\partial p_{i,t}^2} = \frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t}^2} - \frac{1}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*})^2} < 0,$$

and

$$|\frac{\partial^2 \log \Pi_{i,t}^{sc}(\gamma_t, p_t)}{\partial p_{i,t}^2}| = |\frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t}^2}| + \frac{1}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*})^2}.$$

Since $\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial \gamma_{j,t}} = 0$ for $j \neq i$, and

$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial p_{i,t}\partial \gamma_{i,t}} = \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2},$$

we have

$$\begin{aligned}
|\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t, p_t)}{\partial p^2_{i,t}}| &= |\frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p^2_{i,t}}| + \frac{1}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2} \\
&> \sum_{j\neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial p_{i,t}\partial p_{j,t}} + \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2} \\
&= \sum_{j\neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial p_{i,t}\partial p_{j,t}} + \sum_{j=1}^{N} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial p_{i,t}\partial \gamma_{j,t}},
\end{aligned}$$

where the inequality follows from (4) and condition (i). Hence, (29) holds for all i and all (γ_t, p_t).

Since $\nu''_{i,t}(\cdot) \geq 0$ and (3), we have

$$\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t, p_t)}{\partial \gamma^2_{i,t}} = \frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma^2_{i,t}} - \frac{\nu''_{i,t}(\gamma_t)(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}) + (\nu'_{i,t}(\gamma_t))^2}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2} < 0,$$

and

$$|\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t, p_t)}{\partial \gamma^2_{i,t}}| = |\frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma^2_{i,t}}| + \frac{\nu''_{i,t}(\gamma_t)(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}) + (\nu'_{i,t}(\gamma_t))^2}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2}.$$

Since $\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial \gamma_{i,t}\partial p_{j,t}} = 0$ for $j \neq i$, and

$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial \gamma_{i,t}\partial p_{i,t}} = \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2},$$

we have

$$\begin{aligned}
|\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t, p_t)}{\partial \gamma^2_{i,t}}| &= |\frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma^2_{i,t}}| + \frac{\nu''_{i,t}(\gamma_t)(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}) + (\nu'_{i,t}(\gamma_t))^2}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2} \\
&> \sum_{j\neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial \gamma_{i,t}\partial \gamma_{j,t}} + \frac{\nu''_{i,t}(\gamma_t)(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \underline{c}_{i,t}) + (\nu'_{i,t}(\gamma_t))^2}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2} \\
&\geq \sum_{j\neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial \gamma_{i,t}\partial \gamma_{j,t}} + \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2} \\
&= \sum_{j\neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial \gamma_{i,t}\partial \gamma_{j,t}} + \sum_{j=1}^{N} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial \gamma_{i,t}\partial p_{j,t}},
\end{aligned}$$

where the first inequality follows from (4) and $\pi^{sc*}_{i,t} \geq \underline{c}_{i,t}$, and the second from condition (ii). Hence, (30) holds for all i and all (γ_t, p_t).

We now show that if (29) and (30) hold, $\mathcal{G}^{sc,1}_t$ has a unique Nash equilibrium. Recall that the set of Nash equilibria in $\mathcal{G}^{sc,1}_t$ forms a complete lattice (see Theorem 2 in Zhou, 1994). If, to the contrary, there exist two distinct equilibria (γ^*_t, p^*_t) and $(\hat{\gamma}^*_t, \hat{p}^*_t)$, where $\hat{p}^*_{i,t} \geq p^*_{i,t}$ for all i and $\hat{\gamma}^*_{j,t} \geq \gamma^*_{j,t}$ for all j, with the inequality being strict for some i or j. If, for some i, $\hat{p}^*_{i,t} > p^*_{i,t}$, $\hat{p}^*_{i,t} - p^*_{i,t} \geq \hat{p}^*_{l,t} - p^*_{l,t}$ for all l, and $\hat{p}^*_{i,t} - p^*_{i,t} \geq \hat{\gamma}^*_{l,t} - \gamma^*_{l,t}$ for all l, without loss of generality, we assume that $i = 1$. Lemma 1 suggests that

$$\partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\hat{\gamma}^*_t, \hat{p}^*_t)) \geq \partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^*_t, p^*_t)). \tag{31}$$

On the other hand, by Newton-Leibniz formula, we have

$$\begin{aligned}
& \partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\hat{\gamma}^*_t, \hat{p}^*_t)) - \partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^*_t, p^*_t)) \\
= & \int_{s=0}^{1} [\sum_{j=1}^{N} (\hat{p}^*_{j,t} - p^*_{j,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial p_{1,t} \partial p_{j,t}} \\
& + \sum_{j=1}^{N} (\hat{\gamma}^*_{j,t} - \gamma^*_{j,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial p_{1,t} \partial \gamma_{j,t}}] \, \mathrm{d}s \\
\leq & \int_{s=0}^{1} [\sum_{j=1}^{N} (\hat{p}^*_{1,t} - p^*_{1,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial p_{1,t} \partial p_{j,t}} \\
& + \sum_{j=1}^{N} (\hat{p}^*_{1,t} - p^*_{1,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial p_{1,t} \partial \gamma_{j,t}}] \, \mathrm{d}s \\
< & \ 0,
\end{aligned}$$

where the first inequality follows from $\hat{p}^*_{1,t} - p^*_{1,t} \geq \hat{p}^*_{l,t} - p^*_{l,t}$ for all l and $\hat{p}^*_{1,t} - p^*_{1,t} \geq \hat{\gamma}^*_{l,t} - \gamma^*_{l,t}$ for all l, and the second from $\hat{p}^*_{1,t} - p^*_{1,t} > 0$ and (29). This contradicts (31).

If, for some j, $\hat{\gamma}^*_{j,t} > \gamma^*_{j,t}$, $\hat{\gamma}^*_{j,t} - \gamma^*_{j,t} \geq \hat{p}^*_{l,t} - p^*_{l,t}$ for all l, and $\hat{\gamma}^*_{j,t} - \gamma^*_{j,t} \geq \hat{\gamma}^*_{l,t} - \gamma^*_{l,t}$ for all l, without loss of generality, we assume that $j = 1$. Lemma 1 suggests that

$$\partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\hat{\gamma}^*_t, \hat{p}^*_t)) \geq \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^*_t, p^*_t)). \tag{32}$$

On the other hand, by Newton-Leibniz formula, we have

$$\begin{aligned}
& \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\hat{\gamma}^*_t, \hat{p}^*_t)) - \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^*_t, p^*_t)) \\
= & \int_{s=0}^{1} [\sum_{j=1}^{N} (\hat{\gamma}^*_{j,t} - \gamma^*_{j,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial \gamma_{1,t} \partial \gamma_{j,t}} \\
& + \sum_{j=1}^{N} (\hat{p}^*_{j,t} - p^*_{j,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial \gamma_{1,t} \partial p_{j,t}}] \, \mathrm{d}s \\
\leq & \int_{s=0}^{1} [\sum_{j=1}^{N} (\hat{\gamma}^*_{1,t} - \gamma^*_{1,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial \gamma_{1,t} \partial \gamma_{j,t}} \\
& + \sum_{j=1}^{N} (\hat{\gamma}^*_{1,t} - \gamma^*_{1,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial \gamma_{1,t} \partial p_{j,t}}] \, \mathrm{d}s \\
< & \ 0,
\end{aligned}$$

where the first inequality follows from $\hat{\gamma}^*_{1,t} - \gamma^*_{1,t} \geq \hat{p}^*_{l,t} - p^*_{l,t}$ for all l and $\hat{\gamma}^*_{1,t} - \gamma^*_{1,t} \geq \hat{\gamma}^*_{l,t} - \gamma^*_{l,t}$ for all l, and the second from $\hat{\gamma}^*_{1,t} - \gamma^*_{1,t} > 0$ and (30). This contradicts (32). Therefore, the Nash equilibrium in $\mathcal{G}^{sc,1}_t$ is unique, if conditions (i) and (ii) in Theorem 1(c) hold.

If $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, we have $\nu'_{i,t}(\gamma_{i,t}) = 1$ and $\nu''_{i,t}(\gamma_{i,t}) = 0$ for all $\gamma_{i,t} \in [0, \bar{\gamma}_{i,t}]$. Thus, if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, conditions (i) and (ii) in Theorem 1(c) hold.

Note that for any $\lambda \in [0,1]$ and $(\gamma_{i,t}, p_{i,t}), (\hat{\gamma}_{i,t}, \hat{p}_{i,t}) \in [0, \bar{\gamma}_{1,t}] \times [0, \bar{\gamma}_{2,t}] \times \cdots \times [0, \bar{\gamma}_{N,t}] \times [\underline{p}_{1,t}, \bar{p}_{1,t}] \times [\underline{p}_{2,t}, \bar{p}_{2,t}] \times \cdots \times [\underline{p}_{N,t}, \bar{p}_{N,t}]$,

$$\begin{aligned}
& \lambda \log(\hat{p}_{i,t} - \delta_i w_{i,t} - \nu_{i,t}(\hat{\gamma}_{i,t}) + \pi^{sc*}_{i,t}) + (1-\lambda) \log(p_{i,t} - \delta_i w_{i,t} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}) \\
\leq \ & \log(\lambda \hat{p}_{i,t} + (1-\lambda) p_{i,t} - \delta_i w_{i,t} - \lambda \nu_{i,t}(\hat{\gamma}_{i,t}) - (1-\lambda) \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}) \\
\leq \ & \log(\lambda \hat{p}_{i,t} + (1-\lambda) p_{i,t} - \delta_i w_{i,t} - \nu_{i,t}(\lambda \hat{\gamma}_{i,t} + (1-\lambda)\gamma_{i,t}) + \pi^{sc*}_{i,t}),
\end{aligned}$$

where the first inequality follows from the concavity of $\log(\cdot)$, and the second from that $\log(\cdot)$ is an increasing function and $\nu_{i,t}(\cdot)$ is a convex function. Thus, $\log(p_{i,t} - \delta_i w_{i,t} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*})$ is jointly concave in $(\gamma_{i,t}, p_{i,t})$. Hence, the diagonal dominance condition (3) and (4) implies that $\log(\Pi_{i,t}^{sc}(\gamma_t, p_t))$ is jointly concave in $(\gamma_{i,t}, p_{i,t})$ for any given $(\gamma_{-i,t}, p_{-i,t})$. Therefore, the first-order conditions with respect to $\gamma_{i,t}$ and $p_{i,t}$ is the necessary and sufficient condition for $(\gamma_t^{sc*}, p_t^{sc*})$ to be the unique Nash equilibrium in $\mathcal{G}_t^{sc,1}$. Since

$$\partial_{\gamma_{i,t}} \log(\Pi_{i,t}^{sc}(\gamma_t, p_t)) = \frac{\partial_{\gamma_{i,t}} \psi_{i,t}(\gamma_t)}{\psi_{i,t}(\gamma_t)} - \frac{\nu'_{i,t}(\gamma_t)}{p_{i,t} - \delta w_{i,t} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*}},$$

and

$$\partial_{p_{i,t}} \log(\Pi_{i,t}^{sc}(\gamma_t, p_t)) = \frac{\partial_{p_{i,t}} \rho_{i,t}(p_t)}{\rho_{i,t}(p_t)} + \frac{1}{p_{i,t} - \delta w_{i,t} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*}},$$

the Nash equilibrium of $\mathcal{G}_t^{sc,1}$ is a solution to the system of equations (15). Since $\mathcal{G}_t^{sc,1}$ has a unique equilibrium, (15) has a unique solution, which coincides with the unique pure strategy Nash equilibrium of $\mathcal{G}_t^{sc,1}$. As shown above, for all i,

$$\Pi_{i,t}^{sc}(\gamma_t^{sc*}, p_t^{sc*}) \geq (\bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\bar{\gamma}_{i,t}) - b_{i,t})\underline{\epsilon}_{i,t} > 0.$$

Hence, $\Pi_{i,t}^{sc*} = \Pi_{i,t}^{sc}(\gamma_t^{sc*}, p_t^{sc*}) > 0$ for all i.

Next, we show that $\{(\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}, \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*}) \psi_{i,t}(\gamma_t^{sc*})) : 1 \leq i \leq N\}$ is an equilibrium in the subgame of period t. Since $y_{i,t}^{sc*} > 0$, $\Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*}) \psi_{i,t}(\gamma_t^{sc*}) > 0$ for all i. Therefore, regardless of the starting inventory in period t, $I_{i,t}$, firm i could adjust its inventory to $x_{i,t}^{sc*}(I_t, \Lambda_t) = \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*}) \psi_{i,t}(\gamma_t^{sc*})$. Thus, by Propositions 1-2, $\{(\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}, \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*}) \psi_{i,t}(\gamma_t^{sc*})) : 1 \leq i \leq N\}$ forms an equilibrium in the subgame of period t. In particular, if conditions (i) and (ii) hold, $\{(\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}, \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*}) \psi_{i,t}(\gamma_t^{sc*})) : 1 \leq i \leq N\}$ is the unique equilibrium in the subgame of period t.

Next, we show that there exists a positive vector $\beta_t^{sc} = (\beta_{1,t}^{sc}, \beta_{2,t}^{sc}, \cdots, \beta_{N,t}^{sc})$, such that $V_{i,t}(I_t, \Lambda_t | \sigma_t^{sc*}) = w_{i,t} I_{i,t} + \beta_{i,t}^{sc} \Lambda_{i,t}$. By (12), we have that

$$V_{i,t}(I_t, \Lambda_t | \sigma_t^{sc*}) = J_{i,t}(\gamma_t^{sc*}, p_t^{sc*}, \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*}) \psi_{i,t}(\gamma_t^{sc*}), I_t, \Lambda_t | \sigma_{t-1}^{sc*}) = w_{i,t} I_{i,t} + (\sigma_i \beta_{i,t-1}^{sc} \mu_{i,t} + \Pi_{i,t}^{sc*}) \Lambda_{i,t}.$$

Since $\beta_{i,t-1}^{sc} \geq 0$ and $\Pi_{i,t}^{sc*} > 0$, $\beta_{i,t}^{sc} = \delta_i \beta_{i,t-1}^{sc} \mu_{i,t} + \Pi_{i,t}^{sc*} > 0$. This completes the induction and, thus, the proof of Theorem 1, Proposition 1, Proposition 2, and Theorem 2. $\square$

Proof of Proposition 3: By Theorems 1-2, and Propositions 1-2, it suffices to show that, if there exists a constant $\beta_{s,t-1}^{sc} \geq 0$, such that $V_{i,t-1}(I_{t-1}, \Lambda_{t-1} | \sigma_{t-1}^{sc*}) = w_{s,t} I_{i,t-1} + \beta_{s,t-1}^{sc} \Lambda_{i,t-1}$ for all i, we have: (a) the unique Nash equilibrium in $\mathcal{G}_t^{sc,2}$ is symmetric, i.e., $y_{i,t}^{sc*} = y_{j,t}^{sc*}$ for all i, j; (b) the unique Nash equilibrium in $\mathcal{G}_t^{sc,1}$ is symmetric, i.e., $(\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}) = (\gamma_{j,t}^{sc*}, p_{j,t}^{sc*})$ for all $i \neq j$, and (c) there exists a constant $\beta_{s,t}^{sc} > 0$, such that $V_{i,t}(I_t, \Lambda_t | \sigma_{s,t}^{sc*}) = w_{s,t} I_{i,t} + \beta_{s,t}^{sc} \Lambda_{i,t}$ for all i. Since $V_{i,0}(I_t, \Lambda_t) = w_{s,0} I_{i,0}$ for all i, the initial condition is satisfied with $\beta_{s,0}^{sc} = 0$.

Since $V_{i,t-1}(I_{t-1}, \Lambda_{t-1} | \sigma_{t-1}^{sc*}) = w_{s,t} I_{i,t-1} + \beta_{s,t}^{sc} \Lambda_{i,t-1}$ for all i, by (12),

$$\pi_{i,t}^{sc}(y_t) = (\delta_s w_{s,t-1} - w_{s,t}) y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta_{s,t-1}^{sc} (\kappa_{sa,t}(\mathbb{E}(y_{i,t}^+ \wedge \xi_{i,t})) - \sum_{j \neq i} \kappa_{sb,t}(\mathbb{E}(y_{j,t}^+ \wedge \xi_{j,t}))).$$

Hence,

$$\zeta_{i,t}^{sc}(y_{i,t}) = (\delta_s w_{s,t-1} - w_{s,t}) y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta_{s,t-1}^{sc} \kappa_{sa,t}(\mathbb{E}(y_{i,t}^+ \wedge \xi_{i,t})).$$

Thus, $\zeta^{sc}_{i,t}(\cdot) \equiv \zeta^{sc}_{j,t}(\cdot)$ for all i and j. Therefore, for all i and j,

$$y^{sc*}_{i,t} = \text{argmax}_y \zeta^{sc}_{i,t}(y) = \text{argmax}_y \zeta^{sc}_{j,t}(y) = y^{sc*}_{j,t}$$

and, hence,

$$\pi^{sc*}_{i,t} = \pi^{sc}_{i,t}(y^{sc*}_t) = \pi^{sc}_{j,t}(y^{sc*}_t) = \pi^{sc*}_{j,t}.$$

We denote $y^{sc*}_{s,t} = y^{sc*}_{i,t}$ for each i, and $\pi^{sc*}_{s,t} = \pi^{sc*}_{i,t}$ for each i. Observe that, the objective functions of $\mathcal{G}^{sc,1}_t$,

$$\{\Pi^{sc}_{i,t}(\gamma_t, p_t) = \rho_{s,t}(p_t)\psi_{s,t}(\gamma_t)[p_{i,t} - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{i,t}) + \pi^{sc*}_{s,t}] : 1 \le i \le N\}$$

are symmetric. Hence, if there exists an asymmetric Nash equilibrium $(\gamma^{sc*}_t, p^{sc*}_t)$, there exists another Nash equilibrium $(\underline{\gamma}^{sc*}_t, \underline{p}^{sc*}_t) \neq (\gamma^{sc*}_t, p^{sc*}_t)$, where $\underline{\gamma}^{sc*}_t$ is a permutation of γ^{sc*}_t and $\underline{p}^{sc*}_t$ is a permutation of p^{sc*}_t. This contradicts the uniqueness of the Nash equilibrium in $\mathcal{G}^{sc,1}_t$. Thus, the unique Nash equilibrium in $\mathcal{G}^{sc,1}_t$ is symmetric. Hence,

$$\Pi^{sc*}_{i,t} = \Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}, p^{sc*}_{ss,t}) = \rho_{s,t}(p^{sc*}_{ss,t})\psi_{s,t}(\gamma^{sc*}_{ss,t})[p^{sc*}_{s,t} - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}) + \pi^{sc*}_{s,t}] = \Pi^{sc}_{j,t}(\gamma^{sc*}_{ss,t}, p^{sc*}_{ss,t}) = \Pi^{sc*}_{j,t} > 0.$$

Thus, we denote the payoff of each firm i as $\Pi^{sc*}_{s,t}$. By Theorem 2(a),

$$\beta^{sc}_{i,t} = \delta_s \beta^{sc}_{s,t-1}\mu_{s,t} + \Pi^{sc*}_{i,t} = \delta_s \beta^{sc}_{s,t-1}\mu_{s,t} + \Pi^{sc*}_{j,t} = \beta^{sc}_{j,t} > 0.$$

Thus, we denote the SC market size coefficient of each firm i as $\beta^{sc}_{s,t}$. This completes the induction and, thus, the proof of Proposition 3. □

Proof of Theorem 3: Part (a). Clearly, by (13), $y^{sc*}_{i,t}$ is independent of $\beta^{sc}_{j,t-1}$ for all $j \neq i$. Moreover, because

$$\frac{\partial^2 \zeta^{sc}_{i,t}(y_{i,t})}{\partial y_{i,t} \partial \beta^{sc}_{i,t-1}} = \begin{cases} \delta_i \bar{F}_{i,t}(y_{i,t})\kappa'_{ii,t}(\mathbb{E}(y_{i,t} \wedge \xi_{i,t})) \ge 0, & \text{if } y_{i,t} \ge 0; \\ 0, & \text{otherwise,} \end{cases}$$

$\zeta^{sc}_{i,t}(y_{i,t})$ is supermodular in $(y_{i,t}, \beta^{sc}_{i,t-1})$. Therefore, $y^{sc*}_{i,t} = \text{argmax}_{y_{i,t} \in \mathbb{R}} \zeta^{sc}_{i,t}(y_{i,t})$ is increasing in $\beta^{sc}_{i,t-1}$. The continuity of $y^{sc*}_{i,t}$ in $\beta^{sc}_{i,t-1}$ follows directly from the continuous differentiability of $\zeta^{sc}_{i,t}(\cdot)$ in $(y_{i,t}, \beta^{sc}_{i,t-1})$. This completes the proof of part (a).

Part (b). Note that, by part (a), $\sum_{l \neq i} \kappa_{il,t}(\mathbb{E}((y^{sc*}_{l,t})^+ \wedge \xi_{l,t}))$ is independent of $\beta^{sc}_{i,t-1}$ and continuously increasing in $\beta^{sc}_{j,t-1}$ for $j \neq i$. Moreover,

$$\zeta^{sc}_{i,t}(y_{i,t}) = (\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i \beta^{sc}_{i,t-1}\kappa_{ii,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}])$$

is continuously increasing in $\beta^{sc}_{i,t-1}$ and independent of $\beta^{sc}_{j,t-1}$ for all $j \neq i$. Thus,

$$\pi^{sc*}_{i,t} = [\max_{y_{i,t} \ge 0} \zeta^{sc}_{i,t}(y_{i,t})] - \sum_{j \neq i} \kappa_{ij,t}(\mathbb{E}(y^{sc*}_{j,t} \wedge \xi_{j,t}))$$

is continuously increasing in $\beta^{sc}_{i,t-1}$ and continuously decreasing in $\beta^{sc}_{j,t-1}$ for all $j \neq i$. This completes the proof of part (b).

Part (c). We denote the objective function of each firm i in $\mathcal{G}^{sc,1}_{s,t}$ as $\Pi^{sc}_{i,t}(\cdot, \cdot | \pi^{sc*}_{s,t})$ to capture the dependence of the objective functions on $\pi^{sc*}_{s,t}$. The unique symmetric Nash equilibrium in $\mathcal{G}^{sc,1}_{s,t}$ is denoted as $(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))$, where $\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) = (\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}), \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}), \cdots, \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$ and $p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) = (p^{sc*}_{s,t}(\pi^{sc*}_{s,t}), p^{sc*}_{s,t}(\pi^{sc*}_{s,t}), \cdots, p^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. It suffices to show that, if $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \ge \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$, and $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \le p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$.

We first show that $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \leq p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ for all $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$. Assume, to the contrary, that $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) > p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$. Lemma 1 implies that

$$\partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})) \geq \partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t})),$$

i.e.,

$$\begin{aligned} &\partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) + \frac{1}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} \\ \geq &\partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) + \frac{1}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}. \end{aligned} \tag{33}$$

By (4) and Newton-Leibniz formula, we have

$$\begin{aligned} & \partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) - \partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) \\ = & \int_{s=0}^{1} [\sum_{j=1}^{N} (p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - p^{sc*}_{s,t}(\pi^{sc*}_{s,t})) \frac{\partial^2 \log \rho_{s,t}((1-s)p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) + s p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))}{\partial p_{1,t} \partial p_{j,t}}] \, \mathrm{d}s \\ < & \ 0. \end{aligned}$$

Hence, inequality (33) suggests that

$$p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t} < p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}. \tag{34}$$

Since $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) > p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ and $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, $\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) > \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. Thus, $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) > \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$. Lemma 1 yields that $\partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})) \geq \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t}))$, i.e.,

$$\begin{aligned} &\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} \\ \geq &\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}} \end{aligned} \tag{35}$$

Since $\nu_{s,t}(\cdot)$ is convexly increasing, $\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \geq \nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. Thus, inequality (34) implies that

$$-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} < -\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}.$$

Hence, (35) suggests that

$$\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) > \partial_{\gamma_{1,t}} \log \psi_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})). \tag{36}$$

By (3) and Newton-Leibniz formula, we have

$$\begin{aligned} & \partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) - \partial_{\gamma_{1,t}} \log \psi_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) \\ = & \int_{s=0}^{1} [\sum_{j=1}^{N} (\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) \frac{\partial^2 \log \psi_{s,t}((1-s)\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) + s\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))}{\partial \gamma_{1,t} \partial \gamma_{j,t}}] \, \mathrm{d}s \\ < & \ 0, \end{aligned}$$

which contradicts (36). Therefore, for all $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, we have $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \leq p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$.

We now show that $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \geq \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ for all $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$. Assume, to the contrary, that $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) < \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$. Lemma 1 implies that

$$\partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})) \leq \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t})),$$

i.e.,

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} \\
\leq &\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}.
\end{aligned} \tag{37}$$

By (3) and Newton-Leibniz formula, we have

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) - \partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) \\
= &\int_{s=0}^{1} [\sum_{j=1}^{N} (\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) \frac{\partial^2 \log \psi_{s,t}(s\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) + (1-s)\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))}{\partial \gamma_{1,t} \partial \gamma_{j,t}}] \, \mathrm{d}s < 0.
\end{aligned}$$

Hence, inequality (37) implies that

$$-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} < -\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}.$$

Since $\nu_{s,t}(\cdot)$ is convexly increasing, $\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) \leq \nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. Hence,

$$p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t} < p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}.$$

Since $\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) \leq \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$ and $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) < p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$. Lemma 1 implies that $\partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})) \leq \partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t}))$, i.e.,

$$\begin{aligned}
&\partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) + \frac{1}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} \\
\leq &\partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) + \frac{1}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}.
\end{aligned} \tag{38}$$

Because

$$\frac{1}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} > \frac{1}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}},$$

we have that

$$\partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) < \partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})). \tag{39}$$

By (4) and Newton-Leibniz formula, we have

$$\begin{aligned}
&\partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) - \partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) \\
= &\int_{s=0}^{1} [\sum_{j=1}^{N} (p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) \frac{\partial^2 \log \rho_{s,t}(sp^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) + (1-s)p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))}{\partial p_{1,t} \partial p_{j,t}}] \, \mathrm{d}s \\
< &\ 0,
\end{aligned}$$

which contradicts (39). Therefore, for all $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, we have $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \leq \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$. The continuity of $\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ and $p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ in $\pi^{sc*}_{s,t}$ follows directly from that $\Pi^{sc}_{i,t}(\gamma_t, p_t|\pi^{sc*}_{s,t})$ is twice continuously differentiable and the implicit function theorem. This completes the proof of part (c).

Part (d). By Theorem 2(a), $\beta^{sc}_{s,t} = \delta_s \beta^{sc}_{s,t} \mu_{s,t} + \Pi^{sc*}_{s,t}$, it suffices to show that $\Pi^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ is continuously increasing in $\pi^{sc*}_{s,t}$, where $\Pi^{sc*}_{s,t}(\pi^{sc*}_{s,t}) := \Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))$.

Assume that $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$. Since part (c) implies that $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \leq p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ and $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \geq \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$, the monotonicity condition (17) implies that

$$\rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) \geq \rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) \text{ and } \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) \geq \psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})). \tag{40}$$

If $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) = p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ and $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) = \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$, by $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, we have

$$p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t} > p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}.$$

Thus,

$$\begin{aligned}
\Pi^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) &= \Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t}) \\
&= (p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) \\
&> (p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) \\
&= \Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t}) \\
&= \Pi^{sc*}_{s,t}(\pi^{sc*}_{s,t}).
\end{aligned}$$

If $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) < p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$, Lemma 1 yields that

$$\partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), \gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})) \le \partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), \gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t})),$$

i.e.,

$$\begin{aligned}
&\partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) + \frac{1}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} \\
\le&\partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) + \frac{1}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}.
\end{aligned} \tag{41}$$

By (4) and Newton-Leibniz formula, we have

$$\begin{aligned}
&\partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) - \partial_{p_{1,t}} \log \rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) \\
=& \int_{s=0}^{1} [\sum_{j=1}^{N} (p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) \frac{\partial^2 \log \rho_{s,t}((1-s)p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}) + s p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))}{\partial p_{1,t} \partial p_{j,t}}] \,\mathrm{d}s < 0.
\end{aligned}$$

Hence, (41) implies that

$$p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t} > p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}.$$

Therefore,

$$\begin{aligned}
\Pi^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) &= \Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t}) \\
&= (p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) \\
&> (p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) \\
&= \Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t}) \\
&= \Pi^{sc*}_{s,t}(\pi^{sc*}_{s,t}).
\end{aligned}$$

If $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) = p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ and $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) > \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$, Lemma 1 yields that

$$\partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), \gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})) \ge \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), \gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t})),$$

i.e.,

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} \\
\ge&\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}.
\end{aligned} \tag{42}$$

By (4) and Newton-Leibniz formula, we have

$$\begin{aligned}
&\partial_{\gamma_{1,t}}\log\psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))-\partial_{\gamma_{1,t}}\log\psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\\
=\ &\int_{s=0}^{1}[\sum_{j=1}^{N}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})-\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))\frac{\partial^2\log\psi_{s,t}(s\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})+(1-s)\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}]\,\mathrm{d}s<0.
\end{aligned}$$

Hence, (42) implies that

$$-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))+\bar{\pi}^{sc*}_{s,t}}>-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}((\pi^{sc*}_{s,t}))-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))+\pi^{sc*}_{s,t}}. \tag{43}$$

Since $\nu_{s,t}(\cdot)$ is convexly increasing, $\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))\geq\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. Hence, (43) implies that

$$p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))+\bar{\pi}^{sc*}_{s,t}>p^{sc*}_{s,t}((\pi^{sc*}_{s,t}))-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))+\pi^{sc*}_{s,t}.$$

Therefore,

$$\begin{aligned}
\Pi^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) &= \Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}),p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})\\
&= (p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))+\bar{\pi}^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))\\
&> (p^{sc*}_{s,t}(\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))+\pi^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\\
&= \Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}),p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t})\\
&= \Pi^{sc*}_{s,t}(\pi^{sc*}_{s,t}).
\end{aligned}$$

Thus, we have shown that, if $\bar{\pi}^{sc*}_{s,t}>\pi^{sc*}_{s,t}$, $\Pi^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})>\Pi^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ and, hence, by Theorem 2(a), $\beta^{sc}_{s,t}(\bar{\pi}^{sc*}_{s,t})>\beta^{sc}_{s,t}(\pi^{sc*}_{s,t})$. The continuity of $\beta^{sc}_{s,t}$ in $\pi^{sc*}_{s,t}$ follows directly from the continuous differentiability of $\Pi^{sc}_{i,t}(\gamma_t,p_t|\pi^{sc*}_{s,t})$ in $(\gamma_t,p_t,\pi^{sc*}_{s,t})$ and the continuity of $(\gamma^{sc*}_{ss,t},p^{sc*}_{ss,t})$ in $\pi^{sc*}_{s,t}$. This completes the proof of part (d).

Part (e). By part (c), it suffices to show that, $\pi^{sc*}_{s,t}$ is continuously increasing in $\beta^{cs}_{s,t-1}$. The monotonicity follows from the assumption, whereas the continuity follows directly from part (a) and that the compound function is continuous if each individual function is continuous. This completes the proof of part (e).

Part (f). By the proof of part (e), $\pi^{sc*}_{s,t}$ is continuously increasing in $\beta^{cs}_{s,t-1}$. By part (d), $\beta^{sc}_{s,t}$ is continuously increasing in $\beta^{sc}_{s,t-1}$. □

Proof of Theorem 4: Part (a). Because $\beta^{sc}_{i,t-1}\geq\tilde{\beta}^{sc}_{i,t-1}=0$ for each i and t, Theorem 3(a) implies that $y^{sc*}_{i,t}\geq\tilde{y}^{sc*}_{i,t}$ for all i and t. Thus,

$$z^{sc*}_{i,t}=\mathbb{E}[(y^{sc*}_{i,t})^+\wedge\xi_{i,t}]\geq\mathbb{E}[(\tilde{y}^{sc*}_{i,t})^+\wedge\xi_{i,t}]=z^{sc*}_{i,t},\ \text{for all } i \text{ and } t.$$

Moreover, since $\tilde{\beta}^{sc}_{i,t-1}=0$, $\tilde{\pi}^{sc}_{i,t}(y_t)=(\delta_i w_{i,t-1}-w_{i,t})y_{i,t}-L_{i,t}(y_{i,t})$. Moreover, if $y_{i,t}\leq 0$, $\tilde{\pi}^{sc}_{i,t}(y_t)$ is strictly increasing in $y_{i,t}$. Hence, $\tilde{\pi}^{sc*}_{i,t}=\max\{(\delta_i w_{i,t-1}-w_{i,t})y_{i,t}-L_{i,t}(y_{i,t}):y_{i,t}\geq 0\}$. Thus,

$$\begin{aligned}
\pi^{sc*}_{i,t} &= \max\{(\delta_i w_{i,t-1}-w_{i,t})y_{i,t}-L_{i,t}(y_{i,t})+\delta_i\beta^{sc}_{i,t-1}(\kappa_{ii,t}(\mathbb{E}[y_{i,t}\wedge\xi_{i,t}])-\sum_{j\neq i}\kappa_{ij,t}(\mathbb{E}[y^*_{j,t}\wedge\xi_{j,t}])):y_{i,t}\geq 0\}\\
&\geq \max\{(\delta_i w_{i,t-1}-w_{i,t})y_{i,t}-L_{i,t}(y_{i,t})+\delta_i\beta^{sc}_{i,t-1}(\kappa_{ii,t}(0)-\sum_{j\neq i}\kappa_{ij,t}(1)):y_{i,t}\geq 0\}\\
&\geq \max\{(\delta_i w_{i,t-1}-w_{i,t})y_{i,t}-L_{i,t}(y_{i,t}):y_{i,t}\geq 0\}\\
&= \tilde{\pi}^{sc*}_{i,t},
\end{aligned}$$

where the first inequality follows from that $\kappa_{ii,t}(\cdot)$ is increasing in $y_{i,t}$ and $\kappa_{ij,t}(\cdot)$ is increasing in $y_{j,t}$, and the second from that $\alpha_{i,t}(\cdot) \geq 0$ for all i, t, and z_t. This proves part (a).

Part (b-i). Part (a) suggests that $\pi^{sc*}_{s,t} \geq \tilde{\pi}^{sc*}_{s,t}$ for all t. Thus, by Theorem 3(c), $\gamma^{sc*}_{s,t} \geq \tilde{\gamma}^{sc*}_{s,t}$ for all t. By Theorem 2(b), $\gamma^{sc*}_{i,t}(I_t, \Lambda_t) = \gamma^{sc*}_{s,t} \geq \tilde{\gamma}^{sc*}_{s,t} = \tilde{\gamma}^{sc*}_{i,t}(I_t, \Lambda_t)$ for all t and $(I_t, \Lambda_t) \in \mathcal{S}$. This proves part (b-i).

Part (b-ii). Part (a) suggests that $\pi^{sc*}_{s,t} \geq \tilde{\pi}^{sc*}_{s,t}$ for all t. Thus, by Theorem 3(c), $p^{sc*}_{s,t} \leq \tilde{p}^{sc*}_{s,t}$ for all t. By Theorem 2(b), $p^{sc*}_{i,t}(I_t, \Lambda_t) = p^{sc*}_{s,t} \leq \tilde{p}^{sc*}_{s,t} = \tilde{p}^{sc*}_{i,t}(I_t, \Lambda_t)$ for all t and $(I_t, \Lambda_t) \in \mathcal{S}$. This proves part (b-ii).

Part (b-iii). By Proposition 3(d), $x^{sc*}_{i,t}(I_t, \Lambda_t) = y^{sc*}_{s,t}\rho_{s,t}(p^{sc*}_{ss,t})\psi_{s,t}(\gamma^{sc*}_{ss,t})\Lambda_{i,t}$ and $\tilde{x}^{sc*}_{i,t}(I_t, \Lambda_t) = \tilde{y}^{sc*}_{s,t}\rho_{s,t}(\tilde{p}^{sc*}_{ss,t})\psi_{s,t}(\tilde{\gamma}^{sc*}_{ss,t})\Lambda_{i,t}$. Part (a) implies that $y^{sc*}_{s,t} \geq \tilde{y}^{sc*}_{s,t}$. Since, by parts (b-i) and (b-ii), $p^{sc*}_{s,t} \leq \tilde{p}^{sc*}_{s,t}$ and $\gamma^{sc*}_{s,t} \geq \tilde{\gamma}^{sc*}_{s,t}$, the monotonicity condition (17) yields that $\rho_{s,t}(p^{sc*}_{ss,t}) \geq \rho_{s,t}(\tilde{p}^{sc*}_{ss,t})$, and $\psi_{s,t}(\gamma^{sc*}_{ss,t}) \geq \psi_{s,t}(\tilde{\gamma}^{sc*}_{ss,t})$. Therefore, for each $(I_t, \Lambda_t) \in \mathcal{S}$,

$$x^{sc*}_{i,t}(I_t, \Lambda_t) = y^{sc*}_{s,t}\rho_{s,t}(p^{sc*}_{ss,t})\psi_{s,t}(\gamma^{sc*}_{ss,t})\Lambda_{i,t} \geq \tilde{y}^{sc*}_{s,t}\rho_{s,t}(\tilde{p}^{sc*}_{ss,t})\psi_{s,t}(\tilde{\gamma}^{sc*}_{ss,t})\Lambda_{i,t} = \tilde{x}^{sc*}_{i,t}(I_t, \Lambda_t).$$

This completes the proof of part (b-iii). $\square$

Proof of Theorem 5: Part (a). We show part (a) by backward induction. More specifically, we show that if $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t and $\hat{\beta}^{sc}_{s,t-1} \geq \beta^{sc}_{s,t-1}$, (i) $\hat{\pi}^{sc*}_{s,t} \geq \pi^{sc*}_{s,t}$, (ii) $\hat{\gamma}^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t}$, (iii) $\hat{\gamma}^{sc*}_{i,t}(I_t, \Lambda_t) \geq \gamma^{sc*}_{s,t}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$, (iv) $\hat{p}^{sc*}_{s,t} \leq p^{sc*}_{s,t}$, (v) $\hat{p}^{sc*}_{i,t}(I_t, \Lambda_t) \leq p^{sc*}_{i,t}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$, and (vi) $\hat{\beta}^{sc}_{s,t} \geq \beta^{sc}_{s,t}$. Since $\hat{\beta}^{sc}_{s,0} = \beta^{sc}_{s,0} = 0$, the initial condition is satisfied.

Since $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t,

$$\hat{\kappa}_{sa,t}(y_{i,t}) - (N-1)\hat{\kappa}^0_{sb,t} \geq \kappa_{sa,t}(y_{i,t}) - (N-1)\kappa^0_{sb,t} \geq 0, \text{ for all } y_{i,t} \geq 0.$$

Therefore,

$$\begin{aligned}
\hat{\pi}^{sc*}_{s,t} &= \max\{(\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\hat{\beta}^{sc}_{s,t-1}(\hat{\kappa}_{sa,t}(\mathbb{E}[y_{i,t} \wedge \xi_{i,t}]) - (N-1)\hat{\kappa}^0_{sb,t}) : y_{i,t} \geq 0\} \\
&\geq \max\{(\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\beta^{sc}_{s,t-1}(\kappa_{sa,t}(\mathbb{E}[y_{i,t} \wedge \xi_{i,t}]) - (N-1)\kappa^0_{sb,t}) : y_{i,t} \geq 0\} \\
&= \pi^{sc*}_{s,t}.
\end{aligned}$$

Since $\hat{\pi}^{sc*}_{s,t} \geq \pi^{sc*}_{s,t}$, Theorem 3(c) implies that $\hat{\gamma}^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t}$ and $\hat{p}^{sc*}_{s,t} \leq p^{sc*}_{s,t}$. Thus, $\hat{\gamma}^{sc*}_{i,t}(I_t, \Lambda_t) = \hat{\gamma}^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t} = \gamma^{sc*}_{i,t}(I_t, \Lambda_t)$ for each i and all $(I_t, \Lambda_t) \in \mathcal{S}$. Analogously, $\hat{p}^{sc*}_{i,t}(I_t, \Lambda_t) = \hat{p}^{sc*}_{s,t} \leq p^{sc*}_{s,t} = p^{sc*}_{i,t}(I_t, \Lambda_t)$ for each i and all $(I_t, \Lambda_t) \in \mathcal{S}$. By Theorem 3(d), $\hat{\pi}^{sc*}_{s,t} \geq \pi^{sc*}_{s,t}$ implies that $\hat{\beta}^{sc}_{s,t} \geq \beta^{sc}_{s,t}$. This completes the induction and, thus, the proof of part (a).

Part (b). By part (a), it suffices to show that, if $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t, $\hat{\kappa}'_{sa,t}(z_{i,t}) \geq \kappa'_{sa,t}(z_{i,t})$ for all $z_{i,t}$, and $\hat{\beta}^{sc}_{s,t-1} \geq \beta^{sc}_{s,t-1}$, we have (i) $\hat{y}^{sc*}_{s,t} \geq y^{sc*}_{s,t}$ and (ii) $\hat{x}^{sc*}_{i,t}(I_t, \Lambda_t) \geq x^{sc*}_{i,t}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$.

First, we show that $\hat{y}^{sc*}_{s,t} \geq y^{sc*}_{s,t}$. If, to the contrary, $\hat{y}^{sc*}_{s,t} < y^{sc*}_{s,t}$, Lemma 1 yields that

$$\begin{aligned}
&\partial_{y_{i,t}}[(\delta_s w_{s,t-1} - w_{s,t})\hat{y}^{sc*}_{s,t} - L_{s,t}(\hat{y}^{sc*}_{s,t}) + \delta_s\hat{\beta}^{sc}_{s,t-1}(\hat{\kappa}_{sa,t}(\mathbb{E}[\hat{y}^{sc*}_{s,t} \wedge \xi_{i,t}]) - (N-1)\hat{\kappa}^0_{sb,t})] \\
\leq\ &\partial_{y_{i,t}}[(\delta_s w_{s,t-1} - w_{s,t})y^{sc*}_{s,t} - L_{s,t}(y^{sc*}_{s,t}) + \delta_s\beta^{sc}_{s,t-1}(\kappa_{sa,t}(\mathbb{E}[y^{sc*}_{s,t} \wedge \xi_{i,t}]) - (N-1)\kappa^0_{sb,t})],
\end{aligned}$$

i.e.,

$$\begin{aligned}
&(\delta_s w_{s,t-1} - w_{s,t}) - L'_{s,t}(\hat{y}^{sc*}_{s,t}) + \delta_s\hat{\beta}^{sc}_{s,t-1}\bar{F}_{s,t}(\hat{y}^{sc*}_{s,t})\hat{\kappa}'_{sa,t}(\mathbb{E}[\hat{y}^{sc*}_{s,t} \wedge \xi_{i,t}]) \\
\leq\ &(\delta_s w_{s,t-1} - w_{s,t}) - L'_{s,t}(y^{sc*}_{s,t}) + \delta_s\beta^{sc}_{s,t-1}\bar{F}_{s,t}(y^{sc*}_{s,t})\kappa'_{sa,t}(\mathbb{E}[y^{sc*}_{s,t} \wedge \xi_{i,t}]).
\end{aligned} \tag{44}$$

Since $-L_{s,t}(\cdot)$ is strictly concave in $y_{i,t}$ and $\hat{y}_{s,t}^{sc*} < y_{s,t}^{sc*}$, (44) implies that

$$\delta_s \hat{\beta}_{s,t-1}^{sc} \bar{F}_{s,t}(\hat{y}_{s,t}^{sc*}) \hat{\kappa}'_{sa,t}(\mathbb{E}[\hat{y}_{s,t}^{sc*} \wedge \xi_{i,t}]) < \delta_s \beta_{s,t-1}^{sc} \bar{F}_{s,t}(y_{s,t}^{sc*}) \kappa'_{sa,t}(\mathbb{E}[y_{s,t}^{sc*} \wedge \xi_{i,t}]). \tag{45}$$

However, since $\hat{\kappa}'_{sa,t}(z_{i,t}) \geq \kappa'_{sa,t}(z_{i,t})$ for all $z_{i,t}$ and $\hat{y}_{s,t}^{sc*} < y_{s,t}^{sc*}$, we have $\hat{\kappa}'_{sa,t}(\mathbb{E}[\hat{y}_{s,t}^{sc*} \wedge \xi_{i,t}]) \geq \kappa'_{sa,t}(\mathbb{E}[y_{s,t}^{sc*} \wedge \xi_{i,t}])$ and $\bar{F}_{s,t}(\hat{y}_{s,t}^{sc*}) \geq \bar{F}_{s,t}(y_{s,t}^{sc*})$. Because $\hat{\beta}_{s,t-1}^{sc} \geq \beta_{s,t-1}^{sc}$,

$$\delta_s \hat{\beta}_{s,t-1}^{sc} \bar{F}_{s,t}(\hat{y}_{s,t}^{sc*}) \hat{\kappa}'_{sa,t}(\mathbb{E}[\hat{y}_{s,t}^{sc*} \wedge \xi_{i,t}]) \geq \delta_s \beta_{s,t-1}^{sc} \bar{F}_{s,t}(y_{s,t}^{sc*}) \kappa'_{sa,t}(\mathbb{E}[y_{s,t}^{sc*} \wedge \xi_{i,t}]),$$

which contradicts (45). The inequality $\hat{y}_{s,t}^{sc*} \geq y_{s,t}^{sc*}$ then follows immediately.

Now we show that $\hat{x}_{i,t}^{sc*}(I_t, \Lambda_t) \geq x_{i,t}^{sc*}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$. By Proposition 3(d), $\hat{x}_{i,t}^{sc*}(I_t, \Lambda_t) = \hat{y}_{s,t}^{sc*} \rho_{s,t}(\hat{p}_{ss,t}^{sc*}) \psi_{s,t}(\hat{\gamma}_{ss,t}^{sc*}) \Lambda_{i,t}$ and $x_{i,t}^{sc*}(I_t, \Lambda_t) = y_{s,t}^{sc*} \rho_{s,t}(p_{ss,t}^{sc*}) \psi_{s,t}(\gamma_{ss,t}^{sc*}) \Lambda_{i,t}$. We have shown that $\hat{y}_{s,t}^{sc*} \geq y_{s,t}^{sc*}$. Since (17) holds for period t, $\rho_{s,t}(\hat{p}_{ss,t}^{sc*}) \geq \rho_{s,t}(p_{ss,t}^{sc*})$, and $\psi_{s,t}(\hat{\gamma}_{ss,t}^{sc*}) \geq \psi_{s,t}(\gamma_{ss,t}^{sc*})$. Therefore, for each i and $(I_t, \Lambda_t) \in \mathcal{S}$,

$$\hat{x}_{i,t}^{sc*}(I_t, \Lambda_t) = \hat{y}_{s,t}^{sc*} \rho_{s,t}(\hat{p}_{ss,t}^{sc*}) \psi_{s,t}(\hat{\gamma}_{ss,t}^{sc*}) \Lambda_{i,t} \geq y_{s,t}^{sc*} \rho_{s,t}(p_{ss,t}^{sc*}) \psi_{s,t}(\gamma_{ss,t}^{sc*}) \Lambda_{i,t} = x_{i,t}^{sc*}(I_t, \Lambda_t).$$

This completes the proof of part (b). $\square$

Proof of Theorem 6: We show **parts (a)-(b)** together by backward induction. More specifically, we show that if $\beta_{s,t-1}^{sc} \geq \beta_{s,t-2}^{sc}$, (i) $y_{s,t}^{sc*} \geq y_{s,t-1}^{sc*}$, (ii) $\gamma_{s,t}^{sc*} \geq \gamma_{s,t-1}^{sc*}$, (iii) $\gamma_{i,t}^{sc*}(I, \Lambda) \geq \gamma_{i,t-1}^{sc*}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$, (iv) $p_{s,t}^{sc*} \leq p_{s,t-1}^{sc*}$, (v) $p_{i,t}^{sc*}(I, \Lambda) \leq p_{i,t-1}^{sc*}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$, (vi) $x_{i,t}^{sc*}(I, \Lambda) \geq x_{i,t-1}^{sc*}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$, and (vii) $\beta_{s,t}^{sc} \geq \beta_{s,t-1}^{sc}$. Since, by Proposition 3(a), $\beta_{s,1}^{sc} \geq \beta_{s,0}^{sc} = 0$. Thus, the initial condition is satisfied.

Since the model is stationary, by Theorem 3(a), $\beta_{s,t-1}^{sc} \geq \beta_{s,t-2}^{sc}$ suggests that $y_{s,t}^{sc*} \geq y_{s,t-1}^{sc*}$. Analogously, Theorem 3(e) yields that $\gamma_{s,t}^{sc*} \geq \gamma_{s,t-1}^{sc*}$ and $p_{s,t}^{sc*} \leq p_{s,t-1}^{sc*}$. Hence, $\gamma_{i,t}^{sc*}(I, \Lambda) = \gamma_{s,t}^{sc*} \geq \gamma_{s,t-1}^{sc*} = \gamma_{i,t-1}^{sc*}(I, \Lambda)$ and $p_{i,t}^{sc*}(I, \Lambda) = p_{s,t}^{sc*} \leq p_{s,t-1}^{sc*} = p_{i,t-1}^{sc*}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$. Because the monotonicity condition (17) holds, we have $\rho_{s,t}(p_{ss,t}^{sc*}) \geq \rho_{s,t-1}(p_{ss,t-1}^{sc*})$, and $\psi_{s,t}(\gamma_{ss,t}^{sc*}) \geq \psi_{s,t-1}(\gamma_{ss,t-1}^{sc*})$. Therefore, for each i and $(I, \Lambda) \in \mathcal{S}$,

$$x_{i,t}^{sc*}(I, \Lambda) = y_{s,t}^{sc*} \rho_{s,t}(p_{ss,t}^{sc*}) \psi_{s,t}(\gamma_{ss,t}^{sc*}) \Lambda_i \geq y_{s,t-1}^{sc*} \rho_{s,t-1}(p_{ss,t-1}^{sc*}) \psi_{s,t-1}(\gamma_{ss,t-1}^{sc*}) \Lambda_i = x_{i,t-1}^{sc*}(I, \Lambda).$$

Finally, $\beta_{s,t}^{sc} \geq \beta_{s,t-1}^{sc}$ follows immediately from Theorem 3(f) and $\beta_{s,t-1}^{sc} \geq \beta_{s,t-2}^{sc}$. This completes the induction and, thus, the proof of Theorem 6. $\square$

Before presenting the proofs of the results in the PF model, we give the following lemma that is used throughout the rest of our proofs.

LEMMA **2** *Let A_t be an $N \times N$ matrix with entries defined by $A_{ii,t} = 2\theta_{ii,t}$ and $A_{ij,t} = -\theta_{ij,t}$ where $i \neq j$. The following statements hold:*

(a) A_t is invertible. Moreover, $(A_t^{-1})_{ij} \geq 0$ for all $1 \leq i, j \leq N$.

(b) $\frac{1}{2} \leq \theta_{ii,t}(A_t^{-1})_{ii} < 1$.

(c) $\frac{1}{2} \leq \sum_{j=1}^{N} \theta_{jj,t}(A_t^{-1})_{ij} < 1$.

Proof: Part (a) follows from Lemma 2(a) in Bernstein and Federgruen (2004c) and **Part (b)** follows from Lemma 2(c) in Bernstein and Federgruen (2004c).

Part (c). Let $\mathcal{I}$ be the $N \times N$ identity matrix, B_t be the $N \times N$ matrix with

$$(B_t)_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \frac{\theta_{ij,t}}{\theta_{ii,t}} & \text{if } i \neq j; \end{cases}$$

and C_t be the $N \times N$ diagonal matrix with

$$(C_t)_{ij} = \begin{cases} 2\theta_{ii,t} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Because $\theta_{ii,t} > \sum_{j \neq i} \theta_{ij,t}$, B_t is a substochastic matrix.

Observe that, $A_t = C_t(\mathcal{I} - \frac{1}{2}B_t)$ and, hence, $A_t^{-1} = (\mathcal{I} - \frac{1}{2}B_t)^{-1}C_t^{-1}$. Let $\theta_t = (\theta_{11,t}, \theta_{22,t} \cdots, \theta_{NN,t})'$ be the $N-$dimensinal vector. Thus, $\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij} = (A_t^{-1}\theta_t)_i$. Moreover,

$$A_t^{-1}\theta_t = (\mathcal{I} - \frac{1}{2}B_t)^{-1}C_t^{-1}\theta_t = (\mathcal{I} - \frac{1}{2}B_t)^{-1}(C_t^{-1}\theta_t) = \frac{1}{2}(\mathcal{I} - \frac{1}{2}B_t)^{-1},$$

where the last equality follows from $C_t^{-1}\theta_t = \frac{1}{2}\mathcal{I}$. Therefore,

$$\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij} = \frac{1}{2}\sum_{j=1}^N [(\mathcal{I} - \frac{1}{2}B_t)^{-1}]_{ij} = \frac{1}{2}\sum_{j=1}^N [\mathcal{I} + \sum_{l=1}^{+\infty} \left(\frac{1}{2}\right)^l (B_t)^l]_{ij},$$

where the second equality follows from the fact that $\mathcal{I} - \frac{1}{2}B_t$ is a diagonal dominant matrix. Thus, for all i, $\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij} \geq \frac{1}{2}\sum_{j=1}^N \mathcal{I}_{ij} = \frac{1}{2}$. On the other hand, for all i,

$$\frac{1}{2}\sum_{j=1}^N [\mathcal{I} + \sum_{l=1}^{+\infty} \left(\frac{1}{2}\right)^l (B_t)^l]_{ij} = \frac{1}{2}\sum_{j=1}^N [\sum_{l=0}^{+\infty} \left(\frac{1}{2}\right)^l (B_t)^l]_{ij} = \frac{1}{2}\sum_{l=0}^{+\infty} [\left(\frac{1}{2}\right)^l \sum_{j=1}^N (B_t)^l_{ij}] < \frac{1}{2}\sum_{l=0}^{+\infty} \left(\frac{1}{2}\right)^l = 1,$$

where the inequality follows from that B_t is a sub-stochastic matrix. This completes the proof of part (c). $\square$

Proof of Theorems 7-8 and Propositions 4-6: We show Theorem 7, Proposition 4, Proposition 5, Proposition 6, and Theorem 8 together by backward induction. More specifically, we show that, if $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{pf*}) = w_{i,t-1}I_{i,t-1} + \beta_{i,t-1}^{pf}\Lambda_{i,t-1}$ for all i, (a) Proposition 4 holds for period t, (b) Proposition 5 holds for period t, (c) Proposition 6 holds for period t, (d) there exists a Markov strategy profile $\{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \leq i \leq N\}$, which forms an equilibrium in the subgame of period t, (e) if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for all i and $\gamma_{i,t}$, the equilibrium in the subgame of period t, $\{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \leq i \leq N\}$, is unique, and (f) there exists a positive vector $\beta_t^{pf} = (\beta_{1,t}^{pf}, \beta_{2,t}^{pf}, \cdots, \beta_{N,t}^{pf})$, such that $V_{i,t}(I_t, \Lambda_t|\sigma_t^{pf*}) = w_{i,t}I_{i,t} + \beta_{i,t}^{pf}\Lambda_{i,t}$ for all i. Because $V_{i,0}(I_0, \Lambda_0) = w_{i,0}I_{i,0}$ for all i, the initial condition is satisfied.

First, we observe that Proposition 4 follows directly from the same argument as the proof of Proposition 1. We now show Proposition 5 holds in period t. Because $\partial_{p_{i,t}}^2 \Pi_{i,t}^{pf,2}(p_t|\gamma_t) = -2\theta_{ii,t} < 0$, $\Pi_{i,t}^{pf,2}(\cdot, p_{-i,t}|\gamma_t)$ is strictly concave in $p_{i,t}$ for any given $p_{-i,t}$. Hence, by Theorem 1.2 in Fudenberg and Tirole (1991), $\mathcal{G}_t^{pf,2}$ has a pure strategy Nash equilibrium $p_t^{pf*}(\gamma_t)$. Since, for each i and t, $\underline{p}_{i,t}$ is sufficiently low whereas $\bar{p}_{i,t}$ is sufficiently high so that they will not affect the equilibrium behaviors of all firms, $p_t^{pf*}(\gamma_t)$ can be characterized by first-order conditions $\partial_{p_{i,t}}\Pi_{i,t}^{pf,2}(p_t^{pf*}(\gamma_t)|\gamma_t) = 0$ for each i, i.e.,

$$\begin{aligned} &-\theta_{ii,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}(\gamma_t)) + \rho_{i,t}(p_t^{pf*}(\gamma_t)) \\ =& -2\theta_{ii,t}p_{i,t}^{pf*}(\gamma_t) + \sum_{j \neq i} \theta_{ij,t}p_{j,t}^{pf*}(\gamma_t) + f_{i,t}(\gamma_t) = 0, \text{ for all } i. \end{aligned} \tag{46}$$

In terms of the matrix language, we have $A_t p_t^{pf*}(\gamma_t) = f_t(\gamma_t)$. By Lemma 2(a), A_t is invertible and, thus, $p_t^{pf*}(\gamma_t)$ is uniquely determined by $p_t^{pf*}(\gamma_t) = A_t^{-1} f_t(\gamma_t)$. To show that $p_{i,t}^{pf*}(\gamma_t) = \sum_j (A_t^{-1})_{ij} f_{j,t}(\gamma_t)$ is continuously increasing in $\gamma_{j,t}$, we observe that

$$\frac{\partial p_{i,t}^{pf*}(\gamma_t)}{\partial \gamma_{j,t}} = (A_t^{-1})_{ij}\theta_{jj,t}\nu'_{j,t}(\gamma_{j,t}).$$

Since, by Lemma 2(a), $(A_t^{-1})_{ij} \geq 0$ for all i and j, we have $\partial_{\gamma_{j,t}} p_{i,t}^{pf*}(\gamma_t) \geq 0$ and, thus, $p_{i,t}^{pf*}(\gamma_t)$ is continuously increasing in $\gamma_{j,t}$ for each j.

Now, we compute $\Pi_{i,t}^{pf*,2}(\gamma_t)$.

$$\begin{aligned}
\Pi_{i,t}^{pf*,2}(\gamma_t) &= \rho_{i,t}(p_t^{pf*}(\gamma_t))(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}) \\
&= (\phi_{i,t} - \theta_{ii,t} p_{i,t}^{pf*}(\gamma_t) + \sum_{j\neq i}\theta_{ij,t} p_{j,t}^{pf*}(\gamma_t))(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}) \\
&= (\theta_{ii,t} p_{i,t}^{pf*}(\gamma_t) - f_{i,t}(\gamma_t) + \phi_{i,t})(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}) \\
&= \theta_{ii,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})^2,
\end{aligned}$$

where the third equality follows from (46) and the last from $f_{i,t}(\gamma_t) = \phi_{i,t} + \theta_{ii,t}(\delta_i w_{i,t-1} + \nu_{i,t}(\gamma_{i,t}) - \pi_{i,t}^{pf*})$. The above computation also implies that $\rho_{i,t}(p_t^{pf*}(\gamma_t)) = \theta_{ii,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})$. We now show that $\Pi_{i,t}^{pf*,2}(\gamma_t) > 0$. Note that $\Pi_{i,t}^{pf*,2}(\gamma_t) = \frac{1}{\theta_{ii,t}}[\rho_{i,t}(p_t^{pf*}(\gamma_t))]^2 > 0$, where the inequality follows from the assumption that $\rho_{i,t}(\cdot) > 0$ for all p_t. This completes the proof of Proposition 5.

Next, we show Proposition 6. Since $\Pi_{i,t}^{pf*,2}(\gamma_t) > 0$ for all γ_t, $\Pi_{i,t}^{pf,1}(\gamma_t) = \Pi_{i,t}^{pf*,2}(\gamma_t)\psi_{i,t}(\gamma_t) > 0$ and, hence, $\log(\Pi_{i,t}^{pf,1}(\cdot))$ is well defined. Therefore,

$$\log(\Pi_{i,t}^{pf,1}(\gamma_t)) = \log(\theta_{ii,t}) + 2\log(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}) + \log(\psi_{i,t}(\gamma_t)). \tag{47}$$

Since

$$p_{j,t}^{pf*}(\gamma_t) = \sum_{l=1}^{N}(A_t^{-1})_{jl} f_{l,t}(\gamma_t) = \sum_{l=1}^{N}[(A_t^{-1})_{jl}(\phi_{l,t} + \theta_{ll,t}(\delta_l w_{l,t-1} + \nu_{l,t}(\gamma_{l,t}) - \pi_{l,t}^{pf*}))], \text{ for all } j,$$

by direct computation,

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} = \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})\theta_{jj,t}(A_t^{-1})_{ij}\nu'_{i,t}(\gamma_{i,t})\nu'_{j,t}(\gamma_{j,t})}{(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})^2} + \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}}, \text{ for all } j \neq i. \tag{48}$$

By Lemma 2(a,b), $1 - \theta_{ii,t}(A_t^{-1})_{ii} > 0$ and $(A_t^{-1})_{ij} \geq 0$. Thus, the first term of (48) is non-negative. Because $\psi_{i,t}(\cdot)$ satisfies (3),

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} \geq \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} \geq 0, \text{ for all } j \neq i.$$

and, thus, $\mathcal{G}_t^{pf,1}$ is a log-supermodular game. The feasible action set of player i, $[0, \bar{\gamma}_{i,t}]$, is a compact subset of $\mathbb{R}$. Therefore, by Theorem 2 in Zhou (1994), the pure strategy Nash equilibria of $\mathcal{G}_t^{pf,1}$ is a nonempty complete sublattice of $\mathbb{R}^N$

We now show that if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, the Nash equilibrium of $\mathcal{G}_t^{pf,1}$ is unique. We first show that

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}^2} < 0, \text{ and } \left|\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}^2}\right| > \sum_{j\neq i}\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}}, \text{ for all } i \text{ and } \gamma_t. \tag{49}$$

Since $\nu_{l,t}(\gamma_{l,t}) = \gamma_{l,t}$ for all l (i.e., $\nu'_{l,t}(\cdot) \equiv 1$ for all l), direct computation yields that

$$\frac{\partial^2 \log(\Pi^{pf,1}_{i,t}(\gamma_t))}{\partial \gamma^2_{i,t}} = \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma^2_{i,t}} - \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})^2}{(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2}.$$

Inequality (3) implies that $\partial^2_{\gamma_{i,t}} \log(\psi_{i,t}(\gamma_t)) < 0$ and, thus, $\partial^2_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma_t)) < 0$. Moreover,

$$|\frac{\partial^2 \log(\Pi^{pf,1}_{i,t}(\gamma_t))}{\partial \gamma^2_{i,t}}| = |\frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma^2_{i,t}}| + \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})^2}{(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2}$$

and

$$\sum_{j\neq i} \frac{\partial^2 \log(\Pi^{pf,1}_{i,t}(\gamma_t))}{\partial \gamma_{i,t}\partial \gamma_{j,t}} = \sum_{j\neq i} \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma_{i,t}\partial \gamma_{j,t}} + \sum_{j\neq i} \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})\theta_{jj,t}(A_t^{-1})_{ij}}{(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2}.$$

Inequality (3) implies that

$$|\frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma^2_{i,t}}| > \sum_{j\neq i} \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma_{i,t}\partial \gamma_{j,t}}.$$

Lemma 2(b) implies that $1-\theta_{ii,t}(A_t^{-1})_{ii} > 0$. Moreover, Lemma 2(c) suggests that $1-(A_t^{-1})_{ii}\theta_{ii,t} > \sum_{j\neq i}(A_t^{-1})_{ij}\theta_{jj,t}$ and, hence,

$$\frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})^2}{(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2} > \sum_{j\neq i} \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})\theta_{jj,t}(A_t^{-1})_{ij}}{(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2}.$$

Therefore, inequality (49) holds for all γ_t.

Because $\mathcal{G}^{pf,1}_t$ is a log-supermodular game, by Theorem 5 in Milgrom and Roberts (1990), if there are two distinct pure strategy Nash equilibria $\hat{\gamma}^{pf*}_t \neq \gamma^{pf*}_t$, we must have $\hat{\gamma}^{pf*}_{i,t} \geq \gamma^{pf*}_{i,t}$ for each i, with the inequality being strict for some i. Without loss of generality, we assume that $\hat{\gamma}^{pf*}_{1,t} > \gamma^{pf*}_{1,t}$ and $\hat{\gamma}^{pf*}_{1,t} - \gamma^{pf*}_{1,t} \geq \hat{\gamma}^{pf*}_{i,t} - \gamma^{pf*}_{i,t}$ for each i. Lemma 1 yields that

$$\frac{\partial \log(\Pi^{pf,1}_{1,t}(\hat{\gamma}^{pf*}_t))}{\partial \gamma_{1,t}} \geq \frac{\partial \log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_t))}{\partial \gamma_{1,t}} \tag{50}$$

Since $\partial_{\gamma_{1,t}}\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{1,t}(\gamma_t))$ is Lebesgue integrable for all $i \neq 1$ and γ_t, Newton-Leibniz formula implies that

$$\begin{aligned}
\frac{\partial \log(\Pi^{pf,1}_{1,t}(\hat{\gamma}^{pf*}_t))}{\partial \gamma_{1,t}} - \frac{\partial \log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_t))}{\partial \gamma_{1,t}} &= \int_{s=0}^1 \sum_{j=1}^N (\hat{\gamma}^{pf*}_{j,t} - \gamma^{pf*}_{j,t}) \frac{\partial^2 \log(\Pi^{pf,1}_{1,t}((1-s)\gamma^{pf*}_t + s\hat{\gamma}^{pf*}_t))}{\partial \gamma_{1,t}\partial \gamma_{j,t}}\, \mathrm{d}s \\
&\leq \int_{s=0}^1 \sum_{j=1}^N (\hat{\gamma}^{pf*}_{1,t} - \gamma^{pf*}_{1,t}) \frac{\partial^2 \log(\Pi^{pf,1}_{1,t}((1-s)\gamma^{pf*}_t + s\hat{\gamma}^{pf*}_t))}{\partial \gamma_{1,t}\partial \gamma_{j,t}}\, \mathrm{d}s \\
&< 0,
\end{aligned}$$

where the first inequality follows from $\hat{\gamma}^{pf*}_{1,t} - \gamma^{pf*}_{1,t} \geq \hat{\gamma}^{pf*}_{i,t} - \gamma^{pf*}_{i,t}$ for all i, and the second from (49), and $\hat{\gamma}^{pf*}_{1,t} - \gamma^{pf*}_{1,t} > 0$. This contradicts (50). Thus, $\mathcal{G}^{pf,1}_t$ has a unique pure strategy Nash equilibrium γ^{pf*}_t.

We now show that the unique pure strategy Nash equilibrium γ^{pf*}_t can be characterized by the system of first-order conditions (26). First, (49) implies that $\log(\Pi^{pf,1}_{i,t}(\cdot, \gamma_{-i,t}))$ is strictly concave in $\gamma_{i,t}$ for any i and any fixed $\gamma_{-i,t}$. Hence, γ^{pf*}_t must satisfy the system of first-order conditions, i.e., for each i, $\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma^{pf*}_t)) \leq 0$ if $\gamma^{pf*}_{i,t} = 0$; $\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma^{pf*}_t)) = 0$ if $\gamma^{pf*}_{i,t} \in (0, \bar{\gamma}_{i,t})$; and $\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma^{pf*}_t)) \geq 0$ if $\gamma^{pf*}_{i,t} = \bar{\gamma}_{i,t}$. Differentiate (47), and we have

$$\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma_t)) = \frac{\partial_{\gamma_{i,t}} \psi_{i,t}(\gamma_t)}{\psi_{i,t}(\gamma_t)} - \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})\nu'_{i,t}(\gamma_{i,t})}{p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t}}.$$

Thus, γ_t^{pf*} satisfies the system of first-order conditions (26). Since, by Proposition 5(c), $\Pi_{i,t}^{pf*,2}(\gamma_t^{pf*}) > 0$ and $\psi_{i,t}(\gamma_t^{pf*}) > 0$, we have $\Pi_{i,t}^{pf*,1} = \Pi_{i,t}^{pf*,2}(\gamma_t^{pf*})\psi_{i,t}(\gamma_t^{pf*}) > 0$ for all i. This completes the proof of Proposition 6.

Next, we show that $\{(\gamma_{i,t}^{pf*}, p_{i,t}^{pf*}(\gamma_t), \Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t))\psi_{i,t}(\gamma_t)) : 1 \le i \le N\}$ is an equilibrium in the subgame of period t. By Proposition 4, $y_{i,t}^{pf*} > 0$, $\Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t))\psi_{i,t}(\gamma_t) > 0$ for all i. Therefore, regardless of the starting inventory level in period t, $I_{i,t}$, firm i could adjust its inventory to $x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t) = \Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t))\psi_{i,t}(\gamma_t)$. Thus, $\{(\gamma_{i,t}^{pf*}, p_{i,t}^{pf*}(\gamma_t), \Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t))\psi_{i,t}(\gamma_t)) : 1 \le i \le N\}$ forms an equilibrium in the subgame of period t. In particular, this equilibrium is the unique one, if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for all i.

Finally, we show that there exists a positive vector $\beta_t^{pf} = (\beta_{1,t}^{pf}, \beta_{2,t}^{pf}, \cdots, \beta_{N,t}^{pf})$, such that $V_{i,t}(I_t, \Lambda_t|\sigma_t^{pf*}) = w_{i,t}I_{i,t} + \beta_{i,t}^{pf}\Lambda_{i,t}$. By (22), we have that

$$\begin{aligned} V_{i,t}(I_t, \Lambda_t|\sigma_t^{pf*}) =& J_{i,t}(\gamma_{i,t}^{pf*}, p_{i,t}^{pf*}(\gamma_t^{pf*}), \Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t^{pf*}))\psi_{i,t}(\gamma_t^{pf*}), I_t, \Lambda_t|\sigma_{t-1}^{pf*}) \\ =& w_{i,t}I_{i,t} + (\sigma_i\beta_{i,t-1}^{pf}\mu_{i,t} + \Pi_{i,t}^{pf*,1})\Lambda_{i,t}. \end{aligned}$$

Since $\beta_{i,t-1}^{pf} > 0$, $\beta_{i,t}^{pf} = \delta_i\beta_{i,t-1}^{pf}\mu_{i,t} + \Pi_{i,t}^{pf*,1} > 0$. This completes the induction and, thus, the proof of Theorem 7, Proposition 4, Proposition 5, Proposition 6, and Theorem 8. □

Proof of Proposition 7: By Theorems 7-8, and Propositions 4-6, it suffices to show that, if there exists a constant $\beta_{s,t-1}^{pf} \ge 0$, such that $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{pf*}) = w_{s,t}I_{i,t-1} + \beta_{s,t-1}^{pf}\Lambda_{i,t-1}$ for all i, we have: (a) the unique Nash equilibrium in $\mathcal{G}_t^{pf,3}$ is symmetric, i.e., $y_{i,t}^{pf*} = y_{j,t}^{pf*}$ for all i, j; (b) the unique Nash equilibrium in $\mathcal{G}_t^{pf,2}(\gamma_t)$ is symmetric if $\gamma_{i,t} = \gamma_{j,t}$ for all i and j, (c), the unique Nash equilibrium in $\mathcal{G}_t^{pf,1}$, γ_t^{pf*} is symmetric, and (d) there exists a constant $\beta_{s,t}^{pf} > 0$, such that $V_{i,t}(I_t, \Lambda_t|\sigma_{s,t}^{pf*}) = w_{s,t}I_{i,t} + \beta_{s,t}^{pf}\Lambda_{i,t}$ for all i. Since $V_{i,0}(I_t, \Lambda_t) = w_{i,0}I_{i,0}$ for all i, the initial condition is satisfied with $\beta_{s,0}^{pf} = 0$.

First, we observe that $y_{i,t}^{pf*} = y_{j,t}^{pf*}$ and $\pi_{i,t}^{pf*} = \pi_{j,t}^{pf*}$ for all i and j follow directly from the same proof of Proposition 3. Thus, we omit their proofs for brevity, and denote $y_{s,t}^{pf*} := y_{i,t}^{pf*}$ and $\pi_{s,t}^{pf*} = \pi_{i,t}^{pf*}$ for each firm i in $\mathcal{G}_t^{pf,3}$.

Next, we show that if $\gamma_{i,t} = \gamma_{j,t}$ for all i and j, $p_{i,t}^{pf*}(\gamma_t) = p_{j,t}^{pf*}(\gamma_t)$. Direct computation yields that, for the symmetric PF model, $\sum_{j=1}^N (A_t^{-1})_{ij}$ is independent of i. Thus, if the value of $\gamma_{j,t}$ is independent of j,

$$\begin{aligned} p_{i,t}^{pf*}(\gamma_t) =& \sum_{j=1}^N (A_t^{-1})_{ij}f_{j,t}(\gamma_t) = \sum_{j=1}^N [(A_t^{-1})_{ij}(\phi_{s,t} + \theta_{sa,t}(\delta_s w_{s,t-1} + \nu_{s,t}(\gamma_{j,t}) - \pi_{s,t}^{pf*}))] \\ =& (\phi_{s,t} + \theta_{sa,t}(\delta_s w_{s,t-1} + \nu_{s,t}(\gamma_{j,t}) - \pi_{s,t}^{pf*})) \sum_{j=1}^N (A_t^{-1})_{ij}, \end{aligned} \tag{51}$$

which is independent of firm i, which we denote as $p_{s,t}^{pf*}(\gamma_t)$.

Note that the objective functions of $\mathcal{G}_t^{pf,1}$,

$$\{\Pi_{i,t}^{pf,1}(\gamma_t) = \theta_{sa,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{i,t}) + \pi_{s,t}^{pf*})\psi_{s,t}(\gamma_t) : 1 \le i \le N\}$$

are symmetric. Thus, if there exists an asymmetric Nash equilibrium γ_t^{pf*}, there exists another Nash equilibrium $\underline{\gamma}_t^{pf*} \neq \gamma_t^{pf*}$, where $\underline{\gamma}_t^{pf*}$ is a permutation of γ_t^{pf*}. This contradicts the uniqueness of the Nash equilibrium in $\mathcal{G}_t^{pf,1}$. Thus, the unique Nash equilibrium in $\mathcal{G}_t^{pf,1}$ is symmetric, which we denote as $\gamma_{ss,t}^{pf*} = (\gamma_{s,t}^{pf*}, \gamma_{s,t}^{pf*}, \cdots, \gamma_{s,t}^{pf*})$. Hence,

$$\Pi_{i,t}^{pf*,1} = \Pi_{i,t}^{pf,1}(\gamma_{ss,t}^{pf*}) = \Pi_{j,t}^{pf,1}(\gamma_{ss,t}^{pf*}) = \Pi_{j,t}^{pf*,1} > 0.$$

Thus, we denote the payoff of each firm i in $\mathcal{G}_t^{pf,1}$ as $\Pi_{s,t}^{pf*,1}$. By Theorem 8(a),

$$\beta_{i,t}^{pf} = \delta_s \beta_{s,t-1}^{pf} \mu_{s,t} + \Pi_{i,t}^{pf*,1} = \delta_s \beta_{s,t-1}^{pf} \mu_{s,t} + \Pi_{j,t}^{pf*,1} = \beta_{j,t}^{pf} > 0.$$

Thus, we denote the PF market size coefficient of each firm i as $\beta_{s,t}^{pf}$. This completes the induction and, thus, the proof of Proposition 7. $\square$

Proof of Theorem 9: Parts (a)-(b). The proof of parts (a)-(b) follows from the same argument as that of Theorem 3(a)-(b) and is, hence, omitted.

Part (c). Because

$$p_{i,t}^{pf*}(\gamma_t) = \sum_{j=1}^{N} (A_t^{-1})_{ij} f_{j,t}(\gamma_t) = \sum_{j=1}^{N} [(A_t^{-1})_{ij} (\phi_{j,t} + \theta_{jj,t}(\delta_j w_{j,t-1} + \nu_{j,t}(\gamma_{j,t}) - \pi_{j,t}^{pf*}))],$$

we have

$$\partial_{\pi_{j,t}^{pf*}} p_{i,t}^{pf*}(\gamma_t) = -\theta_{jj,t}(A_t^{-1})_{ij} \leq 0,$$

where the inequality follows from Lemma 2(a). Thus, $p_{i,t}^{pf*}(\gamma_t)$ is continuously decreasing in $\pi_{j,t}^{pf*}$ for each j. Part (c) follows.

Part (d). We denote the objective function of each firm i in $\mathcal{G}_{s,t}^{pf,1}$ as $\Pi_{i,t}^{pf,1}(\cdot|\pi_{s,t}^{pf*})$ to capture its dependence on $\pi_{s,t}^{pf*}$. The unique symmetric pure strategy Nash equilibrium in $\mathcal{G}_{s,t}^{pf,1}$ is denoted as $\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*})$ to capture the dependence of the equilibrium on $\pi_{s,t}^{pf*}$, where

$$\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*}) = (\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}), \gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}), \cdots, \gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})).$$

We first show that, if $\bar{\pi}_{s,t}^{pf*} > \pi_{s,t}^{pf*}$, $\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}) \geq \gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})$.

If, to the contrary, $\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}) < \gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})$, Lemma 1 yields that $\partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{pf,1}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})|\bar{\pi}_{s,t}^{pf*})) \leq \partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{pf,1}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})|\pi_{s,t}^{pf*}))$, i.e.,

$$\begin{aligned}
& \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) - \frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}))}{p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) + \bar{\pi}_{s,t}^{pf*}} \\
\leq \quad & \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) - \frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}))}{p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) + \pi_{s,t}^{pf*}}.
\end{aligned}$$

Note that

$$\begin{aligned}
& [p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) + \bar{\pi}_{s,t}^{pf*}] - [p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) + \pi_{s,t}^{pf*}] \\
=& (1 - \sum_{j=1}^{N} (A_t^{-1})_{1j}\theta_{sa,t})(\nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) - \nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}))) + (1 - \sum_{j=1}^{N} (A_t^{-1})_{1j}\theta_{sa,t})(\bar{\pi}_{s,t}^{pf*} - \pi_{s,t}^{pf*}) \\
>& 0
\end{aligned} \tag{52}$$

where the inequality follows from Lemma 2(c). Thus,

$$p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) + \bar{\pi}_{s,t}^{pf*} > p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) + \pi_{s,t}^{pf*} > 0.$$

Lemma 2(b) implies that $1 - \theta_{sa,t}(A_t^{-1})_{ii} > 0$. Hence,

$$-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}))}{p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) + \bar{\pi}_{s,t}^{pf*}} \geq -\frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}))}{p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) + \pi_{s,t}^{pf*}}.$$

Thus, we have

$$\partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) \leq \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})). \tag{53}$$

By (3) and Newton-Leibniz formula,

$$\begin{aligned}
& \partial_{\gamma_{1,t}} \log(\psi_{1,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))) - \partial_{\gamma_{1,t}} \log(\psi_{1,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}))) \\
= & \int_{s=0}^{1} \sum_{j=1}^{N} (\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}) - \gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}))[\frac{\partial^2 \log(\psi_{s,t}(s\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}) + (1-s)\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}] \,\mathrm{d}s \\
< & \ 0,
\end{aligned}$$

which contradicts (53). Therefore, $\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})$ is increasing in $\pi^{pf*}_{s,t}$. The continuity of $\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})$ in $\pi^{pf*}_{s,t}$ follows directly from that $\Pi^{pf,1}_{i,t}(\gamma_t|\pi^{pf*}_{s,t})$ is twice continuously differentiable in $(\gamma_t, \pi^{pf*}_{s,t})$ and the implicit function theorem.

Next we show that if (17) holds, $\beta^{pf}_{s,t}(\pi^{pf*}_{s,t})$ is increasing in $\pi^{pf*}_{s,t}$. By Theorem 8(a), it suffices to show that $\Pi^{pf*,1}_{s,t}(\pi^{pf*}_{s,t}) := \Pi^{pf*,1}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})|\pi^{pf*}_{s,t})$ is increasing in $\pi^{pf*}_{s,t}$. Assume that $\bar{\pi}^{pf*}_{s,t} > \pi^{pf*}_{s,t}$. Since we have just shown $\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}) \geq \gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})$, (17) implies that $\psi_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) \geq \psi_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t}))$.

If $\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}) = \gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})$,

$$p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) + \bar{\pi}^{pf*}_{s,t} > p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t},$$

and, hence,

$$\begin{aligned}
\Pi^{pf*,1}_{s,t}(\bar{\pi}^{pf*}_{s,t}) &= \theta_{sa,t}(p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) + \bar{\pi}^{pf*}_{s,t})^2 \psi_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) \\
&> \theta_{sa,t}(p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t})^2 \psi_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) \\
&= \Pi^{pf*,1}_{s,t}(\pi^{pf*}_{s,t}).
\end{aligned}$$

If $\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t}) > \gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})$, Lemma 1 implies that

$$\partial_{\gamma_{1,t}} \log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})|\bar{\pi}^{pf*}_{s,t})) \geq \partial_{\gamma_{1,t}} \log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})|\pi^{pf*}_{s,t})),$$

i.e.,

$$\begin{aligned}
& \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) - \frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) + \bar{\pi}^{pf*}_{s,t}} \\
\geq & \ \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) - \frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t}}.
\end{aligned}$$

By (3) and Newton-Leibniz formula,

$$\begin{aligned}
& \partial_{\gamma_{1,t}} \log(\psi_{1,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}))) - \partial_{\gamma_{1,t}} \log(\psi_{1,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))) \\
= & \int_{s=0}^{1} \sum_{j=1}^{N} (\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}) - \gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))[\frac{\partial^2 \log(\psi_{s,t}((1-s)\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}) + s\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}] \,\mathrm{d}s \\
< & \ 0,
\end{aligned}$$

Hence,

$$-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) + \bar{\pi}^{pf*}_{s,t}} > -\frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t}}.$$

Because, by Lemma 2(b) and the convexity of $\nu_{s,t}(\cdot)$, $1-\theta_{sa,t}(A_t^{-1})_{ii} > 0$ and $\nu'_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) \geq \nu'_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}))$, we have

$$p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}))-\delta_s w_{s,t}-\nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}))+\bar{\pi}_{s,t}^{pf*} > p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*}))-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}))+\pi_{s,t}^{pf*}.$$

Therefore,

$$\begin{aligned}
\Pi_{s,t}^{pf*,1}(\bar{\pi}_{s,t}^{pf*}) &= \theta_{sa,t}(p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}))-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}))+\bar{\pi}_{s,t}^{pf*})^2\psi_{s,t}(\gamma_{ss,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) \\
&> \theta_{sa,t}(p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*}))-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}))+\pi_{s,t}^{pf*})^2\psi_{s,t}(\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*})) \\
&= \Pi_{s,t}^{pf*,1}(\pi_{s,t}^{pf*}).
\end{aligned}$$

We have, thus, shown that $\beta_{s,t}^{pf}(\pi_{s,t}^{pf*})$ is increasing in $\pi_{s,t}^{pf*}$. The continuity of $\beta_{s,t}^{pf}(\pi_{s,t}^{pf*})$ in $\pi_{s,t}^{pf*}$ follows directly from that of $\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})$ and that $\Pi_{i,t}^{pf,1}(\gamma_t|\pi_{s,t}^{pf*})$ is continuous in $(\gamma_t,\pi_{s,t}^{pf*})$. This concludes the proof of part (d).

Part (e). By part (d), we have that $\gamma_{s,t}^{pf*}$ is continuously increasing in $\pi_{s,t}^{pf*}$ and, thus, $\beta_{s,t-1}^{pf}$. By part (c), we have that $p_{i,t}^{pf*}(\gamma_t)$ is continuously decreasing in $\pi_{s,t}^{pf*}$ and, thus, $\beta_{s,t-1}^{pf}$. Moreover, if (17) holds, part (d) yields that $\beta_{s,t}^{pf}$ is continuously increasing in $\pi_{s,t}^{pf*}$ and, thus, $\beta_{s,t-1}^{pf}$ as well. This completes the proof of part (e). □

Proof of Theorem 10: Part (a). Part (a) follows from the same argument as the proof of Theorem 4(a) and is, hence, omitted.

Part (b). By part (a), $\pi_{i,t}^{pf*} \geq \tilde{\pi}_{i,t}^{pf*}$ for each i. Hence, Theorem 9(c) yields that $p_{i,t}^{pf*}(\gamma_t) \leq \tilde{p}_{i,t}^{pf*}(\gamma_t)$ for each firm i and each γ_t.

When the PF model is symmetric, $\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij}$ is independent of i. Direct computation yields that

$$\tilde{p}_{i,t}^{pf*}(\gamma_t)-p_{i,t}^{pf*}(\gamma_t) = (\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij})(\pi_{s,t}^{pf*}-\tilde{\pi}_{s,t}^{pf*}) \geq 0, \text{ for all } \gamma_t,$$

which is independent of i. Thus, (17) and Newton-Leibniz formula imply that

$$\rho_{s,t}(\tilde{p}_t^{pf*}(\gamma_t))-\rho_{s,t}(p_t^{pf*}(\gamma_t)) = \int_{s=0}^1 \sum_{i=1}^N (\tilde{p}_{i,t}^{pf*}(\gamma_t)-p_{i,t}^{pf*}(\gamma_t))\frac{\partial\rho_{s,t}((1-s)p_t^{pf*}(\gamma_t)+s\tilde{p}_t^{pf*}(\gamma_t))}{\partial p_{i,t}}\,\mathrm{d}s \leq 0.$$

Hence, $\rho_{s,t}(p_t^{pf*}(\gamma_t)) \geq \rho_{s,t}(\tilde{p}_t^{pf*}(\gamma_t))$. Since $y_{s,t}^{pf*} \geq \tilde{y}_{s,t}^{pf*}$, Theorem 8(b) implies that, for any $(I_t,\Lambda_t)\in\mathcal{S}$ and $\gamma_t\in[0,\bar{\gamma}_{s,t}]^N$,

$$x_{i,t}^{pf*}(I_t,\Lambda_t,\gamma_t) = y_{s,t}^{pf*}\rho_{s,t}(p_t^{pf*}(\gamma_t))\psi_{s,t}(\gamma_t) \geq \tilde{y}_{s,t}^{pf*}\rho_{s,t}(\tilde{p}_t^{pf*}(\gamma_t))\psi_{s,t}(\gamma_t) = \tilde{x}_{i,t}^{pf*}(I_t,\Lambda_t,\gamma_t).$$

This completes the proof of part (b).

Part (c). Because $\pi_{s,t}^{pf*} \geq \tilde{\pi}_{s,t}^{pf*}$, Theorem 9(d) yields that $\gamma_{s,t}^{pf*} \geq \tilde{\gamma}_{s,t}^{pf*}$ and, hence, $\gamma_{i,t}^{pf*}(I_t,\Lambda_t) = \gamma_{s,t}^{pf*} \geq \tilde{\gamma}_{s,t}^{pf*} = \tilde{\gamma}_{s,t}^{pf*}(I_t,\Lambda_t)$ for each i and $(I_t,\Lambda_t)\in\mathcal{S}$. This completes the proof of part (c). □

Proof of Theorem 11: Part (a). We show part (a) by backward induction. More specifically, we show that if $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t and $\hat{\beta}_{s,t-1}^{pf} \geq \beta_{s,t-1}^{pf}$, (i) $\hat{\pi}_{s,t}^{pf*} \geq \pi_{s,t}^{pf*}$, (ii) $\hat{p}_{s,t}^{pf*}(\gamma_t) \leq p_{s,t}^{pf*}(\gamma_t)$, (iii) $\hat{p}_{i,t}^{pf*}(I_t,\Lambda_t,\gamma_t) \leq p_{i,t}^{pf*}(I_t,\Lambda_t,\gamma_t)$ for each i, $(I_t,\Lambda_t)\in\mathcal{S}$, and $\gamma_t\in[0,\bar{\gamma}_{s,t}]^N$, (iv) $\hat{\gamma}_{s,t}^{pf*} \geq \gamma_{s,t}^{pf*}$, (v) $\hat{\gamma}_{i,t}^{pf*}(I_t,\Lambda_t) \geq \gamma_{s,t}^{pf*}(I_t,\Lambda_t)$ for each i and $(I_t,\Lambda_t)\in\mathcal{S}$, and (vi) $\hat{\beta}_{s,t}^{pf} \geq \beta_{s,t}^{pf}$. Since $\hat{\beta}_{s,0}^{pf} = \beta_{s,0}^{pf} = 0$, the initial condition is satisfied.

The same argument as the proof of Theorem 5(a) implies that $\hat{\pi}^{pf*}_{s,t} \geq \pi^{pf*}_{s,t}$. Hence, Theorem 9(c) implies that $\hat{p}^{pf*}_{i,t}(\gamma_t) \leq p^{pf*}_{i,t}(\gamma_t)$ for all i and γ_t. Thus, $\hat{p}^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t) = \hat{p}^{pf*}_{i,t}(\gamma_t) \leq p^{pf*}_{i,t}(\gamma_t) = p^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t)$ for each i, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$. Analogously, Theorem 9(d) implies that $\hat{\gamma}^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t}$. Hence, $\hat{\gamma}^{pf*}_{i,t}(I_t, \Lambda_t) = \hat{\gamma}^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t} = \gamma^{pf*}_{i,t}(I_t, \Lambda_t)$ for each i and all $(I_t, \Lambda_t) \in \mathcal{S}$. By Theorem 9(d), under inequality (17), $\hat{\pi}^{pf*}_{s,t} \geq \pi^{pf*}_{s,t}$ implies that $\hat{\beta}^{pf}_{s,t} \geq \beta^{pf}_{s,t}$. This completes the induction and, thus, the proof of part (a).

Part (b). By part (a), it suffices to show that, if $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t, $\hat{\kappa}'_{sa,t}(z_{i,t}) \geq \kappa'_{sa,t}(z_{i,t})$ for all $z_{i,t}$, and $\hat{\beta}^{pf}_{s,t-1} \geq \beta^{pf}_{s,t-1}$, we have (i) $\hat{y}^{pf*}_{s,t} \geq y^{pf*}_{s,t}$ and (ii) $\hat{x}^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t) \geq x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t)$ for each i, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$.

The same argument as the proof of Theorem 5(b) suggests that $\hat{y}^{pf*}_{s,t} \geq y^{pf*}_{s,t}$. We now show that $\hat{x}^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t) \geq x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t)$ for each i, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$. Because the PF model is symmetric, $\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij}$ is independent of i. Direct computation yields that

$$p^{pf*}_{i,t}(\gamma_t) - \hat{p}^{pf*}_{i,t}(\gamma_t) = (\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij})(\hat{\pi}^{pf*}_{s,t} - \pi^{pf*}_{s,t}) \geq 0, \text{ for all } \gamma_t,$$

which is independent of i. Thus, (17) and Newton-Leibniz formula implies that

$$\rho_{s,t}(p^{pf*}_t(\gamma_t)) - \rho_{s,t}(\hat{p}^{pf*}_t(\gamma_t)) = \int_{s=0}^1 \sum_{i=1}^N (p^{pf*}_{i,t}(\gamma_t) - \hat{p}^{pf*}_{i,t}(\gamma_t)) \frac{\partial \rho_{s,t}((1-s)\hat{p}^{pf*}_t(\gamma_t) + s p^{pf*}_t(\gamma_t))}{\partial p_{i,t}} \, \mathrm{d}s \leq 0.$$

Hence, $\rho_{s,t}(\hat{p}^{pf*}_t(\gamma_t)) \geq \rho_{s,t}(p^{pf*}_t(\gamma_t))$ for all γ_t. Since $\hat{y}^{pf*}_{s,t} \geq y^{pf*}_{s,t}$, Theorem 8(b) implies that, for any $(I_t, \Lambda_t) \in \mathcal{S}$ and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$,

$$\hat{x}^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t) = \hat{y}^{pf*}_{s,t} \rho_{s,t}(\hat{p}^{pf*}_t(\gamma_t))\psi_{s,t}(\gamma_t) \geq y^{pf*}_{s,t} \rho_{s,t}(p^{pf*}_t(\gamma_t))\psi_{s,t}(\gamma_t) = x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t).$$

This completes the proof of part (b). □

Proof of Theorem 12: We show **parts (a)-(b)** together by backward induction. More specifically, we show that if $\beta^{pf}_{s,t-1} \geq \beta^{pf}_{s,t-2}$, (i) $y^{pf*}_{s,t} \geq y^{pf*}_{s,t-1}$, (ii) $p^{pf*}_{i,t}(\gamma) \leq p^{pf*}_{i,t-1}(\gamma)$ for all $\gamma \in [0, \bar{\gamma}_{s,t}]^N$, (iii) $p^{pf*}_{i,t}(I, \Lambda, \gamma) \leq p^{pf*}_{i,t-1}(I, \Lambda, \gamma)$ for each i, $(I, \Lambda) \in \mathcal{S}$, and $\gamma \in [0, \bar{\gamma}_{s,t}]^N$, (iv) $\gamma^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t-1}$, (v) $\gamma^{pf*}_{i,t}(I, \Lambda) \geq \gamma^{pf*}_{i,t-1}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$, (vi) $x^{pf*}_{i,t}(I, \Lambda, \gamma) \geq x^{pf*}_{i,t-1}(I, \Lambda, \gamma)$ for each i, $(I, \Lambda) \in \mathcal{S}$, and $\gamma \in [0, \bar{\gamma}_{s,t}]^N$, and (vii) $\beta^{pf}_{s,t} \geq \beta^{pf}_{s,t-1}$. Since, by Theorem 8(a), $\beta^{pf}_{s,1} \geq \beta^{pf}_{s,0} = 0$. Thus, the initial condition is satisfied.

Since the model is stationary, by Theorem 9(a), $\beta^{pf}_{s,t-1} \geq \beta^{pf}_{s,t-2}$ suggests that $y^{pf*}_{s,t} \geq y^{pf*}_{s,t-1}$. Since $\pi^{pf*}_{s,t}$ is increasing in $\beta^{pf}_{s,t-1}$, $\beta^{pf}_{s,t-1} \geq \beta^{pf}_{s,t-2}$ implies that $\pi^{pf*}_{s,t} \geq \pi^{pf*}_{s,t-1}$. Theorem 9(c) yields that $p^{pf*}_{s,t}(\gamma) \leq p^{pf*}_{s,t-1}(\gamma)$ for all $\gamma \in [0, \bar{\gamma}_{s,t}]^N$. Theorem 9(e) implies that $\gamma^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t-1}$. Hence, $p^{pf*}_{i,t}(I, \Lambda, \gamma) = p^{pf*}_{i,t}(\gamma) \leq p^{pf*}_{i,t-1}(\gamma) = p^{pf*}_{i,t-1}(I, \Lambda, \gamma)$ for each i, $(I, \Lambda) \in \mathcal{S}$, and $\gamma \in [0, \bar{\gamma}_{s,t}]^N$, and $\gamma^{pf*}_{i,t}(I, \Lambda) = \gamma^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t-1} = \gamma^{pf*}_{i,t-1}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$. We now show that $x^{pf*}_{i,t}(I, \Lambda, \gamma) \geq x^{pf*}_{i,t-1}(I, \Lambda, \gamma)$ for each i, $(I, \Lambda) \in \mathcal{S}$, and $\gamma \in [0, \bar{\gamma}]^N$. Because the PF model is symmetric, $\sum_{j=1}^N \theta_{jj,t}(A^{-1})_{ij}$ is independent of i. Direct computation yields that

$$p^{pf*}_{i,t-1}(\gamma) - p^{pf*}_{i,t}(\gamma) = (\sum_{j=1}^N \theta_{jj}(A^{-1})_{ij})(\pi^{pf*}_{s,t} - \pi^{pf*}_{s,t-1}) \geq 0, \text{ for all } \gamma,$$

which is independent of i. Thus, (17) and Newton-Leibniz formula implies that

$$\rho_s(p^{pf*}_{t-1}(\gamma)) - \rho_s(p^{pf*}_t(\gamma)) = \int_{s=0}^1 \sum_{i=1}^N (p^{pf*}_{i,t-1}(\gamma) - p^{pf*}_{i,t}(\gamma)) \frac{\partial \rho_s((1-s)p^{pf*}_t(\gamma) + s p^{pf*}_{t-1}(\gamma))}{\partial p_i} \, \mathrm{d}s \leq 0.$$

Hence, $\rho_s(p_t^{pf*}(\gamma)) \geq \rho_s(p_{t-1}^{pf*}(\gamma))$ for all γ. Since $y_{s,t}^{pf*} \geq y_{s,t-1}^{pf*}$, Theorem 8(b) implies that, for any $(I,\Lambda) \in \mathcal{S}$ and $\gamma \in [0, \bar{\gamma}_{s,t}]^N$,

$$x_{i,t}^{pf*}(I,\Lambda,\gamma) = y_{s,t}^{pf*}\rho_s(p_t^{pf*}(\gamma))\psi_s(\gamma_t) \geq y_{s,t-1}^{pf*}\rho_s(p_{t-1}^{pf*}(\gamma_t))\psi_{s,t}(\gamma_t) = x_{i,t-1}^{pf*}(I,\Lambda,\gamma).$$

Finally, we show that $\beta_{s,t}^{pf} \geq \beta_{s,t-1}^{pf}$. Since the model is stationary and $\pi_{s,t}^{pf*} \geq \pi_{s,t-1}^{pf*}$, $\beta_{s,t}^{pf} \geq \beta_{s,t-1}^{pf}$ follows from Theorem 9(d) immediately. This completes the induction and, thus, the proof of Theorem 12. $\square$

Proof of Theorem 13: Part (a). Because $\beta_{s,t-1}^{pf} \geq \beta_{s,t-1}^{sc}$, $\pi_{s,t}^{pf*} \geq \pi_{s,t}^{sc*}$. The same argument as the proof of Theorem 3(a) implies that $y_{s,t}^{pf*} \geq y_{s,t}^{sc*}$.

We now show that, if $\pi_{s,t}^{pf*} \geq \pi_{s,t}^{sc*}$, $\gamma_{s,t}^{pf*} \geq \gamma_{s,t}^{sc*}$. Proposition 5 implies that $p_t^{pf*}(\gamma_{ss,t}^{pf*}) = A_t^{-1}f_t(\gamma_{ss,t}^{pf*})$. By Proposition 2, the equilibrium sales prices, $p_{ss,t}^{sc*}$, satisfy the system of first-order equations (15). Equivalently, $p_{ss,t}^{sc*} = A_t^{-1}f_t(\gamma_{ss,t}^{sc*})$.

We assume, to the contrary, that $\gamma_{s,t}^{pf*} < \gamma_{s,t}^{sc*}$. Lemma 1 implies that $\partial_{\gamma_{1,t}}\log(\Pi_{1,t}^{pf,1}(\gamma_{ss,t}^{pf*})) \leq \partial_{\gamma_{1,t}}\log(\Pi_{1,t}^{sc}(\gamma_{ss,t}^{sc*}, p_{ss,t}^{sc*}))$, i.e.,

$$\begin{aligned} &-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu_{s,t}'(\gamma_{s,t}^{pf*})}{\sum_{j=1}^N (A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{pf*})-\pi_{s,t}^{pf*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*}}+\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma_{ss,t}^{pf*})) \\ \leq &-\frac{\nu_{s,t}'(\gamma_{s,t}^{sc*})}{\sum_{j=1}^N (A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{sc*})-\pi_{s,t}^{sc*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*}}+\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma_{ss,t}^{sc*})). \end{aligned} \tag{54}$$

Inequality (3) and the Newton-Leibniz formula imply that

$$\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma_{ss,t}^{sc*}))-\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma_{ss,t}^{pf*})) = \int_{s=0}^1 \sum_{j=1}^N (\gamma_{s,t}^{sc*}-\gamma_{s,t}^{pf*})[\frac{\partial^2\log(\psi_{s,t}((1-s)\gamma_{s,t}^{pf*}+s\gamma_{s,t}^{sc*}))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}]\,\mathrm{d}s < 0.$$

By (54),

$$\begin{aligned} &-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu_{s,t}'(\gamma_{s,t}^{pf*})}{\sum_{j=1}^N (A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{pf*})-\pi_{s,t}^{pf*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*}} \\ < &-\frac{\nu_{s,t}'(\gamma_{s,t}^{sc*})}{\sum_{j=1}^N (A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{sc*})-\pi_{s,t}^{sc*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*}}. \end{aligned}$$

Lemma 2(b) suggests that $0 \leq 2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu_{s,t}'(\gamma_{s,t}^{pf*}) \leq \nu_{s,t}'(\gamma_{s,t}^{sc*})$. Hence,

$$\begin{aligned} &\sum_{j=1}^N (A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{pf*})-\pi_{s,t}^{pf*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*} \\ < &\sum_{j=1}^N (A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{sc*})-\pi_{s,t}^{sc*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*}. \end{aligned} \tag{55}$$

Since $\pi_{s,t}^{pf*} \geq \pi_{s,t}^{sc*}$ and $\nu_{s,t}(\gamma_{s,t}^{pf*}) \leq \nu_{s,t}(\gamma_{s,t}^{sc*})$, $\pi_{s,t}^{pf*} - \nu_{s,t}(\gamma_{s,t}^{pf*}) \geq \pi_{s,t}^{sc*} - \nu_{s,t}(\gamma_{s,t}^{sc*})$. Lemma 2(c) implies that $1-\sum_{j=1}^{N}(A_t^{-1})_{1j}\theta_{sa,t}>0$. Therefore,

$$\begin{aligned}&\sum_{j=1}^{N}(A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{pf*})-\pi_{s,t}^{pf*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*}\\ =&\sum_{j=1}^{N}(A_t^{-1})_{1j}(\phi_{sa,t}+\theta_{sa,t}\delta_s w_{s,t-1})-\delta_s w_{s,t-1}+(1-\sum_{j=1}^{N}(A_t^{-1})_{1j}\theta_{sa,t})(\pi_{s,t}^{pf*}-\nu_{s,t}(\gamma_{s,t}^{pf*}))\\ \geq&\sum_{j=1}^{N}(A_t^{-1})_{1j}(\phi_{sa,t}+\theta_{sa,t}\delta_s w_{s,t-1})-\delta_s w_{s,t-1}+(1-\sum_{j=1}^{N}(A_t^{-1})_{1j}\theta_{sa,t})(\pi_{s,t}^{sc*}-\nu_{s,t}(\gamma_{s,t}^{sc*}))\\ =&\sum_{j=1}^{N}(A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{sc*})-\pi_{s,t}^{sc*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*},\end{aligned}$$

which contradicts the inequality (55). Therefore, $\gamma_{s,t}^{pf*}\geq\gamma_{s,t}^{sc*}$. This completes the proof of part (a).

Part (b). We first show, by backward induction, that, if $\theta_{sb,t}=0$ for each t, $\beta_{s,t}^{pf}\geq\beta_{s,t}^{sc}$ for each t. Since $\beta_{s,0}^{pf}=\beta_{s,0}^{sc}=0$, the initial condition is satisfied. Now we prove that if $\beta_{s,t-1}^{pf}\geq\beta_{s,t-1}^{sc}$ and $\theta_{sb,t}=0$, we have $\beta_{s,t}^{pf}\geq\beta_{s,t}^{sc}$.

First, we observe that if $\theta_{sb,t}=0$, $(A_t^{-1})_{11}\theta_{sa,t}=\frac{1}{2}$ and, thus, $2(1-\theta_{sa,t}(A_t^{-1})_{11})=1$. Part (a) shows that $\gamma_{s,t}^{pf*}\geq\gamma_{s,t}^{sc*}$. If $\gamma_{s,t}^{pf*}=\gamma_{s,t}^{sc*}$,

$$\begin{aligned}\Pi_{s,t}^{pf*,1}=&\theta_{sa,t}((A_t^{-1}f_t(\gamma_{ss,t}^{pf*}))_i-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*})^2\psi_{s,t}(\gamma_{ss,t}^{pf*})\\ \geq&\theta_{sa,t}((A_t^{-1}f_t(\gamma_{ss,t}^{sc*}))_i-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*})^2\psi_{s,t}(\gamma_{ss,t}^{sc*})\\ =&\Pi_{s,t}^{sc*},\end{aligned}$$

where the inequality follows from $\pi_{s,t}^{pf*}\geq\pi_{s,t}^{sc*}$.

If $\gamma_{s,t}^{pf*}>\gamma_{s,t}^{sc*}$, Lemma 1 implies that $\partial_{\gamma_{1,t}}\log(\Pi_{1,t}^{pf,1}(\gamma_{ss,t}^{pf*}))\geq\partial_{\gamma_{1,t}}\log(\Pi_{1,t}^{sc}(\gamma_{ss,t}^{sc*},p_{ss,t}^{sc*}))$, i.e.,

$$\begin{aligned}&-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu'_{s,t}(\gamma_{s,t}^{pf*})}{(A_t^{-1}f_t(\gamma_{ss,t}^{pf*}))_1-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*}}+\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma_{ss,t}^{pf*}))\\ \geq&-\frac{\nu'_{s,t}(\gamma_{s,t}^{sc*})}{(A_t^{-1}f_t(\gamma_{ss,t}^{sc*}))_1-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*}}+\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma_{ss,t}^{sc*})).\end{aligned}\tag{56}$$

Inequality (3) and the Newton-Leibniz formula imply that

$$\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma_{ss,t}^{pf*}))-\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma_{ss,t}^{sc*}))=\int_{s=0}^{1}\sum_{j=1}^{N}(\gamma_{s,t}^{pf*}-\gamma_{s,t}^{sc*})[\frac{\partial^2\log(\psi_{s,t}((1-s)\gamma_{s,t}^{sc*}+s\gamma_{s,t}^{pf*}))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}]\,\mathrm{d}s<0.$$

By (56), we have

$$-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu'_{s,t}(\gamma_{s,t}^{pf*})}{(A_t^{-1}f_t(\gamma_{ss,t}^{pf*}))_1-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*}}>-\frac{\nu'_{s,t}(\gamma_{s,t}^{sc*})}{(A_t^{-1}f_t(\gamma_{ss,t}^{sc*}))_1-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*}}.$$

Because $2(1-\theta_{sa,t}(A_t^{-1})_{11})=1$ and $\gamma_{s,t}^{pf*}>\gamma_{s,t}^{sc*}$, $2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu'_{s,t}(\gamma_{s,t}^{pf*})\geq\nu'_{s,t}(\gamma_{s,t}^{sc*})$. Therefore,

$$(A_t^{-1}f_t(\gamma_{ss,t}^{pf*}))_1-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*}>(A_t^{-1}f_t(\gamma_{ss,t}^{sc*}))_1-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*}>0.$$

By inequality (17), $\gamma_{s,t}^{pf*}>\gamma_{s,t}^{sc*}$ implies that $\psi_{s,t}(\gamma_{ss,t}^{pf*})>\psi_{s,t}(\gamma_{ss,t}^{sc*})$. Thus, we have

$$\begin{aligned}\Pi_{s,t}^{pf*,1}=&\theta_{sa,t}((A_t^{-1}f_t(\gamma_{ss,t}^{pf*}))_1-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*})^2\psi_{s,t}(\gamma_{ss,t}^{pf*})\\ >&\theta_{sa,t}((A_t^{-1}f_t(\gamma_{ss,t}^{sc*}))_1-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*})^2\psi_{s,t}(\gamma_{ss,t}^{sc*})\\ =&\Pi_{s,t}^{sc*}.\end{aligned}$$

We have thus shown that if $\beta^{pf}_{s,t-1} \geq \beta^{sc}_{s,t-1}$, $\Pi^{pf*,1}_{s,t} \geq \Pi^{sc*}_{s,t}$. By Theorem 2(a) and Theorem 8(a),

$$\beta^{pf}_{s,t} = \delta_s \beta^{pf}_{s,t-1}\mu_{s,t} + \Pi^{pf*,1}_{s,t} \geq \delta_s \beta^{sc}_{s,t-1}\mu_{s,t} + \Pi^{sc*}_{s,t} = \beta^{sc}_{s,t}.$$

This completes the induction and, by part (a), the proof of part (b) for the case $\theta_{sb,t} = 0$.

For any fixed $\theta_{sa,t}$, both $\beta^{pf}_{s,t}$ and $\beta^{sc}_{s,t}$ are continuous in $\theta_{sb,t}$. Thus, for each period t, there exists a $\epsilon_t \geq 0$, such that, if $\theta_{sb,t} \leq \epsilon_t \theta_{sa,t}$, $\beta^{pf}_{s,t} \geq \beta^{sc}_{s,t}$. It remains to show that $\epsilon_t \leq \frac{1}{N-1}$. This inequality follows from the diagonal dominance condition that $\theta_{sa,t} > (N-1)\theta_{sb,t}$. This completes the proof of part (b). □

Appendix B: Sufficient Conditions for the Monotonicity of $\pi^{sc*}_{s,t}$ $[\pi^{pf*}_{s,t}]$ in $\beta^{sc}_{s,t-1}$ $[\beta^{pf}_{s,t-1}]$

In this section, we give some sufficient conditions under which $\pi^{sc*}_{s,t}$ $[\pi^{pf*}_{s,t}]$ is increasing in $\beta^{sc}_{s,t-1}$ $[\beta^{pf}_{s,t-1}]$. Observe that, if $t = 1$, $\beta^{sc}_{s,t-1} = \beta^{pf}_{s,t-1} = 0$. So we only consider the case $t \geq 2$.

We define the $N-$player noncooperative game, $\mathcal{G}_{s,t}$, as the symmetric game with each player i's payoff function given by

$$\pi_{i,t}(y_t) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}]) - \sum_{j\neq i} \kappa_{sb,t}(\mathbb{E}[y^+_{j,t} \wedge \xi_{j,t}])),$$

and feasible set given by $\mathbb{R}^+$. Hence, $\mathcal{G}^{sc,2}_{s,t}$ $[\mathcal{G}^{pf,3}_{s,t}]$ can be viewed as $\mathcal{G}_{s,t}$ with $\beta = \beta^{sc}_{s,t-1}$ $[\beta = \beta^{pf}_{s,t-1}]$. By Propositions 3 and 7, $\mathcal{G}_{s,t}$ has a unique symmetric pure strategy Nash equilibrium. Thus, we use $y^*_{s,t}(\beta)$ and $\pi^*_{s,t}(\beta)$ to denote the equilibrium strategy and payoff of each player in the game $\mathcal{G}_{s,t}$ with parameter β.

Let $y^*_{s,t}(\beta;\lambda,1)$ and $\pi^*_{s,t}(\beta;\lambda,1)$ $(\lambda > 0)$ be the equilibrium strategy and payoff of each firm in $\mathcal{G}_{s,t}(\lambda,1)$, where $\mathcal{G}_{s,t}(\lambda,1)$ is identical to $\mathcal{G}_{s,t}$ except that $\alpha_{s,t}(z_t)$ is replaced with $\kappa_{sa,t}(z_{i,t}) - \frac{1}{\lambda}(\sum_{j\neq i}\kappa_{sb,t}(z_{j,t}))$ in the objective function $\pi_{i,t}(\cdot)$, i.e.,

$$\pi_{i,t}(y_t) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}]) - \frac{1}{\lambda}(\sum_{j\neq i} \kappa_{sb,t}(\mathbb{E}[y^+_{j,t} \wedge \xi_{j,t}]))).$$

Analogously, let $y^*_{s,t}(\beta;\lambda,2)$ and $\pi^*_{s,t}(\beta;\lambda,2)$ $(\lambda \geq 0)$ be the equilibrium strategy and payoff of each firm in $\mathcal{G}_{s,t}(\lambda,2)$, where $\mathcal{G}_{s,t}(\lambda,2)$ is identical to $\mathcal{G}_{s,t}$ except that with $\alpha_{s,t}(z_t)$ is replaced with $\alpha_{s,t}(z_t) + \lambda$ in the objective function $\pi_{i,t}(\cdot)$, i.e.,

$$\pi_{i,t}(y_t) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}]) - \sum_{j\neq i} \kappa_{sb,t}(\mathbb{E}[y^+_{j,t} \wedge \xi_{j,t}]) + \lambda).$$

Finally, let $y^*_{s,t}(\beta;\lambda,3)$ and $\pi^*_{s,t}(\beta;\lambda,3)$ $(\lambda > 0)$ be the equilibrium strategy and payoff of each firm in $\mathcal{G}_{s,t}(\lambda,3)$, where $\mathcal{G}_{s,t}(\lambda,3)$ is identical to $\mathcal{G}_{s,t}$ except that $\alpha_{s,t}(z_t)$ is replaced with $\lambda\alpha_{s,t}(z_t)$ in the objective function $\pi_{s,t}(\cdot)$, i.e.,

$$\pi_{i,t}(y_t) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta\lambda(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}]) - \sum_{j\neq i} \kappa_{sb,t}(\mathbb{E}[y^+_{j,t} \wedge \xi_{j,t}])).$$

In some of our analysis below, we assume that $\alpha_{s,t}(\cdot)$ satisfies the monotonicity condition similar to (17),

$$\sum_{i=1}^{N} \frac{\partial \alpha_{s,t}(z_t)}{\partial z_{i,t}} > 0. \tag{57}$$

i.e., a uniform increase in the current expected fill rates gives rise to a higher expected market size of each firm in the next period.

First, we give a lower bound for the value of $\beta^{sc}_{s,t-1}$ and $\beta^{pf}_{s,t-1}$. By Theorem 2(a) and Theorem 8(a), $\beta^{sc}_{s,t-1} \geq \underline{\beta}_{s,t-1}$ and $\beta^{pf}_{s,t-1} \geq \underline{\beta}_{s,t-1}$, where

$$\underline{\beta}_{s,t-1} := \underline{\Pi}_{s,1} \prod_{\tau=1}^{t-1} (\delta_s \mu_{s,\tau}),$$

with $\underline{\Pi}_{s,1} := \min\{\Pi^{sc*}_{s,1}, \Pi^{pf*,1}_{s,1}\} > 0$. Thus, we assume in this section that $\beta \geq \underline{\beta}_{s,t-1} > 0$.

Let the density of $\xi_{s,t}$ be defined as $q_{s,t}(\cdot) = F'_{s,t}(\cdot)$ and its failure rate defined as $r_{s,t}(\cdot) := q_{s,t}(\cdot)/\bar{F}_{s,t}(\cdot)$. We have the following lemma on the Lipschitz continuity of $y^*_{s,t}(\beta)$ and $y^*_{s,t}(\beta; \lambda, i)$ $(i = 1, 2, 3)$.

Lemma 3 If $\kappa_{sa,t}(\cdot)$ is twice continuously differentiable and the failure rate of $\xi_{s,t}$ is bounded from below by $\underline{r}_{s,t} > 0$ on its support, there exists a constant $K_{s,t} > 0$, independent of λ, i, and β, such that $|y^*_{s,t}(\hat{\beta}) - y^*_{s,t}(\beta)| \leq K_{s,t}|\hat{\beta} - \beta|$ and $|y^*_{s,t}(\hat{\beta}; \lambda, i) - y^*_{s,t}(\beta; \lambda, i)| \leq K_{s,t}|\hat{\beta} - \beta|$ for all $\lambda > 0$, $i = 1, 2, 3$, and $\hat{\beta}, \beta \geq 0$.

Proof: Since $\kappa_{sa,t}(\cdot)$ is twice continuously differentiable, by the implicit function theorem, $y^*_{s,t}(\beta)$ and $y^*_{s,t}(\beta; \lambda, i)$ $(i = 1, 2, 3)$ are continuously differentiable in β with the derivatives given by:

$$\begin{aligned}&\frac{\partial y^*_{s,t}(\beta)}{\partial \beta} = \frac{\partial y^*_{s,t}(\beta; \lambda, 1)}{\partial \beta} = \frac{\partial y^*_{s,t}(\beta; \lambda, 2)}{\partial \beta} \\ =&\frac{\delta_s \bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])}{L''(y^*_{s,t}(\beta)) + \delta_s \beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}]) - \delta_s \beta \bar{F}^2_{s,t}(y^*_{s,t}(\beta))\kappa''_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])},\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial y^*_{s,t}(\beta; \lambda, 3)}{\partial \beta} \\ =&\frac{\lambda \delta_s \bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])}{L''(y^*_{s,t}(\beta)) + \lambda \delta_s \beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}]) - \lambda \delta_s \beta \bar{F}^2_{s,t}(y^*_{s,t}(\beta))\kappa''_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])}.\end{aligned}$$

Observe that

$$\begin{aligned}&\frac{\delta_s \bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])}{L''(y^*_{s,t}(\beta)) + \delta_s \beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}]) - \delta_s \beta \bar{F}^2_{s,t}(y^*_{s,t}(\beta))\kappa''_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])} \\ \leq&\frac{\delta_s \bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])}{\delta_s \beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])} \leq \frac{1}{\underline{\beta}_{s,t-1} r_{s,t}(y^*_{s,t}(\beta))} \leq \frac{1}{\underline{\beta}_{s,t-1}\underline{r}_{s,t}},\end{aligned}$$

where the first inequality follows from the convexity of $L_{s,t}(\cdot)$ and the concavity of $\kappa_{sa,t}(\cdot)$, the second from $\kappa'_{sa,t}(\cdot) \geq 0$, and the last from $r_{s,t}(\cdot) \geq \underline{r}_{s,t}$. Analogously, we have

$$\begin{aligned}&\frac{\lambda \delta_s \bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])}{L''(y^*_{s,t}(\beta)) + \lambda \delta_s \beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}]) - \lambda \delta_s \beta \bar{F}^2_{s,t}(y^*_{s,t}(\beta))\kappa''_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])} \\ \leq&\frac{\lambda \delta_s \bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])}{\lambda \delta_s \beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta) \wedge \xi_{s,t}])} \leq \frac{1}{\underline{\beta}_{s,t-1} r_{s,t}(y^*_{s,t}(\beta))} \leq \frac{1}{\underline{\beta}_{s,t-1}\underline{r}_{s,t}}.\end{aligned}$$

By the mean value theorem,

$$|y^*_{s,t}(\hat{\beta}) - y^*_{s,t}(\beta)| = |\hat{\beta} - \beta||\frac{\partial y^*_{s,t}(\tilde{\beta})}{\partial \beta}| \leq K_{s,t}|\hat{\beta} - \beta|,$$

where $\tilde{\beta}$ is a real number that lies between β and $\hat{\beta}$, and $K_{s,t} := \frac{1}{\underline{\beta}_{s,t-1}\underline{r}_{s,t}}$. The inequality $|y^*_{s,t}(\hat{\beta};\lambda,i) - y^*_{s,t}(\beta;\lambda,i)| \leq K_{s,t}|\hat{\beta}-\beta|$ for all $\lambda > 0$ and $i = 1,2,3$ follows from exactly the same argument. □

We remark that the assumption that the failure rate $r_{s,t}(\cdot)$ is uniformly bounded away from 0 is not a restrictive assumption, and can be satisfied by, e.g., all the distributions that satisfy (i) the increasing failure rate property, and (ii) the density $q_{s,t}(\cdot)$ being positive on the lower bound of its support. The same argument as the proof of Theorem 3(a) and Theorem 9(a) imply that, for all $\hat{\beta} > \beta$, $y^*_{s,t}(\hat{\beta}) \geq y^*_{s,t}(\beta)$ and $y^*_{s,t}(\hat{\beta};\lambda,i) \geq y^*_{s,t}(\beta;\lambda,i)$ $(i = 1,2,3)$. We now characterize sufficient conditions for $\pi^*_{s,t}(\beta)$ and $\pi^*_{s,t}(\beta;\lambda,i)$ $(i = 1,2,3)$ to be increasing in β.

Lemma 4 The following statements hold:

(a) If $\kappa_{sb,t}(\cdot) \equiv \kappa^0_{sb,t}$ for some constant $\kappa^0_{sb,t}$, $\pi^*_{s,t}(\beta)$ is increasing in β.

(b) Assume that $\alpha_{s,t}(\cdot) > 0$ for all z_t and that the conditions of Lemma 3 hold, we have:

(i) If $\kappa_{sb,t}(\cdot)$ is Lipschitz continuous, there exists an $M^1_{s,t} < +\infty$, such that for all $\lambda \geq M^1_{s,t}$, $\pi^*_{s,t}(\beta;\lambda,1)$ is increasing in β.

(ii) If the monotonicity condition (57) holds, there exists an $M^2_{s,t} < +\infty$, such that for all $\lambda \geq M^2_{s,t}$, $\pi^*_{s,t}(\beta;\lambda,2)$ is increasing in β.

(iii) If the monotonicity condition (57) holds, there exists an $M^3_{s,t} < +\infty$, such that for all $\lambda \geq M^3_{s,t}$, $\pi^*_{s,t}(\beta;\lambda,3)$ is increasing in β.

Proof: Part (a). Observe that, $\delta_s\beta\kappa_{sa,t}(\mathbb{E}[y^+_{i,t}\wedge\xi_{i,t}])$ is increasing in β for any $y_{i,t}$. Therefore,

$$\pi^*_{s,t}(\beta) = \max\{(\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\beta\kappa_{sa,t}(\mathbb{E}[y^+_{i,t}\wedge\xi_{i,t}]) - (N-1)\kappa^0_{sb,t} : y_{i,t} \geq 0\}$$

is increasing in β. This completes the proof of part (a).

Part (b-i). Let $\hat{\beta} > \beta$, and $k_t < +\infty$ be the Lipschitz constant for $\kappa_{sb,t}(\cdot)$. Since $\alpha_{s,t}(\cdot)$ is a continuous function on a compact support, $\alpha_{s,t}(\cdot) > 0$ for all z_t implies that $\alpha_{s,t}(\cdot) \geq \underline{\alpha}_{s,t} > 0$ for some constant $\underline{\alpha}_{s,t}$. We define

$$\zeta_{i,t}(y_{i,t}) := (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\beta\kappa_{sa,t}(\mathbb{E}[y_{i,t}\wedge\xi_{i,t}]).$$

By the envelope theorem,

$$\frac{\partial\zeta_{i,t}(y^*_{s,t}(\beta;\lambda,1))}{\partial\beta} = \delta_s\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,1)\wedge\xi_{i,t}]) \geq \delta_s\underline{\alpha}_{s,t} > 0,$$

where the first inequality follows from $\kappa_{sa,t}(z_{i,t}) \geq \alpha_{s,t}(z_t) \geq \underline{\alpha}_{s,t}$. By the mean value theorem and $\hat{\beta} > \beta$,

$$\zeta_{i,t}(y^*_{s,t}(\hat{\beta};\lambda,1)) - \zeta_{i,t}(y^*_{s,t}(\beta;\lambda,1)) \geq \delta_s\underline{\alpha}_{s,t}(\hat{\beta}-\beta). \tag{58}$$

At the same time, since $\alpha_{s,\tau}(\cdot)$, $\rho_{s,\tau}(\cdot)$, and $\psi_{s,\tau}(\cdot)$ are all uniformly bounded from above for $\tau \leq t-1$, $\beta^{sc}_{s,t-1}$ and $\beta^{pf}_{s,t-1}$ have a uniform upper bound, which we denote as $\bar{\beta}_{s,t-1} < +\infty$. On the other hand,

$$\begin{aligned}
&\frac{\delta_s}{\lambda}(N-1)[\hat{\beta}\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,1)\wedge\xi_{s,t}]) - \beta\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,1)\wedge\xi_{s,t}])]\\
=&\frac{\delta_s}{\lambda}(N-1)[\hat{\beta}\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,1)\wedge\xi_{s,t}]) - \hat{\beta}\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,1)\wedge\xi_{s,t}])\\
&+\hat{\beta}\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,1)\wedge\xi_{s,t}]) - \beta\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,1)\wedge\xi_{s,t}])]\\
\leq&\frac{\delta_s}{\lambda}(N-1)[\bar{\beta}_{s,t-1}k_t(y^*_{s,t}(\hat{\beta};\lambda,1) - y^*_{s,t}(\beta;\lambda,1)) + (\hat{\beta}-\beta)\bar{\kappa}_{sb,t}]\\
\leq&\frac{\delta_s}{\lambda}(N-1)(\bar{\beta}_{s,t-1}k_tK_{s,t} + \bar{\kappa}_{sb,t})(\hat{\beta}-\beta),
\end{aligned} \tag{59}$$

where the first inequality follows from the Lipschitz continuity of $\kappa_{sb,t}(\cdot)$, $y^*_{s,t}(\hat{\beta};\lambda,1) \geq y^*_{s,t}(\beta;\lambda,1)$, and $\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,1)\wedge\xi_{s,t}] - \mathbb{E}[y^*_{s,t}(\beta;\lambda,1)\wedge\xi_{s,t}] \leq y^*_{s,t}(\hat{\beta};\lambda,1) - y^*_{s,t}(\beta;\lambda,1)$, with $\bar{\kappa}_{sb,t} := \max\{\kappa_{sb,t}(z_{i,t}) : z_{i,t}\in[0,1]\} < +\infty$, and the second from Lemma 3. Define

$$M^1_{s,t} := \frac{(N-1)(\bar{\beta}_{s,t-1}k_tK_{s,t} + \bar{\kappa}_{sb,t})}{\underline{\alpha}_{s,t}} < +\infty.$$

If $\lambda \geq M^1_{s,t}$,

$$\begin{aligned}
\pi^*_{s,t}(\hat{\beta};\lambda,1) - \pi^*_{s,t}(\beta;\lambda,1) &= \zeta_{i,t}(y^*_{s,t}(\hat{\beta};\lambda,1)) - \zeta_{i,t}(y^*_{s,t}(\beta;\lambda,1)) \\
&\quad - \frac{(N-1)\delta_s}{\lambda}[\hat{\beta}\kappa_{sb,t}(y^*_{s,t}(\hat{\beta};\lambda,1)) - \beta\kappa_{sb,t}(y^*_{s,t}(\beta;\lambda,1))] \\
&\geq (\delta_s\underline{\alpha}_{s,t} - \frac{\delta_s}{\lambda}(N-1)(\bar{\beta}_{s,t-1}k_tK_{s,t} + \bar{\kappa}_{sb,t}))(\hat{\beta}-\beta) \\
&\geq (\delta_s\underline{\alpha}_{s,t} - \delta_s\underline{\alpha}_{s,t})(\hat{\beta}-\beta) \\
&= 0,
\end{aligned}$$

where the first inequality follows from (58) and (59), and the second from $\lambda \geq M^1_{s,t}$. This establishes part (b-i).

Part (b-ii). Let $H_{s,t}(y_{i,t}) := (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t})$. Since

$$\delta_s w_{s,t-1} - w_{s,t} - h_{s,t} \leq H'_{s,t}(y_{i,t}) \leq b_{s,t} + \delta_s w_{s,t-1} - w_{s,t},$$

$H_{s,t}(\cdot)$ is Lipschitz continuous with the Lipschitz constant equal to $l_t := \max\{|\delta_s w_{s,t-1} - w_{s,t} - h_{s,t}|, |b_{s,t} + \delta_s w_{s,t-1} - w_{s,t}|\} < +\infty$. Thus,

$$H_{s,t}(y^*_{s,t}(\beta;\lambda,2)) - H_{s,t}(y^*_{s,t}(\hat{\beta};\lambda,2)) \leq l_t(y^*_{s,t}(\hat{\beta};\lambda,2) - y^*_{s,t}(\beta;\lambda,2)) \leq l_tK_{s,t}(\hat{\beta}-\beta), \tag{60}$$

where the second inequality follows from Lemma 3 and $y^*_{s,t}(\hat{\beta};\lambda,2) \geq y^*_{s,t}(\beta;\lambda,2)$. On the other hand,

$$\begin{aligned}
&\delta_s\hat{\beta}(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2)\wedge\xi_{s,t}]) + \lambda) \\
&\quad - \delta_s\beta(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,2)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,2)\wedge\xi_{s,t}]) + \lambda) \\
&\geq \delta_s\hat{\beta}(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2)\wedge\xi_{s,t}]) + \lambda) \\
&\quad - \delta_s\beta(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2)\wedge\xi_{s,t}]) + \lambda) \\
&\geq \delta_s\lambda(\hat{\beta}-\beta) + \delta_s\underline{\alpha}_{s,t}(\hat{\beta}-\beta) \\
&= \delta_s(\lambda + \underline{\alpha}_{s,t})(\hat{\beta}-\beta),
\end{aligned} \tag{61}$$

where the first inequality follows from (57) and the second from the definition of $\underline{\alpha}_{s,t}$. Define

$$M^2_{s,t} := \frac{l_tK_{s,t}}{\delta_s} - \underline{\alpha}_{s,t} < +\infty.$$

If $\lambda \geq M^2_{s,t}$,

$$\begin{aligned}
\pi^*_{s,t}(\hat{\beta};\lambda,2) - \pi^*_{s,t}(\beta;\lambda,2) &= \delta_s\hat{\beta}(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2)\wedge\xi_{s,t}]) + \lambda) \\
&\quad - \delta_s\beta(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,2)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,2)\wedge\xi_{s,t}]) + \lambda) \\
&\quad - (H_{s,t}(y^*_{s,t}(\beta;\lambda,2)) - H_{s,t}(y^*_{s,t}(\hat{\beta};\lambda,2))) \\
&\geq (\delta_s\lambda + \delta_s\underline{\alpha}_{s,t} - l_tK_{s,t})(\hat{\beta}-\beta) \\
&\geq (l_tK_{s,t} - \delta_s\underline{\alpha}_{s,t} + \delta_s\underline{\alpha}_{s,t} - l_tK_{s,t})(\hat{\beta}-\beta) \\
&= 0,
\end{aligned}$$

where the first inequality follows from (60) and (61), and the second from $\lambda \geq M^2_{s,t}$. This establishes part (b-ii).

Part (b-iii). As shown in part (b-ii), $H_{s,t}(\cdot)$ is a Lipschitz function with the Lipschitz constant l_t. Thus,

$$H_{s,t}(y^*_{s,t}(\beta;\lambda,3)) - H_{s,t}(y^*_{s,t}(\hat{\beta};\lambda,3)) \leq l_t(y^*_{s,t}(\hat{\beta};\lambda,3) - y^*_{s,t}(\beta;\lambda,3)) \leq l_t K_{s,t}(\hat{\beta}-\beta), \tag{62}$$

where the second inequality follows from Lemma 3 and $y^*_{s,t}(\hat{\beta};\lambda,3) \geq y^*_{s,t}(\beta;\lambda,3)$. The monotonicity condition (57) and $y^*_{s,t}(\hat{\beta};\lambda,3) \geq y^*_{s,t}(\beta;\lambda,3)$ implies that

$$\begin{aligned} & \kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) \\ \geq\ & \kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}]). \end{aligned}$$

Therefore,

$$\begin{aligned} & \delta_s\hat{\beta}\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}])) \\ & - \delta_s\beta\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}])) \\ \geq & \delta_s\hat{\beta}\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}])) \\ & - \delta_s\beta\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}])) \\ \geq & \delta_s\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]))(\hat{\beta}-\beta) \\ \geq & \delta_s\lambda\underline{\alpha}_{s,t}(\hat{\beta}-\beta), \end{aligned} \tag{63}$$

where the last inequality follows from the definition of $\underline{\alpha}_{s,t}$. Define

$$M^3_{s,t} := \frac{l_t K_{s,t}}{\delta_s\underline{\alpha}_{s,t}} < +\infty.$$

If $\lambda \geq M^3_{s,t}$,

$$\begin{aligned} \pi^*_{s,t}(\hat{\beta};\lambda,3) - \pi^*_{s,t}(\beta;\lambda,3) &= \delta_s\hat{\beta}\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}])) \\ & \quad -\delta_s\beta\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}])) \\ & \quad -(H_{s,t}(y^*_{s,t}(\beta;\lambda,3)) - H_{s,t}(y^*_{s,t}(\hat{\beta};\lambda,3))) \\ &\geq (\delta_s\lambda\underline{\alpha}_{s,t} - l_t K_{s,t})(\hat{\beta}-\beta) \\ &\geq (l_t K_{s,t} - l_t K_{s,t})(\hat{\beta}-\beta) \\ &= 0, \end{aligned}$$

where the first inequality follows from (62) and (63), and the second from $\lambda \geq M^3_{s,t}$. This establishes part (b-iii). □

Lemma 4 has several economical interpretations. Parts (a) and (b-i) imply that, if the adverse effect of a firm's competitors' service level upon its future market size is not strong, $\pi^{sc*}_{s,t}$ [$\pi^{pf*}_{s,t}$] is increasing in $\beta^{sc}_{s,t-1}$ [$\beta^{pf}_{s,t-1}$]. Part (b-ii) implies that if the network effect is sufficiently strong, $\pi^{sc*}_{s,t}$ [$\pi^{pf*}_{s,t}$] is increasing in $\beta^{sc}_{s,t-1}$ [$\beta^{pf}_{s,t-1}$]. Finally, part (b-iii) implies that if the both the service effect and the network effect are sufficiently strong, $\pi^{sc*}_{s,t}$ [$\pi^{pf*}_{s,t}$] is increasing in $\beta^{sc}_{s,t-1}$ [$\beta^{pf}_{s,t-1}$].

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