Competition and Coopetition for Two-Sided Platforms

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Two-sided platforms have become omnipresent (e.g., ride-sharing and on-demand delivery services). In this context, firms compete not only for customers but also for flexible self-scheduling workers who can work for multiple platforms. We consider a setting where two-sided platforms simultaneously choose prices and wages to compete for both sides of the market. We assume that customers and workers each follow an endogenous Multinomial Logit choice model that accounts for network effects. In our model, the behavior of an agent depends not only on the price or wage set by the platform, but also on the strategic interactions among agents on both sides of the market. We show that a unique equilibrium exists and that it can be computed using a *tatônnement* scheme. The proof technique for the competition between two-sided platforms is not a simple extension of the traditional (one-sided) setting and involves different arguments. Armed with this result, we study the impact of *coopetition* between two-sided platforms, that is, the business strategy of cooperating with competitors. Motivated by recent practice in the ride-sharing industry, we analyze a setting where two competing platforms engage in a profit sharing contract by introducing a new joint service. We show that a well-designed profit sharing contract (e.g., under Nash bargaining) will benefit every party in the market (both platforms, riders, and drivers).

Key words: Two-sided platforms, ride-sharing, coopetition, choice models

1. Introduction

The service industry has significantly evolved in recent years. Thanks to the emergence of online platforms, several types of services are now offered on-demand. Specifically, customers can use their smartphones to request services from anywhere and at any time. These services include: ride-sharing, food delivery, cleaning, and repair works, just to name a few. According to a survey run by the National Technology Readiness Survey, in October 2015, the on-demand economy was attracting more than 22.4 million consumers annually and $57.6 billion in spending.[1] This new trend has also made the market increasingly competitive. In each sector, several competing firms offer the same type of service (a list of companies that offer on-demand services can be found at https://theondemandeconomy.org/participants/). For example, in the U.S. ride-sharing market, one can find several competitors including Uber, Lyft, Via, Gett, and Curb. These platforms

[1] https://hbr.org/2016/04/the-on-demand-economy-is-growing-and-not-just-for-the-young-and-wealthy

compete not only for customers (riders) but also for workers (drivers). They often send enticing monetary incentives to attract both sides of the market. Traditionally, firms were competing only for customers while hiring permanent workers. In two-sided markets, platforms also compete for workers who can work for multiple platforms and seemingly switch back and forth between companies. As of 2017, 70% of on-demand U.S. drivers work for both Uber and Lyft, and 25% drive for more than just those two, according to a survey by The Rideshare Guy.[2]

The first part of this paper (Sections 2 and 3) studies the competition between two-sided platforms that compete for both customers and workers (e.g., Uber and Lyft). We model this problem using an endogenous Multinomial Logit (MNL) choice model that accounts for network effects across both sides of the market. In our model, the utilities of both customers and workers are endogenously determined by the total demand and supply of each platform. Consequently, the behavior of an agent depends not only on the price or wage set by the platform, but also on the strategic interactions among agents on both sides of the market. Given that firms compete for customers and workers, the standard equilibrium analysis in the choice model literature needs to be revisited. The two-sidedness nature of our setting makes the objective function non-differentiable, and hence traditional arguments from the literature are not applicable. Instead, we use an approach based on analyzing the best-response strategy to characterize the equilibrium. We ultimately show the existence and uniqueness of the equilibrium market outcome.

Within the ride-sharing industry, a recent trend of partnerships has emerged. One such example is the partnership between Curb and Via in NYC. Curb[3] is an online platform that allows taxi rides to be ordered from a smartphone application and the payment can be completed either via the app or in person. Via[4] is a ride-sharing platform that allows riders heading in the same direction to carpool and share a ride. One can definitely view these two platforms as competitors. Yet, they decided to collaborate and engage in a joint partnership. Specifically, on June 6, 2017, they started offering a joint service through a *profit sharing contract*, under which Curb and Via each earn a portion of the net profit from the joint service. This type of partnership is often referred to as *coopetition*, a term coined to describe cooperative competition (see, e.g., Brandenburger and Nalebuff 2011). The new service introduced by Curb and Via allows users to book a shared taxi from either platform (this is called Shared Taxi). Shared Taxi fares are calculated using the meter

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[2] https://docs.google.com/document/d/1QSUFSqasfjM9b9UsqBwZlpa8EgqNj6EBfWybFBSHj3o/edit

[3] https://gocurb.com/

[4] https://ridewithvia.com/

price and paid directly to the driver. If the matching algorithm finds another rider heading in the same direction, the two riders will carpool and save 40% on any shared portion of the trip.[5]

The recent partnership between Curb and Via is not an exception. Below, we report four additional similar examples:

1. In December 2016, Uber partnered with Indonesia's second largest taxi operator PT Express Transindo Utama Tbk. This partnership gave Uber access to Express fleet of taxis and drivers. Express drivers who participate in the program can now serve requests from Uber.

2. In October 2014, Uber partnered with For Hire taxis to expand pick-up availability in Seattle. In this partnership, riders can select multiple options directly from the Uber app (UberX, UberXL, Black Car, SUV, and For Hire).

3. In March 2017, Grab partnered with SMRT Taxis with the goal of building the largest car fleet (taxi and private-hire) in Southeast Asia. In this partnership, all SMRT drivers will use only Grab's application for third-party bookings (to complement street-hail pickups).

4. On January 31, 2017, Go-Jek partnered with PT Blue Bird Tbk in Indonesia. In this partnership, riders will simply be served by the closest driver.[6]

It is clear that both platforms have their own incentives to engage in such partnerships. For example, it allows ride-sharing platforms to expand their number of drivers and increase their market share. Platforms can also benefit from technological advances developed by other firms (e.g., efficient matching algorithms). On the other hand, such partnerships can cannibalize the original market shares (customers who were riding with one of the platforms may now switch to the new service).

The second part of this paper (Sections 4 and 5) is motivated by the type of partnerships described above. We study the implications of introducing a new joint service between two competing platforms via a profit sharing contract. Our goal is to examine the impact of the new service on both platforms, riders, and drivers. In conformance with recent partnerships (e.g., Curb and Via), we assume that the workers are coming from two separate labor pools. Although it is not a-priori obvious whether coopetition will benefit the platforms, our analysis shows that—under the Nash bargaining framework— a well-designed contract is beneficial for both platforms, riders, and drivers (i.e., yields a Pareto improvement).

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[5] The partnership between Curb and Via in NYC was the topic of extensive media coverage. See for example: https://www.nytimes.com/2017/06/06/nyregion/new-york-yellow-taxis-ride-sharing.html, https://techcrunch.com/2017/06/06/curb-and-via-bring-ride-sharing-to-nycs-yellow-taxis/ and https://qz.com/999132/can-shared-rides-save-the-iconic-new-york-city-yellow-cab/

[6] https://www.techinasia.com/go-jek-launches-blue-bird-partnership-now-on-iphone

1.1. Contributions

Given the recent popularity of two-sided platforms, this paper extends our understanding of competition and coopetition models in this context. We next summarize our main contributions.

Equilibrium in a two-sided MNL setting with network effects. This paper is among the first to study the (price and wage) competition between two-sided platforms. We use an endogenous MNL choice model with network effects to capture the decision process of potential customers and workers. We prove the existence and uniqueness of equilibrium under general price and wage (Theorem 1) and under a fixed-commission rate (Theorem 3). We also convey that the equilibrium outcome can be computed efficiently using a *tatônnement scheme.* Interestingly, the proof technique for two-sided markets is not a simple extension of the traditional (one-sided) setting. Instead, we show that the best-response strategy is a monotone contraction mapping, allowing us to prove the existence and uniqueness of equilibrium.

Win-win coopetition using a profit sharing contract. Motivated by recent practice in ride-sharing, we study how introducing a new joint service affects the competing platforms. We first show that there exists a unique equilibrium even after introducing the coopetition partnership. We then identify conditions under which the coopetition partnership is *strictly* beneficial for both platforms. We also capture the strategic interactions between the platforms using the Nash bargaining framework. We then show that the platforms will agree on a profit sharing contract that increases the profit of each platform. As a result, engaging in a coopetition is a win-win strategy. Finally, we identify three main effects induced by introducing the partnership: new market share, cannibalization, and wage variation.

Pareto improvement under coopetition. As expected, riders will benefit from coopetition. We show that one can design a profit sharing contract that also benefits drivers. Consequently, when the coopetition terms are carefully designed, it will benefit every party (both platforms, riders, and drivers).

1.2. Related Literature

This paper is related to three streams of literature: price competition under choice models, economics of ride-sharing platforms, and coopetition models.

Price competition under choice models: The first stream of relevant literature is related to choice models (for a review on this topic, see Train 2009, and the references therein), and in particular price competition under the MNL model and its extensions (see, e.g., Anderson et al. 1992, Gallego et al. 2006, Konovalov and Sándor 2010, Li and Huh 2011, Aksoy-Pierson et al. 2013, Gallego and Wang 2014). Talluri and Van Ryzin (2004) consider a revenue management context under a general discrete choice model. The authors propose an exact solution and derive

several structural properties. Using the MNL model, Gallego et al. (2006) show that a unique equilibrium exists when costs are increasing and convex in sales. In Li and Huh (2011), the authors consider the problem of pricing multiple products under the nested-MNL model and show that characterizing the equilibrium is analytically tractable. In this literature, the main focus is on showing the existence and/or uniqueness of the equilibrium outcome. In this paper, we extend the results of Gallego et al. (2006) and Li and Huh (2011) to show that a unique equilibrium exists in a two-sided market where firms compete for both customers and workers. As mentioned, the proof technique for two-sided markets is not a simple extension of the traditional (one-sided) setting and involves different arguments. To capture the fact that the decision of an agent depends on other agents' decisions, we consider an MNL model that accounts for network effects. Specifically, our choice model is constructed in a similar fashion as the MNL model with endogenous network effects. Wang and Wang (2016) and Du et al. (2016) incorporate demand-side network effects into the standard MNL model and study the optimal pricing and assortment strategies. We extend this framework by endogenizing both demand and supply network effects to examine the two-sided competition between platforms.

Economics of ride-sharing platforms: The popularity of ride-sharing platforms triggered a great interest in studying pricing decisions in this context. Several papers consider the problem of designing incentives on prices and wages to coordinate supply with demand for on-demand service platforms (see, e.g., Chen and Hu 2016, Tang et al. 2017, Taylor 2017, Hu and Zhou 2017, Bimpikis et al. 2016, Yu et al. 2017, Benjaafar et al. 2018). Our work has a similar motivation but is among the first to explicitly capture the competition between platforms using an MNL choice model for each side of the market. The recent work by Nikzad (2018) also analyzes the competition between ride-sharing platforms but with a different focus. The author shows that the effect of competition on prices and wages crucially depends on market thickness (i.e., the number of potential workers). The author identifies an underlying mechanism which is quite similar to the one we advocate in our work: monopoly may soften competition (which may hurt workers and customers) but given the resource pooling on the supply side, it may actually benefit all parties. In Hu and Zhou (2017), the authors study the pricing decisions of an on-demand platform and demonstrate the good performance of a flat-commission contract. We will also consider the special case of flat-commission contracts.

Coopetition models: As mentioned, when two competitors cooperate, this is often referred to as *coopetition* (see, e.g., Brandenburger and Nalebuff 2011). Closer to our work, there are several papers on coopetition in operations management. For example, Nagarajan and Sošić (2007) propose a model for coalition formation among competitors and characterize the equilibrium behavior of the resulting strategic alliances. Casadesus-Masanell and Yoffie (2007) study the simultaneously

competitive and cooperative relationship between two manufacturers of complementary products, such as Intel and Microsoft, on their R&D investment, pricing, and timing of new product releases. In a strategic alliance setting with capacity sharing, Roels and Tang (2017) show that an ex-ante capacity reservation contract will benefit both firms. In the revenue management literature, several papers have studied a common form of coopetition among airline companies, called airline alliances (see, e.g., Netessine and Shumsky 2005, Wright et al. 2010). Coopetition and its related contractual issues have also been studied in the context of service operations (as opposed to manufacturing and supply chain). For example, Roels et al. (2010) analyze the contracting issues that arise in collaborative services and identify the optimal contracts. In a recent work, Yuan et al. (2019) show that as price competition increases, the service providers may surprisingly charge higher prices under coopetition. Our contribution with respect to this literature lies in the fact that, motivated by recent partnerships, we are the first to study coopetition in ride-sharing.

Finally, our work is related to the economics literature on competition between two-sided platforms (see Rochet and Tirole 2003, Armstrong 2006, and the references therein). Our paper differs from this literature in two important ways. First, we explicitly consider a supply-constrained setting where the total sales (matches between customers and workers) are truncated by both demand and supply. Second, we focus on a setting where each side of the market (customers and workers) follow a choice model to decide which platform to use and to work. As a result, our model is especially applicable to the increasingly competitive environment between on-demand service platforms.

Structure of the paper. Section 2 presents our model of competition between two-sided platforms (in the absence of coopetition), and Section 3 reports our equilibrium analysis for this model. We next consider introducing the coopetition partnership: Section 4 presents our coopetition model, and Section 5 studies the impact of coopetition. Finally, we consider an extension of our model with endogenous waiting times in Section 6, and we report our conclusions in Section 7. The proofs of the technical results are relegated to the Appendix.

2. Competition Between Two-sided Platforms: Model

In this section, we present our model of competition between two-sided platforms. We consider two competing online platforms denoted as P_1 and P_2 (most of our results extend to $n > 2$ platforms, as we will discuss later). Each platform P_i ($i = 1, 2$) offers a service via its mobile or online application. Let q_i be the perceived value/quality (e.g., safety, reputation) of the service offered by P_i, and $p_i \geq 0$ the price charged by P_i. A summary of the notation can be found in Appendix A.

Demand side: We assume that customers follow an endogenous MNL discrete choice model that accounts for network effects. More specifically, a customer can choose between three alternatives: P_1, P_2, and the outside option. Although the service quality of each platform, q_i, is exogenous and

determined by factors outside the scope of our model (e.g., customer preferences), the utility a customer derives from the service offered by P_i *endogenously* depends on the aggregate behavior of all customers (captured by P_i's total demand, d_i) and on the aggregate behavior of all workers (captured by P_i's total supply, s_i). If P_i's demand is dominated by its supply (i.e., $d_i \leq s_i$), then every customer who opts for P_i will be served. In this case, a customer who selects P_i earns a utility of $q_i - p_i$. If $d_i > s_i$, then the supply capacity has to be rationed. In this case, we assume that the platform randomly allocates its supply s_i to the customers who choose its service. If a customer chooses P_i's service and successfully receives it, his/her utility is $q_i - p_i$. A customer who opts for P_i but does not get served will choose the outside option, whose utility is u_0. More precisely, if $d_i > s_i$, a customer will have a probability of s_i/d_i to be served and a probability of $1 - s_i/d_i$ to go with the outside option. Consequently, the expected utility of a customer who opts for P_i is $u_i = \min\{1, s_i/d_i\}(q_i - p_i) + [1 - \min\{1, s_i/d_i\}]u_0$. The actual utility of a customer choosing P_i is $u_i + \xi_i$, where ξ_i represents the random unobserved utility terms for using P_i. The expected utility of the outside option is u_0. Opting for the outside option is also associated with a random unobserved utility ξ_0, which makes the total utility of the outside option equal to $u_0 + \xi_0$. For each customer, ξ_1, ξ_2, and ξ_0 are assumed to be independent and identically distributed with a Gumbel distribution. Therefore, the total demand for P_i is given by:

$$d_i = \frac{\Lambda \exp(u_i)}{\exp(u_0) + \exp(u_1) + \exp(u_2)}$$
$$= \frac{\Lambda \exp[u_0 + \min\{1, s_i/d_i\}(q_i - p_i - u_0)]}{\exp(u_0) + \exp[u_0 + \min\{1, s_1/d_1\}(q_1 - p_1 - u_0)] + \exp[u_0 + \min\{1, s_2/d_2\}(q_2 - p_2 - u_0)]},$$

where Λ is the total customer arrival rate, that is, the maximum number of potential customers arriving per unit time. We assume that Λ is deterministic and known to both platforms.

For simplicity, we normalize $u_0 = 0$. Note, however, that all our results and insights continue to hold for any $u_0 \neq 0$. Thus, the demand for P_i satisfies

$$d_i = \frac{\Lambda \exp[\min\{1, s_i/d_i\}(q_i - p_i)]}{1 + \exp[\min\{1, s_1/d_1\}(q_1 - p_1)] + \exp[\min\{1, s_2/d_2\}(q_2 - p_2)]}. \tag{1}$$

Supply side: Workers also follow an endogenous MNL model with network effects. Similar to the demand side, the utility a worker derives from working for P_i depends endogenously on the aggregate behavior of all customers (captured by the total demand d_i) and on the aggregate behavior of all workers (captured by the total supply s_i). Each worker chooses one of three alternatives: P_1, P_2, and the outside option. We denote the attractiveness of P_i by a_i, while a_0 represents the attractiveness of the outside option. Although the attractiveness of each platform to a worker (i.e., a_i) is exogenous and determined by factors outside the scope of our model (e.g., worker preferences), the utility a worker derives from working for P_i *endogenously* depends on the aggregate

behavior of all customers and all workers. More precisely, the expected utility of a worker who chooses to work for P_i equals the platform's attractiveness plus the expected wage s/he earns. Let w_i be the wage per service distributed by P_i to its workers. If demand dominates supply ($d_i \geq s_i$), any worker who opts for P_i can serve a customer and receive the wage w_i. If $s_i > d_i$, demand will be randomly rationed to the workers who opt for P_i. In this case, a worker will have a probability of d_i/s_i to be matched with a demand request (and earning w_i) and a probability of $1 - d_i/s_i$ to not be matched with any customer (and earning 0). Thus, the expected utility of a worker choosing P_i is $v_i = a_i + \min\{1, d_i/s_i\}w_i + [1 - (\min\{1, d_i/s_i\})] \cdot 0 = a_i + \min\{1, d_i/s_i\}w_i$. The actual utility a worker earns from working for P_i is $v_i + \omega_i$, where ω_i represents the random unobserved utility terms of working for P_i. The actual utility of working for the outside option is $v_0 = a_0 + \omega_0$. For each worker, ω_1, ω_2, and ω_0 are assumed to be independent and identically distributed with a Gumbel distribution. Therefore, the total supply for P_i is given by:

$$s_i = \frac{K\exp(v_i)}{\exp(v_0) + \exp(v_1) + \exp(v_2)} = \frac{K\exp(a_i + \min\{1, d_i/s_i\}w_i)}{\exp(a_0) + \exp(a_1 + \min\{1, d_1/s_1\}w_1) + \exp(a_2 + \min\{1, d_2/s_2\}w_2)},$$

where K is the total number of workers on the market, which we normalize to 1.

Without loss of generality, we normalize $a_0 = 0$. As before, our results and insights continue to hold for any $a_0 \neq 0$. Thus, the supply of P_i satisfies

$$s_i = \frac{\exp(a_i + \min\{1, d_i/s_i\}w_i)}{1 + \exp(a_1 + \min\{1, d_1/s_1\}w_1) + \exp(a_2 + \min\{1, d_2/s_2\}w_2)}. \tag{2}$$

As is clear from our model, the utilities of both customers and workers are endogenously determined by the total demand and supply of each platform. Consequently, customers and workers account for the strategic interactions among themselves, which give rise to endogenous market outcomes. Our endogenous two-sided MNL model is constructed in a similar fashion as the MNL model with endogenous network effects (see Wang and Wang 2016, Du et al. 2016). In these papers, the utility (and the purchase probability) of choosing one product is endogenously determined by the demand of each product through network effects. In our model, the utility of a customer will increase if his/her chance of being served increases. Analogously, the utility of a worker will increase if his/her chance of working for the platform increases. Ultimately, the choice behavior of a customer or a worker depends not only on the price or wage set by the platform, but also on the strategic interactions among agents on both sides on the market.

In equilibrium, each two-sided platform exhibits *positive* cross-side network effects and *negative* same-side network effects. Namely, if we add one agent (a customer or a worker) to either platform, the utility of other agents from the same side of the market will decrease; conversely, the utility of agents from the opposite side of the market will increase. This extends the framework proposed in

Wang and Wang (2016) and Du et al. (2016) which only considers positive network effects on the demand side.

The total sales of P_i are truncated by both demand and supply, that is, $\min\{d_i, s_i\}$. Thus, the profit earned by P_i is given by:

$$\pi_i(p_1, w_1, p_2, w_2) = (p_i - w_i)\min\{d_i, s_i\},$$
$$\text{where } d_i = \frac{\Lambda \exp[\min\{1, s_i/d_i\}(q_1 - p_1)]}{1 + \exp[\min\{1, s_1/d_1\}(q_1 - p_1)] + \exp[\min\{1, s_2/d_2\}(q_2 - p_2)]}$$

and

$$s_i = \frac{\exp(a_i + \min\{1, d_i/s_i\}w_i)}{1 + \exp(a_1 + \min\{1, d_1/s_1\}w_1) + \exp(a_2 + \min\{1, d_2/s_2\}w_2)}.$$

In the special case of a fixed-commission rate, each platform allocates a fixed proportion $0 < \beta < 1$ of the price paid by customers to its workers, that is, $w_i = \beta p_i$. In this case, the profit earned by P_i can be calculated as

$$\pi_i^c(p_1, p_2) = (p_i - \beta p_i)\min\{d_i, s_i\} = (1 - \beta)p_i \min\{d_i, s_i\},$$
$$\text{where } d_i = \frac{\Lambda \exp[\min\{1, s_i/d_i\}(q_i - p_i)]}{1 + \exp[\min\{1, s_1/d_1\}(q_1 - p_1)] + \exp[\min\{1, s_2/d_2\}(q_2 - p_2)]}$$

and

$$s_i = \frac{\exp(a_i + \min\{1, d_i/s_i\}\beta p_i)}{1 + \exp(a_1 + \min\{1, d_1/s_1\}\beta p_1) + \exp(a_2 + \min\{1, d_2/s_2\}\beta p_2)}.$$

3. Competition Between Two-sided Platforms: Equilibrium Analysis

Recall that P_1 and P_2 compete on both price and wage. More specifically, they engage in a simultaneous game in which P_i sets p_i and w_i to maximize $\pi_i(p_1, w_1, p_2, w_2)$. After observing the platforms' price and wage decisions, customers and workers engage in a game where every agent takes an action (i.e., decide between both platforms and the outside option) which best responds to the aggregate decisions of all other players. By the Brouwer's fixed-point theorem, given (p_1, p_2, w_1, w_2), there always exists an equilibrium (d_1, d_2, s_1, s_2) that satisfies equations (1) and (2).

In this section, we characterize the equilibrium outcome of this game which we call the *two-sided competition game*. A strategy profile of both platforms is an equilibrium, if each platform maximizes its own profit given the competitor's strategy, that is,

$$(p_i^*, w_i^*) \in \underset{(p_i, w_i)}{\arg\max}\, \pi_i(p_i, w_i, p_{-i}^*, w_{-i}^*),$$

where (p_{-i}^*, w_{-i}^*) is the equilibrium price and wage of the other platform. We also denote the equilibrium demand and supply of P_i by d_i^* and s_i^*, respectively, where $d_i^* = \frac{\Lambda \exp[\min\{1, s_i^*/d_i^*\}(q_i - p_i^*)]}{1 + \exp[\min\{1, s_1^*/d_1^*\}(q_1 - p_1^*)] + \exp[\min\{1, s_2^*/d_2^*\}(q_2 - p_2^*)]}$ and $s_i^* = \frac{\exp(a_i + \min\{1, d_i^*/s_i^*\}w_i^*)}{1 + \exp(a_1 + \min\{1, d_1^*/s_1^*\}w_1^*) + \exp(a_2 + \min\{1, d_2^*/s_2^*\}w_2^*)}$. The following theorem shows that a unique equilibrium exists and that at equilibrium, supply should match with demand.

THEOREM 1. *Consider the above two-sided competition game. Then, the following holds:*

1. *Under equilibrium, supply matches with demand, i.e., $s_i^* = d_i^*$ for $i = 1, 2$.*

2. *The two-sided competition game admits a unique equilibrium $(p_1^*, w_1^*, p_2^*, w_2^*)$. Furthermore, the equilibrium can be computed using a tatônnement scheme.*

In the two-sided competition game, if supply does not match with demand, one can always find a profitable unilateral deviation by increasing the price (when demand exceeds supply) or by decreasing the wage (when supply exceeds demand). See more details in the proof of Theorem 1 in the Appendix. Furthermore, based on the second part of Theorem 1, the equilibrium can be computed using a *tatônnement* scheme: if each platform uses the best-response strategies based on the price and wage of its competitor in the previous iteration, the sequence of price and wage strategies converge to the unique equilibrium $(p_1^*, w_1^*, p_2^*, w_2^*)$.

When establishing the existence and uniqueness of equilibrium in one-sided competition with logit type demand models without network effects (e.g., MNL, nested-MNL), existing results in the literature typically leverage the first-order optimality condition (FOC) of the profit function. A common approach is to show that the system of equations that characterize the FOC has a unique solution (see, e.g., Gallego et al. 2006, Li and Huh 2011, Gallego and Wang 2014, Aksoy-Pierson et al. 2013). In the two-sided competition game with an endogenous MNL model considered in this paper, the FOC turns out to be difficult to analyze. This is driven by the fact that the platforms have more flexibility in decisions (price and wage). Furthermore, the total sales of each platform is endogenously determined by the choice behaviors of all customers and workers in the market, and is truncated by its demand and supply. As a result, the objective function becomes non-differentiable, making traditional arguments not applicable to our model. To overcome this technical challenge, we resort to a different approach that directly exploits the structural properties of the best-response mapping of each platform. Leveraging the fact that supply must equal to demand under equilibrium, we show that the best-response mapping is increasing, which implies the existence of equilibrium by Tarski's Theorem. To prove the uniqueness of equilibrium, we first try to establish the contraction mapping property of the best-response mapping. However, this approach does not work due to the two-sidedness nature of our model. Fortunately, we find that the $k-$fold best-response mapping is a contraction under the ℓ_1 norm when k is sufficiently large (see more details in the proof of Theorem 1). Consequently, a *tatônnement* scheme converges to the unique equilibrium. An important insight of price competition under MNL or nested-MNL models from the literature is the optimality of the so-called equal or adjusted markup policy (see, Li and Huh 2011, Gallego and Wang 2014). This property, however, no longer holds in our two-sided competition setting where demand and supply are endogenized and the platforms have the flexibility to adjust both price and wage.

The proof of Theorem 1 is based on directly analyzing the best-response strategy and establishing the convergence of a *tatônnement* scheme to the unique equilibrium. Besides characterizing the equilibrium, the iterative procedure of the *tatônnement* scheme allows us to derive structural properties of the equilibrium. Specifically, we inductively characterize the desired properties of the *tatônnement* scheme in each iteration, implying that the same properties hold under equilibrium by taking the limit. We next exploit this technique to (i) compare the equilibrium outcome of our two-sided competition game to a monopoly market (i.e., both platforms are owned by a single entity) and (ii) characterize how the equilibrium strategy reacts to real-time demand changes.

We denote the prices and wages under the monopolistic setting by $(\mathbf{p}^{m*}, \mathbf{w}^{m*}) = (p_1^{m*}, w_1^{m*}, p_2^{m*}, w_2^{m*})$.

PROPOSITION 1. *We have $p_i^* < p_i^{m*}$ and $w_i^* > w_i^{m*}$ for $i = 1, 2$.*

As stated in Proposition 1, in a competitive market, each platform will decrease (resp. increase) its price (resp. wage) to attract customers (resp. workers). Traditionally, it was shown that price competition decreases the price of each firm relative to a monopoly (see, e.g., Li and Huh 2011). Proposition 1 generalizes this result to a two-sided market by showing that competition not only decreases the price, but also raises the wage of each platform. We note that the method we use to prove Proposition 1 is different from the typical argument used in the literature. In the literature, the main argument relies on analyzing the FOC (see, e.g., Li and Huh 2011), whereas in our model, we directly exploit the properties of the best response in each iteration of the *tatônnement* scheme. Specifically, we input the monopoly prices and wages as the initial variables of the *tatônnement* scheme and show that the prices (resp. wages) will be lower (resp. higher) relative to the monopolistic setting, for each iteration of the scheme. At the limit of the *tatônnement* scheme, the equilibrium prices and wages under two-sided platform competition will be higher relative to the monopoly.

Our next result characterizes how platforms adjust their prices and wages in response to demand variations under equilibrium.

PROPOSITION 2. *Under equilibrium, p_i^* and w_i^* are increasing in Λ for $i = 1, 2$.*

Consistent with the ride-sharing business practice, Proposition 2 suggests that both platforms adopt the surge pricing strategy under equilibrium, that is, they react to real-time peak demands by increasing their price and wage. This result generalizes the well-known optimality of surge pricing for a monopoly (see, e.g., Tang et al. 2017) to a competitive two-sided setting with endogenized supply and demand.

We remark that Theorem 1, Proposition 1, and Proposition 2 can be generalized to a model with $n \geq 2$ platforms. The following result generalizes Theorem 1 to $n \geq 2$ platforms.

THEOREM 2. *For the two-sided competition game with n platforms, there exists a unique equilibrium for the two-sided competition game. Furthermore, the equilibrium can be computed using a tatônnement scheme.*

The proof of Theorem 2 follows a similar argument as the proof of Theorem 1 and is briefly summarized in the Appendix. Analogously, Propositions 1 and 2 can also be extended to the model with n platforms under the endogenous two-sided MNL model.

3.1. Fixed-Commission Rate

Platforms often use a fixed-commission rate to pay their workers. Namely, they allocate a fixed share $0 < \beta < 1$ of the price paid by customers to workers, that is, $w_i = \beta p_i$ (see, e.g., Hu and Zhou 2017), where β is a pre-specified parameter that does not change with the state of the market. For example, for Lyft drivers who applied before 12 AM on January 1, 2016, they earn 80% of the passenger's time, distance, and base rates in each trip.[7] In the model with a fixed-commission rate, the equilibrium (p_1^{c*}, p_2^{c*}) is defined as follows:

$$p_i^{c*} \in \arg\max_{p_i} \pi_i^c(p_i, p_{-i}^{c*}),$$

where p_{-i}^{c*} is the equilibrium price of the other platform under a fixed-commission rate. We also denote the equilibrium demand and supply of P_i by $d_i^{c*} = \frac{\Lambda \exp[\min\{1, s_i^{c*}/d_i^{c*}\}(q_i - p_i^{c*})]}{1 + \exp[\min\{1, s_1^{c*}/d_1^{c*}\}(q_1 - p_1^{c*})] + \exp[\min\{1, s_2^{c*}/d_2^{c*}\}(q_2 - p_2^{c*})]}$ and $s_i^{c*} = \frac{\exp(a_i + \min\{1, d_i^{c*}/s_i^{c*}\}\beta p_i^{c*})}{1 + \exp(a_1 + \min\{1, d_1^{c*}/s_1^{c*}\}\beta p_1^{c*}) + \exp(a_2 + \min\{1, d_2^{c*}/s_2^{c*}\}\beta p_2^{c*})}$, respectively. For simplicity, we assume the same commission rate β for both platforms. Nevertheless, our result extends when both commission rates are different but not too far apart (using a continuity argument in the proof of Theorem 3).

THEOREM 3. *Consider the two-sided competition game under a fixed-commission rate. Then, the following holds:*

1. *Under equilibrium, supply exceeds demand, i.e., $s_i^{c*} \geq d_i^{c*}$ for $i = 1, 2$.*

2. *The two-sided competition game under a fixed-commission rate admits a unique equilibrium (p_1^{c*}, p_2^{c*}). Furthermore, the equilibrium can be computed using a tatônnement scheme.*

Note that the first part of Theorem 3 is different from Theorem 1. Indeed, under a fixed-commission rate, the platforms have less flexibility in the decision-making process since the wage is tied to the price. Consequently, the argument of finding a profitable unilateral deviation does not hold anymore when $d_i < s_i$. When $d_i > s_i$, by increasing p_i (and thus also $w_i = \beta p_i$ and the profit margin $(1 - \beta)p_i$), this will raise d_i and s_i so that P_i's profit increases. However, when $d_i < s_i$, such an approach does not necessarily increase the profit of the platform: by decreasing p_i (and thus

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[7] https://thehub.lyft.com/pay-breakdown/

w_i and the margin), d_i will increase while s_i will decrease so that the impact on P_i's profit is not clear. Consequently, as shown in the proof of Theorem 3, the model with a fixed-commission rate requires a different equilibrium analysis to carefully examine the case when $d_i < s_i$.

3.2. Separate Pools of Workers

To conclude this section, we consider a two-sided competition model where workers come from two different labor pools (e.g., Curb's taxi drivers and Via's self-employed drivers). This is different from our base model in which all workers belong to the same pool. In this case, a worker faces only two choices: working for the focal platform or choosing the outside option. We denote by K_i the total pool size of P_i's workers. As before, we normalize the utility of the outside option to 0. The expected utility earned from working for P_i is $a_i + \min\{1, d_i^s/s_i^s\}w_i$, where d_i^s and s_i^s are the total demand and supply for P_i respectively (under separate pools of workers). As before, both customers and workers follow an endogenous MNL model. Thus, the total demand and supply for P_i are given by:

$$d_i^s = \frac{\Lambda \exp[\min\{1, s_i^s/d_i^s\}(q_i - p_i)]}{1 + \exp[\min\{1, s_1^s/d_1^s\}(q_1 - p_1)] + \exp[\min\{1, s_2^s/d_2^s\}(q_2 - p_2)]}, \ i = 1, 2$$

and

$$s_i^s = \frac{K_i \exp(a_i + \min\{1, d_i^s/s_i^s\}w_i)}{1 + \exp(a_i + \min\{1, d_i^s/s_i^s\}w_i)}, \ i = 1, 2.$$

As a corollary of Theorem 1, we show that a unique equilibrium exists for the model with separate pools of workers.

COROLLARY 1. *Consider the two-sided competition game with separate pools of workers. Then, the following holds:*

  1. *Under equilibrium, supply matches with demand, i.e., $s_i^{s*} = d_i^{s*}$ for $i = 1, 2$.*

  2. *The two-sided competition game admits a unique equilibrium $(p_1^{s*}, w_1^{s*}, p_2^{s*}, w_2^{s*})$. Furthermore, the equilibrium can be computed efficiently using a tatônnement scheme or a binary search.*

4. Coopetition Between Two-sided Platforms: Model

Inspired by recent practice in ride-sharing, we model the setting where a coopetition partnership is introduced via a profit sharing contract between two platforms. In particular, the two competing platforms P_1 and P_2 collaborate and offer a new joint service, which is available to riders from either platform. As mentioned before, one such recent example is the partnership between Curb and Via with the introduction of a taxi sharing service in NYC in June 6, 2017. For the rest of this paper, we use the terms "new joint service," "new service," and "coopetition" interchangeably. Since the coopetition partnership is mainly adopted in the ride-sharing market, we refer to customers as riders and to workers as drivers in the model with coopetition.

In this section, our main inspiration is the coopetition partnership between Curb and Via. We use the superscript ~ to denote the different variables in the presence of coopetition. To be consistent with the business practice of Curb (P_1) and Via (P_2), we assume that P_1 and P_2 have separate pools of drivers. We denote the prices of the original services offered by P_1 and P_2, after introducing the new service by $\tilde{p}_1$ and $\tilde{p}_2$. The quality and price of the new service are denoted by q_n and $\tilde{p}_n$, respectively. In addition, we propose to capture the pooling effect of the new service by the parameter $\tilde{n}$. This parameter corresponds to the (average) number of customers per service (i.e., riders per ride) for the new service. If the new joint service does not offer a pooling option (i.e., only private rides), $\tilde{n} = 1$ and otherwise, $\tilde{n} > 1$ (which is the case for the Curb-Via partnership). Since the new service is a combination of the original taxi-hailing (Curb) and carpooling (Via) services, its quality q_n can be interpreted as a convex combination of the qualities of the original services, that is, $q_n = \eta q_1 + (1 - \eta) q_2$ (where $\eta \in [0, 1]$). Furthermore, as in the competition model, the utility earned by a customer or by a worker depends on the decisions of all customers and workers. Namely, in the presence of coopetition, the demand and supply of each service are also endogenous market outcomes.

Inspired by the coopetition between Curb and Via, we assume that the new service is solely provided by P_1's drivers. Nevertheless, our results and insights extend to the situation where the new service is served by workers from both platforms. Motivated from practice, we assume that Curb's drivers have no choice whether to accept requests for the new service. More generally, in most carpooling platforms, drivers need to satisfy all incoming requests. Moreover, both Curb and Via prioritize the new service in the same as they do for their original services.

The total demand for P_1's drivers is the sum of the demand of its original service and of the new service: $\tilde{\lambda}_1 := \tilde{d}_1 + \tilde{d}_n / \tilde{n}$. We denote the total demand for P_2's drivers as $\tilde{\lambda}_2 = \tilde{d}_2$. We assume that P_1's drivers are randomly assigned to either the original service or the new service. More specifically, if $\tilde{s}_1 > \tilde{\lambda}_1$, where $\tilde{s}_1$ is the total number of workers who choose to work for P_1, then P_1 has enough drivers to fulfill all demand requests. Otherwise, we assume that drivers are proportionally allocated to the original service and to the new service (considering random arrivals and a first-come-first-serve allocation). Recall that the demand of P_1's original service (resp. new service) is $\tilde{d}_1$ (resp. $\frac{\tilde{d}_n}{\tilde{n}}$). Thus, P_1 will allocate $\tilde{s}_1 \cdot \frac{\tilde{d}_1}{\tilde{\lambda}_1}$ to its original service and $\tilde{s}_n = \tilde{s}_1 \cdot \frac{\tilde{d}_n / \tilde{n}}{\tilde{\lambda}_1}$ to the new service.

As in the competition model, the utility earned by a consumer from choosing one of the services is endogenously determined by the aggregate demand and supply. The expected utility a customer derives from P_1's original service is $\tilde{u}_1 = \min\{1, \tilde{s}_1 \cdot \frac{\tilde{d}_1}{\tilde{\lambda}_1} / \tilde{d}_1\}(q_1 - \tilde{p}_1) = \min\{1, \tilde{s}_1 / \tilde{\lambda}_1\}(q_1 - \tilde{p}_1)$, from P_2's original service is $\tilde{u}_2 = \min\{1, \tilde{s}_2 / \tilde{\lambda}_2\}(q_2 - \tilde{p}_2)$, from the new service is $\tilde{u}_n = \min\{1, \tilde{s}_n \tilde{n} / \tilde{d}_n\}(q_n - \tilde{p}_n) = \min\{1, \tilde{s}_1 / \tilde{\lambda}_1\}(q_n - \tilde{p}_n)$, and from the outside option is $\tilde{u}_0 = 0$. After introducing the new

service, a customer faces four different alternatives (P_1, P_2, the new service, and the outside option) and decides according to the following endogenous MNL model:

$$\tilde{d}_i = \frac{\Lambda \exp(\tilde{u}_i)}{\exp(\tilde{u}_0) + \exp(\tilde{u}_1) + \exp(\tilde{u}_2) + \exp(\tilde{u}_n)}, \text{ for } i = 0, 1, 2, n.$$

Consequently, in the presence of coopetition, the demand for P_1's service is

$$\tilde{d}_1 = \frac{\Lambda \exp[\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_1 - \tilde{p}_1)]}{1 + \exp[\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_1 - \tilde{p}_1)] + \exp[\min\{1, \tilde{s}_2/\tilde{\lambda}_2\}(q_2 - \tilde{p}_2)] + \exp[\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_n - \tilde{p}_n)]},$$

the demand for P_2's service is

$$\tilde{d}_2 = \frac{\Lambda \exp[\min\{1, \tilde{s}_2/\tilde{\lambda}_2\}(q_2 - \tilde{p}_2)]}{1 + \exp[\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_1 - \tilde{p}_1)] + \exp[\min\{1, \tilde{s}_2/\tilde{\lambda}_2\}(q_2 - \tilde{p}_2)] + \exp[\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_n - \tilde{p}_n)]},$$

and the demand for the new joint service is

$$\tilde{d}_n = \frac{\Lambda \exp[\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_n - \tilde{p}_n)]}{1 + \exp[\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_1 - \tilde{p}_1)] + \exp[\min\{1, \tilde{s}_2/\tilde{\lambda}_2\}(q_2 - \tilde{p}_2)] + \exp[\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_n - \tilde{p}_n)]}.$$

Analogously, a driver working for P_i will earn an expected utility of $\tilde{v}_i = a_i + \min\{1, \tilde{\lambda}_i/\tilde{s}_i\}w_i$, whereas the expected utility of the outside option is $\tilde{v}_0 = 0$. Thus, the supply of P_i's drivers is

$$\tilde{s}_i = \frac{K_i \exp(\tilde{v}_i)}{\exp(\tilde{v}_0) + \exp(\tilde{v}_i)} = \frac{K_i \exp(a_i + \min\{1, \tilde{\lambda}_i/\tilde{s}_i\}w_i)}{1 + \exp(a_i + \min\{1, \tilde{\lambda}_i/\tilde{s}_i\}w_i)}.$$

As before, the utility earned by a worker is endogenously determined by the aggregate demand and supply. Note that the introduction of the new service impacts the realized utility of the platforms as well as the perceived wage of the workers. Moreover, the value of $\tilde{n}$ affects the equilibrium price, wage, and demand/supply. Note also that the wage for the new service is the same as in the original service. This follows from the fact that Curb's drivers (who fulfill the new service) are compensated according to the meter price.

We consider a *profit sharing contract* under which P_1 and P_2 split the net profit generated by the new service. More precisely, P_1 receives a fraction $\gamma_1 = \gamma \in (0, 1)$ of the profit generated by the new service, and P_2 receives $\gamma_2 = 1 - \gamma$.

Under coopetition, P_i's profit comprises two parts: (i) the profit from its original service and (ii) the profit from the new service allocated to P_i. Specifically, P_i's profit from its original service is $(\tilde{p}_i - \tilde{w}_i) \min\{\tilde{d}_i, \frac{\tilde{s}_i \tilde{d}_i}{\tilde{\lambda}_i}\} = (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i \min\left\{1, \frac{\tilde{s}_i}{\tilde{\lambda}_i}\right\}$ and P_i's profit from the new service is $\gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1) \min\left\{\frac{\tilde{d}_n}{\tilde{n}}, \tilde{s}_1 \cdot \frac{\tilde{d}_n}{\tilde{n}\tilde{\lambda}_1}\right\} = \gamma_i\left(\tilde{p}_n - \frac{\tilde{w}_1}{\tilde{n}}\right)\tilde{d}_n \min\left\{1, \frac{\tilde{s}_1}{\tilde{\lambda}_1}\right\}$. Putting everything together, the expression for the total profit earned by P_i is given by:

$$\tilde{\pi}_i(\tilde{p}_1, \tilde{w}_1, \tilde{p}_2, \tilde{w}_2) = (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i \min\left\{1, \frac{\tilde{s}_i}{\tilde{\lambda}_i}\right\} + \gamma_i\left(\tilde{p}_n - \frac{\tilde{w}_1}{\tilde{n}}\right)\tilde{d}_n \min\left\{1, \frac{\tilde{s}_1}{\tilde{\lambda}_1}\right\}, \text{ for } i = 1, 2.$$

5. Impact of Coopetition

In this section, we analyze the impact of coopetition (i.e., introducing the new joint service). We first consider the profit implications on both platforms, and then examine the impact on riders and drivers. We next show that even in the presence of coopetition, there still exists a unique equilibrium. The sequence of events unfolds as follows:

1. Both platforms agree upon the price of the new service $\tilde{p}_n$ and the profit-sharing parameter γ (see more details below).

2. Given $(\tilde{p}_n, \gamma)$, each platform simultaneously decides the price and wage of its original service $\tilde{p}_i$ and $\tilde{w}_i$ to maximize its own profit.

Using backward induction, we start by characterizing the equilibrium of the second step. Given $(\tilde{p}_n, \gamma)$, the platforms engage in a price and wage competition using the model presented in Section 4. As before, the equilibrium outcome $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ should satisfy: $(\tilde{p}_i^*, \tilde{w}_i^*) \in \arg\max_{(p_i, w_i)} \tilde{\pi}_i(p_i, w_i, p_{-i}^*, w_{-i}^*)$. We next extend the result on existence and uniqueness of equilibrium in the presence of coopetition. Recall that the supply and demand of P_i are denoted by $\tilde{s}_i$ and $\tilde{\lambda}_i$, respectively.

THEOREM 4. *Consider the two-sided competition game in the presence of coopetition. Then, the following holds:*

1. *Under equilibrium, supply matches with demand for each platform, i.e., $\tilde{s}_i^* = \tilde{\lambda}_i^*$ for $i = 1, 2$.*

2. *For any $(\tilde{p}_n, \gamma)$, there exists a unique equilibrium $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ that can be computed using a tatônnement scheme or a binary search.*

Note that when $\tilde{p}_n \uparrow +\infty$, the model with coopetition converges to the original model (without coopetition) for any $\gamma \in (0, 1)$. We also remark that one can construct examples of coopetition partnerships that are detrimental to both platforms. In other words, if the platforms do not carefully decide $\tilde{p}_n$ and γ, introducing the new service may lead to an undesirable lose-lose outcome.

5.1. Impact on Platforms' Profits

At a high level, the coopetition will induce three effects: (i) a new market share effect (i.e., capturing new riders who were previously choosing the outside option), (ii) a cannibalization effect (i.e., losing some existing market share to the new service), and (iii) a wage variation (i.e., adapting the wage to match supply with demand). Our goal is to study how the platforms could use well-designed profit-sharing contracts to balance these effects and benefit from coopetition. We first show that when the price of the new service $\tilde{p}_n$ and the profit-sharing parameter γ are carefully chosen, coopetition will increase the profits of both platforms.

To unlock the largest potential of coopetition, we consider the case where $\tilde{p}_n$ is jointly set by both platforms to maximize the total profit, that is,

$$\tilde{p}_n^* \in \arg\max_{\tilde{p}_n}\{\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) + \tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)\}.$$

Since the equilibrium outcome $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ depends on γ, $\tilde{p}_n^*$ will also depend on γ.

THEOREM 5. *If $\tilde{p}_n$ is set to maximize the total profit of both platforms, then there exists an interval $(\underline{\gamma}, \bar{\gamma}) \subset (0,1)$ such that if $\gamma \in (\underline{\gamma}, \bar{\gamma})$, $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_i(p_1^*, w_1^*, p_2^*, w_2^*)$ for $i = 1, 2$.*

Theorem 5 implies that in the presence of coopetition, P_1 and P_2 can set $\tilde{p}_n$ and γ so that introducing the new service will lead to a profit increase for each platform. As mentioned, when the terms of the coopetition (i.e., $\tilde{p}_n$, γ) are not carefully designed, introducing the new service can yield lower profits for each platform. Recall that the coopetition induces three effects: (i) a new market share, (ii) an adverse cannibalization, and (iii) a wage variation. Theorem 5 shows that under a well-designed profit sharing contract, the new market share effect dominates the cannibalization and wage variation effects for each platform. We will discuss in greater detail the implications of these effects at the end of this section.

We next elaborate on how $(\tilde{p}_n, \gamma)$ can be determined. In practice (e.g., the Curb-Via partnership), P_1 and P_2 negotiate to decide the values of $\tilde{p}_n$ and γ. To model the negotiation process, we use the Nash bargaining framework (see, e.g., Nash Jr 1950, Osborne and Rubinstein 1990). We define $\theta_i \in (0,1)$ as the bargaining power of P_i ($i = 1, 2$) with $\theta_1 + \theta_2 = 1$. Then, the equilibrium price and profit-sharing parameter $(\tilde{p}_n^{**}, \gamma^{**})$ satisfy:

$$(\tilde{p}_n^{**}, \gamma^{**}) \in \arg\max_{\tilde{p}_n \geq 0, \gamma \in (0,1)} [\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) - \pi_1(p_1^*, w_1^*, p_2^*, w_2^*)]^{\theta_1} \cdot [\tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) - \pi_2(p_1^*, w_1^*, p_2^*, w_2^*)]^{\theta_2}$$

$$\text{s.t. } \tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) \geq \pi_i(p_1^*, w_1^*, p_2^*, w_2^*) \text{ for } i = 1, 2. \tag{3}$$

We know from Theorem 5 that there exists $(\tilde{p}_n, \gamma)$ such that $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_i(p_1^*, w_1^*, p_2^*, w_2^*)$ for $i = 1, 2$. Thus, for any (θ_1, θ_2), $(\tilde{p}_n^{**}, \gamma^{**})$ is well defined. We next show that the equilibrium profit of each platform under Nash bargaining increases under coopetition.

PROPOSITION 3. *Under Nash bargaining, that is, when the platforms set $(\tilde{p}_n^{**}, \gamma^{**})$, we have $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_i(p_1^*, w_1^*, p_2^*, w_2^*)$ for $i = 1, 2$.*

So far, we have shown that the platforms can set the price of the new service and the profit-sharing parameter to ensure that coopetition is beneficial. Nevertheless, we are interested in avoiding extreme cases and in identifying conditions under which coopetition yields a strict benefit for both platforms in the presence of price constraint for the new service. As mentioned, it is always possible to set $\tilde{p}_n$ to a large value so that no customer will opt for the new service, and we are back to the original setting. The following proposition shows that when the value of $\tilde{p}_n$ is bounded (i.e., the platforms cannot set the price of the new service arbitrarily high), both platforms will be strictly better off only when the demand-supply ratio (captured by Λ) is not too high.

PROPOSITION 4. *The following statements hold:*

1. *If $\Lambda \uparrow +\infty$, then $\tilde{p}_n^* \uparrow +\infty$ and $\tilde{p}_n^{**} \uparrow +\infty$.*

2. *Assume that there is an upper bound on the price of the new service, that is, $\tilde{p}_n \leq \bar{p}$ for some $\bar{p} < +\infty$. There exists a threshold $\bar{\Lambda}(\bar{p})$ such that (i) if $\bar{p}$ is sufficiently large and $\Lambda < \bar{\Lambda}(\bar{p})$, then $\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_1(p_1^*, w_1^*, p_2^*, w_2^*)$ and $\tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_2(p_1^*, w_1^*, p_2^*, w_2^*)$ for some $(\tilde{p}_n, \gamma)$ with $\tilde{p}_n \leq \bar{p}$, and (ii) if $\Lambda > \bar{\Lambda}(\bar{p})$, then either $\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < \pi_1(p_1^*, w_1^*, p_2^*, w_2^*)$ or $\tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < \pi_2(p_1^*, w_1^*, p_2^*, w_2^*)$ for any $(\tilde{p}_n, \gamma)$ with $\tilde{p}_n \leq \bar{p}$.*

Proposition 4 shows that, when $\tilde{p}_n$ is bounded, the demand-supply ratio has a crucial implication on the impact of coopetition on the platforms' profits. specifically, if the demand-supply ratio is not too high (i.e., $\Lambda < \bar{\Lambda}(\bar{p})$), the platforms can design a profit sharing contract (by setting $\tilde{p}_n$ and γ) that will make the coopetition partnership strictly beneficial for both platforms (i.e., a Pareto improvement in both profits). However, if the demand-supply ratio becomes too high, at least one of the platforms will be hurt by coopetition (assuming $\tilde{p}_n$ is bounded). In this case, introducing the new service will make (at least) one of the platforms over-demanded. This will in turn induce the platform(s) to increase their wage, and hence reduce profit. On the other hand, when the demand-supply ratio is not too high, introducing the new service expands the market share of both platforms, thus increasing revenues without imposing high additional wages.

We next revisit the three effects induced by the coopetition and discuss how a well-designed profit sharing contract can help balance these effects to benefit both platforms.

New market share. The new service may attract customers who would otherwise leave the market. Mathematically, the new market share effect for P_i can be quantified as the profit portion generated by the new service which is allocated to P_i, that is, $\frac{\gamma_i \tilde{n} \tilde{p}_n \tilde{d}_n^*}{\tilde{n}} = \gamma_i \tilde{p}_n \tilde{d}_n^*$. It can be shown that the new market share effect for P_i is quasi-concave in $\tilde{p}_n$ and increasing in γ_i.

Cannibalization. Introducing the new service will also cannibalize demand since customers may switch from the original services to the new one. The cannibalization effect for P_i is captured by $p_i^* d_i^* - \tilde{p}_i^* \tilde{d}_i^*$, which is always positive unless $\tilde{p}_n = +\infty$. We can show that the cannibalization effect is decreasing in $\tilde{p}_n$, as expected.

Wage variation. To match supply with demand in the presence of coopetition (which is the equilibrium condition), the platforms will adjust their wages. While the new market share effect is beneficial and the cannibalization effect is harmful to the platforms, the wage variation effect may go either way. Specifically, the wage variation for P_i is $\tilde{w}_i^* \tilde{d}_i^* + \gamma_i \tilde{w}_1^* \frac{\tilde{d}_n^*}{\tilde{n}} - w_i^* d_i^*$. We can show that the wage variation effect shrinks to zero as $\tilde{p}_n \uparrow +\infty$ and is strengthened as γ_i increases.

To summarize, our results show that a well-designed profit sharing contract can successfully balance these effects and lead to an overall positive benefit for both platforms, regardless of whether the coopetition parameters (p_n, γ) are jointly set by both platforms or determined through bargaining. We next turn our attention to the impact of coopetition on riders and drivers.

5.2. Surpluses of Riders and Drivers

We investigate the impact of coopetition on riders and drivers. Note that the surpluses of riders and drivers are not (explicitly) dependent on the profit sharing parameter γ. We use RS to denote the expected rider surplus of the benchmark setting (i.e., without coopetition):

$$RS = \Lambda \mathbb{E}\Big[\max\{\min\{1, s_1/d_1\}(q_1 - p_1) + \xi_1, \min\{1, s_2/d_2\}(q_2 - p_2) + \xi_2, \xi_0\}\Big].$$

Let $\tilde{RS}$ denote the expected rider surplus after introducing the new service:

$$\tilde{RS} = \Lambda \mathbb{E}\Big[\max\{\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_1 - \tilde{p}_1) + \xi_1, \min\{1, \tilde{s}_2/\tilde{\lambda}_2\}(q_2 - \tilde{p}_2) + \xi_2, \min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_n - \tilde{p}_n) + \xi_n, \xi_0\}\Big].$$

Note that the rider surpluses, RS and $\tilde{RS}$, are unique up to an additive constant. Indeed, for any rider, if the random utility terms $(\xi_1, \xi_2, \xi_n, \xi_0)$ are shifted to $(\xi_1 + c, \xi_2 + c, \xi_n + c, \xi_0 + c)$ for any constant c, then the probabilities that this rider will choose any of the four alternatives (P_1, P_2, the new service, and the outside option) remain the same. Nevertheless, the change in the expected rider surplus generated by introducing the new service, $\tilde{RS}(\tilde{p}_n, \tilde{p}_1, \tilde{p}_2) - RS(p_1, p_2)$, is independent of c. One can derive the following expressions: $RS = \log[1 + \exp(\min\{1, s_1/d_1\}(q_1 - p_1)) + \exp(\min\{1, s_2/d_2\}(q_2 - p_2))] + c$ and $\tilde{RS} = \log[1 + \exp(\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_1 - \tilde{p}_1)) + \exp(\min\{1, \tilde{s}_2/\tilde{\lambda}_2\}(q_2 - \tilde{p}_2)) + \exp(\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_n - \tilde{p}_n))] + c$. For more details on consumer surplus under the MNL model and on the derivation of the above expressions, see, e.g., Chapter 3.5 of Train (2009). We use $\tilde{RS}^*$ and RS^* to denote the equilibrium rider surplus with and without coopetition, respectively.

PROPOSITION 5. *For any $(\tilde{p}_n, \gamma)$, $\tilde{RS}^* > RS^*$.*

Proposition 5 shows that introducing the new service will increase the expected rider surplus, regardless of the price of the new service and of the profit-sharing parameter. This result is expected as riders can now enjoy an additional alternative of service.

We next examine the impact of coopetition on drivers, which appears to be more subtle. Since P_1 and P_2 have separate pools of drivers, we need to evaluate the effect of coopetition separately. Let DS_i denote the expected surplus of P_i's drivers before the coopetition partnership:

$$DS_i = \mathbb{E}\Big[\max\{a_i + \min\{1, d_i/s_i\}w_i + \eta_i, \eta_0\}\Big], \ i = 1, 2.$$

Analogously, we define the expected surplus of P_i's drivers with coopetition:

$$\tilde{DS}_i = \mathbb{E}\Big[\max\{a_i + \min\{1, \tilde{\lambda}_i/\tilde{s}_i\}\tilde{w}_i + \eta_i, \eta_0\}\Big], \ i = 1, 2.$$

Finally, $\tilde{DS}_i^*$ and DS_i^* denote the equilibrium surplus of P_i's drivers with and without coopetition, respectively.

PROPOSITION 6. *For any* $(\tilde{p}_n, \gamma)$, *the following holds:*

1. *There exists a threshold* $\bar{n}_d > 1$ *such that* $\tilde{DS}_1^* > DS_1^*$ *if and only if* $\tilde{n} < \bar{n}_d$.

2. $\tilde{DS}_2^* < DS_2^*$.

As shown in Proposition 6, P_1's drivers may not necessarily benefit from coopetition. When the average number of riders per trip for the new service is not too high (i.e., $\tilde{n} < \bar{n}_d$), there exist profit sharing contracts that will strictly benefit P_1's drivers. Indeed, when $\tilde{n}$ is small, the platform needs to increase its wage to attract additional P_1's drivers to satisfy the demand of the new service. When $\tilde{n}$ is large, however, P_1's drivers will be worse off in the presence of coopetition. In this case, fewer drivers are needed, so that the platform can reduce its wage. This finding explains partially why several coopetition partnerships either have no carpooling option for the new service (i.e., $\tilde{n} = 1$) or impose a restriction on the number of riders per trip. For example, in the case of Curb and Via, the platforms imposed a limit of at most two riders who can share a ride for the new taxi-sharing service (i.e., $\tilde{n} \leq 2$). Note that when $\tilde{n} = 1$, P_1's drivers will always benefit from coopetition. Proposition 6 also shows that introducing the new service will always decrease the surplus of P_2's drivers. This follows from the fact that P_2's drivers are directly affected by the market share reduction ($\tilde{s}_2^* < s_2^*$) induced by the cannibalization effect.

We next propose a simple and realistic way to address the issue that some drivers may be hurt by coopetition. In particular, P_1 and P_2 can reallocate some of their profit gains to their drivers. In practice, the incentives are provided to the drivers through promotions/bonuses or other monetary compensations. For example, Grab began to subsidize trip fares on June 19, 2018, to ensure decent driver earnings.[8] In fact, bonuses are widely used in practice as a competitive lever for two-sided platforms (see, e.g., Liu et al. 2019). We denote the total platform and driver surplus of P_i with and without coopetition as $\tilde{\pi}_i + \tilde{DS}_i$ and $\pi_i + DS_i$, respectively. We next show that the platforms can reach an agreement on $(\tilde{p}_n, \gamma)$ that will guarantee a strict total surplus gain for each platform.

PROPOSITION 7. *For any* $\tilde{n}$, *there exist* $(\tilde{p}_n, \gamma)$ *such that under equilibrium,* $\tilde{\pi}_i^* + \tilde{DS}_i^* > \pi_i^* + DS_i^*$ *for* $i = 1, 2$.

Proposition 7 shows that under a well-designed profit sharing contract, the coopetition partnership can strictly increase the total surplus (i.e., platform's profit and driver surplus) of each platform. Consequently, if each platform redistributes a portion of its profit gain to its drivers (e.g., by offering bonuses), both platforms and all the drivers will be better off. This will also result in more drivers joining each platform. Ultimately, the platforms can establish a profit sharing contract $(\tilde{p}_n, \gamma)$ that will benefit every party in the market (i.e., both platforms, riders, and drivers).

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[8] https://www.rappler.com/business/205322-grab-philippines-drivers-subsidy-fares-earnings

5.3.   Computational Experiments

We investigate computationally how three market features affect the impact of coopetition between ride-sharing platforms: (a) Product differentiation, measured by q_1/q_2, (b) Demand-supply ratio of P_2, measured by $\Lambda/(n_2 K_2)$ (n_2 is the number of riders per trip for P_2's original service), and (c) The expected number of riders per trip in the new joint service, $\tilde{n}$. To this end, we set $q_2 = 1$ and vary q_1 so that $q_1/q_2 \in \{1.1, 1.4, 1.8, 2.2, 2.6, 3\}$. To illustrate the coopetition partnership between Curb (P_1) and Via (P_2), we assume that K_1 is very large, and that the wage per trip for P_1's drivers is normalized to $w_1 = 1$. This is consistent with the business practice that Curb has abundant taxi drivers, and that the wage of taxi drivers is determined by the meter price. Recall that the new service is fulfilled exclusively by P_1's drivers. For the original taxi-hailing service of Curb, the average riders per trip is $n_1 = 1$. We fix $n_2 = 3$, $K_2 = 500$, and vary Λ so that $\Lambda/(n_2 k_2) \in \{0.5, 1, 1.5, 2, 5, 7\}$. Finally, we consider several values of $\tilde{n} \in \{1, 1.3, 1.7, 2, 2.5, 3\}$. Note that $\tilde{n} = 1$ is the extreme case in which there is no carpooling for the new service. Recall that the Via-Curb partnership is such that $\tilde{n} \leq 2$. However, we still consider the case where $\tilde{n}$ can be larger than 2 to test the robustness of our results. Note that the set of parameters used in this section encompasses a wide range of realistic instances and hence, this allows us to quantify the practical impact of the coopetition partnership.

It is natural to assume that the quality of the new service, q_n, increases with q_1 and decreases with $\tilde{n}$. To capture this behavior, we use $q_n = q_2 + (q_1 - q_2)(n_2 + 1 - \tilde{n})/n_2$. Note that $q_n = q_1$ when $\tilde{n} = 1$ (in this case, the new service is equivalent to P_1's original service) and q_n is slightly larger than q_2 when $\tilde{n} = n_2$ (in this case, the new service is slightly better than P_2's original service). For all problem instances, we use a profit sharing contract with $(\tilde{p}_n^{**}, \gamma^{**})$.

Table 1 summarizes the impact of coopetition on P_1, P_2, drivers, and riders for the problem instances discussed above. We compute the relative impact of introducing the new service for each party. For example, the relative profit difference of P_i ($i = 1, 2$) is given by: $\Delta \pi_i / \pi_i = [\tilde{\pi}_i(\tilde{p}_n^*, p_1^*, p_2^* | \gamma^*) - \pi_i(p_1^*, p_2^*)]/\pi_i(p_1^*, p_2^*)$.[10] Our computational tests convey that for the parameter values we consider, introducing the new service will in general substantially benefit all stakeholders. In particular, we can see from Table 1 that the average relative profit improvements for P_1 and P_2 are 25.38% and 23.45% respectively (and even in the worst case instances, the relative improvements amount to 13.37% and 13.13%). The average benefits for drivers and riders seem also to be significant. The only exception is a slight decrease in the expected surplus of P_1's drivers when $\tilde{n} > 2$ (i.e., every trip is shared by more than 2 riders on average) and q_1/q_2 is large. In this case, one can see from Table 1 that the surplus of P_1's drivers can be reduced by 3.78% in the worst case

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[10] Since the rider surplus is unique up to an additive constant (see Chapter 3.5 of Train 2009), we report here the absolute (instead of the relative) differences for the expected rider surplus. The same comment applies to Tables 2-4.

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(this occurs when $q_1/q_2 = 3$ and $\tilde{n} = 3$). This is consistent with Proposition 6 that shows that if $\tilde{n}$ is large, P_1's drivers are negatively affected by coopetition. However, as shown in Proposition 7, P_1 can redistribute a portion of its profit gain to its drivers so that the coopetition will benefit both the platform and its drivers.

Table 1   Summary statistics of the impact of coopetition (%)

	Average	Min	25th Percentile	Median	75th Percentile	Max
$\Delta\pi_1/\pi_1$	25.38	13.37	22.76	25.00	27.75	42.17
$\Delta\pi_2/\pi_2$	23.45	13.13	21.32	23.82	25.45	35.99
$\Delta DS_1/DS_1$	17.22	-3.78	5.94	14.85	27.36	48.02
$\Delta(\pi_2 + DS_2)/(\pi_2 + DS_2)$	20.30	9.38	18.37	20.96	22.69	27.93
ΔRS	1429.35	143.49	440.28	736.67	2627.44	4669.56

In Tables 2, 3, and 4, we report the average values of the relative impact when a single parameter is varied and the other two are set to specific values. This allows us to isolate the impact of a single market feature. One can see that in all cases, all surpluses are increasing, suggesting that everyone benefits from the introduction of the new service. In Table 2, we study the effect of quality differentiation. We observe that as q_1/q_2 increases, the impact on the profits earned by both platforms is quite stable (the relative improvement remains around 20-30%). On the other hand, increasing the quality ratio will hurt P_1's drivers which will benefit less from coopetition. In Table 3, we study the effect of the demand-supply ratio. When increasing $\Lambda/(n_2 k_2)$, the impact on the profits of both platforms and on drivers are quite stable, while riders will benefit more from coopetition. This follows from the fact that under high demand, introducing a new alternative will yield a larger benefit to riders, as expected. In Table 4, we examine the effect of the expected number of riders per trip in the new service. In this case, increasing $\tilde{n}$ does not have a significant impact on profits, on P_2's drivers, and on riders. However, it has a strong effect on P_1's drivers, who exclusively serve the new service. In summary, even though the impact of the coopetition partnership may be sensitive to different market conditions, it seems to be beneficial for all parties (both platforms, riders, and drivers) in the vast majority of instances we considered.

We observe in Tables 2-4 that there is not a clear monotonicity pattern, as when we vary a single parameter, the profit sharing parameter γ^{**} changes as well (since it is endogenously determined).

Table 2   Impact of the service quality ratio q_1/q_2 when $\Lambda/(n_2 k_2) = 5$ and $\tilde{n} = 2$ (%)

q_1/q_2	1.1	1.4	1.8	2.2	2.6	3
$\Delta\pi_1/\pi_1$	34.67	34.07	32.33	29.52	26.27	22.78
$\Delta\pi_2/\pi_2$	29.92	29.82	28.56	26.45	23.77	21.02
$\Delta DS_1/DS_1$	25.55	19.87	13.68	8.77	5.00	2.10
$\Delta(\pi_2 + DS_2)/(\pi_2 + DS_2)$	24.04	23.92	22.68	20.83	18.44	16.13
ΔRS	2817.74	2826.93	2774.94	2688.51	2569.04	2438.02

Table 3   **Impact of the demand-supply ratio $\Lambda/(n_2 k_2)$ when $q_1/q_2 = 2$ and $\tilde{n} = 2$ (%)**

$\Lambda/(n_2 k_2)$	0.5	1	1.5	2	5	7
$\Delta\pi_1/\pi_1$	24.69	25.80	26.85	27.81	32.33	34.52
$\Delta\pi_2/\pi_2$	24.20	24.89	25.49	26.05	28.56	29.73
$\Delta DS_1/DS_1$	14.17	14.14	14.05	14.01	13.68	13.51
$\Delta(\pi_2 + DS_2)/(\pi_2 + DS_2)$	23.35	23.26	23.18	23.08	22.68	22.49
ΔRS	220.21	456.14	708.88	972.93	2774.94	4123.17

Table 4   **Impact of $\tilde{n}$ when $q_1/q_2 = 2$ and $\Lambda/(n_2 k_2) = 5$ (%)**

$\tilde{n}$	1	1.3	1.7	2	2.5	3
$\Delta\pi_1/\pi_1$	26.82	30.28	32.10	32.33	31.51	29.94
$\Delta\pi_2/\pi_2$	23.34	26.61	28.38	28.56	27.73	26.13
$\Delta DS_1/DS_1$	38.56	29.03	19.15	13.68	7.21	2.96
$\Delta(\pi_2 + DS_2)/(\pi_2 + DS_2)$	18.39	21.07	22.53	22.68	21.99	20.67
ΔRS	2358.98	2622.90	2759.60	2774.94	2712.94	2591.45

6. Extension: Endogenous Waiting Times

We extend our model by explicitly considering a key feature of ride-sharing platforms: the waiting time experienced by riders. We assume that the expected waiting time depends on the number of available drivers. Specifically, the expected waiting time for P_i (without coopetition) is given by:

$$T_i = \kappa(s_i - d_i),$$

where $s_i - d_i$ is the number of available (or idle) drivers, and $\kappa(\cdot) > 0$ is a strictly decreasing and convex function on $(0, +\infty)$ with $\lim_{x \downarrow 0} \kappa(x) = +\infty$ and $\lim_{x \uparrow +\infty} \kappa(x) = 0$.[9] Note that this includes as special cases the $M/M/k$ queuing system and the situation where idle drivers are uniformly distributed on a circle so that the expected travel time to pick up a new rider is $c/(s_i - d_i)$ for some constant $c > 0$. Similar modeling approaches have been used in the literature on ride-sharing platforms (see, e.g., Tang et al. 2017, Benjaafar et al. 2018, Nikzad 2018).

Following a similar approach as Cachon and Harker (2002), we assume that the platforms compete on the *total price*, $f_i = p_i + g_i$, where g_i is the operational performance of P_i, which we define as $g_i = T_i$. Hence, the actual price charged by P_i to its riders is $p_i = f_i - g_i = f_i - \kappa(s_i - d_i)$.

Since $\kappa(\cdot)$ satisfies $\lim_{x \downarrow 0} \kappa(x) = +\infty$, we must have $s_i > d_i$ under equilibrium, that is, $\min\{s_i, d_i\} = d_i$. Thus, the profit earned by P_i when waiting times are endogenous is

$$\pi_i^e(f_1, w_1, f_2, w_2) = [f_i - \kappa(s_i - d_i) - w_i]d_i,$$

where $d_i = \frac{\Lambda \exp[\min\{1, s_i/d_i\}(q_i - f_i)]}{1 + \exp[\min\{1, s_1/d_1\}(q_1 - f_1)] + \exp[\min\{1, s_2/d_2\}(q_2 - f_2)]} = \frac{\Lambda \exp(q_i - f_i)}{1 + \exp(q_1 - f_1) + \exp(q_2 - f_2)}$, $s_i = \frac{\exp[\min\{1, d_i/s_i\}(a_i + w_i)]}{1 + \exp[\min\{1, d_1/s_1\}(a_1 + w_1)] + \exp[\min\{1, d_2/s_2\}(a_2 + w_2)]}$,   and (s_i, d_i) satisfy $s_i > d_i$. An equilibrium $(f_1^{e*}, f_2^{e*}, w_1^{e*}, w_2^{e*})$ should then satisfy

$$(f_i^{e*}, w_i^{e*}) \in \underset{(f_i, w_i)}{\arg\max} \ \pi_i^e(f_i, w_i, f_{-i}^{e*}, w_{-i}^{e*}).$$

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[9] The expected waiting time is defined only when $s_i > d_i$, as otherwise, the system is not stable.

We next extend Theorem 1 to the setting with endogenous waiting times.

THEOREM 6. *The two-sided competition game with endogenous waiting times admits a unique equilibrium* $(f_1^{e*}, f_2^{e*}, w_1^{e*}, w_2^{e*})$ *that can be computed using a tatônnement scheme.*

As in the original setting, we can study the impact of coopetition for the model with endogenous waiting times. We denote the total prices of the original services offered by P_1 and P_2 after introducing the new service by $\tilde{f}_1$ and $\tilde{f}_2$, respectively. We also denote by $\tilde{f}_n$ the total price of the new service. Under coopetition, P_i's demand is $\tilde{d}_i = \Lambda \tilde{d}_i'$, where (since $\tilde{d}_i < \tilde{s}_i$)

$$\tilde{d}_i' = \frac{\exp(q_i - \tilde{f}_i)}{1 + \exp(q_1 - \tilde{f}_1) + \exp(q_2 - \tilde{f}_2) + \exp(q_n - \tilde{f}_n)}.$$

Recall that the total requests for P_1 and P_2 drivers are $\tilde{\lambda}_1 = \tilde{d}_1 + \tilde{d}_n/\tilde{n}$ and $\tilde{\lambda}_2 = \tilde{d}_2$, respectively. Since the number of idle drivers in P_i is $\tilde{s}_i - \tilde{\lambda}_i > 0$, the expected waiting time on this platform is $\kappa(\tilde{s}_i - \tilde{\lambda}_i)$. Consequently, the actual price charged by P_i for its original service under coopetition is $\tilde{p}_i = \tilde{f}_i - \kappa(\tilde{s}_i - \tilde{d}_i)$. Note that the expected waiting time of a customer who requests the new service is the same as in the original P_1's service, that is, $\kappa(\tilde{s}_1 - \tilde{\lambda}_1)$. As before, the actual price of the new service is the difference between the full price and the expected waiting time: $\tilde{p}_n = \tilde{f}_n - \kappa(\tilde{s}_1 - \tilde{\lambda}_1)$. We can now write P_i's profit as

$$\begin{aligned}
\tilde{\pi}_i^e(\tilde{f}_1, \tilde{w}_1, \tilde{f}_2, \tilde{w}_2) &= (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i + \gamma_i(\tilde{p}_n - \tilde{w}_1)\tilde{d}_n \\
&= [\tilde{f}_i - \kappa(\tilde{s}_i - \tilde{d}_i) - \tilde{w}_i]\tilde{d}_i + \gamma_i\left[\tilde{f}_n - \kappa(\tilde{s}_1 - \tilde{\lambda}_1) - \frac{\tilde{w}_1}{\tilde{n}}\right]\tilde{d}_n,
\end{aligned}$$

where $\tilde{d}_i = \frac{\Lambda \exp(q_i - \tilde{f}_i)}{1 + \exp(q_1 - \tilde{f}_1) + \exp(q_2 - \tilde{f}_2) + \exp(q_n - \tilde{f}_n)}$, $\tilde{s}_i = \frac{\exp(a_i + \tilde{w}_i)}{1 + \exp(a_1 + \tilde{w}_1) + \exp(a_2 + \tilde{w}_2)}$, and $(\tilde{s}_i, \tilde{\lambda}_i)$ satisfy $\tilde{s}_i > \tilde{\lambda}_i$ for $i = 1, 2$.

As in the model without coopetition, the platforms first jointly decide $\tilde{f}_n$ and γ. They then engage in a competition game to maximize their profits by setting the equilibrium $(\tilde{f}_1^{e*}, \tilde{w}_1^{e*}, \tilde{f}_2^{e*}, \tilde{w}_2^{e*})$, which satisfies $(\tilde{f}_i^{e*}, \tilde{w}_i^{e*}) \in \arg\max_{(f_i, w_i)} \tilde{\pi}_i^e(f_i, w_i, \tilde{f}_{-i}^{e*}, \tilde{w}_{-i}^{e*})$. We can show the existence and uniqueness of equilibrium in the model under coopetition with endogenous waiting times. Furthermore, all the results of Section 5 also extend to this model (the proofs are omitted for conciseness).

7. Conclusions

The ubiquity of two-sided platforms has increased significantly over the past few years. These platforms compete not only for customers but also for flexible workers. In the first part of this paper, we study the problem of competition between two-sided platforms. We propose to model this problem using an endogenous Multinomial Logit (MNL) choice model that accounts for network effects across both sides of the market. In our model, the behavior of a customer or a worker depends not only on the price or wage set by the platform, but also on the strategic interactions among

agents on both sides of the market. The two-sidedness nature of our setting makes the objective function non-differentiable, and hence traditional arguments from the literature are not applicable. Instead, we use an approach based on analyzing the best-response strategy to characterize the equilibrium. We ultimately show the existence and uniqueness of equilibrium.

Recently, several coopetition partnerships emerged in the ride-sharing industry. Examples include Curb and Via in NYC and Uber and PT Express in Indonesia. The second part of this paper is motivated by such partnerships that can be implemented via a profit sharing contract. It is not clear a-priori whether the competing platforms will benefit from coopetition. We present a rigorous analysis to show that—when properly designed (e.g., using the Nash bargaining framework)—such coopetition partnerships are beneficial for both platforms. We convey that riders and drivers can also benefit from coopetition. In summary, our results suggest that when the coopetition terms are carefully designed, it will benefit every party (both platforms, riders, and drivers).

This paper is among the first to propose a tractable model to study competition and partnerships in the ride-sharing industry. It allows us to draw practical insights on the impact of some recent partnerships observed in practice. Several interesting extensions are left for future research. For example, what is the long-term impact of such partnerships? Shall the platforms consider more complicated contracts such as two-part piecewise linear agreements (i.e., allowing two different profit portions depending on the scale of the new service)? A second direction for future research is to study an alternative form of coopetition, known as joint ownership of a subsidiary. For example, Uber and a Russian taxi-sharing platform Yandex.Taxi merged their businesses in Russia under a new company.[11] It could be interesting to compare the two different forms of coopetition.

Acknowledgments

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[11] https://www.nytimes.com/2017/07/13/technology/uber-russia-yandex.html

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Appendix A: Summary of Notation

Table 5   Summary of Notation

P_1 :	Platform 1
P_2 :	Platform 2
q_i :	Perceived quality of Platform i $(i = 1, 2)$
q_n :	Perceived quality of the new joint service
p_i :	Price of P_i without the new joint service
$\tilde{p}_i$:	Price of P_i with the new joint service
$\tilde{p}_n$:	Price of the new joint service
Λ :	Total customer arrival rate
d_i :	Customer arrival rate of P_i without the new joint service
$\tilde{d}_i$:	Customer arrival rate of P_i with the new joint service
$\tilde{d}_n$:	Customer arrival rate of the new joint service
a_i :	Attractiveness of Platform i
K :	Total number of workers on the market, normalized to 1
w_i :	Wage of P_i's workers
s_i :	Number of workers working for P_i without the new joint service
$\tilde{s}_i$:	Number of workers working for P_i with the new joint service
γ :	Fraction of profit generated by the new joint service allocated to P_1
$\tilde{\lambda}_i$:	Total number of workers needed by P_i (with coopetition)
β :	Fixed share of the price allocated to workers under a fixed-commission rate
$\tilde{n}$:	Number of customers per service for the new joint service

Appendix B: Proof of Statements

Auxiliary Lemma

Before presenting the proofs of our results, we state and prove an auxiliary lemma which is extensively used throughout this Appendix.

LEMMA 1. *Define* $\bar{d}_i := \frac{\exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)}$. *We have:* $\partial_{p_i} \bar{d}_i = -(1 - \bar{d}_i)\bar{d}_i$ *and* $\partial_{p_j} \bar{d}_i = \bar{d}_i \bar{d}_j$ *($i = 1, 2$ and $j \neq i$).*

Proof. Since $\bar{d}_i = \frac{\exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)}$, we have:

$$\partial_{p_i} \bar{d}_i = \frac{-\exp(q_i - p_i)[1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)] + [\exp(q_i - p_i)]^2}{[1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)]^2}$$

$$= -\frac{\exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)} + \left(\frac{\exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)} \right)^2 = -d_i + (\bar{d}_i)^2 = -(1 - \bar{d}_i)\bar{d}_i$$

and

$$\partial_{p_j} \bar{d}_i = \frac{\exp(q_i - p_i) \exp(q_j - p_j)}{[1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)]^2}$$

$$= \frac{\exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)} \times \frac{\exp(q_j - p_j)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)} = \bar{d}_i \bar{d}_j. \quad \square$$

Proof of Theorem 1

We first introduce some notations that will prove useful in our analysis. Given the competitor's strategy (p_{-i}, w_{-i}), we define $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ as P_i's best price and wage responses. We also define the best-response mapping of the two-sided competition game as

$$T(p_1, w_1, p_2, w_2) := \Big(p_1(p_1, w_1, p_2, w_2), w_1(p_1, w_1, p_2, w_2), p_2(p_1, w_1, p_2, w_2), w_2(p_1, w_1, p_2, w_2) \Big)$$
$$= \Big(p_1(p_2, w_2), w_1(p_2, w_2), p_2(p_1, w_1), w_2(p_1, w_1) \Big).$$

We then iteratively define the $k-$fold best-response mapping ($k \geq 2$) as

$$T^{(k)}(p_1, w_1, p_2, w_2) = \Big(p_1^{(k)}(p_1, w_1, p_2, w_2), w_1^{(k)}(p_1, w_1, p_2, w_2), p_2^{(k)}(p_1, w_1, p_2, w_2), w_2^{(k)}(p_1, w_1, p_2, w_2) \Big),$$

where for $i = 1, 2$

$$p_i^{(k)}(p_1, w_1, p_2, w_2) = p_i \Big(p_1^{(k-1)}(p_1, w_1, p_2, w_2), w_1^{(k-1)}(p_1, w_1, p_2, w_2), p_2^{(k-1)}(p_1, w_1, p_2, w_2), w_2^{(k-1)}(p_1, w_1, p_2, w_2) \Big),$$
$$w_i^{(k)}(p_1, w_1, p_2, w_2) = w_i \Big(p_1^{(k-1)}(p_1, w_1, p_2, w_2), w_1^{(k-1)}(p_1, w_1, p_2, w_2), p_2^{(k-1)}(p_1, w_1, p_2, w_2), w_2^{(k-1)}(p_1, w_1, p_2, w_2) \Big).$$

We use $||\cdot||_1$ to represent the ℓ_1 norm, that is, $||x||_1 = \sum_{i=1}^{n} |x_i|$ for $x \in \mathbb{R}^n$. The proof of Theorem 1 is based on the following four steps:

- <u>Step I</u>. Under equilibrium, $s_i^* = d_i^*$ for $i = 1, 2$.

- <u>Step II</u>. The best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are continuously increasing in p_{-i} and w_{-i}. This will imply that an equilibrium exists.

- <u>Step III</u>. There exists a k^*, such that the k^*-fold best response is a contraction mapping under the ℓ_1 norm, i.e., there exists a constant $\theta \in (0, 1)$, such that

$$||T^{(k^*)}(p_1, w_1, p_2, w_2) - T^{(k^*)}(p_1', w_1', p_2', w_2')||_1 \leq \theta ||(p_1, w_1, p_2, w_2) - (p_1', w_1', p_2', w_2')||_1.$$

This will imply that the equilibrium is unique.

- <u>Step IV</u>. For any (p_1, w_1, p_2, w_2), the sequence $T^{(k)}(p_1, w_1, p_2, w_2)$ converges to the unique equilibrium $(p_1^*, w_1^*, p_2^*, w_2^*)$ as $k \uparrow +\infty$. This will imply that the equilibrium can be computed using a *tatônnement* scheme.

<u>Step I</u> is proved by contradiction (Lemma 2 below). We show that if $s_i^* > d_i^*$, then P_i can unilaterally decrease w_i to increase its profit; and if $s_i^* < d_i^*$, then P_i can unilaterally increase p_i to increase its profit. This implies that we must have $s_i^* = d_i^*$ under equilibrium.

<u>Step II</u> is proved by exploiting structural properties of the best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$, and by using the fact that $d_i^* = s_i^*$ under equilibrium (Lemma 3 below). Since the feasible region of (p_{-i}, w_{-i}) is a lattice, Step II immediately implies that an equilibrium exists by Tarski's Theorem.

Step III is proved by bounding the ℓ_1 norm of $T(p_1, p_2, w_1, w_2)$. We note that $T(\cdot, \cdot, \cdot, \cdot)$ is not necessarily a contraction mapping, but $T^{(k*)}(\cdot, \cdot, \cdot, \cdot)$ for some $k^* > 1$ is (Lemma 4 below). Using the result of Step III, a standard contradiction argument will show that the equilibrium is unique.

Step IV is proved by exploiting the contraction mapping property of $T^{(k)}(\cdot, \cdot, \cdot, \cdot)$ (Lemma 5 below). Putting Steps I-IV together concludes the proof of Theorem 1. $\square$

The following lemma proves Step I in the proof of Theorem 1.

LEMMA 2. *Under equilibrium, $d_i^* = s_i^*$ for $i = 1, 2$.*

Proof. Assume by contradiction that $s_i^* < d_i^*$. This implies that $d_i^* > \min\{d_i^*, s_i^*\} = s_i^*$,

$$d_i^* = \frac{\Lambda \exp[s_i^*/d_i^*(q_i - p_i^*)]}{1 + \exp[s_i^*/d_i^*(q_i - p_i^*)] + \exp[\min\{1, s_{-i}^*/d_{-i}^*\}(q_{-i} - p_{-i}^*)]}$$

and

$$s_i^* = \frac{\exp(a_i + w_i^*)}{1 + \exp(a_i + w_i^*) + \exp(a_{-i} + \min\{1, d_{-i}^*/s_{-i}^*\}w_{-i}^*)}.$$

Consequently, P_i can increase its price to $p_i^*(\epsilon) = p_i^* + \epsilon$ (for a sufficiently small $\epsilon > 0$) and $(w_i^*, p_{-i}^*, w_{-i}^*)$ remain unchanged, with the induced market outcome $(d_i^*(\epsilon), s_i^*(\epsilon), d_{-i}^*(\epsilon), s_{-i}^*(\epsilon))$, which satisfies

$$d_i^*(\epsilon) = \frac{\Lambda \exp[s_i^*(\epsilon)/d_i^*(\epsilon)(q_i - p_i^* - \epsilon)]}{1 + \exp[s_i^*(\epsilon)/d_i^*(\epsilon)(q_i - p_i^* - \epsilon)] + \exp[\min\{1, s_{-i}^*(\epsilon)/d_{-i}^*(\epsilon)\}(q_{-i} - p_{-i}^*)]},$$

$$s_i^*(\epsilon) = \frac{\exp(a_i + w_i^*)}{1 + \exp(a_i + w_i^*) + \exp(a_{-i} + \min\{1, d_{-i}^*(\epsilon)/s_{-i}^*(\epsilon)\}w_{-i}^*)},$$

$$d_{-i}^*(\epsilon) = \frac{\Lambda \exp[\min\{1, s_{-i}^*(\epsilon)/d_{-i}^*(\epsilon)\}(q_{-i} - p_{-i}^*)]}{1 + \exp[s_i^*(\epsilon)/d_i^*(\epsilon)(q_i - p_i^* - \epsilon)] + \exp[\min\{1, s_{-i}^*(\epsilon)/d_{-i}^*(\epsilon)\}(q_{-i} - p_{-i}^*)]},$$

and

$$s_{-i}^*(\epsilon) = \frac{\exp(a_{-i} + \min\{1, d_{-i}^*(\epsilon)/s_{-i}^*(\epsilon)\}w_{-i}^*)}{1 + \exp(a_i + w_i^*) + \exp(a_{-i} + \min\{1, d_{-i}^*(\epsilon)/s_{-i}^*(\epsilon)\}w_{-i}^*)}.$$

One can check that, for a sufficiently small $\epsilon > 0$, $s_i^* < d_i^*(\epsilon) < d_i^*$, $s_i^*(\epsilon) = s_i^*$, and thus, $\min\{d_i^*(\epsilon), s_i^*(\epsilon)\} = s_i^*(\epsilon) = s_i^*$, where the inequality follows from the fact that $d_i(\epsilon)$ and $s_i(\epsilon)$ are continuous in ϵ. Hence, $\pi_i(\epsilon) = (p_i^* + \epsilon - w_i^*) \min\{d_i^*(\epsilon), s_i^*(\epsilon)\} > (p_i^* - w_i^*)s_i^* = \pi_i^*$, which contradicts the fact that $(p_i^*, w_i^*, p_{-i}^*, w_{-i}^*)$ is an equilibrium. Therefore, we must have $s_i^* \geq d_i^*$.

Assume by contradiction that $s_i^* > d_i^*$. This implies that $s_i^* > \min\{d_i^*, s_i^*\} = d_i^*$,

$$d_i^* = \frac{\Lambda \exp(q_i - p_i^*)}{1 + \exp(q_i - p_i^*) + \exp[\min\{1, s_{-i}^*/d_{-i}^*\}(q_{-i} - p_{-i}^*)]}$$

and

$$s_i^* = \frac{\exp(a_i + d_i^* w_i^*/s_i^*)}{1 + \exp(a_i + d_i^* w_i^*/s_i^*) + \exp(a_{-i} + \min\{1, d_{-i}^*/s_{-i}^*\}w_{-i}^*)}.$$

Consequently, P_i can decrease its wage to $w_i^*(\epsilon) = w_i^* - \epsilon$ (for a sufficiently small $\epsilon > 0$) and (p_i^*, w_i^*, p_{-i}^*) remain unchanged, with the induced market outcome $(d_i^*(\epsilon), s_i^*(\epsilon), d_{-i}^*(\epsilon), s_{-i}^*(\epsilon))$, which satisfies

$$d_i^*(\epsilon) = \frac{\Lambda \exp(q_i - p_i^*)}{1 + \exp(q_i - p_i^*) + \exp[\min\{1, s_{-i}^*(\epsilon)/d_{-i}^*(\epsilon)\}(q_{-i} - p_{-i}^*)]},$$

$$s_i^*(\epsilon) = \frac{\exp(a_i + d_i^*(\epsilon)(w_i^* - \epsilon)/s_i^*(\epsilon))}{1 + \exp(a_i + d_i^*(\epsilon)(w_i^* - \epsilon)/s_i^*(\epsilon)) + \exp(a_{-i} + \min\{1, d_{-i}^*(\epsilon)/s_{-i}^*(\epsilon)\}w_{-i}^*(\epsilon))},$$

$$d_{-i}^*(\epsilon) = \frac{\Lambda \exp[\min\{1, s_{-i}^*(\epsilon)/d_{-i}^*(\epsilon)\}(q_{-i} - p_{-i}^*)]}{1 + \exp(q_i - p_i^*) + \exp[\min\{1, s_{-i}^*(\epsilon)/d_{-i}^*(\epsilon)\}(q_{-i} - p_{-i}^*)]},$$

and

$$s_{-i}^*(\epsilon) = \frac{\exp(a_{-i} + \min\{1, d_{-i}^*(\epsilon)/s_{-i}^*(\epsilon)\}w_{-i}^*(\epsilon))}{1 + \exp(a_i + d_i^*(\epsilon)(w_i^* - \epsilon)/s_i^*(\epsilon)) + \exp(a_{-i} + \min\{1, d_{-i}^*(\epsilon)/s_{-i}^*(\epsilon)\}w_{-i}^*(\epsilon))}.$$

One can check that, for a sufficiently small $\epsilon > 0$, $s_i^* > s_i^*(\epsilon) > d_i^*$, $d_i^*(\epsilon) = d_i^*$, and thus, $\min\{d_i^*(\epsilon), s_i^*(\epsilon)\} = d_i^*(\epsilon) = d_i^*$, where the inequality follows from the fact that $d_i(\epsilon)$ and $s_i(\epsilon)$ are continuous in ϵ. Hence, $\pi_i(\epsilon) = (p_i^* - w_i^* + \epsilon) \min\{d_i^*(\epsilon), s_i^*(\epsilon)\} > (p_i^* - w_i^*)d_i^* = \pi_i^*$, contradicting that $(p_i^*, w_i^*, p_{-i}^*, w_{-i}^*)$ is an equilibrium. Thus, we have $s_i^* \leq d_i^*$. Putting together $s_i^* \geq d_i^*$ and $s_i^* \leq d_i^*$, we conclude that $s_i^* = d_i^*$. $\square$

The following lemma establishes Step II in the proof of Theorem 1.

LEMMA 3. $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are continuously increasing in p_{-i} and w_{-i}. Hence, an equilibrium exists in the two-sided competition model.

Proof. Since $s_i^* = d_i^*$, we denote $s = s_i = d_i$ as the demand/supply of P_i. Given (p_{-i}, w_{-i}, s), we can write $p_i(p_{-i}, w_{-i}, s) = q_i - \log\left(\frac{s/\Lambda}{1 - s/\Lambda}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))]$ and $w_i(p_{-i}, w_{-i}, s) = -a_i + \log\left(\frac{s}{1-s}\right) + \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})]$. Thus, given (p_{-i}, w_{-i}), P_i's price and wage optimization problem can be formulated as the following one-dimensional convex program:

$$\max_s \pi_i(s|p_{-i}, w_{-i})$$

$$\text{where } \pi_i(s|p_{-i}, w_{-i}) = [p_i(p_{-i}, w_{-i}, s) - w_i(p_{-i}, w_{-i}, s)]s$$

$$= \left\{ q_i + a_i - \log\left(\frac{s/\Lambda}{1 - s/\Lambda}\right) - \log\left(\frac{s}{1-s}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] \right.$$

$$\left. - \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})] \right\}s.$$

One can check that $\pi_i(s|p_{-i}, w_{-i})$ is concave in s and supermodular in (p_{-i}, s) (by calculating the cross-derivative). Therefore, $s^* := \arg\max_s \pi_i(s|p_{-i}, w_{-i})$ is increasing in p_{-i}, which implies that $w_i(p_{-i}, w_{-i}) = w_i(p_{-i}, w_{-i}, s^*) = -a_i + \log\left(\frac{s^*}{1-s^*}\right) + \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})]$ is also increasing in p_{-i}.

We define $m(p_{-i}, w_{-i}, s) := p_i(p_{-i}, w_{-i}, s) - w_i(p_{-i}, w_{-i}, s)$ as P_i's profit margin. Thus, $\pi_i'(s|p_{-i}, w_{-i}) = \partial_s m(p_{-i}, w_{-i}, s)s + m(p_{-i}, w_{-i}, s)$. Since $\pi_i'(s^*|p_{-i}, w_{-i}) = 0$, we have $\partial_s m(p_{-i}, w_{-i}, s^*)s^* + m(p_{-i}, w_{-i}, s^*) = 0$. As a result, $\partial_s m(p_{-i}, w_{-i}, s)s$ is strictly decreasing in s and independent of (p_{-i}, w_{-i}). Assume that $\hat{p}_{-i} > p_{-i}$, so we have $\hat{s}^* > s^*$. Thus, $\partial_s m(\hat{p}_{-i}, w_{-i}, \hat{s}^*)\hat{s}^* < \partial_s m(p_{-i}, w_{-i}, s^*)s^*$. By the first-order condition, $\pi_i'(\hat{s}^*|\hat{p}_{-i}, w_{-i}) = \pi_i'(s^*|p_{-i}, w_{-i}) = 0$, i.e., $\partial_s m(\hat{p}_{-i}, w_{-i}, \hat{s}^*)\hat{s}^* + m(\hat{p}_{-i}, w_{-i}, \hat{s}^*) = \partial_s m(p_{-i}, w_{-i}, s^*)s^* + m(p_{-i}, w_{-i}, s^*) = 0$. Hence, $m(\hat{p}_{-i}, w_{-i}, \hat{s}^*) > m(p_{-i}, w_{-i}, s^*)$, i.e.,

$$-\log\left(\frac{\hat{s}^*/\Lambda}{1 - \hat{s}^*/\Lambda}\right) - \log\left(\frac{\hat{s}^*}{1 - \hat{s}^*}\right) - \log[1 + \exp(\min\{1, \hat{s}_{-i}/\hat{d}_{-i}\}(q_{-i} - \hat{p}_{-i}))] >$$
$$-\log\left(\frac{s^*/\Lambda}{1 - s^*/\Lambda}\right) - \log\left(\frac{s^*}{1 - s^*}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))].$$

Since we also have $\hat{s}^* > s^*$, we obtain that

$$p_i(p_{-i}, w_{-i}) = q_i - \log\left(\frac{s^*/\Lambda}{1 - s^*/\Lambda}\right) - \log\left(\frac{s^*}{1 - s^*}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] + \log\left(\frac{s^*}{1 - s^*}\right)$$

is increasing in p_{-i}, i.e., both $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in p_{-i}. With a similar argument, we can show that s^* is decreasing in w_{-i}, which further implies that $p_i(p_{-i}, w_{-i}) = p_i(p_{-i}, w_{-i}, s^*) = q_i - \log\left(\frac{s^*/\Lambda}{1 - s^*/\Lambda}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))]$ is increasing in w_{-i}. Moreover, the profit margin $m(p_{-i}, w_{-i}, s^*)$ is decreasing in w_{-i}. Thus, $-\log\left(\frac{s^*/\Lambda}{1 - s^*/\Lambda}\right) - \log\left(\frac{s^*}{1 - s^*}\right) - \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})]$ and s^* are both decreasing in w_{-i}. Therefore,

$$w_i(p_{-i}, w_{-i}) = -a_i + \log\left(\frac{s^*}{1 - s^*}\right) + \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})]$$
$$= -a_i - \left\{-\log\left(\frac{s^*/\Lambda}{1 - s^*/\Lambda}\right) - \log\left(\frac{s^*}{1 - s^*}\right) - \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})]\right\} -$$
$$-\log\left(\frac{s^*/\Lambda}{1 - s^*/\Lambda}\right)$$

is increasing in w_{-i}. We have thus shown that both $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in p_{-i} and in w_{-i}. The continuity of $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ follows from the facts that $\pi_i(s|p_{-i}, w_{-i})$ is concave in s and continuous in p_{-i} and in w_{-i}. This completes the proof of Step II. By Tarski's Fixed Point Theorem (see, e.g., Milgrom and Roberts 1990), the continuity and monotonicity of $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$, together with the fact that the feasible sets of $p_i(\cdot, \cdot)$ and $w_i(\cdot, \cdot)$ are latices, imply that an equilibrium exists.   □

The following lemma establishes Step III in the proof of Theorem 1.

LEMMA 4. *There exists a k^*, such that the k^*-fold best response is a contraction mapping under the ℓ_1 norm, i.e., there exists a constant $\theta \in (0, 1)$, such that*

$$||T^{(k^*)}(p_1, w_1, p_2, w_2) - T^{(k^*)}(p_1', w_1', p_2', w_2')||_1 \leq \theta ||(p_1, w_1, p_2, w_2) - (p_1', w_1', p_2', w_2')||_1.$$

Furthermore, the equilibrium is unique.

Proof. We denote $\hat{p}_{-i} = p_{-i} + \delta$. We observe that $\partial_{p_{-i}}\{-\log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))]\} \leq \frac{\exp(q_{-i} - p_{-i})}{1 + \exp(q_{-i} - p_{-i})} < \frac{\exp(q_{-i})}{1 + \exp(q_{-i})}$, for $p_{-i} > w_{-i} \geq 0$. By the mean-value theorem, for $\delta > 0$, $|\log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] - \log[1 + \exp(\min\{1, s_{-i}(\delta)/d_{-i}(\delta)\}(q_{-i} - p_{-i} - \delta))]| \leq \frac{\exp(q_{-i})}{1 + \exp(q_{-i})}\delta \leq C\delta$, where $C := \max\left\{\frac{\exp(a_{-i})}{1 + \exp(a_{-i})}, \frac{\exp(q_{-i})}{1 + \exp(q_{-i})}\right\} < 1$. As shown in the proof of Step II of Theorem 1, $\hat{s}^* > s^*$ and $m_i(\hat{p}_{-i}, w_{-i}, \hat{s}^*) > m_i(p_{-i}, w_{-i}, s^*)$, that is,

$$0 < [p_i(\hat{p}_{-i}, w_{-i}) - w_i(\hat{p}_{-i}, w_{-i})] - [p_i(p_{-i}, w_{-i}) - w_i(p_{-i}, w_{-i})] <$$

$$\log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] - \log[1 + \exp(\min\{1, s_{-i}(\delta)/d_{-i(\delta)}\}(q_{-i} - p_{-i} - \delta))] < C\delta. \quad (4)$$

We denote $\delta_2 = \log\left(\frac{\hat{s}^*/\Lambda}{1 - \hat{s}^*/\Lambda}\right) - \log\left(\frac{s^*/\Lambda}{1 - s^*/\Lambda}\right) > 0$ and $\delta_3 = \log\left(\frac{\hat{s}^*}{1 - \hat{s}^*}\right) - \log\left(\frac{s^*}{1 - s^*}\right) > 0$, so that inequality (4) implies that $\delta_2 + \delta_3 < C\delta$. Therefore, we obtain

$$p_i(\hat{p}_{-i}, w_{-i}) - p_i(p_{-i}, w_{-i}) = -\log\left(\frac{\hat{s}^*/\Lambda}{1 - \hat{s}^*/\Lambda}\right) + \log\left(\frac{s^*/\Lambda}{1 - s^*/\Lambda}\right) +$$

$$+ \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] - \log[1 + \exp(\min\{1, s_{-i}(\delta)/d_{-i}(\delta)\}(q_{-i} - p_{-i} - \delta))] < C\delta - \delta_2,$$

and $w_i(\hat{p}_{-i}, w_{-i}) - w_i(p_{-i}, w_{-i}) = \log\left(\frac{\hat{s}^*}{1 - \hat{s}^*}\right) - \log\left(\frac{s^*}{1 - s^*}\right) = \delta_3 < C\delta - \delta_2$. As a result,

$$|p_i(\hat{p}_{-i}, w_{-i}) - p_i(p_{-i}, w_{-i})| + |w_i(\hat{p}_{-i}, w_{-i}) - w_i(p_{-i}, w_{-i})| < 2(C\delta - \delta_2) < 2C\delta,$$

that is, $||T(p_i, w_i, \hat{p}_{-i}, w_{-i}) - T(p_i, w_i, p_{-i}, w_{-i})||_1 \leq 2C\delta$.

Let $\delta_p := p_i(\hat{p}_{-i}, w_{-i}) - p_i(p_{-i}, w_{-i}) > 0$ and $\delta_w := w_i(\hat{p}_{-i}, w_{-i}) - w_i(p_{-i}, w_{-i}) > 0$. Recall that inequality (4) implies that $\delta_p + \delta_w < C\delta$. Hence, $\delta_p, \delta_w \in (0, \delta)$ and $|\delta_p - \delta_w| < C\delta$. Thus, we have $-\log[1 + \exp(q_i - p_i(\hat{p}_{-i}, w_{-i}))] - \{-\log[1 + \exp(q_i - p_i(p_{-i}, w_{-i}))]\} < C[p_i(\hat{p}_{-i}, w_{-i}) - p_i(p_{-i}, w_{-i})] = C\delta_p$ and $\log[1 + \exp(a_i + w_i(p_{-i}, w_{-i}))] - \log[1 + \exp(a_i + w_i(\hat{p}_{-i}, w_{-i}))] > -C\delta_w$. Therefore,

$$F_i(p_i(\hat{p}_{-i}, w_{-i}), w_i(\hat{p}_{-i}, w_{-i})) - F_i(p_i(p_{-i}, w_{-i}), w_i(p_{-i}, w_{-i})) < C(\delta_p - \delta_w) < C^2\delta,$$

where $F_i(p, w) := -\log[1 + \exp(q_i - p_i)] - \log[1 + \exp(a_i + w_i)]$. By repeating the same argument, we obtain the following inequality:

$$||T^{(2)}(p_i, w_i, \hat{p}_{-i}, w_{-i}) - T^{(2)}(p_i, w_i, p_{-i}, w_{-i})||_1$$

$$= |p_{-i}^{(2)}(p_i, w_i, \hat{p}_{-i}, w_{-i}) - p_{-i}^{(2)}(p_i, w_i, p_{-i}, w_{-i})| + |w_{-i}^{(2)}(p_i, w_i, \hat{p}_{-i}, w_{-i}) - w_{-i}^{(2)}(p_i, w_i, p_{-i}, w_{-i})| < 2C^2\delta.$$

We define $\hat{w}_{-i} = w_{-i} + \delta$ (for $\delta > 0$). Using the same argument once again, we obtain $||T(p_i, w_i, p_{-i}, \hat{w}_{-i}) - T(p_i, w_i, p_{-i}, w_{-i})||_1 < 2C\delta$ and $||T^{(2)}(p_i, w_i, p_{-i}, \hat{w}_{-i}) - T^{(2)}(p_i, w_i, p_{-i}, w_{-i})||_1 < 2C^2\delta$.

By using the standard induction argument, we can write

$$|(p_1^{(k)}(\hat{p}_1,w_1,p_2,w_2) - w_1^{(k)}(\hat{p}_1,w_1,p_2,w_2)) - (p_1^{(k)}(p_1,w_1,p_2,w_2) - w_1^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta$$

$$|(p_2^{(k)}(\hat{p}_1,w_1,p_2,w_2) - w_2^{(k)}(\hat{p}_1,w_1,p_2,w_2)) - (p_2^{(k)}(p_1,w_1,p_2,w_2) - w_2^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta$$

$$|(p_1^{(k)}(p_1,\hat{w}_1,p_2,w_2) - w_1^{(k)}(p_1,\hat{w}_1,p_2,w_2)) - (p_1^{(k)}(p_1,w_1,p_2,w_2) - w_1^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta$$

$$|(p_2^{(k)}(p_1,\hat{w}_1,p_2,w_2) - w_2^{(k)}(p_1,\hat{w}_1,p_2,w_2)) - (p_2^{(k)}(p_1,w_1,p_2,w_2) - w_2^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta$$

$$|(p_1^{(k)}(p_1,w_1,\hat{p}_2,w_2) - w_1^{(k)}(p_1,w_1,\hat{p}_2,w_2)) - (p_1^{(k)}(p_1,w_1,p_2,w_2) - w_1^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta$$

$$|(p_2^{(k)}(p_1,w_1,\hat{p}_2,w_2) - w_2^{(k)}(p_1,w_1,\hat{p}_2,w_2)) - (p_2^{(k)}(p_1,w_1,p_2,w_2) - w_2^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta$$

$$|(p_1^{(k)}(p_1,w_1,p_2,\hat{w}_2) - w_1^{(k)}(p_1,w_1,p_2,\hat{w}_2)) - (p_1^{(k)}(p_1,w_1,p_2,w_2) - w_1^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta$$

$$|(p_2^{(k)}(p_1,w_1,p_2,\hat{w}_2) - w_2^{(k)}(p_1,w_1,p_2,\hat{w}_2)) - (p_2^{(k)}(p_1,w_1,p_2,w_2) - w_2^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta,$$

and

$$||T^{(k)}(\hat{p}_1,w_1,p_2,w_2) - T^{(k)}(p_1,w_1,p_2,w_2)||_1 \leq 2C^k\delta$$

$$||T^{(k)}(p_1,\hat{w}_1,p_2,w_2) - T^{(k)}(p_1,w_1,p_2,w_2)||_1 \leq 2C^k\delta$$

$$||T^{(k)}(p_1,w_1,\hat{p}_2,w_2) - T^{(k)}(p_1,w_1,p_2,w_2)||_1 \leq 2C^k\delta$$

$$||T^{(k)}(p_1,w_1,p_2,\hat{w}_2) - T^{(k)}(p_1,w_1,p_2,w_2)||_1 \leq 2C^k\delta,$$

where $\hat{p}_i = p_i + \delta$ and $\hat{w}_i = w_i + \delta$. We define k^* as the smallest integer k such that $2C^k < 1$ (i.e., the smallest integer k such that $k > -\log(2)/\log(C)$). Therefore, we obtain

$$||T^{(k^*)}(p_1,w_1,p_2,w_2) - T^{(k^*)}(p_1',w_1',p_2',w_2')||_1$$

$$\leq ||T^{(k^*)}(p_1,w_1,p_2,w_2) - T^{(k^*)}(p_1',w_1,p_2,w_2)||_1 + ||T^{(k^*)}(p_1',w_1,p_2,w_2) - T^{(k^*)}(p_1',w_1',p_2,w_2)||_1$$

$$+ ||T^{(k^*)}(p_1',w_1',p_2,w_2) - T^{(k^*)}(p_1',w_1',p_2',w_2)||_1 + ||T^{(k^*)}(p_1',w_1',p_2',w_2) - T^{(k^*)}(p_1',w_1',p_2',w_2')||_1$$

$$< 2C^{k^*}|p_1 - p_1'| + 2C^{k^*}|w_1 - w_1'| + 2C^{k^*}|p_2 - p_2'| + 2C^{k^*}|w_2 - w_2'| = \theta||(p_1,w_1,p_2,w_2) - (p_1',w_1',p_2',w_2')||_1,$$

where the first inequality follows from the triangle inequality. Since $\theta := 2C^{(k^*)} < 1$, we conclude that $T^{(k^*)}(\cdot,\cdot,\cdot,\cdot)$ is a contraction mapping under the ℓ_1 norm.

We next show that the equilibrium is unique. Assume by contradiction that there are two distinct equilibria $(p_1^*,w_1^*,p_2^*,w_2^*)$ and $(\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*)$. Then, by the equilibrium definition, we have $T(p_1^*,w_1^*,p_2^*,w_2^*) = (p_1^*,w_1^*,p_2^*,w_2^*)$ and $T(\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*) = (\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*)$. Therefore, $T^{(k^*)}(p_1^*,w_1^*,p_2^*,w_2^*) = (p_1^*,w_1^*,p_2^*,w_2^*)$ and $T^{(k^*)}(\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*) = (\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*)$. Hence, we have

$$||T^{(k^*)}(p_1^*,w_1^*,p_2^*,w_2^*) - T^{(k^*)}(\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*)||_1 = ||(p_1^*,w_1^*,p_2^*,w_2^*) - (\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*)||_1. \quad (5)$$

Since $T^{(k^*)}(\cdot,\cdot,\cdot,\cdot)$ is a contraction mapping, we have

$$||T^{(k^*)}(p_1^*,w_1^*,p_2^*,w_2^*) - T^{(k^*)}(\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*)||_1 < \theta||(p_1^*,w_1^*,p_2^*,w_2^*) - (\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*)||_1,$$

contradicting equation (5) if $(p_1^*,w_1^*,p_2^*,w_2^*) \neq (\bar{p}_1^*,\bar{w}_1^*,\bar{p}_2^*,\bar{w}_2^*)$. Thus, a unique equilibrium exists.
$\square$

The following lemma establishes Step IV in the proof of Theorem 1.

LEMMA 5. $T^{(k)}(p_1, w_1, p_2, w_2)$ *converges to the unique equilibrium as* $k \uparrow +\infty$.

Proof. As shown in Step III, $||T^{(k)}(p_1, w_1, p_2, w_2) - T^{(k)}(p_1', w_1', p_2', w_2')||_1 \leq 2C^k ||(p_1, w_1, p_2, w_2) - (p_1', w_1', p_2', w_2')||_1$ for any (p_1, w_1, p_2, w_2) and (p_1', w_1', p_2', w_2'). We define $x_k := T^{(k)}(p_1, w_1, p_2, w_2)$ for $k \geq 1$ and $x_0 = (p_1, w_1, p_2, w_2)$. For any k and $l > 0$,

$$||T^{(k)}(p_1, w_1, p_2, w_2) - T^{(k+l)}(p_1', w_1', p_2', w_2')||_1 \leq \sum_{i=1}^{l} ||x_{k+i} - x_{k+i-1}||_1$$

$$\leq \sum_{i=1}^{l} 2C^{(k+i-1)} ||x_1 - x_0||_1 \leq \sum_{i=1}^{+\infty} 2C^{(k+i-1)} ||x_1 - x_0||_1 = \frac{2||x_1 - x_0||_1 C^k}{1 - C},$$

where the first inequality follows from the triangle inequality, and the second from $x_{k+i} = T^{(k+i-1)}(x_1)$ and $x_{k+i-1} = T^{(k+i-1)}(x_0)$. Thus, $||x_k - x_{k+l}|| \to 0$ uniformly with respect to l as $k \uparrow +\infty$, that is, $\{x_k : k \geq 1\}$ is a Cauchy sequence, and hence x_k converges to x^*, which is a fixed point of $T(\cdot, \cdot, \cdot, \cdot)$, i.e., $T(x^*) = x^*$ so that x^* is the unique equilibrium. Hence, the unique equilibrium can be obtained using a *tatônnement* scheme, and this concludes the proof of Theorem 1. $\square$

Proof of Proposition 1

As shown in the proof of Theorem 1, the sequence $\{T^{(k)}(p_1^{m*}, w_1^{m*}, p_2^{m*}, w_2^{m*}) : k \geq 1\}$ converges to the equilibrium $(p_1^*, w_1^*, p_2^*, w_2^*)$. We define:

$$(p_1^{(k)}, w_1^{(k)}, p_2^{(k)}, w_2^{(k)}) := T^{(k)}(p_1^{m*}, w_1^{m*}, p_2^{m*}, w_2^{m*}) \text{ for } k \geq 1,$$

and $(p_1^{(0)}, w_1^{(0)}, p_2^{(0)}, p_2^{(0)}) := (p_1^{m*}, w_1^{m*}, p_2^{m*}, w_2^{m*})$. We also define $s_i^{(k)}$ as the optimal supply of P_i in the k-th iteration of the *tatônnement* scheme. Then, it suffices to show that $p_i^{(k)} < p_i^{(m*)}$ and $w_i^{(k)} > w_i^{(m*)}$ for $k \geq 1$ and $i = 1, 2$.

Note that for a monopoly, $d_i^{m*} = s_i^{m*}$ for $i = 1, 2$. Indeed, following the same argument as in the proof of Step I of Theorem 1, if $d_i^{m*} > s_i^{m*}$, we can increase p_i and strictly increase the profit of each platform. Analogously, if $d_i^{m*} < s_i^{m*}$, we can increase w_i and strictly increase the profit of each platform. As a result, under the optimal price and wage policies, $d_i^{m*} = s_i^{m*}$ for $i = 1, 2$.

We next show that $p_i^{(1)} < p_i^{(0)}$ and $w_i^{(1)} > w_i^{(0)}$. As shown in the proof of Theorem 1, $(p_i^{(1)}, w_i^{(1)})$ can be represented by $\left(p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(1)}), w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(1)})\right)$, where $p_i(\cdot, \cdot, \cdot)$ (resp. $w_i(\cdot, \cdot, \cdot)$) is the price (resp. wage) policy of P_i given (p_{-i}, w_{-i}, s) and $s_i^{(1)}$ is the optimal supply (which is equal to demand) obtained by solving the following one-dimensional convex program:

$$\max_s \pi_i(s)$$
$$\text{where } \pi_i(s) = (p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s) - w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s))s$$
$$= \Big\{ q_i + a_i - \log\left(\frac{s/\Lambda}{1 - s/\Lambda}\right) - \log\left(\frac{s}{1 - s}\right) - \log[1 + \exp(\min\{1, s_{-i}^{(0)}/d_{-i}^{(0)}\}(q_{-i} - p_{-i}^{(0)}))]$$
$$- \log[1 + \exp(a_{-i} + \min\{1, d_{-i}^{(0)}/s_{-i}^{(0)}\}w_{-i}^{(0)})] \Big\} s.$$

Given that under the optimal policy, $s_i^{m*} = d_i^{m*}$, the optimal price and wage of a monopoly (p_1^{m*}, w_1^{m*}) can be obtained by $\left(p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{m*}), w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{m*}) \right)$, where s_i^{m*} is the solution of the following one-dimensional optimization problem:

$$\max_s [\pi_i(s) + \pi_{-i}(s)]$$

$$\text{where } \pi_{-i}(s) = (p_{-i}^{(0)} - w_{-i}^{(0)}) \min\{d_{-i}, s_{-i}\},$$

$$\text{with } d_{-i} = \frac{\exp(\min\{1, s_{-i}^{(0)}/d_{-i}^{(0)}\}(q_{-i} - p_{-i}^{(0)}))}{1 + \exp[q_i - p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s)] + \exp(\min\{1, s_{-i}^{(0)}/d_{-i}^{(0)}\}(q_{-i} - p_{-i}^{(0)}))}$$

$$s_{-i} = \frac{\exp(a_{-i} + \min\{1, d_{-i}^{(0)}/s_{-i}^{(0)}\}w_{-i}^{(0)})}{1 + \exp[a_i + w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s)] + \exp(a_{-i} + \min\{1, d_{-i}^{(0)}/s_{-i}^{(0)}\}w_{-i}^{(0)})}.$$

One can check that d_{-i}, s_{-i}, and thus $\pi_{-i}(\cdot)$ are all strictly decreasing in s. Since $s_i^{(1)}$ is the maximizer of $\pi_i(s)$, we must have $s_i^{m*} < s_i^{(1)}$. Since $p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s)$ is strictly decreasing in s, whereas $w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s)$ is strictly increasing, we have $p_i^{(1)} = p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(1)}) < p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(m*)})$ and $w_i^{(1)} = w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(1)}) > w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(m*)})$. Then, we have shown that $p_i^{(1)} < p_i^{(0)}$ and $w_i^{(1)} > w_i^{(0)}$.

Next, we show that if $p_{-i}^{(k)} < p_{-i}^{(m*)}$ and $w_{-i}^{(k)} > w_{-i}^{(m*)}$, then $p_i^{(k+1)} < p_i^{(m*)}$ and $w_i^{(k+1)} > w_i^{(m*)}$. Assume by contradiction that either $p_i^{(k+1)} \geq p_i^{(m*)}$ or $w_i^{(k+1)} \leq w_i^{(m*)}$. Then, we have $s_i^{(k+1)} < s_i^{m*}$ and $m_i^{(k+1)} := p_i^{(k+1)} - w_i^{(k+1)} > m_i^{m*} := p_i^{(m*)} - w_i^{(m*)}$. As shown in the proof of Theorem 1, $\partial_s m(p_{-i}, w_{-i}, s)s$ is independent of (p_{-i}, w_{-i}) and decreasing in s. Thus, we have:

$$\pi_i'(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) = \partial_s m_i^{(k+1)} s_i^{(k+1)} + m_i^{(k+1)} > \partial_s m_i^{(m*)} s_i^{(m*)} + m_i^{(m*)} = \pi_i'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}),$$

where the inequality follows from $s_i^{(k+1)} < s_i^{m*}$ and $m_i^{(k+1)} > m_i^{m*}$. By the FOC of the monopoly model, $\pi_i'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) + \pi_{-i}'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) = 0$, so that $\pi_i'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) = -\pi_{-i}'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) > 0$, where the inequality follows from the fact that $\pi_{-i}(\cdot)$ is strictly decreasing in s. This implies that $\pi_i'(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) > 0$, which contradicts the FOC $\pi_i'(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) = 0$. Thus, we must have $p_i^{(k+1)} < p_i^{(m*)}$ and $w_i^{(k+1)} < w_i^{(m*)}$. Proposition 1 then follows from $p_i^* = \lim_{k \to +\infty} p_i^{(k)} < p_i^{(0)} = p^{m*}$ and $w_i^* = \lim_{k \to +\infty} w_i^{(k)} > w_i^{(0)} = w_i^{m*}$ for $i = 1, 2$. $\square$

Proof of Proposition 2

We first show that the best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in Λ. Recall from the proof of Theorem 1 that $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ can be characterized as the solution to the following one-dimensional convex program:

$$s^* = \arg\max_s \pi_i(s|p_{-i}, w_{-i}, \Lambda)$$

$$\text{where } \pi_i(s|p_{-i}, w_{-i}, \Lambda) = [p_i(p_{-i}, w_{-i}, s) - w_i(p_{-i}, w_{-i}, s)]s$$

$$= \left\{ q_i + a_i - \log\left(\frac{s/\Lambda}{1 - s/\Lambda}\right) - \log\left(\frac{s}{1-s}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] \right.$$

$$\left. - \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})] \right\} s.$$

We then have $p_i(p_{-i}, w_{-i}) = q_i - \log\left(\frac{s^*/\Lambda}{1 - s^*/\Lambda}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))]$ and $w_i(p_{-i}, w_{-i}) = -a_i + \log\left(\frac{s^*}{1-s^*}\right) + \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})]$. By computing the

cross derivative, one can see that $\pi_i(s|p_{-i}, w_{-i}, \Lambda)$ is supermodular in (s, Λ). Therefore, s^* and $w_i(p_{-i}, w_{-i}) = -a_i + \log\left(\frac{s^*}{1-s^*}\right) + \log[1 + \exp(a_{-i} + w_{-i})]$ are increasing in Λ.

We define $t := \frac{s/\Lambda}{1-s/\Lambda} = \frac{s}{\Lambda - s}$. We then have $s = \frac{\Lambda t}{1-t}$. Optimizing $\pi_i(\cdot|p_{-i}, w_{-i}, \Lambda)$ over s is equivalent to optimizing $\psi_i(t|\Lambda)$ over t, where

$$\psi_i(t|\Lambda) := \Big\{ q_i + a_i - \log(t) - \log\left(\frac{\Lambda t}{1 - t - \Lambda t}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))]$$
$$- \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})] \Big\} \frac{\Lambda t}{1-t}.$$

We define $t^* := \arg\max_t \psi_i(t|\Lambda)$. We have $t^* = \frac{s^*/\Lambda}{1-s^*/\Lambda}$. By computing the cross derivative, one can see that $\psi_i(t|\Lambda)$ is submodular in (t, Λ), and hence t^* is decreasing in Λ. Thus, $p_i(p_{-i}, w_{-i}) = q_i - \log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] = q_i - \log(t^*) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))]$ is increasing in Λ. We then have proved that both $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in Λ. We define $(p_1^{(k)}, w_1^{(k)}, p_2^{(k)}, w_2^{(k)}) = T^{(k)}(p_1, w_1, p_2, w_2)$ for $k \geq 1$ under a given initial strategy (p_1, w_1, p_2, w_2). Since $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are both increasing in p_{-i} and w_{-i}, then $p_1^{(k)}$, $w_1^{(k)}$, $p_2^{(k)}$, and $w_2^{(k)}$ are increasing in Λ for any k. By Theorem 1, $(p_1^*, w_1^*, p_2^*, w_2^*) = \lim_{k\uparrow+\infty}(p_1^{(k)}, w_1^{(k)}, p_2^{(k)}, w_2^{(k)})$. Thus, $p_i^* = \lim_{k\uparrow+\infty} p_i^{(k)}$ and $w_i^* = \lim_{k\uparrow+\infty} w_i^{(k)}$, $i = 1, 2$ are increasing in Λ. This concludes the proof of Proposition 2. $\quad\square$

Proof of Theorem 2

Since the proof follows a similar argument as Theorem 1, we only present its sketch.

Similar to Theorem 1, we can show that, in equilibrium, the supply and demand of each platform should match. Then, as in Theorem 1, we show that given other platforms' price and wage decisions, $(\mathbf{p_{-i}}, \mathbf{w_{-i}}) = (p_1, w_1, p_2, w_2, \cdots, p_{i-1}, w_{i-1}, p_{i+1}, w_{i+1}, \cdots, p_n, w_n)$, P_i's best-response price $p_i(\mathbf{p_{-i}}, \mathbf{w_{-i}})$ and wage $w_i(\mathbf{p_{-i}}, \mathbf{w_{-i}})$ are continuously increasing in p_j and w_j for any $j \neq i$. By Tarski's Fixed Point Theorem, a pure equilibrium strategy exists.

To show the equilibrium uniqueness, it suffices to show that the $k-$fold best-response mapping, $T^{(k)}(p_1, w_1, p_2, w_2, \cdots, p_n, w_n)$, is a contraction under the ℓ_1 norm for k sufficiently large. This would follow from a similar argument as in the proof of Lemma 4. More specifically, for any $(\mathbf{p}, \mathbf{w})$ and $(\mathbf{p}', \mathbf{w}')$, we have

$$||T^{(k)}(\mathbf{p}, \mathbf{w}) - T^{(k)}(\mathbf{p}', \mathbf{w}')||_1 \leq 2(n-1)C^k ||(\mathbf{p}, \mathbf{w}) - (\mathbf{p}', \mathbf{w}')||_1.$$

Therefore, if $k > -[\log(2) + \log(n-1)]/\log(C)$, then $T^{(k)}(\cdot, \cdot)$ is a contraction mapping under ℓ_1. Following the same argument as in Lemma 4 and Theorem 1, we obtain that the sequence $\{T^{(k)}(\mathbf{p}, \mathbf{w}) : k \geq 1\}$ converges to the unique equilibrium of the two-sided competition game. This also implies that the equilibrium can be computed using a *tatônnement* scheme. $\quad\square$

Proof of Theorem 3

Similar to the proof of Theorem 1, we prove Theorem 3 using the following three steps:

- Under equilibrium, $s_i^{c*} \geq d_i^{c*}$, i.e., supply dominates demand.

- The best-response price $p_i^c(p_{-i})$ is continuously increasing in p_{-i}. This implies that an equilibrium exists.

- The best-response price $p_i^c(\cdot)$ is a contraction mapping, i.e., $|p_i^c(p_{-i}) - p_i^c(p'_{-i})| \leq q_c |p_{-i} - p'_{-i}|$ for some $q_c \in (0,1)$. This will imply that the equilibrium is unique and can be computed using a *tatônnement* scheme.

Step I. $s_i^{c*} \geq d_i^{c*}$

If $s_i^{c*} < d_i^{c*}$, then P_i can increase its price from p_i^{c*} to $\hat{p}_i^{c*} = p_i^{c*} + \epsilon$ (for a small $\epsilon > 0$), and accordingly its wage from βp_i^{c*} to $\beta p_i^{c*} + \beta \epsilon$, where ϵ is small enough so that $\hat{s}_i^{c*} \leq \hat{d}_i$. With this price adjustment, P_i's profit increases by at least $(1-\beta)\epsilon s_i^{c*} > 0$, contradicting that (p_i^{c*}, p_{-i}^{c*}) is an equilibrium. Therefore, we must have $s_i^{c*} \geq d_i^{c*}$ for $i = 1,2$.

Step II. $p_i^c(p_{-i})$ is continuously increasing in p_{-i}

Since $s_i^{c*} \geq d_i^{c*}$, the price/wage optimization of P_i can be formulated as follows:

$$\max_{p_i} (1-\beta) p_i d_i$$

$$\text{s.t. } d_i = \frac{\Lambda \exp(q_i - p_i)}{1 + \exp(q_i - p_i) + \exp(q_{-i} - p_{-i})}$$

$$s_i = \frac{\exp(a_i + \beta d_i p_i / s_i)}{1 + \exp(a_i + \beta d_i p_i / s_i) + \exp(a_{-i} + \beta \min\{1, d_{-i}/s_{-i}\} p_{-i})}$$

$$s_i \geq d_i.$$

One can show that the objective function is supermodular in (p_i, p_{-i}) and that the feasible set is a lattice. Thus, the best-response price $p_i^c(p_{-i})$ is continuously increasing in p_{-i}, so that using Tarski's Fixed Point Theorem, an equilibrium exists.

Step III. $p_i^c(\cdot)$ is a contraction mapping

As shown in the proof of Step II above,

$$p_i^c(p_{-i}) = \arg\max_{p_i} (1-\beta) p_i d_i$$

$$\text{s.t. } d_i = \frac{\Lambda \exp(q_i - p_i)}{1 + \exp(q_i - p_i) + \exp(q_{-i} - p_{-i})}$$

$$s_i = \frac{\exp(a_i + \beta d_i p_i / s_i)}{1 + \exp(a_i + \beta d_i p_i / s_i) + \exp(a_{-i} + \beta \min\{1, d_{-i}/s_{-i}\} p_{-i})}$$

$$s_i \geq d_i.$$

We define $p(p_{-i})$ as the unconstrained optimizer of $p_i d_i$, which is increasing in p_{-i}. We also define $\bar{p}(p_{-i})$ as the unique p_i such that $s_i = d_i$, which is also increasing in p_{-i}. We have $p_i^c(p_{-i}) = \max\{p(p_{-i}), \bar{p}(p_{-i})\}$. It suffices to show that there exists a constant $C \in (0,1)$ such that $p(p_{-i} + \delta) - p(p_{-i}) \leq C\delta$ and $\bar{p}(p_{-i} + \delta) - \bar{p}(p_{-i}) \leq C\delta$ for $\delta > 0$.

Since the MNL demand model satisfies the diagonal dominance condition, that is,

$$0 > \frac{\partial^2 \log\left(\frac{\exp(q_i - p_i)}{1 + \exp(q_i - p_i) + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))}\right)}{\partial (p_i)^2} > -\frac{\partial^2 \log\left(\frac{\exp(q_i - p_i)}{1 + \exp(q_i - p_i) + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))}\right)}{\partial p_i \partial p_{-i}},$$

we have:

$$\frac{\partial p(p_{-i})}{\partial p_{-i}} = -\frac{\partial^2 \log\left(\frac{\exp(q_i - p_i)}{1 + \exp(q_i - p_i) + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))}\right)}{\partial p_i \partial p_{-i}} \Bigg/ \left[\frac{\partial^2 \log\left(\frac{\exp(q_i - p_i)}{1 + \exp(q_i - p_i) + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))}\right)}{\partial (p_i)^2} - \frac{1}{(p_{-i})^2}\right]$$

$$< \frac{\exp(q_{-i})}{1 + \exp(q_{-i})} < 1.$$

Hence, by the mean value theorem, $p(p_{-i} + \delta) - p(p_{-i}) < C\delta$ for $C := \frac{\exp(q_{-i})}{1 + \exp(q_{-i})} < 1$ and any $\delta > 0$. Note that $\bar{p}(p_{-i})$ satisfies the following

$$\frac{\Lambda \exp[q_i - \bar{p}(p_{-i})]}{1 + \exp[q_i - \bar{p}(p_{-i})] + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))} = \frac{\exp[a_i + \beta \bar{p}(p_{-i})]}{1 + \exp[a_i + \beta \bar{p}(p_{-i})] + \exp(a_{-i} + \beta \min\{1, d_{-i}/s_{-i}\}p_{-i})} := s.$$

Note also that, if $\hat{p}_{-i} = p_{-i} + \delta$, then

$$\frac{\Lambda \exp[q_i - \bar{p}(p_{-i})]}{1 + \exp[q_i - \bar{p}(p_{-i})] + \exp(\min\{1, \hat{s}_{-i}/\hat{d}_{-i}\}(q_{-i} - \hat{p}_{-i}))} > s, \text{ whereas}$$

$$\frac{\exp[a_i + \beta \bar{p}(p_{-i})]}{1 + \exp[a_i + \beta \bar{p}(p_{-i})] + \exp(a_{-i} + \beta \min\{1, \hat{d}_{-i}/\hat{s}_{-i}\}\hat{p}_{-i})} < s.$$

Furthermore, we have

$$\frac{\Lambda \exp[q_i - \bar{p}(\hat{p}_{-i})]}{1 + \exp[q_i - \bar{p}(\hat{p}_{-i})] + \exp(\min\{1, \hat{s}_{-i}/\hat{d}_{-i}\}(q_{-i} - \hat{p}_{-i}))} < \frac{\Lambda \exp[q_i - \bar{p}(p_{-i})]}{1 + \exp[q_i - \bar{p}(p_{-i})] + \exp(\min\{1, \hat{s}_{-i}/\hat{d}_{-i}\}(q_{-i} - \hat{p}_{-i}))} \text{ and}$$

$$\frac{\exp[a_i + \beta \bar{p}(\hat{p}_{-i})]}{1 + \exp[a_i + \beta \bar{p}(\hat{p}_{-i})] + \exp(a_{-i} + \beta \min\{1, \hat{d}_{-i}/\hat{s}_{-i}\}\hat{p}_{-i})} > \frac{\exp[a_i + \beta \bar{p}(p_{-i})]}{1 + \exp[a_i + \beta \bar{p}(p_{-i})] + \exp(a_{-i} + \beta \min\{1, \hat{d}_{-i}/\hat{s}_{-i}\}\hat{p}_{-i})}.$$

Therefore,

$$\hat{s} := \frac{\Lambda \exp[q_i - \bar{p}(\hat{p}_{-i})]}{1 + \exp[q_i - \bar{p}(\hat{p}_{-i})] + \exp(\min\{1, \hat{s}_{-i}/\hat{d}_{-i}\}(q_{-i} - \hat{p}_{-i}))} = \frac{\exp[a_i + \beta \bar{p}(\hat{p}_{-i})]}{1 + \exp[a_i + \beta \bar{p}(\hat{p}_{-i})] + \exp(a_{-i} + \beta \min\{1, \hat{d}_{-i}/\hat{s}_{-i}\}\hat{p}_{-i})}$$

$$\in \left(\frac{\exp[a_i + \beta \bar{p}(p_{-i})]}{1 + \exp[a_i + \beta \bar{p}(p_{-i})] + \exp(a_{-i} + \beta \min\{1, \hat{d}_{-i}/\hat{s}_{-i}\}\hat{p}_{-i})}, \frac{\Lambda \exp[q_i - \bar{p}(p_{-i})]}{1 + \exp[q_i - \bar{p}(p_{-i})] + \exp(\min\{1, \hat{s}_{-i}/\hat{d}_{-i}\}(q_{-i} - \hat{p}_{-i}))}\right).$$

If $\hat{s} < s$, assume that p' satisfies $\frac{\exp(a_i + \beta p')}{1 + \exp(a_i + \beta p') + \exp(a_{-i} + \beta \min\{1, d_{-i}/s_{-i}\}\hat{p}_{-i})} = s > \hat{s}$. Since $\hat{s} < s$, we have $\bar{p}(\hat{p}_{-i}) < p'$. Since the MNL model satisfies the diagonal dominance condition, we have $0 < \bar{p}(\hat{p}_{-i}) - \bar{p}(p_{-i}) < p' - \bar{p}(p_{-i}) < q_s(\hat{p}_{-i} - p_{-i}) = q_s\delta$.

Analogously, if $\hat{s} > s$, assume that p'' satisfies $\frac{\Lambda \exp(q_i - p'')}{1 + \exp(q_i - p'') + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - \hat{p}_{-i}))} = s < \hat{s}$. Since $\hat{s} > s$, we have $\bar{p}(\hat{p}_{-i}) < p''$. By the diagonal dominance condition of the MNL model, we have $0 < \bar{p}(\hat{p}_{-i}) - \bar{p}(p_{-i}) < p'' - \bar{p}(p_{-i}) < q_d(\hat{p}_{-i} - p_{-i}) = q_d\delta$.

We define $q_c := \max\{q_d, q_s\} < 1$. The above analysis implies that $0 < \bar{p}(p_{-i} + \delta) - \bar{p}(p_{-i}) < q_c\delta$, for $\delta > 0$. Therefore, $p_i^c(p_{-i} + \delta) - p_i^c(p_{-i}) \leq \max\{C, q_s\}\delta$ for $\delta > 0$, where $\max\{C, q_s\} < 1$.

We have established that under fixed commission, the best-response mapping is a contraction mapping over the strategy space. Then, using Banach's Fixed Point Theorem, a unique equilibrium exists and can be computed using a *tatônnement* scheme. $\qquad\square$

Proof of Corollary 1

The first part follows from the same argument as in the proof of Theorem 1. If $s_i^{s*} < d_i^{s*}$, then P_i can increase its price and strictly increase its profit. If $s_i^{s*} > d_i^{s*}$, then P_i can decrease its price and strictly increase its profit. As a result, under equilibrium, we must have $s_i^{s*} = d_i^{s*}$ for $i = 1, 2$.

Similarly, the equilibrium existence and uniqueness follow from the same argument as in the proof of Theorem 1. To show how to compute the equilibrium $(p_1^{s*}, w_1^{s*}, p_2^{s*}, w_2^{s*})$, we note that $s_i^{s*} = d_i^{s*}$ implies that P_i's profit is equal to

$$\pi_i^s(p_i, p_{-i}) = \frac{\Lambda p_i \exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)} - C_i(s_i),$$

where $C_i(s_i) := \frac{s_i}{K_i} \left(\log \left(\frac{s_i}{1 - s_i} \right) - a_i \right)$ represents the total cost of P_i when the supply level is s_i and $s_i = d_i = \frac{\Lambda \exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)} = \frac{K_i \exp(a_i + w_i)}{1 + \exp(a_i + w_i)}$. It is clear that $\pi_i^s(\cdot, \cdot)$ is continuously differentiable for $i = 1, 2$. As a result, the equilibrium prices p_1^{s*} and p_2^{s*} should satisfy the first-order condition: $\partial_{p_i} \pi_i^s(p_i^{s*}, p_{-i}^{s*}) = 0$, that is, $\Lambda(\bar{d}_i^{s*} + p_i^{s*} \partial_{p_i} \bar{d}_i^{s*}) - \Lambda C_i'(\bar{d}_i^{s*}) \partial_{p_i} \bar{d}_i^{s*} = 0$. By using Lemma 1, we know that $\partial_{p_i} \bar{d}_i = -\bar{d}_i + (\bar{d}_i)^2$, and hence $\Lambda[\bar{d}_i^{s*} - p_i^{s*} \bar{d}_i^{s*}(1 - \bar{d}_i^{s*})] + \Lambda C_i'(\Lambda \bar{d}_i^{s*}) \bar{d}_i^{s*}(1 - \bar{d}_i^{s*}) = 0$, where $C_i'(s_i) = \frac{1}{K_i} \left(\log \left(\frac{s_i}{1 - s_i} \right) - a_i \right) + \frac{s_i}{K_i} \left(\frac{1}{s_i} + \frac{1}{1 - s_i} \right)$. Thus, we have

$$p_i^{s*} = \frac{1}{1 - \bar{d}_i^{s*}} + \frac{1}{K_i} \left(\log \left(\frac{\Lambda \bar{d}_i^{s*}}{1 - \Lambda \bar{d}_i^{s*}} \right) - a_i \right) + \frac{\Lambda \bar{d}_i^{s*}}{K_i} \left(\frac{1}{\Lambda \bar{d}_i^{s*}} + \frac{1}{1 - \Lambda \bar{d}_i^{s*}} \right).$$

On the other hand, using the definition of the MNL model, $\exp(q_i - p_i^{s*}) = \bar{d}_i^{s*} / \bar{d}_0^{s*}$ (using again $d_i^{s*} = s_i^{s*}$), i.e., $p_i^{s*} = q_i + \log(\bar{d}_0^{s*} / \bar{d}_i^{s*})$. Therefore,

$$\frac{1}{1 - d_i^{s*}} + \frac{1}{K_i} \left(\log \left(\frac{\Lambda \bar{d}_i^{s*}}{1 - \Lambda \bar{d}_i^{s*}} \right) - a_i \right) + \frac{\Lambda \bar{d}_i^{s*}}{K_i} \left(\frac{1}{\Lambda \bar{d}_i^{s*}} + \frac{1}{1 - \Lambda \bar{d}_i^{s*}} \right) = q_i + \log \left(\frac{\bar{d}_0^{s*}}{\bar{d}_i^{s*}} \right),$$

or equivalently,

$$\bar{d}_i^{s*} \exp \left(\frac{\bar{d}_i^{s*}}{1 - \bar{d}_i^{s*}} \right) \exp \left[\frac{1}{K_i} \left(\log \left(\frac{\Lambda \bar{d}_i^{s*}}{1 - \Lambda \bar{d}_i^{s*}} \right) - a_i \right) + \frac{\Lambda \bar{d}_i^{s*}}{K_i} \left(\frac{1}{\Lambda \bar{d}_i^{s*}} + \frac{1}{1 - \Lambda \bar{d}_i^{s*}} \right) \right] = \bar{d}_0^{s*} \exp(q_i - 1).$$

We define $U_i(x) := x \exp \left(\frac{x}{1 - x} \right) \exp \left[\frac{1}{K_i} \left(\log \left(\frac{\Lambda x}{1 - \Lambda x} \right) - a_i \right) + \frac{\Lambda x}{K_i} \left(\frac{1}{\Lambda x} + \frac{1}{1 - \Lambda x} \right) \right]$. The function $U_i(x)$ is continuous and strictly increasing in x, so we denote its inverse by $U_i^{-1}(\cdot)$. We then have $\bar{d}_i^{s*} = U_i^{-1}(\bar{d}_0^{s*} \exp(q_i - 1))$. Since $U_i^{-1}(\cdot)$ is strictly increasing, $U_i^{-1}(0+) = 0$, and $U_i^{-1}(+\infty) = 1$, then there exists a unique $\bar{d}_0^{s*} \in (0, 1)$ that satisfies the following:

$$\bar{d}_0^{s*} + U_1^{-1}(\bar{d}_0^{s*} \exp(q_1 - 1)) + U_2^{-1}(\bar{d}_0^{s*} \exp(q_2 - 1)) = 1, \tag{6}$$

that is, $\bar{d}_0^{s*} + \bar{d}_1^{s*} + \bar{d}_2^{s*} = 1$, where $\bar{d}_i^{s*} = U_i^{-1}(\bar{d}_0^{s*} \exp(q_i - 1))$ for $i = 1, 2$. Since the left-hand side of equation (6), $d_0 + U_1^{-1}(d_0 \exp(q_1 - 1)) + U_2^{-1}(d_0 \exp(q_2 - 1))$, is strictly increasing in d_0, one can solve for $\bar{d}_0^{s*}$ efficiently (e.g., using binary search).

Since $p_i^{s*} = q_i + \log(\bar{d}_0^{s*} / \bar{d}_i^{s*})$, then $p_i^{s*} = q_i + \log \left[\bar{d}_0^{s*} / U_i^{-1}(\bar{d}_0^{s*} \exp(q_i - 1)) \right]$ for $i = 1, 2$. In addition, w_i^{s*} can be computed by solving the equation $d_i^{s*} = \Lambda \bar{d}_i^{s*} = s_i^{s*}$, for $i = 1, 2$. $\qquad \square$

Proof of Theorem 4

We first observe that the same argument as in the proof of Step I of Theorem 1 implies that, in equilibrium, the supply and demand of each platform should match. More specifically, if $\tilde{s}_i^* > \tilde{\lambda}_i^*$ (resp. $\tilde{s}_i^* < \tilde{\lambda}_i^*$), P_i can decrease (resp. increase) its wage $\tilde{w}_i$ (resp. price $\tilde{p}_i$) by a sufficiently small amount to strictly increase its profit. Here, $\tilde{s}_i^*$ is the equilibrium supply of P_i, $\tilde{\lambda}_1^* = \tilde{d}_1^* + \tilde{d}_n^*/\tilde{n}$ is the total equilibrium demand for P_1's workers, and $\tilde{\lambda}_2^* = \tilde{d}_2^*$ is the total equilibrium demand for P_2's workers. Using $\tilde{s}_i^* = \tilde{\lambda}_i^*$, we can write P_i's profit function as follows

$$\tilde{\pi}_i(\tilde{p}_1, \tilde{w}_1, \tilde{p}_2, \tilde{w}_2) = (\tilde{p}_i - \tilde{w}_i) \min\left\{ \tilde{d}_i, \frac{\tilde{s}_i \tilde{d}_i}{\tilde{\lambda}_i} \right\}$$

$$= (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1)\frac{\tilde{d}_n}{\tilde{n}} = (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i + \gamma_i\left(\tilde{p}_n - \frac{\tilde{w}_1}{\tilde{n}}\right)\tilde{d}_n.$$

Given P_{-i}'s strategy, $(\tilde{p}_{-i}, \tilde{w}_{-i})$, we use $\tilde{p}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ and $\tilde{w}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ to denote the best-response price and wage of P_i under coopetition. Given $(\tilde{p}_{-i}, \tilde{w}_{-i}, \tilde{p}_n)$, we define d as the demand of P_i's original service and λ as the total number of requests for P_i's workers. For $i = 2$ and $\lambda = d$, the proof reduces to that of Theorem 1. For $i = 1$, it should be noted that, under equilibrium, d and λ satisfy $\frac{\tilde{n}(\lambda - d)}{d} = \frac{\exp(q_n - \tilde{p}_n)}{\exp(q_1 - \tilde{p}_1)}$. We next write $(\tilde{p}_1, \tilde{w}_1)$ in terms of d and λ: $\tilde{p}_1 = q_1 - \frac{d/\Lambda}{1-d/\Lambda} - \log[1 + \exp(\min\{1, \tilde{s}_{-2}/\tilde{\lambda}_2\}(q_2 - \tilde{p}_2)) + \exp(q_n - \tilde{p}_n)]$ and $\tilde{w}_1 = -a_1 + \frac{\lambda}{1-\lambda} + \log[1 + \exp(a_2 + \min\{1, \tilde{\lambda}_2/\tilde{s}_2\}\tilde{w}_2)]$. As in the proof of Theorem 1, solving for the best-response functions $\tilde{p}_1(\tilde{p}_{-2}, \tilde{w}_{-2})$ and $\tilde{w}_1(\tilde{p}_{-2}, \tilde{w}_{-2})$ is equivalent to solving for the optimal d and λ. Specifically, after some algebraic manipulations, P_i's profit can be computed using the following two-dimensional optimization formulation, the solution to which we denote as (d^*, λ^*):

$$\max_{(d,\lambda)} \tilde{\pi}_1(d, \lambda | \tilde{p}_2, \tilde{w}_2)$$

$$\text{s.t. } \tilde{\pi}_1(d, \lambda | \tilde{p}_2, \tilde{w}_2) = \left\{ q_1 - \log\left(\frac{d/\Lambda}{1-d/\Lambda}\right) - \log[1 + \exp(\min\{1, \tilde{s}_2/\tilde{\lambda}_2\}(q_2 - \tilde{p}_2)) + \exp(q_n - \tilde{p}_n)] \right\} d$$

$$- \left\{ -a_i + \log\left(\frac{\lambda}{1-\lambda}\right) + \log[\exp(a_{-i} + \min\{1, \tilde{\lambda}_2/\tilde{s}_2\}\tilde{w}_2)] \right\} \lambda + \frac{\gamma_1 \tilde{p}_n(\Lambda - d)\exp(q_n - \tilde{p}_n)}{1 + \exp(q_2 - \tilde{p}_2) + \exp(q_n - \tilde{p}_n)} \quad (7)$$

$$\frac{\lambda - d}{d} = \frac{\exp(q_n - \tilde{p}_n)}{\tilde{n}\exp(q_1 - \tilde{p}_1)}$$

$$\tilde{p}_1 = q_1 - \log\left(\frac{d/\Lambda}{1-d/\Lambda}\right) - \log[1 + \exp(\min\{1, \tilde{s}_{-2}/\tilde{\lambda}_2\}(q_2 - \tilde{p}_2)) + \exp(q_n - \tilde{p}_n)].$$

Following the same argument as in the proof of Step II of Theorem 1, we can show that both $\tilde{p}_i = q_i - \log\left(\frac{d/\Lambda}{1-d/\Lambda}\right) - \log[1 + \exp(\min\{1, \tilde{s}_{-i}/\tilde{\lambda}_{-i}\}(q_{-i} - \tilde{p}_{-i})) + \exp(\min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_n - \tilde{p}_n))]$ and $\tilde{w}_i = -a_i + \log\left(\frac{\lambda}{1-\lambda}\right) + \log[1 + \exp(a_{-i} + \min\{1, \tilde{\lambda}_{-i}/\tilde{s}_{-i}\}\tilde{w}_{-i})]$ are continuously increasing in $\tilde{p}_{-i}$ and $\tilde{w}_{-i}$ for $i = 1, 2$. Therefore, by Tarski's Fixed Point Theorem, an equilibrium exists for the model with coopetition.

To show that the equilibrium is unique, we follow the same argument as in the proof of Lemma 4. It suffices to show that for some k, the k-fold best-response mapping, $\tilde{T}^{(k)}(p_1, w_1, p_2, w_2)$, (defined

in a similar fashion as $T^{(k)}(\cdot,\cdot,\cdot,\cdot)$, but for the model with coopetition) is a contraction mapping. The same argument as in the proof of Lemma 4 implies that for any (p_1,w_1,p_2,w_2) and any $\delta > 0$,

$$||T^{(k)}(p_1+\delta,w_1,p_2,w_2) - T^{(k)}(p_1,w_1,p_2,w_2)||_1 \le 2C^k\delta$$

$$||T^{(k)}(p_1,w_1+\delta,p_2,w_2) - T^{(k)}(p_1,w_1,p_2,w_2)||_1 \le 2C^k\delta$$

$$||T^{(k)}(p_1,w_1,p_2+\delta,w_2) - T^{(k)}(p_1,w_1,p_2,w_2)||_1 \le 2C^k\delta$$

$$||T^{(k)}(p_1,w_1,p_2,w_2+\delta) - T^{(k)}(p_1,w_1,p_2,w_2)||_1 \le 2C^k\delta,$$

which together with the triangle inequality, leads to

$$||\tilde{T}^{(k)}(p_1,w_1,p_2,w_2) - \tilde{T}^{(n)}(p_1',w_1',p_2',w_2')||_1 \le 2C^k||(p_1,w_1,p_2,w_2) - (p_1',w_1',p_2',w_2')||_1,$$

where $C := \max\left\{\frac{\exp(a_i)}{1+\exp(a_i)}, \frac{\exp(q_i)}{1+\exp(q_i)} : i = 1,2\right\} < 1$. Consequently, $\tilde{T}^{(k^*)}$ is a contraction mapping under the ℓ_1 norm, where k^* is the smallest integer satisfying $2C^{(k^*)} < 1$ (i.e., $k^* > -\log(2)/\log(C)$). The contraction mapping property of $T^{(k^*)}(\cdot,\cdot,\cdot,\cdot)$, as shown in the proof of Theorem 1, implies that the equilibrium is unique in the presence of coopetition, and that it can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 4. $\qquad\square$

Proof of Theorem 5

We first show that if $\tilde{p}_n \uparrow +\infty$, then $(\tilde{p}_i^*, \tilde{w}_i^*)$ converges to (p_i^*, w_i^*) for $i = 1,2$. For given (p_1,w_1,p_2,w_2), we define the two-dimensional sequence $\{(\tilde{p}_1(k,j), \tilde{w}_1(k,j), \tilde{p}_2(k,j), \tilde{w}_2(k,j)) : k \ge 1, j \ge 1\}$, where $(\tilde{p}_1(k,j), \tilde{w}_1(k,j), \tilde{p}_2(k,j), \tilde{w}_2(k,j)) = \tilde{T}^{(k)}(p_1,w_1,p_2,w_2)$ with $\tilde{p}_n = j$. From the proof of Lemma 4, we know that $\lim_{j\uparrow+\infty}(\tilde{p}_1(k,j), \tilde{w}_1(k,j), \tilde{p}_2(k,j), \tilde{w}_2(k,j)) = T^{(k)}(p_1,w_1,p_2,w_2)$.

Therefore, as shown in the proof of Theorem 4, the equilibrium strategies with $\tilde{p}_n = j$ satisfy $(\tilde{p}_1^*(j), \tilde{w}_1^*(j), \tilde{p}_2^*(j), \tilde{w}_2^*(j)) = \lim_{k\uparrow+\infty}(\tilde{p}_1(k,j), \tilde{w}_1(k,j), \tilde{p}_2(k,j), \tilde{w}_2(k,j))$. Using the proof of Theorem 4, we have $||T^{(k)}(p_1,w_1,p_2,w_2) - T^{(k)}(p_1',w_1',p_2',w_2')||_1 \le 2C^k||(p_1,w_1,p_2,w_2) - (p_1',w_1',p_2',w_2')||_1$ for $k \ge 1$. Thus,

$$|\tilde{p}_1(k+1,j) - \tilde{p}_1(k,j)| \le 2C^k||(\tilde{p}_1(1,j), \tilde{w}_1(1,j), \tilde{p}_2(1,j), \tilde{w}_2(1,j)) - (p_1,w_1,p_2,w_2)||_1$$

$$|\tilde{w}_1(k+1,j) - \tilde{w}_1(k,j)| \le 2C^k||(\tilde{p}_1(1,j), \tilde{w}_1(1,j), \tilde{p}_2(1,j), \tilde{w}_2(1,j)) - (p_1,w_1,p_2,w_2)||_1$$

$$|\tilde{p}_2(k+1,j) - \tilde{p}_2(k,j)| \le 2C^k||(\tilde{p}_1(1,j), \tilde{w}_1(1,j), \tilde{p}_2(1,j), \tilde{w}_2(1,j)) - (p_1,w_1,p_2,w_2)||_1$$

$$|\tilde{w}_2(k+1,j) - \tilde{w}_2(k,j)| \le 2C^k||(\tilde{p}_1(1,j), \tilde{w}_1(1,j), \tilde{p}_2(1,j), \tilde{w}_2(1,j)) - (p_1,w_1,p_2,w_2)||_1.$$

As a result, $\sum_{k=1}^{+\infty}|\tilde{p}_1(k+1,j) - \tilde{p}_1(k,j)| < +\infty$, $\sum_{k=1}^{+\infty}|\tilde{w}_1(k+1,j) - \tilde{w}_1(k,j)| < +\infty$, $\sum_{k=1}^{+\infty}|\tilde{p}_2(k+1,j) - \tilde{p}_2(k,j)| < +\infty$, and $\sum_{k=1}^{+\infty}|\tilde{w}_2(k+1,j) - \tilde{w}_2(k,j)| < +\infty$. Using the dominated convergence theorem, we obtain

$$\lim_{j\uparrow+\infty}\lim_{k\uparrow+\infty}(\tilde{p}_1(k,j), \tilde{w}_1(k,j), \tilde{p}_2(k,j), \tilde{w}_2(k,j)) = \lim_{k\uparrow+\infty}\lim_{j\uparrow+\infty}(\tilde{p}_1(k,j), \tilde{w}_1(k,j), \tilde{p}_2(k,j), \tilde{w}_2(k,j))$$

$$\text{that is, } \lim_{j\uparrow+\infty}(\tilde{p}_1^*(j), \tilde{w}_1^*(j), \tilde{p}_2^*(j), \tilde{w}_2^*(j)) = \lim_{j\uparrow+\infty}\lim_{k\uparrow+\infty}(\tilde{p}_1(k,j), \tilde{w}_1(k,j), \tilde{p}_2(k,j), \tilde{w}_2(k,j))$$

$$= \lim_{k\uparrow+\infty}\lim_{j\uparrow+\infty}(\tilde{p}_1(k,j), \tilde{w}_1(k,j), \tilde{p}_2(k,j), \tilde{w}_2(k,j))$$

$$= \lim_{k\uparrow+\infty}T^{(k)}(p_1,w_1,p_2,w_2) = (p_1^*,w_1^*,p_2^*,w_2^*),$$

which states that if $\tilde{p}_n \uparrow +\infty$, then $(\tilde{p}_i^*, \tilde{w}_i^*)$ converges to (p_i^*, w_i^*) for $i = 1, 2$.

We next show that $\tilde{\pi}(\tilde{p}_n) := \tilde{\pi}_1(\tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), w_2^*(\tilde{p}_n)) + \tilde{\pi}_2(\tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), w_2^*(\tilde{p}_n))$ is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, where $(\tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), w_2^*(\tilde{p}_n))$ is the equilibrium outcome under coopetition when the price of the new service is $\tilde{p}_n$.

We first show that, under a given equilibrium price and wage vector $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ associated with $\tilde{p}_n$, the total profit of both platforms, $\tilde{\pi}(\tilde{p}_n | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, where

$$\tilde{\pi}(\tilde{p}_n | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = (\tilde{p}_1^* - \tilde{w}_1^*) \min\left\{\tilde{d}_1, \frac{\tilde{s}_1 \tilde{d}_1}{\tilde{\lambda}_1}\right\} + (\tilde{p}_2^* - \tilde{w}_2^*) \min\left\{\tilde{d}_2, \frac{\tilde{s}_2 \tilde{d}_2}{\tilde{\lambda}_2}\right\} + (\tilde{n}\tilde{p}_n - \tilde{w}_1^*) \min\left\{\frac{\tilde{d}_n}{\tilde{n}}, \frac{\tilde{s}_1 \tilde{d}_n}{\tilde{n}\tilde{\lambda}_1}\right\}$$

$$= (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1 + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2 + \left(\tilde{p}_n - \frac{\tilde{w}_1^*}{\tilde{n}}\right)\tilde{d}_n,$$

with $\tilde{\lambda}_1 = \tilde{d}_1 + \tilde{d}_n/\tilde{n} = \tilde{s}_1$ and $\tilde{\lambda}_2 = \tilde{d}_2 = \tilde{s}_2$ under equilibrium. We then obtain

$$\tilde{d}_i = \Lambda \bar{d}_i = \frac{\Lambda \exp(q_i - \tilde{p}_i^*)}{1 + \exp(q_1 - \tilde{p}_1^*) + \exp(q_2 - \tilde{p}_2^*) + \exp(q_n - \tilde{p}_n)}, \text{ for } i = 1, 2$$

and

$$\tilde{d}_n = \Lambda \bar{d}_n = \frac{\Lambda \exp(q_n - \tilde{p}_n)}{1 + \exp(q_1 - \tilde{p}_1^*) + \exp(q_2 - \tilde{p}_2^*) + \exp(q_n - \tilde{p}_n)}.$$

By Lemma 1, we have

$$\partial_{\tilde{p}_n} \tilde{\pi}(\tilde{p}_n | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = (\tilde{p}_1^* - \tilde{w}_1^*)\partial_{\tilde{p}_n}\tilde{d}_1 + (\tilde{p}_2^* - \tilde{w}_2^*)\partial_{\tilde{p}_n}\tilde{d}_2 + \tilde{d}_n + \left(\tilde{p}_n - \frac{\tilde{w}_1^*}{\tilde{n}}\right)\partial_{\tilde{p}_n}\tilde{d}_n$$

$$= \Lambda(\tilde{p}_1^* - \tilde{w}_1^*)\bar{d}_1\bar{d}_n + \Lambda(\tilde{p}_2^* - \tilde{w}_2^*)\bar{d}_2\bar{d}_n + \Lambda\bar{d}_n - \left(\tilde{p}_n - \frac{\tilde{w}_1^*}{\tilde{n}}\right)\Lambda(1 - \bar{d}_n)\bar{d}_n.$$

Hence, $\partial_{\tilde{p}_n} \tilde{\pi}(\tilde{p}_n | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = 0$ implies that

$$\tilde{p}_n = \frac{(\tilde{p}_1^* - \tilde{w}_1^*)\bar{d}_1}{1 - \bar{d}_n} + \frac{(\tilde{p}_2^* - \tilde{w}_2^*)\bar{d}_2}{1 - \bar{d}_n} + \frac{1}{1 - \bar{d}_n} + \frac{\tilde{w}_1^*}{\tilde{n}} = (\tilde{p}_1^* - \tilde{w}_1^*)\bar{d}_1^* + (\tilde{p}_2^* - \tilde{w}_2^*)\bar{d}_2^* + \frac{1}{1 - \bar{d}_n} + \frac{\tilde{w}_1^*}{\tilde{n}}, \quad (8)$$

where $\bar{d}_1^*$ (resp. $\bar{d}_2^*$) is the equilibrium market share of P_1 (resp. P_2), when $\tilde{p}_n$ satisfies (8). We observe that the right-hand side of equation (8) is decreasing with respect to $\tilde{p}_n$. Therefore, there exists a unique $\tilde{p}_n^*$ such that $\tilde{p}_n^* = (\tilde{p}_1^* - \tilde{w}_1^*)\bar{d}_1^* + (\tilde{p}_2^* - \tilde{w}_2^*)\bar{d}_2^* + \frac{1}{1 - \bar{d}_n} + \frac{\tilde{w}_1^*}{\tilde{n}}$. Furthermore, one can check that $\partial_{\tilde{p}_n} \tilde{\pi}(\tilde{p}_n | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > 0$ (resp. < 0) if $\tilde{p}_n < \tilde{p}_n^*$ (resp. $\tilde{p}_n > \tilde{p}_n^*$). As a result, $\tilde{\pi}(\cdot | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ is decreasing in $\tilde{p}_n$ for $\tilde{p}_n \geq \tilde{p}_n^*$. Note that $\tilde{p}_n^*$ is uniformly bounded from above by an upper bound on the right-hand side of equation (8), say $\bar{p}^* := (p_1^* - w_1^* + p_2^* - w_2^*) + w_1^* + \frac{1}{1 - \bar{d}_0'}$, where $\bar{d}_0'$ is the market share of the new joint service with $\tilde{p}_n = 0$. It then follows that, when $\tilde{p}_n \geq \bar{p}^*$, $\tilde{\pi}(\tilde{p}_n) = \tilde{\pi}(\tilde{p}_n | \tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), \tilde{w}_2^*(\tilde{p}_n))$ is strictly decreasing in $\tilde{p}_n$.

We observe that as $\tilde{p}_n \uparrow +\infty$, $\tilde{d}_n \downarrow 0$. Since $(\tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), \tilde{w}_2^*(\tilde{p}_n))$ approaches $(p_1^*, w_1^*, p_2^*, w_2^*)$ when $\tilde{p}_n \uparrow +\infty$, then $\tilde{\pi}(\tilde{p}_n) = \tilde{\pi}(\tilde{p}_n | \tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), \tilde{w}_2^*(\tilde{p}_n))$ approaches the equilibrium total profit of P_1 and P_2 without coopetition, that is, $\pi^* := \pi_1(p_1^*, w_1^*, p_2^*, w_2^*) +$

$\pi_2(p_1^*, w_1^*, p_2^*, w_2^*)$. Since we have shown that $\tilde{\pi}(\cdot)$ is strictly decreasing in $\tilde{p}_n \geq \bar{p}^*$ and $\lim_{\tilde{p}_n \to +\infty} \tilde{\pi}(\tilde{p}_n) = \pi^*$, then $\tilde{\pi}^* := \max_{\tilde{p}_n} \tilde{\pi}(\tilde{p}_n) > \pi^*$, that is, the maximum total profit of P_1 and P_2 with coopetition dominates the maximum total profit without coopetition for any $\gamma \in (0, 1)$. In other words, for $\tilde{p}_n \geq \bar{p}^*$, $\tilde{\pi}_1(\tilde{p}_n | \tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), \tilde{w}_2^*(\tilde{p}_n)) + \tilde{\pi}_2(\tilde{p}_n | \tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), \tilde{w}_2^*(\tilde{p}_n)) > \pi_1(p_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) + \pi_2(p_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$. Therefore, there exists a range of profit sharing parameters $(\underline{\gamma}, \bar{\gamma}) \subset (0, 1)$, such that when $\gamma \in (\underline{\gamma}, \bar{\gamma})$, $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_i(p_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ for $i = 1, 2$. This concludes the proof of Theorem 5. $\qquad\square$

Proof of Proposition 3

By Theorem 5, we can select $\gamma_0 \in (\underline{\gamma}, \bar{\gamma})$ and $\tilde{p}_n^* = \arg\max_{\tilde{p}_n} \{\pi_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) + \pi_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)\}$ that maximizes the total profit of both platforms, so that $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^* | \tilde{p}_n^*, \gamma_0) > \pi_i(p_1^*, w_1^*, p_2^*, w_2^*)$, for $i = 1, 2$. Thus, for any $\theta_1 + \theta_2 = 1$ $(\theta_i > 0)$, $(\tilde{p}_n^*, \gamma_0)$ is a feasible solution to the optimization in problem (3). Therefore, an optimal solution to (3), $(\tilde{p}_n^{**}, \gamma^{**})$, exists with

$$(\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^* | \tilde{p}_n^{**}, \gamma^{**}) - \pi_1(p_1^*, w_1^*, p_2^*, w_2^*))^{\theta_1} \cdot (\tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^* | \tilde{p}_n^{**}, \gamma^{**}) - \pi_2(p_1^*, w_1^*, p_2^*, w_2^*))^{\theta_2}$$

$$\geq (\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^* | \tilde{p}_n^*, \gamma_0) - \pi_1(p_1^*, w_1^*, p_2^*, w_2^*))^{\theta_1} \cdot (\tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^* | \tilde{p}_n^*, \gamma_0) - \pi_2(p_1^*, w_1^*, p_2^*, w_2^*))^{\theta_2} > 0.$$

As a result, we have $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^* | \tilde{p}_n^{**}, \gamma^{**}) > \pi_i(p_1^*, w_1^*, p_2^*, w_2^*)$ for $i = 1, 2$, which concludes the proof of Proposition 3. $\qquad\square$

Proof of Proposition 4

One can check using equation (7) that the best-response mapping sequences, $(\tilde{p}_1(k, j), \tilde{w}_1(k, j), \tilde{p}_2(k, j), \tilde{w}_2(k, j))$, satisfy the following: as $\Lambda \uparrow +\infty$, $\lambda_i^*(k, j) \uparrow 1$ for all (k, j) and $i = 1, 2$. Here, $\lambda_i^*(k, j)$ is the optimal total requests for P_i's workers in the k−th iteration of the *tatônnement* scheme with $\tilde{p}_n = j$. Thus, if $\Lambda \uparrow +\infty$, then $\tilde{w}_1(k, j) = -a_1 + \log\left(\frac{\lambda_1^*(k, j)}{1 - \lambda_1^*(k, j)}\right) + \log[a_2 + w_2(k - 1, j)] \uparrow +\infty$ and $\tilde{w}_2(k, j) = -a_2 + \log\left(\frac{\lambda_2^*(k, j)}{1 - \lambda_2^*(k, j)}\right) + \log[a_1 + w_1(k - 1, j)] \uparrow +\infty$ for any k and j. As a result, for any $\tilde{p}_n$, the equilibrium strategy under coopetition, $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$, satisfies $\lim_{\Lambda \uparrow +\infty} \tilde{w}_1^* = +\infty$ and $\lim_{\Lambda \uparrow +\infty} \tilde{w}_2^* = +\infty$. Thus, using equation (8), we have $\lim_{\Lambda \uparrow +\infty} \tilde{p}_n^* = +\infty$. To show that $\lim_{\Lambda \uparrow +\infty} \tilde{p}_n^{**} \uparrow +\infty$, we note that $\lim_{\Lambda \uparrow +\infty} \tilde{w}_1^* / \tilde{n} = +\infty$. Under the Nash Bargaining equilibrium, we must have $\tilde{p}_n > \tilde{w}_1 / \tilde{n}$, which together with $\lim_{\Lambda \uparrow +\infty} \tilde{w}_1^* / \tilde{n} = +\infty$ leads to $\lim_{\Lambda \uparrow +\infty} \tilde{p}_n^{**} \uparrow +\infty$. This concludes the proof of the first part.

We next show that the total profit under coopetition increases when $\tilde{p}_n = \bar{p}$ and Λ is sufficiently small. A similar argument as in the previous paragraph shows that as $\Lambda \downarrow 0$, $w_i^* \downarrow 0$ and $\tilde{w}_i^* \downarrow 0$ $(i = 1, 2)$. Therefore, for $\tilde{p}_n = \bar{p}$, the equilibrium profit from the new service $(\tilde{p}_n - \tilde{w}_1^* / \tilde{n}) \tilde{d}_n^* > 0$. This implies that the total profit under coopetition increases when $\tilde{p}_n = \bar{p}$ and Λ is sufficiently small. Consequently, we can find a profit sharing parameter γ such that $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{p}_2^*, \tilde{p}_2^*, \tilde{w}_2^* | \bar{p}, \gamma) > \pi_i(p_1^*, w_1^*, p_2^*, w_2^*)$ for $i = 1, 2$. This concludes the proof of Part (i) of the second bullet point.

Using the proof of Theorem 5, the total profit of both platforms under coopetition, $\tilde{\pi}_1 + \tilde{\pi}_2$, is higher relative to the setting without coopetition, when $\tilde{p}_n$ is sufficiently large. Hence, if $\bar{p}$ is sufficiently large, then $\tilde{\pi}_1 + \tilde{\pi}_2 > \pi_1 + \pi_2$ and we can find a parameter γ such that $\tilde{\pi}_1 > \pi_1$ and $\tilde{\pi}_2 > \pi_2$ under equilibrium. This concludes the proof of the second part of Proposition 4.

We next show that if there is a finite upper bound on the prices set by the platforms, $\bar{p}$, at least one platform would be worse off under coopetition, i.e., $\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < \pi_1(p_1^*, w_1^*, p_2^*, w_2^*)$ or $\tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < \pi_2(p_1^*, w_1^*, p_2^*, w_2^*)$, when Λ is sufficiently large. From the proof of Step II of Theorem 1, given any P_{-i}'s strategy, (p_{-i}, w_{-i}), the best-response profit of P_i can be written as

$$\max_s \pi_i(s)$$

where $\pi_i(s) = (p_i(p_{-i}, w_{-i}, s) - w_i(p_{-i}, w_{-i}, s))s$

$$= \left\{ q_i + a_i - \log\left(\frac{s/\Lambda}{1 - s/\Lambda}\right) - \log\left(\frac{s}{1-s}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] \right.$$
$$\left. - \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})] \right\}s$$
$$= \left\{ q_i + a_i + \log(\Lambda - s) + \log(1 - s) - 2\log(s) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] \right.$$
$$\left. - \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})] \right\}s.$$

We thus have $\lim_{\Lambda \uparrow +\infty} \pi_i(s) = +\infty$ for any feasible s, which implies that $\lim_{\Lambda \uparrow +\infty} \max_s \pi_i(s) = +\infty$. Note that the optimal sales s^* should satisfy the first-order condition $\pi'(s^*) = 0$, that is,

$$q_i + a_i - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] - \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})] - \frac{s^*}{\Lambda - s^*} +$$
$$+ \log(\Lambda - s^*) - \frac{s^*}{1 - s^*} + \log(1 - s^*) - 2\log(s^*) - 2 = 0.$$

Therefore, s^* is increasing in Λ, and as $\Lambda \uparrow +\infty$, $s^* \uparrow 1$. Thus, for any (p_{-i}, w_{-i}), when $\Lambda \uparrow +\infty$, we have $w_i(p_{-i}, w_{-i}) = -a_i + \log\left(\frac{s^*}{1-s^*}\right) + \log[1 + \exp(a_{-i} + \min\{1, d_{-i}/s_{-i}\}w_{-i})] \uparrow +\infty$, which implies that $\lim_{\Lambda \uparrow +\infty} w_i^* = +\infty$ for $i = 1, 2$. From equation (7), for any $\tilde{p}_{-i}, \tilde{w}_{-i}$, and $\tilde{p}_n$, the best response satisfies $\lim_{\Lambda \uparrow +\infty} \tilde{w}_i(\tilde{p}_{-i}, \tilde{w}_{-i}) = +\infty$ for $i = 1, 2$. This implies that the equilibrium wage satisfies $\lim_{\Lambda \uparrow +\infty} \tilde{w}_i^* = +\infty$ for $i = 1, 2$. If $\tilde{p}_n \le \bar{p} < +\infty$, then the profit margin of the new joint service is negative when Λ is sufficiently large, i.e., $\tilde{n}\tilde{p}_n - \tilde{w}_i^* < 0$.

On the other hand, in the presence of coopetition, when P_{-i} sets the price of its original service (p_{-i}) and given the sales of P_i (s), P_i's price satisfies $\tilde{p}_i = q_i - \log\left(\frac{s/\Lambda}{1 - s/\Lambda}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i})) + \exp(\min\{1, s_1/d_1\}(q_n - \tilde{p}_n))] < q_i - \log\left(\frac{s/\Lambda}{1 - s/\Lambda}\right) - \log[1 + \exp(\min\{1, s_{-i}/d_{-i}\}(q_{-i} - p_{-i}))] = p_i$, i.e., P_i needs to charge a lower price to induce the same sales level under coopetition assuming its competitor offers the same price. As a result, for any (p_{-i}, w_{-i}), P_i's optimal profit from its original service is lower under coopetition. Using a limit argument, this implies that P_i's equilibrium profit from its original service is lower under coopetition for $i = 1, 2$. Since we have shown that for sufficiently large Λ, the total profit from the new service is negative, the total profit of both platforms is lower under coopetition. Consequently, at least one of the platforms is worse off for any γ, when Λ is sufficiently large, and this concludes the proof of Proposition 4. $\square$

Proof of Proposition 5

First, since $d_i^* = s_i^*$ without coopetition and $\tilde{\lambda}_i^* = \tilde{s}_i^*$ with coopetition for $i = 1, 2$, we have

$$RS^* = \Lambda \mathbb{E}\Big[\max\{q_1 - p_1^* + \xi_1, q_2 - p_2^* + \xi_2, \xi_0\}\Big]$$

and

$$\tilde{RS}^* = \Lambda \mathbb{E}\Big[\max\{q_1 - \tilde{p}_1^* + \xi_1, q_2 - \tilde{p}_2^* + \xi_2, q_2 - \tilde{p}_n^* + \xi_n, \xi_0\}\Big].$$

We observe that if $\tilde{p}_i^* \le p_i^*$ for $i = 1, 2$, then we have

$$\begin{aligned}
\tilde{RS}^* &= \Lambda \mathbb{E}\Big[\max\{q_1 - \tilde{p}_1^* + \xi_1, q_2 - \tilde{p}_2^* + \xi_2, q_2 - \tilde{p}_n^* + \xi_n, \xi_0\}\Big] \\
&> \Lambda \mathbb{E}\Big[\max\{q_1 - p_1^* + \xi_1, q_2 - p_2^* + \xi_2, \xi_0\}\Big] = RS^*.
\end{aligned}$$

Consequently, it suffices to show that $\tilde{p}_i^* \le p_i^*$ for $i = 1, 2$.

We define $(\tilde{p}_1^*(k, \tilde{p}_n), \tilde{w}_1^*(k, \tilde{p}_n), \tilde{p}_2^*(k, \tilde{p}_n), \tilde{w}_2^*(k, \tilde{p}_n)) := \tilde{T}^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$, where $\tilde{T}^{(k)}(\cdot, \cdot, \cdot, \cdot)$ is the $k-$fold best-response mapping when the price of the new service is $\tilde{p}_n$. On the other hand, we know that $(p_1^*, w_1^*, p_2^*, w_2^*) = T^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$ for any $k \ge 1$, where $T^{(k)}(\cdot, \cdot, \cdot, \cdot)$ is the $k-$fold best-response mapping of the model without coopetition, which can also be viewed as a special case of $\tilde{T}^{(k)}(\cdot, \cdot, \cdot, \cdot)$ with $\tilde{p}_n = +\infty$. Comparing the best-response formulations of $\tilde{T}^{(1)}$ and $T^{(1)}$, one can show that given (p_{-i}^*, w_{-i}^*), the best-response price $\tilde{p}_i^*(1, \tilde{p}_n)$ is increasing in $\tilde{p}_n$. Since the model without coopetition can be viewed as a special case of the model with coopetition when $\tilde{p}_n = +\infty$, we have $\tilde{p}_i^*(1, \tilde{p}_n) < \tilde{p}_i^*(1, +\infty) = p_i^*$ for $i = 1, 2$. Then, following the same argument as in the proof of Theorem 4, we conclude that $\tilde{p}_i^*(k+1, \tilde{p}_n)$ is strictly increasing in both $\tilde{p}_n$ and $\tilde{p}_{-i}^*(k)$ for $i = 1, 2$. Using a standard induction argument, we thus obtain $\tilde{p}_i^*(k, \tilde{p}_n) < \tilde{p}_i^*(k, +\infty) = p_i^*$ for $k \ge 1$ and $i = 1, 2$. Therefore, $\tilde{p}_i^* = \lim_{k \uparrow +\infty} \tilde{p}_i^*(k, \tilde{p}_n) < p_i^*$ for $i = 1, 2$, and this concludes the proof of Proposition 5. □

Proof of Proposition 6

We first rewrite DS_i^* and $\tilde{DS}_i^*$ as functions of (s_1^*, s_2^*, s_0^*) and $(\tilde{s}_1^*, \tilde{s}_2^*, \tilde{s}_0^*)$. Specifically, one can derive the following expressions: $DS_i^* = K_i \log\left(\frac{1}{1-s_i^*}\right) + c$ and $\tilde{DS}_i^* = K_i \log\left(\frac{1}{1-\tilde{s}_i^*}\right) + c$. For more details on the consumer surplus under the MNL model and on the derivation of the above expressions, we refer the reader to the literature (see, e.g., Chapter 3.5 of Train 2009).

We next show the first bullet point. Specifically, we show that (a) if $\tilde{n} = 1$, then $\tilde{s}_1^* > s_1^*$; (b) if $\tilde{n}$ is sufficiently large, then $\tilde{s}_1^* < s_1^*$; and (c) $\tilde{s}_1^*$ is continuously decreasing in $\tilde{n}$. Then, claims (a), (b), and (c) below would imply the first bullet point of Proposition 6.

Claim (a): If $\tilde{n} = 1$, from the proof of Theorem 4, we have $\tilde{s}_1^* = \tilde{\lambda}_1^* = \tilde{d}_1^* + \tilde{d}_n^*/\tilde{n} = \tilde{d}_1^* + \tilde{d}_n^*$. Therefore, $\tilde{s}_1^* = \tilde{d}_1^* + \tilde{d}_n^*$ and $\tilde{s}_2^* = \tilde{d}_2^*$. As shown in the proof of Proposition 5, $\tilde{p}_i^* < p_i^*$ for $i = 1, 2$, and hence $\tilde{s}_1^* = \tilde{d}_1^* + \tilde{d}_n^* > d_1^* = s_1^*$. This concludes the proof of claim (a).

Claim (b): As $\tilde{n} \uparrow +\infty$, we have $\tilde{s}_1^* = \tilde{\lambda}_1^* = \tilde{d}_1^* + \tilde{d}_n^*/\tilde{n} = \tilde{d}_1^*$. To prove claim (b), it then suffices to show that $\tilde{d}_1^* < d_1^*$, or equivalently $\tilde{s}_1^* < s_1^*$. Using equation (7), one can see that when $\tilde{s}_i^* = \tilde{d}_i^*$, the optimization problem to characterize the best-response price and wage functions under coopetition reduces to the case without coopetition.

We define $(\tilde{p}_1^*(k), \tilde{w}_1^*(k), \tilde{p}_2^*(k), \tilde{w}_2^*(k)) := \tilde{T}^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$, where $\tilde{T}^{(k)}(\cdot, \cdot, \cdot, \cdot)$ is the $k-$fold best-response mapping when the price of the new service is $\tilde{p}_n^*$ and $\tilde{n} = +\infty$. On the other hand, we know that $(p_1^*, w_1^*, p_2^*, w_2^*) = T^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$ for $k \geq 1$, where $T^{(k)}(\cdot, \cdot, \cdot, \cdot)$ is the $k-$fold best-response mapping of the model without coopetition, which can also be viewed as a special case of $\tilde{T}^{(k)}(\cdot, \cdot, \cdot, \cdot)$ with $\tilde{p}_n = +\infty$. Comparing the best-response formulations of $\tilde{T}^{(1)}$ and $T^{(1)}$, we obtain that given (p_{-i}^*, w_{-i}^*), $\tilde{w}_i^*(1) < w_i^*$ for $i = 1, 2$. Furthermore, the best-response mapping is increasing in w_{-i} (see the proof of Theorem 1), and hence using the standard induction argument, we obtain $\tilde{w}_i^*(k) < w_i^*$ for $k \geq 1$ and $i = 1, 2$. Thus, the equilibrium wage satisfies $\tilde{w}_i^* = \lim_{k \uparrow +\infty} \tilde{w}_i^*(k) < w_i^*$ for $i = 1, 2$; implying that $\tilde{s}_1^* = \frac{K_1 \exp(a_1 + \tilde{w}_1^*)}{1 + \exp(a_1 + \tilde{w}_1^*)} > \frac{K_1 \exp(a_1 + w_1^*)}{1 + \exp(a_1 + w_1^*)} = s_1^*$, and hence concluding the proof of claim (b).

Claim (c): We first show that $\tilde{w}_i^*$ is decreasing in $\tilde{n}$. We define $(\tilde{p}_1^*(k, \tilde{n}), \tilde{w}_1^*(k, \tilde{n}), \tilde{p}_2^*(k, \tilde{n}), \tilde{w}_2^*(k, \tilde{n})) := \tilde{T}^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$, where $\tilde{T}^{(k)}(\cdot, \cdot, \cdot, \cdot)$ is the $k-$fold best-response mapping when the price of the new service is $\tilde{p}_n^*$ and the pooling parameter is $\tilde{n}$. By examining the best-response mapping $\tilde{T}^{(1)}$ (see the proof of Theorem 5), we obtain that given (p_{-i}^*, w_{-i}^*), $\tilde{w}_i^*(1, \tilde{n})$ is decreasing in $\tilde{n}$ for $i = 1, 2$. Furthermore, the best-response mapping is increasing in $\tilde{w}_{-i}$ (see the proof of Theorem 1). Using the standard induction argument, we obtain that $\tilde{w}_i^*(k, \tilde{n})$ is increasing in $\tilde{w}_{-i}^*(k - 1, \tilde{n})$, which is decreasing in $\tilde{n}$. Thus, $\tilde{w}_i^*(k, \tilde{n})$ is decreasing in $\tilde{n}$ for $k \geq 1$ and $i = 1, 2$. As a result, the equilibrium wage is such that $\tilde{w}_i^* = \lim_{k \uparrow +\infty} \tilde{w}_i^*(k, \tilde{n})$ is decreasing in $\tilde{n}$ for $i = 1, 2$, implying that $\tilde{s}_1^* = \frac{K_1 \exp(a_1 + \tilde{w}_1^*)}{1 + \exp(a_1 + \tilde{w}_1^*)}$ is decreasing in $\tilde{n}$. This concludes the proof of claim (c), and thus the proof of the first bullet point of Proposition 6.

We next show the second bullet point of Proposition 6. As shown in claim (b) above, $\tilde{w}_2^* < w_2^*$, so we must have $\tilde{s}_2^* = \frac{K_2 \exp(a_2 + \tilde{w}_2^*)}{1 + \exp(a_2 + \tilde{w}_2^*)} < s_2^* = \frac{K_2 \exp(a_2 + w_2^*)}{1 + \exp(a_2 + w_2^*)} = s_2^*$. Therefore, $\tilde{DS}_2^* = K_i \log(1/(1 - \tilde{s}_2^*)) + c < K_i \log(1/(1 - s_2^*)) + c = DS_2^*$. $\qquad\square$

Proof of Proposition 7

Following the same argument as in the proof of Theorem 5, we know that if $\tilde{p}_n \to +\infty$, then $\lim_{\tilde{p}_n \uparrow +\infty}(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = (p_1^*, w_1^*, p_2^*, w_2^*)$, $\lim_{\tilde{p}_n \uparrow +\infty}(\tilde{d}_1^*, \tilde{d}_2^*) = (d_1^*, d_2^*)$, and $\lim_{\tilde{p}_n \uparrow +\infty}(\tilde{s}_1^*, \tilde{s}_2^*) = (s_1^*, s_2^*)$. Therefore, we have $\lim_{\tilde{p}_n \uparrow +\infty} \tilde{\pi}_i^* + \tilde{DS}_i^* = \pi_i^* + DS_i^*$ for $i = 1, 2$.

We next show that $\tilde{R}(\tilde{p}_n) := \tilde{\pi}_i^* + \tilde{DS}_i^*$ is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, where $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ is the equilibrium under coopetition with $\tilde{p}_n$. Given the equilibrium price and wage vector $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$, we define the total platform and driver surplus of both platforms:

$$\tilde{R}(\tilde{p}_n | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = (\tilde{p}_1^* - \tilde{w}_1^*)d_1 + (\tilde{p}_2^* - \tilde{w}_2^*)d_2 + (\tilde{p}_n - \tilde{w}_1^*/n)\tilde{d}_n + K_1 \log[1/(1 - \tilde{s}_1)] + K_2 \log[1/(1 - \tilde{s}_2)],$$

where $\tilde{s}_1 = \tilde{d}_1 + \tilde{d}_n/\tilde{n}$ and $\tilde{s}_2 = \tilde{d}_2$. Following the same argument as in the proof of Theorem 5, we have $\partial_{\tilde{p}_n} R(\tilde{p}_n | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < 0$ for a sufficiently large $\tilde{p}_n$. This also shows that $R(\tilde{p}_n)$ is strictly decreasing in $\tilde{p}_n$ for a sufficiently large $\tilde{p}_n$. Furthermore, we have also shown that $\lim_{\tilde{p}_n \uparrow +\infty} \tilde{R}(\tilde{p}_n) = \lim_{\tilde{p}_n \uparrow +\infty} (\tilde{\pi}_1^* + \tilde{DS}_1^* + \tilde{\pi}_2^* + \tilde{DS}_2^*) = \pi_1^* + DS_1^* + \pi_2^* + DS_2^*$. Since $\tilde{R}(\cdot)$ is strictly decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, one can find a value of $\tilde{p}_n$ such that $\tilde{R}(\tilde{p}_n) > \pi_1^* + DS_1^* + \pi_2^* + DS_2^*$. Since $\tilde{R}(\tilde{p}_n) = \tilde{\pi}_1^* + \tilde{DS}_1^* + \tilde{\pi}_2^* + \tilde{DS}_2^*$, one can find a value of γ such that $\tilde{\pi}_i^* + \tilde{DS}_i^* > \pi_i^* + DS_i^*$ for $i = 1, 2$. This concludes the proof of Proposition 7. $\qquad\square$

Proof of Theorem 6

Given that $\kappa(0+) = +\infty$, we must have $s_i^{e*} > d_i^{e*}$ for $i = 1, 2$ under equilibrium. Hence, P_i's profit under equilibrium can be written as $[f_i - \kappa(s_i - d_i) - w_i]d_i$. Given (p_{-i}, w_{-i}), we rewrite P_i's profit as a function of d_i and s_i:

$$\pi_i^e(d_i, s_i | p_{-i}, w_{-i}) = \Big\{ q_i + a_i - \log\left(\frac{d_i/\Lambda}{1 - d_i/\Lambda} \right) - \log[1 + \exp(a_{-i} + d_{-i}w_{-i}/s_{-i})] - \log[1 + \exp(q_{-i} - p_{-i})]$$
$$- \log\left(\frac{s_i}{1 - s_i} \right) - \kappa(s_i - d_i) \Big\} d_i$$
$$s.t. \quad d_i < s_i.$$

Hence, given (f_{-i}, w_{-i}), P_i's best-response can be characterized as follows:

$$\max_{(d_i, s_i)} \pi_i^e(d_i, s_i | f_{-i}, w_{-i})$$
$$s.t. \ d_i < s_i.$$

Given P_i's demand, d_i, the best-response supply of P_i should be such that $s_i \in \arg\max_{s \in (d_i, 1)}[\log\left(\frac{s}{1-s} \right) + \kappa(s - d_i)]$. As a result, we can reduce $\pi_i^e(d_i, s_i | f_{-i}, w_{-i})$ to a single variable function: $\pi_i^e(d_i | f_{-i}, w_{-i}) = \Big\{ q_i + a_i - \log\left(\frac{d_i/\Lambda}{1 - d_i/\Lambda} \right) - \log[1 + \exp(a_{-i} + d_{-i}w_{-i}/s_{-i})] - \log[1 + \exp(q_{-i} - f_{-i})] - h(d_i) \Big\} d_i$, where $h(d_i) := \max_{s \in (d_i, 1)}[\log\left(\frac{s}{1-s} \right) + \kappa(s - d_i)]$.

We denote by $(f_i^e(f_{-i}, w_{-i}), w_i^e(f_{-i}, w_{-i}))$ P_i's best-response price and wage functions given (f_{-i}, w_{-i}). Following the same argument as in Step II of the proof of Theorem 1, we can show that $(f_i^e(f_{-i}, w_{-i}), w_i^e(f_{-i}, w_{-i}))$ is continuously increasing in f_{-i} and w_{-i}. Therefore, an equilibrium $(f_1^{e*}, w_1^{e*}, f_2^{e*}, w_2^{e*})$ exists.

To show that the equilibrium is unique, we denote by $T_e(\cdot, \cdot, \cdot, \cdot)$ the best-response mapping of the model with endogenous waiting times, that is, $T_e(f_1, w_1, f_2, w_2) = \big(f_1^e(f_2, w_2), w_1^e(f_2, w_2), f_2^e(f_1, w_1), w_2^e(f_1, w_1) \big)$. Using the same argument as in the proof of Lemma 4, we obtain that there exists a constant $C = \max\left\{ \frac{\exp(q_i)}{1+\exp(q_i)}, \frac{\exp(a_i)}{1+\exp(a_i)} : i = 1, 2 \right\} \in (0, 1)$, such that

$$||T_e^{(k)}(f_1, w_1, f_2, w_2) - T_e^{(k)}(f_1', w_1', f_2', w_2')||_1 \leq 2C^{(k)}||(f_1, w_1, f_2, w_2) - (f_1', w_1', f_2', w_2')||_1,$$

and thus, the k^*-fold best-response mapping, $T_e^{(k^*)}(\cdot, \cdot, \cdot, \cdot)$, is a contraction mapping, where $k^* > -\log(2)/\log(C)$. Consequently, using the same argument as in the proof of Lemma 4, the equilibrium is unique and can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 6. $\quad\square$