Competition and Coopetition for Two-sided Platforms

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Two-sided platforms have become omnipresent (e.g., ride-sharing and on-demand delivery services). In this context, firms compete not only for customers but also for flexible self-scheduling workers who can work for multiple platforms. We consider a setting where two-sided platforms simultaneously choose their prices and wages to compete for both sides of the market. We assume that customers and workers each follow a Multinomial Logit choice model (our results also extend to more general models), and show that a unique equilibrium exists and can be obtained using a *tatônnement* scheme. The proof technique for the competition between two-sided platforms is not a simple extension of the traditional (one-sided) setting and involves different arguments. Armed with this result, we study the impact of *coopetition* between two-sided platforms, i.e., the business strategy of cooperating with competitors. Motivated by recent practice in the ride-sharing industry, we analyze a setting where two competing platforms engage in a profit sharing contract by introducing a new joint service. We show that a well-designed profit sharing contract will benefit every single party in the market (riders, drivers, and both platforms).

Key words: Two-sided platforms, ride-sharing, coopetition, choice models

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1. Introduction

The service industry has significantly evolved in recent years. Thanks to the emergence of online platforms, several types of services are now offered in an on-demand basis. More specifically, customers can use their smartphones to request services from anywhere and at any time. These services include: ride-sharing, food delivery, maintenance, cleaning, repair works, and dry-cleaning, just to name a few. According to a survey run by the National Technology Readiness Survey, in October 2015, the on-demand economy is attracting more than 22.4 million consumers annually and $57.6 billion in spending.[1] This new trend has also made the market increasingly competitive. In each sector, several competing firms offer the same type of service (a long list of companies that offer on-demand services can be found at https://theondemandeconomy.org/participants/). In the U.S. ride-sharing market, for example, one can find several competitors including Uber, Lyft, Via, Gett, and Curb. Interestingly, these platforms compete not only for customers (e.g., riders) but also

[1] https://hbr.org/2016/04/the-on-demand-economy-is-growing-and-not-just-for-the-young-and-wealthy

for workers (e.g., drivers). They are often sending enticing monetary incentives to attract both sides of the market. Traditionally, firms were competing only for customers while hiring permanent workers. In two-sided markets, platforms often also compete for workers who can work for multiple platforms and seemingly switch back and forth between companies. As of 2017, nearly 70% of on-demand U.S. drivers work for both Uber and Lyft, and 25% drive for more than just those two, according to a survey by The Rideshare Guy.[2]

Within the ride-sharing industry, a recent trend of partnerships has emerged. One such example is the partnership between Curb and Via in NYC. Curb[3] is an online platform that allows taxi rides to be ordered from a smartphone application and the payment can be completed either via the app or in person. On their website, one can read: "Curb is the #1 taxi app in the U.S. that connects you to fast, convenient and safe rides across the U.S. (50,000 Cabs – 100,000 Drivers)." Via[4] is a ride-sharing platform that allows riders heading in the same direction to carpool and share a ride. Via offers an affordable fare for riders who are willing to carpool, whereas Curb offers a traditional private taxi ride while charging the meter price plus an additional fixed fee per trip. One can definitely view these two platforms as competitors. Yet, they decided to collaborate and engage in a joint partnership. More precisely, on June 6, 2017, both platforms started to offer a joint service through a *profit sharing contract*, under which Curb and Via each earn a certain portion of the net profit from the joint service. This type of partnership is sometimes referred to as *coopetition*, a term coined to describe cooperative competition (see, e.g., Brandenburger and Nalebuff 2011). The new joint service introduced by Curb and Via in NYC allows users to book a shared taxi from either platform. For example, when a user requests a ride through the Via smartphone application, s/he may be offered to ride with a nearby available taxi (this option is called Shared Taxi). Then, the rider can either accept the Shared Taxi or decline by requesting a regular Via ride. Shared Taxi fares are calculated using the meter price and paid directly to the driver. If the matching algorithm finds another rider heading in the same direction, the two riders will carpool and save 40% on any shared portion of the trip. When introducing this new service, Via sent an e-mail advertising campaign to several NYC users (parts of the e-mail content can be seen in Figure 1).[5]

The recent partnership between Curb and Via in NYC is definitely not an exception. Below, we report other similar examples:

[2] https://docs.google.com/document/d/1QSUFSqasfjM9b9UsqBwZlpa8EgqNj6EBfWybFBSHj3o/edit

[3] https://gocurb.com/

[4] https://ridewithvia.com/

[5] The partnership between Curb and Via in NYC was the topic of extensive media coverage. See for example: https://www.nytimes.com/2017/06/06/nyregion/new-york-yellow-taxis-ride-sharing.html, https://techcrunch.com/2017/06/06/curb-and-via-bring-ride-sharing-to-nycs-yellow-taxis/ and https://qz.com/999132/can-shared-rides-save-the-iconic-new-york-city-yellow-cab/

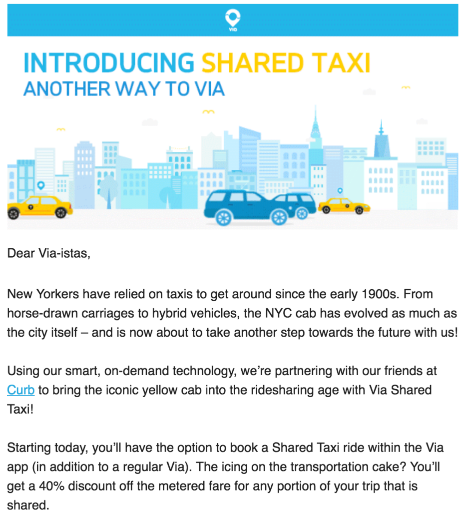

Figure 1 Advertising e-mail sent on June 6, 2017 on the new partnership between Curb and Via in NYC.

- In December 2016, Uber partnered with Indonesia's second largest taxi operator PT Express Transindo Utama Tbk. This partnership allows for ride-sharing, and gave Uber access to Express fleet of more than 11,000 taxis and 17,000 drivers. Express drivers who participate in the program can now serve requests from the Uber app. On the other hand, Uber drivers can lease vehicles by making monthly payments, partly from the income generated from working for Uber.
- In October 2014, Uber partnered with For Hire taxis to expand pick-up availability in Seattle. In this partnership, riders can select multiple options directly from the Uber app (UberX, UberXL, Black Car, SUV, and For Hire). The prices and availability of each option are different.
- In March 2017, Grab partnered with SMRT Taxis with the goal of building the largest car fleet (taxi and private-hire) in Southeast Asia. SMRT, Singapore's third largest taxi operator, was reported in April 2017 to have a fleet of 3,400 taxis. As of October 2017, Grab claims that its current market share is 95% in third-party taxi hailing and 72% in private vehicle hailing.[6] Since January 2017, the number of Grab drivers has nearly tripled (exceeding 1.8 million), making it the largest land transportation fleet in Southeast Asia. In this partnership, all SMRT drivers will use only Grab's application for third-party bookings (to complement street-hail pickups).
- On January 31, 2017, Go-Jek partnered with PT Blue Bird Tbk in Indonesia. In this partnership, riders will simply be served by the closest driver. Therefore, the new service uses both types of drivers (one platform has 23,000 vehicles, whereas the other has at least 11,000 vehicles).[7]

It is clear that both parties have their own incentives to engage in this type of partnerships. For example, it allows ride-sharing platforms to expand their number of drivers and complement their

[6] http://www.straitstimes.com/business/companies-markets/grab-secures-record-us700m-in-debt-facilities-partnering-smrt-to-build

[7] https://www.techinasia.com/go-jek-launches-blue-bird-partnership-now-on-iphone

market share. In addition, platforms can potentially benefit from technological advances developed by other platforms (e.g., efficient matching algorithms and online secured payments). Nevertheless, such partnerships can cannibalize the original market shares (customers who were initially riding with one of the platforms may now switch to the new service).

This paper is motivated by the type of partnerships described above. In particular, we are interested in studying the implications of introducing a new joint service between two competing ride-sharing platforms via a profit sharing contract. Our goal is to examine the impact of the new joint service on both platforms (e.g., Curb and Via), riders, and drivers. We propose to model this problem using the Multinomial Logit (MNL) choice model to capture the fact that riders *and* drivers face several alternatives. Although it is not obvious a-priori whether such coopetition will benefit the platforms, our analysis shows that a well-designed contract is beneficial for both platforms, riders, and drivers.

Importantly, when examining the impact of coopetition in ride-sharing, we observed that the setting with competing two-sided platforms was not well-studied (even in the absence of coopetition). Given that firms compete for both customers and workers, the equilibrium analysis needs to be revisited. Since one side of the market affects demand and the other affects supply, the objective function becomes non-differentiable, making traditional arguments based on the first-order condition not applicable. Before studying the impact of coopetition, we first rigorously analyze a setting where platforms compete for both customers and workers. We undertake a new approach to perform general equilibrium analysis for competing two-sided platforms where each side of the market follows an MNL choice model.

1.1. Contributions

Given the recent popularity of two-sided platforms and ride-sharing in particular, this paper extends the current understanding of competition and coopetition models in the context of two-sided markets. We summarize our main contributions as follows:

- **Showing that a two-sided MNL setting has a unique equilibrium.** To the best of our knowledge, this paper is among the first to study the (price and wage) competition between two-sided platforms. We use the MNL choice model (our results also extend to more general models) to capture the decision process of potential customers and workers. We prove the existence and uniqueness of equilibrium under general price and wage decisions (Theorem 1) and under a fixed-commission rate (Theorem 3). We also convey that the equilibrium outcome can be computed efficiently using a *tatônnement scheme*. Interestingly, the proof technique for two-sided markets is not a simple extension of the traditional (one-sided) setting. We instead show that the best-response strategy is a monotone contraction mapping, allowing us to prove the existence and uniqueness of equilibrium and derive structural properties.

- **Studying the impact of coopetition under a profit sharing contract.** Motivated by recent practice in the ride-sharing industry, we study how introducing a new joint service affects the competing platforms. First, we show that there exists a unique equilibrium even after introducing the coopetition partnership. Second, we find conditions under which the coopetition is strictly beneficial for both platforms. Finally, we identify three main effects induced by introducing the partnership: new market share, cannibalization, and wage variation.
- **Demonstrating that a profit sharing contract can yield a win-win outcome.** We show that regardless of which platform sets the price of the new service, there always exists a profit sharing contract that increases the profits of each platform. As a result, engaging in a coopetition could be a win-win strategy for both platforms.
- **Showing that drivers (and riders) can also benefit.** As expected, riders also benefit from introducing the new service because they have more service options. Furthermore, we show that one can design a profit sharing contract that also benefits drivers. Consequently, when the coopetition terms are carefully designed, every single party (riders, drivers, and both platforms) will benefit.

1.2. Related Literature

This paper is related to at least three streams of literature: price competition under choice models, economics of ride-sharing platforms, and coopetition models.

Price competition under choice models: The first stream of relevant literature is related to choice models (for a review on this topic, see Train 2009, and the references therein), and in particular price competition under the MNL model and its extensions (see, e.g., Anderson et al. 1992, Gallego et al. 2006, Konovalov and Sándor 2010, Li and Huh 2011, Aksoy-Pierson et al. 2013, Gallego and Wang 2014). Using the MNL model, Gallego et al. (2006) show that a unique equilibrium exists when costs are increasing and convex in sales. In Li and Huh (2011), the authors consider the problem of pricing multiple products under the nested-MNL model and show that characterizing the equilibrium outcome is analytically tractable. In this literature, the main focus is on showing the existence and/or uniqueness of the equilibrium outcome. In this paper, we extend the results of Gallego et al. (2006) and Li and Huh (2011) to show that a unique equilibrium exists in a two-sided market where firms compete for both customers and workers. As mentioned, the proof technique for two-sided markets is not a simple extension of the traditional (one-sided) setting and involves different arguments (more details can be found in Section 3).

Economics of ride-sharing platforms: The recent popularity of ride-sharing platforms triggered a great interest in studying pricing decisions in this context. Several papers consider the problem of designing incentives on prices and wages to coordinate supply with demand for on-demand service platforms (see, e.g., Chen and Hu 2016, Tang et al. 2017, Taylor 2017, Hu and

Zhou 2017, Bimpikis et al. 2016, Yu et al. 2017, Benjaafar et al. 2018). Our work has a similar motivation but is among the first to explicitly capture the competition between platforms using an MNL choice model for each side of the market. The recent work of Nikzad (2018) also analyzes the competition between ride-sharing platforms but with a different focus. The author shows that the effect of competition on prices and wages crucially depends on market thickness (i.e., the number of potential workers). In Hu and Zhou (2017), the authors study the pricing decisions of an on-demand platform and demonstrate the good performance of a flat-commission contract. We will also consider the special case of flat-commission contracts in this paper.

Coopetition models: As mentioned, when two competitors cooperate, this is often referred to as *coopetition*, a term coined to describe cooperative competition (see, e.g., Brandenburger and Nalebuff 2011). Closer to our work, there are several papers on coopetition in operations management. For example, Nagarajan and Sošić (2007) propose a model for coalition formation among competitors, and characterize the equilibrium behavior of the resulting strategic alliances. Casadesus-Masanell and Yoffie (2007) study the simultaneously competitive and cooperative relationship between two manufacturers of complementary products, such as Intel and Microsoft, on their R&D investment, pricing, and timing of new product releases. In a strategic alliance setting with capacity sharing, Roels and Tang (2017) show that an ex-ante capacity reservation contract always benefits both firms. In the revenue management literature, several papers have studied a commonly adopted form of coopetition among airline companies, called airline alliances (see, e.g., Netessine and Shumsky 2005, Wright et al. 2010). Coopetition and its related contractual issues have also been studied in the context of service operations (as opposed to manufacturing and supply chain). For example, Roels et al. (2010) analyze the contracting issues that arise in collaborative services and identify the optimal contracts. In a recent work, Yuan et al. (2016) show that as price competition increases, the service providers may surprisingly charge higher prices under coopetition. Our contribution with respect to this literature lies in the fact that, motivated by recent partnerships, we are the first to study coopetition in the ride-sharing industry.

Finally, our work is related to the economics literature on competition between two-sided platforms (see the seminal papers Rochet and Tirole 2003, Armstrong 2006, and the references therein). Our paper differs from this literature in two important ways. First, we explicitly consider a supply-constrained setting where the total sales (matches between customers and workers) are truncated by both demand and supply. Second, we focus on a setting where each side of the market, i.e., customers and workers, follow a choice model to decide which platform to use and to work for respectively. As a result, our model is especially applicable to the increasingly competitive environment between on-demand service platforms.

Structure of the paper. Section 2 presents our model of competition between two-sided platforms (in the absence of coopetition), and Section 3 reports our equilibrium analysis for this model. We next consider introducing a coopetition partnership between two platforms: Section 4 presents our coopetition model, and Section 5 studies the impact of coopetition on both platforms' profits, on riders, and on drivers. We present computational experiments using a case study based on the Curb-Via partnership in Section 6. Finally, we consider an extension of our model with endogenous waiting times in Section 7, and report our conclusions in Section 8. Most proofs of the technical results are relegated to the Appendix.

2. Competition Between Two-sided Platforms: Model

In this section, we present our model of competition between two-sided platforms. We consider two competing online platforms denoted by P_1 and P_2 (most of our results extend to $n > 2$ platforms, as we will discuss later). Each platform P_i $(i = 1, 2)$ offers a service via its mobile or online application. Let q_i be the perceived value/quality (e.g., safety, reputation) of the service offered by P_i, and $p_i \geq 0$ the price charged by P_i. A summary of the notation can be found in Appendix A.

Demand side: We assume that customers follow the MNL discrete choice model (we discuss the extension under more general models in Section 3). More specifically, a customer can choose between three alternatives: P_1, P_2, and the outside option. The utility a customer derives from the service offered by P_i is $u_i = q_i - p_i + \xi_i$, where ξ_i represents the random unobserved utility terms for using P_i. The utility of the outside option is normalized so that $u_0 = \xi_0$. For each customer, ξ_1, ξ_2, and ξ_0 are assumed to be independent and identically distributed with a Gumbel distribution. Therefore, after observing the prices (p_1, p_2), a customer selects P_1 with probability

$$d_1' = \frac{\exp(q_1 - p_1)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)},$$

selects P_2 with probability

$$d_2' = \frac{\exp(q_2 - p_2)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)},$$

and selects the outside option with probability

$$d_0' = \frac{1}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)}.$$

Let Λ be the total customer arrival rate, i.e., the maximum number of potential customers arriving per unit time. We assume that Λ is deterministic and known to both platforms. Our model and results also apply to the setting where the platforms can adapt their price and wage in response to the realized stochastic arrival rate. This is consistent with the "surge pricing" strategy under which ride-sharing platforms increase their price and wage to react to real-time peak demands (see Proposition 2 in Section 3).

Supply side: We assume that workers also follow the MNL model. In particular, a worker chooses one of the three alternatives: P_1, P_2, and the outside option. The utility a worker earns from working for P_i is $v_i = a_i + w_i + \eta_i$, where a_i is the attractiveness of P_i, $w_i \geq 0$ is the wage (per service) distributed by P_i, and η_i represents the random unobserved utility terms of working for P_i. As before, the utility of the outside option is normalized so that $v_0 = \eta_0$. For each worker, η_1, η_2, and η_0 are assumed to be independent and identically distributed with a Gumbel distribution. Therefore, after observing the wages (w_1, w_2), a worker chooses to work for P_1 with probability

$$s_1 = \frac{\exp(a_1 + w_1)}{1 + \exp(a_1 + w_1) + \exp(a_2 + w_2)},$$

selects P_2 with probability

$$s_2 = \frac{\exp(a_2 + w_2)}{1 + \exp(a_1 + w_1) + \exp(a_2 + w_2)},$$

and selects the outside option with probability

$$s_0 = \frac{1}{1 + \exp(a_1 + w_1) + \exp(a_2 + w_2)}.$$

Without loss of generality, we normalize the total number of workers to 1. The supply (i.e., number of workers) of P_i per unit time is then simply s_i.

The total sales of platform P_i is truncated by both demand and supply, that is, $\min\{d_i, s_i\}$. Therefore, the profit earned by P_i is given by:

$$\pi_i(p_1, w_1, p_2, w_2) = (p_i - w_i) \min\{d_i, s_i\},$$
$$\text{where } d_i = \frac{\Lambda \exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)} \text{ and } s_i = \frac{\exp(a_i + w_i)}{1 + \exp(a_1 + w_1) + \exp(a_2 + w_2)}.$$

In the special case of a fixed-commission rate, each platform allocates a fixed proportion $0 < \beta < 1$ of the price paid by customers to its workers, i.e., $w_i = \beta p_i$. In this case, the profit earned by P_i can be calculated as

$$\pi_i^c(p_1, p_2) = (p_i - \beta p_i) \min\{d_i, s_i\} = (1 - \beta) p_i \min\{d_i, s_i\},$$
$$\text{where } d_i = \frac{\Lambda \exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)} \text{ and } s_i = \frac{\exp(a_i + \beta p_i)}{1 + \exp(a_1 + \beta p_1) + \exp(a_2 + \beta p_2)}.$$

3. Competition Between Two-sided Platforms: Equilibrium Analysis

In our model, the platforms P_1 and P_2 compete on both price and wage. More specifically, they engage in a simultaneous game in which P_i sets p_i and w_i to maximize $\pi_i(p_1, w_1, p_2, w_2)$. In this section, we characterize the equilibrium outcome of this game which we call the *two-sided competition game.* A strategy profile of the two platforms is an equilibrium, if each platform maximizes its own profit given the competitor's strategy, that is,

$$(p_i^*, w_i^*) \in \arg\max_{(p_i, w_i)} \pi_i(p_i, w_i, p_{-i}^*, w_{-i}^*),$$

where (p_{-i}^*, w_{-i}^*) is the equilibrium price and wage of the other platform. We also denote the equilibrium demand and supply of P_i by $d_i^* = \frac{\Lambda \exp(q_i - p_i^*)}{1+\exp(q_1-p_1^*)+\exp(q_2-p_2^*)}$ and $s_i^* = \frac{\exp(a_i+w_i^*)}{1+\exp(a_1+w_1^*)+\exp(a_2+w_2^*)}$ respectively. The following theorem shows that a unique equilibrium exists and that under equilibrium, supply should match with demand.

THEOREM 1. *Consider the two-sided competition game described above. Then, the following holds:*

1. *Under equilibrium, supply matches with demand, i.e.,* $s_i^* = d_i^*$ *for* $i = 1, 2$.
2. *The two-sided competition game admits a unique equilibrium* $(p_1^*, w_1^*, p_2^*, w_2^*)$. *Furthermore, the equilibrium can be computed using a tatônnement scheme.*

In the two-sided competition game, if supply does not match with demand, one can always find a profitable unilateral deviation by increasing the price (when demand exceeds supply) or decreasing the wage (when supply exceeds demand). See more details in the proof of Theorem 1 below. Based on the second part of Theorem 1, the equilibrium can be computed using a *tatônnement* scheme, i.e., if each platform uses the best-response price and wage strategies based on the price and wage of its competitor in the previous iteration, the sequence of price and wage strategies converge to the unique equilibrium $(p_1^*, w_1^*, p_2^*, w_2^*)$.

When establishing the existence and uniqueness of equilibrium in one-sided competition with logit type demand models (e.g., MNL, nested-MNL, and mixed-MNL), existing results in the literature typically leverage the first-order optimality condition (FOC) of the profit function. A common approach is to show that the system of equations that characterize the FOC has a unique solution (see, e.g., Gallego et al. 2006, Li and Huh 2011, Gallego and Wang 2014, Aksoy-Pierson et al. 2013). In the two-sided competition game considered in this paper, the first-order condition turns out to be difficult to analyze. This is because the platforms have more flexibility in decisions (price and wage). Furthermore, the total sales of each platform is truncated by both demand and supply. As a result, the objective function becomes non-differentiable, making traditional arguments not applicable. To overcome this issue, we resort to a different approach that directly exploits the structural properties of the best-response mapping of each platform. Due to the two-sidedness nature of our model, the best-response mapping is *not* a contraction mapping. However, the $k-$fold best-response mapping is a contraction under the ℓ_1 norm when k is sufficiently large (see more details in the proof of Theorem 1). Consequently, a *tatônnement* scheme converges to the unique equilibrium $(p_1^*, w_1^*, p_2^*, w_2^*)$. An important insight of price competition under MNL or nested-MNL models from the literature is the optimality of the so-called equal or adjusted markup policy (see, Li and Huh 2011, Gallego and Wang 2014). This property, however, no longer holds in our two-sided competition setting where the platforms have the flexibility to adjust both price and wage.

Proof of Theorem 1. Before presenting the proof, we first introduce some notation that will prove useful for our analysis. Given the competitor's strategy (p_{-i}, w_{-i}), we define $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ as P_i's best price and wage responses. We also define the best-response mapping of the two-sided competition game as

$$\begin{aligned} T(p_1, w_1, p_2, w_2) :=& \Big(p_1(p_1, w_1, p_2, w_2), w_1(p_1, w_1, p_2, w_2), p_2(p_1, w_1, p_2, w_2), w_2(p_1, w_1, p_2, w_2)\Big) \\ =& \Big(p_1(p_2, w_2), w_1(p_2, w_2), p_2(p_1, w_1), w_2(p_1, w_1)\Big). \end{aligned}$$

Then, we iteratively define the $k-$fold best-response mapping $(k \geq 2)$ as

$$T^{(k)}(p_1, w_1, p_2, w_2) = \Big(p_1^{(k)}(p_1, w_1, p_2, w_2), w_1^{(k)}(p_1, w_1, p_2, w_2), p_2^{(k)}(p_1, w_1, p_2, w_2), w_2^{(k)}(p_1, w_1, p_2, w_2)\Big),$$

where for $i = 1, 2$

$$\begin{aligned} p_i^{(k)}(p_1, w_1, p_2, w_2) =& p_i\Big(p_1^{(k-1)}(p_1, w_1, p_2, w_2), w_1^{(k-1)}(p_1, w_1, p_2, w_2), p_2^{(k-1)}(p_1, w_1, p_2, w_2), w_2^{(k-1)}(p_1, w_1, p_2, w_2)\Big), \\ w_i^{(k)}(p_1, w_1, p_2, w_2) =& w_i\Big(p_1^{(k-1)}(p_1, w_1, p_2, w_2), w_1^{(k-1)}(p_1, w_1, p_2, w_2), p_2^{(k-1)}(p_1, w_1, p_2, w_2), w_2^{(k-1)}(p_1, w_1, p_2, w_2)\Big). \end{aligned}$$

We use $||\cdot||_1$ to represent the ℓ_1 norm, i.e., $||x||_1 = \sum_{i=1}^n |x_i|$ for $x \in \mathbb{R}^n$.

The proof of Theorem 1 is based on the following four steps:

- Under equilibrium, $s_i^* = d_i^*$ for $i = 1, 2$.
- The best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are continuously increasing in p_{-i} and w_{-i}. This will imply that an equilibrium exists.
- There exists a k^*, such that the k^*-fold best response is a contraction mapping under the ℓ_1 norm, i.e., there exists a constant $q \in (0, 1)$, such that

$$||T^{(k^*)}(p_1, w_1, p_2, w_2) - T^{(k^*)}(p_1', w_1', p_2', w_2')||_1 \leq q||(p_1, w_1, p_2, w_2) - (p_1', w_1', p_2', w_2')||_1.$$

This will imply that the equilibrium is unique.

- For any (p_1, w_1, p_2, w_2), the sequence $T^{(k)}(p_1, w_1, p_2, w_2)$ converges to the unique equilibrium $(p_1^*, w_1^*, p_2^*, w_2^*)$ as $k \uparrow +\infty$. This will imply that the equilibrium can be computed using a *tatônnement* scheme.

Step I. $s_i^* = d_i^*$.

Assume by contradiction that $s_i^* < d_i^*$. This implies that $d_i^* > \min\{d_i^*, s_i^*\} = s_i^*$. As a result, P_i can increase its price to $p_i^*(\epsilon) = p_i^* + \epsilon$ (for some $\epsilon > 0$), so that $d_i^*(\epsilon) > s_i^*$ and $\min\{d_i^*(\epsilon), s_i^*\} = s_i^*$.

Hence, $\pi_i(\epsilon) = (p_i^* + \epsilon - w_i^*)\min\{d_i^*(\epsilon), s_i^*\} > (p_i^* - w_i^*)\min\{d_i^*, s_i^*\} = \pi_i^*$, which contradicts the fact that s_i^* and d_i^* are the equilibrium supply and demand. Hence, $s_i^* \geq d_i^*$.

Assume by contradiction that $s_i^* > d_i^*$. This implies that $s_i^* > \min\{d_i^*, s_i^*\} = d_i^*$. As a result, P_i can decrease its wage to $w_i^*(\epsilon) = w_i^* - \epsilon$ (for some $\epsilon > 0$), so that $s_i^*(\epsilon) > d_i^*$ and $\min\{d_i^*, s_i^*(\epsilon)\} = d_i^*$. Hence, $\pi_i(\epsilon) = (p_i^* - w_i^* + \epsilon)\min\{d_i^*, s_i^*(\epsilon)\} > (p_i^* - w_i^*\}\min\{d_i^*, s_i^*\} = \pi_i^*$, which contradicts the fact that s_i^* and d_i^* are the equilibrium supply and demand. Hence, $s_i^* \leq d_i^*$. Putting together $s_i^* \geq d_i^*$ and $s_i^* \leq d_i^*$, we conclude that $s_i^* = d_i^*$.

Step II. $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are continuously increasing in p_{-i} and w_{-i}

Since $s_i^* = d_i^*$, we denote $s = s_i = d_i$ as the demand/supply of P_i. Given (p_{-i}, w_{-i}, s), we can write $p_i(p_{-i}, w_{-i}, s) = q_i - \log\left(\frac{s/\Lambda}{1-s/\Lambda}\right) - \log[1 + \exp(q_{-i} - p_{-i})]$ and $w_i(p_{-i}, w_{-i}, s) = -a_i + \log\left(\frac{s}{1-s}\right) + \log[1 + \exp(a_{-i} + w_{-i})]$. Thus, given (p_{-i}, w_{-i}), P_i's price and wage optimization problem can be formulated as the following one-dimensional convex program:

$$\max_s \; \pi_i(s|p_{-i}, w_{-i})$$

$$\begin{aligned} \text{where } \pi_i(s|p_{-i}, w_{-i}) &= [p_i(p_{-i}, w_{-i}, s) - w_i(p_{-i}, w_{-i}, s)]s \\ &= \left\{q_i + a_i - \log\left(\frac{s/\Lambda}{1-s/\Lambda}\right) - \log\left(\frac{s}{1-s}\right) - \log[1 + \exp(q_{-i} - p_{-i})] - \log[1 + \exp(a_{-i} + w_{-i})]\right\}s. \end{aligned}$$

One can check that $\pi_i(s|p_{-i}, w_{-i})$ is concave in s and, by calculating the cross-derivative, supermodular in (p_{-i}, s). Therefore, $s^* := \arg\max_s \pi_i(s|p_{-i}, w_{-i})$ is increasing in p_{-i}, which implies that $w_i(p_{-i}, w_{-i}) = w_i(p_{-i}, w_{-i}, s^*) = -a_i + \log\left(\frac{s^*}{1-s^*}\right) + \log[1 + \exp(a_{-i} + w_{-i})]$ is also increasing in p_{-i}.

We define $m(p_{-i}, w_{-i}, s) := p_i(p_{-i}, w_{-i}, s) - w_i(p_{-i}, w_{-i}, s)$ as P_i's profit margin. Thus, $\pi_i'(s|p_{-i}, w_{-i}) = \partial_s m(p_{-i}, w_{-i}, s)s + m(p_{-i}, w_{-i}, s)$. Since $\pi_i'(s^*|p_{-i}, w_{-i}) = 0$, we have $\partial_s m(p_{-i}, w_{-i}, s^*)s^* + m(p_{-i}, w_{-i}, s^*) = 0$. Straightforward calculation implies that $\partial_s m(p_{-i}, w_{-i}, s)s$ is strictly decreasing in s and independent of (p_{-i}, w_{-i}). Assume that $\hat{p}_{-i} > p_{-i}$, so we have $\hat{s}^* > s^*$. Thus, $\partial_s m(\hat{p}_{-i}, w_{-i}, \hat{s}^*)\hat{s}^* < \partial_s m(p_{-i}, w_{-i}, s^*)s^*$. By the first-order condition, $\pi_i'(\hat{s}^*|\hat{p}_{-i}, w_{-i}) = \pi_i'(s^*|p_{-i}, w_{-i}) = 0$, i.e., $\partial_s m(\hat{p}_{-i}, w_{-i}, \hat{s}^*)\hat{s}^* + m(\hat{p}_{-i}, w_{-i}, \hat{s}^*) = \partial_s m(p_{-i}, w_{-i}, s^*)s^* + m(p_{-i}, w_{-i}, s^*) = 0$. Hence, $m(\hat{p}_{-i}, w_{-i}, \hat{s}^*) > m(p_{-i}, w_{-i}, s^*)$, i.e.,

$$\begin{aligned} &-\log\left(\frac{\hat{s}^*/\Lambda}{1-\hat{s}^*/\Lambda}\right) - \log\left(\frac{\hat{s}^*}{1-\hat{s}^*}\right) - \log[1 + \exp(q_{-i} - \hat{p}_{-i})] > \\ &-\log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) - \log\left(\frac{s^*}{1-s^*}\right) - \log[1 + \exp(q_{-i} - p_{-i})]. \end{aligned}$$

Since we also have $\hat{s}^* > s^*$, it follows that

$$p_i(p_{-i}, w_{-i}) = q_i - \log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) - \log\left(\frac{s^*}{1-s^*}\right) - \log[1 + \exp(q_{-i} - p_{-i})] + \log\left(\frac{s^*}{1-s^*}\right)$$

is increasing in p_{-i}, i.e., both $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in p_{-i}. With a similar argument, we can show that s^* is decreasing in w_{-i}, which further implies that $p_i(p_{-i}, w_{-i}) = p_i(p_{-i}, w_{-i}, s^*) = q_i - \log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) - \log[1 + \exp(q_{-i} - p_{-i})]$ is increasing in w_{-i}. Moreover, the profit margin $m(p_{-i}, w_{-i}, s^*)$ is decreasing in w_{-i}. Thus, $-\log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) - \log\left(\frac{s^*}{1-s^*}\right) - \log[1+\exp(a_{-i}+w_{-i})]$ and s^* are both decreasing in w_{-i}. Therefore,

$$\begin{aligned} w_i(p_{-i}, w_{-i}) &= -a_i + \log\left(\frac{s^*}{1-s^*}\right) + \log[1+\exp(a_{-i}+w_{-i})] \\ &= -a_i - \left\{-\log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) - \log\left(\frac{s^*}{1-s^*}\right) - \log[1+\exp(a_{-i}+w_{-i})]\right\} - \log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) \end{aligned}$$

is increasing in w_{-i}. We have thus shown that $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in p_{-i} and w_{-i}. The continuity of $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ follows immediately from the facts that $\pi_i(s|p_{-i}, w_{-i})$ is concave in s and continuous in p_{-i} and w_{-i}. This completes the proof of Step II. By Tarski's Fixed Point Theorem (see, e.g., Milgrom and Roberts 1990), the continuity and monotonicity of $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$, together with that the feasible sets of $p_i(\cdot,\cdot)$ and $w_i(\cdot,\cdot)$ are clearly latices, imply that an equilibrium exists in the two-sided competition game.

Step III. $T^{(k*)}$ is a contraction mapping for some $k^* \geq 1$ under the ℓ_1 norm

For conciseness, we defer the proof of Step III to the Appendix (see Lemma 2). We next show that the equilibrium is unique. Assume, to the contrary, that there are two distinct equilibria $(p_1^*, w_1^*, p_2^*, w_2^*)$ and $(\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*)$. Then, by the equilibrium definition, we have $T(p_1^*, w_1^*, p_2^*, w_2^*) = (p_1^*, w_1^*, p_2^*, w_2^*)$ and $T(\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*) = (\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*)$. Therefore, $T^{(k^*)}(p_1^*, w_1^*, p_2^*, w_2^*) = (p_1^*, w_1^*, p_2^*, w_2^*)$ and $T^{(k^*)}(\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*) = (\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*)$. Hence, we have

$$||T^{(k^*)}(p_1^*, w_1^*, p_2^*, w_2^*) - T^{(k^*)}(\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*)||_1 = ||(p_1^*, w_1^*, p_2^*, w_2^*) - (\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*)||_1. \tag{1}$$

Since $T^{(k^*)}(\cdot,\cdot,\cdot,\cdot)$ is a contraction mapping, we have

$$||T^{(k^*)}(p_1^*, w_1^*, p_2^*, w_2^*) - T^{(k^*)}(\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*)||_1 < q||(p_1^*, w_1^*, p_2^*, w_2^*) - (\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*)||_1,$$

which contradicts equation (1) if $(p_1^*, w_1^*, p_2^*, w_2^*) \neq (\bar{p}_1^*, \bar{w}_1^*, \bar{p}_2^*, \bar{w}_2^*)$. Thus, a unique equilibrium exists.

Step IV. $T^{(k)}(p_1, w_1, p_2, w_2)$ converges to the unique equilibrium

As shown in Step III, $||T^{(k)}(p_1, w_1, p_2, w_2) - T^{(k)}(p_1', w_1', p_2', w_2')||_1 \leq 2C^k||(p_1, w_1, p_2, w_2) - (p_1', w_1', p_2', w_2')||_1$ for any (p_1, w_1, p_2, w_2) and (p_1', w_1', p_2', w_2'). We define $x_k := T^{(k)}(p_1, w_1, p_2, w_2)$ for $k \geq 1$ and $x_0 = (p_1, w_1, p_2, w_2)$. For any k and $l > 0$,

$$\begin{aligned} &||T^{(k)}(p_1, w_1, p_2, w_2) - T^{(k+l)}(p_1', w_1', p_2', w_2')||_1 \leq \sum_{i=1}^{l} ||x_{k+i} - x_{k+i-1}||_1 \\ &\leq \sum_{i=1}^{l} 2C^{(k+i-1)}||x_1 - x_0||_1 \leq \sum_{i=1}^{+\infty} 2C^{(k+i-1)}||x_1 - x_0||_1 = \frac{2||x_1 - x_0||_1 C^k}{1-C}, \end{aligned}$$

where the first inequality follows from the triangular inequality, and the second from $x_{k+i} = T^{(k+i-1)}(x_1)$ and $x_{k+i-1} = T^{(k+i-1)}(x_0)$. Therefore, $||x_k - x_{k+l}|| \to 0$ uniformly with respect to l as $k \uparrow +\infty$, i.e., $\{x_k : k \geq 1\}$ is a Cauchy sequence, and hence x_k converges to x^*, which is a fixed point of $T(\cdot,\cdot,\cdot,\cdot)$, i.e., $T(x^*)=x^*$ so that x^* is the unique equilibrium. As a result, the unique equilibrium can be obtained using a *tatônnement* scheme, and this concludes the proof of Theorem 1. $\square$

The proof of Theorem 1 is based on directly analyzing the best-response strategy and establishing the convergence of a *tatônnement* scheme to the unique equilibrium. We highlight that, besides characterizing the equilibrium, the iterative procedure of the *tatônnement* scheme allows us to derive structural properties of the equilibrium. Specifically, we inductively characterize the desired properties of the *tatônnement* scheme in each iteration, implying that the same properties hold under equilibrium. We next exploit this technique to (i) compare the equilibrium outcome of our two-sided competition game to a monopoly market (i.e., both platforms are owned by a single entity) and (ii) characterize how the equilibrium strategy reacts to real-time peak demands.

To compare the equilibrium prices and wages of our two-sided competition game with a monopoly market, we denote the prices and wages under the monopolistic setting by $(\mathbf{p}^{m*},\mathbf{w}^{m*})=(p_1^{m*},w_1^{m*},p_2^{m*},w_2^{m*})$.

Proposition 1. *We have that $p_i^*<p_i^{m*}$ and $w_i^*>w_i^{m*}$ for $i=1,2$.*

As shown in Proposition 1, in a competitive market, each platform will decrease (resp. increase) its price (resp. wage) to attract customers (resp. workers). Traditionally, it was shown that price competition will decrease the price of each firm relative to a monopoly market (see, e.g., Li and Huh 2011). Proposition 1 generalizes this result to a two-sided market by showing that competition not only decreases the price, but also raises the wage of each platform. We note that the method we use to prove Proposition 1 is different from the typical argument used in the literature. In the literature, the main argument relies on analyzing the FOCs (see, e.g., Li and Huh 2011), whereas in our model, we directly exploit the properties of the best response in each iteration of the *tatônnement* scheme, as illustrated in the proof below.

Proof of Proposition 1. As shown in the proof of Theorem 1, the sequence $\{T^{(k)}(p_1^{m*},w_1^{m*},p_2^{m*},w_2^{m*}):k\geq 1\}$ converges to the equilibrium $(p_1^*,w_1^*,p_2^*,w_2^*)$. We define:

$$(p_1^{(k)},w_1^{(k)},p_2^{(k)},w_2^{(k)}):=T^{(k)}(p_1^{m*},w_1^{m*},p_2^{m*},w_2^{m*}) \text{ for } k\geq 1,$$

and $(p_1^{(0)},w_1^{(0)},p_2^{(0)},p_2^{(0)}):=(p_1^{m*},w_1^{m*},p_2^{m*},w_2^{m*})$. We also define $s_i^{(k)}$ as the optimal supply of P_i in the kth iteration of the *tatônnement* scheme. It suffices to show that $p_i^{(k)}<p_i^{(m*)}$ for $k\geq 1$ and $i=1,2$, as well as $w_i^{(k)}>w_i^{(m*)}$ for $k\geq 1$ and $i=1,2$.

We first observe that, for a monopoly market, $d_i^{m*}=s_i^{m*}$ for $i=1,2$. Indeed, if $d_i^{m*}>s_i^{m*}$, we can increase p_i and strictly increase the profit of each platform. Analogously, if $d_i^{m*}<s_i^{m*}$, we can increase w_i and strictly increase the profit of each platform. Therefore, under the optimal price and wage policies, $d_i^{m*}=s_i^{m*}$ for $i=1,2$.

We next show that $p_i^{(1)} < p_i^{(0)}$ and $w_i^{(1)} > w_i^{(0)}$. As shown in the proof of Theorem 1, $(p_i^{(1)}, w_i^{(1)})$ can be represented by $\big(p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(1)}), w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(1)})\big)$, where $p_i(\cdot,\cdot,\cdot)$ (resp. $w_i(\cdot,\cdot,\cdot)$) is the price (resp. wage) policy of P_i given (p_{-i}, w_{-i}, s) and $s_i^{(1)}$ is the optimal supply (which is equal to demand) obtained by solving the following one-dimensional convex program:

$$\max_s \pi_i(s)$$

$$\begin{aligned}\text{where } \pi_i(s) &= (p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s) - w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s))s \\ &= \Big\{q_i + a_i - \log\left(\frac{s/\Lambda}{1-s/\Lambda}\right) - \log\left(\frac{s}{1-s}\right) - \log[1+\exp(q_{-i} - p_{-i}^{(0)})] - \log[1+\exp(a_{-i} + w_{-i}^{(0)})]\Big\}s.\end{aligned}$$

Given that under the optimal policy, $s_i^{m*} = d_i^{m*}$, the optimal price and wage of a monopoly market (p_1^{m*}, w_1^{m*}) can be obtained by $\big(p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{m*}), w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{m*})\big)$, where s_i^{m*} is the solution of the following one-dimensional optimization problem:

$$\begin{aligned}&\max_s [\pi_i(s) + \pi_{-i}(s)] \\ &\qquad \text{where } \pi_{-i}(s) = (p_{-i}^{(0)} - w_{-i}^{(0)}) \min\{d_{-i}, s_{-i}\}, \\ &\qquad\qquad \text{with } d_{-i} = \frac{\exp(q_{-i} - p_{-i}^{(0)})}{1 + \exp[q_i - p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s)] + \exp(q_{-i} - p_{-i}^{(0)})} \\ &\qquad\qquad\qquad s_{-i} = \frac{\exp(a_{-i} + w_{-i}^{(0)})}{1 + \exp[a_i + w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s)] + \exp(a_{-i} + w_{-i}^{(0)})}.\end{aligned}$$

It is easy to check that d_{-i}, s_{-i}, and thus $\pi_{-i}(\cdot)$ are strictly decreasing in s. Since $s_i^{(1)}$ is the maximizer of $\pi_i(s)$, we must have $s_i^{m*} < s_i^{(1)}$. Since $p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s)$ is strictly decreasing in s, whereas $w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s)$ is strictly increasing, we have $p_i^{(1)} = p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(1)}) < p_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(m*)})$ and $w_i^{(1)} = w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(1)}) > w_i(p_{-i}^{(0)}, w_{-i}^{(0)}, s_i^{(m*)})$. We then have shown that $p_i^{(1)} < p_i^{(0)}$ and $w_i^{(1)} > w_i^{(0)}$.

Next, we show that if $p_{-i}^{(k)} < p_{-i}^{(m*)}$ and $w_{-i}^{(k)} > w_{-i}^{(m*)}$, then $p_i^{(k+1)} < p_i^{(m*)}$ and $w_i^{(k+1)} > w_i^{(m*)}$. Assume, to the contrary, that $p_i^{(k+1)} \geq p_i^{(m*)}$ or $w_i^{(k+1)} \leq w_i^{(m*)}$. It can be easily checked that $s_i^{(k+1)} < s_i^{m*}$ and that $m_i^{(k+1)} := p_i^{(k+1)} - w_i^{(k+1)} > m_i^{m*} := p_i^{(m*)} - w_i^{(m*)}$. As shown in the proof of Theorem 1, $\partial_s m(p_{-i}, w_{-i}, s)s$ is independent of (p_{-i}, w_{-i}) and decreasing in s. Thus, we have

$$\pi_i'(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) = \partial_s m_i^{(k+1)} s_i^{(k+1)} + m_i^{(k+1)} > \partial_s m_i^{(m*)} s_i^{(m*)} + m_i^{(m*)} = \pi_i'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}),$$

where the inequality follows from $s_i^{(k+1)} < s_i^{m*}$ and $m_i^{(k+1)} > m_i^{m*}$. By the FOC of the monopoly model, $\pi_i'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) + \pi_{-i}'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) = 0$, so $\pi_i'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) = -\pi_{-i}'(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) > 0$, where the inequality follows from the fact that $\pi_{-i}(\cdot)$ is strictly decreasing in s. This implies that $\pi_i'(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) > 0$, which contradicts the FOC $\pi_i'(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) = 0$. Thus, $p_i^{(k+1)} < p_i^{(m*)}$ and $w_i^{(k+1)} < w_i^{(m*)}$ must hold. Proposition 1 then follows from $p_i^* = \lim_{k\to+\infty} p_i^{(k)} < p_i^{(0)} = p_i^{m*}$ and $w_i^* = \lim_{k\to+\infty} w_i^{(k)} > w_i^{(0)} = w_i^{m*}$ for $i = 1, 2$. $\square$

Our next result characterizes how the platforms adjust their prices and wages in response to demand variations under equilibrium.

Proposition 2. *Under equilibrium, p_i^* and w_i^* are increasing in Λ for $i=1,2$.*

Consistent with the business practice in ride-sharing markets, Proposition 2 suggests that both platforms adopt the surge pricing strategy under equilibrium, i.e., they react to real-time peak demands by increasing their price and wage. This result generalizes the well-known optimality of surge pricing for a monopoly (see, e.g., Tang et al. 2017) to a competitive two-sided setting.

We remark that Theorem 1, Proposition 2, and Proposition 1 can be generalized to a model with $n \geq 2$ platforms and under more general demand and supply functions. Specifically, we assume that there are n platforms on the market, $P_1, P_2, \ldots, P_n$, each choosing p_i and w_i. Given the price and wage vectors $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, we define the demand and supply of P_i as $d_i(\mathbf{p})$ and $s_i(\mathbf{w})$ respectively. Then, the profit earned by P_i $(i = 1, 2, \ldots, n)$ is given by:

$$\pi_i(\mathbf{p}, \mathbf{w}) = (p_i - w_i) \min\{d_i(\mathbf{p}), s_i(\mathbf{w})\},$$

and an equilibrium $(\mathbf{p}^*, \mathbf{w}^*) = (p_1^*, w_1^*, p_2^*, w_2^*, \ldots, p_n^*, w_n^*)$ is defined as $(p_i^*, w_i^*) \in \arg\max_{(p_i, w_i)} \pi_i(p_i, w_i, \mathbf{p}_{-i}^*, \mathbf{w}_{-i}^*)$, where $\mathbf{p}_{-i}^* = (p_1^*, p_2^*, \ldots, p_{i-1}^*, p_{i+1}^*, \ldots, p_n^*)$ and $\mathbf{w}_{-i}^* = (w_1^*, w_2^*, \ldots, w_{i-1}^*, w_{i+1}^*, \ldots, w_n^*)$. We next impose the following assumption on $d_i(\mathbf{p})$ and $s_i(\mathbf{w})$.

Assumption 1. *For each $i = 1, \ldots, n$, $d_i(\mathbf{p})$ and $s_i(\mathbf{w})$ satisfy the following:*

(a) $d_i(\mathbf{p})$ is strictly continuously decreasing (resp. increasing) in p_i (resp. p_j, for all $j \neq i$), and $s_i(\mathbf{w})$ is continuously increasing (resp. decreasing) in w_i (resp. w_j, for all $j \neq i$).

(b) Suppose $d_i(\mathbf{p}) = d_i(\hat{\mathbf{p}})$, where $\mathbf{p}_j \leq \hat{\mathbf{p}}_j$ for all j. Then, there exists a constant $q_d < 1$ independent of i, $\mathbf{p}$, and $\hat{\mathbf{p}}$, such that $|p_i - \hat{p}_i| \leq q_d \sum_{j \neq i} |p_j - \hat{p}_j|$.

(c) Suppose $s_i(\mathbf{w}) = \mathbf{s_i}(\hat{\mathbf{w}})$, where $\mathbf{w}_j \leq \hat{\mathbf{w}}_j$ for all j. Then, there exists a constant $q_s < 1$ independent of i, $\mathbf{w}$, and $\hat{\mathbf{w}}$, such that $|w_i - \hat{w}_i| \leq q_s \sum_{j \neq i} |w_j - \hat{w}_j|$.

Assumption 1(a) states that the demand of P_i is decreasing in its own price and increasing in the prices of its competitors; whereas the supply of P_i is increasing in its own wage and decreasing in the wages of its competitors. Assumption 1(b) states that the demand of P_i is more sensitive to its own price changes than to those of its competitors (a special case of this assumption is the widely-used diagonal dominance condition). Analogously, Assumption 1(c) states that the supply of P_i is more sensitive to its own wage changes than to those of its competitors. It is worth noting that Assumption 1 is satisfied by several commonly-used models, such as the MNL, the nested-MNL, the mixed-MNL, the attraction model, and the linear model with the diagonal dominance condition. The following result generalizes Theorem 1 to $n \geq 2$ platforms and under more general demand and supply models.

Theorem 2. *The following statements hold for the two-sided competition game with demand and supply functions $\{d_i(\cdot) : i = 1, 2, \ldots, n\}$ and $\{s_i(\cdot) : i = 1, 2, \ldots, n\}$:*

- *If $\{d_i(\cdot): i=1,2,\ldots,n\}$ and $\{s_i(\cdot): i=1,2,\ldots,n\}$ satisfy Assumption 1(a), then there exists an equilibrium for the two-sided competition game with n platforms.*
- *If $\{d_i(\cdot): i=1,2,\ldots,n\}$ and $\{s_i(\cdot): i=1,2,\ldots,n\}$ satisfy Assumptions 1(a,b,c), then there exists a unique equilibrium for the two-sided competition game between n platforms. Furthermore, the equilibrium can be computed using a tatônnement scheme.*

The proof of Theorem 2 follows from a similar argument as the proof of Theorem 1 and is sketched in the Appendix for conciseness. Analogously, Propositions 1 and 2 can also be extended to the model with n platforms and general demand and supply functions that satisfy Assumption 1.

3.1. Fixed-commission Rate

Very often, platforms use a fixed-commission rate to pay their workers. Namely, they allocate a fixed share $0<\beta<1$ of the price paid by customers to its workers, i.e., $w_i=\beta p_i$ (see, e.g., Hu and Zhou 2017), where β is a pre-specified parameter that does not change with the state of the market. For example, for Lyft drivers who applied before 12 AM on January 1, 2016, the rates are 80% of the passenger's time, distance, and base rates.[8] In the model with a fixed-commission rate, the equilibrium (p_1^{c*}, p_2^{c*}) is defined as follows:

$$p_i^{c*} \in \arg\max_{p_i} \pi_i^c(p_i, p_{-i}^{c*}),$$

where p_{-i}^{c*} is the equilibrium price of the other platform under a fixed-commission rate. We also denote the equilibrium demand and supply of P_i by $d_i^{c*} = \frac{\Lambda \exp(q_i - p_i^{c*})}{1+\exp(q_1-p_1^{c*})+\exp(q_2-p_2^{c*})}$ and $s_i^{c*} = \frac{\exp(a_i+\beta p_i^{c*})}{1+\exp(a_1+\beta p_1^{c*})+\exp(a_2+\beta p_2^{c*})}$ respectively. For simplicity, we assume the same commission rate β for both platforms. Nevertheless, our result extends when both commission rates are different but not too far apart (using a continuity argument in the proof of Theorem 3).

Theorem 3. *Consider the two-sided competition game under a fixed-commission rate. Then, the following holds:*

1. *Under equilibrium, supply exceeds demand, i.e., $s_i^{c*} \geq d_i^{c*}$ for $i=1,2$.*
2. *The two-sided competition game under a fixed-commission rate admits a unique equilibrium (p_1^{c*}, p_2^{c*}). Furthermore, the equilibrium can be computed using a tatônnement scheme.*

Note that the first part of Theorem 3 is different than Theorem 1. Indeed, under a fixed-commission rate, the platforms have less flexibility in the decision-making process since the wage is tied to the price. Consequently, the argument of finding a profitable unilateral deviation does not hold anymore for the case when $d_i<s_i$. When $d_i>s_i$, by increasing p_i (and thus w_i too), this will decrease d_i and increase s_i so that P_i's profit will increase. However, when $d_i<s_i$, such an

[8] https://thehub.lyft.com/pay-breakdown/

approach does not work: by decreasing p_i (and w_i), d_i will increase and s_i will decrease so that the impact on P_i's profit is not clear. Consequently, as shown in the proof of Theorem 3, the setting with a fixed-commission rate requires a different equilibrium analysis to carefully examine the case when $d_i < s_i$.

4. Coopetition Between Two-sided Platforms: Model

Inspired by recent practice in the ride-sharing industry, we model the setting where a coopetition partnership is introduced through a profit sharing contract between the two platforms. In particular, the two competing platforms P_1 and P_2 collaborate and offer a new joint service, which is available to riders from either platform. As mentioned, one such recent example is the partnership between Curb and Via with the introduction of a taxi sharing service in NYC as of June 6, 2017. For the rest of this paper, we use the terms "new joint service", "new service", and "coopetition" interchangeably. Because the coopetition partnership is mainly adopted in the ride-sharing market, we refer to customers as riders and workers as drivers in the model with coopetition.

We use the superscript $\tilde{}$ to denote the different variables in the presence of coopetition. More specifically, we denote the prices of the original services offered by P_1 and P_2, after introducing the new joint service by $\tilde{p}_1$ and $\tilde{p}_2$. The quality and price of the new service are denoted by q_n and $\tilde{p}_n$ respectively. In addition, we propose to capture the pooling effect by the parameter $\tilde{n}$. This parameter corresponds to the (average) number of customers per service (i.e., riders per ride) for the new joint service. If the new joint service does not offer a pooling option (i.e., only private rides), $\tilde{n} = 1$ and otherwise, $\tilde{n} > 1$.

As for the original services, the utility derived by a customer from choosing the new joint service is $u_n = q_n - \tilde{p}_n + \xi_n$, where ξ_n represents the random unobserved utility terms for using the new service. As before, we assume that for each customer, ξ_1, ξ_2, ξ_n, and ξ_0 are independent and identically distributed with a Gumbel distribution. After introducing the new joint service, a customer faces four different alternatives (P_1, P_2, the new joint service, and the outside option), and chooses the one that yields the highest utility. Therefore, in the presence of coopetition, the demand of P_1 is $\tilde{d}_1 = \Lambda \tilde{d}'_1$, where

$$\tilde{d}'_1 = \frac{\exp(q_1 - \tilde{p}_1)}{1 + \exp(q_1 - \tilde{p}_1) + \exp(q_2 - \tilde{p}_2) + \exp(q_n - \tilde{p}_n)}.$$

Analogously, in the presence of coopetition, the demand of P_2 is $\tilde{d}_2 = \Lambda \tilde{d}'_2$, where

$$\tilde{d}'_2 = \frac{\exp(q_2 - \tilde{p}_2)}{1 + \exp(q_1 - \tilde{p}_1) + \exp(q_2 - \tilde{p}_2) + \exp(q_n - \tilde{p}_n)}.$$

Finally, the demand of the new service is equal to $\tilde{d}_n = \Lambda \tilde{d}'_n$, where $\tilde{d}'_n$ is given by:

$$\tilde{d}'_n = \frac{\exp(q_n - \tilde{p}_n)}{1 + \exp(q_1 - \tilde{p}_1) + \exp(q_2 - \tilde{p}_2) + \exp(q_n - \tilde{p}_n)}.$$

We assume that $\alpha_1 = \alpha \in [0,1]$ (resp. $\alpha_2 = 1-\alpha$) percentage of the new service is completed by P_1's (resp. P_2's) drivers. Hence, $\alpha = 1$ (resp. $\alpha = 0$) refers to the case where P_1's (resp. P_2's) drivers solely provide the new joint service. For example, in the coopetition partnership between Curb and Via, the new taxi-sharing service is fulfilled only by Curb's drivers. The total number of P_i's drivers needed to satisfy all the demand is the sum of the demand of its original service and the demand of the new joint service allocated to its drivers, denoted by $\tilde{\lambda}_i := \tilde{d}_i + \alpha_i \tilde{d}_n/\tilde{n}$ $(i=1,2)$. As before, given $\tilde{w}_1$ and $\tilde{w}_2$, the supply curve of each platform is

$$\tilde{s}_i = \frac{\exp(a_i + \tilde{w}_i)}{1+\exp(a_1+\tilde{w}_1)+\exp(a_2+\tilde{w}_2)}, \text{ for } i=1,2.$$

We consider a *profit sharing contract* in which P_1 and P_2 split the net profit generated by the new joint service. More precisely, P_1 receives a fraction $\gamma_1 = \gamma \in (0,1)$ of the profit generated by the new service and P_2 receives $\gamma_2 = 1-\gamma$.

Under coopetition, the profit of P_i comprises three parts: (i) the profit from its original service, (ii) the profit from the new joint service completed by its own drivers, and (iii) the profit from the new joint service completed by the drivers of the other platform. To compute these three terms, we first need to understand how P_i allocates its supply. If $\tilde{s}_i > \tilde{\lambda}_i$, then P_i has enough drivers to fulfill all the demand requests. Otherwise, we assume that the drivers are proportionally allocated to the original service and to the joint service (considering random arrivals and a first-come-first-serve allocation). Recall that the demand of P_i's original service (resp. new joint service) is $\tilde{d}_i$ (resp. $\frac{\alpha_i \tilde{d}_n}{\tilde{n}}$). Therefore, P_i allocates $\tilde{s}_i \cdot \frac{\tilde{d}_i}{\tilde{\lambda}_i}$ to its original service and $\tilde{s}_i \cdot \frac{\alpha_i \tilde{d}_n/\tilde{n}}{\tilde{\lambda}_i}$ to the new joint service. As a result, P_i's profit from its original service is $(\tilde{p}_i - \tilde{w}_i)\min\{\tilde{d}_i, \frac{\tilde{s}_i\tilde{d}_i}{\tilde{\lambda}_i}\}$ and P_i's profit from the new joint service completed by its own drivers is $\gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_i)\min\left\{\frac{\alpha_i\tilde{d}_n}{\tilde{n}}, \frac{\tilde{s}_i\alpha_i\tilde{d}_n}{\tilde{n}\tilde{\lambda}_i}\right\}$. Analogously, P_i's profit from the new joint service completed by its competitor's drivers is: $\gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_{-i})\min\left\{\frac{\alpha_{-i}\tilde{d}_n}{\tilde{n}}, \frac{\tilde{s}_{-i}\alpha_{-i}\tilde{d}_n}{\tilde{n}\tilde{\lambda}_{-i}}\right\}$. Putting everything together, the expression for the total profit earned by P_i is given by:

$$\begin{aligned}\tilde{\pi}_i(\tilde{p}_1,\tilde{w}_1,\tilde{p}_2,\tilde{w}_2) = {} & (\tilde{p}_i - \tilde{w}_i)\min\left\{\tilde{d}_i, \frac{\tilde{s}_i\tilde{d}_i}{\tilde{\lambda}_i}\right\} \\ & + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1)\min\left\{\frac{\alpha_1\tilde{d}_n}{\tilde{n}}, \frac{\tilde{s}_1\alpha_1\tilde{d}_n}{\tilde{n}\tilde{\lambda}_1}\right\} + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_2)\min\left\{\frac{\alpha_2\tilde{d}_n}{\tilde{n}}, \frac{\tilde{s}_2\alpha_2\tilde{d}_n}{\tilde{n}\tilde{\lambda}_2}\right\}.\end{aligned}$$

5. Impact of Coopetition

In this section, we analyze the impact of the coopetition partnership (i.e., introducing the new joint service). We first consider the profit implications on both platforms, and then examine the impact on riders and drivers. Before doing so, we first show that even in the presence of coopetition, there still exists a unique equilibrium.

We consider a setting with given values of $\tilde{p}_n$ (price of the new service), γ (profit sharing parameter), and α (demand allocation parameter). Then, the platforms engage in a price and wage competition using the model presented in Section 4. As before, the equilibrium outcome $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ should satisfy: $(\tilde{p}_i^*, \tilde{w}_i^*) \in \arg\max_{(p_i,w_i)} \tilde{\pi}_i(p_i, w_i, p_{-i}^*, w_{-i}^*)$. We next extend the result of Section 3 by showing the existence and uniqueness of the equilibrium even in the presence of coopetition. Recall that the supply and the total demand of P_i are denoted by $\tilde{s}_i$ and $\tilde{\lambda}_i$ respectively.

Theorem 4. *Consider the two-sided competition game in the presence of coopetition. Then, the following holds:*

1. *Under equilibrium, supply matches with demand, i.e., $\tilde{s}_i^* = \tilde{\lambda}_i^*$ for $i = 1, 2$.*
2. *For any $(\tilde{p}_n, \gamma, \alpha)$, there exists a unique equilibrium $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ which can be computed using a tatônnement scheme.*

It is worth noting that when $\tilde{p}_n \uparrow +\infty$, the model with coopetition will converge to the competition model (without coopetition). We also remark that one can easily find examples of coopetition partnerships that are detrimental to both platforms. In other words, if the platforms do not carefully decide the values of $\tilde{p}_n$, γ, and α, introducing the new service may lead to an undesirable lose-lose outcome for both platforms.

5.1. Impact on Platforms' Profits

In this subection, we examine the impact of the coopetition on the profits of both platforms. At a high level, the coopetition will induce three effects: (i) a new market share effect (i.e., capturing new riders who were previously choosing the outside option), (ii) a cannibalization effect (i.e., losing some existing market share to the new service), and (iii) a wage variation (i.e., adapting the wage so as to match supply with demand). Under the two-sided competition game, the impact of coopetition on both platforms is not clear. In this subsection, our goal is to study how the platforms could use well-designed profit-sharing contracts to balance these effects and benefit from coopetition.

We first consider the case where the price $\tilde{p}_n$ is jointly decided by both platforms. Then, we extend the analysis to the case where $\tilde{p}_n$ is decided by a single platform. Finally, we identify a necessary and sufficient condition on the demand-supply ratio under which the coopetition is strictly beneficial for both platforms.

We consider that $\tilde{p}_n$ is jointly set by both platforms to maximize the total profits, that is,

$$\tilde{p}_n^* \in \arg\max_{\tilde{p}_n} \{\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) + \tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)\}.$$

Theorem 5. *For any given α, if $\tilde{p}_n$ is jointly decided by both platforms, then there exists an interval $(\underline{\gamma}, \bar{\gamma}) \subset (0, 1)$, such that if $\gamma \in (\underline{\gamma}, \bar{\gamma})$, $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_i(p_1^*, w_1^*, p_2^*, w_2^*)$ for $i = 1, 2$.*

Theorem 5 shows that if the price of the new service is jointly decided by P_1 and P_2, a well-designed profit sharing contract (i.e., $\gamma \in (\underline{\gamma}, \bar{\gamma})$) will benefit both platforms. As discussed above, when the terms of the coopetition (i.e., $\tilde{p}_n$, γ, and α) are not carefully designed, introducing the new joint service can yield lower profits for each platform. However, as we show in Theorem 5, if the profit split parameter γ is properly-chosen, introducing the new service will lead to a win-win outcome for both platforms. Engaging in the coopetition under a profit sharing contract induces three effects: (i) a new market share that will be split by the platforms, (ii) an adverse cannibalization effect, and (iii) a wage variation. Theorem 5 shows that under a well-designed profit sharing contract, the new market share effect dominates the cannibalization and wage variation effects for each platform. We will discuss in greater detail the implications of these effects at the end of this section.

Next, we consider the case where one of the platforms has more pricing power and selfishly decides the price $\tilde{p}_n$. If for example, $\tilde{p}_n$ is set by P_i to maximize its own profit, we have:

$$\tilde{p}_n^{i*} \in \arg\max_{\tilde{p}_n} \tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*).$$

Proposition 3. *For any given α, there exists an interval $(\underline{\gamma}', \bar{\gamma}')$, such that if $\gamma \in (\underline{\gamma}', \bar{\gamma}')$, then $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_i(p_1^*, w_1^*, p_2^*, w_2^*)$ for $i = 1, 2$ regardless of which platform sets the price of the new service.*

The results of Theorem 5 and Proposition 3 show that regardless of the decision-making process of the price of the joint service, the coopetition leads to at least the same profits as before. Nevertheless, we are interested in avoiding extreme cases and identify conditions under which the coopetition partnership yields a *strict* benefit for both platforms. Indeed, it is always possible to set $\tilde{p}_n$ very high so that no customer will opt for the new service, and we are back to the original setting. The following proposition shows that when we have a limit on the value of $\tilde{p}_n$ (i.e., the platforms cannot set the price of the new joint service arbitrarily high), both platforms can be strictly better off only when the demand-supply ratio is not too high.

Proposition 4. *The following statements hold:*

- *If $\Lambda \uparrow +\infty$, then $\tilde{p}_n^* \uparrow +\infty$ and $\tilde{p}_n^{i*} \uparrow +\infty$ for $i = 1, 2$.*
- *Assume that there is an upper bound for the price of the new joint service, i.e., $\tilde{p}_n \le \bar{p}$ for some $\bar{p} < +\infty$. Then, there exists a threshold $\bar{\Lambda}$, such that if $\Lambda > \bar{\Lambda}$, $\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < \pi_1(p_1^*, w_1^*, p_2^*, w_2^*)$ or $\tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < \pi_2(p_1^*, w_1^*, p_2^*, w_2^*)$ regardless of who decides the price of the new service.*

Proposition 4 shows that the demand-supply ratio (captured by Λ) has a crucial implication on how the coopetition affects the platforms' profits. More specifically, if the demand-supply ratio is not too high (i.e., $\Lambda < \bar{\Lambda}$), the platforms can design a profit sharing contract (by setting $\tilde{p}_n$, γ, and

α) that will make the coopetition partnership strictly beneficial for both platforms (i.e., a Pareto improvement in both profits). However, if the demand-supply ratio becomes too high, at least one of the platforms will be hurt by the coopetition (assuming $\tilde{p}_n$ is bounded). In this case, introducing the new service will make (at least) one of the platforms over-demanded. This will in turn drive the platform(s) to increase the wage, and hence reduce the profit. On the other hand, when the demand-supply ratio is not too high, introducing the new joint service expands the market share of both platforms, thus increasing the revenues without imposing high additional wages.

To conclude this subsection, we revisit the three effects induced by the coopetition partnership (i.e., new market share, cannibalization, and wage variation) and discuss how a well-designed profit sharing contract could help balance these effects to benefit both platforms.

- ***New market share***. The new joint service may attract customers who would otherwise leave the market. Mathematically, the new market share effect for P_i can be quantified as the portion of the profit generated by the new service which is allocated to P_i, i.e., $\frac{\gamma_i \tilde{n} \tilde{p}_n \tilde{d}_n^*}{\tilde{n}} = \gamma_i \tilde{n} \tilde{p}_n$. It can be shown that the new market share effect for P_i is quasi-concave in $\tilde{p}_n$ and increasing in γ_i.
- ***Cannibalization***. Introducing the new service will also cannibalize the demand of the original services since customers may switch from the original services to the new one. The cannibalization effect for P_i is captured by $p_i^* d_i^* - \tilde{p}_i^* \tilde{d}_i^*$, which is always positive unless $\tilde{p}_n = +\infty$. We can show that the cannibalization effect for each platform is decreasing in $\tilde{p}_n$, as expected.
- ***Wage variation***. To still match supply with demand in the presence of coopetition, the platforms will adjust their wages. It is worth noting that while the new market share effect is beneficial and the cannibalization effect is harmful to the platforms, the wage variation effect may go either way. Specifically, we can quantify the wage variation for P_i as $\tilde{w}_i^* \tilde{d}_i^* + \gamma_i(\alpha_1 \tilde{w}_1^* + \alpha_2 \tilde{w}_2^*) \frac{\tilde{d}_n^*}{\tilde{n}} - w_i^* d_i^*$. We can show that the wage variation will shrink to zero if $\tilde{p}_n$ approaches infinity, and this effect gets strengthened when γ_i is larger.

To summarize, our analysis in this subsection allows us to draw practical insights on the interplay of the above three effects. Our results show that a well-designed profit sharing contract can successfully balance these effects and lead to an overall positive benefit for both platforms, regardless of which platform is setting the price of the new service. We next turn our attention to investigate the impact of coopetition on riders and drivers.

5.2. Surpluses of Riders and Drivers

So far, we focused on the impact of coopetition on the profits earned by the platforms. In this section, we investigate the impact of coopetition on the surpluses of riders and drivers. It is worth noting that the surpluses of riders and drivers are not (explicitly) dependent on the profit sharing parameter γ. We use $RS(p_1,p_2)$ to denote the rider surplus of the benchmark setting (i.e., without coopetition), when P_1 (resp. P_2) sets p_1 (resp. p_2) for its original service. We have:

$$RS(p_1,p_2) = \Lambda\mathbb{E}\Big[\max\{q_1 - p_1 + \xi_1, q_2 - p_2 + \xi_2, \xi_0\}\Big].$$

Let $\tilde{RS}(\tilde{p}_n,\tilde{p}_1,\tilde{p}_2)$ denote the expected rider surplus after introducing the new joint service:

$$\tilde{RS}(\tilde{p}_1,\tilde{p}_2,\tilde{p}_n) = \Lambda\mathbb{E}\Big[\max\{q_1 - \tilde{p}_1 + \xi_1, q_2 - \tilde{p}_2 + \xi_2, q_n - \tilde{p}_n + \xi_n, \xi_0\}\Big].$$

We remark that the rider surpluses $RS(\cdot,\cdot)$ and $\tilde{RS}(\cdot,\cdot)$ are unique up to an additive constant. Indeed, for any rider, if the random utility terms $(\xi_1,\xi_2,\xi_n,\xi_0)$ are shifted to $(\xi_1+c,\xi_2+c,\xi_n+c,\xi_0+c)$ for any constant c, then the probabilities that this rider will choose any of the four alternatives (P_1, P_2, the new service, and the outside option) remain the same. Nevertheless, the change in the expected rider surplus generated by introducing the new joint service, $\tilde{RS}(\tilde{p}_n,\tilde{p}_1,\tilde{p}_2) - RS(p_1,p_2)$, is independent of the constant c. More specifically, one can derive the following expressions: $RS(p_1,p_2) = \log[1+\exp(q_1-p_1)+\exp(q_2-p_2)]+c$ and $\tilde{RS}(\tilde{p}_n,\tilde{p}_1,\tilde{p}_2) = \log[1+\exp(q_1-\tilde{p}_1)+\exp(q_2-\tilde{p}_2)+\exp(q_n-\tilde{p}_n)]+c$. For more details on the consumer surplus under the MNL model and on the derivation of the above expressions, we refer the reader to the literature on discrete choice models (see, e.g., Chapter 3.5 of Train 2009).

Proposition 5. *For any $(\tilde{p}_n,\gamma,\alpha)$, $\tilde{RS}(\tilde{p}_1^*,\tilde{p}_2^*,\tilde{p}_n) > RS(p_1^*,p_2^*)$.*

Proposition 5 shows that introducing the new joint service will increase the expected rider surplus, regardless of the specifics of the profit sharing contract and of the new service price. This result is expected as riders can now enjoy an additional alternative for service.

We next examine the impact of coopetition on drivers, which appears to be more subtle. Let $DS(w_1,w_2)$ denote the expected surplus of drivers before the coopetition partnership. We have:

$$DS(w_1,w_2) = \mathbb{E}\Big[\max\{a_1 + w_1 + \eta_1, a_2 + w_2 + \eta_2, \eta_0\}\Big].$$

Analogously, the expected surplus of drivers after introducing the new joint service is:

$$\tilde{DS}(\tilde{w}_1,\tilde{w}_2) = \mathbb{E}\Big[\max\{a_1 + \tilde{w}_1 + \eta_1, a_2 + \tilde{w}_2 + \eta_2, \eta_0\}\Big].$$

Proposition 6. *For any $(\tilde{p}_n,\gamma,\alpha)$, there exists a threshold $\bar{n}_d > 1$ such that $\tilde{DS}(\tilde{w}_1^*,\tilde{w}_2^*) > DS(w_1^*,w_2^*)$ if and only if $\tilde{n} < \bar{n}_d$.*

As shown in Proposition 6, the drivers may not necessarily benefit from the coopetition. When the average number of riders per trip for the new service is not too high (i.e., $\tilde{n} < \bar{n}_d$), there exists profit sharing contracts that will make the drivers strictly better off under coopetition. Indeed, when $\tilde{n}$ is small, the platforms need to increase the wages to attract additional drivers to fulfill the demand of the new joint service. When $\tilde{n}$ is large, however, drivers will be worse off in the presence of coopetition. In this case, fewer drivers are needed, so that the platforms can reduce their wages. This finding explains partially why several coopetition partnerships either have no ride-sharing option for the new service (i.e., $\tilde{n}=1$), or impose a restriction on the number of riders per trip. For example, in the case of Curb and Via, the platforms imposed a limit of at most two riders who can share a ride for the new taxi-sharing service (i.e., $\tilde{n}\leq 2$). Note that when $\tilde{n}=1$, drivers will always benefit from the coopetition. In this case, the new service creates more need for drivers.

We next propose a simple and realistic way to address the issue that some drivers may be hurt by coopetition. More specifically, P_1 and P_2 can reallocate some of their revenue gains to their drivers through promotions/bonuses or other monetary compensations. We define the equilibrium revenue of P_i in the absence of coopetition as $R_i^*=p_i^*d_i^*$, and in the presence of coopetition as $\tilde{R}_i^*=\tilde{p}_i^*\tilde{d}_i^*+\gamma_i\tilde{p}_n\tilde{d}_n^*$. We next show that one can always find a profit sharing contract that will guarantee a strict revenue gain for each platform.

Proposition 7. *For any Λ, $\tilde{n}$, and α, there exists $(\tilde{p}_n,\gamma)$ such that $\tilde{R}_i^*>R_i^*$ for $i=1,2$.*

Proposition 7 shows that regardless of the demand-supply ratio and the pooling parameter value, the coopetition can strictly increase the revenue of each platform with a well-designed profit-sharing contract. Consequently, if each platform redistributes a portion of the revenue gain to its drivers (e.g., by offering bonuses), both the platforms and the drivers will be better off. Therefore, by Theorem 5, Propositions 3, 5, and 7, one can design a profit sharing contract that will benefit every single party in the market (i.e., both platforms, drivers, and riders).

6. Case Study: Curb-Via Partnership

In this section, inspired by the Curb-Via coopetition partnership, we draw insights on the impact of their coopetition in the ride-sharing market. To this end, we consider a special case of our general two-sided competition model where the two platforms have *separate* pools of drivers (taxi drivers for Curb and independent drivers for Via). We first present the analytical results for this special case, and then run computational experiments to quantify our insights.

6.1. Model

As mentioned, we consider a setting where the platforms only compete for riders and not for drivers (i.e., each platform has access to a different pool of drivers). Given that the drivers of such platforms are self-scheduled, each platform will set the wage so as to attract enough drivers. Let K_i be the total number of drivers working for P_i, $r_i\sim G_i(\cdot)$ be the reservation wage of P_i's drivers per unit time (with CDF $G_i(\cdot)$), and w_i be the wage (per unit time) offered by P_i to its drivers.

Hence, the fraction of drivers working for P_i is given by $\mathbb{P}(r_i \leq w_i) = G_i(w_i)$ and the total number of active drivers in P_i is $s_i^s = K_i G_i(w_i) n_i$, where n_i is the number of riders per trip for P_i (if there is no car-pooling for the original service of P_i, $n_i = 1$).[9]

When $G_i(\cdot)$ satisfies the log-concave condition,[10] a unique equilibrium exists. Next, we compute the expected profit per unit time of each platform. Given the price vector (p_1, w_1, p_2, w_2), let $\pi_i^s(p_1, w_1, p_2, w_2)$ be the expected profit of P_i (in the absence of coopetition). We have:

$$\pi_i^s(p_1, w_1, p_2, w_2) = (p_i - w_i)\min\{d_i^s, s_i^s\},$$
$$\text{where } d_i^s = \frac{\Lambda \exp(q_i - p_i)}{1 + \exp(q_1 - p_1) + \exp(q_2 - p_2)} \text{ and } s_i^s = K_i G_i(w_i) n_i.$$

As a corollary of Theorem 1, there exists a unique equilibrium for the special case with separate pools of drivers (assuming the functions $G_i(\cdot)$ satisfy the log-concave condition for $i = 1, 2$).

Corollary 1. *For the model with separate pools of drivers, the following hold:*

- *Under equilibrium, supply matches with demand, i.e.,* $s_i^{s*} = d_i^{s*}$ *for* $i = 1, 2$.
- *The two-sided competition game admits a unique equilibrium* $(p_1^{s*}, w_1^{s*}, p_2^{s*}, w_2^{s*})$. *Furthermore, the equilibrium can be computed efficiently using binary search.*

Note that the special case of our model with separate pools of drivers reduces to a traditional one-sided competition setting, and hence can be solved using the standard first-order condition argument from the literature.

In the presence of coopetition, the demand for the original service of P_i is

$$\tilde{d}_i^s = \frac{\Lambda \exp(q_i - \tilde{p}_i)}{1 + \exp(q_1 - \tilde{p}_1) + \exp(q_2 - \tilde{p}_2) + \exp(q_n - \tilde{p}_n)}.$$

To fit the setting of the Curb-Via partnership, we assume that the new service is fulfilled only by Curb drivers, i.e., $\alpha_1 = 1$ and $\alpha_2 = 0$. Recall that the total number of drivers needed by P_i is $\tilde{\lambda}_i^s := \tilde{d}_i^s + \alpha_i \tilde{d}_n^s / \tilde{n}$ $(i = 1, 2)$. As in the base model, we consider a profit sharing contract in which P_1 and P_2 split the net profit generated by the new joint service using $\gamma_1 + \gamma_2 = 1$. We can write the profit function of each platform as

$$\tilde{\pi}_i^s(\tilde{p}_1, \tilde{w}_1, \tilde{p}_2, \tilde{w}_2) = (\tilde{p}_i - \tilde{w}_i)\min\left\{\tilde{d}_i^s, \frac{\tilde{s}_i^s \tilde{d}_i^s}{\tilde{\lambda}_i^s}\right\} +$$
$$+ \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1)\min\left\{\frac{\alpha_1 \tilde{d}_n^s}{\tilde{n}}, \frac{\tilde{s}_1^s \alpha_1 \tilde{d}_n^s}{\tilde{n}\tilde{\lambda}_1^s}\right\} + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_2)\min\left\{\frac{\alpha_2 \tilde{d}_n^s}{\tilde{n}}, \frac{\tilde{s}_2^s \alpha_2 \tilde{d}_n^s}{\tilde{n}\tilde{\lambda}_2^s}\right\},$$

[9] Note that the MNL model considered in Section 2 under separate pools of drivers is a special case of this model when we assume that r_i follows the logistic distribution.

[10] A distribution satisfies the log-concave condition, if the logarithm of its CDF is concave. Several commonly-used distributions (e.g., uniform, normal, Gamma, Weibull, logistic) satisfy this condition (see Bagnoli and Bergstrom 2005).

where $\tilde{d}_i^s = \frac{\Lambda \exp(q_i - \tilde{p}_i)}{1+\exp(q_1-\tilde{p}_1)+\exp(q_2-\tilde{p}_2)+\exp(q_n-\tilde{p}_n)}$ and $\tilde{s}_i^s = K_i G_i(w_i) n_i$ for $i = 1, 2$.

As before, the platforms first jointly decide $\tilde{p}_n$ and γ. Then, they engage in a competition game to maximize their profits, i.e., the equilibrium $(\tilde{p}_1^{s*}, \tilde{w}_1^{s*}, \tilde{p}_2^{s*}, \tilde{w}_2^{s*})$ satisfies: $(\tilde{p}_i^{s*}, \tilde{w}_i^{s*}) \in \arg\max_{(p_i, w_i)} \tilde{\pi}_i^s(p_i, w_i, \tilde{p}_{-i}^{s*}, \tilde{w}_{-i}^{s*})$. We can show the existence and uniqueness of equilibrium $(\tilde{p}_1^{s*}, \tilde{w}_1^{s*}, \tilde{p}_2^{s*}, \tilde{w}_2^{s*})$ in the model under coopetition with separate pools of drivers (since it is a special case of the model with a single pool of drivers). Furthermore, all the following results hold.

Corollary 2. *For the model with separate pools of drivers, the following holds:*

1. *There exist profit sharing contracts (i.e., values of γ) such that the profit of each platform increases under coopetition, regardless of which platform sets $\tilde{p}_n$.*

2. *Under coopetition, the rider surplus always increases for any $\tilde{p}_n$.*

3. *In the case tailored to the Curb-Via partnership (i.e., $\alpha_1 = 1$ and $\alpha_2 = 0$): (a) P_1's (Curb's) driver surplus increases in the presence of coopetition if and only if $\tilde{n}$ is not too high, for any $\tilde{p}_n$; (b) P_2's (Via's) driver surplus always decreases for any $\tilde{p}_n$.*

4. *Assume that P_i modifies its objective function to account for its drivers' surplus (i.e., maximizes the sum of its own profit and driver surplus). Then, there exist profit sharing contracts (i.e., values of γ) such that the profit and the driver surplus of each platform will increase in the presence of coopetition, regardless of which platform sets $\tilde{p}_n$.*

When comparing the results of Corollary 2 to the results of the general model with a single pool of drivers in Section 5, we make the following observations. The results on the platforms' profits and on the rider surplus are identical. The result of P_1's driver surplus is also the same. However, the impact of coopetition on P_2's driver surplus is different and is now always negative. The new joint service which is solely provided by P_1's drivers, so P_2's drivers are not involved therein and hence suffer the cannibalization effect. Interestingly, each platform can overcome this issue by protecting its drivers and modify the objective to account for the driver surplus. In this case, one can guarantee that every single party (drivers, riders, and both platforms) will benefit from the coopetition partnership under a well-designed profit sharing contract.

6.2. Computational Experiments

We investigate computationally how three market features affect the impact of coopetition: (a) Product differentiation, measured by q_1/q_2, (b) Demand-supply ratio of P_2, measured by $\Lambda/(n_2 K_2)$, and (c) The expected number of riders per trip in the new joint service, $\tilde{n}$. To this end, we set $q_2 = 1$ and vary q_1 so that $q_1/q_2 \in \{1.1, 1.4, 1.8, 2.2, 2.6, 3\}$. To illustrate the coopetition partnership between Curb (P_1) and Via (P_2), we assume that K_1 is very large, the distribution $G_1(\cdot)$ is concentrated at $r_1 = 1$, and the new joint service is fulfilled exclusively by P_1's drivers (i.e., $\alpha = 1$). This is consistent with the business practice that Curb has abundant taxi drivers, and that the wage of taxi drivers is determined by the meter price. For the original taxi-hailing service of Curb, the average riders per trip is $n_1 = 1$. We fix $n_2 = 3$, $K_2 = 500$, assume that r_2 is uniformly distributed on $[0,1]$, and vary Λ so that $\Lambda/(n_2 k_2) \in \{0.5, 1, 1.5, 2, 5, 7\}$. Finally, we consider several values of $\tilde{n} \in \{1, 1.3, 1.7, 2, 2.5, 3\}$. Note that $\tilde{n} = 1$ is the extreme case in which there is no carpooling in the new joint service. Recall that the partnership between Curb and Via in NYC is such that $\tilde{n} \leq 2$. However, we still consider the case where $\tilde{n}$ can be larger than 2 to test the robustness of our results. Note that the set of parameters we are using in this section encompasses a wide range of realistic instances and hence, this allows us to quantify the practical impact of the coopetition partnership.

It is natural to assume that the quality of the new joint service q_n increases with q_1 and decreases with $\tilde{n}$. To capture this behavior, we use $q_n = q_2 + (q_1 - q_2)(n_2 + 1 - \tilde{n})/n_2$. Note that $q_n = q_1$ when $\tilde{n} = 1$ (in this case, the new service is equivalent to P_1's original service) and q_n is slightly larger than q_2 when $\tilde{n} = n_2$ (in this case, the new service is slightly better than P_2's original service). For all problem instances, we use a profit sharing contract with parameter γ^*. Here, γ^* is defined as the profit sharing parameter under which the price of the new service would be independent of which platform sets it.

Table 1 summarizes the impact of the coopetition partnership on P_1, P_2, drivers, and riders for the problem instances discussed above. We compute the relative impact of introducing the new joint service for each party. For example, the relative profit difference of P_i $(i = 1, 2)$ is given by: $\Delta\pi_i/\pi_i = [\tilde{\pi}_i(\tilde{p}_n^*, p_1^*, p_2^*|\gamma^*) - \pi_i(p_1^*, p_2^*)]/\pi_i(p_1^*, p_2^*)$.[10] Our computational tests convey that for the parameter values we consider, introducing the new joint service will in general substantially benefit all the stakeholders in the market. In particular, we can see from Table 1 that the average relative profit improvements for P_1 and P_2 are 25.38% and 23.45% respectively (and even in the worst case instances under consideration, the relative improvements amount to 13.37% and 13.13%). In addition, the average benefits of the drivers and the riders seem also to be significant. The only exception is a slight decrease in the expected surplus of P_1's drivers when $\tilde{n} > 2$ (i.e., every trip is shared by more than 2 riders on average) and q_1/q_2 is large. In this case, one can see from Table 1 that the surplus of P_1's drivers can be reduced by 3.78% in the worst case (this occurs for the instance with $q_1/q_2 = 3$ and $\tilde{n} = 3$). This is consistent with Proposition 6 and Corollary 2, which shows that if $\tilde{n}$ is large, at least one platform's drivers will not necessarily benefit from the introduction of the new joint service. However, in such cases, P_1 can still redistribute its profit gain to its drivers so that the coopetition will benefit the platform and its drivers together (see the last part of Corollary 2). We also note that even though we use a profit sharing contract with γ^*

[10] Since the rider surplus is unique up to an additive constant (see Chapter 3.5 of Train 2009), we report here the absolute (instead of the relative) differences in the expected rider surplus. The same comment applies to Tables 2-4.

and $\tilde{p}_n^*$ (designed to maximize the platforms' profits), the total surplus of P_2 and its drivers still increases. Consequently, P_2 can also decide to redistribute some of its profit gain to its drivers, so that both the platform and the drivers benefit from the coopetition.

Table 1 Summary statistics of the impact of introducing the new joint service (%)

	Average	Min	25th Percentile	Median	75th Percentile	Max
$\Delta\pi_1/\pi_1$	25.38	13.37	22.76	25.00	27.75	42.17
$\Delta\pi_2/\pi_2$	23.45	13.13	21.32	23.82	25.45	35.99
$\Delta DS_1/DS_1$	17.22	-3.78	5.94	14.85	27.36	48.02
$\Delta(\pi_2+DS_2)/(\pi_2+DS_2)$	20.30	9.38	18.37	20.96	22.69	27.93
ΔRS	1429.35	143.49	440.28	736.67	2627.44	4669.56

In Tables 2, 3, and 4, we report the average values of the relative impact when a single parameter is varied and the other two are set to specific values. This allows us to isolate the impact of a single market feature. One can see that in all cases, all the surpluses are increasing, suggesting that everyone benefits from the introduction of the new joint service. In Table 2, we study the effect of quality differentiation. We observe that as q_1/q_2 increases, the impact on the profits earned by both platforms is quite stable (the relative improvement remains around 20-30%). On the other hand, increasing the quality ratio will hurt P_1's drivers which will benefit less from the coopetition. In Table 3, we study the effect of the demand-supply ratio. When increasing $\Lambda/(n_2k_2)$, we can see that the impact on the profits of both platforms and on drivers are quite stable, while riders will benefit more from the coopetition. This follows from the fact that under high demand, introducing a new alternative will yield a larger benefit to the riders, as expected. In Table 4, we examine the effect of the expected number of riders per trip in the new joint service. In this case, increasing $\tilde{n}$ does not have a significant impact on the profits, on P_2's drivers, and on the riders. However, it has a strong effect on P_1's drivers, who exclusively provide the new joint service in our example. In summary, even though the impact of the coopetition partnership may be sensitive with respect to the different market conditions, it seems to be beneficial for all parties (both platforms, drivers, and riders) in the vast majority of the instances we considered.

We observe in Tables 2-4 that there is not a clear monotonicity pattern, as when we vary a single parameter, the profit sharing parameter γ^* changes as well (since it is endogenously decided).

Table 2 Impact of the service quality ratio q_1/q_2 when $\Lambda/(n_2k_2)=5$ and $\tilde{n}=2$ (%)

q_1/q_2	1.1	1.4	1.8	2.2	2.6	3
$\Delta\pi_1/\pi_1$	34.67	34.07	32.33	29.52	26.27	22.78
$\Delta\pi_2/\pi_2$	29.92	29.82	28.56	26.45	23.77	21.02
$\Delta DS_1/DS_1$	25.55	19.87	13.68	8.77	5.00	2.10
$\Delta(\pi_2+DS_2)/(\pi_2+DS_2)$	24.04	23.92	22.68	20.83	18.44	16.13
ΔRS	2817.74	2826.93	2774.94	2688.51	2569.04	2438.02

Table 3 **Impact of the demand-supply ratio $\Lambda/(n_2k_2)$ when $q_1/q_2=2$ and $\tilde{n}=2$ (%)**

$\Lambda/(n_2k_2)$	0.5	1	1.5	2	5	7
$\Delta\pi_1/\pi_1$	24.69	25.80	26.85	27.81	32.33	34.52
$\Delta\pi_2/\pi_2$	24.20	24.89	25.49	26.05	28.56	29.73
$\Delta DS_1/DS_1$	14.17	14.14	14.05	14.01	13.68	13.51
$\Delta(\pi_2+DS_2)/(\pi_2+DS_2)$	23.35	23.26	23.18	23.08	22.68	22.49
ΔRS	220.21	456.14	708.88	972.93	2774.94	4123.17

Table 4 **Impact of $\tilde{n}$ when $q_1/q_2=2$ and $\Lambda/(n_2k_2)=5$ (%)**

$\tilde{n}$	1	1.3	1.7	2	2.5	3
$\Delta\pi_1/\pi_1$	26.82	30.28	32.10	32.33	31.51	29.94
$\Delta\pi_2/\pi_2$	23.34	26.61	28.38	28.56	27.73	26.13
$\Delta DS_1/DS_1$	38.56	29.03	19.15	13.68	7.21	2.96
$\Delta(\pi_2+DS_2)/(\pi_2+DS_2)$	18.39	21.07	22.53	22.68	21.99	20.67
ΔRS	2358.98	2622.90	2759.60	2774.94	2712.94	2591.45

7. Extension: Endogenous Waiting Times

In this section, we extend our model by explicitly considering a key feature of ride-sharing platforms: the waiting time that riders may experience. We assume that the expected waiting time depends on the number of vacant drivers. Specifically, the expected waiting time for P_i (in the absence of coopetition) is

$$T_i = \kappa(s_i - d_i),$$

where $s_i - d_i$ is the number of vacant/idle drivers, and $\kappa(\cdot) > 0$ is a strictly decreasing and convex function on $(0,+\infty)$ with $\lim_{x\downarrow 0}\kappa(x) = +\infty$ and $\lim_{x\uparrow+\infty}\kappa(x) = 0$.[11] Note that this includes as special cases the $M/M/k$ queuing system and the situation where idle drivers are uniformly distributed across a circle so that the expected travel time to pick up a new rider is $c/(s_i - d_i)$ from some constant $c > 0$. Similar modeling approaches have also been adopted in the literature on ride-sharing platforms (see, e.g., Tang et al. 2017, Benjaafar et al. 2018, Nikzad 2018).

Following a similar approach as Cachon and Harker (2002), we assume that the platforms compete on the *total price* $f_i = p_i + g_i$, where g_i is the operational performance of P_i, which we define as $g_i = T_i$. Hence, the actual price charged by P_i to its riders is $p_i = f_i - g_i = f_i - \kappa(s_i - d_i)$.

Since $\kappa(\cdot)$ satisfies $\lim_{x\downarrow 0}\kappa(x) = +\infty$, we must have $s_i > d_i$ under equilibrium, i.e., $\min\{s_i, d_i\} = d_i$. As a result, the profit earned by P_i when waiting times are endogenous can be written as

$$\pi_i^e(f_1, w_1, f_2, w_2) = [f_i - \kappa(s_i - d_i) - w_i]d_i,$$

where $d_i = \frac{\Lambda\exp(q_i - f_i)}{1+\exp(q_1-f_1)+\exp(q_2-f_2)}$, $s_i = \frac{\exp(a_i+w_i)}{1+\exp(a_1+w_1)+\exp(a_2+w_2)}$, and (s_i, d_i) satisfies $s_i > d_i$. Thus, an equilibrium $(f_1^{e*}, f_2^{e*}, w_1^{e*}, w_2^{e*})$ should satisfy

$$(f_i^{e*}, w_i^{e*}) \in \arg\max_{(f_i,w_i)} \pi_i^e(f_i, w_i, f_{-i}^{e*}, w_{-i}^{e*}).$$

[11] The expected waiting time is defined only when $s_i > d_i$, as otherwise, the system is not stable.

We next extend Theorem 1 to the setting with endogenous waiting times.

THEOREM 6. *The two-sided competition game with endogenous waiting times admits a unique equilibrium* $(f_1^{e*}, f_2^{e*}, w_1^{e*}, w_2^{e*})$. *Furthermore, the equilibrium can be computed using a tatônnement scheme.*

As we did in the base setting, we can also study the impact of coopetition for the model with endogenous waiting times. We denote the total prices of the original services offered by P_1 and P_2 after introducing the new joint service by $\tilde{f}_1$ and $\tilde{f}_2$ respectively. We also denote by $\tilde{f}_n$ the total price of the new service. In the presence of coopetition, the demand of P_i is $\tilde{d}_i = \Lambda \tilde{d}'_i$, where

$$\tilde{d}'_i = \frac{\exp(q_i - \tilde{f}_i)}{1 + \exp(q_1 - \tilde{f}_1) + \exp(q_2 - \tilde{f}_2) + \exp(q_n - \tilde{f}_n)}.$$

Recall that the total number of P_i's drivers needed to satisfy all the demand is $\tilde{\lambda}_i := \tilde{d}_i + \alpha_i \tilde{d}_n / \tilde{n}$ $(i = 1, 2)$. Since there are $\tilde{s}_i - \tilde{\lambda}_i > 0$ idle P_i's drivers, the expected waiting time on this platform is $\kappa(\tilde{s}_i - \tilde{\lambda}_i)$. Consequently, the actual price charged by P_i for its original service under coopetition is $\tilde{p}_i = \tilde{f}_i - \kappa(\tilde{s}_i - \tilde{d}_i)$. Note that the average waiting time of a customer who requests the new service is given by $\alpha_1 \kappa(\tilde{s}_1 - \tilde{\lambda}_1) + \alpha_2 \kappa(\tilde{s}_2 - \tilde{\lambda}_2)$. As before, the actual price of the new service is the difference between the full price and the expected waiting time, that is, $\tilde{p}_n = \tilde{f}_n - \alpha_1 \kappa(\tilde{s}_1 - \tilde{\lambda}_1) - \alpha_2 \kappa(\tilde{s}_2 - \tilde{\lambda}_2)$. We can now write P_i's profit as

$$\begin{aligned}
\tilde{\pi}_i^e(\tilde{f}_1, \tilde{w}_1, \tilde{f}_2, \tilde{w}_2) =& (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i + \gamma_i(\tilde{p}_n - \tilde{w}_1)\alpha_1 \tilde{d}_n + \gamma_i(\tilde{p}_n - \tilde{w}_2)\alpha_2 \tilde{d}_n \\
=& [\tilde{f}_i - \kappa(\tilde{s}_i - \tilde{d}_i) - \tilde{w}_i]\tilde{d}_i + \gamma_i[\tilde{f}_n - \alpha_1 \kappa(\tilde{s}_1 - \tilde{\lambda}_1) - \alpha_2 \kappa(\tilde{s}_2 - \tilde{\lambda}_2) - \tilde{w}_1]\alpha_1 \tilde{d}_n \\
&+ \gamma_i[\tilde{f}_n - \alpha_1 \kappa(\tilde{s}_1 - \tilde{\lambda}_1) - \alpha_2 \kappa(\tilde{s}_2 - \tilde{\lambda}_2) - \tilde{w}_2]\alpha_2 \tilde{d}_n,
\end{aligned}$$

where $\tilde{d}_i = \frac{\Lambda \exp(q_i - \tilde{f}_i)}{1 + \exp(q_1 - \tilde{f}_1) + \exp(q_2 - \tilde{f}_2) + \exp(q_n - \tilde{f}_n)}$, $\tilde{s}_i = \frac{\exp(a_i + \tilde{w}_i)}{1 + \exp(a_1 + \tilde{w}_1) + \exp(a_2 + \tilde{w}_2)}$, and $(\tilde{s}_i, \tilde{\lambda}_i)$ satisfies $\tilde{s}_i > \tilde{\lambda}_i$ for $i = 1, 2$.

As in the model without coopetition, the platforms first jointly decide $\tilde{f}_n$, γ, and α. Then, they engage in a competition game to maximize their profits, i.e., the equilibrium $(\tilde{f}_1^{e*}, \tilde{w}_1^{e*}, \tilde{f}_2^{e*}, \tilde{w}_2^{e*})$ satisfies: $(\tilde{f}_i^{e*}, \tilde{w}_i^{e*}) \in \arg\max_{(f_i, w_i)} \tilde{\pi}_i^e(f_i, w_i, \tilde{f}_{-i}^{e*}, \tilde{w}_{-i}^{e*})$. We can show the existence and uniqueness of equilibrium $(\tilde{f}_1^{e*}, \tilde{w}_1^{e*}, \tilde{f}_2^{e*}, \tilde{w}_2^{e*})$ in the model under coopetition with endogenous waiting times. Furthermore, all the results presented in Section 5 can also be extended to the model with endogenous waiting times (the proofs are omitted for conciseness).

8. Conclusions

The number of two-sided platforms has increased significantly over the past few years. Recently, several coopetition partnerships emerged in the ride-sharing industry. Examples include Curb and Via in NYC and Uber and PT Express in Indonesia. This paper is motivated by such coopetition partnerships that can be implemented through a profit sharing contract. It is not clear a-priori whether the competing platforms will benefit from coopetition. This paper presents a rigorous analysis to show that when properly designed, such coopetition partnerships are indeed beneficial.

To characterize the impact of coopetition on different stakeholders of the market, we first formally examine the setting with n two-sided platforms who compete for both customers and workers. Assuming that each side of the market follows an MNL model, we show the existence and uniqueness of equilibrium under general price and wage decisions and under a fixed commission rate. The two-sidedness nature of our setting makes the objective function non-differentiable, and hence traditional arguments from the literature are not applicable. We then develop a new approach based on directly analyzing the best-response strategy to characterize the existence and uniqueness of the equilibrium. We also extend our results to more general choice models (e.g., nested-MNL, mixed-MNL, and attraction models).

Armed with the existence and uniqueness of equilibrium and our new approach, we study the impact of coopetition. We show that regardless of which platform sets the price of the new service, there always exists a profit sharing contract that increases the profit of each platform—a win-win strategy. We also convey that riders and drivers can benefit from the coopetition partnership. In summary, the analysis and results in this paper suggest that when the coopetition terms are carefully designed, every single party will benefit (riders, drivers, and both platforms).

This paper is among the first to propose a tractable model to study competition and partnerships in the ride-sharing industry. It allows us to draw practical insights on the impact of some recent partnerships observed in practice. Several interesting extensions are left for future research. For example, what is the long-term impact of such partnerships? Shall the platforms consider more complicated contracts such as two-part piecewise linear agreements (i.e., allowing two different profit portions depending on the scale of the joint service)? A second potential direction for future research is to study an alternative form of coopetition, known as joint ownership of a subsidiary. For example, Uber and a Russian taxi-sharing platform Yandex.Taxi merged their businesses in Russia under a new company.[11] It could be interesting to compare the two different forms of coopetition.

Acknowledgments

We would like to thank several people at Via's NYC office for insightful discussions and feedback. In particular, we are thankful to Michael-David Fiszer, Saar Golde, and Ori Klein. The authors also thank Tarek Abdallah, Ming Hu, Alexandre Jacquillat, and Georgia Perakis for their valuable feedback, which has helped improve this paper.

[11] https://www.nytimes.com/2017/07/13/technology/uber-russia-yandex.html

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Appendix A: Summary of Notation

Table 5 Summary of Notation

P_1 :	Platform 1
P_2 :	Platform 2
q_i :	Perceived quality of Platform i $(i=1,2)$
q_n :	Perceived quality of the new joint service
p_i :	Price of P_i without the new joint service
$\tilde{p}_i$:	Price of P_i with the new joint service
$\tilde{p}_n$:	Price of the new joint service
Λ :	Total customer arrival rate
d_i :	Customer arrival rate of P_i without the new joint service
$\tilde{d}_i$:	Customer arrival rate of P_i with the new joint service
$\tilde{d}_n$:	Customer arrival rate of the new joint service
a_i :	Attractiveness of Platform i
K :	Total number of workers on the market, normalized to 1
w_i :	Wage of P_i's workers
s_i :	Number of workers working for P_i without the new joint service
$\tilde{s}_i$:	Number of workers working for P_i with the new joint service
γ :	Fraction of profit generated by the new joint service allocated to P_1
α :	Fraction of the new joint service allocated to P_1's drivers
$\tilde{\lambda}_i$:	Total number of workers needed by P_i (with coopetition)
β :	Fixed share of the price allocated to workers under a fixed-commission rate
$\tilde{n}$:	Number of customers per service for the new joint service

Appendix B: Proof of Statements

Auxiliary Lemma

Before presenting the proofs of our results, we state and prove an auxiliary lemma that is extensively used throughout this Appendix.

Lemma 1. *For the model without coopetition, we have:* $\partial_{p_i} d'_i = -(1-d'_i)d'_i$ *and* $\partial_{p_j} d'_i = d'_i d'_j$ $(i=1,2$ *and* $j \neq i)$. *For the model with coopetition, we have* $\partial_{\tilde{p}_i} \tilde{d}'_i = -(1-\tilde{d}'_i)\tilde{d}'_i$ *and* $\partial_{\tilde{p}_j} \tilde{d}'_i = \tilde{d}'_i \tilde{d}'_j$.

Proof. We only present the proof for the case without coopetition as the case with coopetition follows the exact same argument. Since $d'_i = \frac{\exp(q_i-p_i)}{1+\exp(q_1-p_1)+\exp(q_2-p_2)}$, we have:

$$\begin{aligned}
\partial_{p_i} d'_i &= \frac{-\exp(q_i-p_i)[1+\exp(q_1-p_1)+\exp(q_2-p_2)]+[\exp(q_i-p_i)]^2}{[1+\exp(q_1-p_1)+\exp(q_2-p_2)]^2} \\
&= -\frac{\exp(q_i-p_i)}{1+\exp(q_1-p_1)+\exp(q_2-p_2)} + \left(\frac{\exp(q_i-p_i)}{1+\exp(q_1-p_1)+\exp(q_2-p_2)}\right)^2 \\
&= -d'_i + (d'_i)^2 = -(1-d'_i)d'_i
\end{aligned}$$

and

$$\begin{aligned}
\partial_{p_j} d'_i &= \frac{\exp(q_i-p_i)\exp(q_j-p_j)}{[1+\exp(q_1-p_1)+\exp(q_2-p_2)]^2} \\
&= \frac{\exp(q_i-p_i)}{1+\exp(q_1-p_1)+\exp(q_2-p_2)} \times \frac{\exp(q_j-p_j)}{1+\exp(q_1-p_1)+\exp(q_2-p_2)} = d'_i d'_j. \quad \square
\end{aligned}$$

The following lemma establishes Step III in the proof of Theorem 1.

LEMMA 2. *There exists a k^*, such that the k^*-fold best response is a contraction mapping under the ℓ_1 norm, i.e., there exists a constant $q \in (0,1)$, such that*

$$||T^{(k^*)}(p_1,w_1,p_2,w_2) - T^{(k^*)}(p_1',w_1',p_2',w_2')||_1 \leq q||(p_1,w_1,p_2,w_2) - (p_1',w_1',p_2',w_2')||_1.$$

Proof. We denote $\hat{p}_{-i} = p_{-i} + \delta$. Observe that $\big\{-\log[1+\exp(q_{-i}-p_{-i})]\big\}' = \frac{\exp(q_{-i}-p_{-i})}{1+\exp(q_{-i}-p_{-i})} < \frac{\exp(q_{-i})}{1+\exp(q_{-i})}$ for $p_{-i} > w_{-i} \geq 0$. By the mean-value theorem, for $\delta > 0$, $\log[1+\exp(q_{-i}-p_{-i})] - \log[1+\exp(q_{-i}-p_{-i}-\delta)] \leq \frac{\exp(q_{-i})}{1+\exp(q_{-i})}\delta \leq C\delta$, where $C := \max\big\{\frac{\exp(a_{-i})}{1+\exp(a_{-i})}, \frac{\exp(q_{-i})}{1+\exp(q_{-i})}\big\} < 1$. As shown in the proof of Step II of Theorem 1, $\hat{s}^* > s^*$ and $m_i(\hat{p}_{-i}, w_{-i}, \hat{s}^*) > m_i(p_{-i}, w_{-i}, s^*)$, i.e.,

$$\begin{aligned} 0 < &[p_i(\hat{p}_{-i},w_{-i}) - w_i(\hat{p}_{-i},w_{-i})] - [p_i(p_{-i},w_{-i}) - w_i(p_{-i},w_{-i})] < \\ &\log[1+\exp(q_{-i}-p_{-i})] - \log[1+\exp(q_{-i}-p_{-i}-\delta)] < C\delta. \end{aligned} \tag{2}$$

We denote $\delta_2 = \log\left(\frac{\hat{s}^*/\Lambda}{1-\hat{s}^*/\Lambda}\right) - \log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) > 0$ and $\delta_3 = \log\left(\frac{\hat{s}^*}{1-\hat{s}^*}\right) - \log\left(\frac{s^*}{1-s^*}\right) > 0$, so inequality (2) implies $\delta_2 + \delta_3 < C\delta$. Therefore, we have

$$\begin{aligned} p_i(\hat{p}_{-i},w_{-i}) - p_i(p_{-i},w_{-i}) = &-\log\left(\frac{\hat{s}^*/\Lambda}{1-\hat{s}^*/\Lambda}\right) + \log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) + \\ &+ \log[1+\exp(q_{-i}-p_{-i})] - \log[1+\exp(q_{-i}-p_{-i}-\delta)] < C\delta - \delta_2, \end{aligned}$$

and $w_i(\hat{p}_{-i},w_{-i}) - w_i(p_{-i},w_{-i}) = \log\left(\frac{\hat{s}^*}{1-\hat{s}^*}\right) - \log\left(\frac{s^*}{1-s^*}\right) = \delta_3 < C\delta - \delta_2$. As a result,

$$|p_i(\hat{p}_{-i},w_{-i}) - p_i(p_{-i},w_{-i})| + |w_i(\hat{p}_{-i},w_{-i}) - w_i(p_{-i},w_{-i})| < 2(C\delta - \delta_2) < 2C\delta,$$

that is, $||T(p_i,w_i,\hat{p}_{-i},w_{-i}) - T(p_i,w_i,p_{-i},w_{-i})||_1 \leq 2C\delta$.

Let $\delta_p := p_i(\hat{p}_{-i},w_{-i}) - p_i(p_{-i},w_{-i}) > 0$ and $\delta_w := w_i(\hat{p}_{-i},w_{-i}) - w_i(p_{-i},w_{-i}) > 0$. Recall that inequality (2) implies $\delta_p + \delta_w < C\delta$. Hence, $\delta_p, \delta_w \in (0,\delta)$ and $|\delta_p - \delta_w| < C\delta$. Thus, we have $-\log[1+\exp(q_i - p_i(\hat{p}_{-i},w_{-i}))] - \{-\log[1+\exp(q_i - p_i(p_{-i},w_{-i}))]\} < C[p_i(\hat{p}_{-i},w_{-i}) - p_i(p_{-i},w_{-i})] = C\delta_p$ and $\log[1+\exp(a_i + w_i(p_{-i},w_{-i}))] - \log[1+\exp(a_i + w_i(\hat{p}_{-i},w_{-i}))] > -C\delta_w$. Therefore,

$$F_i(p_i(\hat{p}_{-i},w_{-i}), w_i(\hat{p}_{-i},w_{-i}))) - F_i(p_i(p_{-i},w_{-i}), w_i(p_{-i},w_{-i}))) < C(\delta_p - \delta_w) < C^2\delta,$$

where $F_i(p,w) := -\log[1+\exp(q_i - p_i)] - \log[1+\exp(a_i + w_i)]$. By repeating the same argument, we obtain the following inequality:

$$\begin{aligned} &||T^{(2)}(p_i,w_i,\hat{p}_{-i},w_{-i}) - T^{(2)}(p_i,w_i,p_{-i},w_{-i})||_1 \\ =&|p_{-i}^{(2)}(p_i,w_i,\hat{p}_{-i},w_{-i}) - p_{-i}^{(2)}(p_i,w_i,p_{-i},w_{-i})| + |w_{-i}^{(2)}(p_i,w_i,\hat{p}_{-i},w_{-i}) - w_{-i}^{(2)}(p_i,w_i,p_{-i},w_{-i})| < 2C^2\delta. \end{aligned}$$

We define $\hat{w}_{-i} = w_{-i} + \delta$ (for $\delta > 0$). Using the same argument once again, we obtain $||T(p_i,w_i,p_{-i},\hat{w}_{-i}) - T(p_i,w_i,p_{-i},w_{-i})||_1 < 2C\delta$ and $||T^{(2)}(p_i,w_i,p_{-i},\hat{w}_{-i}) - T^{(2)}(p_i,w_i,p_{-i},w_{-i})||_1 < 2C^2\delta$.

By using the standard induction argument, we can write

$$\begin{aligned} &|(p_1^{(k)}(\hat{p}_1,w_1,p_2,w_2) - w_1^{(k)}(\hat{p}_1,w_1,p_2,w_2)) - (p_1^{(k)}(p_1,w_1,p_2,w_2) - w_1^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta \\ &|(p_2^{(k)}(\hat{p}_1,w_1,p_2,w_2) - w_2^{(k)}(\hat{p}_1,w_1,p_2,w_2)) - (p_2^{(k)}(p_1,w_1,p_2,w_2) - w_2^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta \\ &|(p_1^{(k)}(p_1,\hat{w}_1,p_2,w_2) - w_1^{(k)}(p_1,\hat{w}_1,p_2,w_2)) - (p_1^{(k)}(p_1,w_1,p_2,w_2) - w_1^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta \\ &|(p_2^{(k)}(p_1,\hat{w}_1,p_2,w_2) - w_2^{(k)}(p_1,\hat{w}_1,p_2,w_2)) - (p_2^{(k)}(p_1,w_1,p_2,w_2) - w_2^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta \\ &|(p_1^{(k)}(p_1,w_1,\hat{p}_2,w_2) - w_1^{(k)}(p_1,w_1,\hat{p}_2,w_2)) - (p_1^{(k)}(p_1,w_1,p_2,w_2) - w_1^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta \\ &|(p_2^{(k)}(p_1,w_1,\hat{p}_2,w_2) - w_2^{(k)}(p_1,w_1,\hat{p}_2,w_2)) - (p_2^{(k)}(p_1,w_1,p_2,w_2) - w_2^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta \\ &|(p_1^{(k)}(p_1,w_1,p_2,\hat{w}_2) - w_1^{(k)}(p_1,w_1,p_2,\hat{w}_2)) - (p_1^{(k)}(p_1,w_1,p_2,w_2) - w_1^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta \\ &|(p_2^{(k)}(p_1,w_1,p_2,\hat{w}_2) - w_2^{(k)}(p_1,w_1,p_2,\hat{w}_2)) - (p_2^{(k)}(p_1,w_1,p_2,w_2) - w_2^{(k)}(p_1,w_1,p_2,w_2))| \leq C^k\delta, \end{aligned}$$

and

$$||T^{(k)}(\hat{p}_1, w_1, p_2, w_2) - T^{(k)}(p_1, w_1, p_2, w_2)||_1 \le 2C^k\delta$$
$$||T^{(k)}(p_1, \hat{w}_1, p_2, w_2) - T^{(k)}(p_1, w_1, p_2, w_2)||_1 \le 2C^k\delta$$
$$||T^{(k)}(p_1, w_1, \hat{p}_2, w_2) - T^{(k)}(p_1, w_1, p_2, w_2)||_1 \le 2C^k\delta$$
$$||T^{(k)}(p_1, w_1, p_2, \hat{w}_2) - T^{(k)}(p_1, w_1, p_2, w_2)||_1 \le 2C^k\delta,$$

where $\hat{p}_i = p_i + \delta$ and $\hat{w}_i = w_i + \delta$. We define k^* as the smallest integer k such that $2C^k < 1$ (i.e., the smallest integer k such that $k > -\log(2)/\log(C)$). Therefore, we have

$$\begin{aligned}
&||T^{(k^*)}(p_1, w_1, p_2, w_2) - T^{(k^*)}(p_1', w_1', p_2', w_2')||_1 \\
\le& ||T^{(k^*)}(p_1, w_1, p_2, w_2) - T^{(k^*)}(p_1', w_1, p_2, w_2)||_1 + ||T^{(k^*)}(p_1', w_1, p_2, w_2) - T^{(k^*)}(p_1', w_1', p_2, w_2)||_1 \\
&+ ||T^{(k^*)}(p_1', w_1', p_2, w_2) - T^{(k^*)}(p_1', w_1', p_2', w_2)||_1 + ||T^{(k^*)}(p_1', w_1', p_2', w_2) - T^{(k^*)}(p_1', w_1', p_2', w_2')||_1 \\
&< 2C^{k^*}|p_1 - p_1'| + 2C^{k^*}|w_1 - w_1'| + 2C^{k^*}|p_2 - p_2'| + 2C^{k^*}|w_2 - w_2'| = q||(p_1, w_1, p_2, w_2) - (p_1', w_1', p_2', w_2')||_1,
\end{aligned}$$

where the first inequality follows from the triangle inequality. Since $q := 2C^{(k^*)} < 1$, we conclude that $T^{(k^*)}(\cdot,\cdot,\cdot,\cdot)$ is a contraction mapping under the ℓ_1 norm. $\square$

Proof of Proposition 2

We first show that the best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in Λ. Recall from the proof of Theorem 1 that $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ can be characterized as the solution to the following one-dimensional convex program:

$$\begin{aligned}
s^* = \arg\max_s \ & \pi_i(s|p_{-i}, w_{-i}, \Lambda) \\
\text{where } \pi_i(s|p_{-i}, w_{-i}, \Lambda) &= [p_i(p_{-i}, w_{-i}, s) - w_i(p_{-i}, w_{-i}, s)]s \\
&= \left\{q_i + a_i - \log\left(\frac{s/\Lambda}{1 - s/\Lambda}\right) - \log\left(\frac{s}{1-s}\right) - \log[1 + \exp(q_{-i} - p_{-i})] - \log[1 + \exp(a_{-i} + w_{-i})]\right\}s.
\end{aligned}$$

We then have $p_i(p_{-i}, w_{-i}) = q_i - \log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) - \log[1 + \exp(q_{-i} - p_{-i})]$ and $w_i(p_{-i}, w_{-i}) = -a_i + \log\left(\frac{s^*}{1-s^*}\right) + \log[1 + \exp(a_{-i} + w_{-i})]$. One can easily check, by computing the cross-derivative, that $\pi_i(s|p_{-i}, w_{-i}, \Lambda)$ is supermodular in (s, Λ). Therefore, s^* and $w_i(p_{-i}, w_{-i}) = -a_i + \log\left(\frac{s^*}{1-s^*}\right) + \log[1 + \exp(a_{-i} + w_{-i})]$ are increasing in Λ.

We define $t := \frac{s/\Lambda}{1-s/\Lambda} = \frac{s}{\Lambda - s}$. Then, we have $s = \frac{\Lambda t}{1-t}$. Optimizing $\pi_i(\cdot|p_{-i}, w_{-i}, \Lambda)$ over s is equivalent to optimizing $\psi_i(t|\Lambda)$ over t, where

$$\psi_i(t|\Lambda) := \left\{q_i + a_i - \log(t) - \log\left(\frac{\Lambda t}{1 - t - \Lambda t}\right) - \log[1 + \exp(q_{-i} - p_{-i})] - \log[1 + \exp(a_{-i} + w_{-i})]\right\}\frac{\Lambda t}{1-t}.$$

We define $t^* := \arg\max_t \psi_i(t|\Lambda)$. We have $t^* = \frac{s^*/\Lambda}{1-s^*/\Lambda}$. By calculating the cross-derivative, one can easily check that $\psi_i(t|\Lambda)$ is submodular in (t, Λ), and hence t^* is decreasing in Λ. Thus, $p_i(p_{-i}, w_{-i}) = q_i - \log\left(\frac{s^*/\Lambda}{1-s^*/\Lambda}\right) - \log[1 + \exp(q_{-i} - p_{-i})] = q_i - \log(t^*) - \log[1 + \exp(q_{-i} - p_{-i})]$ is increasing in Λ. We then have proved that $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in Λ. We define $(p_1^{(k)}, w_1^{(k)}, p_2^{(k)}, w_2^{(k)}) = T^{(k)}(p_1, w_1, p_2, w_2)$ for $k \ge 1$ and for a given initial strategy (p_1, w_1, p_2, w_2). Since $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are both increasing in p_{-i} and w_{-i}, then $p_1^{(k)}$, $w_1^{(k)}$, $p_2^{(k)}$, and $w_2^{(k)}$ are increasing in Λ for each k. By Theorem 1, $(p_1^*, w_1^*, p_2^*, w_2^*) = \lim_{k\uparrow+\infty}(p_1^{(k)}, w_1^{(k)}, p_2^{(k)}, w_2^{(k)})$. Hence, $p_i^* = \lim_{k\uparrow+\infty} p_i^{(k)}$, $w_i^* = \lim_{k\uparrow+\infty} w_i^{(k)}$, $i = 1, 2$ are all increasing in Λ. This concludes the proof of Proposition 2. $\square$

Proof of Theorem 2

Since the proof follows from a similar argument as Theorem 1, we only present its sketch.

Under Assumption 1(a), similar to Theorem 1, we can show that, in equilibrium, the supply and demand of each platform should match. Then, as in Theorem 1, we show that given other platforms' price and wage decisions, $(\mathbf{p_{-i}}, \mathbf{w_{-i}}) = (p_1, w_1, p_2, w_2, \cdots, p_{i-1}, w_{i-1}, p_{i+1}, w_{i+1}, \cdots, p_n, w_n)$, P_i's best-response price $p_i(\mathbf{p_{-i}}, \mathbf{w_{-i}})$ and wage $w_i(\mathbf{p_{-i}}, \mathbf{w_{-i}})$ are continuously increasing in p_j and w_j for any $j \neq i$. By Tarski's Fixed Point Theorem, a pure equilibrium strategy exists under Assumption 1(a).

If Assumptions 1(a,b,c) hold, to show the equilibrium uniqueness, it suffices to show that the $k-$fold best-response mapping, $T^{(k)}(p_1, w_1, p_2, w_2, \cdots, p_n, w_n)$, is a contraction mapping under the ℓ_1 norm when k is sufficiently large. This would follow from a similar argument as the proof of Lemma 2. More specifically, under Assumptions 1(a,b,c), for any $(\mathbf{p}, \mathbf{w})$ and $(\mathbf{p}', \mathbf{w}')$, we have

$$||T^{(k)}(\mathbf{p}, \mathbf{w}) - T^{(k)}(\mathbf{p}', \mathbf{w}')||_1 \leq 2(n-1)C^k ||(\mathbf{p}, \mathbf{w}) - (\mathbf{p}', \mathbf{w}')||_1.$$

Therefore, if $k > -[\log(2) + \log(n-1)]/\log(C)$, then $T^{(k)}(\cdot, \cdot)$ is a contraction mapping under the ℓ_1 norm. Following the same argument as in Lemma 2 and Theorem 1, we obtain that the sequence $\{T^{(k)}(\mathbf{p}, \mathbf{w}) : k \geq 1\}$ converges to the unique equilibrium of the two-sided competition game. This also implies that the equilibrium can be computed using a *tatônnement* scheme. $\square$

Proof of Theorem 3

Similar to the proof of Theorem 1, we prove Theorem 3 using the following three steps:

- Under equilibrium, $s_i^{c*} \geq d_i^{c*}$, i.e., supply dominates demand.
- The best-response price $p_i^c(p_{-i})$ is continuously increasing in p_{-i}. This will imply that an equilibrium exists.
- The best-response price $p_i^c(\cdot)$ is a contraction mapping, i.e., $|p_i^c(p_{-i}) - p_i^c(p'_{-i})| \leq q_c |p_{-i} - p'_{-i}|$ for some $q_c \in (0,1)$. This will imply that the equilibrium is unique and can be computed using a *tatônnement* scheme.

Step I. $s_i^{c*} \geq d_i^{c*}$

If $s_i^{c*} < d_i^{c*}$, then P_i can increase its price from p_i^{c*} to $\hat{p}_i^{c*} = p_i^{c*} + \epsilon$, and accordingly its wage from βp_i^{c*} to $\beta p_i^{c*} + \beta\epsilon$, where ϵ is small enough so that $\hat{s}_i^{c*} \leq \hat{d}_i$. With this price adjustment, P_i's profit increases by at least $\epsilon s_i^{c*} > 0$, contradicting that (p_i^{c*}, p_{-i}^{c*}) is an equilibrium. Therefore, we must have $s_i^{c*} \geq d_i^{c*}$ for $i = 1, 2$.

Step II. $p_i^c(p_{-i})$ *is continuously increasing in* p_{-i}

Since $s_i^{c*} \geq d_i^{c*}$, the price/wage optimization of P_i can be formulated as follows:

$$\begin{aligned} &\max_{p_i} (1-\beta) p_i d_i \\ &\text{s.t. } d_i = \frac{\Lambda \exp(q_i - p_i)}{1 + \exp(q_i - p_i) + \exp(q_{-i} - p_{-i})} \\ &\qquad s_i = \frac{\exp(a_i + \beta p_i)}{1 + \exp(a_i + \beta p_i) + \exp(a_{-i} + \beta p_{-i})} \\ &\qquad s_i \geq d_i. \end{aligned}$$

It is clear that the objective function is supermodular in (p_i, p_{-i}) and that the feasible set is a lattice. Hence, the best-response price $p_i^c(p_{-i})$ is continuously increasing in p_{-i}. Using Tarski's Fixed Point Theorem, an equilibrium exists.

<u>*Step III. $p_i^c(\cdot)$ is a contraction mapping*</u>

As shown in the proof of Step II above,

$$\begin{aligned} p_i^c(p_{-i}) &= \arg\max_{p_i} (1-\beta) p_i d_i \\ \text{s.t. } d_i &= \frac{\Lambda \exp(q_i - p_i)}{1+\exp(q_i-p_i)+\exp(q_{-i}-p_{-i})} \\ s_i &= \frac{\exp(a_i+\beta p_i)}{1+\exp(a_i+\beta p_i)+\exp(a_{-i}+\beta p_{-i})} \\ s_i &\geq d_i. \end{aligned}$$

We define $p(p_{-i})$ as the unconstrained optimizer of $p_i d_i$, which is increasing in p_{-i}. We also define $\bar{p}(p_{-i})$ as the unique p_i such that $s_i = d_i$, which is also increasing in p_{-i}. It is clear that $p_i^c(p_{-i}) = \max\{p(p_{-i}), \bar{p}(p_{-i})\}$. It suffices to show that there exists a constant $C \in (0,1)$ such that $p(p_{-i}+\delta) - p(p_{-i}) \leq C\delta$ and $\bar{p}(p_{-i}+\delta) - \bar{p}(p_{-i}) \leq C\delta$ for $\delta > 0$.

Since the MNL demand model satisfies the diagonal dominance condition, i.e.,

$$0 > \frac{\partial^2 \log(\frac{\exp(q_i-p_i)}{1+\exp(q_i-p_i)+\exp(q_{-i}-p_{-i})})}{\partial (p_i)^2} > -\frac{\partial^2 \log(\frac{\exp(q_i-p_i)}{1+\exp(q_i-p_i)+\exp(q_{-i}-p_{-i})})}{\partial p_i \partial p_{-i}},$$

we have:

$$\frac{\partial p(p_{-i})}{\partial p_{-i}} = -\frac{\partial^2 \log(\frac{\exp(q_i-p_i)}{1+\exp(q_i-p_i)+\exp(q_{-i}-p_{-i})})}{\partial p_i \partial p_{-i}} \Big/ \left[\frac{\partial^2 \log(\frac{\exp(q_i-p_i)}{1+\exp(q_i-p_i)+\exp(q_{-i}-p_{-i})})}{\partial (p_i)^2} - \frac{1}{(p_{-i})^2}\right] < \frac{\exp(q_{-i})}{1+\exp(q_{-i})} < 1.$$

Hence, by the mean value theorem, $p(p_{-i}+\delta) - p(p_{-i}) < C\delta$ for $C := \frac{\exp(q_{-i})}{1+\exp(q_{-i})} < 1$ and any $\delta > 0$. Note that $\bar{p}(p_{-i})$ satisfies the following

$$\frac{\Lambda \exp(q_i - \bar{p}(p_{-i}))}{1+\exp[q_i-\bar{p}(p_{-i})]+\exp(q_{-i}-p_{-i})} = \frac{\exp[a_i+\beta\bar{p}(p_{-i})]}{1+\exp[a_i+\beta\bar{p}(p_{-i})]+\exp(a_{-i}+\beta p_{-i})} := s.$$

Note also that, if $\hat{p}_{-i} = p_{-i} + \delta$,

$$\frac{\Lambda \exp[q_i - \bar{p}(p_{-i})]}{1+\exp[q_i-\bar{p}(p_{-i})]+\exp(q_{-i}-\hat{p}_{-i})} > s, \text{ whereas } \frac{\exp[a_i+\beta\bar{p}(p_{-i})]}{1+\exp[a_i+\beta\bar{p}(p_{-i})]+\exp(a_{-i}+\beta \hat{p}_{-i})} < s.$$

Furthermore, we have

$$\frac{\Lambda \exp[q_i - \bar{p}(\hat{p}_{-i})]}{1+\exp[q_i-\bar{p}(\hat{p}_{-i})]+\exp(q_{-i}-\hat{p}_{-i})} < \frac{\Lambda \exp[q_i - \bar{p}(p_{-i})]}{1+\exp[q_i-\bar{p}(p_{-i})]+\exp(q_{-i}-\hat{p}_{-i})} \text{ and}$$
$$\frac{\exp[a_i+\beta\bar{p}(\hat{p}_{-i})]}{1+\exp[a_i+\beta\bar{p}(\hat{p}_{-i})]+\exp(a_{-i}+\beta \hat{p}_{-i})} > \frac{\exp[a_i+\beta\bar{p}(p_{-i})]}{1+\exp[a_i+\beta\bar{p}(p_{-i})]+\exp(a_{-i}+\beta \hat{p}_{-i})}.$$

Therefore,

$$\begin{aligned} \hat{s} :=& \frac{\Lambda \exp[q_i - \bar{p}(\hat{p}_{-i})]}{1+\exp[q_i-\bar{p}(\hat{p}_{-i})]+\exp(q_{-i}-\hat{p}_{-i})} = \frac{\exp[a_i+\beta\bar{p}(\hat{p}_{-i})]}{1+\exp[a_i+\beta\bar{p}(\hat{p}_{-i})]+\exp(a_{-i}+\beta \hat{p}_{-i})} \\ &\in \left(\frac{\exp[a_i+\beta\bar{p}(p_{-i})]}{1+\exp[a_i+\beta\bar{p}(p_{-i})]+\exp(a_{-i}+\beta \hat{p}_{-i})}, \frac{\Lambda \exp[q_i - \bar{p}(p_{-i})]}{1+\exp[q_i-\bar{p}(p_{-i})]+\exp(q_{-i}-\hat{p}_{-i})}\right). \end{aligned}$$

If $\hat{s} < s$, assume that p' satisfies $\frac{\exp(a_i+\beta p')}{1+\exp(a_i+\beta p')+\exp(a_{-i}+\beta\hat{p}_{-i})} = s > \hat{s}$. Since $\hat{s} < s$, we have $\bar{p}(\hat{p}_{-i}) < p'$. Since the MNL model satisfies Assumption 1(c), $0 < \bar{p}(\hat{p}_{-i}) - \bar{p}(p_{-i}) < p' - \bar{p}(p_{-i}) < q_s(\hat{p}_{-i} - p_{-i}) = q_s\delta$.

Analogously, if $\hat{s} > s$, assume that p'' satisfies $\frac{\Lambda\exp(q_i-p'')}{1+\exp(q_i-p'')+\exp(q_{-i}-\hat{p}_{-i})} = s < \hat{s}$. Since $\hat{s} > s$, $\bar{p}(\hat{p}_{-i}) < p''$. Since the MNL model satisfies Assumption 1(b), $0 < \bar{p}(\hat{p}_{-i}) - \bar{p}(p_{-i}) < p'' - \bar{p}(p_{-i}) < q_d(\hat{p}_{-i} - p_{-i}) = q_d\delta$.

We define $q_c := \max\{q_d, q_s\} < 1$. Our analysis above implies that $0 < \bar{p}(p_{-i}+\delta) - \bar{p}(p_{-i}) < q_c\delta$ for $\delta > 0$. Therefore, $p_i^c(p_{-i}+\delta) - p_i^c(p_{-i}) \leq \max\{C, q_s\}\delta$ for $\delta > 0$.

We have established that the best-response mapping is a contraction mapping on the strategy space. Then, using Banach's Fixed Point Theorem, a unique equilibrium exists and can be computed using a *tatônnement* scheme. $\square$

Proof of Theorem 4

First, we observe that the same argument as in the proof of Step I of Theorem 1 implies that, in equilibrium, the supply and demand of each platform should match. More specifically, if $\tilde{s}_i^* > \tilde{\lambda}_i^*$ (resp. $\tilde{s}_i^* < \tilde{\lambda}_i^*$), P_i can decrease (resp. increase) its wage $\tilde{w}_i$ (resp. price $\tilde{p}_i$) by a sufficiently small amount to strictly increase its profit, where $\tilde{s}_i^*$ is the equilibrium supply of P_i and $\tilde{\lambda}_i^* = \tilde{d}_i^* + \alpha_i \tilde{d}_n^*/\tilde{n}$ is the total equilibrium demand for P_i's workers. With $\tilde{s}_i^* = \tilde{\lambda}_i^*$, we can write P_i's profit function as follows

$$\begin{aligned}\tilde{\pi}_i(\tilde{p}_1, \tilde{w}_1, \tilde{p}_2, \tilde{w}_2) = &(\tilde{p}_i - \tilde{w}_i) \min\left\{\tilde{d}_i, \frac{\tilde{s}_i \tilde{d}_i}{\tilde{\lambda}_i}\right\} + \\ &+ \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1) \min\left\{\frac{\alpha_1 \tilde{d}_n}{\tilde{n}}, \frac{\tilde{s}_1 \alpha_1 \tilde{d}_n}{\tilde{n}\tilde{\lambda}_1}\right\} + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_2) \min\left\{\frac{\alpha_2 \tilde{d}_n}{\tilde{n}}, \frac{\tilde{s}_2 \alpha_2 \tilde{d}_n}{\tilde{n}\tilde{\lambda}_2}\right\} \\ &= (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1)\frac{\alpha_1 \tilde{d}_n}{\tilde{n}} + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_2)\frac{\alpha_2 \tilde{d}_n}{\tilde{n}}.\end{aligned}$$

Given P_{-i}'s strategy, $(\tilde{p}_{-i}, \tilde{w}_{-i})$, we use $\tilde{p}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ and $\tilde{w}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ to denote the best-response price and wage of P_i in the model with coopetition. Given $(\tilde{p}_{-i}, \tilde{w}_{-i}, \tilde{p}_n)$, we define d as the demand of P_i's original service and λ as the total request for P_i's workers. It should be noted that d and λ satisfy the following identity: $\frac{\tilde{n}(\lambda-d)}{\alpha_i d} = \frac{\tilde{d}_n}{\tilde{d}_i} = \frac{\exp(q_n - \tilde{p}_n)}{\exp(q_i - \tilde{p}_i)}$. We next write $(\tilde{p}_i, \tilde{w}_i)$ in terms of d and λ: $\tilde{p}_i = q_i - \frac{d/\Lambda}{1-d/\Lambda} - \log[1 + \exp(q_{-i} - \tilde{p}_{-i}) + \exp(q_n - \tilde{p}_n)]$ and $\tilde{w}_i = -a_i + \frac{\lambda}{1-\lambda} + \log[\exp(a_{-i} + \tilde{w}_{-i})]$. As in the proof of Theorem 1, solving for the best-response functions $\tilde{p}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ and $\tilde{w}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ is equivalent to solving for the optimal d and λ. Specifically, after some algebraic manipulations, P_i's profit can be computed using the following two-dimensional convex program, the solution to which we denote as (d^*, λ^*):

$$\begin{aligned}&\max_{(d,\lambda)} \tilde{\pi}_i(d, \lambda | \tilde{p}_{-i}, \tilde{w}_{-i}) \\ &\text{s.t. } \tilde{\pi}_i(d, \lambda | \tilde{p}_{-i}, \tilde{w}_{-i}) = \left\{q_i - \log\left(\frac{d/\Lambda}{1-d/\Lambda}\right) - \log[1 + \exp(q_{-i} - \tilde{p}_{-i}) + \exp(q_n - \tilde{p}_n)]\right\} d \\ &\quad - \left\{-a_i + \log\left(\frac{\lambda}{1-\lambda}\right) + \log[1 + \exp(a_{-i} + \tilde{w}_{-i})]\right\} \lambda + \frac{\gamma_i \tilde{p}_n (\Lambda - d)\exp(q_n - \tilde{p}_n)}{1 + \exp(q_{-i} - \tilde{p}_{-i}) + \exp(q_n - \tilde{p}_n)} \\ &\quad \lambda - d = \frac{\alpha_i (\lambda - d)\exp(q_n - \tilde{p}_n)}{\tilde{n}[1 + \exp(q_{-i} - \tilde{p}_{-i}) + \exp(q_n - \tilde{p}_n)]}.\end{aligned} \tag{3}$$

Following the same argument as in the proof of Step II of Theorem 1, we can show that $\tilde{p}_i(\tilde{p}_{-i}, \tilde{w}_{-i}) = q_i - \log\left(\frac{d^*/\Lambda}{1-d^*/\Lambda}\right) - \log[1 + \exp(q_{-i} - \tilde{p}_{-i}) + \exp(q_n - \tilde{p}_n)]$ and $\tilde{w}_i(\tilde{p}_{-i}, \tilde{w}_{-i}) = -a_i + \log\left(\frac{\lambda^*}{1-\lambda^*}\right) + \log[1 + \exp(a_{-i} + \tilde{w}_{-i})]$ are both continuously increasing in $\tilde{p}_{-i}$ and $\tilde{w}_{-i}$ for $i = 1, 2$. Therefore, by Tarski's Fixed Point Theorem, an equilibrium exists for the model with coopetition.

To show that the equilibrium is unique, we follow the same argument as in the proof of Lemma 2. It suffices to show that for some k, the k-fold best-response mapping, $\tilde{T}^{(k)}(p_1, w_1, p_2, w_2)$, (defined in a similar fashion as $T^{(k)}(\cdot,\cdot,\cdot,\cdot)$, but for the model with coopetition) is a contraction mapping. The exact same argument as in the proof of Lemma 2 implies that for any (p_1, w_1, p_2, w_2) and any $\delta > 0$,

$$||T^{(k)}(p_1 + \delta, w_1, p_2, w_2) - T^{(k)}(p_1, w_1, p_2, w_2)||_1 \le 2C^k \delta$$

$$||T^{(k)}(p_1, w_1 + \delta, p_2, w_2) - T^{(k)}(p_1, w_1, p_2, w_2)||_1 \le 2C^k \delta$$

$$||T^{(k)}(p_1, w_1, p_2 + \delta, w_2) - T^{(k)}(p_1, w_1, p_2, w_2)||_1 \le 2C^k \delta$$

$$||T^{(k)}(p_1, w_1, p_2, w_2 + \delta) - T^{(k)}(p_1, w_1, p_2, w_2)||_1 \le 2C^k \delta,$$

which together with the triangle inequality, leads to

$$||\tilde{T}^{(k)}(p_1,w_1,p_2,w_2)-\tilde{T}^{(n)}(p'_1,w'_1,p'_2,w'_2)||_1\le 2C^k||(p_1,w_1,p_2,w_2)-(p'_1,w'_1,p'_2,w'_2)||_1,$$

where $C:=\max\left\{\frac{\exp(a_i)}{1+\exp(a_i)},\frac{\exp(q_i)}{1+\exp(q_i)}:i=1,2\right\}<1$. Consequently, $\tilde{T}^{(k^*)}$ is a contraction mapping under the ℓ_1 norm, where k^* is the smallest integer satisfying $2C^{(k^*)}<1$ (i.e., $k^*>-\log(2)/\log(C)$). The contraction mapping property of $T^{(k^*)}(\cdot,\cdot,\cdot,\cdot)$, as shown in the proof of Theorem 1, implies that the equilibrium is unique in the presence of coopetition, and that it can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 4. □

Proof of Theorem 5

First, we show that if $\tilde{p}_n\uparrow+\infty$, then $(\tilde{p}_i^*,\tilde{w}_i^*)$ converges to (p_i^*,w_i^*) for $i=1,2$. For a given (p_1,w_1,p_2,w_2), we define a two-dimensional sequence $\{\big(\tilde{p}_1(k,j),\tilde{w}_1(k,j),\tilde{p}_2(k,j),\tilde{w}_2(k,j)\big):k\ge1,j\ge1\}$, where $\big(\tilde{p}_1(k,j),\tilde{w}_1(k,j),\tilde{p}_2(k,j),\tilde{w}_2(k,j)\big)=\tilde{T}^{(k)}(p_1,w_1,p_2,w_2)$ with $\tilde{p}_n=j$. By the proof of Lemma 2, we know that $\lim_{j\uparrow+\infty}\big(\tilde{p}_1(k,j),\tilde{w}_1(k,j),\tilde{p}_2(k,j),\tilde{w}_2(k,j)\big)=T^{(k)}(p_1,w_1,p_2,w_2)$.

Therefore, as shown in the proof of Theorem 4, the equilibrium strategies with $\tilde{p}_n=j$ satisfy $\big(\tilde{p}_1^*(j),\tilde{w}_1^*(j),\tilde{p}_2^*(j),\tilde{w}_2^*(j)\big)=\lim_{k\uparrow+\infty}\big(\tilde{p}_1(k,j),\tilde{w}_1(k,j),\tilde{p}_2(k,j),\tilde{w}_2(k,j)\big)$. By the proof of Theorem 4, we have $||T^{(k)}(p_1,w_1,p_2,w_2)-T^{(k)}(p'_1,w'_1,p'_2,w'_2)||_1\le 2C^k||(p_1,w_1,p_2,w_2)-(p'_1,w'_1,p'_2,w'_2)||_1$ for $k\ge1$. Thus,

$$\begin{aligned}
|\tilde{p}_1(k+1,j)-\tilde{p}_1(k,j)|&\le 2C^k||(\tilde{p}_1(1,j),\tilde{w}_1(1,j),\tilde{p}_2(1,j),\tilde{w}_2(1,j))-(p_1,w_1,p_2,w_2)||_1\\
|\tilde{w}_1(k+1,j)-\tilde{w}_1(k,j)|&\le 2C^k||(\tilde{p}_1(1,j),\tilde{w}_1(1,j),\tilde{p}_2(1,j),\tilde{w}_2(1,j))-(p_1,w_1,p_2,w_2)||_1\\
|\tilde{p}_2(k+1,j)-\tilde{p}_2(k,j)|&\le 2C^k||(\tilde{p}_1(1,j),\tilde{w}_1(1,j),\tilde{p}_2(1,j),\tilde{w}_2(1,j))-(p_1,w_1,p_2,w_2)||_1\\
|\tilde{w}_2(k+1,j)-\tilde{w}_2(k,j)|&\le 2C^k||(\tilde{p}_1(1,j),\tilde{w}_1(1,j),\tilde{p}_2(1,j),\tilde{w}_2(1,j))-(p_1,w_1,p_2,w_2)||_1.
\end{aligned}$$

As a result, $\sum_{k=1}^{+\infty}|\tilde{p}_1(k+1,j)-\tilde{p}_1(k,j)|<+\infty$, $\sum_{k=1}^{+\infty}|\tilde{w}_1(k+1,j)-\tilde{w}_1(k,j)|<+\infty$, $\sum_{k=1}^{+\infty}|\tilde{p}_2(k+1,j)-\tilde{p}_2(k,j)|<+\infty$, and $\sum_{k=1}^{+\infty}|\tilde{w}_2(k+1,j)-\tilde{w}_2(k,j)|<+\infty$. By the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{j\uparrow+\infty}\lim_{k\uparrow+\infty}(\tilde{p}_1(k,j),\tilde{w}_1(k,j),\tilde{p}_2(k,j),\tilde{w}_2(k,j))&=\lim_{k\uparrow+\infty}\lim_{j\uparrow+\infty}(\tilde{p}_1(k,j),\tilde{w}_1(k,j),\tilde{p}_2(k,j),\tilde{w}_2(k,j)),\\
\text{i.e., }\lim_{j\uparrow+\infty}(\tilde{p}_1^*(j),\tilde{w}_1^*(j),\tilde{p}_2^*(j),\tilde{w}_2^*(j))&=\lim_{j\uparrow+\infty}\lim_{k\uparrow+\infty}(\tilde{p}_1(k,j),\tilde{w}_1(k,j),\tilde{p}_2(k,j),\tilde{w}_2(k,j))\\
&=\lim_{k\uparrow+\infty}\lim_{j\uparrow+\infty}(\tilde{p}_1(k,j),\tilde{w}_1(k,j),\tilde{p}_2(k,j),\tilde{w}_2(k,j))\\
&=\lim_{k\uparrow+\infty}T^{(k)}(p_1,w_1,p_2,w_2)=(p_1^*,w_1^*,p_2^*,w_2^*),
\end{aligned}$$

which states that if $\tilde{p}_n\uparrow+\infty$, then $(\tilde{p}_i^*,\tilde{w}_i^*)$ converges to (p_i^*,w_i^*) for $i=1,2$.

We next show that $\tilde{\pi}(\tilde{p}_n):=\tilde{\pi}_1(\tilde{p}_1^*(\tilde{p}_n),\tilde{w}_1^*(\tilde{p}_n),\tilde{p}_2^*(\tilde{p}_n),w_2^*(\tilde{p}_n))+\tilde{\pi}_2(\tilde{p}_1^*(\tilde{p}_n),\tilde{w}_1^*(\tilde{p}_n),\tilde{p}_2^*(\tilde{p}_n),w_2^*(\tilde{p}_n))$ is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, where $(\tilde{p}_1^*(\tilde{p}_n),\tilde{w}_1^*(\tilde{p}_n),\tilde{p}_2^*(\tilde{p}_n),w_2^*(\tilde{p}_n))$ is the equilibrium of the model with coopetition under $\tilde{p}_n$.

We first show that, for a given equilibrium price and wage vector $(\tilde{p}_1^*,\tilde{w}_1^*,\tilde{p}_2^*,\tilde{w}_2^*)$ associated with the price of the new service $\bar{p}_n$, the total profit of both platforms,

$$\begin{aligned}
\tilde{\pi}(\tilde{p}_n|\tilde{p}_1^*,\tilde{w}_1^*,\tilde{p}_2^*,\tilde{w}_2^*)=(\tilde{p}_1^*-\tilde{w}_1^*)\min\left\{\tilde{d}_1,\frac{\tilde{s}_1\tilde{d}_1}{\tilde{\lambda}_1}\right\}+(\tilde{p}_2^*-\tilde{w}_2^*)\min\left\{\tilde{d}_2,\frac{\tilde{s}_2\tilde{d}_2}{\tilde{\lambda}_2}\right\}+(\tilde{n}\tilde{p}_n-\tilde{w}_1^*)\min\left\{\frac{\alpha_1\tilde{d}_n}{\tilde{n}},\frac{\tilde{s}_1\alpha_1\tilde{d}_n}{\tilde{n}\tilde{\lambda}_1}\right\}\\
+(\tilde{n}\tilde{p}_n-\tilde{w}_2^*)\min\left\{\frac{\alpha_2\tilde{d}_n}{\tilde{n}},\frac{\tilde{s}_2\alpha_2\tilde{d}_n}{\tilde{n}\tilde{\lambda}_2}\right\},
\end{aligned}$$

is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, where $\tilde{\lambda}_i = \tilde{d}_i + \alpha_i \tilde{d}_n/\tilde{n}$. Since $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ is the equilibrium strategy associated with the price $\bar{p}_n$, by the proof of Theorem 4, $\tilde{d}_1 \le \frac{\tilde{s}_1 \tilde{d}_1}{\tilde{\lambda}_1}$, $\tilde{d}_2 \le \frac{\tilde{s}_2 \tilde{d}_2}{\tilde{\lambda}_2}$, $\frac{\alpha_1 \tilde{d}_n}{\tilde{n}} \le \frac{\tilde{s}_1 \alpha_1 \tilde{d}_n}{\tilde{n}\tilde{\lambda}_1}$, and $\frac{\alpha_2 \tilde{d}_n}{\tilde{n}} \le \frac{\tilde{s}_2 \alpha_2 \tilde{d}_n}{\tilde{n}\tilde{\lambda}_2}$. Therefore, when $\tilde{p}_n$ is sufficiently large, $\tilde{d}_1 < \frac{\tilde{s}_1 \tilde{d}_1}{\tilde{\lambda}_1}$, $\tilde{d}_2 < \frac{\tilde{s}_2 \tilde{d}_2}{\tilde{\lambda}_2}$, $\frac{\alpha_1 \tilde{d}_n}{\tilde{n}} < \frac{\tilde{s}_1 \alpha_1 \tilde{d}_n}{\tilde{n}\tilde{\lambda}_1}$, and $\frac{\alpha_2 \tilde{d}_n}{\tilde{n}} < \frac{\tilde{s}_2 \alpha_2 \tilde{d}_n}{\tilde{n}\tilde{\lambda}_2}$. Thus, when $\tilde{p}_n \ge \bar{p}_n$,

$$\tilde{\pi}(\tilde{p}_n|\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1 + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2 + (\tilde{n}\tilde{p}_n - \tilde{w}_1^*)\frac{\alpha_1 \tilde{d}_n}{\tilde{n}} + (\tilde{n}\tilde{p}_n - \tilde{w}_2^*)\frac{\alpha_2 \tilde{d}_n}{\tilde{n}}.$$

By Lemma 1, we have

$$\begin{aligned}\partial_{\tilde{p}_n}\tilde{\pi}(\tilde{p}_n|\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) &= (\tilde{p}_1^* - \tilde{w}_1^*)\partial_{\tilde{p}_n}\tilde{d}_1 + (\tilde{p}_2^* - \tilde{w}_2^*)\partial_{\tilde{p}_n}\tilde{d}_2 + \tilde{d}_n + (\tilde{n}\tilde{p}_n - \tilde{w}_1^*)\frac{\alpha_1 \partial_{\tilde{p}_n}\tilde{d}_n}{\tilde{n}} + (\tilde{n}\tilde{p}_n - \tilde{w}_2^*)\frac{\alpha_2 \partial_{\tilde{p}_n}\tilde{d}_n}{\tilde{n}} \\ &= \Lambda(\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1'\tilde{d}_n' + \Lambda(\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2'\tilde{d}_n' + \Lambda\tilde{d}_n' - (\tilde{n}\tilde{p}_n - \tilde{w}_1^*)\frac{\Lambda\alpha_1(1-\tilde{d}_n')\tilde{d}_n'}{\tilde{n}} - (\tilde{n}\tilde{p}_n - \tilde{w}_2^*)\frac{\Lambda\alpha_2(1-\tilde{d}_n')\tilde{d}_n'}{\tilde{n}}.\end{aligned}$$

Hence, $\partial_{\tilde{p}_n}\tilde{\pi}(\tilde{p}_n|\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = 0$ implies that

$$\begin{aligned}\tilde{p}_n =& \frac{(\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1'}{1-\tilde{d}_n'} + \frac{(\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2'}{1-\tilde{d}_n'} + \frac{1}{1-\tilde{d}_n'} + \frac{\alpha_1\tilde{w}_1^* + \alpha_2\tilde{w}_2^*}{\tilde{n}} \\ =& (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1'^* + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2'^* + \frac{1}{1-\tilde{d}_n'} + \frac{\alpha_1\tilde{w}_1^* + \alpha_2\tilde{w}_2^*}{\tilde{n}},\end{aligned} \tag{4}$$

where $d_1'^*$ (resp. $d_2'^*$) is the equilibrium market share of P_1 (resp. P_2), when $\tilde{p}_n = \bar{p}_n$. It is clear that the right-hand side of equation (4) is decreasing in $\tilde{p}_n$. Therefore, there exists a unique $\tilde{p}_n^*$ such that $\tilde{p}_n^* = (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1'^* + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2'^* + \frac{1}{1-\tilde{d}_n'} + \frac{\alpha_1\tilde{w}_1^* + \alpha_2\tilde{w}_2^*}{\tilde{n}}$. Furthermore, it is easy to check that $\partial_{\tilde{p}_n}\tilde{\pi}(\tilde{p}_n|\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > 0$ (resp. < 0) if $\tilde{p}_n < \tilde{p}_n^*$ (resp. $\tilde{p}_n > \tilde{p}_n^*$). As a result, $\tilde{\pi}(\cdot|\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ is decreasing in $\tilde{p}_n$ for $\tilde{p}_n \ge \max\{\bar{p}_n, \tilde{p}_n^*\}$. Since $\tilde{p}_n^*$ is uniformly bounded from above by using an upper bound on the right-hand side of equation (4), say $\bar{p}^* := (p_1^* - w_1^* + p_2^* - w_2^*) + w_1^* + w_2^* - \frac{1}{1-d_0'}$, where d_0' is the market share of the new joint service with $\tilde{p}_n = 0$, it immediately follows that, when $\tilde{p}_n \ge \bar{p}^*$, then $\tilde{\pi}(\tilde{p}_n) = \tilde{\pi}(\tilde{p}_n|\tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), \tilde{w}_2^*(\tilde{p}_n))$ is strictly decreasing in $\tilde{p}_n$.

Note that as $\tilde{p}_n \uparrow +\infty$, $\tilde{d}_n \downarrow 0$. Since $(\tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), \tilde{w}_2^*(\tilde{p}_n))$ approaches $(p_1^*, w_1^*, p_2^*, w_2^*)$ when $\tilde{p}_n \uparrow +\infty$, then $\tilde{\pi}(\tilde{p}_n) = \tilde{\pi}(\tilde{p}_n|\tilde{p}_1^*(\tilde{p}_n), \tilde{w}_1^*(\tilde{p}_n), \tilde{p}_2^*(\tilde{p}_n), \tilde{w}_2^*(\tilde{p}_n))$ approaches the equilibrium total profit of P_1 and P_2 without coopetition, i.e., $\pi^* := \pi_1(p_1^*, w_1^*, p_2^*, w_2^*) + \pi_2(p_1^*, w_1^*, p_2^*, w_2^*)$. Since we have shown that $\tilde{\pi}(\cdot)$ is strictly decreasing in $\tilde{p}_n \ge \bar{p}^*$ and $\lim_{\tilde{p}_n \to +\infty}\tilde{\pi}(\tilde{p}_n) = \pi^*$, $\tilde{\pi}^* := \max_{\tilde{p}_n}\tilde{\pi}(\tilde{p}_n) > \pi^*$, i.e., the maximum total profit of P_1 and P_2 with coopetition dominates that without coopetition for any $\gamma \in (0,1)$. Therefore, there exists an interval $(\underline{\gamma}, \bar{\gamma}) \subset (0,1)$, such that when $\gamma \in (\underline{\gamma}, \bar{\gamma})$, $\tilde{\pi}_i(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_i(p_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ for $i = 1,2$. This concludes the proof of Theorem 5. □

Proof of Proposition 3

We first show that $\tilde{\pi}_i(\tilde{p}_n) := \tilde{\pi}_i(\tilde{p}_i^*(\tilde{p}_n), \tilde{w}_i^*(\tilde{p}_n), \tilde{p}_{-i}^*(\tilde{p}_n), w_{-i}^*(\tilde{p}_n))$ is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, where $(\tilde{p}_i^*(\tilde{p}_n), \tilde{w}_i^*(\tilde{p}_n), \tilde{p}_{-i}^*(\tilde{p}_n), w_{-i}^*(\tilde{p}_n))$ is the equilibrium strategy of the model with coopetition under $\tilde{p}_n$.

As in the proof of Theorem 5, for a given equilibrium price and wage vector $(\tilde{p}_i^*, \tilde{w}_i^*, \tilde{p}_{-i}^*, \tilde{w}_{-i}^*)$ associated with the price of the new service $\bar{p}_n$, the profit of P_i,

$$\begin{aligned}\tilde{\pi}_i(\tilde{p}_n|\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = (\tilde{p}_1^* - \tilde{w}_1^*)\min\left\{\tilde{d}_1, \frac{\tilde{s}_1\tilde{d}_1}{\tilde{\lambda}_1}\right\} + \\ + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1^*)\min\left\{\frac{\alpha_1\tilde{d}_n}{\tilde{n}}, \frac{\tilde{s}_1\alpha_1\tilde{d}_n}{\tilde{n}\tilde{\lambda}_1}\right\} + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_2^*)\min\left\{\frac{\alpha_2\tilde{d}_n}{\tilde{n}}, \frac{\tilde{s}_2\alpha_2\tilde{d}_n}{\tilde{n}\tilde{\lambda}_2}\right\},\end{aligned}$$

is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, where $\tilde{\lambda}_i = \tilde{d}_i + \alpha_i \tilde{d}_n/\tilde{n}$. Similar to the proof of Theorem 5, when $\tilde{p}_n$ is sufficiently large, $\tilde{d}_i < \frac{\tilde{s}_i \tilde{d}_i}{\tilde{\lambda}_i}$, $\frac{\alpha_1 \tilde{d}_n}{\tilde{n}} < \frac{\tilde{s}_1 \alpha_1 \tilde{d}_n}{\tilde{n}\tilde{\lambda}_1}$, and $\frac{\alpha_2 \tilde{d}_n}{\tilde{n}} < \frac{\tilde{s}_2 \alpha_2 \tilde{d}_n}{\tilde{n}\tilde{\lambda}_2}$. Thus, when $\tilde{p}_n \geq \bar{p}_n$,

$$\tilde{\pi}_i(\tilde{p}_n | \tilde{p}_i^*, \tilde{w}_i^*, \tilde{p}_{-i}^*, \tilde{w}_{-i}^*) = (\tilde{p}_i^* - \tilde{w}_i^*)\tilde{d}_i + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1^*)\frac{\alpha_1 \tilde{d}_n}{\tilde{n}} + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_2^*)\frac{\alpha_2 \tilde{d}_n}{\tilde{n}}.$$

By Lemma 1, we have

$$\begin{aligned}
&\partial_{\tilde{p}_n} \tilde{\pi}_i(\tilde{p}_n | \tilde{p}_i^*, \tilde{w}_i^*, \tilde{p}_{-i}^*, \tilde{w}_{-i}^*) = (\tilde{p}_i^* - \tilde{w}_i^*)\partial_{\tilde{p}_n}\tilde{d}_i + \tilde{d}_n + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1^*)\frac{\alpha_1 \partial_{\tilde{p}_n}\tilde{d}_n}{\tilde{n}} + \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_2^*)\frac{\alpha_2 \partial_{\tilde{p}_n}\tilde{d}_n}{\tilde{n}} \\
&= \Lambda(\tilde{p}_i^* - \tilde{w}_i^*)\tilde{d}_i'\tilde{d}_n' + \Lambda(\tilde{p}_i^* - \tilde{w}_i^*)\tilde{d}_i'\tilde{d}_i' + \Lambda\tilde{d}_n' - \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_1^*)\frac{\Lambda\alpha_1(1-\tilde{d}_n')\tilde{d}_n'}{\tilde{n}} - \gamma_i(\tilde{n}\tilde{p}_n - \tilde{w}_2^*)\frac{\Lambda\alpha_2(1-\tilde{d}_n')\tilde{d}_n'}{\tilde{n}}.
\end{aligned}$$

Hence, $\partial_{\tilde{p}_n}\tilde{\pi}_i(\tilde{p}_n | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = 0$ implies that

$$\begin{aligned}
\tilde{p}_n &= \frac{(\tilde{p}_i^* - \tilde{w}_i^*)\tilde{d}_i'}{\gamma_i(1-\tilde{d}_n')} + \frac{1}{1-\tilde{d}_n'} + \frac{\alpha_1\tilde{w}_1^* + \alpha_2\tilde{w}_2^*}{\tilde{n}} \\
&= \frac{(\tilde{p}_i^* - \tilde{w}_i^*)\tilde{d}_i'^*}{\gamma_i} + \frac{1}{1-\tilde{d}_n'} + \frac{\alpha_1\tilde{w}_1^* + \alpha_2\tilde{w}_2^*}{\tilde{n}},
\end{aligned} \tag{5}$$

where $d_1'^*$ (resp. $d_2'^*$) is the equilibrium market share of P_1 (resp. P_2) when $\tilde{p}_n = \bar{p}_n$. It is clear that the right-hand side of equation (5) is decreasing in $\tilde{p}_n$. As a result, there exists a unique $\tilde{p}_n^{i*}(\gamma_i)$ such that equation (5) holds. Furthermore, it is easy to check that $\partial_{\tilde{p}_n}\tilde{\pi}_i(\tilde{p}_n | \tilde{p}_i^*, \tilde{w}_i^*, \tilde{p}_{-i}^*, \tilde{w}_{-i}^*) > 0$ (resp. < 0) if $\tilde{p}_n < \tilde{p}_n^{i*}(\gamma_i)$. Then, $\tilde{\pi}_i(\cdot | \tilde{p}_i^*, \tilde{w}_i^*, \tilde{p}_{-i}^*, \tilde{w}_{-i}^*)$ is decreasing in $\tilde{p}_n$ for $\tilde{p}_n \geq \max\{\bar{p}_n, \tilde{p}_n^{i*}(\gamma_i)\}$. Since $\tilde{p}_n^{i*}(\gamma_i)$ is uniformly bounded from above using an upper bound on the right-hand side of equation (5), say $\bar{p}_i^* := (p_i^* - w_i^*)/\gamma_i + w_1^* + w_2^* - \frac{1}{1-d_0'}$, where d_0' is the market share of the new joint service with $\tilde{p}_n = 0$, it follows that, when $\tilde{p}_n \geq \bar{p}_i^*$, $\tilde{\pi}_i(\tilde{p}_n) = \tilde{\pi}_i(\tilde{p}_n | \tilde{p}_i^*(\tilde{p}_n), \tilde{w}_i^*(\tilde{p}_n), \tilde{p}_{-i}^*(\tilde{p}_n), \tilde{w}_{-i}^*(\tilde{p}_n))$ is strictly decreasing in $\tilde{p}_n$.

Note that as $\tilde{p}_n \uparrow +\infty$, $\tilde{d}_n \downarrow 0$. Since $(\tilde{p}_i^*(\tilde{p}_n), \tilde{w}_i^*(\tilde{p}_n), \tilde{p}_{-i}^*(\tilde{p}_n), \tilde{w}_{-i}^*(\tilde{p}_n))$ approaches $(p_i^*, w_i^*, p_{-i}^*, w_{-i}^*)$ when $\tilde{p}_n \uparrow +\infty$, then $\tilde{\pi}_i(\tilde{p}_n) = \tilde{\pi}_i(\tilde{p}_n | \tilde{p}_i^*(\tilde{p}_n), \tilde{w}_i^*(\tilde{p}_n), \tilde{p}_{-i}^*(\tilde{p}_n), \tilde{w}_{-i}^*(\tilde{p}_n))$ approaches P_i's equilibrium profit without coopetition, i.e., $\pi_i^* := \pi_i(p_i^*, w_i^*, p_{-i}^*, w_{-i}^*)$. Thus, $\tilde{\pi}_i^*(\gamma_i) := \max_{\tilde{p}_n} \tilde{\pi}_i(\tilde{p}_n) > \pi_i^*$, that is, P_i's maximal profit with coopetition dominates that without coopetition when P_i sets $\tilde{p}_n$.

By equation (5), the profit-maximizing price of the new joint service for P_i, $\tilde{p}_n^{i*}(\gamma_i)$, is decreasing in γ_i (and hence increasing in $\gamma_{-i} = 1 - \gamma_i$), i.e., $\tilde{p}_n^{1*}(\gamma_1)$ is decreasing in γ_1 and $\tilde{p}_n^{2*}(\gamma_2)$ is increasing in $\gamma_1 = 1 - \gamma_2$.

We now show the existence of $(\underline{\gamma}', \bar{\gamma}')$. We use $\tilde{\pi}_i^*(\gamma)$ to denote P_i's equilibrium profit when the profit split share of P_1 (resp. P_2) is γ (resp. $1-\gamma$), and P_i sets the price of the new service. One can see that $\tilde{\pi}_1^*(\gamma)$ is strictly increasing in γ, whereas $\tilde{\pi}_2^*(\gamma)$ is strictly decreasing in γ.

Note that, from equation (5), $\tilde{p}_n^{1*}(0) = +\infty$ and $\tilde{p}_n^{2*}(1) = +\infty$. Since $\tilde{p}_n^{1*}(\gamma)$ (resp. $\tilde{p}_n^{2*}(\gamma)$) is decreasing (resp. increasing) in γ, then $\tilde{p}_n^{1*}(\gamma) - \tilde{p}_n^{2*}(\gamma)$ is decreasing in γ with $\tilde{p}_n^{1*}(0) - \tilde{p}_n^{2*}(0) > 0$ and $\tilde{p}_n^{1*}(1) - \tilde{p}_n^{2*}(1) < 0$. As a result, there exists a $\gamma' \in [0,1]$ such that $\tilde{p}_n^{1*}(\gamma') = \tilde{p}_n^{2*}(\gamma')$. We also denote $\tilde{p}_n' := \tilde{p}_n^{1*}(\gamma') = \tilde{p}_n^{2*}(\gamma')$, i.e., if $\gamma = \gamma'$, regardless of which platform sets the price of the new service, this platform would set it at $\tilde{p}_n'$. Therefore, with $\gamma = \gamma'$, P_i's profit in the presence of coopetition is given by $\tilde{\pi}_i(\tilde{p}_n')$, which dominates π_i^* by our analysis above, i.e., $\tilde{\pi}_i(\tilde{p}_n') > \pi_i^*$ for $i = 1, 2$. Since $\tilde{\pi}_i(\tilde{p}_i^*, \tilde{w}_i^*, \tilde{p}_{-i}^*, \tilde{w}_{-i}^* | \tilde{p}_n, \gamma)$ is continuous in $\tilde{p}_n$ and γ, there exists an interval $(\underline{\gamma}', \bar{\gamma}')$, which contains γ', such that $\tilde{\pi}_i(\tilde{p}_i^*, \tilde{w}_i^*, \tilde{p}_{-i}^*, \tilde{w}_{-i}^*) > \pi_i(p_i^*, w_i^*, p_{-i}^*, w_{-i}^*)$ for $i = 1, 2$ and $\gamma \in (\underline{\gamma}', \bar{\gamma}')$ regardless of which platform sets $\tilde{p}_n$. This concludes the proof of Proposition 3. $\square$

Proof of Proposition 4

One can easily check using equation (3) that the best-response mapping sequences, $(\tilde{p}_1(k,j), \tilde{w}_1(k,j), \tilde{p}_2(k,j), \tilde{w}_2(k,j))$, satisfy the following: as $\Lambda \uparrow +\infty$, $\lambda_i^*(k,j) \uparrow 1$ for all (k,j) and $i = 1,2$. Here, $\lambda_i^*(k,j)$ is the optimal total requests for P_i's workers in the $k-$th iteration of the *tatônnement* scheme with $\tilde{p}_n = j$. Thus, if $\Lambda \uparrow +\infty$, $\tilde{w}_1(k,j) = -a_1 + \log\left(\frac{\lambda_1^*(k,j)}{1-\lambda_1^*(k,j)}\right) + \log[a_2 + w_2(k-1,j)] \uparrow +\infty$ and $\tilde{w}_2(k,j) = -a_2 + \log\left(\frac{\lambda_2^*(k,j)}{1-\lambda_2^*(k,j)}\right) + \log[a_1 + w_1(k-1,j)] \uparrow +\infty$ for any k and j. As a result, for any $\tilde{p}_n$, the equilibrium strategy in the presence of coopetition, $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$, satisfies $\lim_{\Lambda\uparrow+\infty} \tilde{w}_1^* = +\infty$ and $\lim_{\Lambda\uparrow+\infty} \tilde{w}_2^* = +\infty$. Therefore, by (4) and (5), we have $\lim_{\Lambda\uparrow+\infty} \tilde{p}_n^* = +\infty$ and $\lim_{\Lambda\uparrow+\infty} \tilde{p}_n^{i*} = +\infty$ for $i = 1,2$. This concludes the proof of the first part.

We next show that if there is a finite upper bound on the prices set by the platforms, $\bar{p}$, at least one platform is worse off in the presence of coopetition, i.e., $\tilde{\pi}_1(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < \pi_1(p_1^*, w_1^*, p_2^*, w_2^*)$ or $\tilde{\pi}_2(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < \pi_2(p_1^*, w_1^*, p_2^*, w_2^*)$ when Λ is sufficiently large. By the proof of Step II of Theorem 1, given any P_{-i}'s strategy, (p_{-i}, w_{-i}), the best-response profit of P_i can be written as

$$\max_s \pi_i(s)$$

$$\begin{aligned}
\text{where } \pi_i(s) &= (p_i(p_{-i}, w_{-i}, s) - w_i(p_{-i}, w_{-i}, s))s \\
&= \left\{ q_i + a_i - \log\left(\frac{s/\Lambda}{1-s/\Lambda}\right) - \log\left(\frac{s}{1-s}\right) - \log[1+\exp(q_{-i} - p_{-i})] - \log[1+\exp(a_{-i} + w_{-i})] \right\} s \\
&= \left\{ q_i + a_i + \log(\Lambda - s) + \log(1-s) - 2\log(s) - \log[1+\exp(q_{-i} - p_{-i})] - \log[1+\exp(a_{-i} + w_{-i})] \right\} s.
\end{aligned}$$

Hence, it is clear that $\lim_{\Lambda\uparrow+\infty} \pi_i(s) = +\infty$ for any feasible s, which implies that $\lim_{\Lambda\uparrow+\infty} \max_s \pi_i(s) = +\infty$. Note that the optimal sales s^* should satisfy the first-order condition $\pi'(s^*) = 0$, i.e.,

$$\begin{aligned}
&q_i + a_i - \log[1+\exp(q_{-i} - p_{-i})] - \log[1+\exp(a_{-i} + w_{-i})] - \frac{s^*}{\Lambda - s^*} + \\
&+ \log(\Lambda - s^*) - \frac{s^*}{1-s^*} + \log(1-s^*) - 2\log(s^*) - 2 = 0.
\end{aligned}$$

Therefore, s^* is increasing in Λ, and as $\Lambda \uparrow +\infty$, $s^* \uparrow 1$. Thus, for any (p_{-i}, w_{-i}), when $\Lambda \uparrow +\infty$, we have $w_i(p_{-i}, w_{-i}) = -a_i + \log\left(\frac{s^*}{1-s^*}\right) + \log[1+\exp(a_{-i} + w_{-i})] \uparrow +\infty$, which implies that $\lim_{\Lambda\uparrow+\infty} w_i^* = +\infty$ for $i = 1,2$. By equation (3), for any $\tilde{p}_{-i}, \tilde{w}_{-i}$, and $\tilde{p}_n$, the best response satisfies $\lim_{\Lambda\uparrow+\infty} \tilde{w}_i(\tilde{p}_{-i}, \tilde{w}_{-i}) = +\infty$ for $i = 1,2$. This implies that the equilibrium wage satisfies $\lim_{\Lambda\uparrow+\infty} \tilde{w}_i^* = +\infty$ for $i = 1,2$. If $\tilde{p}_n \le \bar{p} < +\infty$, the profit margin of the new joint service is negative when Λ is sufficiently large, i.e., $\tilde{n}\tilde{p}_n - \tilde{w}_i^* < 0$.

On the other hand, in the presence of coopetition, when P_{-i} sets the price of its original service (p_{-i}) and given the sales of P_i (s), P_i's price satisfies $\tilde{p}_i = q_i - \log\left(\frac{s/\Lambda}{1-s/\Lambda}\right) - \log[1+\exp(q_{-i} - p_{-i}) + \exp(q_n - \tilde{p}_n)] < q_i - \log\left(\frac{s/\Lambda}{1-s/\Lambda}\right) - \log[1+\exp(q_{-i} - p_{-i})] = p_i$, i.e., P_i needs to charge a lower price to induce the same level of sales in the presence of coopetition assuming its competitor uses the same price. Therefore, for any (p_{-i}, w_{-i}), P_i's optimal profit from its original service is lower in the presence of coopetition. Using a limit argument, this implies that P_i's equilibrium profit from its original service is lower in the presence of coopetition for $i = 1,2$. Since we have shown that for sufficiently large Λ, the total profit from the new joint service is negative, this concludes the proof of the second part of Proposition 4. $\square$

Proof of Proposition 5

First, we observe that if $\tilde{p}_i^* \le p_i^*$ for $i=1,2$, we have

$$\begin{aligned} RS(\tilde{p}_1^*, \tilde{p}_2^*, \tilde{p}_n^*) =& \Lambda \mathbb{E}\Big[\max\{q_1 - \tilde{p}_1^* + \xi_1, q_2 - \tilde{p}_2^* + \xi_2, q_2 - \tilde{p}_n^* + \xi_n, \xi_0\}\Big] \\ >& \Lambda \mathbb{E}\Big[\max\{q_1 - p_1^* + \xi_1, q_2 - p_2^* + \xi_2, \xi_0\}\Big] = RS(p_1^*, p_2^*). \end{aligned}$$

Consequently, it suffices to show that $\tilde{p}_i^* \le p_i^*$ for $i=1,2$.

We define $(\tilde{p}_1^*(k,\tilde{p}_n), \tilde{w}_1^*(k,\tilde{p}_n), \tilde{p}_2^*(k,\tilde{p}_n), \tilde{w}_2^*(k,\tilde{p}_n)) := \tilde{T}^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$, where $\tilde{T}^{(k)}(\cdot,\cdot,\cdot,\cdot)$ is the $k-$fold best-response mapping when the price of the new joint service is $\tilde{p}_n$. On the other hand, we know that $(p_1^*, w_1^*, p_2^*, w_2^*) = T^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$ for any $k \ge 1$, where $T^{(k)}(\cdot,\cdot,\cdot,\cdot)$ is the $k-$fold best-response mapping of the model without coopetition, which can also be viewed as a special case of $\tilde{T}^{(k)}(\cdot,\cdot,\cdot,\cdot)$ with $\tilde{p}_n = +\infty$. Comparing the best-response formulations of $\tilde{T}^{(1)}$ and $T^{(1)}$, it is easy to check that, given (p_{-i}^*, w_{-i}^*), the best-response price $\tilde{p}_i^*(1,\tilde{p}_n)$ is increasing in $\tilde{p}_n$. Since the model without coopetition can be viewed as a special case of the model with coopetition under $\tilde{p}_n = +\infty$, we have $\tilde{p}_i^*(1,\tilde{p}_n) < \tilde{p}_i^*(1,+\infty) = p_i^*$ for $i=1,2$. Then, following the same argument as in the proof of Theorem 4, we conclude that $\tilde{p}_i^*(k+1,\tilde{p}_n)$ is strictly increasing in both $\tilde{p}_n$ and $\tilde{p}_{-i}^*(k)$ for $i=1,2$. Then, using a standard induction argument we obtain $\tilde{p}_i^*(k,\tilde{p}_n) < \tilde{p}_i^*(k,+\infty) = p_i^*$ for $k \ge 1$ and $i=1,2$. Therefore, $\tilde{p}_i^* = \lim_{k\uparrow+\infty} \tilde{p}_i^*(k,\tilde{p}_n) < p_i^*$ for $i=1,2$, and this concludes the proof of Proposition 5. $\square$

Proof of Proposition 6

First, we rewrite $DS(\cdot,\cdot)$ and $\tilde{DS}(\cdot,\cdot)$ as a function of (s_1^*, s_2^*, s_0^*) and $(\tilde{s}_1^*, \tilde{s}_2^*, \tilde{s}_1^*, \tilde{s}_0^*)$. Specifically, one can derive the following expressions: $DS(p_1^*, p_2^*) = \log(1/s_0^*) + c$ and $\tilde{DS}(\tilde{p}_1^*, \tilde{p}_2^*, \tilde{p}_n^*) = \log(1/\tilde{s}_0^*) + c$. For more details on the consumer surplus under the MNL model and on the derivation of the above expressions, we refer the reader to the literature (see, e.g., Chapter 3.5 of Train 2009). We will show that (a) if $\tilde{n}=1$, then $\tilde{s}_0^* < s_0^*$; (b) if $\tilde{n}$ is sufficiently large, then $\tilde{s}_0^* > s_0^*$; and (c) $\tilde{s}_0^*$ is continuously increasing in $\tilde{n}$. Then, claims (a), (b), and (c) would imply the result of Proposition 6.

Claim (a): If $\tilde{n}=1$, from the proof of Theorem 4, we have $\tilde{s}_i^* = \tilde{\lambda}_i^* = \tilde{d}_i^* + \alpha_i \tilde{d}_n^*/\tilde{n} = \tilde{d}_i^* + \alpha_i \tilde{d}_n^*$. Therefore, $\tilde{s}_i^* + \tilde{s}_{-i}^* = \tilde{d}_1^* + \tilde{d}_2^* + \tilde{d}_n^*$. As shown in the proof of Proposition 5, $\tilde{p}_i^* < p_i^*$ for $i=1,2$, and hence $\tilde{d}_1^* + \tilde{d}_2^* + \tilde{d}_n^* > d_1^* + d_2^*$. This concludes the proof of claim (a).

Claim (b): As $\tilde{n} \uparrow +\infty$, we have $\tilde{s}_i^* = \tilde{\lambda}_i^* = \tilde{d}_i^* + \alpha_i \tilde{d}_n^*/\tilde{n} = \tilde{d}_i^*$. As a result, to prove claim (b), it suffices to show that $\tilde{d}_1^* + \tilde{d}_2^* < d_1^* + d_2^*$, or equivalently $\tilde{s}_1^* + \tilde{s}_2^* < s_1^* + s_2^*$. Using equation (3), one can see that when $\tilde{s}_i^* = \tilde{d}_i^*$, the optimization problem to characterize the best-response price and wage functions of the model with coopetition reduces to the case without coopetition.

We define $(\tilde{p}_1^*(k), \tilde{w}_1^*(k), \tilde{p}_2^*(k), \tilde{w}_2^*(k)) := \tilde{T}^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$, where $\tilde{T}^{(k)}(\cdot,\cdot,\cdot,\cdot)$ is the $k-$fold best-response mapping when the price of the new joint service is $\tilde{p}_n^*$ and $\tilde{n} = +\infty$. On the other hand, we know that $(p_1^*, w_1^*, p_2^*, w_2^*) = T^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$ for $k \ge 1$, where $T^{(k)}(\cdot,\cdot,\cdot,\cdot)$ is the $k-$fold best-response mapping of the model without coopetition, which can also be viewed as a special case of $\tilde{T}^{(k)}(\cdot,\cdot,\cdot,\cdot)$ with $\tilde{p}_n = +\infty$. Comparing the best-response formulations of $\tilde{T}^{(1)}$ and $T^{(1)}$, it is easy to check that, given (p_{-i}^*, w_{-i}^*), we have $\tilde{w}_i^*(1) < w_i^*$ for $i=1,2$. Furthermore, the best-response mapping is increasing in w_{-i}^* (see the proof of Theorem 1), and hence using the standard induction argument, we obtain $\tilde{w}_i^*(k) < w_i^*$ for $k \ge 1$ and

$i=1,2$. Therefore, the equilibrium wage satisfies $\tilde{w}_i^* = \lim_{k\uparrow+\infty} \tilde{w}_i^*(k) < w_i^*$ for $i=1,2$. This implies that $\tilde{s}_0^* = \frac{1}{1+\exp(a_1+\tilde{w}_1^*)+\exp(a_2+\tilde{w}_2^*)} > \frac{1}{1+\exp(a_1+w_1^*)+\exp(a_2+w_2^*)} = s_0^*$, and hence concludes the proof of claim (b).

Claim (c): First, we show that $\tilde{w}_i^*$ is decreasing in $\tilde{n}$. We define $(\tilde{p}_1^*(k,\tilde{n}), \tilde{w}_1^*(k,\tilde{n}), \tilde{p}_2^*(k,\tilde{n}), \tilde{w}_2^*(k,\tilde{n})) := \tilde{T}^{(k)}(p_1^*, w_1^*, p_2^*, w_2^*)$, where $\tilde{T}^{(k)}(\cdot,\cdot,\cdot,\cdot)$ is the $k-$fold best-response mapping when the price of the new joint service is $\tilde{p}_n^*$ and the pooling parameter is $\tilde{n}$. By examining the best-response mapping $\tilde{T}^{(1)}$ (see the proof of Theorem 5), it is easy to check that, given (p_{-i}^*, w_{-i}^*), $\tilde{w}_i^*(1,\tilde{n})$ is decreasing in $\tilde{n}$ for $i=1,2$. Furthermore, the best-response mapping is increasing in $\tilde{w}_{-i}^*$ (see the proof of Theorem 1). Using the standard induction argument, we obtain that $\tilde{w}_i^*(k,\tilde{n})$ is increasing in $\tilde{w}_{-i}^*(k-1,\tilde{n})$, which is decreasing in $\tilde{n}$. Thus, $\tilde{w}_i^*(k,\tilde{n})$ is decreasing in $\tilde{n}$ for $k\geq 1$ and $i=1,2$. Therefore, the equilibrium wage is such that $\tilde{w}_i^* = \lim_{k\uparrow+\infty} \tilde{w}_i^*(k,\tilde{n})$ is decreasing in $\tilde{n}$ for $i=1,2$, implying that $\tilde{s}_0^* = \frac{1}{1+\exp(a_1+\tilde{w}_1^*)+\exp(a_2+\tilde{w}_2^*)}$ is increasing in $\tilde{n}$. This concludes the proof of claim (c), and thus the proof of Proposition 6.

Proof of Proposition 7

From Theorem 5, we know that if $\tilde{p}_n \to +\infty$, then $\lim_{\tilde{p}_n\uparrow+\infty}(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = (p_1^*, w_1^*, p_2^*, w_2^*)$ and $\lim_{\tilde{p}_n\uparrow+\infty}(\tilde{d}_1^*, \tilde{d}_2^*) = (d_1^*, d_2^*)$. Furthermore, we have $\lim_{\tilde{p}_n\uparrow+\infty} \tilde{p}_n\tilde{d}_n^* = 0$. As a result, $\lim_{\tilde{p}_n\uparrow+\infty} \tilde{R}_i^* = p_i^* d_i^* = R_i^*$ for $i=1,2$.

We next show that $\tilde{R}(\tilde{p}_n) := \tilde{R}_1 + \tilde{R}_2 = \tilde{p}_1^*\tilde{d}_1^* + \tilde{p}_2^*\tilde{d}_2^* + \tilde{p}_n^*\tilde{d}_n^*$ is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, where $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, w_2^*)$ is the equilibrium of the model with coopetition under $\tilde{p}_n$. Following the same argument as in the proof of Theorem 5, for a given equilibrium price and wage vector $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$, the total revenue of both platforms,

$$\tilde{R}(\tilde{p}_n|\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = \tilde{p}_1^* \min\left\{\tilde{d}_1, \frac{\tilde{s}_1\tilde{d}_1}{\tilde{\lambda}_1}\right\} + \tilde{p}_2^* \min\left\{\tilde{d}_2, \frac{\tilde{s}_2\tilde{d}_2}{\tilde{\lambda}_2}\right\} + \tilde{p}_n\tilde{d}_n = \tilde{p}_1^*\tilde{d}_1 + \tilde{p}_2^*\tilde{d}_2 + \tilde{p}_n\tilde{d}_n,$$

is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, where $\tilde{\lambda}_i = \tilde{d}_i + \alpha_i\tilde{d}_n/\tilde{n}$ $(i=1,2)$.

We have also shown that $\lim_{\tilde{p}_n\uparrow+\infty} \tilde{R}(\tilde{p}_n) = R^* := R_1^* + R_2^*$. Since $\tilde{R}(\cdot)$ is strictly decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_n$, we can find a value of $\tilde{p}_n$ such that $\tilde{R}(\tilde{p}_n) > R^*$. As shown in Theorem 5, we have $\tilde{R}_1(\tilde{p}_n) + \tilde{R}_2(\tilde{p}_n) = \tilde{R}(\tilde{p}_n)$ and $R_1^* + R_2^* = R^*$. This implies that we can find a value of γ such that $\tilde{R}_i(\tilde{p}_n) = \tilde{p}_i^*\tilde{d}_i^* + \gamma_i\tilde{p}_n\tilde{d}_n > p_i^* d_i^* = R_i^*$ for $i=1,2$. This concludes the proof of Proposition 7. $\square$

Proof of Corollary 1

The first part follows from the same argument as in the proof of Theorem 1. If $s_i^{s*} < d_i^{s*}$, P_i can increase its price to make its profit strictly higher. If $s_i^{s*} > d_i^{s*}$, P_i can decrease its price to make its profit strictly higher. Therefore, under equilibrium, we must have $s_i^{s*} = d_i^{s*}$ for $i=1,2$.

Similarly, the equilibrium existence and uniqueness follows from the same argument as in the proof of Theorem 1. To show how to compute the equilibrium $(p_1^{s*}, w_1^{s*}, p_2^{s*}, w_2^{s*})$, we note that $s_i^{s*} = d_i^{s*}$ implies that P_i's profit is equal to

$$\pi_i^s(p_i, p_{-i}) = \frac{\Lambda p_i \exp(q_i - p_i)}{1+\exp(q_1-p_1)+\exp(q_2-p_2)} - C_i(s_i),$$

where $C_i(s_i) := \frac{s_i}{n_i} G_i^{-1}\left(\frac{s_i}{K_i n_i}\right)$ represents the total cost of P_i when the supply level is s_i and $s_i = d_i = \frac{\Lambda\exp(q_i-p_i)}{1+\exp(q_1-p_1)+\exp(q_2-p_2)}$. It is clear that $\pi_i^s(\cdot,\cdot)$ is continuously differentiable for $i=1,2$, whereas $C_i(\cdot)$ is convex with respect to s_i. Therefore, the equilibrium prices p_1^{s*} and p_2^{s*} should satisfy the first-order condition:

$\partial_{p_i}\pi_i^s(p_i^{s*}, p_{-i}^{s*}) = 0$, that is, $\Lambda(d_i^{'s*} + p_i^{s*}\partial_{p_i}d_i^{'s*}) - \Lambda C_i'\left(\Lambda d_i^{'s*}\right)\partial_{p_i}d_i^{'s*} = 0$. By using Lemma 1, we know that $\partial_{p_i}d_i' = -d_i' + (d_i')^2$, and hence $\Lambda[d_i^{'s*} - p_i^{s*}d_i^{'s*}(1 - d_i^{'s*})] + \Lambda C_i'\left(\Lambda d_i^{'s*}\right)d_i^{'s*}(1 - d_i^{'s*}) = 0$, i.e.,

$$p_i^{s*} = \frac{1}{1 - d_i^{'s*}} + C_i'\big(\Lambda d_i^{'s*}\big).$$

On the other hand, by the definition of the MNL model, $\exp(q_i - p_i^{s*}) = d_i^{'s*}/d_0^{'s*}$, i.e., $p_i^{s*} = q_i + \log(d_0^{'s*}/d_i^{'s*})$. Therefore,

$$\frac{1}{1 - d_i^{'s*}} + C_i'\big(\Lambda d_i^{'s*}\big) = q_i + \log(d_0^{'s*}/d_i^{'s*}),$$

or equivalently,

$$d_i^{'s*}\exp\left(\frac{d_i^{'s*}}{1 - d_i^{'s*}}\right)\exp\left[C_i'\big(\Lambda d_i^{'s*}\big)\right] = d_0^{'s*}\exp(q_i - 1).$$

We define $U_i(x) := x\exp\left(\frac{x}{1-x}\right)\exp\left[C_i'(\Lambda x)\right]$. Clearly, $U_i(x)$ is continuous and strictly increasing in x, so we denote its inverse by $U_i^{-1}(\cdot)$. We then have $d_i^{'s*} = U_i^{-1}\big(d_0^{'s*}\exp(q_i - 1)\big)$. Since $U_i^{-1}(\cdot)$ is strictly increasing, $U_i^{-1}(0+) = 0$, and $U_i^{-1}(+\infty) = 1$, then there exists a unique $d_0^{'s*} \in (0, 1)$ that satisfies the following:

$$d_0^{'s*} + U_1^{-1}\big(d_0^{'s*}\exp(q_1 - 1)\big) + U_2^{-1}\big(d_0^{'s*}\exp(q_2 - 1)\big) = 1, \tag{6}$$

i.e., $d_0^{'s*} + d_1^{'s*} + d_2^{'s*} = 1$, where $d_i^{'s*} = U_i^{-1}\big(d_i^{'s*}\exp(q_i - 1)\big)$ for $i = 1, 2$. Since the left-hand side of equation (6), $d_0 + U_1^{-1}\big(d_0\exp(q_1 - 1)\big) + U_2^{-1}\big(d_0\exp(q_2 - 1)\big)$, is strictly increasing in d_0, one can solve for $d_0^{'s*}$ efficiently (e.g., using binary search).

Since $p_i^{s*} = q_i + \log(d_0^{'s*}/d_i^{'s*})$, we have $p_i^{s*} = q_i + \log\left[d_0^{'s*}/U_i^{-1}\big(d_0^{'s*}\exp(q_i - 1)\big)\right]$ for $i = 1, 2$. In addition, w_i^{s*} can be computed by solving the equation $d_i^{s*} = s_i^{s*}$ for $i = 1, 2$. □

Proof of Corollary 2

Since the proof of Corollary 2 is similar to the counterpart results for the model with a common pool of workers, we only present its sketch.

Part 1 follows from the same argument as in the proofs of Theorems 3 and 5. We first show that as $\tilde{p}_n \uparrow +\infty$, the model with coopetition reduces to the model without coopetition. We then argue that the total profit of both platforms is decreasing in $\tilde{p}_n$, for sufficiently large $\tilde{p}_n$. Finally, one can carefully select a value of γ so that each platform benefits from the coopetition.

Part 2 follows from the same argument as in the proof of Proposition 5. We first show that both $\tilde{p}_1^{s*}$ and $\tilde{p}_2^{s*}$ are increasing in $\tilde{p}_n$. Since the model without coopetition is a special case of the model with coopetition with $\tilde{p}_n = +\infty$, we have $\tilde{p}_i^{s*} < p_i^{s*}$ for $i = 1, 2$. This implies that the rider surplus will increase in the presence of coopetition.

Part 3(a) follows from the same argument as in the proof of Proposition 6. Specifically, we show that if $\tilde{n} = 1$, then $\tilde{w}_1^{s*} > w_1^{s*}$, so that P_1's driver surplus is higher in the presence of coopetition. If $\tilde{n} = +\infty$, then $\tilde{w}_1^{s*} < w_1^{s*}$. In this case, P_1's driver surplus is lower in the presence of coopetition. Finally, following the same argument as in the proof of Proposition 6 claim (c), we show that $\tilde{w}_1^{s*}$ is continuously decreasing in $\tilde{n}$. As a result, P_1's drivers will benefit from the coopetition if and only if $\tilde{n}$ is not too large.

Part 3(b) uses the fact that $\alpha_2 = 1 - \alpha_1 = 0$. In this case, one can see that $\tilde{d}_2^{s*} < d_2^{s*}$ and hence $\tilde{s}_2^{s*} < s_2^{s*}$, implying that $\tilde{w}_2^{s*} < w_2^{s*}$. Hence, P_2's driver surplus is always lower in the presence of coopetition.

Part (4) uses the following modified objective:

$$\pi_i^s(p_i,p_{-i}) = \frac{\Lambda p_i \exp(q_i - p_i)}{1+\exp(q_1-p_1)+\exp(q_2-p_2)} - E_i(s_i),$$

where $E_i(s_i) := C_i(s_i) - K_i\mathbb{E}\left[\min\left\{r_i, G_i^{-1}\left(\frac{s_i}{K_i n_i}\right)\right\}\right]$ is the cost of P_i adjusted by the driver surplus, when the supply level is s_i and $s_i = d_i = \frac{\Lambda\exp(q_i-p_i)}{1+\exp(q_1-p_1)+\exp(q_2-p_2)}$ ($C_i(s_i)$ is defined in the proof of Corollary 1 above). By taking the derivative, we obtain $E_i'(s_i) = \frac{1}{n_i}G_i^{-1}\left(\frac{s_i}{K_i n_i}\right) \geq 0$. Thus, $E_i(\cdot)$ is convexly increasing in s_i, preserving the same property as $C_i(\cdot)$. Therefore, if we view P_i's modified objective as its adjusted profit, the conclusion of part (4) follows from the same argument as in the proof of part (1), which is based on the proofs of Theorems 3 and 5 (the details are omitted for conciseness). □

Proof of Theorem 6

Given that $\kappa(0+) = +\infty$, we must have $s_i^{e*} > d_i^{e*}$ for $i = 1,2$ under equilibrium. Hence, P_i's profit under equilibrium can be written as $[f_i - \kappa(s_i - d_i) - w_i]d_i$. Given (p_{-i}, w_{-i}), we rewrite P_i's profit as a function of d_i and s_i:

$$\begin{aligned}\pi_i^e(d_i,s_i|p_{-i},w_{-i}) = &\Big\{q_i + a_i - \log\left(\frac{d_i/\Lambda}{1-d_i/\Lambda}\right) - \log[1+\exp(a_{-i}+w_{-i})] - \log[1+\exp(q_{-i}-p_{-i})] \\ &- \log\left(\frac{s_i}{1-s_i}\right) - \kappa(s_i - d_i)\Big\}d_i.\end{aligned}$$

Hence, given (p_{-i}, w_{-i}), P_i's best-response mapping can be characterized by the following optimization problem:

$$\begin{aligned}&\max_{(d_i,s_i)} \pi_i^e(d_i,s_i|p_{-i},w_{-i}) \\ &s.t.\ d_i < s_i\end{aligned}$$

Given P_i's demand, d_i, the best-response supply of P_i should be such that $s_i \in \arg\max_{s\in(d_i,1)}[\log\left(\frac{s}{1-s}\right) + \kappa(s - d_i)]$. As a result, we can reduce $\pi_i^e(d_i,s_i|p_{-i},w_{-i})$ to a single variable function: $\pi_i^e(d_i|p_{-i},w_{-i}) = \big\{q_i + a_i - \log\left(\frac{d_i/\Lambda}{1-d_i/\Lambda}\right) - \log[1+\exp(a_{-i}+w_{-i})] - \log[1+\exp(q_{-i}-p_{-i})] - h(d_i)\big\}d_i$, where $h(d_i) := \max_{s\in(d_i,1)}[\log\left(\frac{s}{1-s}\right) + \kappa(s-d_i)]$.

We denote $(p_i^e(p_{-i},w_{-i}), w_i^e(p_{-i},w_{-i}))$ P_i's best-response price and wage functions given (p_{-i},w_{-i}). Following the same argument as in Step II of the proof of Theorem 1, we can show that $(p_i^e(p_{-i},w_{-i}), w_i^e(p_{-i},w_{-i}))$ is continuously increasing in p_{-i} and w_{-i}. Therefore, an equilibrium $(p_1^{e*}, w_1^{e*}, p_2^{e*}, w_2^{e*})$ exists.

To show that the equilibrium is unique, we denote by $T_e(\cdot,\cdot,\cdot,\cdot)$ the best-response mapping of the model with endogenous waiting times, i.e.,

$$T_e(p_1,w_1,p_2,w_2) = \big(p_1^e(p_2,w_2), w_1^e(p_2,w_2), p_2^e(p_1,w_1), w_2^e(p_1,w_1)\big).$$

Using the same argument as in the proof of Lemma 2, we obtain that there exists a constant $C = \max\left\{\frac{\exp(q_i)}{1+\exp(q_i)}, \frac{\exp(a_i)}{1+\exp(a_i)} : i = 1,2\right\} \in (0,1)$, such that

$$||T_e^{(k)}(p_1,w_1,p_2,w_2) - T_e^{(k)}(p_1',w_1',p_2',w_2')||_1 \leq 2C^{(k)}||(p_1,w_1,p_2,w_2) - (p_1',w_1',p_2',w_2')||_1,$$

and thus, the k^*-fold best-response mapping, $T_e^{(k^*)}(\cdot,\cdot,\cdot,\cdot)$, is a contraction mapping, where $k^* > -\log(2)/\log(C)$. Consequently, using the same argument as in the proof of Lemma 2, the equilibrium is unique and can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 6. □