Dynamic Competition in Online Retailing: Implications of Customer-to-Customer Interactions

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Problem Definition: Online retailing has increasingly become socially embedded, with customers frequently engaging in customer-to-customer (C2C) interactions such as reviews, social media posts, and recommendations, which significantly shape their purchase decisions. In a competitive context, we distinguish two types of C2C interactions: *Within-brand* interactions expand the future market specifically for the originating retailer, whereas *cross-brand* interactions generate a shared market benefiting competing retailers. This paper aims to investigate how these different types of C2C interactions influence dynamic competition among online retailers. *Methodology/Results:* We formulate a two-period duopoly game between newsvendor-type online retailers with asymmetric costs. Stock-out based substitution occurs within periods, and C2C interactions regulate market evolution across periods. We derive equilibrium outcomes under various scenarios and obtain three major findings. First, under cross-brand C2C interactions, the higher-cost retailer tends to stock less initially compared to the within-brand scenario, strategically leveraging the lower-cost retailer's larger stockpile and benefiting from market expansion at reduced inventory cost, a form of free-riding behavior. Second, equilibrium profit comparisons across scenarios can yield non-intuitive results. Specifically, the lower-cost (advantaged) retailer might achieve higher profits under cross-brand interactions, where competition is softened by shared market growth. Conversely, the higher-cost (disadvantaged) retailer can benefit more under within-brand interactions, which intensify competition but allow it to retain control over its customer base. Third, when retailers have pricing flexibility, equilibrium prices exhibit contrasting trends. Under within-brand interactions, prices increase over time as firms leverage their exclusive market growth. In contrast, under cross-brand interactions, prices decline across periods as retailers aggressively compete to secure larger shares of the jointly expanded market. *Managerial Implications:* Our results provide valuable managerial insights: Online retailers must carefully identify the dominant form of C2C interactions in their markets, as these interactions shape distinct market diffusion processes and competitive outcomes. Moreover, effective inventory and pricing strategies must align with the prevailing type of social interactions to maximize performance in dynamic competitive environments.

1. Introduction

Ever since online retailing was born, it has been constantly redefined by business and technology innovations. In recent years, with the rapid growth of social media platforms (e.g., Facebook, YouTube, and X), online retailing has evolved into a more interactive and socially embedded mode, where consumer behavior is increasingly shaped by interpersonal influences and shared experiences. One notable phenomenon is the prevalence of customer-to-customer (C2C) interactions, such as product reviews, outfit posts, unboxing videos, and other user-generated content across social platforms. These interactions, both verbal and visual, can significantly impact consumer demand and product visibility across brands and categories.

A particularly illustrative example comes from the fashion sector. Consider two online retailers, such as Amazon Fashion and Shein, both selling seasonal clothing. In the fall, they run promotional campaigns that include influencer endorsements and live video shopping events. After customers receive their purchases, many begin to share styling tips and outfit hauls through TikTok and Instagram. These posts, often in informal and brand-agnostic formats, influence not just the visibility of one retailer, but also stimulate demand for the fashion category at large. When winter arrives, a new group of customers who were exposed to this content may enter the market, with some drawn to the brand they saw and others simply attracted to the category. This social contagion process expands the market for both retailers, though not always equally, depending on the nature of the interactions. Such behavior is now widespread across many sectors of online retailing and is especially amplified during short but intense selling periods, such as those surrounding e-commerce holidays like Singles Day and Cyber Monday.

To better understand this dynamic, we take a novel perspective and distinguish between two types of C2C interactions. When prospective customers are influenced specifically by the past buyers of a particular retailer, we refer to this as a *within-brand* C2C interaction. In this case, the market expansion is exclusive to that retailer. This typically occurs when the brand has a loyal customer base, strong identity, or collaborates with well-known influencers. On the other hand, when the influence is more diffuse and prospective buyers are swayed by non-brand-specific content, e.g., general excitement around certain product functionality, we refer to it as a *cross-brand* C2C interaction. This cross-brand spillover expands the common market shared by all retailers selling substitutable products. Both types of C2C interactions have been widely documented in the marketing literature (see, e.g., Libai et al. 2009, Krishnan et al. 2012, Peres and Van den Bulte 2014, Chae et al. 2017, Godes 2017), and in both cases, the social contagions among customers directly influence the size and composition of future demand, creating a dynamic market growth process.

Within-brand or cross-brand, C2C interactions create a dependence between current decisions and future demand for retailers, because current sales, which are determined by pricing and stocking decisions, affect the future market size and structure. Importantly, recognizing the nature of these interactions, retailers may respond differently in managing the trade-off between current profitability and future growth, potentially leading to contrasting equilibrium strategies and managerial insights. Therefore, the central objective of this paper is to understand how the two types of C2C interactions shape dynamic competition among online retailers. To that end, we develop a two-period duopoly model in which two newsvendor-type retailers with asymmetric costs compete in each period over price and inventory (via stock-out-based substitution), and the market evolves over time periods due to C2C interactions. As a key feature, our model explicitly incorporates both within-brand and cross-brand interactions and studies their respective implications on competitive dynamics. Then, we address the following research questions. Treating prices as exogenously given, how do the system parameters such as social contagion strength and competition intensity jointly determine the retailers' strategic stocking decisions under each type of C2C interactions? How does each retailer's profit change and compare under different types of C2C interactions? When prices are endogenously and dynamically decided, how do the retailers' pricing strategies affect earlier results?

To start, we examine the retailers' stocking strategies under exogenously given retail prices. We find that the low-cost retailer orders more aggressively when competition intensifies, as this expands its future market and improves profitability. The high-cost retailer, in contrast, exhibits more nuanced behavior: Under within-brand C2C interactions (Scenario S), it orders a large quantity regardless of competition; under cross-brand C2C interactions (Scenario N), it strategically reduces its order quantity as competition intensifies. This behavior reflects a form of operational free-riding. By ordering less, the high-cost retailer allows unmet demand to spill over to its low-cost rival, which in turn helps grow the shared market under Scenario N. Such a result highlights an important managerial insight: A high-cost firm may benefit from deliberately relying on its rival's sales to stimulate market growth, rather than investing in costly inventory itself.

Next, we compare the retailers' equilibrium profits under the two types of C2C interactions. While it may be intuitive to expect that the low-cost retailer performs better under the more competitive within-brand interactions, and the high-cost retailer prefers the collaborative nature of cross-brand interactions, our results show that this intuition does not always hold. Depending on key parameters such as market share split, cost asymmetry, social interaction strength, and competition intensity, each retailer's profit can be higher or lower in either scenario. For example, when the social interactions are strong, the change in the price difference between the two retailers across periods can serve as the main driver of the profit comparison. If the price difference stays the same, other parameters start to drive the comparison results: When the cost advantage and competition level are substantial, the low-cost retailer may actually earn more under Scenario N, because the joint market expansion outweighs the costs of intensified competition. Conversely, when the high-cost retailer is not at a significant disadvantage, it may benefit more from Scenario S, where it retains full control over the demand it generates. Since the two types of C2C interactions give rise to a blend of competitive and cooperative forces, the above results contribute to the understanding of the notion of coopetition (Brandenburger and Nalebuff 1996, Gnyawali and Madhavan 2001). From a strategic management standpoint, we offer key managerial insights into how firms can navigate the tension between rivalry and collaboration, and recognize when it is in their best interest to pursue a more balanced coopetitive relationship.

Finally, we extend the model to a two-period price-inventory game where both retailers endogenously choose their prices and order quantities. This richer framework reveals new strategic dynamics in how pricing interacts with the two types of C2C interactions. We find that under Scenario S, equilibrium prices increase over time; i.e., retailers initially lower prices to grow their own customer base, and later exploit this advantage. In contrast, under Scenario N, prices decline over time as firms compete for a greater share of the jointly expanded market. Notably, the downward pricing trend under Scenario N reflects a strategic effort to capture a larger share of the common market, particularly in the second period when the value created by demand expansion in the first period is realized. These pricing dynamics, together with our earlier findings, suggest that the strategic use of price as a competitive lever depends critically on the structure of customers' social interactions. Finally, our numerical analysis shows that the qualitative comparisons of profits across scenarios largely persist even with pricing flexibility, provided that the strength of social interactions is not excessively large. From a marketing perspective, our findings suggest that firms operating in socially connected environments must account for how peer influence propagates across brands versus within brands, and adapt pricing strategies accordingly.

The rest of this paper is organized as follows. We position this paper in the related literature in Section 2. Section 3 describes the essential elements of the model framework. We show our main results in Section 4. In Section 5, we extend our model to allow pricing flexibility. Lastly, Section 6 concludes this paper. All proofs are relegated to the Appendix.

2. Literature Review

Caro et al. (2020) have recently provided a comprehensive research survey on retail operations and discussed its new trends; our work belongs to this vast literature at large. As retailing continues to be reshaped by technological advances and novel business models, a growing body of research has begun to examine the impact of customer engagement, influencer marketing, and social media-driven behaviors on firm performance. For example, recent studies on live-streaming e-commerce, a popular format in the retail industry, have investigated important aspects such as customer trust and engagement, influencer roles, and platforms mechanism design, and their impacts on various stakeholders; see, e.g., Wongkitrungrueng and Assarut (2020), Qi et al. (2022), Hou et al. (2022), and Chen et al. (2020). Although our model is not limited to live-streaming e-commerce, this setting serves as a vivid example of our broader online retailing framework, where large-scale, real-time C2C interactions are highly prevalent. In this sense, our work complements the growing literature on socially embedded retail models by identifying two types of social interactions and analyzing their impact on dynamic competition.

A salient feature of our paper is the explicit consideration of C2C interactions, which has been an important topic in marketing literature. The distinction between within-brand and cross-brand C2C interactions in our model draws directly on the taxonomy proposed by Libai et al. (2009), whose pioneering work provides a theoretical foundation for analyzing social contagion in competitive settings. Since then, a growing stream of research has extended this framework to study various managerial issues under different modeling approaches and contexts (see, e.g., Krishnan et al. 2012, Peres and Van den Bulte 2014, Chae et al. 2017, Godes 2017, Sanchez et al. 2020, Geng et al. 2022, Yan and Hu 2023). In our setting, the within-brand C2C interaction resembles traditional market diffusion where a firm's past sales expand its own market. By contrast, the cross-brand C2C interaction highlights spillovers and shared growth: When multiple firms' customers jointly expand the common market, firms may benefit from one another's activities without coordination. This gives rise to potential free-riding behavior, where a firm with limited market-generating ability may still enjoy the shared expansion driven by its rival. Such interactions are especially likely when social influence takes the form of brand-agnostic word-of-mouth, as seen in customer discussions, short posts, or user-generated content that focuses more on product functionality than on brand identity (e.g., Godes and Mayzlin 2004, Krishnan et al. 2012). Related studies include Hu et al. (2020), who examine innovation spillovers when an innovator outsources production to a contract manufacturer that may also be a downstream competitor; Peres and Van den Bulte (2014) demonstrate how word-of-mouth spillovers from rival firms can make a reseller worse off when product exclusivity is enforced; Haviv et al. (2020) empirically quantify intertemporal spillover effects between competing sellers on console video game platforms. Our paper contributes to this stream by incorporating these interaction mechanisms into a dynamic pricing-inventory game, where competitive outcomes are endogenously determined.

Lastly, the analytical model in our paper is based on the competitive newsvendors framework, which has long been studied in the inventory literature; see, e.g., Parlar (1988), Lippman and McCardle (1997), and Netessine and Rudi (2003). The inventory competition due to stock-out based substitution has been used to study firms' strategic stocking decisions in various settings, some of which share similar flavor to our work. For instances, we assume the competing retailers have asymmetric costs; Jiang et al. (2011) and Güler et al. (2018) focus on asymmetric information regarding demand and cost, respectively. Our work is motivated by online retailing; Straubert and Sucky (2023) study a problem from online marketplaces. Our results yield a discussion on the competition-collaboration relationship between the retailers; Dong et al. (2023) develop a coopetitive newsvendor model in a supply chain setting. Furthermore, a rich body of research has explored price-setting newsvendor problems in both monopolistic and competitive settings (see, e.g., Petruzzi and Dada 1999, Zhao and Atkins 2008, Salinger and Ampudia 2011), which provides useful methodological tools for analyzing our endogenous pricing model. Departing from all the above prior works with the single-period assumption, we investigate a multi-period newsvendor competition. In this regard, our paper is more closely related to dynamic inventory competition model (see, e.g., Liu et al. 2007, Hall and Porteus 2000, Nagarajan and Rajagopalan 2008, 2009, Olsen and Parker 2008, 2014). Aside from having different scopes and research agendas, our paper differs from these prior works in an important way: In their settings, the demand evolution is shaped by product availability or substitution, whereas in ours, the market growth is driven by C2C interactions. Thus, we contribute to the literature on competing newsvendors by incorporating an important marketing consideration into a dynamic setting. Being closely tied to inventory decisions on the one hand and regulating market diffusion on the other, C2C interactions play a pivotal role in online retailing that merits careful consideration.

3. General Model Framework

We study a two-period duopoly competition where the market dynamic is regulated by the C2C interactions. Below, we describe the key model elements and formulate the problem.

3.1. Firms

Consider two online retailers, A and B, selling substitutable perishable products in two consecutive sales seasons (periods). For example, Amazon Fashion and Shein would both host live shopping events when they launch their fall collections (e.g., plaid jackets and Halloween-themed apparel) and winter collections (e.g., fur jackets and holiday-themed clothes), respectively. In our general model setup, the two retailers compete in price and inventory in each period, facing a random demand and possible stock-out based substitution. Specifically, in period t, retailer i $(i = A, B)$ sets the price $p_{i,t}$ and then decides the stocking level $y_{i,t}$; both the prices and the inventories will influence the demand allocation and substitution. The two retailers are assumed to be asymmetric in unit ordering cost. Without loss of generality, let retailer A's unit ordering cost c_A be smaller than retailer B's cost c_B.

Given both firms' prices and stocking levels in period t, retailer i receives a random initial demand allocation $M_{i,t}$ (to be detailed in a moment). Any leftovers after each period ends are salvaged through other non-profitable channels; we normalize the salvage value to zero. When a shortage occurs at one retailer, part of its excess demand will spillover to the other retailer. As such, the effective demand for retailer i is given by

$$D_{i,t} = M_{i,t} + \theta(M_{j,t} - y_{j,t})^+, \quad i, j = A, B; i \neq j.$$

Here, parameter $\theta \in [0, 1]$ measures the degree of substitution due to stock-out. Larger θ means higher substitution rate, which implies more intensified competition between the retailers. Finally, in period t, retailer i's realized sales $R_{i,t} = \min\{D_{i,t}, y_{i,t}\}$ and its profit $\pi_{i,t} = p_{i,t}R_{i,t} - c_i y_{i,t}$.

3.2. Demand and Market Growth

Next, we detail the assumptions about the demand process and how the market evolves across periods. First, in every period t, there is a seed demand X_t that characterizes the common market for both retailers. The seed demand $\{X_t\}$ form an i.i.d. sequence of random variables, with distribution $F(\cdot)$ and density $f(\cdot)$. Back to the Amazon Fashion vs Shein example, the seed demand comes from the consumers with consistent behavioral patterns of the seasonal fashion purchases out of necessity and/or social norms, such as updating cold-weather or replacing holiday-themed clothing appropriate to each season. Moreover, since the product categories between fall and winter are also distinct, demand substitution across seasons is unlikely. Hence, both the unsold products and the unmet demand are perishable in our setting.

Second, the impact of C2C interactions from the first period drives an additive market growth component on top of the seed demand. That is, customers attracted via social interactions will become part of the demand in the second period. This market diffusion process is dictated by the C2C interactions commonly observed in the online retailing industry. For example, both Amazon Fashion and Shein would host livestreaming sales and utilize influencers and user-generated content (e.g., try-on hauls); moreover, word-of-mouth and social sharing from satisfied customers permeate the social media platforms. As a result, there will be increased brand/category visibility and potential new customers. In our model, we assume that the new market induced by the C2C interactions is based on the realized sales (i.e., satisfied customers) of the retailers.

Therefore, for retailer i in period t, its initial allocation of the demand, $M_{i,t}$, consists of two parts:

$$M_{i,t} = \alpha_{i,t} X_t + Z_{i,t}. \tag{1}$$

The first part represents a split (initial allocation) of the seed demand, where $\alpha_{i,t}$ is the split ratio satisfying $\alpha_{A,t} + \alpha_{B,t} = 1$. The initial allocation is modeled similarly to the deterministic splitting rule from Lippman and McCardle (1997), but our split ratio is price-dependent. Specifically, since the posted price is the foremost influential factor to customers' purchase decision, we assume that

$$\alpha_{i,t} = \alpha_{i,t}(p_{i,t}, p_{j,t}) = \frac{1}{2} + \beta(p_{j,t} - p_{i,t}), \quad i, j = A, B; i \neq j.$$

Here, parameter $\beta > 0$ captures the impact of price differentiation on the consumer attraction. For simplicity, we will just write $\alpha_{i,t}$ in the following discussions, but its dependence on the retailers' prices should be kept in mind at all times.

The second part of retailer i's initial allocation, $Z_{i,t}$, represents retailer i's demand induced by the C2C interactions. In general, the induced demand is a non-decreasing function of firms' sales in the previous period, i.e., $Z_{i,t} = z_i(R_{A,t-1}, R_{B,t-1})$. Hence, more past sales tend to generate more potential demand. In the duopoly online retailing context, we identify two types of C2C interactions, which give rise to different functional forms of $Z_{i,t}$, as delineated next.

3.3. Two Types of C2C Interactions

At the center of our model is the market growth process for competing retailers driven by the C2C interactions, which include word-of-mouth communications, social media posts/reviews/recommendations, and other verbal or non-verbal interactions. Indeed, the interpersonal communications among customers (i.e., social contagions) have long been recognized by the marketing field as having substantial impacts on business performance, including market growth and profitability increase (see, e.g., Godes 2017, Krishnan et al. 2012). However, one notable aspect of C2C interactions should be particularly highlighted when multiple firms are competing for one market. Since not all interpersonal communications are brand specific, the C2C interactions can transpire at either the *brand level* or the *category level*. As a consequence, prospective buyers may interact with the satisfied customers of one brand but eventually purchase from the competing brand.

Therefore, we distinguish two types of C2C interactions (*within-brand* and *cross-brand*) in this paper, which exert fundamentally different influences on the market growth dynamics: Within-brand influence comes in effect when prospective buyers obtain brand specific information from that brand's satisfied customers; cross-brand influence, on the other hand, is in effect when brand non-specific information is communicated from any satisfied customers to prospective ones. The above taxonomy of C2C interactions is built directly upon the theoretical foundation established by some important prior works in marketing literature (e.g., Libai et al. 2009, Krishnan et al. 2012, Peres and Van den Bulte 2014). Moreover, our study on the impact of different types of C2C interactions is also in line with many research works tackling different issues in similar settings (see, e.g., Chae et al. 2017, Godes 2017, Sanchez et al. 2020, Geng et al. 2022, Yan and Hu 2023).

From a practical perspective, the type of C2C interaction that dominates in a given market can be influenced, at least partially, by firms' marketing strategies on social media platforms. For example, a brand-focused campaign may foster within-brand interactions, while category level, product-feature-based content may give rise to cross-brand interactions. Our goal, therefore, is to provide strategic guidance to firms on how different types of C2C interactions influence market evolution and competitive outcomes. By understanding which type of interaction is more favorable under what conditions, retailers can better tailor their campaigns and platform engagement strategies.

3.3.1. Within-Brand C2C Interactions. First, the interpersonal communications between satisfied customers and prospective buyers may promote a specific brand. Practically, this happens when existing customers of one brand share their experiences and recommend that brand on their social media platforms, influencing others to purchase from the same brand. For example, after shopping at Shein for fall outfits, satisfied customers may post "fall fashion hauls" clips on TikTok or YouTube Shorts showcasing their sweaters and coats and recommending Shein. Then, followers and viewers, many of whom did not buy in fall, are influenced to try the brand in winter, because they now trust Shein's quality, fit, or trendiness. Hence, satisfied customers become micro-influencers, growing winter demand for Shein via social contagions.

We use "Scenario S" to refer to the case of within-brand C2C interactions, which is brand specific, and give the notations a superscript "S" whenever necessary. In Scenario S, past sales of retailer i ($i = A, B$) induce new customers exclusively for retailer i. Thus, the demand induced by social interactions in period t for retailer i can be written as $Z^S_{i,t} = \gamma R^S_{i,t-1}$, a linear function of its previous sales. The parameter $\gamma > 0$ represents the strength of the C2C interactions (e.g., it could measure how contagious the social interactions are). Finally, the market dynamic equation (1) can be further written as

$$M^S_{i,t} = \alpha_{i,t} X_t + \gamma R^S_{i,t-1}. \tag{2}$$

One direct observation from (2) is that, each retailer's initial allocation only depends on the seed demand and its *own* sales from the previous period.

3.3.2. Cross-Brand C2C Interactions. Second, the C2C interactions may be non-specific in brand (we refer to this case as Scenario N). Here, all the satisfied customers' experiences shape the prospective buyers' purchase propensity equally for the two retailers. In practice, cross-brand C2C interactions may be observed when existing customers of both brands promote the product or the trend it has created, generating general excitement that benefits the competing brands by increasing the common market size. For example, after fall purchases, both Amazon Fashion and Shein customers participate in a viral TikTok trend "How I Style My Fall Looks," tagging hashtags such as "#HolidayLooks" This collective content would create buzz around the seasonal style category. Going viral on social media platforms, the shared trend elevates consumers' interest in winter fashion as a whole, encouraging new consumers to shop for winter clothes. Since the trend contains both Amazon Fashion and Shein products, the two retailers both benefit from the expanding winter demand, even if not all customers would buy from the same brand.

We model the generated demand in this case as a linear function of the retailers' *total* past sales, denoted by $R^N_{t-1} = R^N_{A,t-1} + R^N_{B,t-1}$ (similar to before, superscript "N" indicates Scenario N). Then, γR^N_{t-1} is the social interactions induced market that is to be shared by both firms; the parameter γ again denotes the strength of the social contagions. We further assume that customers attracted by social interactions will choose from the two retailers according to the same splitting rule of the seed demand; then, retailer i initially receives $Z^N_{i,t} = \alpha_{i,t}\gamma R^N_{t-1}$ as its allocated induced demand. Therefore, we can write the initial allocation equation in this scenario as

$$M^N_{i,t} = \alpha_{i,t}\left(X_t + \gamma R^N_{t-1}\right). \tag{3}$$

Finally, it is worth mentioning that, unlike Scenario S, retailer i's initial allocation in (3) depends on not only its own but also its rival's previous sales.

3.4. Problem Formulation

The aforementioned key components of our model yield a dynamic competitive newsvendor problem facing the two online retailers: They decide retail prices and stock up to compete for demand from a common market in each period, and the market growth across periods is regulated by the C2C interactions. We formulate this problem as a two-period two-person Nash game with complete information. Since the impact of the C2C interactions, which is our primary research focus, can fully manifest as soon as the period moves forward once, concentrating on a two-period model suffices to serve the purpose. Besides, the two-period model admits tractable analysis, which allows derivation of clean and useful managerial insights.

The sequence of events in each period t is as follows. (1) Both retailers observe the previous sales of each firm, $(R_{A,t-1}, R_{B,t-1})$, and thus obtain the distribution of the initial allocation $M_{i,t}$; if $t = 1$, we set $R_{i,0} = 0$. (2) The retailers first decide the retail prices $p_{i,t}$ and then the order quantity $y_{i,t}$ based on the observed previous sales; each decision is made simultaneously by the two retailers. (3) The seed demand X_t is realized, and so is the initial allocation. For each retailer, if there is an excess demand, a fixed proportion (θ) of the unsatisfied customers will attempt to purchase from the other retailer; otherwise, any unused products are salvaged with zero value. (4) The sales of each retailer, $R_{i,t}$, is realized and the profit is collected. The objective of each retailer is to maximize its total (discounted) profit of all periods. As such, retailer i solves the following problem:

$$(\mathcal{P}) \qquad \max_{p_{i,1},y_{i,1};p_{i,2},y_{i,2}} \mathbb{E}_{X_1,X_2}\left[(p_{i,1}R_{i,1} - c_i y_{i,1}) + \rho\left(p_{i,2}R_{i,2} - c_i y_{i,2}\right)\right],$$

subject to

$$R_{i,t} = \min\{D_{i,t}, y_{i,t}\}, D_{i,t} = M_{i,t} + \theta(M_{j,t} - y_{j,t})^+, \text{ and } M_{i,t} = \alpha_{i,t}X_t + Z_{i,t}; \quad t = 1,2.$$

The parameter $0 < \rho \leq 1$ is the discount factor. The equilibrium concept adopted is feedback Nash equilibrium. We treat the sales $R_{i,t}$ as state variables in each period, and solve for the prices $p_{i,t}$ and the order quantities $y_{i,t}$ via backward induction.

This two-period model highlights the complexities online retailers face in pricing and inventory management when different types of C2C interactions have significant influences on demand. However, such a price-inventory competition in multiple periods is analytically difficult to solve. Hence, in this paper, we consider two models. First, we assume that the prices in both periods are fixed and study the Exogenous Price Model in Section 4. Then, in Section 5, we investigate the retailers' pricing decisions in the Endogenous Price Model and derive relevant insights into the impact of pricing flexibility. Lastly, we will set the discount factor $\rho = 1$ and assume uniform distribution for the seed demand for the rest of the paper; that is, let $X_t \sim U[\mu - \sigma, \mu + \sigma]$ for $t = 1,2$. Note that we have obtained the equilibrium of the Exogenous Price Model with any $\rho \in (0,1]$ and general seed demand distribution (see details in proof). Moreover, we have numerically tested that all our results remain qualitatively the same when other distribution is used.

4. Exogenous Price Model

In this section, we assume that retail prices in each period, $p_{i,t}$ ($i = A, B; t = 1,2$), are predetermined and fixed. Such an exogenous price model may be suitable considering the practical constraints in the digital selling environment. Livestreaming campaigns, promotional commitments, platform policies, and the seasonal nature of fashion and holiday-themed apparel may discourage short-term price adjustments. Moreover, competitive environments often lead to price convergence at the start of the season, leaving inventory as the primary decision variable for retailers. In the following, we scrutinize the two-period inventory competition between the two retailers; i.e., in period t, after the retailers observe the previous sales volumes, they only decide their stocking levels $y_{i,t}$. We first derive the equilibrium of the competition in different scenarios and then present our main findings regarding the equilibrium comparisons between scenarios. To preserve the structural properties of the objective functions, we make a mild technical assumption[1] that $p_{B,t} \geq p_{A,t} \geq \frac{c_A}{c_B} p_{B,t}$ for $t = 1,2$. Since $c_A < c_B$, it is intuitive that retailer A can set a lower price due to its cost efficiency; our assumption simply requires that the price advantage is not too large.

4.1. Equilibrium Analysis

4.1.1. Benchmark Scenario: No C2C Interactions. In the benchmark scenario, the market growth across the two periods disappears, and the model is reduced to repeated static games. We directly apply the results from Lippman and McCardle (1997) here.

Lemma 1. *Suppose there is no C2C interaction. A unique Nash equilibrium exists:* $y^0_{A,t} = (\alpha_{A,t} + \theta\alpha_{B,t})\zeta^0_{A,t} - \theta\alpha_{B,t}\zeta^0_{B,t}$ *and* $y^0_{B,t} = \alpha_{B,t}\zeta^0_{B,t}$*, where* $\zeta^0_{i,t} = F^{-1}(1 - \frac{c_i}{p_{i,t}})$ $(i = A, B; t = 1, 2)$.

Note that $\zeta^0_{i,t}$ is exactly the classic newsvendor's critical z-score for retailer i, and $\zeta^0_{A,t} > \zeta^0_{B,t}$ because $c_A < c_B$. Recall that $F(\cdot)$ is the cdf of the seed demand distribution; hence, for uniform distribution $X_t \sim U[\mu - \sigma, \mu + \sigma]$, $F^{-1}(x) = 2\sigma \cdot x + (\mu - \sigma)$. From Lemma 1, we have $y^0_{A,t} > y^0_{B,t}$ for all θ (see Lemma 4). Thus, the more cost-efficient retailer A always maintains a higher stocking level; retailer B, on the other hand, is in a disadvantageous position in the duopoly relationship due to the cost asymmetry.

4.1.2. Scenario S. Next, we include within-brand C2C interactions in our analysis. This type of interactions among customers is brand specific, and the market dynamic equation is given by (2). In this scenario, it is each retailer's individual past sales that determines the expansion of its own market, so they take advantage of the C2C interactions independently.

Lemma 2. *Suppose the C2C interactions are within-brand. A unique Nash equilibrium exists: In the first period,* $y^S_{A,1} = (\alpha_{A,1} + \theta\alpha_{B,1})\zeta^S_{A,1} - \theta\alpha_{B,1}\zeta^S_{B,1}$ *and* $y^S_{B,1} = \alpha_{B,1}\zeta^S_{B,1}$*, where* $\zeta^S_{i,1} = F^{-1}\left(1 - \frac{c_i}{p_{i,1} + \gamma(p_{i,2} - c_i)}\right)$*; in the second period, given the two retailers' sales,* $R^S_{A,1}$ *and* $R^S_{B,1}$*,* $y^S_{A,2} = y^0_{A,2} + \gamma R^S_{A,1}$ *and* $y^S_{B,2} = y^0_{B,2} + \gamma R^S_{B,1}$*. Moreover, the retailers' equilibrium sales and profits are specified in appendix.*

In Scenario S, the retailers' first-period order quantity has the same form as in benchmark, but with different critical z-scores. Moreover, we have $\zeta^S_{A,1} > \zeta^S_{B,1}$ due to the cost asymmetry $c_A < c_B$;

[1] This assumption is made only to obtain structured results in this section; the model can still be solved without it (see appendix). Moreover, we will drop the assumption in the next section for the Endogenous Price Model.

as a result, $y^S_{A,1} > y^S_{B,1}$. For the second-period order quantity $y^S_{i,2}$, it must first satisfy the induced demand $\gamma R^S_{i,1}$ (contingent on the previous sales), and the rest is the same as the benchmark scenario, due to the end-of-horizon effect. Furthermore, note that the within-brand C2C interactions have effectively risen the underage cost for the retailers in the first period, because stock-out means not only losing current customers, but also potential future customers, to the rival. Hence, retailers would order more in the first period compared to the benchmark case (see details in proof).

4.1.3. Scenario N. Now, we turn to the case where customers engage in cross-brand interactions, which are non-specific in brand names. In Scenario N, the market expansion characterized by equation (3) is based on the joint sales of both retailers, and thus the induced new demand is pooled into the seed demand and shared by the two firms. In this regard, the retailers enjoy the benefit of the cross-brand C2C interactions in a collaborative manner.

LEMMA 3. *Suppose the C2C interactions are cross-brand. A unique Nash equilibrium exists: In the first period,* $y^N_{A,1} = (\alpha_{A,1} + \theta\alpha_{B,1})\zeta^N_{A,1} - \theta\alpha_{B,1}\zeta^N_{B,1}$ *and* $y^N_{B,1} = \alpha_{B,1}\zeta^N_{B,1}$, *where*

$$\zeta^N_{A,1} = F^{-1}\left(1 - \frac{c_A}{p_{A,1} + \alpha_{A,2}\gamma(p_{A,2} - c_A)}\right) \text{ and } \zeta^N_{B,1} = F^{-1}\left(1 - \frac{c_B - \alpha_{B,2}\gamma\theta(p_{B,2} - c_B)\frac{c_A}{p_{A,1}+\alpha_{A,2}\gamma(p_{A,2}-c_A)}}{p_{B,1} + \alpha_{B,2}\gamma(1-\theta)(p_{B,2} - c_B)}\right);$$

in the second period, given the two retailers' total sales $R^N_1 = R^N_{A,1} + R^N_{B,1}$, $y^N_{A,2} = y^0_{A,2} + \alpha_{A,2}\gamma R^N_1$ *and* $y^N_{B,2} = y^0_{B,2} + \alpha_{B,2}\gamma R^N_1$. *Moreover, the retailers' equilibrium sales and profits are specified in appendix.*

Since retailers' first-period sales affect each other's second-period customers acquisition, their order quantities are intertwined, as shown in Lemma 3. Compared to Scenario S (Lemma 2), we highlight a couple of interesting differences here. First, for retailer A, we have $\zeta^N_{A,1} < \zeta^S_{A,1}$ (see details in proof). Thus, although the cross-brand C2C interactions also tend to increase the underage cost for retailer A, the increase is not as much as the case of within-brand C2C interactions. After all, the increased sales expand the common, not individual, market. Second, for retailer B, its order quantity becomes quite involved. Particularly, retailer B's objective function contains a part proportional to retailer A's first-period sales, which is *decreasing* in retailer B's order quantity (see details in proof). Indeed, when retailer B orders more, there is less demand spillover to retailer A, negatively affecting retailer A's sales and the market expansion, which essentially increases retailer B's overage cost. Therefore, since the rival's sales is helpful to its profit, retailer B may in a sense "free-ride" retailer A's order quantity — it saves on ordering cost and still enjoys the benefit of an enlarged market due to cross-brand C2C interactions.

4.2. Comparative Analysis and Main Results

Given the equilibrium outcomes obtained above, we now present the main results from the comparative analysis of different scenarios. While the benchmark scenario provides a basic characterization of the duopoly game without C2C interactions, we will mainly examine and compare the equilibrium outcomes in the presence of the within-brand and cross-brand C2C interactions, respectively.[2] Although C2C interactions in general will benefit both retailers, the two types studied in our paper have notable differences: Within-brand C2C interactions are competitive in nature, whereas cross-brand C2C interactions have a cooperative feature. Therefore, it is interesting to investigate in each scenario whether the duopoly competition is intensified or softened and, more importantly, whether each retailer can benefit more or less from the expanded market generated by the C2C interactions. Moreover, our comparative results are intended to serve as strategic guidelines for online retailers operating in socially embedded markets. The distinction between within-brand and cross-brand C2C interactions is not merely a theoretical construct; rather, it has practical implications for how firms design their marketing strategies. In this sense, our results can help firms better understand how their market performance is shaped by the underlying form of social influence within their customer base.

In the following, we concentrate on the retailers' first-period equilibrium stock levels, sales volumes, and profits to highlight the impact of the C2C interactions across periods. The second period is not our focus because it could be characterized by the end-of-horizon effect. Three parameters are of particular interests, namely, the social contagion strength γ (measuring the magnitude of market growth from C2C interactions), the stock-out based substitution factor θ (measuring the degree of competition), and the cost difference $\Delta c := c_B - c_A$ (measuring the asymmetry level of the retailers). Our results and discussions will anchor on the three parameters.

4.2.1. Order Quantities. We start by examining the retailers' equilibrium order quantities from two aspects. Between the two firms, we compare their stocking levels in different scenarios. Across scenarios, we compare each retailer's order quantity, which showcases the impacts of the within-brand and the cross-brand C2C interactions.

Lemma 4. (*a*) *In the benchmark scenario, we have* $y^0_{A,t}/\alpha_{A,t} \geq y^0_{B,t}/\alpha_{B,t}, t = 1, 2$.

(*b*) *In Scenario k* ($k = S, N$), *we have* $y^k_{A,1}/\alpha_{A,1} \geq y^k_{B,1}/\alpha_{B,1}$ *in the first period and* $(y^k_{A,2} - Z^k_{A,2})/\alpha_{A,2} \geq (y^k_{B,2} - Z^k_{B,2})/\alpha_{B,2}$ *in the second.*

[2] In Appendix B, we study the equilibrium for the Exogenous Price Model with both types of interactions in presence.

Note that, since $p_{A,t} \leq p_{B,t}$, by the definition of the split ratio, we have $\alpha_{A,t} \geq \alpha_{B,t}$. Hence, a direct corollary of Lemma 4 is that the two retailers' equilibrium stocking levels in different scenarios can be compared: $y^0_{A,t} \geq y^0_{B,t}$, $y^k_{A,1} \geq y^k_{B,1}$, and $y^k_{A,2} - Z^k_{A,2} \geq y^k_{B,2} - Z^k_{B,2}$ $(k = S, N)$. In other words, retailer A always has more supplies than retailer B in all scenarios. Indeed, the order quantities depend on the critical z-scores, which further depend on the system parameters, especially on the order cost c_i. The retailer with less ordering cost tends to order more, hence the dominance of retailer A's stock level. This feature is unaltered even in the presence of C2C interactions: According to Lemma 4(b), retailer A orders more in the first period, and in the second period, it has more leftover supplies after satisfying the demand induced by social interactions.

In Lemma 4, what is really compared is each retailer's "initial" fill rate — how much seed demand can their inventories satisfy before substitution occurs. From this perspective, we have one interesting implication of Lemma 4: In the first period of all scenarios, any demand spillover can only occur from retailer B to retailer A; similarly, in the second period for Scenarios S and N, after filling the demand induced by social interactions, any stock-out based substitution should only be possible that part of retailer B's customers turn to seek purchase from retailer A. Thus, at the equilibrium in any situation, we can only observe such one-sided substitution. This observation can help us better understand the incentives of each retailer's ordering behavior. Retailer A is likely to increase its order, compared to the non-competitive case, in order to accommodate the excess demand from retailer B. Being more cost efficient, retailer A is in the advantageous position and tends to benefit from the demand spillover. For retailer B, on the other hand, the higher ordering cost is to its disadvantage, and it may decrease the stocking level. However, as we will see later, the impact of the one-sided substitution is not necessarily all negative on retailer B when the C2C interactions are brought into the equation.

Next, we study the equilibrium first-period order quantity for each retailer, highlighting the impact of market growth induced by the C2C interactions. We start with retailer A.

Proposition 1. *Retailer A's equilibrium first-period order quantity in different scenarios, namely y^0_A, $y^S_{A,1}$, and $y^N_{A,1}$, are all increasing in θ. Moreover, when γ is large, there exists a $\underline{\theta}^{SN} \in [0,1]$ such that $y^S_{A,1} \geq y^N_{A,1}$ for $\theta \in [0, \underline{\theta}^{SN}]$ and $y^S_{A,1} \leq y^N_{A,1}$ for $\theta \in [\underline{\theta}^{SN}, 1]$.*

Proposition 1 shows that retailer A's order quantity increases in the substitution rate θ in all three scenarios. Since the stock-out based substitution is always from retailer B to retailer A, as θ increases, retailer A will face more potential demand spillover, leading to a larger order quantity in equilibrium. Interestingly, however, the functional dependence on θ is not the same in different scenarios. As shown in the proof (see appendix for details), y_A^0 and $y_{A,1}^S$ is linear, whereas $y_{A,1}^N$ is convex in θ. Therefore, compared to the benchmark scenario, the within-brand C2C interactions do not change the underlying logic of the duopoly competition. After all, in Scenario S, the substitution independently affects each retailer's individual market expansion, and therefore the benefit of the demand spillover is proportionate to the competition intensity. However, when the cross-brand C2C interactions are present, the demand spillover in the first period has multiple implications. It increases retailer A's current demand on the one hand and expands both retailers' future market on the other. The later sends incentives to retailer B to strategically spill demand over to the rival, which causes retailer A's equilibrium first-period order quantity $y_{A,1}^N$ to have a more involved dependence relationship with the competition intensity θ.

Furthermore, Scenarios S and N are also compared in Proposition 1 regarding retailer A's first-period order quantity, which we find may cross each other depending on the system parameters. Basically, under the condition that the C2C interactions are strong (large γ), retailer A orders more in Scenario S when the substitution rate is small, but it orders more in Scenario N otherwise. This result can be explained as follows. Compared to Scenario S, when the C2C interactions are cross-brand and strong, retailer A's first-period sales are used to expand the total market, only part of which contributes to its own future market. Hence, retailer A faces a smaller underage cost in Scenario N than in Scenario S, especially when there is only limited demand spillover. As a result, $y_{A,1}^N \le y_{A,1}^S$ when θ is small. On the other hand, when θ is large, the overage cost for retailer A also becomes small. Besides, seeing a large substitution rate and the presence of strong cross-brand C2C interactions, retailer B may strategically send its demand to retailer A, hoping the total future market gets enlarged (as shown later in Proposition 2). This will further incentivize retailer A to overstock. Note that retailer B will not do so in Scenario S, because each retailer is responsible for its own market expansion. Therefore, with a large demand spillover rate, we have $y_{A,1}^N \ge y_{A,1}^S$.

Now, we turn to study retailer B's first-period ordering decision in each scenario. The comparison results are given in the next proposition.

Proposition 2. *Consider retailer B's equilibrium first-period order quantity in different scenarios. The following statements hold.*

(*a*) *$y_{B,1}^N$ is decreasing in θ, whereas $y_{B,1}^0$ and $y_{B,1}^S$ are independent of θ.*

(*b*) $y_{B,1}^0 < y_{B,1}^N < y_{B,1}^S$.

(*c*) *As a function of γ, $y_{B,1}^S - y_{B,1}^N$ is first increasing then decreasing in γ.*

First, Proposition 2(a) shows that how retailer B's order quantity depends on θ is in contrast to the counterpart results concerning retailer A. Specifically, only in Scenario N does the substitution rate affect retailer B's order quantity, and the correlation is negative; in other scenarios, however, the substitution rate is irrelevant to retailer B's order decision. In fact, since retailer B will never receive demand spillover, it is not concerned with the possible stock-out based substitution unless the substitution can affect its future demand. In Scenario N, the demand spillover does have an indirect impact on retailer B's second-period market size. By spilling the excess demand over to retailer A, the shared market growth induced by the cross-brand C2C interactions may benefit retailer B. Hence, the larger the demand spillover is, the more incentive retailer B has to order less — so it can enjoy the expanded future market with a smaller cost. This explains why $y^N_{B,1}$ is decreasing in θ.

Second, regardless of the system parameters, we always have the same ordered relationship of retailer B's order quantities across scenarios, as given in Proposition 2(b). To understand the comparison $y^0_{B,1} < y^N_{B,1}$ and $y^0_{B,1} < y^S_{B,1}$, it is intuitive that, within-brand or cross-brand, the C2C interactions always increase the underage cost of retailer B, and thus it tends to increase the order quantity in the first period compared to the benchmark. For the comparison $y^N_{B,1} < y^S_{B,1}$, i.e., retailer B always orders more in Scenario S than in Scenario N, the underlying reason is twofold. On the one hand, retailer B faces a larger underage cost under the within-brand, rather than the cross-brand, C2C interactions. On the other hand, as discussed previously, the cross-brand C2C interactions depress retailer B's order incentive: It will limit its order quantity to utilize the (total) market expansion at a lower cost. Therefore, under the above two driving forces, retailer B's first-period order quantity in Scenario N is dominated by that in Scenario S.

In summary, although both types of C2C interactions increase retailer B's order quantity, the increase is smaller when the C2C interactions are cross-brand. Indeed, when the future market expansion is from both retailers' past sales and is going to be shared, the less cost-efficient firm may want to take advantage of such collaborative feature of the C2C interactions and strategically shift excess demand to the rival who is more capable of stocking up to meet the demand. In this sense, retailer B's behavior of limiting its order quantity to send customers to retailer A can be seen as "free-riding" the rival's more cost-efficient stockpiling. The difference $y^S_{B,1} - y^N_{B,1}$ reflects and measures such a free-riding behavior. Proposition 2(c) characterizes how the order difference depends on the strength of the social contagions: $y^S_{B,1} - y^N_{B,1}$ is first increasing and then decreasing in γ. When the C2C interactions are relatively weak, any incremental growth in the strength enlarges retailer B's free-riding behavior; that is, retailer B's order increases less when the C2C interactions are cross-brand than when they are within-brand. However, when the C2C interactions is already strong, as it gets stronger, retailer B's order will increase in similar magnitudes in both Scenarios S and N, rendering the free-riding behavior diminishing in scale.

4.2.2. Sales. After investigating the retailers' equilibrium order quantities, we now examine their first-period expected sales in different scenarios. The reason why we focus on the sales in the first period is threefold. First, in our two-period dynamic model, the first-period sales are the only effective state variables and thus deserve attention. Second, together with the order quantities, the sales can offer useful insights into retailers' profit generating performance. Third, most importantly, it is through the retailers' sales in the first period that the C2C interactions influence the market growth, so studying the sales helps us understand how retailers' strategies are formed.

We start with each retailer's first-period sales under different types of C2C interactions.

Proposition 3. *Consider retailer i's equilibrium first-period sales in Scenario k, i.e., $R^k_{i,1}$ for $i = A, B$ and $k = S, N$. The following statements hold.*

(*a*) $\mathbb{E}[R^S_{A,1}]$ *and* $\mathbb{E}[R^N_{A,1}]$ *are increasing in* θ *when* γ *is large. Moreover, there exists a* $\underline{\theta}' \in [0,1]$ *such that* $\mathbb{E}[R^S_{A,1}] \geq \mathbb{E}[R^N_{A,1}]$ *for* $\theta \in [0, \underline{\theta}']$ *and* $\mathbb{E}[R^S_{A,1}] < \mathbb{E}[R^N_{A,1}]$ *for* $\theta \in (\underline{\theta}', 1]$.

(*b*) $\mathbb{E}[R^N_{B,1}]$ *is decreasing in* θ*, whereas* $\mathbb{E}[R^S_{B,1}]$ *is independent of* θ*. Moreover,* $\mathbb{E}[R^N_{B,1}] < \mathbb{E}[R^S_{B,1}]$.

It is noteworthy that the monotonicity and the relative magnitude of each retailer's sales in Scenarios S and N are consistent with those of their order quantities. For retailer B, since it will not receive demand spillover, its sales $R^k_{B,1} = \min\{\alpha_{B,1}X_1, y^k_{B,1}\}$ has the same property as its order $y^k_{B,1}$ in Scenario k; this can be verified by comparing Propositions 2 and 3(b). For retailer A, however, it is not straightforward to have the above result. In particular, the threshold in Proposition 3(a) is different from that in Proposition 1(b). Nevertheless, qualitatively, we obtain a similar result that retailer A sells more [less] in Scenario S when the substitution rate θ is small [large]. Therefore, our finding indicates that the retailers' sales are mostly determined by its own order, even in the presence of stock-out based substitution.

With cross-brand C2C interactions, the market expansion is based on the total, rather than individual, sales. Hence, for Scenario N, we scrutinize the retailers' total sales in the first period.

Proposition 4. *Consider retailers' total equilibrium first-period sales in Scenario N, i.e.,* R^N_1*. There exists a* $\gamma^R > 0$ *such that, if* $\gamma > \gamma^R$*,* $\mathbb{E}[R^N_1]$ *is first decreasing and then increasing in* θ.

The above result does not directly follow from Proposition 3, because, with cross-brand C2C interactions, the two retailers' sales change in opposite directions as θ increases: Retailer A's sales are boosted whereas retailer B's sales diminish. As a result, as shown in Proposition 4, the impact of the substitution rate on the total sales R^N_1 is non-monotone if the cross-brand C2C interactions are strong enough. Basically, as θ increases, each retailer receives an incentive to change its order quantity. For retailer A, it tends to order more so that the sales go up to drive the next period demand via social interactions between the satisfied customers and the prospective buyers; moreover, this incentive gets stronger when θ is larger (more demand spillover increases its underage cost). For retailer B, it decides to order less to free-ride the rival's sales and take advantage of the expanded common market; and this incentive is diminishing as θ becomes large (more demand spillover makes the free-riding more effective so it does not have to sacrifice too much first-period demand). Consequently, the total sales first decreases and then increases in θ. In summary, the non-monotone impact of substitution rate on the two retailers' total first-period sales offers a perspective for us to understand how the cross-brand C2C interactions influence the retailers' profits, as further explained in the next subsections.

4.2.3. Profits. We now start to investigate the impact of different types of C2C interactions on the retailers' equilibrium profits. The focus is on the comparison between Scenarios S and N, so that insights could be shed on how the results of the duopoly competition are driven by social interactions with opposing natures. Our findings interestingly pivot on three parameters, namely, the stock-out based substitution rate θ, the retailers' cost difference Δc, and the strength of the C2C interactions γ, all of which reflect the competition level in a way. In the duopoly competition, due to the cost asymmetry, retailer A enjoys an advantage against retailer B; moreover, this advantage is naturally enhanced when θ or Δc increases. Recall that the within-brand C2C interactions tend to ignite fierce competition while the cross-brand ones foster collaboration between firms. Hence, one may intuit that in Scenario S, the competition is intensified, and retailer A will have more advantage and thus better outcomes; by contrast, in Scenario N, the cross-brand C2C interactions will soften the competition, which is beneficial to retailer B. However, our results counter the above intuition by showing that there are exceptions in both cases. That is, under certain conditions, retailer A [B] may be better [worse] off in Scenario N than in Scenario S. In addition to the above three parameters, our results also depend on the retailers' exogenous prices in the two periods; we characterize their influences as well. In the following, we study retailer A and retailer B separately; let Π_i^k represent retailer i's total profit in Scenario k $(i = A, B; k = S, N)$.

For retailer A, the next proposition gives the monotonicity property of its profit and characterizes the situation where Π_A^S is dominated by Π_A^N.

Proposition 5. *Consider retailer A's equilibrium profit under different types of C2C interactions. There exists a $\gamma^A > 0$, such that, if $\gamma \geq \gamma^A$, the following statements hold.*

(a) Π_A^S *is increasing in* θ.

(b) *We have* $\Pi_A^k > \Pi_A^0$ *in Scenario* k $(k = S, N)$.

(c1) If $\alpha_{A,1} < \alpha_{A,2}$ [resp., $\alpha_{A,1} > \alpha_{A,2}$], we have $\Pi_A^S < \Pi_A^N$ [resp., $\Pi_A^S > \Pi_A^N$].

(c2) Suppose $\alpha_{A,1} = \alpha_{A,2}$. There exist two thresholds, $\Delta^A > 0$ and $\theta^A \in [0,1]$, such that, if $\Delta c \geq \Delta^A$, then we have $\Pi_A^S \geq \Pi_A^N$ for $\theta \in [0, \theta^A]$ and $\Pi_A^S < \Pi_A^N$ for $\theta \in (\theta^A, 1]$.

Since retailer A is in the advantageous position in the duopoly competition, the substitution rate θ, which represents the intensity of the competition, is positively correlated with retailer A's total profit, especially when the C2C interactions are within-brand, as stated in Proposition 5(a). The demand spillover from retailer B to the more cost-efficient retailer, is the main reason behind this result. However, when strong cross-brand C2C interactions are present, we find that Π_A^N may decrease in θ when θ takes intermediate values. In this case, θ not only creates spilled demand for retailer A but also may incentivize B's free-riding ordering decision, which reduces the total sales (see Proposition 4) and retailer A's profit.

As for profit comparison, Proposition 5(b) shows that, regardless of the type, C2C interactions would always increase retailer A's profit compared to the benchmark case, where no market expansion exists. But when comparing retailer A's equilibrium profit in Scenarios S and N, we identify counter-intuitive cases where $\Pi_A^S < \Pi_A^N$, as described in Proposition 5(c1)&(c2). Part (c1) states that retailer A earns higher profit in Scenario N when $\alpha_{A,1} < \alpha_{A,2}$ and in Scenario S when $\alpha_{A,1} > \alpha_{A,2}$. Since the split ratio depends solely on the retail prices of both firms, retailer A's profit comparison here does not involve any parameters beyond the exogenous prices. Particularly, if the price difference $p_{B,t} - p_{A,t}$ increases over time, then retailer A's split ratio also increases in t, and its profit in Scenario N dominates that in Scenario S. In fact, when γ is large, Π_A^S and Π_A^N are primarily driven by the first- and second-period split ratios $\alpha_{A,1}$ and $\alpha_{A,2}$, respectively. Thus, the sign of $\Pi_A^S - \Pi_A^N$ is determined by that of the dominant term $\alpha_{A,1} - \alpha_{A,2}$.

The case revealed in Proposition 5(c2) is more subtle, in the sense that Π_A^S and Π_A^N may cross each other and parameters θ and Δc play important roles in retailer A's profit comparison between scenarios. As we mentioned, with strong cross-brand C2C interactions, retailer B tends to order less in the expectation that retailer A gets more demand, leading to larger future market expansion. This effect is enhanced when the substitution rate θ and the cost difference are also large – because retailer B's "free-riding" behavior becomes even more efficient. This behavior in turn benefits retailer A's sales and profit (not only the first-period profit but also the second-period due to a significant market growth). On the other hand, with strong within-brand C2C interactions, retailer B is taking a rather different action, which is to drive up its own sales by ordering more quantity so that its own future market can be enlarged more. As a consequence, retailer A will get less substitution demand and its profit performance will not be as good as in Scenario N. This result demonstrates an interesting relation between stock-out substitution rate and C2C interactions. In particular, when the substitution rate θ is high, a significant portion of retailer B's unmet demand spills over to retailer A. As a result, the cross-brand C2C interaction effectively stimulates additional demand and generates sales just for retailer A. Therefore, the only parameter regime under which retailer A prefers the cross-brand C2C interaction is when this condition holds.

Another angle of viewing the above result in part (c2) is from the competition and collaboration relationship between the two retailers. When $\alpha_{A,1} = \alpha_{A,2}$ and the exogenous prices do not play a role, the conditions on parameters θ and Δc both being large imply that the duopoly competition is intensified and retailer A is in a very advantageous position. With large γ, the retailers also face strong C2C interactions, and in Scenario S [N], strong social contagions among customers means more [less] competition. Hence, when the competition is already intense, an even escalated competition level (e.g., due to within-brand C2C interactions) is not necessarily always beneficial to the advantageous firm; rather, introducing some collaborative element (such as cross-brand C2C interactions) may be a better strategy.

Next, we examine how retailer B's profit in different scenarios depends on the system parameters and how the profit changes across scenarios. Our findings are in parallel with those for retailer A.

Proposition 6. *Consider retailer B's equilibrium profit under different types of C2C interactions. There exists a $\gamma^B > 0$, such that, if $\gamma \geq \gamma^B$, the following statements hold.*

(a) Π_B^S is independent of θ, whereas Π_B^N is increasing in θ.

(b) We have $\Pi_B^k > \Pi_B^0$ in Scenario k $(k = S, N)$.

(c1) If $\alpha_{B,1} > \alpha_{B,2}$ [resp., $\alpha_{B,1} < \alpha_{B,2}$], we have $\Pi_B^S > \Pi_B^N$ [resp., $\Pi_B^S < \Pi_B^N$].

(c2) Suppose $\alpha_{B,1} = \alpha_{B,2}$. There exist two thresholds, $\Delta^B > 0$ and $\theta^B \in [0,1]$, such that, if $0 < \Delta c \leq \Delta^B$, then we have $\Pi_B^S > \Pi_B^N$ for $\theta \in [0, \theta^B)$ and $\Pi_B^S \leq \Pi_B^N$ for $\theta \in [\theta^B, 1]$.

As stated in Proposition 6(a), retailer B's profit is independent of the substitution rate in Scenario S but increases in θ in Scenario N. When the C2C interactions are within-brand, the two retailers are relatively independent in terms of utilizing customers social interactions to generate new demand. In this scenario, since demand spillover never occurs from retailer A to retailer B, retailer B's profit does not depend on the substitution rate. When cross-brand C2C interactions are present, however, the demand spillover from retailer B to retailer A turns out to have an impact back on retailer B, because the future market growth is for both retailers. Furthermore, we show that this positive impact of θ on Π_B^N dominates the negative impact related to demand spillover to the rival, resulting in the increasing relationship between profit and substitution rate.

When comparing retailer B's profit across scenarios, Proposition 6(b) first shows that its profit in the presence of either type of C2C interactions would dominate that in the benchmark, which is again due to the market growth. However, the comparison between Scenarios S&N is not straightforward. In Scenario N, retailer B has the incentive to order less and "free-ride" retailer A's first-period sales to get a larger second-period market. In contrast, in Scenario S, retailer B has to depend on itself, so it tends to order more in the first-period (and thus has to sacrifice its first-period profit) for the sake of the market growth due to customers' social interactions. For the above reason, retailer B always obtains higher first-period profit in Scenario N. However, the benefit of the cross-brand C2C interactions for retailer B does not seem to always extend to the second period. The comparison between Π_B^S and Π_B^N shown in Proposition 6(c1)&(c2) again uncovers counter-intuitive situations where retailer B is worse off in Scenario N.

Part (c1) describes how retailer B's profit comparison may be solely affected by retailers' exogenous prices, which is in fact a counterpart statement of Proposition 5(c1). Similar to the previous case, when the social contagions are strong (γ is large), the sign of $\Pi_B^S - \Pi_B^N$ is determined by the sign of $\alpha_{B,1} - \alpha_{B,2}$. Hence, if the price difference $p_{B,t} - p_{A,t}$ becomes larger along time, retailer B receives a smaller demand split ratio in the second period, i.e., $\alpha_{B,1} > \alpha_{B,2}$; hence, Π_B^S dominates Π_B^N when γ is large. Proposition 6(c2) further characterizes retailer B's profit comparison between scenarios when the exogenous prices do not play a role, i.e., when the price difference stays the same in each period and $\alpha_{B,1} = \alpha_{B,2}$. In this case, retailer B earns higher profit in Scenario S if the substitution rate is relatively small, the two retailers have a small cost difference, and the social contagions are strong. The underlying logic goes as follows. Seeing strong cross-brand C2C interactions, retailer B wants to send many customers to retailer A by ordering a small quantity, but the demand spillover is only moderate due to a small substitution rate. As a result, retailer B's second-period profit is hurt. On the other hand, with strong within-brand C2C interactions, retailer B could achieve a high second-period profit when it orders a large quantity to obtain high sales in the first period, especially when the competition is soft (both substitution rate and the cost difference are small). Therefore, retailer B's total profit turns to be higher in Scenario S than in Scenario N.

From the viewpoint of the duopoly competitive/collaborative relationship, the result in Proposition 6 (c2) can be interpreted as follows. In this case, when θ and Δc are small, the duopoly competition is relaxed. Particularly, retailer B's disadvantage related to the demand spillover becomes alleviated and the retailers' cost gap tends to close up. Given such a circumstance, will retailer B continues to benefit further when the competition intensity is reduced even more? Not necessarily.

In fact, the more collaborative-in-nature cross-brand C2C interactions could actually hurt retailer B's profit, compared to when the competition-oriented within-brand C2C interactions are present.

The above comparison results generate interesting insights into competing firms' relationship. Facing multiple non-price levers to adjust the degrees of competition and collaboration between rivalries, a healthy duopoly relationship should not go to extremities. Adding competitiveness may not always benefit the advantageous firm when the competition is already intense; and, facing already softened competition, the disadvantageous firm may not always prefer collaborative initiatives. The proper level of competition is likely to be a mix of both. Nevertheless, it is worth mentioning that the prices may play a key role that dominates the profit comparison regardless of other parameters such as substitution rate and cost difference; see part (c1) of both Propositions 5&6. To gain more insights on pricing, we examine the endogenous price model in the next section.

5. Endogenous Price Model

Up to now, we have assumed that the retail price $p_{i,t}$ posted by retailer i in period t ($i = A, B; t = 1, 2$) is exogenously given. In our general model framework, however, the retailers are allowed to decide the prices at the beginning of each period. The primary goal of this section, therefore, is to include pricing, an important marketing lever, into the problem and study how the social interactions play a role in shaping the retailers' pricing strategies. In this Endogenous Price Model, we assume that the retailers have the flexibility to dynamically set prices based on the observed previous sales. This assumption reflects the increasing use of adaptive pricing in online retailing, especially in short-cycle and event-driven markets. Retailers may frequently adjust prices between sales events or promotional periods in response to past performance metrics such as engagement data and realized sales.

Price-setting newsvendors' problem, in monopoly or in competitive settings, has been intensively studied in the literature (see, e.g., Petruzzi and Dada 1999, Zhao and Atkins 2008, Salinger and Ampudia 2011). Departing from the prior works, our paper focuses on the impact of the different types of C2C interactions in a dynamic duopoly game. To concentrate on the focal features, we attempt to answer the following three questions. How do the retailers' prices affect their order quantities? Are the retailers' equilibrium price paths increasing or decreasing over time? How do retailers' equilibrium profits under different types of C2C interactions compare with each other?

The game setup in this section follows the model framework introduced in §3, and the retailers' problems are formulated as $(\mathcal{P})$ given in §3.4. Note that the Endogenous Price Model is significantly more difficult to solve analytically, due to two major differences from the Exogenous Price

Model. First, unlike the previous model, we do not apply the constraint $p_{B,t} \geq p_{A,t} \geq \frac{c_A}{c_B} p_{B,t}$ on the endogenous prices; instead, we simply assume all prices are bounded, i.e., $p_{i,t} \in [\underline{p}, \overline{p}]$ $(i, j = A, B; t = 1, 2)$. As a result, the spillover of excess demand can be in either direction now, increasing the complexity of analysis. Second, the demand split ratio $\alpha_{i,t} = \alpha_{i,t}(p_{i,t}, p_{j,t})$ $(i, j = A, B; i \neq j)$ explicitly depends on the retail prices of both retailers, and the backward induction can become quite involved in the first period. To obtain clean results and extract clear insights, we focus our analysis on the case where the social interactions are strong (i.e., γ is large), and we numerically investigate the general situations.

5.1. The Impact of Pricing on Order Quantities

In each period, since the retailers' pricing decisions precede the stocking decisions, their order quantities are affected by the retail prices of both firms. In what follows, we establish the directional relationship between prices and order quantities in different scenarios, starting with the retailers' second-period order quantities.

Proposition 7. *Consider retailer i's* $(i = A, B)$ *second-period order quantity* $y^k_{i,2}$ *as a function of* $(p_{A,2}, p_{B,2})$ *in Scenario* $k = S, N$. *There exists a* $\underline{\beta}^D > 0$ *such that the following statements hold if* $\beta > \underline{\beta}^D$.

(*a*) $y^k_{i,2}$ *either decreases or first increases then decreases in* $p_{i,2}$ $(i = A, B;\ k = S, N)$.

(*b*) $y^k_{i,2}$ *either increases or first decreases then increases in* $p_{j,2}$ $(i, j = A, B, j \neq i;\ k = S, N)$.

Proposition 7 shows that the second-period order quantities are either monotonic or unimodal in the prices (see detailed conditions in proof). Part (a) highlights two opposing forces that shape how a retailer's own price influences its second-period order quantity. On one hand, as retailer i raises its price, it captures a smaller share of the market demand, which naturally discourages it from stocking large quantities. On the other hand, a higher price increases the per-unit profit and the underage cost of stockouts, creating stronger incentives to order more. Additionally, as retailer i loses market share, the competitor gains a larger one, potentially leading to more demand spillover toward retailer i, which further encourages a larger order. The net impact of these conflicting forces leads to a non-monotonic relationship identified in the proposition.

Proposition 7(b) follows a similar logic but in reverse. As the competitor j increases its price, retailer i benefits from a larger market share and is incentivized to stock more. However, the reduced likelihood of spillover demand from a now more cautious competitor can counteract that incentive, potentially leading to non-monotonicity in $y^k_{i,2}$ with respect to $p_{j,2}$. Notably, the results in both parts (a) and (b) hold regardless of whether within-brand or cross-brand C2C interactions are present. This is because, in the second period, the market size is already determined and no further customer-driven diffusion occurs, making the type of the past social interactions irrelevant.

Next, we turn to study the retailers' first-period order quantities.

Proposition 8. *Consider retailer i's first-period order quantity $y^k_{i,1}$ as a function of $(p_{A,1}, p_{B,1})$ in Scenario k ($i = A, B$; $k = S, N$). There exists a $\gamma^D > 0$ such that the following statements hold if $\gamma > \gamma^D$.*

(a) In Scenario S, $y^S_{i,1}$ decreases in $p_{i,1}$ and increases in $p_{j,1}$ ($i, j = A, B, j \neq i$).

(b1) In Scenario N, $y^N_{A,1}$ decreases in $p_{A,1}$ and $y^N_{B,1}$ increases in $p_{A,1}$.

(b2) In Scenario N, there exist two thresholds $\theta^D_2 \in [0, 1]$ and $\overline{\beta}^D > 0$, such that $y^N_{A,1}$ decreases in $p_{B,1}$ and $y^N_{B,1}$ increases in $p_{B,1}$ if $\theta^D \leq \theta \leq 1$ and $0 < \beta \leq \overline{\beta}^D$.

Different from Proposition 7, the first-period stocking decisions now depend crucially on which type of C2C interactions is present. In Scenario S, the within-brand interactions govern each retailer's market expansion, which is driven exclusively by its own customer base. As a result, Proposition 8(a) yields intuitive comparative statics that mirrors the standard logic of demand response, i.e., higher prices suppress its own market share while amplifying the rival's. However, in Scenario N, where market expansion is shaped by the combined influence of both retailers' sales, Proposition 8(b1)&(b2) reveal richer, and sometimes counterintuitive, comparative statics.

According to Proposition 8(b1), in Scenario N, the retailers' order quantities respond to retailer A's price in opposite ways. As retailer A raises its price, it loses market share while retailer B gains more of the common market and faces a lower overage cost (see Lemma 3). These factors incentivize retailer B to order more inventory. Meanwhile, retailer A becomes less motivated to maintain a large stock in anticipation of second-period demand, especially given the intensified price competition in that period. As a result, retailer A's reduced market share dominates any other effects, forcing it to order less. This illustrates how pricing decisions can have ripple effects across both retailers' operational strategies.

Proposition 8(b2) presents a more nuanced scenario. When the free-riding behavior is significant (i.e., large θ) and prices have limited influences on market shares (i.e., small β), retailer B's order quantity increases with its own price. This is driven by a higher underage cost that outweighs the loss in market shares, prompting retailer B to stock more. In turn, this reduces demand spillover, softens B's free-riding behavior, and ultimately reduces retailer A's order quantity. Thus, under these parameter conditions, retailer B's pricing decision indirectly suppresses A's inventory investment. These results underscore how pricing power and C2C dynamics jointly determine inventory strategies in competitive online markets.

5.2. Properties of Equilibrium Prices

Next, we examine the retailers' price-inventory competition under different types of C2C interactions. As previously mentioned, the dynamic game is analytically difficult to obtain the close form solution. However, by focusing on certain ranges of parameters, we are able to derive some structural properties of the equilibrium price paths for each retailer. Specifically, we compare the retailers' equilibrium prices between periods when within-brand and cross-brand C2C interactions are present, respectively, which leads to an interesting finding that the retailers may adopt contrastingly different pricing strategies in Scenarios S versus N.

PROPOSITION 9. *Consider the two-period price-inventory competition under the two types of C2C interactions, where all prices are bounded in the interval* $[\underline{p}, \overline{p}]$. *There exists a* $\gamma^P \geq 0$ *for* $\theta \in [0, 1)$ *such that, when* $\gamma > \gamma^P$, *an equilibrium profile of prices and ordering quantities,* $(p^k_{A,t}, p^k_{B,t})$ *and* $(y^k_{A,t}, y^k_{B,t})$ $(t = 1, 2;$ $k = S, N)$, *exists; moreover, the following statements hold:*

(a) *In Scenario S, we have* $\underline{p} = p^S_{i,1} < p^S_{i,2} = \overline{p}$ $(i = A, B)$.

(b) *In Scenario N, we have* $\underline{p} < p^N_{i,2} < p^N_{i,1} < \overline{p}$ $(i = A, B)$.

Proposition 9 reveals that the equilibrium pricing strategies of both retailers can exhibit opposite intertemporal patterns depending on the type of C2C interactions. When the strength of social contagion is sufficiently large, prices increase over time in the presence of within-brand C2C interactions (Scenario S), but decrease over time under cross-brand C2C interactions (Scenario N). This contrast arises from the different ways in which early sales influence future demand. In Scenario S, since each retailer's market expansion is primarily driven by its own existing customer base, it is optimal to set a low price in the first period to stimulate initial adoption and generate brand-specific social influence. In the second period, the retailer can raise its price to capitalize on the increased demand it has individually cultivated. Hence, the equilibrium pricing path under within-brand C2C interactions aligns with the classic exploration–exploitation logic: Firms price low early to build demand, then price high to harvest it.

In Scenario N, interestingly, the opposite pricing dynamic emerges. Because C2C interactions expand the overall market regardless of which brand the information originates from, early-period sales by one retailer can benefit both players. When cross-brand social contagion is strong, each retailer has an incentive to rely on the shared market expansion, and therefore they do not need to set a very low price initially. However, in the second period, the competition intensifies as both retailers attempt to capture a greater portion of the now-enlarged common market. To do so, they must lower prices to attract more demand. This results in a downward-sloping equilibrium price path, which reflects the strategic use of pricing to mitigate the rival's free-riding behavior and defend one's own market share in the latter stage.

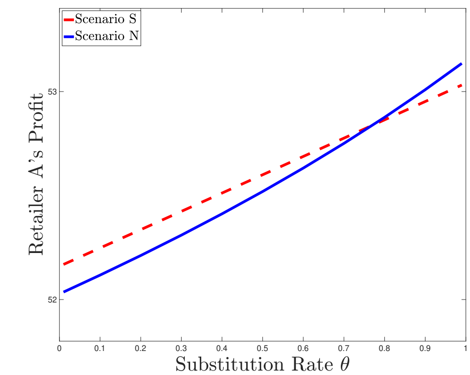
From a broader perspective, this result offers a compelling illustration of the shifting competitive dynamics in response to different forms of customer-driven market growth. Under within-brand C2C interactions, the pricing competition naturally softens over time: retailers start aggressively with low prices and gradually relax competitive pressure as they rely on their individually expanded markets. In contrast, cross-brand C2C interactions foster a more collaborative environment in the early stage, followed by intensified rivalry later on. The equilibrium pricing becomes more aggressive in the second period as retailers seek to outperform each other in capturing demand from the collectively grown market. These findings reinforce one of the central insights of the paper: Effective strategic responses to customer-driven market dynamics often feature a delicate balance between cooperation and competition.

5.3. Comparing Equilibrium Profit Between Scenarios

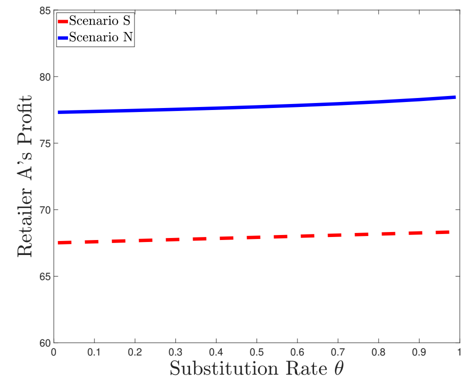
As a parallel to the results in Section 4.2.3, we now compare the retailers' equilibrium profits between Scenarios S and N. Recall that, in the main model with exogenous prices, each retailer's profit may cross under different types of C2C interactions. That is, retailer A may earn higher profit in Scenario N, while retailer B may benefit more in Scenario S. Does this pattern persist under retailers' endogenous pricing competition? The main objective of this subsection is to address this question. As previously mentioned, the price-inventory game is analytically intractable; therefore, we rely on numerical experiments to investigate the equilibrium profit outcomes. Moreover, to simplify computation while preserving the key mechanism behind endogenous pricing, we restrict each retailer's feasible price set to two discrete levels in our numerical studies. Overall, when the retailers endogenously determine their prices, we find that the earlier patterns still hold in some cases, but may change in others. Specifically, Figure 1 uses some representative instances to graphically illustrate the main results.

From these graphs, we gather observations regarding the retailers' total profits under equilibrium pricing across different scenarios, with the strength of C2C interactions varying. Before diving into the specific findings, we highlight an important distinction between the current analysis and that in Section 4.2.3. Previously, both scenarios were evaluated under exogenously fixed retail prices, whereas here, prices are determined endogenously and may differ across scenarios. In general, when prices are continuous and constrained within a bounded interval, the equilibrium market split ratio between the two retailers may vary across periods. However, in our numerical experiments, although the equilibrium prices across the two periods may not be the same, the

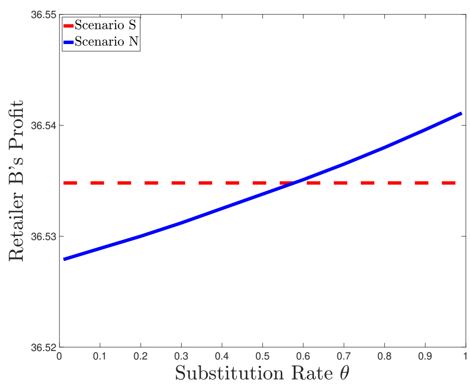
Figure 1 **Retailers' equilibrium profits comparison between Scenarios.**



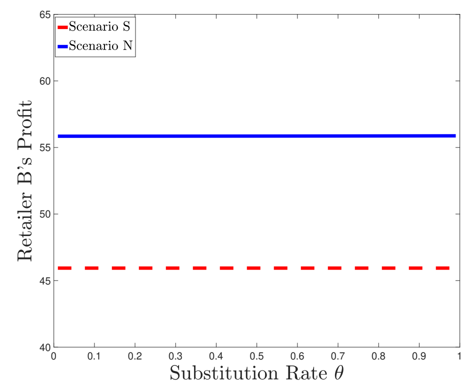
(a) Retailer A comparison; $\gamma = 0.7$



(b) Retailer A comparison; $\gamma = 2$



(c) Retailer B comparison; $\gamma = 0.7$



(d) Retailer B comparison; $\gamma = 2$

Parameters: We pick $p_{i,t} \in \{2,4\}(i = A, B; t = 1,2)$, and fix $c_B = 1$, $\beta = 0.1$, $X_t \sim U[5,15](t = 1,2)$. For retailer A's comparison, we choose $c_A = 0.1$; and for retailer B's comparison, we choose $c_A = 0.95$.

resulting split ratios remain constant over time. This feature simplifies the analysis and enables us to focus on the particularly insightful case, i.e., $\alpha_{i,1} = \alpha_{i,2}$ for $i = A, B$, where the retailers' profit functions in the two scenarios are directly comparable and may intersect.

With this in mind, we summarize two key observations from the numerical results. First, Figure 1(a)&(c) replicate the results in Propositions 5&6 under a moderate γ. For retailer A, as θ increases to 1, Scenario N becomes more profitable than Scenario S. For retailer B, when θ is small, Scenario S yields higher profit. These outcomes align with our earlier findings, as the parameter choices reflect the required conditions, namely, a non-negligible level of social interactions and a large [resp., small] cost difference for retailer A's [resp., B's] comparison. Moreover, although we restrict the feasible price set to two discrete values in the numerical experiments, this setup preserves the core mechanism of endogenous pricing and allows meaningful equilibrium comparisons.

Second, Figures 1(b)&(d) illustrate how the earlier theoretical results may not hold when C2C interactions become very strong in the Endogenous Price Model. Specifically, Figure 1(b) shows that retailer A always earns more in Scenario N, even when θ is small and $\alpha_{A,1} = \alpha_{A,2}$ — an outcome not seen before. This can be explained by the intensified price competition in Scenario S under large γ, which suppresses retailer A's first-period profit. Figure 1(d) further shows that, across all θ values, retailer B consistently earns more in Scenario N, even though $\alpha_{B,1} = \alpha_{B,2}$. In fact, in Scenario S, although retailer B can expand its own market share, it cannot do it effectively due to retailer A's advantageous cost-efficiency and the resulting pricing dominance.

Taken together, these results suggest that when retailers engage in endogenous pricing, strong social interactions can fundamentally alter the profit comparison. In such cases, the crossover pattern predicted in Propositions 5&6 may no longer appear, and one scenario may dominate the other for the entire range of θ. Thus, to extend the previous propositions to this endogenous pricing setting, one might need to impose an upper bound on γ. Intuitively, this is because pricing flexibility enables retailers to make aggressive markdowns that help capture more market share while amplifying the benefit from strong social interactions.

6. Concluding Remarks

Online retailing has grown increasingly dynamic and competitive, with customer behavior shaped not only by product availability and price but also by the social interactions embedded in the shopping experience. This is particularly salient in formats like live video shopping, where past sales and customer engagement can amplify future demand through C2C interactions. In this paper, we take a novel approach to study two distinct types of C2C interactions that influence demand dynamics in a competitive setting. Under within-brand interactions (Scenario S), a retailer's own sales drive its market expansion. In contrast, under cross-brand interactions (Scenario N), both retailers' sales contribute to enlarging a shared market, creating the potential for free-riding behavior. These two mechanisms govern the market growth over time and induce very different competitive dynamics. To investigate these effects, we develop a two-period duopoly model in which online retailers with asymmetric costs compete as newsvendors. Our model allows for both inventory and pricing decisions, enabling us to capture key operational and strategic considerations in the presence of dynamic demand feedback via C2C interactions. Below, we summarize the main findings and highlight their managerial implications.

First, under cross-brand C2C interactions, the high-cost retailer (retailer B) may benefit from the market expansion driven by the low-cost retailer's (retailer A's) sales. In this case, retailer B strategically orders less and allows excess demand to spill over to retailer A, who carries more inventory at lower cost. Retailer B then reaps the rewards of the expanded market in the second period, without incurring much cost itself. This behavior demonstrates the possibility of demand "free-riding" through social interactions and suggests that a high-cost online retailer, when facing a more efficient competitor, can capitalize on its rival's operational advantage to gain market access through cross-brand social contagions. Meanwhile, the inventory decisions of retailer A are shaped by the nature of the C2C interaction, with the intensity of inventory competition playing a central role in determining its strategic response.

Second, the comparison of retailers' total profits under different types of C2C interactions reveals an interesting finding regarding their scenario preferences. While it may seem intuitive that the low-cost [high-cost] retailer benefits more under within-brand [cross-brand] interactions, our results show this intuition does not always hold. When the low-cost retailer has already secured a strong competitive edge, the added intensity from within-brand interactions may hurt rather than help, making cross-brand interactions more profitable. Conversely, if the high-cost retailer is not too disadvantaged, it can perform better under within-brand interactions, which contribute to a more competitive environment. These findings suggest that both competitive and collaborative forces shape the equilibrium outcomes, and online retailers may benefit most from a balanced interplay between the two.

Third, when prices are endogenously determined, the firms' equilibrium pricing strategies differ sharply across scenarios. Under within-brand C2C interactions, both retailers adopt increasing price paths: They use lower prices in the first period to build up their own market and raise prices later to extract value. Under cross-brand interactions, however, prices decline across periods. The firms, especially the low-cost one, lower their second-period prices to capture a larger share of the common market expanded in the first period. This pricing behavior can be seen as a strategic response to the effect of cross-brand interactions; i.e., having contributed to the market growth, the retailers use the pricing lever to claim their fair shares. Notably, our previous results on profit comparisons largely carry over under endogenous pricing, though stronger C2C interactions may change the results in certain cases. These insights underscore how pricing decisions can interact with social dynamics to reshape market outcomes in competitive online retail environments.

To conclude, we point out some interesting avenues for future research, which could extend this paper in multiple directions. (1) In practice, the influence of social interactions may fluctuate due to varying customer engagement levels. Incorporating stochastic interaction strength would allow us to study how uncertainty in social contagion affects retailers' strategic decisions and competitive dynamics. (2) If the retailers sell non-perishable products, inventory may carry over across periods. This would introduce richer system dynamics, where inventory positions evolve over time and the equilibrium order quantities become more intricate. (3) From a supply chain management perspective, the model can be extended to incorporate upstream manufacturers. The presence of C2C interactions may then have cascading effects across the supply chain, prompting new forms of strategic response from different channel members.

References

Brandenburger, A., B. Nalebuff. 1996. *Coopetition*. Currency Doubleday: New York.

Caro, F., A. G. Kök, V. Martínez-de Albéniz. 2020. The future of retail operations. *Manufacturing & Service Operations Management* **22**(1) 47–58.

Chae, I., A. T. Stephen, Y. Bart, D. Yao. 2017. Spillover effects in seeded word-of-mouth marketing campaigns. *Marketing Science* **36**(1) 89–104.

Chen, Y., G. Gallego, P. Gao, Y. Li. 2020. Position auctions with endogenous product information: Why live-streaming advertising is thriving. *Working Paper* .

Dong, B., Y. Ren, C. McIntosh. 2023. A co-opetitive newsvendor model with product substitution and a wholesale price contract. *European Journal of Operational Research* **311**(2) 502–514.

Geng, X., X. Guo, G. Xiao. 2022. Impact of social interactions on duopoly competition with quality considerations. *Management Science* **68**(2) 941–959.

Gnyawali, D. R., R. Madhavan. 2001. Cooperative networks and competitive dynamics: A structural embeddedness perspective. *Academy of Management review* **26**(3) 431–445.

Godes, D. 2017. Product policy in markets with word-of-mouth communication. *Management Science* **63**(1) 267–278.

Godes, D., D. Mayzlin. 2004. Using online conversations to study word-of-mouth communication. *Marketing Science* **23**(4) 545–560.

Güler, K., E. Körpeoğlu, A. Şen. 2018. Newsvendor competition under asymmetric cost information. *European Journal of Operational Research* **271**(2) 561–576.

Hall, J., E. Porteus. 2000. Customer service competition in capacitated systems. *Manufacturing & Service Operations Management* **2**(2) 144–165.

Haviv, A., Y. Huang, N. Li. 2020. Intertemporal demand spillover effects on video game platforms. *Management Science* **66**(10) 4788–4807.

Hou, J., H. Shen, F. Xu. 2022. A model of livestream selling with online influencers. *Working Paper* .

Hu, B., Y. Mai, S. Pekeč. 2020. Managing innovation spillover in outsourcing. *Production and Operations Management* **29**(10) 2252–2267.

Jiang, H., S. Netessine, S. Savin. 2011. Robust newsvendor competition under asymmetric information. *Operations Research* **59**(1) 254–261.

Krishnan, T. V., P. B. Seetharaman, D. Vakratsas. 2012. The multiple roles of interpersonal communication in new product growth. *International Journal of Research in Marketing* **29**(3) 292–305.

Libai, B., E. Muller, R. Peres. 2009. The role of within-brand and cross-brand communications in competitive growth. *Journal of Marketing* **73**(3) 19–34.

Lippman, S. A., K. F. McCardle. 1997. The competitive newsboy. *Operations Research* **45**(1) 54–65.

Liu, L., W. Shang, S. Wu. 2007. Dynamic competitive newsvendors with service-sensitive demands. *Manufacturing & Service Operations Management* **9**(1) 84–93.

Nagarajan, M., S. Rajagopalan. 2008. Inventory models for substitutable products: Optimal policies and heuristics. *Management Science* **54**(8) 1453–1466.

Nagarajan, M., S. Rajagopalan. 2009. A multiperiod model of inventory competition. *Operations Research* **57**(3) 785–790.

Netessine, S., N. Rudi. 2003. Centralized and competitive inventory models with demand substitution. *Operations Research* **51**(2) 329–335.

Olsen, T. L., R. P. Parker. 2008. Inventory management under market size dynamics. *Management Science* **54**(10) 1805–1821.

Olsen, T. L., R. P. Parker. 2014. On markov equilibria in dynamic inventory competition. *Operations Research* **62**(2) 332–344.

Parlar, M. 1988. Game theoretic analysis of the substitutable product inventory problem with random demands. *Naval Research Logistics* **35**(3) 397–409.

Peres, R., C. Van den Bulte. 2014. When to take or forgo new product exclusivity: Balancing protection from competition against word-of-mouth spillover. *Journal of Marketing* **78**(2) 83–100.

Petruzzi, N. C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Operations Research* **47**(2) 183–194.

Qi, A., S. Sethi, L. Wei, J. Zhang. 2022. Top or regular influencer? contracting in live-streaming platform selling. *Available at SSRN: 3668390* .

Salinger, M., M. Ampudia. 2011. Simple economics of the price-setting newsvendor problem. *Management Science* **57**(11) 1996–1998.

Sanchez, J., C. Abril, M. Haenlein. 2020. Competitive spillover elasticities of electronic word of mouth: an application to the soft drink industry. *Journal of the Academy of Marketing Science* **48** 270–287.

Straubert, C., E. Sucky. 2023. Inventory competition on electronic marketplaces–a competitive newsvendor problem with a unilateral sales commission fee. *European Journal of Operational Research* **309**(2) 656–670.

Wongkitrungrueng, A., N. Assarut. 2020. The role of live streaming in building consumer trust and engagement with social commerce sellers. *Journal of Business Research* **117** 543–556.

Yan, X., H. Hu. 2023. New product demand forecasting and production capacity adjustment strategies: Within-product and cross-product word-of-mouth. *Computers & Industrial Engineering* **182** 109394.

Zhao, X., D. R. Atkins. 2008. Newsvendors under simultaneous price and inventory competition. *Manufacturing & Service Operations Management* **10**(3) 539–546.

Online Appendices to "Dynamic Competition in Online Retailing: The Implications of Customer-to-Customer Interactions"

Appendix A: Proof of Statements

A.1. Proof of Statements in Section 4

Proof of Lemma 1: In benchmark scenario, retailer A maximizes its single-period profit in each period t:

$$\max_{y_{A,t}} p_{A,t}\mathbb{E}\left[y_{A,t} \wedge D_{A,t}\right] - c_A y_{A,t} = (p_{A,t} - c_A)y_{A,t} - p_{A,t}\mathbb{E}\left[y_{A,t} - D_{A,t}\right]^+,$$

where $D_{A,t} = \alpha_{A,t}X_t + \theta\left[\alpha_{B,t}X_t - y_{B,t}\right]^+$. Retailer A's profit is concave in $y_{A,t}$ and its derivative w.r.t $y_{A,t}$ is $p_{A,t} - c_A - p_{A,t}\mathbf{Prob}\left(D_{A,t} \le y_{A,t}\right)$, where

$$\mathbf{Prob}\left(D_{A,t} \le y_{A,t}\right) = \int_0^{y_{B,t}/\alpha_{B,t}} \mathbb{1}_{\{z \le y_{A,t}/\alpha_{A,t}\}} f(z)dz + \int_{y_{B,t}/\alpha_{B,t}}^{+\infty} \mathbb{1}_{\left\{z \le \frac{y_{A,t}+\theta y_{B,t}}{\alpha_{A,t}+\theta\alpha_{B,t}}\right\}} f(z)dz. \tag{4}$$

If $y_{A,t}/\alpha_{A,t} \ge y_{B,t}/\alpha_{B,t}$, $\frac{y_{B,t}}{\alpha_{B,t}} \le \frac{y_{A,t}+\theta y_{B,t}}{\alpha_{A,t}+\theta\alpha_{B,t}} \le \frac{y_{A,t}}{\alpha_{A,t}}$ for $\theta \in [0,1]$. Thus, by (4), we have $\mathbf{Prob}\left(D_{A,t} \le y_{A,t}\right) = F\left(\frac{y_{A,t}+\theta y_{B,t}}{\alpha_{A,t}+\theta\alpha_{B,t}}\right)$. If $y_{A,t}/\alpha_{A,t} < y_{B,t}/\alpha_{B,t}$, $\frac{y_{A,t}}{\alpha_{A,t}} < \frac{y_{A,t}+\theta y_{B,t}}{\alpha_{A,t}+\theta\alpha_{B,t}} < \frac{y_{B,t}}{\alpha_{B,t}}$ for $\theta \in [0,1]$. Thus, by (4), we have $\mathbf{Prob}\left(D_{A,t} \le y_{A,t}\right) = F\left(\frac{y_{A,t}}{\alpha_{A,t}}\right)$. Therefore, the best response of retailer A is

$$y^*_{A,t}(y_{B,t}) = \begin{cases} (\alpha_{A,t} + \theta\alpha_{B,t})\zeta^0_{A,t} - \theta y_{B,t}, & \text{if } y_{B,t} \le \alpha_{B,t}\zeta^0_{A,t}, \\ \alpha_{A,t}\zeta^0_{A,t}, & \text{if } y_{B,t} > \alpha_{B,t}\zeta^0_{A,t}, \end{cases} \tag{5}$$

where $\zeta^0_{A,t} := F^{-1}(1 - \frac{c_A}{p_{A,t}})$. On the other hand, retailer B's problem is as follows:

$$\max_{y_{B,t}} p_{B,t}\mathbb{E}\left[y_{B,t} \wedge D_{B,t}\right] - c_B y_{B,t} = (p_{B,t} - c_B)y_{B,t} - p_{B,t}\mathbb{E}\left[y_{B,t} - D_{B,t}\right]^+,$$

where $D_{B,t} = \alpha_{B,t}X_t + \theta\left[\alpha_{A,t}X_t - y_{A,t}\right]^+$. By analogous arguments, the best response of retailer B is

$$y^*_{B,t}(y_{A,t}) = \begin{cases} (\alpha_{B,t} + \theta\alpha_{A,t})\zeta^0_{B,t} - \theta y_{A,t}, & \text{if } y_{A,t} \le \alpha_{A,t}\zeta^0_{B,t}, \\ \alpha_{B,t}\zeta^0_{B,t}, & \text{if } y_{A,t} > \alpha_{A,t}\zeta^0_{B,t}, \end{cases} \tag{6}$$

where $\zeta^0_{B,t} := F^{-1}(1 - \frac{c_B}{p_{B,t}})$.

Since $p_{B,t}c_A/c_B \le p_{A,t}$, we have $\zeta^0_{A,t} \ge \zeta^0_{B,t}$ for $t = 1, 2$. By combining (5) and (6), for each period t, there exists a unique Nash equilibrium $(y^0_{A,t}, y^0_{B,t})$, where $y^0_{A,t} := (\alpha_{A,t} + \theta\alpha_{B,t})\zeta^0_{A,t} - \theta\alpha_{B,t}\zeta^0_{B,t}$ and $y^0_{B,t} := \alpha_{B,t}\zeta^0_{B,t}$.

Last, by substituting $y^0_{A,t}$ and $y^0_{B,t}$ into retailers' profit functions, retailers A's and B's equilibrium profit in benchmark scenario is $\pi^0_{A,t}$ and $\pi^0_{B,t}$, respectively: $\pi^0_{A,t} := (p_{A,t} - c_A)y^0_{A,t} - p_{A,t}\mathbb{E}[L^0_{A,t}]$ and $\pi^0_{B,t} := (p_{B,t} - c_A)y^0_{B,t} - p_{B,t}\mathbb{E}[L^0_{B,t}]$, where $L^0_{A,t} := \left[(\alpha_{A,t} + \theta\alpha_{B,t})\zeta^0_{A,t} - \theta\alpha_{B,t}\zeta^0_{B,t} - \alpha_{A,t}X_t - \theta(\alpha_{B,t}X_t - \alpha_{B,t}\zeta^0_{B,t})^+\right]^+$ and $L^0_{B,t} := \left[\alpha_{B,t}\zeta^0_{B,t} - \alpha_{B,t}X_t - \theta\left\{\alpha_{A,t}X_t - (\alpha_{A,t} + \theta\alpha_{B,t})\zeta^0_{A,t} + \theta\alpha_{B,t}\zeta^0_{B,t}\right\}^+\right]^+$, $t = 1, 2$. Furthermore, retailer i's equilibrium sales in each period are $R^0_{i,t} := y^0_{i,t} - L^0_{i,t}$, $i = A, B$ and $t = 1, 2$. Thus, we have proved Lemma 1. *Q.E.D.*

Proof of Lemma 2: We first study the subgame in the second period. Given the first-period seed demand X_1 and retailers' first-period sales, retailer A maximizes its second-period profit:

$$\max_{y_{A,2}} \mathbb{E}\left[p_{A,2}(y_{A,2} \wedge D_{A,2}) - c_A y_{A,2} | X_1\right]$$

$$= \mathbb{E}\left[(p_{A,2}-c_A)y_{A,2} - p_{A,2}\left(y_{A,2}-\alpha_{A,2}X_2-\gamma R_{A,1}-\theta\left(\alpha_{B,2}X_2+\gamma R_{B,1}-y_{B,2}\right)^+\right)^+\middle|X_1\right]$$
$$= \mathbb{E}\left[(p_{A,2}-c_A)\overline{y}_{A,2} - p_{A,2}\left(\overline{y}_{A,2}-\alpha_{A,2}X_2-\theta\left(\alpha_{B,2}X_2-\overline{y}_{B,2}\right)^+\right)^+ + (p_{A,2}-c_A)\gamma R_{A,1}\middle|X_1\right] \quad (7)$$

where $D_{A,2}=\alpha_{A,2}X_2+\gamma R_{A,1}+\theta\left[\alpha_{B,2}X_2+\gamma R_{B,1}-y_{B,2}\right]^+$, $\overline{y}_{A,2}:=y_{A,2}-\gamma R_{A,1}$, and $\overline{y}_{B,2}:=y_{B,2}-\gamma R_{B,1}$. Given firm i's first-period sales, $\overline{y}_{i,2}$ is uniquely determined by $y_{i,2}$. Thus, we use $\overline{y}_{A,2}$ and $\overline{y}_{B,2}$ as firm A's and firm B's second-period decision, respectively. Note that the profit function in (7) is concave in $\overline{y}_{A,2}$. By analogous arguments in the proof of Lemma 1, retailer A's best response in the second period is:

$$\overline{y}^*_{A,2}(\overline{y}_{B,2}) = \begin{cases}(\alpha_{A,2}+\theta\alpha_{B,2})\zeta^0_{A,2}-\theta\overline{y}_{B,2}, & \text{if } \overline{y}_{B,2}\le\alpha_{B,2}\zeta^0_{A,2},\\ \alpha_{A,2}\zeta^0_{A,2}, & \text{if } \overline{y}_{B,2}>\alpha_{B,2}\zeta^0_{A,2}.\end{cases} \quad (8)$$

On the other hand, given X_1 and retailers' first-period sales, retailer B maximizes its second-period profit:

$$\max_{y_{B,2}} \mathbb{E}\left[p_{B,2}(y_{B,2}\wedge D_{B,2})-c_By_{B,2}|X_1\right]$$
$$= \mathbb{E}\left[(p_{B,2}-c_B)y_{B,2} - p_{B,2}\left(y_{B,2}-\alpha_{B,2}X_2-\gamma R_{B,1}-\theta\left(\alpha_{A,2}X_2+\gamma R_{A,1}-y_{A,2}\right)^+\right)^+\middle|X_1\right]$$
$$= \mathbb{E}\left[(p_{B,2}-c_B)\overline{y}_{B,2} - p_{B,2}\left(\overline{y}_{B,2}-\alpha_{B,2}X_2-\theta\left(\alpha_{A,2}X_2-\overline{y}_{A,2}\right)^+\right)^+ + (p_{B,2}-c_B)\gamma R_{B,1}\middle|X_1\right], \quad (9)$$

where $D_{B,2}=\alpha_{B,2}X_2+\gamma R_{B,1}+\theta\left[\alpha_{A,2}X_2+\gamma R_{A,1}-y_{A,2}\right]^+$. Retailer B's profit function in (9) is concave in $\overline{y}_{B,2}$. By analogous arguments in the proof of Lemma 1, retailer B's best response in the second period is:

$$\overline{y}^*_{B,2}(\overline{y}_{A,2}) = \begin{cases}(\alpha_{B,2}+\theta\alpha_{A,2})\zeta^0_{B,2}-\theta\overline{y}_{A,2}, & \text{if } \overline{y}_{A,2}\le\alpha_{A,2}\zeta^0_{B,2},\\ \alpha_{B,2}\zeta^0_{B,2}, & \text{if } \overline{y}_{A,2}>\alpha_{A,2}\zeta^0_{B,2},\end{cases} \quad (10)$$

By combining (8) and (10), there exists a unique Nash equilibrium in the second period, $(\overline{y}^S_{A,2},\overline{y}^S_{B,2})$, where $\overline{y}^S_{A,2}=y^0_{A,2}$ and $\overline{y}^S_{B,2}=y^0_{B,2}$. That is, retailers' equilibrium order quantities are $y^S_{A,2}=y^0_{A,2}+\gamma R_{A,1}$ and $y^S_{B,2}=y^0_{B,2}+\gamma R_{B,1}$. Furthermore, by substituting $\overline{y}^S_{A,2}$ and $\overline{y}^S_{B,2}$ into (7) and (9), retailer A's second-period profit equals $\pi^0_{A,2}+(p_{A,2}-c_A)\gamma\mathbb{E}\left[R_{A,1}|X_1\right]$ and retailer B's second period profit equals $\pi^0_{B,2}+(p_{B,2}-c_B)\gamma\mathbb{E}\left[R_{B,1}|X_1\right]$.

Next, we study two retailers' competition in the first period. Firm A optimizes its order quantity to maximize its profit in two periods:

$$\max_{y_{A,1}} p_{A,1}\mathbb{E}\left[y_{A,1}\wedge D_{A,1}\right]-c_Ay_{A,1}+\rho(p_{A,2}-c_A)\gamma\mathbb{E}\left[y_{A,1}\wedge D_{A,1}\right]+\rho\pi^0_{A,2}$$
$$=\left[p_{A,1}-c_A+\rho(p_{A,2}-c_A)\gamma\right]y_{A,1}-\left[p_{A,1}+\rho(p_{A,2}-c_A)\gamma\right]\mathbb{E}\left[y_{A,1}-D_{A,1}\right]^+ +\rho\pi^0_{A,2}, \quad (11)$$

where $D_{A,1}=\alpha_{A,1}X_1+\theta\left[\alpha_{B,1}X_1-y_{B,1}\right]^+$. Note that profit function in (11) is concave in $y_{A,1}$ and it derivative w.r.t. $y_{A,1}$ is $p_{A,1}-c_A+\rho\gamma(p_{A,2}-c_A)-\left[p_{A,1}+\rho\gamma(p_{A,2}-c_A)\right]\mathbf{Prob}\left(D_{A,1}\le y_{A,1}\right)$, where $\mathbf{Prob}\left(D_{A,1}\le y_{A,1}\right)$ follows (4). Thus, by analogous arguments in the proof of Lemma 1, the best response of retailer A is

$$y^*_{A,1}(y_{B,1}) = \begin{cases}(\alpha_{A,1}+\theta\alpha_{B,1})\zeta^S_{A,1}-\theta y_{B,1}, & \text{if } y_{B,1}\le\alpha_{B,1}\zeta^S_{A,1},\\ \alpha_{A,1}\zeta^S_{A,1}, & \text{if } y_{B,1}>\alpha_{B,1}\zeta^S_{A,1},\end{cases} \quad (12)$$

where $\zeta^S_{A,1} := F^{-1}(1 - \frac{c_A}{p_{A,1}+\rho\gamma(p_{A,2}-c_A)})$. On the other hand, retailer B optimizes its order quantity to maximize its total profit in two periods:

$$\begin{aligned}&\max_{y_{B,1}} p_{B,1}\mathbb{E}\left[y_{B,1} \wedge D_{B,1}\right] - c_B y_{B,1} + \rho(p_{B,2}-c_B)\gamma\mathbb{E}\left[y_{B,1} \wedge D_{B,1}\right] + \rho\pi^0_{B,2} \\ &= [p_{B,1} - c_B + \rho(p_{B,2}-c_B)\gamma]y_{B,1} - [p_{B,1} + \rho(p_{B,2}-c_B)\gamma]\mathbb{E}\left[y_{B,1} - D_{B,1}\right]^+ + \rho\pi^0_{B,2},\end{aligned} \tag{13}$$

where $D_{B,1} = \alpha_{B,1}X_1 + \theta\left[\alpha_{A,1}X_1 - y_{A,1}\right]^+$. Note that the profit function in (13) is concave in $y_{B,1}$. By analogous arguments, retailer B's best response is

$$y^*_{B,1}(y_{A,1}) = \begin{cases}(\alpha_{B,1} + \theta\alpha_{A,1})\zeta^S_{B,1} - \theta y_{A,1}, & \text{if } y_{A,1} \le \alpha_{A,1}\zeta^S_{B,1}, \\ \alpha_{B,1}\zeta^S_{B,1}, & \text{if } y_{A,1} > \alpha_{A,1}\zeta^S_{B,1},\end{cases} \tag{14}$$

where $\zeta^S_{B,1} := F^{-1}(1 - \frac{c_B}{p_{B,1}+\rho\gamma(p_{B,2}-c_B)})$. Note that $\zeta^S_{i,1} > \zeta^0_{i,1}$ for $\gamma > 0$, when $\rho > 0$ and $p_{i,t} > c_i$, $i = A, B$ and $t = 1, 2$.

Since $p_{B,t}c_A/c_B \le p_{A,t}$, $t = 1, 2$, we have $\frac{c_A}{p_{A,1}+\rho\gamma(p_{A,2}-c_A)} \le \frac{c_B}{p_{B,1}+\rho\gamma(p_{B,2}-c_B)}$ and thus $\zeta^S_{A,1} \ge \zeta^S_{B,1}$. By combining (12) and (14), there exists a unique Nash equilibrium $(y^S_{A,1}, y^S_{B,1})$, where $y^S_{A,1} = (\alpha_{A,1} + \theta\alpha_{B,1})\zeta^S_{A,1} - \theta\alpha_{B,1}\zeta^S_{B,1}$, $y^S_{B,1} = \alpha_{B,1}\zeta^S_{B,1}$, and $\zeta^S_{i,1} := F^{-1}(1 - \frac{c_i}{p_{i,1}+\rho\gamma(p_{i,2}-c_i)})$, $i = A, B$.

Last, by substituting $y^S_{A,1}$ and $y^S_{B,1}$ into (11) and (13), we have retailers' equilibrium profits as follows: $\Pi^S_A = [p_{A,1} - c_A + \rho(p_{A,2}-c_A)\gamma]y^S_{A,1} - [p_{A,1} + \rho(p_{A,2}-c_A)\gamma]\mathbb{E}[L^S_{A,1}] + \rho\pi^0_{A,2}$ and $\Pi^S_B = [p_{B,1} - c_B + \rho(p_{B,2}-c_B)\gamma]y^S_{B,1} - [p_{B,1} + \rho(p_{B,2}-c_B)\gamma]\mathbb{E}[L^S_{B,1}] + \rho\pi^0_{B,2}$, where $L^S_{A,1} := [(\alpha_{A,1} + \theta\alpha_{B,1})\zeta^S_{A,1} - \theta\alpha_{B,1}\zeta^S_{B,1} - \alpha_{A,1}X_1 - \theta(\alpha_{B,1}X_1 - \alpha_{B,1}\zeta^S_{B,1})^+]^+$ and $L^S_{B,1} := [\alpha_{B,1}\zeta^S_{B,1} - \alpha_{B,1}X_1 - \theta\{\alpha_{A,1}X_1 - (\alpha_{A,1} + \theta\alpha_{B,1})\zeta^S_{A,1} + \theta\alpha_{B,1}\zeta^S_{B,1}\}^+]^+$. Furthermore, retailers' equilibrium first-period sales are $R^S_{A,1} := y^S_{A,1} - L^S_{A,1}$ and $R^S_{B,1} := y^S_{B,1} - L^S_{B,1}$. Retailers' second-period order quantities are $y^S_{A,2} = y^0_{A,2} + \gamma R^S_{A,1}$ and $y^S_{B,2} = y^0_{B,2} + \gamma R^S_{B,1}$. Thus, we have proved Lemma 2. *Q.E.D.*

Proof of Lemma 3: We first study the subgame in the second period. Retailers' demand is $D_{A,2} = \alpha_{A,2}X_2 + \alpha_{A,2}\gamma R_1 + \theta\left[\alpha_{B,2}X_2 + \alpha_{B,2}\gamma R_1 - y_{B,2}\right]^+$ and $D_{B,2} = \alpha_{B,2}X_2 + \alpha_{B,2}\gamma R_1 + \theta\left[\alpha_{A,2}X_2 + \alpha_{A,2}\gamma R_1 - y_{A,2}\right]^+$. Let $\overline{\overline{y}}_{A,2} := y_{A,2} - \alpha_{A,2}\gamma R_1$ and $\overline{\overline{y}}_{B,2} := y_{B,2} - \alpha_{B,2}\gamma R_1$, by analogous arguments in the proof of Lemma 2, we have retailer A's second-period best response, which has the same form as (8). Moreover, by analogous arguments in the proof of Lemma 2, we have retailer B's second-period best response, which has the same form as (10).

By combining two retailers' best responses, there exists a unique Nash equilibrium in the second period, $(\overline{\overline{y}}^N_{A,2}, \overline{\overline{y}}^N_{B,2})$, where $\overline{\overline{y}}^N_{A,2} = y^0_{A,2}$ and $\overline{\overline{y}}^N_{B,2} = y^0_{B,2}$. That is, retailer A's equilibrium order quantity is $y^N_{A,2} = y^0_{A,2} + \alpha_{A,2}\gamma R_1$ and $y^N_{B,2} = y^0_{B,2} + \alpha_{B,2}\gamma R_1$. Furthermore, by substituting $\overline{\overline{y}}^N_{A,2}$ and $\overline{\overline{y}}^N_{B,2}$, retailer A's second-period profit equals $\pi^0_{A,2} + (p_{A,2} - c_A)\alpha_{A,2}\gamma\mathbb{E}\left[R_1|X_1\right]$ and retailer B's second-period profit equals $\pi^0_{B,2} + (p_{B,2} - c_B)\alpha_{B,2}\gamma\mathbb{E}\left[R_1|X_1\right]$.

Next, we study two firms' competition in the first period. Retailer A's maximizes its profit in two periods:

$$\begin{aligned}&\max_{y_{A,1}} p_{A,1}\mathbb{E}\left[y_{A,1} \wedge D_{A,1}\right] - c_A y_{A,1} + \rho(p_{A,2}-c_A)\alpha_{A,2}\gamma\mathbb{E}\left[(y_{A,1} \wedge D_{A,1}) + (y_{B,1} \wedge D_{B,1})\right] + \rho\pi^0_{A,2} \\ &= [p_{A,1} - c_A + \rho(p_{A,2}-c_A)\alpha_{A,2}\gamma]y_{A,1} - [p_{A,1} + \rho(p_{A,2}-c_A)\alpha_{A,2}\gamma]\mathbb{E}\left[y_{A,1} - D_{A,1}\right]^+ \\ &\quad + \rho(p_{A,2}-c_A)\alpha_{A,2}\gamma y_{B,1} - \rho(p_{A,2}-c_A)\alpha_{A,2}\gamma\mathbb{E}\left[y_{B,1} - D_{B,1}\right]^+ + \rho\pi^0_{A,2},\end{aligned} \tag{15}$$

where $D_{i,1} = \alpha_{i,1}X_1 + \theta\left[\alpha_{j,1}X_1 - y_{j,1}\right]^+$, $j \neq i$ and $i, j = A, B$. The first-order derivative of the profit function in (15) w.r.t. $y_{A,1}$ is $p_{A,1} - c_A + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma - [p_{A,1} + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma]\mathbf{Prob}(D_{A,1} \leq y_{A,1}) - \rho(p_{A,1} - c_A)\alpha_{A,2}\gamma\frac{\partial\mathbb{E}[y_{B,1}-D_{B,1}]^+}{\partial y_{A,1}}$, where $\mathbf{Prob}(D_{A,1} \leq y_{A,1})$ is same to (4) and

$$\mathbb{E}\left(y_{B,1} - D_{B,1}\right)^+ = \int_0^{y_{A,1}/\alpha_{A,1}} (y_{B,1} - \alpha_{B,1}z)^+ f(z)dz + \int_{y_{A,1}/\alpha_{A,1}}^{+\infty} \left[y_{B,1} + \theta y_{A,1} - (\alpha_{B,1} + \theta\alpha_{A,1})z\right]^+ f(z)dz.$$

If $y_{A,1}/\alpha_{A,1} \geq y_{B,1}/\alpha_{B,1}$, $\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta\alpha_{B,1}} \in [\frac{y_{B,1}}{\alpha_{B,1}}, \frac{y_{A,1}}{\alpha_{A,1}}]$ and $\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}} \in [\frac{y_{B,1}}{\alpha_{B,1}}, \frac{y_{A,1}}{\alpha_{A,1}}]$ for $\theta \in [0,1]$. Thus, $\mathbf{Prob}\left(D_{A,1} \leq y_{A,1}\right) = F\left(\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta\alpha_{B,1}}\right)$ and $\frac{\partial\mathbb{E}[y_{B,1}-D_{B,1}]^+}{\partial y_{A,1}} = \frac{\partial}{\partial y_{A,1}}\int_0^{y_{B,1}/\alpha_{B,1}}(y_{B,1} - \alpha_{B,1}z)f(z)dz = 0$. If $y_{A,1}/\alpha_{A,1} < y_{B,1}/\alpha_{B,1}$, $\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta\alpha_{B,1}} \in [\frac{y_{A,1}}{\alpha_{A,1}}, \frac{y_{B,1}}{\alpha_{B,1}}]$ and $\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}} \in [\frac{y_{A,1}}{\alpha_{A,1}}, \frac{y_{B,1}}{\alpha_{B,1}}]$ for $\theta \in [0,1]$. Thus, $\mathbf{Prob}(D_{A,1} \leq y_{A,1}) = F\left(\frac{y_{A,1}}{\alpha_{A,1}}\right)$ and $\frac{\partial\mathbb{E}[y_{B,1}-D_{B,1}]^+}{\partial y_{A,1}} = \theta\left[F\left(\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}}\right) - F\left(\frac{y_{A,1}}{\alpha_{A,1}}\right)\right]$. Therefore, if $y_{A,1}/\alpha_{A,1} \geq y_{B,1}/\alpha_{B,1}$, the first-order derivative of retailer A's profit function in (15) is $p_{A,1} - c_A + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma - [p_{A,1} + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma]F\left(\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta\alpha_{B,1}}\right)$, and its second-order derivative is $-\frac{1}{\alpha_{A,1}+\theta\alpha_{B,1}}[p_{A,1} + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma]f\left(\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta\alpha_{B,1}}\right)$. If $y_{A,1}/\alpha_{A,1} < y_{B,1}/\alpha_{B,1}$, the first-order derivative of retailer A's profit function in (15) is $p_{A,1} - c_A + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma - [p_{A,1} + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma(1-\theta)]F\left(\frac{y_{A,1}}{\alpha_{A,1}}\right) - \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma\theta F\left(\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}}\right)$, and its second-order derivative is $-\frac{1}{\alpha_{A,1}}[p_{A,1} + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma(1-\theta)]f\left(\frac{y_{A,1}}{\alpha_{A,1}}\right) - \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma\frac{\theta^2}{\alpha_{B,1}+\theta\alpha_{A,1}}f\left(\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}}\right)$. Therefore, given $y_{B,1} \geq 0$, retailer A's profit function in (15) has two concave pieces for $y_{A,1} \geq y_{B,1}\alpha_{A,1}/\alpha_{B,1}$ and $y_{A,1} < y_{B,1}\alpha_{A,1}/\alpha_{B,1}$. Furthermore, the first-order derivative of retailer A's profit function is continuous at $y_{A,1} = y_{B,1}\alpha_{A,1}/\alpha_{B,1}$. Thus, retailer A's profit function is concave in $y_{A,1}$ for $y_{A,1} \geq 0$. Therefore, retailer A's best response is as follows: If $y_{B,1} \leq \alpha_{B,1}\zeta^N_{A,1}$, where $\zeta^N_{A,1} := F^{-1}\left(1 - \frac{c_A}{p_{A,1}+\rho\alpha_{A,2}\gamma(p_{A,2}-c_A)}\right)$, $y^*_{A,1}(y_{B,1}) = (\alpha_{A,1} + \theta\alpha_{B,1})\zeta^N_{A,1} - \theta y_{B,1}$; If $y_{B,1} > \alpha_{B,1}\zeta^N_{A,1}$, the following equation,

$$\begin{aligned} p_{A,1} - c_A + \rho(p_{A,2} - c_A)\gamma\alpha_{A,2} = {} & [p_{A,1} + \rho(p_{A,2} - c_A)\gamma\alpha_{A,2}(1-\theta)]F\left(\frac{y^*_{A,1}(y_{B,1})}{\alpha_{A,1}}\right) \\ & + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma\theta F\left(\frac{y_{B,1} + \theta y^*_{A,1}(y_{B,1})}{\alpha_{B,1} + \theta\alpha_{A,1}}\right), \end{aligned} \tag{16}$$

implicitly characterizes $y^*_{A,1}(y_{B,1})$. By the above equation, $y^*_{A,1}(y_{B,1})$ is decreasing in $y_{B,1}$ and goes to $\alpha_{A,1}F^{-1}\left(1 - \frac{c_A}{p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma(1-\theta)}\right)$ as $y_{B,1} \to +\infty$.

On the other hand, retailer B maximizes its profits in two periods:

$$\begin{aligned} & \max_{y_{B,1}} p_{B,1}\mathbb{E}\left[y_{B,1} \wedge D_{B,1}\right] - c_B y_{B,1} + \rho(p_{B,2} - c_B)\alpha_{B,2}\gamma\mathbb{E}\left[(y_{A,1} \wedge D_{A,1}) + (y_{B,1} \wedge D_{B,1})\right] + \rho\pi^0_{B,2} \\ = {} & [p_{B,1} - c_B + \rho(p_{B,2} - c_B)\alpha_{B,2}\gamma]y_{B,1} - [p_{B,1} + \rho(p_{B,2} - c_B)\alpha_{B,2}\gamma]\mathbb{E}\left[y_{B,1} - D_{B,1}\right]^+ \\ & + \rho(p_{B,2} - c_B)\alpha_{B,2}\gamma y_{A,1} - \rho(p_{B,2} - c_B)\alpha_{B,2}\gamma\mathbb{E}\left[y_{A,1} - D_{A,1}\right]^+ + \rho\pi^0_{B,2}. \end{aligned} \tag{17}$$

By analogous arguments for retailer A's problem, the best response of retailer B is as follows: If $y_{A,1} < \alpha_{A,1}F^{-1}\left(1 - \frac{c_B}{p_{B,1}+\rho\alpha_{B,2}\gamma(p_{B,2}-c_B)}\right)$, $y^*_{B,1}(y_{A,1}) = (\alpha_{B,1} + \theta\alpha_{A,1})F^{-1}\left(1 - \frac{c_B}{p_{B,1}+\rho\alpha_{B,2}\gamma(p_{B,2}-c_B)}\right) - \theta y_{A,1}$; If $y_{A,1} \geq \alpha_{A,1}F^{-1}\left(1 - \frac{c_B}{p_{B,1}+\rho\alpha_{B,2}\gamma(p_{B,2}-c_B)}\right)$, the following equation

$$p_{B,1} - c_B + \rho(p_{B,2} - c_B)\alpha_{B,2}\gamma = [p_{B,1} + \rho(p_{B,2} - c_B)\alpha_{B,2}\gamma(1-\theta)]F\left(\frac{y^*_{B,1}(y_{A,1})}{\alpha_{B,1}}\right)$$

$$+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma\theta F\left(\frac{y_{A,1}+\theta y^*_{B,1}(y_{A,1})}{\alpha_{A,1}+\theta\alpha_{B,1}}\right), \tag{18}$$

implicitly characterizes $y^*_{B,1}(y_{A,1})$. By the above equation, $y^*_{B,1}(y_{A,1})$ is decreasing in $y_{A,1}$ and goes to $\alpha_{B,1}F^{-1}\left(1-\frac{c_B}{p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma(1-\theta)}\right)$ as $y_{A,1}\to+\infty$.

We show that there exists a unique Nash equilibrium of the two-period game, $(y^N_{A,1},y^N_{B,1})$, in area $\{(y_{A,1},y_{B,1}):\frac{y_{A,1}}{\alpha_{A,1}}\geq\frac{y_{B,1}}{\alpha_{B,1}}\}$: We first show $\zeta^N_{A,1}=F^{-1}\left(1-\frac{c_A}{p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma}\right)\geq F^{-1}\left(1-\frac{c_B}{p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma}\right)$: Since $\frac{c_A}{c_B}p_{B,t}\leq p_{A,t}$ and $\alpha_{B,t}\leq\alpha_{A,t}$, we have $\frac{c_A}{c_B}(p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma)\leq p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{B,2}\gamma\leq p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma$, which leads to the above inequality. Next, we then show that there exists no equilibrium in area $\{(y_{A,1},y_{B,1}):\frac{y_{A,1}}{\alpha_{A,1}}<\frac{y_{B,1}}{\alpha_{B,1}}\}$, i.e., two firms' best responses do not intersect each other in the area. We assume, to the contrary, there exists an equilibrium in area $\{(y_{A,1},y_{B,1}):\frac{y_{A,1}}{\alpha_{A,1}}<\frac{y_{B,1}}{\alpha_{B,1}}\}$. Thus, it must satisfy the following system:

$$\begin{cases} p_{A,1}-c_A+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma=[p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma(1-\theta)]F\left(\frac{y_{A,1}}{\alpha_{A,1}}\right) \\ \qquad\qquad\qquad +\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma\theta F\left(\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}}\right), \\ p_{B,1}-c_B+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma=[p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma]F\left(\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}}\right). \end{cases} \tag{19}$$

Since $\frac{y_{A,1}}{\alpha_{A,1}}<\frac{y_{B,1}}{\alpha_{B,1}}$, $\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}}>\frac{y_{A,1}}{\alpha_{A,1}}$ for $\theta\in[0,1]$. Thus, $F\left(\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}}\right)>F\left(\frac{y_{A,1}}{\alpha_{A,1}}\right)$ for $\theta\in[0,1]$. Thus, by the first equation in (19), we have $1-\frac{c_A}{p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma}<F\left(\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}}\right)$. By the second equation in (19), we have $F\left(\frac{y_{B,1}+\theta y_{A,1}}{\alpha_{B,1}+\theta\alpha_{A,1}}\right)=1-\frac{c_B}{p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma}$ and thus $F^{-1}\left(1-\frac{c_A}{p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma}\right)<F^{-1}\left(1-\frac{c_B}{p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma}\right)$, which leads to a contradiction. Therefore, there exists no equilibrium in area $\{(y_{A,1},y_{B,1}):\frac{y_{A,1}}{\alpha_{A,1}}<\frac{y_{B,1}}{\alpha_{B,1}}\}$. Next, by the above arguments, both retailers' best responses are decreasing in $y_{B,1}$ in area $\{(y_{A,1},y_{B,1}):\frac{y_{A,1}}{\alpha_{A,1}}\geq\frac{y_{B,1}}{\alpha_{B,1}}\}$, where retailer A's best response drops from $(\alpha_{A,1}+\theta\alpha_{B,1})\zeta^N_{A,1}$ to $\alpha_{A,1}\zeta^N_{A,1}$ on interval $[0,\alpha_{B,1}\zeta^N_{A,1}]$ and retailer B's best response drops from infinity to $\alpha_{B,1}F^{-1}\left(1-\frac{c_B}{p_{B,1}+\rho\alpha_{B,2}\gamma(p_{B,2}-c_B)}\right)$ on interval $\left(\alpha_{B,1}F^{-1}\left(1-\frac{c_B}{p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma(1-\theta)}\right),\alpha_{B,1}F^{-1}\left(1-\frac{c_B}{p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma}\right)\right]$. Therefore, there exists a unique equilibrium, $(y^N_{A,1},y^N_{B,1})$, which solves the following system:

$$\begin{cases} p_{A,1}-c_A+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma=[p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma]F\left(\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta\alpha_{B,1}}\right) \\ p_{B,1}-c_B+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma=[p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma(1-\theta)]F\left(\frac{y_{B,1}}{\alpha_{B,1}}\right) \\ \qquad\qquad\qquad +\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma\theta F\left(\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta\alpha_{B,1}}\right). \end{cases} \tag{20}$$

That is, $y^N_{A,1}=(\alpha_{A,1}+\theta\alpha_{B,1})\zeta^N_{A,1}-\theta\alpha_{B,1}\zeta^N_{B,1}$ and $y^N_{B,1}=\alpha_{B,1}\zeta^N_{B,1}$, where

$$\zeta^N_{A,1}=F^{-1}\left(1-\frac{c_A}{p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma}\right) \text{ and } \zeta^N_{B,1}=F^{-1}\left(1-\frac{c_B-\rho\alpha_{B,2}\gamma\theta(p_{B,2}-c_B)\frac{c_A}{p_{A,1}+\rho\alpha_{A,2}\gamma(p_{A,2}-c_A)}}{p_{B,1}+\rho\alpha_{B,2}\gamma(1-\theta)(p_{B,2}-c_B)}\right).$$

Moreover, by the definition, we have $\zeta^N_{A,1}<\zeta^S_{A,1}$.

Last, by substituting $y^N_{A,1}$ and $y^N_{B,1}$ into (15) and (17), retailers' equilibrium profits are $\Pi^N_A:=[p_{A,1}-c_A+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma]y^N_{A,1}-[p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma]\mathbb{E}[L^N_{A,1}]+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma y^N_{B,1}-\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma\mathbb{E}[L^N_{B,1}]+\rho\pi^0_{A,2}$ and $\Pi^N_B:=[p_{B,1}-c_B+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma]y^N_{B,1}-[p_{B,1}+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma]\mathbb{E}[L^N_{B,1}]+\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma y^N_{A,1}-\rho(p_{B,2}-c_B)\alpha_{B,2}\gamma\mathbb{E}[L^N_{A,1}]+\rho\pi^0_{B,2}$, where $L^N_{A,1}:=[(\alpha_{A,1}+\theta\alpha_{B,1})\zeta^N_{A,1}-\theta\alpha_{B,1}\zeta^N_{B,1}-\alpha_{A,1}X_1-\theta(\alpha_{B,1}X_1-\alpha_{B,1}\zeta^N_{B,1})^+]^+$ and $L^N_{B,1}:=[\alpha_{B,1}\zeta^N_{B,1}-\alpha_{B,1}X_1-\theta\left\{\alpha_{A,1}X_1-(\alpha_{A,1}+\theta\alpha_{B,1})\zeta^N_{A,1}+\theta\alpha_{B,1}\zeta^N_{B,1}\right\}^+]^+$.

Furthermore, retailers' equilibrium first-period sales are $R^N_{A,1} := y^N_{A,1} - L^N_{A,1}$, $R^N_{B,1} := y^N_{B,1} - L^N_{B,1}$, and $R^N_1 := R^N_{A,1} + R^N_{B,1}$. Retailers' second-period equilibrium order quantities are $y^N_{A,2} = y^0_{A,1} + \alpha_{A,1}\gamma R^N_1$ and $y^S_{B,2} = y^0_{B,1} + \alpha_{B,1}\gamma R^N_1$. Thus, we have proved Lemma 3. *Q.E.D.*

Proof of Lemma 4: (a). By the proof of Lemma 1, $\frac{y^0_{A,t}}{\alpha_{A,t}} - \frac{y^0_{B,t}}{\alpha_{B,t}} = \frac{1}{\alpha_{A,t}}(\alpha_{A,t} + \theta\alpha_{B,t})(\zeta^0_{A,t} - \zeta^0_{B,t}) \geq 0$, $t = 1, 2$, for $\theta \in [0, 1]$.

(b). We first study Scenario S. By the proof of Lemma 2, $\frac{y^S_{A,1}}{\alpha_{A,1}} - \frac{y^S_{B,1}}{\alpha_{B,1}} = \frac{1}{\alpha_{A,1}}(\alpha_{A,1} + \theta\alpha_{B,1})(\zeta^S_{A,1} - \zeta^S_{B,1}) \geq 0$ for $\theta \in [0, 1]$. Furthermore, by part (a) and Lemma 2, $\frac{y^S_{A,2} - Z^S_{A,2}}{\alpha_{A,2}} = \frac{y^0_{A,2}}{\alpha_{A,2}} \geq \frac{y^0_{B,2}}{\alpha_{B,2}} = \frac{y^S_{B,2} - Z^S_{B,2}}{\alpha_{B,2}}$, for $\theta \in [0, 1]$.

Second, we study Scenario N. By the proof of Lemma 3, we have $\frac{y^N_{A,1}}{\alpha_{A,1}} \geq \frac{y^N_{B,1}}{\alpha_{B,1}}$. Furthermore, by analogous arguments, we have $\frac{y^N_{A,2} - Z^N_{A,2}}{\alpha_{A,2}} \geq \frac{y^N_{B,2} - Z^N_{B,2}}{\alpha_{B,2}}$. *Q.E.D.*

Proof of Proposition 1: By Lemma 1 and Lemma 2, for $k = 0, S$, $y^k_{A,1} = \alpha_{A,1}\zeta^k_{A,1} + \theta\alpha_{B,1}(\zeta^k_{A,1} - \zeta^k_{B,1})$. Since $p_{B,1}c_A/c_B \leq p_{A,1}$ and $\zeta^k_{A,1} \geq \zeta^k_{B,1}$, $y^k_{A,1}$ is increasing in $\theta \in [0, 1]$.

By Lemma 3, $y^N_{A,1} = \alpha_{A,1}\zeta^N_{A,1} + \theta\alpha_{B,1}(\zeta^N_{A,1} - \zeta^N_{B,1})$. Thus, $\frac{\partial y^N_{A,1}}{\partial\theta} = \alpha_{B,1}(\zeta^N_A - \zeta^N_B) - \theta\alpha_{B,1}\frac{\partial\zeta^N_{B,1}}{\partial\theta}$. Note that, we have

$$\begin{aligned}\frac{\partial\zeta^N_{B,1}}{\partial\theta} &= \frac{\partial}{\partial\theta}F^{-1}\left(1 - \frac{c_B - \rho\alpha_{B,2}\gamma\theta(p_{B,2} - c_B)\frac{c_A}{p_{A,1} + \rho\alpha_{A,2}\gamma(p_{A,2} - c_A)}}{p_{B,1} + \rho\alpha_{B,2}\gamma(1-\theta)(p_{B,2} - c_B)}\right) \\ &= -\frac{1}{f\left(1 - \frac{c_B - \rho\alpha_{B,2}\gamma\theta(p_{B,2} - c_B)\frac{c_A}{p_{A,1} + \rho\alpha_{A,2}\gamma(p_{A,2} - c_A)}}{p_{B,1} + \rho\alpha_{B,2}\gamma(1-\theta)(p_{B,2} - c_B)}\right)}\frac{\partial}{\partial\theta}\frac{c_B - \rho\alpha_{B,2}\gamma\theta(p_{B,2} - c_B)\frac{c_A}{p_{A,1} + \rho\alpha_{A,2}\gamma(p_{A,2} - c_A)}}{p_{B,1} + \rho\alpha_{B,2}\gamma(1-\theta)(p_{B,2} - c_B)}. \quad (21)\end{aligned}$$

One can check

$$\frac{\partial}{\partial\theta}\frac{c_B - \rho\alpha_{B,2}\gamma\theta(p_{B,2} - c_B)\frac{c_A}{p_{A,1} + \rho\alpha_{A,2}\gamma(p_{A,2} - c_A)}}{p_{B,1} + \rho\alpha_{B,2}\gamma(1-\theta)(p_{B,2} - c_B)} \geq 0 \quad (22)$$

for $\theta \in [0, 1]$, so, we have $\frac{\partial\zeta^N_{B,1}}{\partial\theta} \leq 0$. Thus, $\frac{\partial y^N_{A,1}}{\partial\theta} \geq 0$ and $y^N_{A,1}$ is increasing in $\theta \in [0, 1]$.

Next, we compare $y^S_{A,1}$ and $y^N_{A,1}$. We already show that $y^S_{A,1}$ and $y^N_{A,1}$ are increasing in θ. Furthermore, if $\theta = 0$, we have $y^N_{A,1} - y^S_{A,1} = -\frac{2\alpha_{A,1}\alpha_{B,2}\gamma c_A\sigma(p_{A,2} - c_A)}{(\gamma(p_{A,2} - c_A) + p_{A,1})(\alpha_{A,2}\gamma(p_{A,2} - c_A) + p_{A,1})} < 0$. If $\theta = 1$, we have $y^N_{A,1} - y^S_{A,1} = \frac{2\gamma\sigma}{p_{B,1}(-\gamma c_A + p_{A,1} + \gamma p_{A,2})(-\gamma c_B + p_{B,1} + \gamma p_{B,2})(-\alpha_{A,2}\gamma c_A + p_{A,1} + \alpha_{A,2}\gamma p_{A,2})}g^{SN}_A(\gamma)$, where $g^{SN}_A(\gamma)$ is quadratic function of γ and has the same sign as $y^N_{A,1} - y^S_{A,1}$. Since $\frac{\partial^2 g^{SN}_A(\gamma)}{\partial\gamma^2} = -2\alpha_{B,1}(p_{A,2} - c_A)(p_{B,2} - c_B)(c_A((\alpha_{A,2} - \alpha_{B,2})c_B - \alpha_{A,2}p_{B,2} + p_{B,2}) - \alpha_{A,2}c_Bp_{A,2}) > 0$, thus for large γ, $g^{SN}_A(\gamma) > 0$ and $y^N_{A,1} > y^S_{A,1}$ at $\theta = 1$. By the above arguments, there exists a $\underline{\theta}^{SN} \in [0, 1]$ such that $y^S_{A,1} \geq y^N_{A,1}$ for $\theta \in [0, \underline{\theta}^{SN}]$ and $y^S_{A,1} \leq y^N_{A,1}$ for $\theta \in [\underline{\theta}^{SN}, 1]$. *Q.E.D.*

Proof of Proposition 2: (a). By Lemma 1 and Lemma 2, $y^0_{B,1} = \alpha_{B,1}\zeta^0_{B,1}$ and $y^S_{B,1} = \alpha_{B,1}\zeta^S_{B,1}$ are both independent of θ. By Lemma 3, $y^N_{B,1} = \alpha_{B,1}\zeta^N_{B,1}$. By (21) and (22), we have $\frac{\partial y^N_{B,1}}{\partial\theta} = \alpha_{B,1}\frac{\partial\zeta^N_{B,1}}{\partial\theta} \leq 0$. Thus, $y^N_{B,1}$ is decreasing in θ.

(b). When $\gamma > 0$, $y^0_{B,1} < y^N_{B,1} < y^S_{B,1}$ holds if and only if

$$\frac{c_B}{p_{B,1}} > \frac{c_B - \rho\alpha_{B,2}\gamma\theta(p_{B,2} - c_B)\frac{c_A}{p_{A,1} + \rho\alpha_{A,2}\gamma(p_{A,2} - c_A)}}{p_{B,1} + \rho\alpha_{B,2}\gamma(1-\theta)(p_{B,2} - c_B)} > \frac{c_B}{p_{B,1} + \rho\gamma(p_{B,2} - c_B)} \quad (23)$$

for $\theta \in [0,1]$. By (22), we have

$$\frac{c_B - \rho\alpha_{B,2}\gamma(p - c_B)\frac{c_A}{p_{A,1}+\rho\alpha_{A,2}\gamma(p_{A,2}-c_A)}}{p_{B,1}} \geq \frac{c_B - \rho\alpha_{B,2}\gamma\theta(p_{B,2} - c_B)\frac{c_A}{p_{A,1}+\rho\alpha_{A,2}\gamma(p_{A,2}-c_A)}}{p_{B,1} + \rho\alpha_{B,2}\gamma(1-\theta)(p_{B,2} - c_B)} \geq \frac{c_B}{p_{B,1} + \rho\alpha_{B,2}\gamma(p_{B,2} - c_B)}.$$

Furthermore, since

$$\frac{c_B}{p_{B,1}} > \frac{c_B - \rho\alpha_{B,2}\gamma(p - c_B)\frac{c_A}{p_{A,1}+\rho\alpha_{A,2}\gamma(p_{A,2}-c_A)}}{p_{B,1}} \text{ and } \frac{c_B}{p_{B,1} + \rho\alpha_{B,2}\gamma(p_{B,2} - c_B)} > \frac{c_B}{p_{B,1} + \rho\gamma(p_{B,2} - c_B)},$$

(23) holds for all $\theta \in [0,1]$.

(c). By Lemma 2 and Lemma 3, we have $\frac{\partial}{\partial\gamma}(y^S_{B,1} - y^N_{B,1}) = 2(p_{B,2} - c_B)\alpha_{B,1}\sigma g^{SN}_B(\gamma)/[(\gamma c_B - p_{B,1} - \gamma p_{B,2})^2(\alpha_{A,2}\gamma c_A - p_{A,1} - \alpha_{A,2}\gamma p_{A,2})^2(\alpha_{A,2}\gamma c_B\theta - \alpha_{A,2}\gamma c_B - \gamma c_B\theta + \gamma c_B - p_{B,1} - \alpha\gamma\theta p_{B,2} + \alpha_{A,2}\gamma p_{B,2} + \gamma\theta p_{B,2} - \gamma p_{B,2})^2]$, where $g^{SN}_B(\gamma) = \sum_{j=0}^4 a^{SN}_j \gamma^j$ with $a^{SN}_4 \leq 0$, $a^{SN}_3 \leq 0$, $a^{SN}_1 \geq 0$, and $a^{SN}_0 \geq 0$. By Descartes' Rule of Signs, $g^{SN}_B(\gamma)$ has only one positive real root for $\gamma \geq 0$. So, $g^{SN}_B(\gamma)$ is first positive then negative, as γ increases. Hence, $y^S_{B,1} - y^N_{B,1}$ first increases then decreases in γ. *Q.E.D.*

Proof of Proposition 3: (a). By Lemma 2, we have the closed form of $\mathbb{E}[R^S_{A,1}]$. Since $\frac{\partial\mathbb{E}[R^S_{A,1}]}{\partial\theta} = \frac{\alpha_{B,1}\sigma}{(\gamma(p_{A,2}-c_A)+p_{A,1})^2(\gamma(p_{B,2}-c_B)+p_{B,1})^2} g^S_{R,1}(\gamma)$, where $g^S_{R,1}(\gamma)$ is quadratic function of γ and $\frac{\partial^2 g^S_{R,1}(\gamma)}{\partial\gamma^2} > 0$. Thus, when γ is large, $\mathbb{E}[R^S_{A,1}]$ is increasing in θ. Moreover, by Lemma 3, we have the closed form of $\mathbb{E}[R^N_{A,1}]$. Similarly, $\frac{\partial\mathbb{E}[R^N_{A,1}]}{\partial\theta}$ can also be shown to be positive and $\mathbb{E}[R^N_{A,1}]$ is increasing in θ, when γ is large.

If $\theta = 0$, we have $\mathbb{E}[R^S_{A,1}] - \mathbb{E}[R^N_{A,1}] = \frac{\alpha_{A,1}\alpha_{B,2}\gamma c_A^2\sigma(p_{A,2}-c_A)((\alpha_{A,2}+1)\gamma(p_{A,2}-c_A)+2p_{A,1})}{(\gamma(p_{A,2}-c_A)+p_{A,1})^2(\alpha_{A,2}\gamma(p_{A,2}-c_A)+p_{A,1})^2} > 0$. If $\theta = 1$, we have $\mathbb{E}[R^S_{A,1}] - \mathbb{E}[R^N_{A,1}] = \frac{\gamma\sigma}{p^2_{B,1}(\gamma c_A - p_{A,1} - \gamma p_{A,2})^2(\gamma c_B - p_{B,1} - \gamma p_{B,2})^2(\alpha_{A,2}\gamma c_A - p_{A,1} - \alpha_{A,2}\gamma p_{A,2})^2} g^{SN}_R(\gamma)$, where $g^{SN}_R(\gamma)$ is a function of γ with $\frac{\partial^5 g^{SN}_R(\gamma)}{\partial\gamma^5} < 0$. Thus, since γ is large, $g^{SN}_R(\gamma) < 0$ and thus $\mathbb{E}[R^S_{A,1}] < \mathbb{E}[R^N_{A,1}]$ at $\theta = 1$. Therefore, there exists a $\underline{\theta}' \in [0,1]$ such that $\mathbb{E}[R^S_{A,1}] \geq \mathbb{E}[R^N_{A,1}]$ for $\theta \in [0, \underline{\theta}']$ and $\mathbb{E}[R^S_{A,1}] < \mathbb{E}[R^N_{A,1}]$ for $\theta \in [\underline{\theta}', 1]$.

(b). Since $R^k_{B,1} = \min\{\alpha_{B,1}X_1, y^k_{B,1}\}$, part (b) follows Proposition 2. *Q.E.D.*

Proof of Proposition 4: By Lemma 3, we have the closed form of $\mathbb{E}[R^N_1]$. Moreover, $\frac{\partial}{\partial\theta}\mathbb{E}[R^N_1] = \frac{-\alpha_{B,1}[c_A(p_{B,1}-(\alpha_{A,2}-1)\gamma p_{B,2}) - c_B(\gamma(-2\alpha_{A,2}c_A + c_A + \alpha_{A,2}p_{A,2}) + p_{A,1})]\sigma}{(\alpha_{A,2}\gamma c_A - p_{A,1} - \alpha_{A,2}\gamma p_{A,2})^2(\alpha_{B,2}\gamma(1-\theta)(p_{B,2}-c_B)+p_{B,1})^3} g^N_R(\theta)$, where $g^N_R(\theta)$ has the same sign as $\frac{\partial}{\partial\theta}\mathbb{E}[R^N_1]$ and is a linear function of θ. Note that, $g^N_R(0)$ is a quadratic function of γ with $\frac{\partial^2 g^N_R(\theta)}{\partial\gamma^2} < 0$. Thus, there exists a $\gamma^R > 0$ such that $g^N_R(0) < 0$ if $\gamma > \gamma^R$. Moreover, $g^N_R(1) = p_{B,1}(c_A p_{B,1} + c_B p_{A,1}) + \gamma p_{B,1}(c_B(-2\alpha_{A,2}c_A + c_A + \alpha_{A,2}p_{A,2}) + (\alpha_{A,2} - 1)c_A p_{B,2}) > 0$ for all $\gamma > 0$. Thus, if $\gamma > \gamma^R$, $\frac{\partial}{\partial\theta}\mathbb{E}[R^N_1]$ is first negative then positive and $\mathbb{E}[R^N_1]$ first decreases then increases in θ for $\theta \in [0,1]$. *Q.E.D.*

Proof of Proposition 5: (a). By Lemma 2&3, we have the closed forms of Π^S_A and Π^N_A. Furthermore, we have $\frac{\partial\Pi^S_A}{\partial\theta} = \frac{\sigma}{p_{A,2}p^2_{B,2}(-\gamma c_A + p_{A,1} + \gamma p_{A,2})(-\gamma c_B + p_{B,1} + \gamma p_{B,2})^2} h^S_A(\gamma)$, where $h^S_A(\gamma)$ is cubic function of γ. Note that $\frac{\partial^3 h^S_A(\gamma)}{\partial\gamma^3} = 6\alpha_{B,2}(p_{A,2} - c_A)(p_{B,2} - c_B)^2(c_B p_{A,2} - c_A p_{B,2})^2 > 0$. Thus, there exits a $\gamma^A_1 > 0$ such that $h^S_A(\gamma) \geq 0$ for $\gamma \geq \gamma^A_1$. So $\frac{\partial\Pi^S_A}{\partial\theta} \geq 0$ and Π^S_A is increasing in θ, when $\gamma \geq \gamma^A_1$.

(b). $\Pi^S_A - \Pi^0_A = \gamma h^{S0}_A(\gamma)/[p_{A,1}p^2_{B,1}(-\gamma c_A + p_{A,1} + \gamma p_{A,2})(\gamma c_B - p_{B,1} - \gamma p_{B,2})^2]$, where $h^{S0}_A(\gamma)$ has the same sign as $\Pi^S_A - \Pi^0_A$ and $\frac{\partial^3 h^{S0}_A(\gamma)}{\partial\gamma^3} = 6\alpha_{A,1}\mu p_{A,1}p^2_{B,1}(p_{A,2} - c_A)^2(p_{B,2} - c_B)^2 > 0$. Moreover, we have $\Pi^N_A -$

$\Pi_A^0 = \gamma h_A^{N0}(\gamma)/[p_{A,1}p_{B,1}^2(\alpha_{A,2}\gamma(p_{A,2}-c_A)+p_{A,1})^2(\alpha_{B,2}\gamma(1-\theta)(p_{B,2}-c_B)+p_{B,1})^2]$, where $h_A^{N0}(\gamma)$ has the same sign as $\Pi_A^N-\Pi_A^0$ and $\frac{\partial^4 h_A^{N0}(\gamma)}{\partial\gamma^4} = 24\alpha_{B,2}^2\alpha_{A,2}^3(1-\theta)^2\mu p_{A,1}p_{B,1}^2(p_{A,2}-c_A)^3(p_{B,2}-c_B)^2>0$. Thus, there exists a $\gamma_2^A>0$ such that $\Pi_A^S>\Pi_A^0$ and $\Pi_A^N>\Pi_A^0$ if $\gamma>\gamma_2^A$.

(c1-c2). Note that $\Pi_A^S-\Pi_A^N = \gamma h_A(\theta,\gamma)/[(-\gamma c_A+p_{A,1}+\gamma p_{A,2})(\gamma c_B-p_{B,1}-\gamma p_{B,2})^2(-\alpha_{A,2}\gamma c_A+p_{A,1}+\alpha_{A,2}\gamma p_{A,2})^2(\alpha_{A,2}\gamma c_B\theta-\alpha_{A,2}\gamma c_B-\gamma c_B\theta+\gamma c_B-p_{B,1}-\alpha_{A,2}\gamma\theta p_{B,2}+\alpha_{A,2}\gamma p_{B,2}+\gamma\theta p_{B,2}-\gamma p_{B,2})^2]$, where $h_A(\theta,\gamma)$ is a cubic function of θ and has the same sign as $\Pi_A^S-\Pi_A^N$. Note that, if $\theta<1$, $\frac{\partial^7 h_A(\theta,\gamma)}{\partial\gamma^7} = 5040\alpha_{B,2}^2\alpha_{A,2}^2(1-\theta)^2\mu(\alpha_{A,1}-\alpha_{A,2})(c_A-p_{A,2})^4(c_B-p_{B,2})^4$. If $\theta=1$, $\frac{\partial^5 h_A(1,\gamma)}{\partial\gamma^5} = 120\alpha_{A,2}^2\mu p_{B,1}^2(\alpha_{A,1}-\alpha_{A,2})(c_A-p_{A,2})^4(c_B-p_{B,2})^2$. Thus, there exists a $\gamma_3^A>0$, $h_A(\theta,\gamma)$ and $\Pi_A^S-\Pi_A^N$ have the same sign as $\alpha_{A,1}-\alpha_{A,2}$ for $\alpha_{A,1}\neq\alpha_{A,2}$, if $\gamma\geq\gamma_3^A$.

Next, we study the case $\alpha_{A,1}=\alpha_{A,2}$. In this case, we first show that $\Pi_A^S>\Pi_A^N$ at $\theta=0$. If $\theta=0$, $h_A(0,\gamma)$ is a quadratic function of γ with $\frac{\partial^2 h_A(0,\gamma)}{\partial\gamma^2}>0$. On the other hand, at $\theta=1$, $h_A(1,\gamma)$ is a quartic function of γ and $\frac{\partial^4 h_A(1,\gamma)}{\partial\gamma^4}$ has the same sign as $-2\alpha_{A,1}c_A^2p_{B,1}+\alpha_{B,1}c_Ap_{A,1}p_{B,2}+2\alpha_{A,1}c_Ap_{A,2}p_{B,1}-c_Bp_{A,1}(\alpha_{A,1}p_{A,2}-(2\alpha_{A,1}-1)c_A)$. Note that, since $(\alpha_{A,1}p_{A,2}-(2\alpha_{A,1}-1)c_A)>0$, we have $\frac{\partial^4 h_A(1,\gamma)}{\partial\gamma^4}<0$ if $c_B>\bar{c}_B := [-2\alpha_{A,1}c_A^2p_{B,1}+\alpha_{B,1}c_Ap_{A,1}p_{B,2}+2\alpha_{A,1}c_Ap_{A,2}p_{B,1}]/[p_{A,1}(\alpha_{A,1}p_{A,2}-(2\alpha_{A,1}-1)c_A)]$. Therefore, there exists a $\gamma_4^A>0$ and $\Delta^A>0$ such that $h_A(0,\gamma)>0$ and $h_A(1,\gamma)<0$ if $\gamma\geq\gamma_4^A$ and $\Delta c\geq\Delta^A$.

Second, note that $\frac{\partial^3 h_A(\theta,\gamma)}{\partial\theta^3} = 6\alpha_{B,1}^3\gamma\sigma(p_{B,2}-c_B)^2(\alpha_{A,1}\gamma(p_{A,2}-c_A)+p_{A,1})^2(c_B(p_{A,1}+\gamma p_{A,2})-c_A(p_{B,1}+\gamma p_{B,2}))^2\geq 0$ for $\gamma\geq 0$. If $\Delta c\geq\Delta^A$ and $\gamma\geq\gamma_4^A$, by the above arguments, $h_A(1,\gamma)\leq 0<h_A(0,\gamma)$. Since $h_A(\theta,\gamma)$ is a cubic function of θ, there exists only one root of $h_A(\theta,\gamma)=0$ on $[0,1]$. Let θ^A be the root. Thus, we have $h_A(\theta,\gamma)\geq 0$ on $[0,\theta^A]$ and $h_A(\theta,\gamma)<0$ on $(\theta^A,1]$. Therefore, in the case of $\alpha_{A,1}=\alpha_{A,2}$, $\Pi_A^S-\Pi_A^N$ and $h_A(\theta,\gamma)$ have the same sign at the intervals $[0,\theta^A]$ and $[\theta^A,1]$.

Last, let $\gamma^A := \max\{\gamma_1^A,\gamma_2^A,\gamma_3^A,\gamma_4^A\}$, we have proved Proposition 5(c1-c2). *Q.E.D.*

Proof of Proposition 6: (a). By Lemma 2, we have the closed form of Π_B^S. Furthermore, since $\mathbb{E}[L_B^k] = \int_0^{\zeta_B^k}(\zeta_B^k-z)f(z)dz$, $k=0,S$, Π_B^S is independent of θ. By Lemma 3, we have the closed form of Π_B^N. Since $\frac{\partial\Pi_B^N}{\partial\theta} = \sigma\alpha_{B,1}\alpha_{B,2}\gamma(p_{B,2}-c_B)h_B^N(\gamma)/[(-\alpha_{A,2}\gamma c_A+p_{A,1}+\alpha_{A,2}\gamma p_{A,2})^2[p_{B,1}+\alpha_{B,2}\gamma(1-\theta)(p_{B,2}-c_B)]^2]$, where $h_B^N(\gamma)$ is a quadratic function of γ and $h_B^N(\gamma)>0$ if $\gamma>0$. Since $\frac{\partial\Pi_B^N}{\partial\theta}$ has the same sign as $h_B^N(\gamma)$, Π_B^N is increasing in θ.

(b). By analogous arguments in Proposition 5(b), we can prove part (b).

(c1-c2). $\Pi_B^S-\Pi_B^N = (p_{B,2}-c_B)\gamma h_B(\theta,\gamma)/[(-\gamma c_B+p_{B,1}+\gamma p_{B,2})(-\alpha_{A,2}\gamma c_A+p_{A,1}+\alpha_{A,2}\gamma p_{A,2})^2[(1-\alpha_{A,2})\gamma(1-\theta)(p_{B,2}-c_B)+p_{B,1}]]$, where $h_B(\theta,\gamma)$ is a linear function of θ and has the same sign as $\Pi_B^S-\Pi_B^N$. Note that, if $\theta<1$, $\frac{\partial^4 h_B(\theta,\gamma)}{\partial\gamma^4} = 24\alpha_{B,2}\alpha_{A,2}^2(1-\theta)\mu(\alpha_{B,1}-\alpha_{B,2})(c_A-p_{A,2})^2(c_B-p_{A,2})^2$. If $\theta=1$, $\frac{\partial^3 h_B(1,\gamma)}{\partial\gamma^3} = 6\alpha_{A,2}^2\mu p_{B,1}(\alpha_{B,1}-\alpha_{B,2})(p_{A,2}-c_A)^2(p_{B,2}-c_B)$. Thus, there exists a $\gamma_1^B>0$, $h_B(\theta,\gamma)$ has the same sign as $\alpha_{B,1}-\alpha_{B,2}$ for $\alpha_{B,1}\neq\alpha_{B,2}$, if $\gamma\geq\gamma_1^B$.

Next, we consider the case where $\alpha_{B,1}=\alpha_{B,2}$. In this case, at $\theta=0$, $\frac{\partial^2 h_B(0,\gamma)}{\partial\gamma^2} = (1-\alpha_{A,1})\alpha_{A,1}\sigma[-2c_B^2(p_{A,2}-c_A)^2\alpha_{A,2}^2-2c_A^2(p_{B,2}-c_B)^2\alpha_{A,2}+2c_A^2(p_{B,2}-c_B)^2]$. One can check that $-2c_B^2(p_{A,2}-c_A)^2\alpha_{A,2}^2-2c_A^2(p_{B,2}-c_B)^2\alpha_{A,2}+2c_A^2(p_{B,2}-c_B)^2$ decreases in $\alpha_{A,2}$ for $\alpha_{A,2}\geq\frac{1}{2}$. Furthermore, since $\alpha_{A,2}\leq\frac{1}{2}+\beta(1-\frac{c_A}{c_B})p_{B,2}$,

$\frac{\partial^2 h_B(0,\gamma)}{\partial\gamma^2} > 0$ for all $\alpha_{A,2}$ is ensured by $\frac{\partial^2 h_B(0,\gamma)}{\partial\gamma^2} > 0$ at $\alpha_{A,2} = \frac{1}{2} + \beta(1 - \frac{c_A}{c_B})p_{B,2}$. Note that, there exists a $\underline{c}_A$ such that $\frac{\partial^2 h_B(0,\gamma)}{\partial\gamma^2} > 0$ if $c_A \geq \underline{c}_A$ at $\alpha_{A,2} = \frac{1}{2} + \beta(1 - \frac{c_A}{c_B})p_{B,2}$. Therefore, there exists a $\gamma_2^B > 0$ and $\Delta^B > 0$ such that $h_B(0,\gamma) \geq 0$ if $0 < \Delta c \leq \Delta^B$ and $\gamma \geq \gamma_2^B$.

When $\theta = 1$, $\frac{\partial^2 h_B(1,\gamma)}{\partial\gamma^2} = 2c_A^2((1-2\alpha_{A,1})c_B^2 - 2(\alpha_{A,1}-1)^2 c_B p_{B,2} + (\alpha_{A,1}-1)^2 p_{B,2}^2) + 4\alpha_{A,1}^2 c_A c_B^2 p_{A,2} - 2\alpha^2 c_B^2 p_{A,2}^2 < 0$. Thus, there exists a $\gamma_3^B > 0$ such that $h_B(1,\gamma) < 0$ if $\gamma \geq \gamma_3^B$. Along with part (a), if $0 < \Delta c \leq \Delta^B$ and $\gamma \geq \max\{\gamma_2^B, \gamma_3^B\}$, there exists a θ^B such that $\Pi_B^S > \Pi_B^N$ for $\theta \in [0,\theta^B)$ and $\Pi_B^S \leq \Pi_B^N$ for $\theta \in [\theta^B, 1]$.

Last, let $\gamma^B := \max\{\gamma_1^B, \gamma_2^B, \gamma_3^B\}$, we have proved Proposition 6(c1-c2). *Q.E.D.*

A.2. Proof of Statements in Section 5

To endogenize pricing decisions, we first relax the assumption, $p_{B,t} \geq p_{A,t} \geq \frac{c_A}{c_B}p_{B,t}, t = 1,2$, and study the inventory competition in a more general setting: $p_{i,t} \in [\underline{p}, \overline{p}], i = A,B, t = 1,2$. Given retailers' prices and market split ratios in two periods, Lemma 5 [Lemma 6] characterizes the two retailers' equilibrium order quantities and profits when the C2C interactions are within-brand [cross-brand].

Lemma 5. *Suppose the C2C interactions are within-brand. Given retailers' retail prices $p_{i,t}$ and split ratios $\alpha_{i,t}$ ($i = A,B; t = 1,2$), a unique Nash equilibrium, $(y_{A,t}^S, y_{B,t}^S)$ ($t = 1,2$), exists. Furthermore, the following statements hold.*

(a) In the first period, if $p_{A,1} \geq \frac{c_A}{c_B}p_{B,1} + \gamma\left(\frac{c_A}{c_B}p_{B,2} - p_{A,2}\right)$, $y_{A,1}^S = (\alpha_{A,1} + \theta\alpha_{B,1})\zeta_{A,1}^S - \theta\alpha_{B,1}\zeta_{B,1}^S$ and $y_{B,1}^S = \alpha_{B,1}\zeta_{B,1}^S$; if $p_{A,1} < \frac{c_A}{c_B}p_{B,1} + \gamma\left(\frac{c_A}{c_B}p_{B,2} - p_{A,2}\right)$, $y_{A,1}^S = \alpha_{A,1}\zeta_{A,1}^S$ and $y_{B,1}^S = (\alpha_{B,1} + \theta\alpha_{A,1})\zeta_{B,1}^S - \theta\alpha_{A,1}\zeta_{A,1}^S$.

(b) In the second period, given the two retailers' sales, $R_{A,1}^S$ and $R_{B,1}^S$, if $p_{A,2} \geq \frac{c_A}{c_B}p_{B,2}$, $y_{A,2}^S = (\alpha_{A,2} + \theta\alpha_{B,2})\zeta_{A,2}^0 - \theta\alpha_{B,2}\zeta_{B,2}^0 + \gamma R_{A,1}^S$ and $y_{B,2}^S = \alpha_{B,2}\zeta_{B,2}^0 + \gamma R_{B,1}^S$; if $p_{A,2} < \frac{c_A}{c_B}p_{B,2}$, $y_{A,2}^S = \alpha_{A,2}\zeta_{A,2}^0 + \gamma R_{A,1}^S$ and $y_{B,2}^S = (\alpha_{B,2} + \theta\alpha_{A,2})\zeta_{B,2}^0 - \theta\alpha_{A,2}\zeta_{A,2}^0 + \gamma R_{B,1}^S$.

Proof of Lemma 5: We first study the subgame in the second period. By the proof of Lemma 2, retailers' best responses are characterized by (8) and (10). If $p_{A,2} \geq \frac{c_A}{c_B}p_{B,2}$, we have $\zeta_{A,2}^0 \geq \zeta_{B,2}^0$. In this case, the equilibrium order quantity is already characterized by Lemma 2. If $p_{A,2} \geq \frac{c_A}{c_B}p_{B,2}$, we have $\zeta_{A,2}^0 < \zeta_{B,2}^0$. Thus, there is a unique equilibrium and the equilibrium order quantity is $\overline{y}_{A,2}^S = \alpha_{A,2}\zeta_{A,2}^0$ and $\overline{y}_{B,2}^S = (\alpha_{B,2} + \theta\alpha_{A,2})\zeta_{B,2}^0 - \theta\overline{y}_{A,2} = (\alpha_{B,2} + \theta\alpha_{A,2})\zeta_{B,2}^0 - \theta\alpha_{A,2}\zeta_{A,2}^0$. Since $\overline{y}_{i,2}^S + \gamma R_{i,1}^S = y_{i,2}^S$, $i = A,B$, therefore, we have proved part (b). Given $R_{i,1}^S$, retailer A's [B's] second-period profit is obtained by substituting $(y_{A,2}^S, y_{B,2}^S)$ back to (7) [(9)].

In the first period, by analogous arguments in the proof of Lemma 2, retailers' best responses are characterized by (12) and (14). If $p_{A,1} \geq \frac{c_A}{c_B}p_{B,1} + \gamma\left(\frac{c_A}{c_B}p_{B,2} - p_{A,2}\right)$, that is, $p_{A,1} \geq \frac{c_A}{c_B}p_{B,1} + \frac{c_A}{c_B}\gamma(p_{B,2} - c_B) - \gamma(p_{A,2} - c_A)$, we have $\zeta_{A,1}^S \geq \zeta_{B,1}^S$. Thus, by analogous arguments, the equilibrium order quantity is already characterized by Lemma 2. If $p_{A,1} < \frac{c_A}{c_B}p_{B,1} + \gamma\left(\frac{c_A}{c_B}p_{B,2} - p_{A,2}\right)$, we have $\zeta_{A,1}^S < \zeta_{B,1}^S$. In this case, there is a unique order quantity and the equilibrium order quantity is $y_{A,1}^S = \alpha_{A,1}\zeta_{A,1}^S$ and $y_{B,1}^S = (\alpha_{B,1} + \theta\alpha_{A,1})\zeta_{B,1}^S - \theta y_{A,1}^S = (\alpha_{B,1} + \theta\alpha_{A,1})\zeta_{B,1}^S - \theta\alpha_{A,1}\zeta_{A,1}^S$. Therefore, we have proved part (a). *Q.E.D.*

Lemma 6. *Suppose the C2C interactions are cross-brand. Given retailers' retail prices $p_{i,t}$ and split ratios $\alpha_{i,t}$ ($i = A,B; t = 1,2$), a unique Nash equilibrium, $(y_{A,t}^N, y_{B,t}^N)$ ($t = 1,2$), exists. Furthermore, the following statements hold.*

(*a*) *In the first period, if* $p_{A,1} \geq \frac{c_A}{c_B} p_{B,1} + \gamma \left[\alpha_{B,2} \frac{c_A}{c_B}(p_{B,2} - c_B) - \alpha_{A,2}(p_{A,2} - c_A)\right]$, $y^N_{A,1} = (\alpha_{A,1} + \theta\alpha_{B,1})\zeta^N_{A,1} - \theta\alpha_{B,1}\zeta^N_{B,1}$ *and* $y^N_{B,1} = \alpha_{B,1}\zeta^N_{B,1}$*; if* $p_{A,1} < \frac{c_A}{c_B} p_{B,1} + \gamma \left[\alpha_{B,2} \frac{c_A}{c_B}(p_{B,2} - c_B) - \alpha_{A,2}(p_{A,2} - c_A)\right]$, $y^N_{A,1} = \alpha_{A,1}\widehat{\zeta}^N_{A,1}$ *and* $y^N_{B,1} = (\alpha_{B,1} + \theta\alpha_{A,1})\widehat{\zeta}^N_{B,1} - \theta\alpha_{A,1}\widehat{\zeta}^N_{A,1}$*, where*

$$\widehat{\zeta}^N_{A,1} = F^{-1}\left(1 - \frac{c_A - \alpha_{A,2}\gamma\theta(p_{A,2} - c_A)\frac{c_B}{p_{B,1} + \alpha_{B,2}\gamma(p_{B,2} - c_B)}}{p_{A,1} + \alpha_{A,2}\gamma(1-\theta)(p_{A,2} - c_A)}\right) \textit{ and } \widehat{\zeta}^N_{B,1} = F^{-1}\left(1 - \frac{c_B}{p_{B,1} + \alpha_{B,2}\gamma(p_{B,2} - c_B)}\right).$$

(*b*) *In the second period, given the two retailers' sales,* R^N_1*, if* $p_{A,2} \geq \frac{c_A}{c_B} p_{B,2}$, $y^N_{A,2} = (\alpha_{A,2} + \theta\alpha_{B,2})\zeta^0_{A,2} - \theta\alpha_{B,2}\zeta^0_{B,2} + \gamma\alpha_{A,2}R^N_1$ *and* $y^N_{B,2} = \alpha_{B,2}\zeta^0_{B,2} + \gamma\alpha_{B,2}R^N_1$*; if* $p_{A,2} < \frac{c_A}{c_B} p_{B,2}$, $y^S_{A,2} = \alpha_{A,2}\zeta^0_{A,2} + \gamma\alpha_{A,2}R^N_1$ *and* $y^S_{B,2} = (\alpha_{B,2} + \theta\alpha_{A,2})\zeta^0_{B,2} - \theta\alpha_{A,2}\zeta^0_{A,2} + \gamma\alpha_{B,2}R^N_1$.

Proof of Lemma 6: We first study the subgame in the second period. By analogous arguments in the proof of Lemma 5 and Lemma 3, we can characterize the unique equilibrium order quantity, as shown in part (b).

In the first period, by analogous arguments in the proof of Lemma 3, retailer A's best response is characterized as follows: If $y_{B,1} \leq \alpha_{B,1}\zeta^N_{A,1}$, where $\zeta^N_{A,1} := F^{-1}\left(1 - \frac{c_A}{p_{A,1} + \rho\alpha_{A,2}\gamma(p_{A,2} - c_A)}\right)$, $y^*_{A,1}(y_{B,1}) = (\alpha_{A,1} + \theta\alpha_{B,1})\zeta^N_{A,1} - \theta y_{B,1}$; If $y_{B,1} > \alpha_{B,1}\zeta^N_{A,1}$, the equation (16) implicitly characterizes $y^*_{A,1}(y_{B,1})$. On the other hand, retailer B's best response is characterized as follows: If $y_{A,1} < \alpha_{A,1}\widehat{\zeta}^N_{B,1}$, where $\widehat{\zeta}^N_{B,1} := F^{-1}\left(1 - \frac{c_B}{p_{B,1} + \rho\alpha_{B,2}\gamma(p_{B,2} - c_B)}\right)$, $y^*_{B,1}(y_{A,1}) = (\alpha_{B,1} + \theta\alpha_{A,1})\widehat{\zeta}^N_{B,1} - \theta y_{A,1}$; If $y_{A,1} \geq \alpha_{A,1}\widehat{\zeta}^N_{B,1}$, the equation (18) implicitly characterizes $y^*_{B,1}(y_{A,1})$.

Then, we compare $\zeta^N_{A,1} := F^{-1}\left(1 - \frac{c_A}{p_{A,1} + \rho\alpha_{A,2}\gamma(p_{A,2} - c_A)}\right)$ and $\widehat{\zeta}^N_{B,1} := F^{-1}\left(1 - \frac{c_B}{p_{B,1} + \rho\alpha_{B,2}\gamma(p_{B,2} - c_B)}\right)$. If $p_{A,1} \geq \frac{c_A}{c_B} p_{B,1} + \gamma \left[\alpha_{B,2} \frac{c_A}{c_B}(p_{B,2} - c_B) - \alpha_{A,2}(p_{A,2} - c_A)\right]$, i.e., $p_{A,1} + \gamma\alpha_{A,2}(p_{A,2} - c_A) \geq \frac{c_A}{c_B}[p_{B,1} + \alpha_{B,2}(p_{B,2} - c_B)]$, we have $\zeta^N_{A,1} \geq \widehat{\zeta}^N_{B,1}$. In this case, by analogous arguments, the equilibrium order quantity is characterized by Lemma 3.

On the other hand, if $p_{A,1} < \frac{c_A}{c_B} p_{B,1} + \gamma \left[\alpha_{B,2} \frac{c_A}{c_B}(p_{B,2} - c_B) - \alpha_{A,2}(p_{A,2} - c_A)\right]$, we have $\zeta^N_{A,1} < \widehat{\zeta}^N_{B,1}$. Next, we then show that there exists no equilibrium in area $\{(y_{A,1}, y_{B,1}) : \frac{y_{A,1}}{\alpha_{A,1}} \geq \frac{y_{B,1}}{\alpha_{B,1}}\}$, i.e., two firms' best responses do not intersect each other in the area. We assume, to the contrary, there exists an equilibrium in area $\{(y_{A,1}, y_{B,1}) : \frac{y_{A,1}}{\alpha_{A,1}} \geq \frac{y_{B,1}}{\alpha_{B,1}}\}$. Thus, by the proof of Lemma 3, it must satisfy the system (20). Since $\frac{y_{A,1}}{\alpha_{A,1}} \geq \frac{y_{B,1}}{\alpha_{B,1}}$, $\frac{y_{A,1} + \theta y_{B,1}}{\alpha_{A,1} + \theta\alpha_{B,1}} \geq \frac{y_{B,1}}{\alpha_{B,1}}$ for $\theta \in [0, 1]$. Thus, $F\left(\frac{y_{A,1} + \theta y_{B,1}}{\alpha_{A,1} + \theta\alpha_{B,1}}\right) \geq F\left(\frac{y_{B,1}}{\alpha_{B,1}}\right)$. Thus, by the second equation in (20), we have $1 - \frac{c_B}{p_{B,1} + \rho\alpha_{B,2}\gamma(p_{B,2} - c_B)} \leq F\left(\frac{y_{A,1} + \theta y_{B,1}}{\alpha_{A,1} + \theta\alpha_{B,1}}\right)$. By the first equation in (20), we have $1 - \frac{c_A}{p_{A,1} + \rho\alpha_{A,2}\gamma(p_{A,2} - c_A)} = F\left(\frac{y_{A,1} + \theta y_{B,1}}{\alpha_{A,1} + \theta\alpha_{B,1}}\right)$, which leads to a contradiction. Therefore, there exists no equilibrium in area $\{(y_{A,1}, y_{B,1}) : \frac{y_{A,1}}{\alpha_{A,1}} \geq \frac{y_{B,1}}{\alpha_{B,1}}\}$. Next, by the monotonicity of retailers' best response, there exists a unique equilibrium, in area $\{(y_{A,1}, y_{B,1}) : \frac{y_{A,1}}{\alpha_{A,1}} < \frac{y_{B,1}}{\alpha_{B,1}}\}$, which solves the system (19). Thus, we have $y^N_{A,1} = \alpha_{A,1}\widehat{\zeta}^N_{A,1}$ and $y^N_{B,1} = (\alpha_{B,1} + \theta\alpha_{A,1})\widehat{\zeta}^N_{B,1} - \theta\alpha_{A,1}\widehat{\zeta}^N_{A,1}$. Hence, we have proved part (a). *Q.E.D.*

Proof of Proposition 7: By Lemma 5 and Lemma 6, given first-period sales, $R_{A,1}$ and $R_{B,1}$, firms' second-period prices affect their second-period sales only by changing $\overline{y}^S_{i,2}$ [$\overline{\overline{y}}^N_{i,2}$] when the C2C interactions are within-brand [corss-brand], $i = A, B$. Furthermore, since $\overline{y}^S_{i,2}$ and $\overline{\overline{y}}^N_{i,2}$ both equal $y^0_{i,2}$, we prove their properties by focusing on $\overline{y}^S_{i,2}$ only. The same arguments show that $\overline{\overline{y}}^N_{i,2}$ have the same properties.

(a). By Lemma 5, we have the closed forms of $\overline{y}^S_{A,2}$ and $\overline{y}^S_{B,2}$. Given $p_{B,2} \in [\underline{p}, \overline{p}]$, $\overline{y}^S_{A,2}$ have two pieces. If $p_{A,2} \leq \frac{c_A}{c_B} p_{B,2}$, we have $\overline{y}^S_{A,2} = (\frac{1}{2} + \beta p_{B,2} - \beta p_{A,2})(-\frac{2c_A\sigma}{p_{A,2}} + \mu + \sigma)$ and $\frac{\partial \overline{y}^S_{A,2}}{\partial p_{A,2}} = \frac{c_A\sigma(2\beta p_{B,2}+1)}{p_{A,2}^2} - \beta(\mu+\sigma)$. So, the first piece of $\overline{y}^S_{A,2}$ can decrease, or first increase then decrease, or increase in $p_{A,2}$. If $p_{A,2} > \frac{c_A}{c_B} p_{B,2}$, we have $\frac{\partial \overline{y}^S_{A,2}}{\partial p_{A,2}} = -\beta(\mu+\sigma) + \frac{c_A\sigma(1+\theta+2\beta(1-\theta)p_{B,2})}{p_{A,2}^2} + \frac{2\beta c_B\theta\sigma}{p_{B,2}}$. Note that, $-\beta(\mu+\sigma) + \frac{c_A\sigma(1+\theta+2\beta(1-\theta)p_{B,2})}{p_{A,2}^2} + \frac{2\beta c_B\theta\sigma}{p_{B,2}} = 0$ has one solution, $p_{A,2} = \frac{\sqrt{c_A p_{B,2}\sigma}\sqrt{\theta+2\beta(1-\theta)p_{B,2}+1}}{\sqrt{\beta p_{B,2}(\mu+\sigma)-2\beta c_B\theta\sigma}}$, on $p_{A,2} > 0$. The solution is small than the threshold $\frac{c_A}{c_B} p_{B,2}$ only if $\beta > \beta^D_1 := \frac{c_B^2\theta\sigma + c_B^2\sigma}{-2c_Ac_B\theta\overline{p}\sigma + c_A\mu\underline{p}^2 + c_A\underline{p}^2\sigma + 2c_B^2\theta\underline{p}\sigma - 2c_B^2\overline{p}\sigma}$. Thus, if $\beta > \beta^D_1$, the second piece of $\overline{y}^S_{A,2}$ decreases in $p_{A,2}$. Thus, by the property of $\overline{y}^S_{A,2}$ on the two pieces, $\overline{y}^S_{A,2}$ is proved to either decrease or first increase then decrease in $p_{A,2}$.

By analogous arguments, there exists a $\beta^D_2 > 0$, $\overline{y}^S_{B,2}$ either decreases or first increases then decreases in $p_{B,2}$ if $\beta > \beta^D_2$. Let $\beta^D := \max\{\beta^D_1, \beta^D_2\}$, we have proved part (a).

(b) Given $p_{A,2} \in [\underline{p}, \overline{p}]$, $\overline{y}^S_{A,2}$, as a function of $p_{B,2}$, has two pieces. If $p_{B,2} < \frac{c_B}{c_A} p_{A,2}$, $\frac{\partial \overline{y}^S_{A,2}}{\partial p_{B,2}} = \beta\left(-\frac{2c_A(1-\theta)\sigma}{p_{A,2}} + \mu + \sigma\right) - \frac{2c_B\beta\theta p_{A,2}\sigma}{p_{B,2}^2} - \frac{c_B\theta\sigma}{p_{B,2}^2}$. Thus, the first piece of $\overline{y}^S_{A,2}$ can increase, or first decrease then increase, or decrease in $p_{B,2}$. If $p_{B,2} \geq \frac{c_B}{c_A} p_{A,2}$, $\frac{\partial \overline{y}^S_{A,2}}{\partial p_{B,2}} = \beta\left(-\frac{2c_A\sigma}{p_{A,2}} + \mu + \sigma\right) > 0$. Thus, by the property of $\overline{y}^S_{A,2}$ on the two pieces, $\overline{y}^S_{A,2}$ either increases or first decreases then increases in $p_{B,2}$. By analogous arguments, $\overline{y}^S_{B,2}$ either increases or first decreases then increases in $p_{A,2}$. Thus, we have proved part (b). *Q.E.D.*

Proof of Proposition 8: To prove this proposition, we assume that the conditions in Proposition 9 hold, i.e., $\gamma > \gamma^D_1 := \max\{\gamma^P_1, \gamma^P_2, \gamma^P_4, \gamma(\epsilon_2), \gamma^P_5\}$. Under this condition, by the proof of Proposition 9, we have the closed form of $y^k_{i,1}$ $(i = A, B; k = S, N)$ for all $(p_{A,1}, p_{B,1})$.

(a). We have $\frac{\partial y^S_{A,1}}{\partial p_{A,1}} = \frac{\phi^S_{A,1}(\gamma)}{(-\gamma c_A + p_{A,1} + \gamma p_{A,2})^2(-\gamma c_B + p_{B,1} + \gamma p_{B,2})}$, where $\phi^S_{A,1}(\gamma)$ has the same sign as $\frac{\partial y^S_{A,1}}{\partial p_{A,1}}$ with $\frac{\partial^3 \phi^S_{A,1}(\gamma)}{\partial \gamma^3} = -6\beta(p_{A,2} - c_A)^2(p_{B,2} - c_B)(\mu+\sigma) < 0$. Furthermore, $\frac{\partial y^S_{A,1}}{\partial p_{B,1}} = \frac{\phi^S_{B,1}(\gamma)}{(-\gamma c_A + p_{A,1} + \gamma p_{A,2})(-\gamma c_B + p_{B,1} + \gamma p_{B,2})^2}$, where $\phi^S_{B,1}(\gamma)$ has the same sign as $\frac{\partial y^S_{A,1}}{\partial p_{B,1}}$ with $\frac{\partial^3 \phi^S_{B,1}(\gamma)}{\partial \gamma^3} = 6\beta(p_{A,2} - c_A)(p_{B,2} - c_B)^2(\mu+\sigma) > 0$. Thus, because both $p_{A,1}$ and $p_{B,1}$ are bounded, there exists a $\gamma^D_2 > 0$ such that $\frac{\partial y^S_{A,1}}{\partial p_{A,1}} > 0$ and $\frac{\partial y^S_{A,1}}{\partial p_{B,1}} < 0$ for $\gamma > \gamma^D_2$. By analogous arguments, the monotonicity of $y^S_{B,1}$ in $p_{B,1}$ and $p_{A,1}$ is proved. Thus, we have proved part (a).

(b1). We have $\frac{\partial y^N_{A,1}}{\partial p_{A,1}} = \phi^N_{AA,1}(\theta,\gamma)/[(\gamma(p_{A,2} - c_A)(-2\beta p_{A,2} + 2\beta p_{B,2} + 1) + 2p_{A,1})^2(\gamma(1-\theta)(p_{B,2} - c_B)(2\beta p_{A,2} - 2\beta p_{B,2} + 1) + 2p_{B,1})]$, where $\phi^N_{AA,1}(\theta,\gamma)$ has the same sign as $\frac{\partial y^N_{A,1}}{\partial p_{A,1}}$. On the other hand, $\frac{\partial y^N_{B,1}}{\partial p_{A,1}} = \phi^N_{BA,1}(\theta,\gamma)/[(\gamma(p_{A,2} - c_A)(-2\beta p_{A,2} + 2\beta p_{B,2} + 1) + 2p_{A,1})^2(\gamma(1-\theta)(p_{B,2} - c_B)(2\beta p_{A,2} - 2\beta p_{B,2} + 1) + 2p_{B,1})]$, where $\phi^N_{BA,1}(\theta,\gamma)$ has the same sign as $\frac{\partial y^N_{B,1}}{\partial p_{A,1}}$. If $\theta < 1$, $\frac{\partial^3 \phi^N_{AA,1}(\theta,\gamma)}{\partial \gamma^3} = -6\beta(1-\theta)(p_{A,2} - c_A)^2(p_{B,2} - c_B)(\mu+\sigma)(2\beta p_{A,2} - 2\beta p_{B,2} + 1)(-2\beta p_{A,2} + 2\beta p_{B,2} + 1)^2 < 0$ and $\frac{\partial^3 \phi^N_{BA,1}(\theta,\gamma)}{\partial \gamma^3} = 6\beta(1-\theta)(p_{A,2} - c_A)^2(p_{B,2} - c_B)(\mu+\sigma)(2\beta p_{A,2} - 2\beta p_{B,2} + 1)(-2\beta p_{A,2} + 2\beta p_{B,2} + 1)^2 > 0$. If $\theta = 1$, $\frac{\partial^2 \phi^N_{AA,1}(1,\gamma)}{\partial \gamma^2} = -\frac{\partial^2 \phi^N_{BA,1}(1,\gamma)}{\partial \gamma^2}$. Furthermore, at $(p_{A,2}, p_{B,2}) = (\frac{4\beta c_A + 2\beta c_B + 3}{6\beta}, \frac{2\beta c_A + 4\beta c_B + 3}{6\beta})$, we have $\frac{\partial^2 \phi^N_{AA,1}(1,\gamma)}{\partial \gamma^2} < 0$ and $\frac{\partial^2 \phi^N_{BA,1}(1,\gamma)}{\partial \gamma^2} > 0$. By the proof of Proposition 9 and the continuity of $\frac{\partial^2 \phi^N_{AA,1}(1,\gamma)}{\partial \gamma^2}$ and $\frac{\partial^2 \phi^N_{BA,1}(1,\gamma)}{\partial \gamma^2}$, we have $\frac{\partial^2 \phi^N_{AA,1}(1,\gamma)}{\partial \gamma^2} < 0$ and $\frac{\partial^2 \phi^N_{BA,1}(1,\gamma)}{\partial \gamma^2} > 0$ for the second-period equilibrium price $(p^N_{A,2}, p^N_{B,2})$. Thus, for $\theta \in [0,1]$, there exists a $\gamma^D_3 > 0$ such that $\frac{\partial y^N_{A,1}}{\partial p_{A,1}} < 0$ and $\frac{\partial y^N_{B,1}}{\partial p_{A,1}} > 0$ for all possible $p_{A,1}$ and $p_{B,1}$, if $\gamma > \gamma^D_3$. Thus, we have proved part (b1).

(b2). We have $\frac{\partial y^N_{A,1}}{\partial p_{B,1}} = \phi^N_{AB,1}(\theta,\gamma)/[(\gamma(p_{A,2}-c_A)(-2\beta p_{A,2}+2\beta p_{B,2}+1)+2p_{A,1})(\gamma(1-\theta)(p_{B,2}-c_B)(2\beta p_{A,2}-2\beta p_{B,2}+1)+2p_{B,1})^2]$, where $\phi^N_{AB,1}(\theta,\gamma)$ has the same sign as $\frac{\partial y^N_{A,1}}{\partial p_{B,1}}$. On the other hand, $\frac{\partial y^N_{B,1}}{\partial p_{B,1}} = \phi^N_{BB,1}(\theta,\gamma)/[(\gamma(p_{A,2}-c_A)(-2\beta p_{A,2}+2\beta p_{B,2}+1)+2p_{A,1})(\gamma(1-\theta)(p_{B,2}-c_A)(2\beta p_{A,2}-2\beta p_{B,2}+1)+2p_{B,1})^2]$, where $\phi^N_{BB,1}(\theta,\gamma)$ has the same sign as $\frac{\partial y^N_{B,1}}{\partial p_{B,1}}$. At $\theta=1$, $\frac{\partial \phi^N_{AB,1}(1,\gamma)}{\partial\gamma} = -\frac{\partial \phi^N_{BB,1}(1,\gamma)}{\partial\gamma}$. Furthermore, at $(p_{A,2},p_{B,2}) = (\frac{4\beta c_A+2\beta c_B+3}{6\beta}, \frac{2\beta c_A+4\beta c_B+3}{6\beta})$, we have $\frac{\partial \phi^N_{AB,1}(1,\gamma)}{\partial\gamma} = \frac{2}{9}(p^2_{B,1}(2\beta\Delta c+3)^2(\mu+\sigma) - \frac{\Delta c\sigma(2\beta p_{A,1}+1)(24\beta c_A+(2\beta\Delta c+3)^2)}{\beta})$. Note that, there exists a $\beta^D>0$ such that $\frac{\partial \phi^N_{AB,1}(1,\gamma)}{\partial\gamma}<0$ and $\frac{\partial \phi^N_{BB,1}(1,\gamma)}{\partial\gamma}>0$ for any $0<\beta\le\beta^D$ and the second-period equilibrium $(p^N_{A,2},p^N_{B,2})$. Thus, at $\theta=1$, there exists a $\gamma^D_4>0$ such that $\frac{\partial y^N_{A,1}}{\partial p_{A,1}}<0$ and $\frac{\partial y^N_{B,1}}{\partial p_{A,1}}>0$ for all possible $p_{A,1}$ and $p_{B,1}$, if $\gamma>\gamma^D_4$ and $0<\beta\le\beta^D$. By the continuity of $y^N_{A,1}$ and $y^N_{B,1}$ in θ, there exists a $\theta^D_1\in[0,1]$ such that the statement in part (b2) holds.

Let $\gamma^D := \max\{\gamma^D_1,\gamma^D_2,\gamma^D_3,\gamma^D_4\}$. We have proved Proposition 8. *Q.E.D.*

Proof of Proposition 9: We study the two-period pricing-inventory competition when the C2C interactions are within-brand [cross-brand] in Scenario S [N].

Scenario S: In the second period, by Lemma 5, we have the closed form of retailer i's second-period profit, $\pi^S_{i,2}(p_{A,2},p_{B,2})(i=A,B)$, which equals $\pi^0_{i,2}(p_{A,2},p_{B,2})+\gamma(p_{i,2}-c_i)R^S_{i,1}$. For each retailer i, $\frac{\partial\pi^S_{i,2}(p_{A,2},p_{B,2})}{\partial p_{i,2}} = \frac{\partial\pi^0_{i,2}(p_{A,2},p_{B,2})}{\partial p_{i,2}}+\gamma R^S_{i,1}$. Notice that, by the closed form of $\pi^0_{i,2}$ and $p_{j,2}$ is bounded, $j=A,B$, $\frac{\partial\pi^0_{i,2}(p_{A,2},p_{B,2})}{\partial p_{i,2}}$ is bounded for all $p_{A,2}$ and $p_{B,2}$. Thus, there exists a $\gamma^P_1>0$ such that $\frac{\partial\pi^S_{i,2}(p_{A,2},p_{B,2})}{\partial p_{i,2}}>0$ for all $p_{A,2}$ and $p_{B,2}$, $i=A,B$, if $\gamma>\gamma^P_1$. So, under the condition, there exists a unique equilibrium $(p^S_{A,2},p^S_{B,2})=(\overline{p},\overline{p})$.

In the first period, since $p^S_{A,2}=p^S_{B,2}$, we have $\frac{c_A}{c_B}p^S_{B,2}-p^S_{A,2}<0$. Thus, there exists a $\gamma^P_2>0$ such that $p_{A,1}\ge\frac{c_A}{c_B}p_{B,1}+\gamma\left(\frac{c_A}{c_B}p^S_{B,2}-p^S_{A,2}\right)$ for all $p_{A,1}$ and $p_{B,1}$, if $\gamma>\gamma^P_2$. By Lemma 5, we have the closed form of each firm i's total profit $\Pi^S_i(p_{A,1},p_{B,1}), i=A,B$, as a function of $p_{A,1}$ and $p_{B,1}$. For retailer A, $\frac{\partial\Pi^S_A(p_{A,1},p_{B,1})}{\partial p_{A,1}} = \psi^N_{A,1}(\theta,\gamma)/[2(-\gamma c_A+p_{A,1}+\gamma\overline{p})^2(-\gamma c_B+p_{B,1}+\gamma\overline{p})^2)]$, where $\psi^N_{A,1}(\theta,\gamma)$ has the same sign as $\frac{\partial\Pi^S_A(p_{A,1},p_{B,1})}{\partial p_{A,1}}$ and is a linear function of θ. We have $\frac{\partial^5\psi^N_{A,1}(\theta,\gamma)}{\partial\gamma^5} = -240\beta(\overline{p}-c_A)^3(\overline{p}-c_B)^2<0$. For retailer B, $\frac{\partial\Pi^S_B(p_{A,1},p_{B,1})}{\partial p_{B,1}} = \psi^N_{B,1}(\gamma)/[2(-\gamma c_B+p_{B,1}+\gamma\overline{p})^2]$, where $\psi^N_{B,1}(\gamma)$ has the same sign as $\frac{\partial\Pi^S_B(p_{A,1},p_{B,1})}{\partial p_{B,1}}$. Note that, we have $\frac{\partial^3\psi^N_{B,1}(\gamma)}{\partial\gamma^3} = -12\beta\mu(\overline{p}-c_B)^3<0$. Thus, because $(p_{A,1},p_{B,1})$ is bounded, there exists a $\gamma^P_3>0$ such that $\frac{\partial\Pi^S_A(p_{A,1},p_{B,1})}{\partial p_{A,1}}<0$ and $\frac{\partial\Pi^S_B(p_{A,1},p_{B,1})}{\partial p_{B,1}}<0$ hold for all $(p_{A,1},p_{B,1})$. Therefore, by the above arguments, if $\gamma>\max\{\gamma^P_1,\gamma^P_2,\gamma^P_3\}$, the two-period pricing-inventory competition has an a unique equilibrium $(p^S_{A,1},p^S_{B,1}) = (\underline{p},\underline{p})$, $(p^S_{A,2},p^S_{B,2})=(\overline{p},\overline{p})$, and $(y^S_{A,t},y^S_{B,t})(t=1,2)$ are characterized by Lemma 5. Thus, part (a) holds.

Scenario N: In the second period, by Lemma 6, we have the closed form of retailer i's second-period profit, $\pi^N_{i,2}(p_{A,2},p_{B,2})(i=A,B)$, which equals $\pi^0_{i,2}(p_{A,2},p_{B,2})+\gamma(p_{i,2}-c_i)(\frac{1}{2}+p_{j,2}\beta-p_{i,2}\beta)R^N_1$. For each retailer i, $\frac{\partial\pi^N_{i,2}(p_{A,2},p_{B,2})}{\partial p_{i,2}} = \frac{\partial\pi^0_{i,2}(p_{A,2},p_{B,2})}{\partial p_{i,2}}+\frac{1}{2}\gamma R^N_1(2\beta c_i-4\beta p_{i,2}+2\beta p_{j,2}+1)$, $i,j=A,B,j\neq i$. Furthermore, for retailer A, one can check that $\frac{\partial^2\pi^N_{A,2}(p_{A,2},p_{B,2})}{\partial p^2_{A,2}}<0$ for all $p_{A,1}\ge\frac{c_A}{c_B}p_{B,1}$ when γ is large; furthermore, $\frac{\partial^2\pi^N_{A,2}(p_{A,2},p_{B,2})}{\partial p^2_{A,2}}<0$ for all $p_{A,1}<\frac{c_A}{c_B}p_{B,1}$ when $\gamma>0$. Furthermore, $\frac{\partial\pi^N_{A,2}(p_{A,2},p_{B,2})}{\partial p_{A,2}}$ is continuous at $p_{A,1}=\frac{c_A}{c_B}p_{B,1}$. Thus, by analogous arguments, there exists a $\gamma^P_4>0$ such that $\pi^N_{i,2}(p_{A,2},p_{B,2})$ is concave in $p_{i,2}$, $i=A,B$, if $\gamma>\gamma^P_4$. Therefore, if $\gamma>\gamma^P_4$, there exists a price equilibrium in the second period. Moreover, we show that, for any $\epsilon_2 > 0$, there exists a $\gamma(\epsilon_2) > 0$, such that the distance between any price equilibrium and the point $(\frac{4\beta c_A+2\beta c_B+3}{6\beta}, \frac{2\beta c_A+4\beta c_B+3}{6\beta})$ is smaller than ϵ_2 if $\gamma > \gamma(\epsilon_2)$. Note that $(\frac{4\beta c_A+2\beta c_B+3}{6\beta}, \frac{2\beta c_A+4\beta c_B+3}{6\beta})$ is the solution to the system $\{2\beta c_A - 4\beta p_{A,2} + 2\beta p_{B,2} + 1 = 0, 2\beta c_B - 4\beta p_{B,2} + 2\beta p_{A,2} + 1 = 0\}$. To the contrary, we assume there exists an equilibrium $(\tilde{p}_{A,2}, \tilde{p}_{B,2})$ which satisfies $|2\beta c_A - 4\beta \tilde{p}_{A,2} + 2\beta \tilde{p}_{B,2} + 1| > \epsilon_2$. Because $\frac{\partial \pi^0_{A,2}(p_{A,2},p_{B,2})}{\partial p_{A,2}}$ is bounded, retailer A has incentive to change its price since $\frac{\partial \pi^N_{A,2}(p_{A,2},p_{B,2})}{\partial p_{A,2}} > 0$ or $\frac{\partial \pi^N_{A,2}(p_{A,2},p_{B,2})}{\partial p_{A,2}} < 0$ for large γ. So, this leads to a contradiction. Thus, for $\gamma > \gamma(\epsilon_2)$, any price equilibrium should be close to $(\frac{4\beta c_A+2\beta c_B+3}{6\beta}, \frac{2\beta c_A+4\beta c_B+3}{6\beta})$ with a distance smaller than ϵ_2.

In the first period, by above arguments, the second-period equilibrium prices should satisfy $\alpha^N_{B,2}\frac{c_A}{c_B}(p^N_{B,2} - c_B) - \alpha^N_{A,2}(p^N_{A,2} - c_A) < 0$. Thus, there exists a $\gamma^P_5 > 0$ such that $p_{A,1} \geq \frac{c_A}{c_B}p_{B,1} + \gamma\left[\alpha^N_{B,2}\frac{c_A}{c_B}(p^N_{B,2} - c_B) - \alpha^N_{A,2}(p^N_{A,2} - c_A)\right]$ for all $p_{A,1}$ and $p_{B,1}$, if $\gamma > \gamma^P_5$. By Lemma 6, we have the closed form of each firm i's total profit $\Pi^N_i(p_{A,1}, p_{B,1}), i = A, B$, as a function of $p_{A,1}$ and $p_{B,1}$. Furthermore, we have $\frac{\partial^2 \Pi^N_A(p_{A,1},p_{B,1})}{\partial p^2_{A,1}} = \psi^N_{A,1}(\theta,\gamma)/[(\gamma(c_A - p_{A,2})(2\beta p_{A,2} - 2\beta p_{B,2} - 1) + 2p_{A,1})^4(\gamma(\theta-1)(c_B - p_{B,2})(-2\beta p_{A,2} + 2\beta p_{B,2} - 1) - 2p_{B,1})^2]$, where $\psi^N_{A,1}(\theta,\gamma)$ has the same sign as $\frac{\partial^2 \Pi^N_A(p_{A,1},p_{B,1})}{\partial p^2_{A,1}}$. We have $\frac{\partial^6 \psi^N_{A,1}(\theta,\gamma)}{\partial \gamma^6} = -1440\beta(\theta - 1)^2\mu(c_A - p_{A,2})^4(c_B - p_{B,2})^2(2\beta p_{A,2} - 2\beta p_{B,2} + 1)^2(-2\beta p_{A,2} + 2\beta p_{B,2} + 1)^4 < 0$ since $\theta < 1$. Moreover, we have $\frac{\partial^2 \Pi^N_B(p_{A,1},p_{B,1})}{\partial p^2_{B,1}} = \psi^N_{B,1}(\theta,\gamma)/[(\gamma(c_A - p_{A,2})(2\beta p_{A,2} - 2\beta p_{B,2} - 1) + 2p_{A,1})^2(\gamma(1-\theta)(p_{B,2} - c_B)(2\beta p_{A,2} - 2\beta p_{B,2} + 1) + 2p_{B,1})^3]$, where $\psi^N_{B,1}(\theta,\gamma)$ has the same sign as $\frac{\partial^2 \Pi^N_B(p_{A,1},p_{B,1})}{\partial p^2_{B,1}}$. We have $\frac{\partial^5 \psi^N_{B,1}(\theta,\gamma)}{\partial \gamma^5} = -240\beta(1-\theta)^3\mu(c_A - p_{A,2})^2(p_{B,2} - c_B)^3(2\beta p_{A,2} - 2\beta p_{B,2} + 1)^3(-2\beta p_{A,2} + 2\beta p_{B,2} + 1)^2 < 0$ since $\theta < 1$. Therefore, there exists a $\gamma^P_6 > 0$ such that $\frac{\partial^2 \Pi^N_A(p_{A,1},p_{B,1})}{\partial p^2_{A,1}} < 0$ and $\frac{\partial^2 \Pi^N_B(p_{A,1},p_{B,1})}{\partial p^2_{B,1}} < 0$ for all $p_{A,1}$ and $p_{B,1}$, if $\gamma > \gamma^P_6$. Thus, in this case, $\Pi^N_A(p_{A,1}, p_{B,1})$ and $\Pi^N_B(p_{A,1}, p_{B,1})$ are concave and there exists a pricing equilibrium in the first period. Let $\gamma = \frac{1}{r}$ and substitute γ with $\frac{1}{r}$ in $\Pi^N_i(p_{A,1}, p_{B,1}), i = A, B$. We have $\frac{\partial \Pi^N_A(p_{A,1},p_{B,1})}{\partial p_{A,1}} = \beta c_A\sigma + \mu(\beta c_A - 2\beta p_{A,1} + \beta p_{B,1} + \frac{1}{2}) + r\chi^N_{A,1}(p_{A,1}, p_{B,1})/[(1-\theta)(p_{A,2} - c_A)(p_{B,2} - c_B)^2(-2\beta p_{A,2} + 2\beta p_{B,2} + 1)(2\beta p_{A,2} - 2\beta p_{B,2} + 1)^2] + \mathcal{O}(r^2)$ and $\frac{\partial \Pi^N_B(p_{A,1},p_{B,1})}{\partial p_{B,1}} = \beta c_B\sigma + \mu(\beta c_B + \beta p_{A,1} - 2\beta p_{B,1} + \frac{1}{2}) + r\chi^N_{B,1}(p_{A,1}, p_{B,1})/[(1-\theta)(p_{A,2} - c_A)^2(p_{B,2} - c_B)(2\beta p_{A,2} - 2\beta p_{B,2} + 1)(-2\beta p_{A,2} + 2\beta p_{B,2} + 1)^2] + \mathcal{O}(r^2)$ at $r = 0$. By analogous arguments in the second period, for any $\epsilon_1 > 0$, there exists a $\gamma(\epsilon_1) > 0$, such that the distance between any price equilibrium and the point $(\frac{1}{6}(\frac{3}{\beta} + \frac{2\sigma(2c_A+c_B)}{\mu} + 4c_A + 2c_B), \frac{1}{6}(\frac{3}{\beta} + \frac{2\sigma(c_A+2c_B)}{\mu} + 2c_A + 4c_B))$ is smaller than ϵ_1 if $\gamma > \gamma(\epsilon_1)$. Note that $(\frac{1}{6}(\frac{3}{\beta} + \frac{2\sigma(2c_A+c_B)}{\mu} + 4c_A + 2c_B), \frac{1}{6}(\frac{3}{\beta} + \frac{2\sigma(c_A+2c_B)}{\mu} + 2c_A + 4c_B))$ is the solution to the system $\{\beta c_A\sigma + \mu(\beta c_A - 2\beta p_{A,1} + \beta p_{B,1} + \frac{1}{2}) = 0, \beta c_B\sigma + \mu(\beta c_B + \beta p_{A,1} - 2\beta p_{B,1} + \frac{1}{2}) = 0\}$.

By the above statements, there exists an equilibrium in the two-period pricing-inventory competition, if $\gamma > \max\{\gamma^P_4, \gamma^P_5, \gamma^P_6, \gamma^P(\epsilon_1), \gamma^P(\epsilon_2)\}$. Furthermore, by the properties characterized above, part(b) holds. *Q.E.D.*

Appendix B: Scenario with the Mix of Within-brand and Cross-brand C2C

In this section, we analyze the scenario where within-brand and cross-brand C2C interactions both exist. Recall that, retailer i's initial demand allocation in Scenario k is $M^k_{i,t} = \alpha_{i,t}X_t + Z^k_{i,t}$, $k = S, N$, where $Z^S_{i,t} = \gamma R_{i,t-1}$ and $Z^N_{i,t} = \gamma\alpha_{i,t}(R_{i,t-1} + R_{j,t-1})$. Now, let λ^S $[\lambda^N]$ be the strength of the within-brand [cross-brand] C2C interactions, where both $\lambda^k \in [0,1]$ and $\lambda^S + \lambda^N = 1$. In the general setting where both within-brand

and cross-brand C2C interactions exist, $Z^G_{i,t} = \lambda^S Z^S_{i,t} + \lambda^N Z^N_{i,t} = \gamma_{i,t} R_{i,t-1} + \gamma^C_{i,t} R_{j,t-1}$, where $\gamma_{i,t} := (\lambda^S + \lambda^N \alpha_{i,t})\gamma \in [\alpha_{i,t}\gamma, \gamma]$, $\gamma^C_{i,t} := \lambda^N \gamma \alpha_{i,t} \in [0, \alpha_{i,t}\gamma]$ and satisfying $\gamma_{i,t} \geq \gamma^C_{i,t}$, $i, j = A, B; i \neq j$. Note that Scenario S is a special case of Scenario G if $\lambda^S = 1$ and $\lambda^N = 0$ (i.e., $\gamma_{i,t} = \gamma$ and $\gamma^C_{i,t} = 0$); Scenario N is a special case of Scenario G if $\lambda^S = 0$ and $\lambda^N = 1$ (i.e., $\gamma_{i,t} = \gamma^C_{i,t} = \alpha_{i,t}\gamma$). We characterize the two-period inventory competition in the following lemma.

Lemma 7. *Suppose the C2C interactions are both within-brand and cross-brand. A unique Nash equilibrium exists: In the first period,* $y^G_{A,1} = (\alpha_{A,1} + \theta\alpha_{B,1})\zeta^G_{A,1} - \theta\alpha_{B,1}\zeta^G_{B,1}$ *and* $y^G_{B,1} = \alpha_{B,1}\zeta^G_{B,1}$*, where*

$$\zeta^G_{A,1} = F^{-1}\left(1 - \frac{c_A}{p_{A,1} + \gamma_{A,2}(p_{A,2} - c_A)}\right) \text{ and } \zeta^G_{B,1} = F^{-1}\left(1 - \frac{c_B - \gamma^C_{B,2}\theta(p_{B,2} - c_B)\frac{c_A}{p_{A,1}+\gamma_{A,2}(p_{A,2}-c_A)}}{p_{B,1} + (\gamma_{B,2} - \gamma^C_{B,2}\theta)(p_{B,2} - c_B)}\right);$$

in the second period, given the two retailers' sales $R^G_{A,1}$ *and* $R^G_{B,1}$*,* $y^G_{A,2} = y^0_{A,2} + \gamma_{A,2} R^G_{A,1} + \gamma^C_{A,2} R^G_{B,1}$ *and* $y^G_{B,2} = y^0_{B,2} + \gamma_{B,2} R^G_{B,1} + \gamma^C_{B,2} R^G_{A,1}$.

Proof of Lemma 7: We first study the subgame in the second period. Retailers' demand is $D_{A,2} = \alpha_{A,2} X_2 + \gamma_{A,2} R_{A,1} + \gamma^C_{A,2} R_{B,1} + \theta\left[\alpha_{B,2} X_2 + \gamma_{B,2} R_{B,1} + \gamma^C_{B,2} R_{A,1} - y_{B,2}\right]^+$ and $D_{B,2} = \alpha_{B,2} X_2 + \gamma_{B,2} R_{B,1} + \gamma^C_{B,2} R_{A,1} + \theta\left[\alpha_{A,2} X_2 + \gamma_{A,2} R_{A,1} + \gamma^C_{A,2} R_{B,1} - y_{A,2}\right]^+$. Let $\overline{\overline{y}}_{A,2} := y_{A,2} - \gamma_{A,2} R_{A,1} - \gamma^C_{A,2} R_{B,1}$ and $\overline{\overline{y}}_{B,2} := y_{B,2} - \gamma_{B,2} R_{B,1} - \gamma^C_{B,2} R_{A,1}$, by analogous arguments in the proof of Lemma 3, we have the equilibrium order quantity $\overline{\overline{y}}^G_{A,2} = y^0_{A,2}$ and $\overline{\overline{y}}^G_{B,2} = y^0_{B,2}$. Thus, we have the second-period order quantity $y^G_{A,2}$ and $y^G_{B,2}$. The

In the first period, by analogous arguments, retailer A's best response is: If $y_{B,1} \leq \alpha_{B,1}\zeta^G_{A,1}$, where $\zeta^G_{A,1} := F^{-1}\left(1 - \frac{c_A}{p_{A,1}+\rho\gamma_{A,2}(p_{A,2}-c_A)}\right)$, $y^*_{A,1}(y_{B,1}) = (\alpha_{A,1} + \theta\alpha_{B,1})\zeta^G_{A,1} - \theta y_{B,1}$; If $y_{B,1} > \alpha_{B,1}\zeta^N_{A,1}$, the following equation,

$$\begin{aligned} p_{A,1} - c_A + \rho(p_{A,2} - c_A)\gamma_{A,2} = {} & [p_{A,1} + \rho(p_{A,2} - c_A)(\gamma_{A,2} - \gamma^C_{A,2}\theta)]F\left(\frac{y^*_{A,1}(y_{B,1})}{\alpha_{A,1}}\right) \\ & + \rho(p_{A,2} - c_A)\gamma^C_{A,2}\theta F\left(\frac{y_{B,1} + \theta y^*_{A,1}(y_{B,1})}{\alpha_{B,1} + \theta\alpha_{A,1}}\right), \end{aligned}$$

implicitly characterizes $y^*_{A,1}(y_{B,1})$. Retailer B's best response is: If $y_{A,1} < \alpha_{A,1}F^{-1}\left(1 - \frac{c_B}{p_{B,1}+\rho\gamma_{B,2}(p_{B,2}-c_B)}\right)$, $y^*_{B,1}(y_{A,1}) = (\alpha_{B,1} + \theta\alpha_{A,1})F^{-1}\left(1 - \frac{c_B}{p_{B,1}+\rho\gamma_{B,2}(p_{B,2}-c_B)}\right) - \theta y_{A,1}$; If $y_{A,1} \geq \alpha_{A,1}F^{-1}\left(1 - \frac{c_B}{p_{B,1}+\rho\gamma_{B,2}(p_{B,2}-c_B)}\right)$, the following equation

$$\begin{aligned} p_{B,1} - c_B + \rho(p_{B,2} - c_B)\gamma_{B,2} = {} & [p_{B,1} + \rho(p_{B,2} - c_B)(\gamma_{B,2} - \gamma^C_{B,2}\theta)]F\left(\frac{y^*_{B,1}(y_{A,1})}{\alpha_{B,1}}\right) \\ & + \rho(p_{B,2} - c_B)\gamma^C_{B,2}\theta F\left(\frac{y_{A,1} + \theta y^*_{B,1}(y_{A,1})}{\alpha_{A,1} + \theta\alpha_{B,1}}\right), \end{aligned}$$

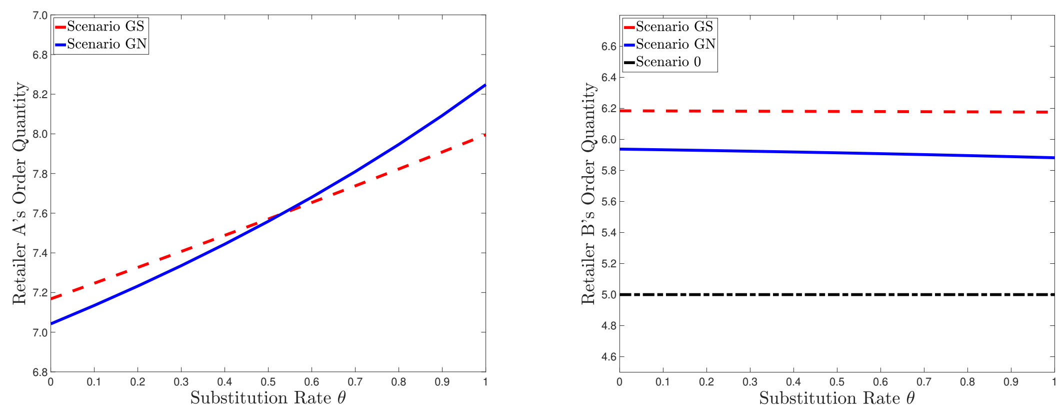
implicitly characterizes $y^*_{B,1}(y_{A,1})$. By analogous arguments, the first-period order quantities satisfy $\frac{y^G_{A,1}}{\alpha_{A,1}} \geq \frac{y^G_{B,1}}{\alpha_{B,1}}$ and are the solution to the system

$$\begin{cases} p_{A,1} - c_A + \rho(p_{A,2} - c_A)\gamma_{A,2} = [p_{A,1} + \rho(p_{A,2} - c_A)\gamma_{A,2}]F\left(\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta\alpha_{B,1}}\right) \\ p_{B,1} - c_B + \rho(p_{B,2} - c_B)\gamma_{B,2} = [p_{B,1} + \rho(p_{B,2} - c_B)(\gamma_{B,2} - \gamma^C_{B,2}\theta)]F\left(\frac{y_{B,1}}{\alpha_{B,1}}\right) \\ \qquad\qquad + \rho(p_{B,2} - c_B)\gamma^C_{B,2}\theta F\left(\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta\alpha_{B,1}}\right). \end{cases}$$

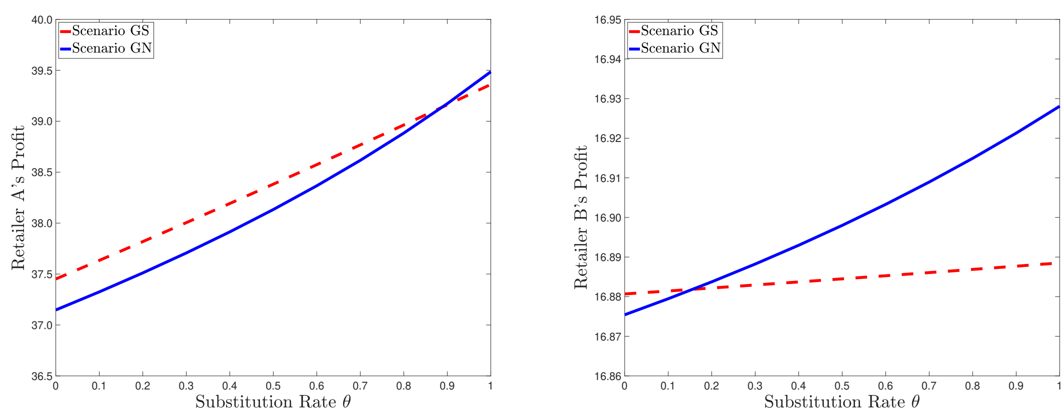
Thus, we have proved Lemma 7. *Q.E.D.*

Next, we test the robustness of our results in the scenario with the mix of within-brand and cross-brand C2C interactions. Figure 2 shows that our main results for the Exogenous Price Model still hold when both within-brand and cross-brand C2C hold. We focus on two general scenarios: In Scenario GS [GN], we choose $\lambda^S = 0.8$ and $\lambda^N = 0.2$ [$\lambda^S = 0.2$ and $\lambda^N = 0.8$]. Figure 2(a) compares retailer A's order quantities between the two scenarios. Figure 2(b) shows that retailer B's free-riding behavior in Scenario GN, i.e., its order quantity in Scenario GN decreases in substitution rate. Figure 2(c) [Figure 2(d)] generalizes the result in Proposition 5 [Proposition 6].

Figure 2 **Retailers' equilibrium order quantities and profits comparison with the mix of within-brand and cross-brand C2C interactions**



(a) Retailer A's order quantity; $c_A = 0.5$ (b) Retailer B's order quantity; $c_A = 0.5$



(c) Retailer A's profit comparison; $c_A = 0.1$ (d) Retailer B's profit comparison; $c_A = 0.92$

Parameters: We fix $p_{i,t} = 2 (i = A, B; t = 1, 2)$, $c_B = 1$, $\beta = 0.1$, $X_t \sim U[5, 15] (t = 1, 2)$. For Scenario GS, we choose $\gamma = 2$, $\lambda^S = 0.8$, and $\lambda^N = 0.2$; and for Scenario GN, we choose $\gamma = 2$, $\lambda^S = 0.2$, and $\lambda^N = 0.8$. For the benchmark (Scenario 0), we choose $\gamma = 0$.

We also test the robustness of our results in the Endogenous Price Model when both within-brand and cross-brand C2C interactions exist. For the above two scenarios, that is, Scenario GS ($\lambda^S = 0.8$ and $\lambda^N = 0.2$) and Scenario GN ($\lambda^S = 0.2$ and $\lambda^N = 0.8$), we adopt the same numerical setting in section 5.3. Our numerical results show that firms' equilibrium price may increase or decrease across periods, that is, the two intertemporal patterns shown in Proposition 9 still exist for the two scenarios. Our numerical experiments show that, if γ is large enough, firms' equilibrium prices may increase in both scenarios.