Personalized Sales Targets with Customer Choices

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Modern firms usually have personalized service/sales targets for different customer segments and products. Such sales target personalization poses additional challenges for firms to optimize their assortment policy. We propose a general modeling framework to study the assortment optimization problem in the presence of personalized sales targets. Our framework integrates both sales target optimization and personalized assortment planning as a two-stage stochastic program. Based on this modeling framework, we develop a family of simple and effective algorithms, the Debt-Weighted Assortment policy, and demonstrate their optimality for assortment planning with personalized sales targets. Our modeling framework is flexible enough to incorporate several important applications that unveil interesting insights. The first application explored in this paper is sales target personalization. In this application, we show that personalized sales targets may require a substantially higher cost to induce the necessary customer traffic. To address this issue, we propose a multi-sourcing strategy that efficiently re-balance customer types and personalized sales targets. Next, we apply our modeling framework to the classic personalized assortment optimization problem with inventory constraints. We propose a family of debt-weighted assortment algorithms that prove to be asymptotically optimal. Furthermore, the proposed algorithms achieve a higher expected revenue and a lower revenue variability than the benchmarks in the existing literature such as choice-based linear program and LP-resolving policies.

Key words: Online Convex Optimization, Personalized Sales Target, Assortment Optimization

1. Introduction

Market segmentation is an important tool to drive a firm's growth. According to the survey conducted by Bain & Company (Markey et al. 2006), 81% of executives treat market segmentation as a critical tool for growing profits. Selling the same product to different segments can bring different values to the firm. As a consequence, firms naturally set different penetration rate targets of different customer segments for a product. For example, the customer segment which consists of a firm's big promoters or experiences high growth rate can generate substantial long-term values and, therefore, will be set a high penetration rate target by the firm. Markey et al. (2006) show that targeting the right products or services to the right segments can help the firm obtain annual profit growth of about 15%. The popularity of online retailing, reaching a record-high global market size of 3535 billion US dollars in 2019[1], provides firms a new opportunity to target their desired segments with the right products. Customers from different segments generally exhibit substantially different sensitivities towards product price and quality (see Arslan et al. 2019, for an example in the sport ticketing industry). Therefore, a firm should customize its operational strategies (e.g., product assortment) to different customer segments, instead of a fitting a single policy to everyone. Furthermore, an online retailer can easily track the sales of each product to each customer, and set personalized sales targets for different customer segment and product mixes accordingly.

The sales target personalization is not limited to online retailing. Similar problems emerge in the online advertising context. A large advertising platform or publisher (e.g., Facebook) generally runs thousands of campaigns from different advertisers simultaneously. Each campaign is usually associated with a budget which the advertiser wishes to spend as much as possible. The platform needs to dynamically allocate its ad spaces (i.e., customer impressions) to these campaigns so that the budget of each advisor is mostly depleted (usually more than 80% of the total budget). Alternatively, the advertising platform may engage into contracts with advertisers under which the platform has to satisfy the click-through or conversion requirement of each advertiser (also called guaranteed display, see Aseri et al. 2017). In either case, an advertiser may require the platform to target his campaign and ads to specific customer segments (location, age, gender, social status, etc.). This will give rise to personalized sales target requirement for the advertising platform.

To meet the (personalized) sales targets, online retailing and advertising platforms face a central operations problem to select a product set (i.e., an assortment) offered to each arriving customer that could generate the highest value. The classical trade-off in assortment optimization with cardinality constraints (e.g. the number of products/ads fit into the first page or a mobile screen) is between the profitability of a product and the relative attractiveness of a product within a set. With the help of real-time data of arriving customers, platforms can utilize more accurate estimation of customers' preference over all products. Therefore, offering personalized assortment can address such trade-off at an individual customer level to both enhance customer experience (Arora et al. 2008) and improve revenue (Golrezaei et al. 2014). For example, Alibaba makes dynamic recommendations of product assortment to each arriving customer based on the features of the products and those of the customers (Feldman et al. 2018).

The (personalized) sales targets pose significant challenges for online retailing or advertising platforms to optimize their personalized assortment decisions. First, on top of classical trade-off faced in

[1] https://www.statista.com/statistics/379046/worldwide-retail-e-commerce-sales/

assortment optimization with cardinality constraints, the sales targets impose additional constraints so that the retailer/advertiser needs to wisely offer assortments to each customer segment so that the sales targets can be guaranteed. Second, the sales targets (possibly together with the customer traffic) are sometimes endogenously set by the decision maker or contracted with the upstream supplier (e.g., advertiser). In this case, the decision maker should wisely decide the sales targets. Higher sales targets will induce greater difficulty in offering personalized assortments to meet them, but, at the same time, may imply the potential to generate a higher revenue.

The main goal of this paper is to explore the personalized assortment planning problem in the presence of sales targets. The key questions we seek to address are: (a) **Sales target optimization**: What are the optimal customer traffic and sales targets for the firm? and (b) **Personalized assortment planning**: How should the firm personalize product assortments to meet the sales targets? We present a general assortment optimization framework with sales targets to study this two-stage decision making problem. We propose a simple randomized assortment policy that ensures the feasibility of the sales target for each product-customer segment pair, which in turn helps optimize the sales targets. Based on our general modeling framework, we study two important applications of personalized assortment optimization in the presence of sales target constraints, one on sales target personalization and the other on assortment optimization with limited inventory. Both applications unveil interesting insights, as introduced in detail below.

1.1. Main Contributions

The contributions of this paper can be summarized as follows:

General framework. We develop in this paper a general modeling framework to study assortment personalization in the presence of sales targets. Specifically, we consider a two-stage stochastic optimization model in which the retailer decides the segmented sales targets and customer traffic in the first stage, and the personalized assortments in the second. What distinguishes our framework from existing models in the literature is that we capture the essential feature of sales target requirement, which enforce the expected sales of each product to each customer segment to meet the pre-specified sales target. Such a new feature enables us to uncover interesting insights on the operational and managerial implications of sales target constraints on assortment optimization. Finally, the assortment personalization framework with sales targets is flexible enough to incorporate different objectives (such as revenue maximization and/or advertisement procurement cost minimization) and sales target constraints (personalized and non-personalized).

Debt-Weighted Assortment algorithms. As our main contribution, we propose a family of simple and effective algorithms, Debt-Weighted Assortment policy, and demonstrate their optimality. This policy assigns a “debt” on each sales target, which measures the difference between the realized total sales (i.e., customer choices) and the pre-set targets, to solve an assortment optimization problem that maximizes a debt-weighted revenue function upon the arrival of each customer, based on the customer type. The "debt" can be either calculated dynamically or randomly sampled from a sufficiently large set constructed offline. By re-weighting the product revenues with the debts that endogenize the sales targets, the algorithms also obtain the optimal solution for the first-stage of our general modeling framework.

Sales target personalization. Our modeling framework is flexible enough to incorporate several important applications that unveil interesting insights. The first application studied in this paper is sales target personalization. It has been well-acknowledged in the literature that target marketing can bring benefit to an individual firm (see, e.g., Rossi et al. 1996). However, when each individual firm requires targeting different customer segments with different sales goals, it may result in substantial burdens on the online retailing or advertising platform. We characterize the minimum customer traffic to secure the (personalized) sales targets and find that if the customer type distribution is highly unbalanced, sales target personalization may require a significantly higher customer traffic and thus a substantially higher cost to meet the sales targets than non-personalized targets. To address this potential drawback of sales target personalization, we show that multi-sourcing could efficiently re-balance customer types and personalized sales targets, thus bringing the total cost down to the same level as the non-personalized sales targets.

Assortment optimization with inventory constraints. Adaptive assortment personalization with inventory capacity constraints have attracted a lot of attention in the recent revenue management literature. We reformulate this problem in our modeling framework and develop a simple debt-weighted assortment algorithm that integrates inventory capacities. We show that this algorithm is asymptotically optimal as the problem size (sales horizon length and inventory) scales up to infinity. Through numerical experiments, we demonstrate that our algorithms lead to *lower revenue variance* than the well-studied linear program based heuristics. Furthermore, we improve the performance of our debt-weighted algorithm by aggregating individualized optimal sales targets together with respect to each product. This enhances the total revenue and reduces its variability. Our approach can be easily adapted to a network revenue management model with multiple products and multiple resources.

In summary, the key message of this paper is that the proposed personalized assortment optimization with sales targets framework together with the associated debt-weighted assortment algorithms could efficiently address a wide variety of problems, ranging from online advertising to online retailing. Our approach is simple and efficient, with provable optimality guarantee and strong numerical performances. The rest of this paper is organized as follows. We review related literature in Section 2. The general modeling framework is introduced in Section 3 and the debt-weighted assortment algorithms are proposed in Section 4. We apply our framework and algorithms to address the personalized sales target problem in Section 5 and the assortment personalization problem with inventory constraints in Section 6. Section 7 concludes this paper. All proofs are relegated to the Appendix.

2. Literature Review

This paper proposes a general modeling framework and efficient algorithms to integrate assortment optimization and personalized sales targets. We are primarily related to two streams of research in the literature: (a) resource allocation with individualized service level constraints and (b) (dynamic) personalized assortment optimization. Papers in the relevant literature generally focus on one perspective of the two topics above, whereas our work studies both decisions jointly.

The resource allocation problem to meet service target constraints in the face of uncertain demand has been extensively studied in the inventory literature (see, e.g., Eppen 1979, Swaminathan and Srinivasan 1999, Alptekinoğlu et al. 2013). A lot of recent progress has been made to tackle this problem using the approaches inspired by online convex optimization (Hazan et al. 2016) and Blackwell's approachability theorem (Blackwell et al. 1956). For example, Hou et al. (2009) study the single-resource allocation in wireless networks with quality of service (QoS) constraints, which are essentially the same as the Type-II service level constraints in the inventory literature. In a similar vein, Zhong et al. (2017) characterize the optimal safety-stock level with individual Type-II service level constraints. Lyu et al. (2019) and Lyu et al. (2017), respectively, extend both the approach and results to the context of Type-I service level constraint and process flexibility. Utilizing a semi-infinite linear program formulation, Jiang et al. (2019) generalize and unify models in this literature and propose a simple randomized rationing policy to meet general service-level constraints, including Type-I and Type-II constraints, and beyond. Our contribution towards this literature is that we generalize the concept of service level constraints to incorporate customer choice uncertainty and indirect resource allocation through assortment planning. We also propose a family of debt-weighted assortment algorithms and demonstrate their optimality to meet the service level constraints.

In the recent decade, online e-commerce platforms typically provide numerous products for customers to choose from (Feldman et al. 2018). Manufacturing firms have also expanded their product lines due to business trends (e.g., fast fashion, Caro and Gallien 2007, Caro et al. 2014) or technology revolution (e.g., 3D printing, Dong et al. 2017). The ever expanding product pool makes personalized assortment more attractive. Therefore, personalized assortment optimization has also received growing attention in the literature. With inventory capacity constraints, Bernstein et al. (2015) demonstrate the optimality of threshold policy and show that it may be optimal to limit the choice set of some customers even if the products are in-stock. Leveraging the competitive ratio framework, Golrezaei et al. (2014) propose the inventory-balancing algorithms that guarantee the worst-case revenue performance without any forecast of the customer type distribution. Bernstein et al. (2018) combine dynamic assortment planning, demand learning, and customer type clustering in a Bayesian framework and propose a prescriptive assortment personalization approach for online retailing. Using re-solving heuristics, Jasin and Kumar (2012) study dynamic personalized assortment planning in a network revenue management framework and show that the proposed heuristics achieve a constant optimality loss. Kallus and Udell (2016) consider a dynamic assortment personalization problem with a large number of items and customer types as a discrete-contextual bandit problem and propose a structural approach with efficient optimization algorithms. Chen et al. (2016) formalize a new checkout recommender system at Walmart's online grocery as an online assortment optimization problem with limited inventory and propose an inventory-protection algorithm with a bounded competitive ratio. A general personalized resource allocation model with customer choices is studied by Gallego et al. (2016). Adopting the column-generation approach to solve the choice based linear program, the authors introduce algorithms with theoretical performance guarantees. Considering the uncertainty in estimating the MNL choice model, Cheung and Simchi-Levi (2017) propose a Thompson Sampling based policy to estimate the latent parameters by offering personalized assortment and demonstrate its near optimality. Our contribution towards this literature is that we propose a new family of debt-weighted assortment policies that prove to be asymptotically optimal and generate revenue with lower variance than the benchmarks in the literature such as choice-based linear program (CBLP) and re-solving heuristics.

3. Model

In this section, we propose a general modeling framework for the assortment optimization problem with personalized sales targets. Consider n products, indexed by $i \in \mathcal{N} := \{1, 2, ..., n\}$, and m customer types, indexed by $j \in \mathcal{M} := \{1, ..., m\}$. Customers arrive in a sequential manner over a $T-$period horizon $\mathcal{T} = \{1, 2, ..., T\}$. We assume at most one customer arrives in period t, and the probability that a customer arrives is $\lambda \in (0, 1]$. Thus, λ parameterizes the customer traffic. The higher the λ, the higher the traffic of arriving customer stream. For each customer, her type $\tilde{\xi}'$ follows a discrete distribution on $\mathcal{M}$, with $\mathbb{P}(\tilde{\xi}' = j) = p_j$ where $j \in \mathcal{M}$ and $\sum_{j\in\mathcal{M}} p_j = 1$. We also define the unconditional customer type $\tilde{\xi} = \tilde{\xi}'$ if a customer of type $\tilde{\xi}' \in \mathcal{M}$ actually arrives. Otherwise, we define $\tilde{\xi} := 0$. Therefore, the unconditional probability that a new-arriving customer is of type $j \in \mathcal{M}$ equals $\mathbb{P}(\tilde{\xi} = j) = \lambda p_j$, and the probability that no customer shows up is $\mathbb{P}(\tilde{\xi} = 0) = 1 - \lambda$. We denote the distribution of $\tilde{\xi}$ on $\mathcal{M} \cup \{0\}$ as $\mathcal{P}$.

For a customer of type $j \in \mathcal{M}$, if a choice set/assortment $S \subset \mathcal{N}$ is offered, she will consume at most one product in the assortment S. Specifically, for each customer type j, each assortment S, and each product $i \in S$, define $\tilde{y}_i^j(S) \in \{0, 1\}$ as the random indicator variable of whether the customer of type j chooses product i. Thus, the probability that a type-j customer chooses product i given assortment S is $\phi_i^j(S) := \mathbb{E}[\tilde{y}_i^j(S)]$. For now, we do not specify any structure of the choice model, except that a customer chooses at most one product, i.e., $\sum_{i \in S} \tilde{y}_i^j(S) \leq 1$ almost surely, which implies that $\sum_{i \in S} \phi_i^j(S) \leq 1$. If $i \notin S$, by convention, $\tilde{y}_i^j(S) = 0$ and, thus, $\phi_i^j(S) = 0$.

The firm is involved in a two-stage decision problem. In the first-stage, knowing the distribution of customer type $\mathcal{P}$ and the choice probabilities $\Phi := \{\phi_i^j(S) : j \in \mathcal{M}, i \in S \subset \mathcal{N}\}$, but not the realizations ξ and the choice of the customer $y_i^j(S)$, the firm decides the total customer traffic λ and the *desired* sales targets $\alpha = \{\alpha_i^j \in [\hat{\alpha}_i^j, \lambda] : i \in \mathcal{N}, j \in \mathcal{M}\}$, where $\hat{\alpha}_i^j$ is the *required* sales target of product i to customer segment j. The required sales target for customer type j and product i, $\hat{\alpha}_i^j$, imposes a constraint that the (unconditional) sales of product i to type-j customer is at least as high $\hat{\alpha}_i^j$. If the firm is an online advertising demand-side platform (DSP), for example, $\hat{\alpha}_i^j$ can be interpreted as the minimum number of per-period click-troughs for Ad-i by type j users. If the firm is an e-commerce retailer, $\hat{\alpha}_i^j$ is the minimum sales of product i to type j customers. If the firm is an advertising platform, $\hat{\alpha}_i^j$ is the guaranteed target set by the advertisers. Based on Markey et al. (2006), targeting the right products or services to the right segments can help the firm obtain annual profit growth of about 15%. The firm sets the desired sales target $\alpha_i^j \geq \hat{\alpha}_i^j$, which prescribes the expected sales the firm wishes to achieve for product i to customer segment j. The payoff of the firm associated with the decision (λ, α) is denoted by $f(\lambda, \alpha)$. The objective function of the firm, $f(\cdot, \cdot)$, can take a very general form. In this paper, we focus on the following specific form of $f(\cdot, \cdot)$:

$$f(\lambda, \alpha) := \sum_{i \in \mathcal{N}, j \in \mathcal{M}} r_i^j \cdot \alpha_i^j - c(\lambda),$$

where $r_i^j > 0$ is the profit margin of selling a product $i \in \mathcal{N}$ to customer $j \in \mathcal{M}$ and $c(\lambda)$ is the marketing cost of securing customer traffic λ, which is strictly increasing in λ. Therefore, $f(\lambda, \alpha)$ is the net profit of the firm associated with the first-stage decisions (λ, α). We denote $\mathcal{D} \subset \mathbb{R}^{1+n \cdot m}$ the set of all feasible marketing investment and desired sales target decisions, i.e., $(\lambda, \alpha) \in \mathcal{D}$.

In the second-stage, the customer type $\tilde{\xi}$ realizes, after which the firm decides the assortment offered to this customer according to a (possibly randomized) general policy $\tilde{G} \in \mathcal{G}$, where $\mathcal{G}$ is the set of all feasible assortment policies. We denote $\tilde{S}^j(\tilde{G}, \xi)$ as the (randomized) assortment, under policy $\tilde{G}$, offered to type j customer if the realized type is ξ. If the realized customer type $\xi \neq j$, $\tilde{S}^j(\tilde{G}, \xi) = \emptyset$. The firm is obliged to meet the *desired* sales targets, i.e., under the assortment policy $\tilde{G}$,

$$\mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G}, \tilde{\xi}))] \geq \alpha_i^j, \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M}.$$

The assortment prescribed by policy $\tilde{G}$ offered to a customer of type j when the realized type is ξ, $\tilde{S}^j(\tilde{G}, \xi)$, can also be equivalently represented by a random vector $\tilde{x}^j(\tilde{G}, \xi) :=$

$(\tilde{x}_1^j(\tilde{G},\xi), \tilde{x}_2^j(\tilde{G},\xi), ..., \tilde{x}_n^j(\tilde{G},\xi))' \in \{0,1\}^n$, where $\tilde{x}_i^j(\tilde{G},\xi) = 1$ if and only if $j = \xi$ and $i \in \tilde{S}^j(\tilde{G},\xi)$. The random vector representation facilitates us to impose some natural constraints for the assortment decision $\tilde{G}$. For example, the cardinality constraint (Rusmevichientong et al. 2010, Wang 2012) can be formulated as: $\sum_{i\in\mathcal{N}} \tilde{x}_i^j(\tilde{G},\xi) \le K$ for all $j \in \mathcal{M}$, where K is the maximum size of an assortment offered to any customer. More generally, the totally unimodular constraint (Davis et al. 2013), which includes the cardinality constraint as a special case, can be incorporated as: $A \cdot \tilde{x}^j(\tilde{G},\xi) \le b$ for all $j, \xi \in \mathcal{M}$, where A is a totally unimodular matrix.

We are now ready to formulate the firm's two-stage decision making problem as a stochastic program. Specifically, to maximize the total profit/value $f(\lambda,\alpha)$, the firm needs to jointly decide the marketing investment and desired sales targets $(\lambda,\alpha) \in \mathcal{D}$ (in the first-stage) and the assortment policy $\tilde{G} \in \mathcal{G}$ (in the second-stage) subject to the sales target constraints:

$$\begin{aligned} \max_{\lambda,\alpha,\tilde{G}} \ & f(\lambda,\alpha) \\ s.t. \ & \mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G},\tilde{\xi}))] \ge \alpha_i^j, \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M} \\ & \alpha_i^j \ge \hat{\alpha}_i^j, \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M} \\ & (\lambda,\alpha) \in \mathcal{D} \\ & \tilde{G} \in \mathcal{G} \end{aligned} \tag{1}$$

Note that although (1) is formulated as a single-period two-stage stochastic optimization, we can also interpret the model as a periodic-review infinite horizon problem, which will prove useful in the characterization of the optimal policy and in the applications of our model. Recall that customers arrive in a sequential manner and at most one customer arrives in period $t \in \mathcal{T}$. Denote the customer type in period t by $\tilde{\xi}_{(t)} \in \mathcal{M} \cup \{0\}$ and assume $\{\tilde{\xi}_{(t)} : t \ge 1\}$ are i.i.d. random variables following distribution $\mathcal{P}$. Denote the realized choice of the customer arriving at period t as $y_i^j(t) \in \{0,1\}$, where $y_i^j(t) = 1$ if and only if $\xi_{(t)} = j$ and the customer chooses product i. The assortment offered to type j customer in period t is denoted by $S_{(t)}^j(\xi_{(t)}, \mathbf{y}(1:t-1))$, where $\mathbf{y}(1:t-1) = (y_i^j(s) : j \in \mathcal{M}, i \in \mathcal{N}, 1 \le s \le t-1)$. The assortment policy $S_{(t)}^j(\xi_{(t)}, \mathbf{y}(1:t-1))$ explicitly formalizes the dependence of the assortment offered to a type j customer in period t on the customer type of this period and on the realized customer choices up to period $t-1$. Clearly, if $j \ne \xi_{(t)}$, $S_{(t)}^j(\xi_{(t)}, \mathbf{y}(1:t-1)) = \emptyset$. Denote $\mathbf{S} := (S_{(t)}^j(\cdot,\cdot) : j \in \mathcal{M}, t \ge 1)$ as the full assortment strategy profile throughout the horizon. The single-period formulation (1) can be interpreted/re-formulated in the following infinite-horizon periodic review sense:

$$\begin{aligned} \max_{\lambda,\alpha,\mathbf{S}} \ & f(\lambda,\alpha) \\ s.t. \ & \liminf_{T\to+\infty} \frac{1}{T} \sum_{t=1}^{T} \tilde{y}_i^j(S_{(t)}^j(\xi_{(t)}, \mathbf{y}(1:t-1))) \ge \alpha_i^j \ a.s., \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M} \\ & \alpha_i^j \ge \hat{\alpha}_i^j, \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M} \\ & (\lambda,\alpha) \in \mathcal{D} \end{aligned} \tag{2}$$

Note that similar interpretation/reformulation has also been adopted in the resource allocation and inventory pooling literature (e.g. Zhong et al. 2017, Jiang et al. 2019).

To conclude this section, we remark that our model is flexible enough to incorporate different objectives of the firm. For example, the firm may be an advertising DSP who seeks to minimize the total marketing investment cost only, which is equivalent to minimizing the total customer traffic λ subject to the sales target constraints. Alternatively, the firm may be an online retailer who optimizes the assortment decisions to maximize its total revenue from selling the products. We will discuss how our general model formulations (1) and (2) can be adapted to these applications in Sections 5 and 6. In the next section, we first characterize the necessary and sufficient condition under which the sales targets can be satisfied. We also propose a family of debt-weighted assortment algorithms with provable optimality guarantee.

4. Debt-Weighted Assortment Algorithm

Our analysis starts with a complete characterization of the necessary and sufficient condition for the feasibility of the customer traffic λ and desired sales targets α. This condition also empowers us to propose a dynamic adaptive assortment policy, which we call the *debt-weighted assortment* algorithms under which the sales targets are met. The thrust of our characterization is to leverage duality and online convex optimization (OCO) to bound the gaps between the realized sales of each product to each customer segment and the respective desired sales targets.

4.1. Necessary Condition

To characterize the feasibility of the desired sales targets, we consider the following formulation with a constant objective function:

$$\begin{aligned} &\max_{\tilde{G}} 0 \\ &s.t.\ \mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G},\tilde{\xi}))] \geq \alpha_i^j, \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M} \\ &\qquad \tilde{G} \in \mathcal{G} \end{aligned} \tag{3}$$

We now seek to understand under what conditions of (λ, α) will the stochastic program (3) have a feasible solution. The formulation (3) is non-linear, so we first reformulate it as a semi-infinite linear program (SLP).

Define the set of all possible deterministic policies as $\hat{G}$, where $G \in \hat{G}$ is a plausible deterministic policy. Under policy G, let $S^j(G,\xi)$ be the assortment prescribed by policy G to a customer of type j if a customer of type ξ arrives. For a randomized policy $\tilde{G}_\mu \in \mathcal{G}$, it is defined by a probability measure $\mu(\cdot)$ on $\hat{G}$. Note that, under a deterministic policy $G \in \hat{G}$, the service level of product i for a customer of type j is given by

$$\lambda p_j \phi_i^j(S^j(G,j))$$

Therefore, (3) can be reformulated as the following SLP.

$$
\begin{aligned}
&\max_{\mu(\cdot)} 0 \\
&s.t. \int_{G\in\hat{G}} \lambda p_j \phi_i^j(S^j(G,j))\mathrm{d}\mu(G) \geq \alpha_i^j, \text{ for each } i\in\mathcal{N} \text{ and } j\in\mathcal{M} \\
&\quad \int_{G\in\hat{G}} \mathrm{d}\mu(G) = 1 \\
&\quad \mathrm{d}\mu(G)\geq 0 \text{ for all } G\in\hat{G}.
\end{aligned} \tag{4}
$$

Taking the dual of the SLP (4), we obtain that:

$$
\begin{aligned}
&\min_{\theta_0,\theta_i^j} \{\theta_0 - \sum_{i\in\mathcal{N},j\in\mathcal{M}} \alpha_i^j\theta_i^j\} \\
&s.t. \sum_{i\in\mathcal{N},j\in\mathcal{M}} \lambda p_j\phi_i^j(S^j(G,j))\theta_i^j - \theta_0 \leq 0, \text{ for all } G\in\hat{G} \\
&\quad \theta_i^j \geq 0 \text{ for all } i\in\mathcal{N} \text{ and } j\in\mathcal{M}.
\end{aligned} \tag{5}
$$

Note that (5) is equivalent to

$$
\min_{\theta_i^j\geq 0}\left\{\max_{G\in\hat{G}} \sum_{i\in\mathcal{N},j\in\mathcal{M}} \lambda p_j\phi_i^j(S^j(G,j))\theta_i^j - \sum_{i\in\mathcal{N},j\in\mathcal{M}} \alpha_i^j\theta_i^j\right\} \geq 0 \tag{6}
$$

It then follows immediately from weak duality that (3) is feasible only if the minimum of the left-hand side of (6) is greater than or equal to 0. Based on this observation, the following theorem establishes the necessary condition for the feasibility of sales target constraints.

THEOREM 1. (NECESSARY CONDITION) *The two-stage decision making problem under sales target constraints* (1) *is feasible only if*

$$
\max_{G\in\hat{G}} \sum_{i\in\mathcal{N},j\in\mathcal{M}} \lambda p_j\phi_i^j(S^j(G,j))\theta_i^j \geq \sum_{i\in\mathcal{N},j\in\mathcal{M}} \alpha_i^j\theta_i^j \textit{ for all } \theta_i^j\geq 0 \ (i\in\mathcal{N}, j\in\mathcal{M}) \tag{7}
$$

The left-hand side of inequality (7) can be viewed as an personalized assortment optimization problem. Specifically, for each customer type j, we seek to offer an assortment S^{j*} that maximizes the total revenue from this customer type with the price of product i equal to $\lambda p_j\theta_i^j$, i.e.,

$$
S^{j*}(\theta) = \arg\max_S \sum_{i\in S} \lambda p_j\theta_i^j\phi_i^j(S) \tag{8}
$$

Given (λ,θ), we define

$$
g(\lambda,\theta) := \sum_{j\in\mathcal{M}} \max_S \sum_{i\in S} \lambda p_j\theta_i^j\phi_i^j(S),
$$

which is the left-hand side of (7). Hence, we obtain an equivalent necessary condition for the feasibility of the desired sales targets (4):

$$
\begin{aligned}
&\min_{\theta_i^j\geq 0} h(\theta|\lambda,\alpha)\geq 0, \\
&\text{where } h(\theta|\lambda,\alpha) := g(\lambda,\theta) - \sum_{i\in\mathcal{N},j\in\mathcal{M}} \alpha_i^j\theta_i^j
\end{aligned} \tag{9}
$$

4.2. Sufficient Condition and Assortment Policy

We have so far established the necessary condition for the feasibility of the desired sales targets. One may wonder whether the necessary condition (7), equivalently (9), is also sufficient. Our next result shows that the answer is yes.

THEOREM 2. (SUFFICIENT CONDITION) *If* (7) *holds, i.e.,*

$$h(\theta|\lambda,\alpha) \geq 0 \text{ for all } \theta_i^j \geq 0 \ i \in \mathcal{N}, j \in \mathcal{M},$$

then (λ,α) *is feasible. Furthermore,* $h(\cdot|\lambda,\alpha)$ *is jointly convex for any* (λ,α).

Theorems 1 and 2 demonstrate that checking the feasibility of the two-stage stochastic program (1) is reduced to minimizing a convex function $h(\cdot|\lambda,\theta)$ over the non-negative quadrant $\{\theta_i^j \geq 0 : i \in \mathcal{N}, j \in \mathcal{M}\}$. Hence, as long as the personalized assortment optimization problem (8) is tractable, one could numerically check the feasibility of (λ,α). Finally, we remark that Theorem 2 essentially establishes the strong duality between the SLP (3) and its dual (5).

Although Theorem 2 does not provide a policy $\tilde{G}$ that meet the sales targets, we could develop an algorithm to generate the (randomized) vector $\{\theta_i^j \geq 0 : i \in \mathcal{N}, j \in \mathcal{M}\}$. Specifically, we consider the infinite-horizon periodic review version of the problem, (2), in which customers arrive sequentially and at most one customer arrives in each period. We present the following Debt-Weighted-Assortment (DWA) policy, denoted as $\tilde{G}_{DWA}$.

Algorithm 1 DEBT-WEIGHTED-ASSORTMENT (DWA) POLICY

Initialize: $d_i^j(1) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

For each period $t \geq 1$**:**

1: Observe the customer type $\xi_{(t)} = \hat{j}$.

2: If $\hat{j} = 0$, assign customer choices $y_i^j(t) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

3: If $\hat{j} \in \mathcal{M}$, offer assortment $S^*(t)$ to the customer, where

$$S^*(t) \leftarrow \arg\max_S \sum_{i \in S} \left(d_i^{\hat{j}}(t)\right)^+ \phi_i^{\hat{j}}(S), \tag{10}$$

which generates customer choices $(y_i^j(t) : i \in \mathcal{N}, j \in \mathcal{M})$.

4: $d_i^j(t+1) \leftarrow d_i^j(t) + \alpha_i^j - y_i^j(t)$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

Similar to checking the feasibility of (λ,α), the DWA policy involves solving an assortment optimization (10) upon the arrival of each customer. The personalized assortment $S^*(t)$ can be efficiently found for a broad class of choice models such as multi-normial logit (MNL) models. We call Algorithm

1 as the *Debt-Weighted-Assortment* policy because the assortment optimization is weighted by the "debt" of each customer-product pair in periods $\{1,2,...,t-1\}$. Note that $(t-1)\alpha_i^j$ is the average sales of product i to customer of type j if the sales target α_i^j is barely satisfied, whereas $\sum_{s=1}^{t-1} y_i^j(s)$ is the realized total sales. Therefore, $\left(d_i^j(t)\right)^+ = \max\left((t-1)\alpha_i^j - \sum_{s=1}^{t-1} y_i^j(s), 0\right)$ is the total "debt" owed by the firm to the desired sales target associated with product i and customer type i in period t. Also note that the debts of period t, $\{d_i^j(t) : i \in \mathcal{N}, j \in \mathcal{M}\}$, only depend on $\{\xi_{(s)} : s = 1,2,...,t-1\}$ and $\{y_i^j(s) : 1,2,...,t-1\}$ and are independent of any information revealed in period t' for $t' \geq t$. Therefore, the DWA policy is non-anticipative. Algorithm 1 can also be implemented as a single-period randomized policy, under which we generate a randomized sample of $\{\xi_{(t)} : t \geq 1\}$ and $\{y_j^i(t) : t \geq 1\}$ with a sufficient large sample size, based on which the (randomized) debts and assortment for each customer type are constructed. The randomized version of the Debt-Weighted-Assortment policy (the RDWA policy) is given as Algorithm 4 in Appendix A.

We now prove the feasibility of the Debt-Weighted-Assortment policy.

THEOREM 3. (FEASIBILITY OF DWA) *If* (λ, α) *is feasible, i.e., inequality* (7) *holds, then we have*

$$\liminf_{T\to+\infty} \frac{1}{T} \sum_{t=1}^{T} \tilde{y}_i^j(\tilde{S}^*(t)) \geq \alpha_i^j \text{ with probability 1 for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M},$$

where the assortments $\{\tilde{S}^*(t) : t \geq 1\}$ *are prescribed by the Debt-Weighted-Assortment policy* $\tilde{G}_{DWA}$.

We remark that the Debt-Weighted-Assortment policy is developed upon the theory of Blackwell's Approachability and online convex optimization. In different contexts of inventory and resource allocation, these techniques have also been adopted by Zhong et al. (2017) and Jiang et al. (2019) to develop resource allocation algorithms to meet service level constraints. A key difference between our approach and theirs is that we cannot fully control the behavior of the customers so that some dedicate analysis of the assortment optimization model is required to develop the Debt-Weighted-Assortment policy. We have now closed the loop for the analysis of the second-stage program for the assortment optimization problem with personalized sales targets. Not only do we offer the necessary and sufficient feasibility condition, but a feasible policy is also constructed under such a condition.

4.3. Feasibility of Sales Targets for the MNL Choice Model

For a general choice model, $\{\phi_i^j(\cdot) : j \in \mathcal{M}, i \in \mathcal{N}\}$, the necessary and sufficient condition for feasible desired sales targets (7), or equivalently (9), may be difficult to check. In this next subsection, we will focus on the Multi-Normial Logit choice model to obtain sharper insights on the feasible set of the first-stage decisions (λ, α).

As explained in Section 3, we could use an equivalent binary variable representation of an assortment policy $G \in \hat{G}$. More specifically, a deterministic policy G can be equivalently represented by an

$(mn)-$dimensional binary vector $\mathbf{x} = (x_i^j \in \{0,1\} : i \in \mathcal{N}, j \in \mathcal{M})$, where $x_i^j = 1$ means that product i is included in the assortment offered to customer type-j. Denote the set of all plausible assortments as X. With a slight abuse of notation, we denote $\phi_i^j(\mathbf{x})$ as the probability that a customer of type j would choose product i if the assortment offered to this customer is represented by $\mathbf{x}$. Under the MNL model, we have

$$\phi_i^j(\mathbf{x}) = \frac{v_i^j x_i^j}{1 + \sum_{i' \in \mathcal{N}} v_{i'}^j x_{i'}^j}, \tag{11}$$

where v_i^j is the attractiveness of product i to customer type j. Applying Theorems 1 and 2 to the MNL choice model, we have the following corollary.

Corollary 1. *If the customer behavior follows the MNL model, (α, λ) is feasible if and only if*

$$\max_{\mathbf{x} \in X} \sum_{i \in \mathcal{N}, j \in \mathcal{M}} \frac{\lambda p_j v_i^j \theta_i^j x_i^j}{1 + \sum_{i' \in \mathcal{N}} v_{i'}^j x_{i'}^j} \geq \sum_{i \in \mathcal{N}, j \in \mathcal{M}} \alpha_i^j \theta_i^j \text{ for all } \theta_i^j \geq 0 \ (i \in \mathcal{N}, j \in \mathcal{M}) \tag{12}$$

Leveraging the structural properties of the MNL model, we can relax the integer constraints on $\mathbf{x}$ and apply a change of variable argument (see the proof of Proposition 1 in Appendix C for details) to give the following sharper and simpler characterization (as the solution to a linear program) for the feasibility of the desired sales targets under the MNL choice model.

Proposition 1. *If customers follow the MNL choice model* (11) *and the size of an assortment cannot exceed K, we have (λ, α) is feasible if and only if there exist $(y_i^j : i \in \mathcal{N}, j \in \mathcal{M})$ and $(z^j : j \in \mathcal{M})$ that satisfy the following:*

$$\begin{aligned}
&\lambda p_j v_i^j y_i^j \geq \alpha_i^j, \quad \forall i, j \\
&\sum_{i=1}^{n} v_i^j y_i^j + z^j = 1, \quad \forall j, \\
&\sum_{i=1}^{n} y_i^j \leq K z^j, \quad \forall j, \\
&y_i^j \leq z^j, \quad \forall i, j, \\
&0 \leq y_i^j \leq 1, \quad \forall i, j,
\end{aligned} \tag{13}$$

where $z^j := 1/(1 + \sum_{i'} v_{i'}^j x_{i'}^j)$ and $y_i^j := x_i^j z^j = x_i^j/(1 + \sum_{i'} v_{i'}^j x_{i'}^j)$.

Note that the feasibility condition (13) involves auxiliary decision variables $(y_i^j : i \in \mathcal{N}, j \in \mathcal{M})$ and $(z^j : j \in \mathcal{M})$. One may question whether we can further streamline the characterization of the feasibility condition by removing these auxiliary decision variables. To this end, we first fix the desired sales targets α and derive the smallest customer traffic $\lambda^*(\alpha)$ under which (λ, α) is feasible. Hence, $\lambda^*(\alpha)$ characterizes the smallest customer traffic to satisfy the desired sales targets α. Reversely, we fix the customer traffic λ and derive the feasible region of the desired sales targets, $\mathcal{A}(\lambda)$. Thus, $\mathcal{A}(\lambda)$ characterizes all the desired sales targets α that could be satisfied under the customer traffic λ.

PROPOSITION 2. *If customers follow the MNL choice model* (11) *and the size of an assortment cannot exceed K, the following statements hold:*

(a) Given α, (λ,α) is feasible if and only if $\lambda \geq \lambda^*(\alpha)$, where

$$\lambda^*(\alpha) := \max_j \Big\{ \frac{1}{p_j} (\max\{\frac{1}{K}\sum_{i=1}^n \frac{\alpha_i^j}{v_i^j}, \max_i\{\frac{\alpha_i^j}{v_i^j}\}\} + \sum_{i=1}^n \alpha_i^j) \Big\} \tag{14}$$

(b) Given λ, (λ,α) is feasible if and only if $\alpha \in \mathcal{A}(\lambda)$, where

$$\mathcal{A}(\lambda) := \Big\{ \alpha \in [0,1]^{n\times m} : \frac{\alpha_i^j}{v_i^j} + \sum_{i=1}^n \alpha_i^j \leq p_j\lambda, \forall i,j, \text{ and } \frac{1}{K}\sum_{i=1}^n \frac{\alpha_i^j}{v_i^j} + \sum_{i=1}^n \alpha_i^j \leq p_j \lambda, \forall j \Big\} \tag{15}$$

The characterization of $\lambda^*(\alpha)$ and $\mathcal{A}(\alpha)$ reveals further insights. To understand the minimum traffic $\lambda^*(\alpha)$ and the feasible desired sales targets $\mathcal{A}(\lambda)$, we observe that (14) and (15) are equivalent to

$$\lambda p_j \geq \max\left\{ \frac{1}{K}\sum_{i=1}^n \frac{\alpha_i^j}{v_i^j}, \max_i\{\frac{\alpha_i^j}{v_i^j}\} \right\} + \sum_{i=1}^n \alpha_i^j, \text{ for all customer type } j.$$

Here, λp_j is the expected traffic of type j customers. Clearly, $\sum_{i=1}^n \alpha_i^j$ is the total required traffic for type-j customers if a customer will choose one of the offered product in the assortment with probability 1. In practice, however, a customer may end up not choosing any product from the assortment, so we need some buffer traffic for type-j customer that accounts for the non-purchase probability.

More specifically, let $\mathcal{S}_i$ denote the set of all assortments containing product i offered to a customer. Since the policy may be random, we define $\mu_j(S)$ as the probability of offering assortment $S \subset \mathcal{N}$ to type j customers. Thus, the desired sales target constraint for product i and type-j customer is

$$\sum_{S\in\mathcal{S}_i} \mu_j(S) \cdot \frac{v_i^j}{1+\sum_{i'\in S} v_{i'}^j} \geq \alpha_i^j.$$

Thus, the non-purchase probability of type j customer when product i is offered satisfies that

$$\alpha_0^j(i) := \sum_{S\in\mathcal{S}_i} \mu_j(S) \cdot \frac{1}{1+\sum_{i'\in S} v_{i'}^j} \geq \frac{\alpha_i^j}{v_i^j}$$

Therefore, to ensure the sales target constraint of customer type j and product i, the traffic of customer type j must satisfy $\lambda p_j \geq \sum_{i'} \alpha_{i'}^j + \alpha_0^j(i) \geq \sum_{i'} \alpha_{i'}^j + \frac{\alpha_i^j}{v_i^j}$ for all $i \in \mathcal{N}$.

The cardinality constraint for assortment size would impose an additional bound on the non-purchase probability of type j customers. Specifically, let $\mathcal{S} := \bigcup_{i=1}^n \mathcal{S}_i$ be the set of all assortments offered to a customer. Because $|S| \leq K$ for any $S \in \mathcal{S}$, $|\{i \in \mathcal{N} : S \in \mathcal{S}_i\}| \leq K$ for all S. We have, given customer type j,

$$K\sum_{S\in\mathcal{S}} \mu_j(S) \cdot \frac{1}{1+\sum_{i\in S} v_i^j} \geq \sum_{i=1}^n \sum_{S\in\mathcal{S}_i} \mu_j(S) \cdot \frac{1}{1+\sum_{i\in S} v_{i'}^j} \geq \sum_{i=1}^n \frac{\alpha_i^j}{v_i^j}.$$

Thus, the non-purchase probability of type j customer satisfies that

$$\alpha_0^j := \sum_{S \in \mathcal{S}} \mu_j(S) \cdot \frac{1}{1 + \sum_{i \in S} v_i^j} \geq \frac{1}{K} \sum_{i=1}^{n} \frac{\alpha_i^j}{v_i^j}.$$

Therefore, given the cardinality constraint of assortments, to ensure the sales target constraint of customer type j with respect to all products, the traffic of customer type j must satisfy $\lambda p_j \geq \sum_i \alpha_i^j + \alpha_0^j \geq \sum_i \alpha_i^j + \frac{1}{K} \sum_i \frac{\alpha_i^j}{v_i^j}$. In sum, the characterization for the feasibility of (λ, α) demonstrates that, to meet the sales target constraints, we should not only care for sales targets themselves, but the assortment policy must take into account the *non-purchase* probabilities of the customers as well.

Finally, we remark that if the customer choice model is MNL, the DWA algorithm can be efficiently implemented. The most time-and-space-consuming step of the algorithm to decide the assortment upon the arrival of each customer (i.e., equation (10)) can be computed easily either through solving a linear program or applying a geometric algorithm (Talluri and Van Ryzin 2004, Rusmevichientong et al. 2010, Davis et al. 2013).

5. Sales Target Personalization

An intriguing question arising from our context and model is how would the personalized sales targets affect the firm's optimal decisions and performances. On the one hand, personalized sales targets can bring the platforms more revenue (Markey et al. 2006). On the other hand, such targets also require the platforms to better manage their consumer traffic. Those giant retail platforms are always seeking more ways to attract consumer traffic for their business when the current channels are saturated. For example, Alibaba sets up various offline pop-up stores to allude more customers to their online platform (Zhang et al. 2019). The analysis in this section reveals another benefit of seeking various consumer traffic sources other than demand expansion: To balance the personalized sales targets with different customer characteristics from various traffic sources.

In this section, we consider a setting where the required sales targets $(\hat{\alpha}_i^j : i \in \mathcal{N}, j \in \mathcal{M})$ are pre-specified by some contractual agreement so that the firm has to satisfy. As a consequence, the objective of the firm is to minimize the marketing investment cost $c(\lambda)$ under the constraint that the required personalized sales targets can be ensured. Therefore, the desired sales target satisfies that $\alpha_i^j = \hat{\alpha}_i^j$ for each product i and customer segment j. The firm's cost minimization problem can be formulated as follows:

$$\begin{aligned} \min_{0 \leq \lambda \leq 1, \tilde{G}} \quad & c(\lambda) \\ \text{s.t.} \quad & \mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G}, \tilde{\xi}))] \geq \hat{\alpha}_i^j, \text{ for each } i \in \mathcal{N}, j \in \mathcal{M} \\ & \tilde{G} \in \mathcal{G} \end{aligned} \tag{16}$$

Sometimes, the supplier/advertiser of the firm may not require personalized sales targets for each customer type. Instead, the firm only needs to meet the total sales target of each product. Specifically, the (total) per-period demand for product i should exceed its (required) non-personalized target $\hat{\alpha}_i$. We remark that, although the sales targets $\hat{\alpha}_i$'s are not personalized to each customer segment, the firm could still offer (random) personalized assortment $\tilde{S}^j$ to each segment j. To understand the managerial implications of personalized sales targets, we compare the minimum customer traffic under personalized sales targets, $\lambda^*(\hat{\alpha})$ with that under non-personalized targets. To make a fair comparison, we assume that $\hat{\alpha}_i = \sum_{j=1}^m \hat{\alpha}_i^j$ for each i, i.e., for each product i, its total required sales target (from all customer types) is the same for both scenarios.

We are now ready to formulate the cost-minimization problem of the firm with non-personalized sales targets.

$$\begin{aligned} \min_{0\leq\lambda\leq 1,\tilde{G}} \quad & c(\lambda) \\ \text{s.t.} \quad & \sum_{j=1}^m \mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G},\tilde{\xi}))] \geq \hat{\alpha}_i, \text{ for each } i\in\mathcal{N} \\ & \tilde{G}\in\mathcal{G} \end{aligned} \tag{17}$$

A similar duality and OCO approach to the model with personalized sales targets would enable us to further simplify (17) and obtain a debt-based algorithm to achieve the targets (see Proposition 7 in Appendix B).

If customer choice follows MNL model, we are able to quantify the impact of personalized sales targets. To this end, we define the minimum (optimal) traffic to satisfy the corresponding non-personalized sales target as $\lambda_N^*(\hat{\alpha})$, where $\hat{\alpha} = (\hat{\alpha}_i^j : i\in\mathcal{N}, j\in\mathcal{M})$ is the personalized sales targets. Clearly, $\lambda_N^*(\hat{\alpha})$ depends on $\hat{\alpha}$ through non-personalized target of each product i $\hat{\alpha}_i = \sum_{j\in\mathcal{M}} \hat{\alpha}_i^j$. Since personalized sales targets are tighter than non-personalized ones, we have $\lambda^*(\hat{\alpha}) \geq \lambda_N^*(\hat{\alpha})$. We define the ratio $\rho(\hat{\alpha}) := \lambda^*(\hat{\alpha})/\lambda_N^*(\hat{\alpha}) \geq 1$ to capture the impact of personalized sales targets. The larger the $\rho(\hat{\alpha})$, the more costly the sales target personalization. The following proposition derives the lower and upper bounds of $\rho(\hat{\alpha})$.

Proposition 3. *If the customer choices follow the MNL model* (11)*, we have the following bounds on* $\rho(\hat{\alpha})$,

$$\frac{\sum_{j=1}^m \frac{p_j \kappa_N^{j*}}{1+\sum_{i=1}^n \hat{\alpha}_i \kappa_N^{j*}}}{\min_j \{\frac{p_j \kappa^{j*}}{1+\sum_{i=1}^n \hat{\alpha}_i^j \kappa^{j*}}\}} \leq \rho(\hat{\alpha}) \leq \frac{\frac{1}{\sum_{i=1}^n \hat{\alpha}_i} \sum_{j=1}^m p_j \frac{\sum_{i=1}^n v_i^j}{1+\sum_{i=1}^n v_i^j}}{\min_j \{\frac{p_j \kappa^{j*}}{1+\sum_{i=1}^n \hat{\alpha}_i^j \kappa^{j*}}\}},$$

where $\kappa^{j*} := \min\left\{\frac{K}{\sum_{i=1}^n \hat{\alpha}_i^j/v_i^j}, \min_i\{\frac{v_i^j}{\hat{\alpha}_i^j}\}\right\}$, *and* $\kappa_N^{j*} := \min\left\{\frac{K}{\sum_{i=1}^n \hat{\alpha}_i/v_i^j}, \min_i\{\frac{v_i^j}{\hat{\alpha}_i}\}\right\}$, *for each* j.

Proposition 3 helps deliver sharper insights on the impact of personalized sales targets. Specifically, we demonstrate that if the customer traffic is extremely unbalanced, personalized sales target constraints may cause a substantially higher total customer traffic and, therefore, a substantially higher marketing cost. We now make this point clear through an example of one product and two customer classes. We assume that $v_1^1 = v_1^2 = v$ and $\hat{\alpha}_1^1 = \hat{\alpha}_1^2 = \epsilon \in \left(0, \frac{v}{1+v}\right)$. Moreover, we assume that $p_2 = \frac{1+v}{v}\epsilon$ and $p_1 = 1 - p_2$. Then, we have $\lambda^*(\hat{\alpha}) = 1$ and $\lambda_N^*(\hat{\alpha}) = 2\epsilon\left(\frac{p_1 v}{1+v} + \epsilon\right)^{-1}$. Therefore,

$$\rho(\hat{\alpha}) = \frac{\lambda^*(\hat{\alpha})}{\lambda_N^*(\hat{\alpha})} = \frac{1}{2\epsilon} \cdot \left(\frac{v}{1+v}\right)^{-1} \tag{18}$$

If $\epsilon \to 0$, we have $p_2 = \frac{1+v}{v} \cdot \epsilon \to 0$ and

$$\rho(\hat{\alpha}) = \frac{\lambda^*(\hat{\alpha})}{\lambda_N^*(\hat{\alpha})} = \frac{1}{2\epsilon} \cdot \left(\frac{v}{1+v}\right)^{-1} \to \infty$$

Therefore, under the extreme imbalance between the traffic of different customer types (i.e., the imbalance between p_1 and p_2 in the example above), personalized sales targets will result in a substantially higher cost compared to the non-personalized sales targets.

The huge gap between $\lambda^*(\hat{\alpha})$ and $\lambda_N^*(\hat{\alpha})$ motivates us to consider the multi-sourcing strategy that can meet personalized sales targets with a much lower total customer traffic. This is because the customer segmentation distribution (i.e., $(p_1, p_2, ..., p_m)$) of different sources may balance each other when procured in a coordinated fashion, thus forming a much more balanced portfolio of the entire customer traffic. More specifically, we consider L different sources of customer traffic arriving at the firm with an intensity of $\lambda_l \in (0, 1]$ for each customer source l. The marketing investment cost of customer source $l \in \mathcal{L} := \{1, 2, ..., L\}$ is $c_l(\lambda_l)$. For simplicity, we assume $c_l(\lambda_l) = \lambda_l$. We use $\tilde{\eta} \in \mathcal{L}$ to denote the source of a customer. Conditioned on the arrival of a source l customer, we denote the probability that this customer is of type j as $p_{l,j} := \mathbb{P}[\tilde{\xi} = j | \tilde{\eta} = l]$.

To make a fair comparison between the single-sourcing and multi-sourcing cases, we first consider the benchmark to minimize the cost of each customer source, $c_l(\lambda_l)$, independently under the sales targets uniformly allocated to each source, i.e.,

$$\mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G}, \tilde{\xi}))] \geq \frac{\hat{\alpha}_i^j}{L}, \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M}.$$

The cost minimization problem for customer source l can be separately formulated as follows:

$$\begin{aligned} &\min_{\lambda \geq 0} \; c_l(\lambda) \\ &\text{s.t. } \mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G}, \tilde{\xi}))] \geq \frac{\hat{\alpha}_i^j}{L}, \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M}. \end{aligned} \tag{19}$$

Hence, under the single-sourcing strategy, the minimum customer traffic of source l is $\lambda_l^*(\hat{\alpha}/L)$. The total traffic is therefore $\sum_{l=1}^L \lambda_l^*(\hat{\alpha}/L)$. By the proof of Proposition 3, we have the minimum total customer traffic is $\sum\limits_{l=1}^L \lambda_l^*(\hat{\alpha}/L) = \frac{1}{L}\sum\limits_{l=1}^L \max\limits_j \{\frac{1}{p_{l,j}}(\frac{1}{\kappa^{j*}} + \sum\limits_{i=1}^n \hat{\alpha}_i^j), \forall j\}$, where $\kappa^{j*} = \min\left\{\frac{K}{\sum_{i=1}^n \frac{\hat{\alpha}_i^j}{v_i^j}}, \min\{\frac{v_i^j}{\hat{\alpha}_i^j}\}\right\}$.

We now study the cost minimization problem in the presence of multiple sources.

PROPOSITION 4. *Assume that customer choices follow the MNL model (11), we have:*

(a) The multi-sourcing problem can be solved by the following optimization:

$$
\begin{aligned}
\min_{y,z,\lambda_l} \quad & \sum_{l=1}^{L} c_l(\lambda_l) \\
s.t. \quad & (\sum_{l=1}^{L} p_{l,j}\lambda_l) v_i^j y_i^j - \hat{\alpha}_i^j \geq 0, \quad \forall i,j, \\
& \sum_{i=1}^{n} v_i^j y_i^j + z^j = 1, \quad \forall j, \\
& \sum_{i=1}^{n} y_i^j \leq K z^j, \quad \forall j, \\
& y_i^j \leq z^j, \quad \forall i,j, \\
& y_i^j \geq 0, \quad \forall i,j, \\
& \sum_{l=1}^{L} \lambda_l \leq 1, \\
& \lambda_l \geq 0, \quad \forall l.
\end{aligned}
$$

(b) If, in addition, $c_l(\lambda_l) = \lambda_l$, the minimal cost and corresponding multi-sourcing strategy can be obtained by the following linear program:

$$
\begin{aligned}
\min_{\lambda_l} \quad & \sum_{l=1}^{L} \lambda_l \\
s.t. \quad & \sum_{l=1}^{L} p_{l,j}\lambda_l \geq \frac{1}{\kappa^{j*}} + \sum_{i=1}^{n} \hat{\alpha}_i^j, \quad \forall j, \\
& \sum_{l=1}^{L} \lambda_l \leq 1, \\
& \lambda_l \geq 0, \quad \forall l,
\end{aligned}
\tag{20}
$$

where $\kappa^{j*} := \min\left\{ \frac{K}{\sum_{i=1}^{n} \frac{\hat{\alpha}_i^j}{v_i^j}}, \min\{\frac{v_i^j}{\hat{\alpha}_i^j}\} \right\}$ *for each* j.

Denote $(\lambda_l^{**}(\hat{\alpha}) : l \in \mathcal{L})$ as the optimal solution to the total cost minimization problem in the presence of multi-sourcing, (20). Because $(\lambda_l^{*}(\hat{\alpha}/L) : l \in \mathcal{L})$ is a feasible solution to (20), we must have $\sum_{l=1}^{L} \lambda_l^{**}(\hat{\alpha}) \leq \sum_{l=1}^{L} \lambda_l^{*}(\hat{\alpha}/L)$. To understand the value of multi-sourcing, we characterize the lower and upper bounds for the ratio between $\sum_{l=1}^{L} \lambda_l^{**}(\hat{\alpha})$ and $\sum_{l=1}^{L} \lambda_l^{*}(\hat{\alpha}/L)$, defined as $\rho_M(\hat{\alpha}) := (\sum_{l=1}^{L} \lambda_l^{**}(\hat{\alpha}))/(\sum_{l=1}^{L} \lambda_l^{*}(\hat{\alpha}/L))$. The smaller the $\rho_M(\hat{\alpha})$, the more valuable the multi-sourcing strategy compared with single-sourcing.

Proposition 5. *Assume that customer choices follow the MNL model* (11). *We have*

$$\frac{L\sum_{j=1}^{m}(\frac{1}{\kappa^{j*}}+\sum_{i=1}^{n}\hat{\alpha}_i^j)}{\sum_{l=1}^{L}\max_j\{\frac{1}{p_{l,j}}(\frac{1}{\kappa^{j*}}+\sum_{i=1}^{n}\hat{\alpha}_i^j)\}}\le\rho_M(\hat{\alpha})\le\max_j\left\{\frac{L(\frac{1}{\kappa^{j*}}+\sum_{i=1}^{n}\hat{\alpha}_i^j)}{\sum_{l=1}^{L}p_{l,j}\max_j\{\frac{1}{p_{l,j}}(\frac{1}{\kappa^{j*}}+\sum_{i=1}^{n}\hat{\alpha}_i^j)\}}\right\},$$

where κ^{j*} is defined in Proposition 4 for each j.

Proposition 5 uncovers insights on the value of multi-sourcing. As shown by Proposition 3, if the traffic of different customer types is extremely unbalanced, personalized sales targets will give rise a substantially higher cost than non-personalized targets. With Proposition 5, we are able to show that, even if the traffic of different customer types is extremely unbalanced, multi-sourcing could significantly reduce the total marketing investment cost.

We consider the same single-product example with 2 customer types, and 2 customer traffic sources as for the comparison between personalized and non-personalized sales targets. The sales target for each class is $\hat{\alpha}_1^1=\hat{\alpha}_1^2=\epsilon\in(0,1)$ and the attractiveness of the product to each customer type is $v_1=v_2=v$. We also assume that $p_{1,2}=p_{2,1}=\frac{1+v}{v}\cdot\epsilon$ and $p_{1,1}=p_{2,2}=1-\frac{1+v}{v}\cdot\epsilon>p_{1,2}=p_{2,1}=\frac{1+v}{v}\cdot\epsilon$, where ϵ is sufficiently small. It is clear from the model setup that if the firm only uses source 1, the problem is equivalent to the single-sourcing case discussed after Proposition 3.

By our discussion following Proposition 3, if $\epsilon\downarrow 0$, personalized sales targets may require a substantially higher customer traffic than non-personalized ones. We now show that, under multi-sourcing, the total customer traffic can be substantially reduced, even if personalized sales targets are required. First, note that $\kappa^{j*}=v/\epsilon$ $(j=1,2)$. By Proposition 4(b), we have $\lambda_1^{**}(\hat{\alpha})=\lambda^{**}(\hat{\alpha})=\epsilon\cdot\left(1+\frac{1}{v}\right)$, which implies that

$$\sum_{l=1}^{2}\lambda_l^{**}(\hat{\alpha})=2\epsilon\cdot\left(1+\frac{1}{v}\right)=\lambda_N^*(\hat{\alpha}).$$

For the single-sourcing strategy, however, we have $\lambda_l^*(\hat{\alpha}/2)=\max\{\frac{\epsilon(1+v)}{2p_{1,1}v},\frac{\epsilon(1+v)}{2p_{1,2}v}\}=\frac{\epsilon(1+v)}{2p_{1,2}v}=\frac{1}{2}$. Thus, the total customer traffic with personalized sales targets and single sourcing is $\sum_{l=1}^{2}\lambda_l^*(\hat{\alpha}/2)=1$. Therefore,

$$\rho_M(\hat{\alpha})=\frac{\sum_{l=1}^{2}\lambda_l^{**}(\hat{\alpha})}{\sum_{l=1}^{2}\lambda_l^*(\hat{\alpha}/2)}\to 0\text{ as }\epsilon\to 0.$$

Putting everything together, we have demonstrated with this simple example that, while personalized sales targets may substantially increase the necessary customer traffic when the distribution of different customer types are extremely unbalanced ($\rho(\hat{\alpha})\uparrow+\infty$ when $\epsilon\to 0$ in our example), multi-sourcing could help the firm re-balance customer types and personalized sales targets so that the necessary customer traffic with the *personalized* sales targets is under control ($\rho_M(\hat{\alpha})\downarrow 0$ as $\epsilon\downarrow 0$). This result transforms into an important actionable insight: In order to enjoy the revenue benefit of target marketing (i.e. personalized sales targets for different product and customer type mixes), the firm should acquire its customer traffic from multiple sources to keep the total marketing cost affordable.

6. Personalized Assortment Optimization with Inventory Constraints

In this section, we consider another application of our assortment optimization model in the presence of personalized sales targets. A salient feature of this application is that the total initial inventory of each product is limited and cannot be replenished during the selling horizon. A special case of the problem without sales targets has been studied by, e.g., Golrezaei et al. (2014) and Bernstein et al. (2015) based on inventory balancing and dynamic programming techniques, respectively. In our setting, whereas inventory balancing cannot handle sales targets, the traditional dynamic program suffers from the notorious curse of dimensionality (thus, analytically and computationally intractable). Based on the DWA algorithm, we develop a new real-time assortment strategy that is easy to implement, asymptotically optimal, and achieves tiny optimality gaps in the non-asymptotic regime. Furthermore, our proposed algorithms generate revenue with smaller variance than the benchmarks proposed in the existing literature, such as CBLP and LP re-solving heuristics.

6.1. Model and Algorithm

We consider a firm that sells n products to m types of customers. Each product $i \in \mathcal{N}$ has an initial inventory $C_i > 0$. Without loss of generality, we assume the revenue to sell a product i to a customer of type j is $r_i^j > 0$. Customers arrive in a sequential manner with type $\xi_t \in \mathcal{M}$ for the customer in period $t \in \mathcal{T}$. For any customer of type $j \in \mathcal{M}$, she follows choice probabilities Φ to select the product to make a purchase. The goal of the firm is to offer an assortment of products $S(t)$ to each arriving customer throughout the sales horizon, in order to maximize the total expected revenue. The firm also has the individualized sales target requirement, i.e., the long-run (per-period) sales of product i to customer type j should be at least as high as $\hat{\alpha}_i^j$.

When the required sales target $\hat{\alpha}_i^j = 0$, the entire problem degenerates to the dynamic assortment personalization problem with inventory capacity constraints, which has been studied in the literature using dynamic program (Bernstein et al. 2015) and inventory balancing (Golrezaei et al. 2014) techniques. At first glance, such dynamic assortment personalization problem has nothing to do with the sales targets studied in this paper. However, we demonstrate that our assortment optimization with sales targets framework could help achieve the optimal expected revenue in the long run. Our approach can easily handle the long-run required sales targets in the dynamic assortment personalization problem with inventory capacity constraints. Note that incorporating the required sales targets worsens the curse of dimensionality for the dynamic program approach because the state variable space needs to account for the accumulative sales of each product to each customer segment. On the other hand, the inventory balancing approach is not easily amenable to individualized sales targets.

To tackle the dynamic assortment personalization problem with inventory capacity and required sales target constraints, we consider an alternative formulation in which the firm is engaged in a two-stage decision problem shown in (21). The *desired* sales targets (α_i^j for product i and customer type j) are determined in the first-stage, whereas, in the second stage, the personalized assortment is offered to each customer upon her arrival. Later, we will utilize the solution obtained in (21) to construct the assortment policy for the original problem.

$$
\begin{aligned}
&\max_{\alpha} \sum_{i,j} r_i^j \alpha_i^j \\
s.t.\ & h(\theta|\lambda,\alpha) \geq 0 \text{ for all } \theta_i^j \geq 0 \text{ for all } i \in \mathcal{N}, j \in \mathcal{M}, \\
& \alpha_i^j \geq \hat{\alpha}_i^j \text{ for all } i \in \mathcal{N}, j \in \mathcal{M}, \\
& \sum_j \alpha_i^j \leq \frac{C_i}{T} \text{ for all } i \in \mathcal{N}, \\
& 0 \leq \alpha_i^j \leq 1 \text{ for all } i \in \mathcal{N}, j \in \mathcal{M},
\end{aligned}
\tag{21}
$$

where the objective is to maximize the expected per-period revenue $\sum_{i,j} r_i^j \alpha_i^j$ (equivalently, the long-run average revenue). According to Theorem 1 and Theorem 2, $h(\theta|\lambda,\alpha)$ defined by (9) being non-negative provides a necessary and sufficient condition for the desired sales targets α to be obtainable in the long-run. Moreover, the desired sales targets should be bounded from below by the respective required sales targets. Due to the inventory capacity constraints, the total sales of each product i should not exceed its initial inventory, i.e., $\sum_j \alpha_i^j \leq \frac{C_i}{T}$. So we have bounded the desired sales targets with the inventory constraints. The convex program formulation (21) characterizes the optimal desired sales targets in the long-run average sense, but offers little insights on how assortments should be offered upon the arrival of each customer. To shed light on the optimal dynamic assortment policy, we propose the DWA-I policy based on the optimal solution α^* to (21).

The DWA-I policy first solves the optimal desired sales target α^* in the presence of limited initial inventory for the products. One should also note that (21) is a convex program so its optimal solution α^* can be obtained through efficient algorithms such as gradient descent. As prescribed by Algorithm 2, our real-time assortment strategy is to implement the DWA policy $\tilde{G}_{DWA}(\lambda,\alpha^*)$ (Algorithm 1) throughout the sales horizon until stockout occurs. Once stockout occurs for a product, any assortment containing this product will no longer be offered. Adopting a coupling argument, Lemma 1 shows that if the problem size (time T and inventory C_i) scales to infinity, the DWA-I policy will *not* give rise to inventory stockout and will secure the pre-set optimal desired sales targets α^*. As a consequence, the sales target constraints $\{\hat{\alpha}_i^j : i \in \mathcal{N}, j \in \mathcal{M}\}$ can be satisfied as well. An interesting feature of the DWA-I algorithm is that it embeds a revenue optimization problem into our sales target framework. As a consequence, the DWA-I policy maximizes revenue in the first-stage, and adaptively offers personalized assortment to satisfy the optimal (desired) sales targets once the sales horizon starts.

Algorithm 2 Debt-Weighted Assortment with Inventory (DWA-I) Policy $\tilde{G}_{DWA-I}$

First-stage optimization: Solve (21) to obtain the optimal desired sales target α^*.

Initialize: $d_i^j(1) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

For each period $t \geq 1$**:**

1: Observe the customer type $\xi_{(t)} = \hat{j}$.

2: If $\hat{j} = 0$, assign customer choices $y_i^j(t) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

3: If $\hat{j} \in \mathcal{M}$, offer assortment $S^*(t)$ to the customer, where

$$S^*(t) \leftarrow \arg\max_S \sum_{i \in S} \left(d_i^{\hat{j}}(t)\right)^+ \phi_i^{\hat{j}}(S), \tag{22}$$

which generates customer choices $(y_i^j(t) : i \in \mathcal{N}, j \in \mathcal{M})$. The revenue, $\sum_i r_i^j y_i^j(t)$, is collected. In the case where product i is stocked out (i.e., $\sum_j \sum_{s \leq t} y_i^j(s) = C_i$), any assortment containing this product will no longer be offered hereafter.

4: $d_i^j(t+1) \leftarrow d_i^j(t) + \alpha_i^{j*} - y_i^j(t)$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

We are now ready to demonstrate the asymptotic optimality of DWA-I. We denote a family of revenue maximization problems with initial inventory for each product i, $C_i(\gamma) := C_i\gamma$, and planning horizon length $T(\gamma) := T\gamma$, as $\mathcal{Q}(\gamma)$, where $\gamma > 0$ is a scaling parameter of problem size. For problem $\mathcal{Q}(\gamma)$ and policy $\tilde{G}$, we denote $Rev(\gamma, \tilde{G})$ as its expected total revenue and $Rev^*(\gamma) = \max_{\tilde{G}} Rev(\gamma, \tilde{G})$ as the optimal expected revenue of $\mathcal{Q}(\gamma)$.

Theorem 4. *The Debt-Weighted Assortment with Inventory policy is asymptotically optimal, i.e.,*

$$\lim_{\gamma \to +\infty} \frac{Rev(\gamma, \tilde{G}_{DWA-I})}{Rev^*(\gamma)} = 1 \tag{23}$$

Theorem 4 proves that the DWA-I policy is optimal when the initial inventory and time horizon length scales up to limit at the same linear rate. The proof of Theorem 4 also implies that the optimality gap of the DWA-I algorithm is of order $\mathcal{O}(\sqrt{\gamma})$. Hence, even if the problem scale γ is small, this algorithm could still generate impressive revenue performances, as shown by our numerical experiments in Section 6.2.

A classical approach in the literature is to formulate the assortment optimization problem in the presence of inventory capacity constraints as a linear program (called choice-based linear program, CBLP, see Liu and Van Ryzin 2008). Throughout the planning horizon, the firm adopts a (stationary) randomized assortment policy based on the solution to the CBLP. Hence, the CBLP approach is not adaptive, which will result in substantial variability in its revenue performance. Our proposed DWA-I algorithm, however, responds to the randomnesses in customer types and choices. That is, if per-period sales of product i to customer type j does not reach the optimal desired sales target α_i^{j*}, product i will be more likely to be included the assortment when the next type j customer arrives. In other words, the assortments offered to customers across periods under the DWA-I algorithm are correlated, so that the debt process exhibits a mean-reverting pattern towards 0. As a consequence, although both CBLP and DWA-I algorithms achieve the asymptotic optimality, the latter policy generates a significantly less variable revenue stream and a higher expected revenue when the loading factor is high (i.e., when demand exceeds supply) in the non-asymptotic regime, as shown by our numeric experiments in Section 6.2.

Note that the DWA-I algorithm may be too conservative at the beginning of the sales horizon, because it places too much emphasis on meeting the (optimal) desired sales targets instead of maximizing the (immediate) revenue. To address this issue, we consider a variant of the DWA-I algorithm, which uses the optimal expected *total revenue* of each product sold to all customer types as the target. This new algorithm on one hand achieves the same asymptotic optimality (and convergence rate) as the DWA-I algorithm, and, on the other hand, performs better when the planning horizon is short. We present the algorithm with aggregate revenue targets as follows, and an analog with individualized revenue targets in Algorithm 6 in Appendix A.

Algorithm 3 Debt-Weighted Assortment with Aggregate Revenue Target (DWA-ART) Policy $\tilde{G}_{DWA-ART}$

First-stage optimization: Solve (21) to obtain the optimal sales target α^*.

Initialize: $d_i(1) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

For each period $t \geq 1$**:**

1: Observe the customer type $\xi_{(t)} = \hat{j}$.

2: If $\hat{j} = 0$, assign customer choices $y_i^j(t) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

3: If $\hat{j} \in \mathcal{M}$, offer assortment $S^*(t)$ to the customer, where

$$S^*(t) \leftarrow \arg\max_S \sum_{i \in S} (d_i(t))^+ \phi_i^{\hat{j}}(S), \tag{24}$$

which generates customer choices $(y_i^j(t) : i \in \mathcal{N}, j \in \mathcal{M})$. The revenue, $\sum_i r_i^j y_i^j(t)$, is collected. In the case where product i is stocked out (i.e., $\sum_j \sum_{s \leq t} y_i^j(s) = C_i$), any assortment containing this product will no longer be offered hereafter.

4: $d_i(t+1) \leftarrow d_i(t) + \sum_{j \in \mathcal{M}} r_i^j \alpha_i^{j*} - \sum_{j \in \mathcal{M}} r_i^j y_i^j(t)$ for all $i \in \mathcal{N}$.

By changing the personalized sales targets of product-customer pairs to aggregate revenue targets of products, the DWA-ART algorithm directly controls the total revenue from each product to optimality. As we show in the following proposition, this algorithm perserves the asymptotic optimality property.

PROPOSITION 6. *The Debt-Weighted Assortment with Aggregate Revenue Target policy is asymptotically optimal, i.e.,*

$$\lim_{\gamma\to+\infty}\frac{Rev(\gamma,\tilde{G}_{DWA-ART})}{Rev^*(\gamma)}=1 \tag{25}$$

By aggregating the revenue of each product to all customer types together, the DWA-ART algorithm further reduces the variability of the realized revenue. As a consequence, a key advantage of DWA-ART over DWA-I is that the former achieves high revenue performance, especially at the beginning of the planning horizon since it directly operates the debt on aggregate revenue. This is an appealing feature when implementing the algorithm in practice where stable revenue is preferred. Finally, we remark that the DWA-ART algorithm guarantees the sales target constraints $(\hat{\alpha}_i^j : j\in\mathcal{M}, i\in\mathcal{N})$, as long as the optimal sales targets α^* is unique. In the case where α^* is not unique, our numerical experiments show that the required sales targets can be satisfied for all of the numerical instances we examined.

6.2. Numerical Experiments

In this subsection, we numerically evaluate the DWA-I and DWA-ART algorithms for assortment optimization in the presence of sales target constraints, benchmarked against the well-established choice-based linear program (CBLP) method and the LP re-solving heuristics. The key takeaways from our numerical experiments are that (1) thanks to the adaptive assortment personalization, our algorithm achieves better revenue performances especially when the load factor is large and/or the sales targets are high; (2) the revenues generated by DWA-I and DWA-ART policies are much more stable (i.e., with lower variability in the realized revenue) than those of CBLP and LP-resolving heuristics, because the debt process directly steers the sales of each product to each customer segment towards the optimal sales target throughout the planning horizon.

Our numerical setup is summarized as follows. The retailer has 10 different products to serve 5 types of customers. The revenue only depends on the product, but not on the customer type. So we denote the revenue of product i as r_i. We sample the product revenues $\{r_1,r_2,...,r_{10}\}$ independently from a uniform distribution on the interval $[10,50]$. The customer type distribution $(p_1,p_2,...,p_5)$ $(\sum_j p_j=1)$ is generated from an m-dimension Dirichlet distribution. We set the arrival rate of the customers in each period as $\lambda=0.9$. We consider the MNL model as the choice decision process of the customers, i.e., for $i\in S\subset\mathcal{S}$,

$$\phi_i^j(S)=\frac{v_i^j}{1+\sum_{i'\in S}v_{i'}^j}.$$

Each product-customer pair is associated with an attraction index v_i^j. For product i and customer type j, we assign $v_i^j := \exp(\mu_i^j)$, where the expected utility of customer type j to consumer product i, μ_i^j, is independently sampled from the uniform distribution on the interval $[-1, 2]$. The sales horizon length is $T = 10{,}000$. Finally, we remark that the above parameter combinations are set without loss of generality. All our results and insights in this subsection are robust if these parameters take different values.

To uncover insights on when our debt-based algorithms will be most valuable relative to the well-established benchmarks, we systematically vary three focal parameters: (a) the concentration parameter (CP) associated with the proportion of each customer type, (b) the loading factor (LF), defined as the ratio between the total expected demand to the total capacity of all products; and (c) the sales target ratio (TR), which parametrizes the ratio between the required sales target $\hat{\alpha}_i^j$ to the *reference* sales target $\alpha_i^j(o)$ (i.e., $\hat{\alpha}_i^j = TR \cdot \alpha_i^j(o)$ for all i and j). The concentration parameters are determined by the parameters of the Dirichlet distribution we use to generate $(p_1, p_2, \cdots, p_5)$. See Appendix B for details. The reference sales target vector $\alpha(o) := (\alpha_i^j(o) : i \in \mathcal{N}, j \in \mathcal{M})$ can be interpreted as the most "stringent" *feasible* sales targets. We give the formal definition of $\alpha(o)$ in Appendix B. As is clear from their definitions, CP measures the uniformness of the customer type distribution, LF measures the tightness of the inventory capacity constraint, and TR measures the tightness of the required sales target. The higher the CP, the more uniform the distribution of customer types; the higher the LF, the tighter the inventory capacity constraint; the higher the TR, the tighter the required sales targets. In our experiments, we vary CP from 0.1 to 100, LF from 1 to 1.8, and TR from 0 to 1.

We consider the CBLP approach and its re-solving variation as the benchmarks to evaluate our debt-based algorithms. Specifically, let $N^j(S)$ denote the total number of periods where assortment $S \in \mathcal{S} \cup \{0\}$ is offered to customer type j. Then, we can formulate the assortment decision of the firm throughout the sales horizon as the following linear program:

$$\begin{aligned} &\max_{N^j(S)} \sum_{i\in\mathcal{N}} r_i \sum_{j\in\mathcal{M}} \sum_{S\in\mathcal{S}} \phi_i^j(S) N^j(S) \\ s.t.\ &\sum_{j,S} \phi_i^j(S) N^j(S) \le C_i \text{ for all } i \in \mathcal{N}, \\ &\sum_{S} N^j(S) \le \lambda p_j T \text{ for all } j \in \mathcal{M}, \\ &\phi_i^j(S) N^j(S) \ge \hat{\alpha}_i^j T, \text{ for all } i \in \mathcal{N}, j \in \mathcal{M}, \\ &N^j(S) \ge 0 \text{ for all } j \in \mathcal{M}, S \in \mathcal{S} \cup \{0\}. \end{aligned} \tag{26}$$

Compared with the standard CBLP approach, the LP formulation (26) has an additional constraint that the total sales of product i to customer type j should meet the corresponding sales target.

Based on the solution to (26), one can construct a randomized algorithm under which, upon the arrival of a type j customer, the assortment S is randomly provided to the customer with probability $\frac{N^j(S)}{\sum_{S'\in\mathcal{S}\cup\{0\}} N^j(S')}$. Note that (26) involves $m2^n$ decision variables and is thus computationally intractable if the number of products n is large.

On top of the CBLP approach, we also adopt an LP-resolving heuristic as another benchmark (Jasin and Kumar 2012). Specifically, for every period $t \in \{1, T/10+1, 2T/10+1, 3T/10+1, ..., 9T/10+1\}$ (which we call the re-solving epoch), we re-solve the CBLP based on the remaining inventory, remaining sales horizon length $T-t+1$, and unfinished targets in place of the initial inventory C_i, total sales horizon length T, and sales targets $(\hat{\alpha}_i^j : i \in \mathcal{N}, j \in \mathcal{M})$. The firm then adopts the randomized algorithm induced by the solution to the re-solving LP until the next re-solving epoch. In the case where the re-solving LP is infeasible or unbounded, we will keep the randomized policy of the previous re-solving epoch unchanged. In both CBLP and LP-resolving approaches, if a product is out-of-stock in some period, we remove this product from the assortment generated by the algorithm until the end of sales horizon.

We report our numerical findings in Table 1, with the ratio between the standard error of the total revenue to the theoretical upper-bound included in the bracket. It is clear from our experiments that the proposed debt-based algorithms achieve near-optimal revenue performances in the non-asymptotic regime. We highlight the following takeaways. First, although the CBLP and LP-resolving approaches could already achieve revenue performances very close to the theoretical upper-bound, the proposed debt-based algorithms could generate even higher revenues especially when the loading factor (LF) is high and/or the target requirement (TR) is high. We emphasize that the CBLP and LP-resolving benchmarks are non-adaptive so that the firm does not adjust the probability of each assortment in real time. When the total inventory is much lower than the total customer demand (i.e., LF is very high), our debt-based algorithms outperform the LP-based heuristics. In this case, our proposed policies aim to clear all the inventory, which proves to dominate the randomized (stationary) assortment policy generated by LP-based heuristics given that demand far exceeds supply. The second important insight from our numerical analysis is that, the debt-based DWA-I and DWA-ART algorithms generate much more stable revenues (i.e., much lower standard deviations) than LP and LP-resolving heuristics. To understand the underlying intuition of this result, we note that the debt-based algorithms assign a higher weight (i.e., debt) to the product-customer pair which is farther away from the optimal target, so as to autonomously drive the system to optimality. As a consequence, the sales (and thus revenue) trajectory would follow a mean-reverting pattern under the DWA-I and DWA-ART algorithms with lower revenue variability than the LP-based benchmarks.

CP	LF	TR	UB	LP	LP-Resolving	DWA-I	DWA-ART
0.1	1.8	0	1.44	85.52% (7.00%)	97.52% (2.55%)	99.72% (0.84%)	99.57% (1.07%)
		0.5	1.44	89.00% (6.51%)	91.64% (4.33%)	99.71% (0.72%)	99.76% (0.71%)
		1.0	1.44	89.54% (5.75%)	89.53% (5.79%)	99.70% (0.84%)	99.78% (0.71%)
	1.5	0	1.72	88.95% (6.85%)	97.67% (1.79%)	98.93% (1.97%)	98.62% (2.17%)
		0.5	1.72	87.63% (5.87%)	93.71% (3.82%)	98.53% (2.96%)	99.03% (1.17%)
		1.0	1.72	91.16% (5.52,%)	92.38% (4.49%)	98.66% (1.5%)	99.34% (0.96%)
	1.2	0	2.23	93.05% (3.68%)	97.09% (3.13%)	93.82% (4.36%)	94.22% (4.75%)
		0.5	2.22	91.73% (4.77%)	94.42% (3.66%)	93.65% (4.80%)	93.15% (4.75%)
		1.0	2.18	92.04% (5.35%)	93.56% (5.11%)	92.18% (4.93%)	93.58% (4.67%)
	1	0	2.37	94.76% (4.38%)	99.11% (3.1%)	93.31% (4.39%)	93.86% (3.56%)
		0.5	2.36	95.67% (5.45%)	96.83% (3.97%)	92.12% (4.27%)	93.73% (3.48%)
		1.0	2.15	93.36% (4.17%)	96.51% (5.12%)	93.84% (4.32%)	95.31% (5.01%)
1	1.8	0	1.44	86.32% (7.52%)	98.15% (1.54%)	99.85% (0.53%)	99.95% (0.20%)
		0.5	1.44	88.60% (6.10%)	92.52% (4.59%)	99.77% (0.98%)	99.94% (0.21%)
		1.0	1.44	87.75% (6.31%)	90.22% (4.32%)	99.82% (0.61%)	99.94% (0.21%)
	1.5	0	1.72	89.47% (5.18%)	98.12% (2.07%)	99.16% (1.49%)	99.89% (0.25%)
		0.5	1.72	90.86% (4.56%)	93.84% (4.28%)	98.67% (2.26%)	99.82% (0.41%)
		1.0	1.72	88.25% (6.48%)	92.20% (5.18%)	98.68% (2.01%)	99.52% (0.80%)
	1.2	0	2.30	92.86% (3.71%)	97.58% (1.60%)	90.43% (4.32%)	91.42% (4.22%)
		0.5	2.30	94.05% (2.96%)	95.15% (2.28%)	88.15% (5.18%)	91.25% (3.28%)
		1.0	2.28	93.60% (2.96%)	94.30% (2.50%)	90.13% (3.96%)	94.12% (3.17%)
	1	0	2.51	94.89% (4.00%)	97.82% (2.26%)	85.88% (4.08%)	89.85% (3.81%)
		0.5	2.49	94.12% (3.54%)	94.68% (3.18%)	87.85% (3.45%)	90.29% (3.59%)
		1.0	2.35	92.92% (5.06%)	93.60% (5.87%)	89.99% (5.38%)	93.22% (3.61%)
10	1.8	0	1.44	86.38% (6.93%)	98.12% (1.40%)	99.73% (0.96%)	99.97% (0.14%)
		0.5	1.44	88.67% (6.51%)	90.71% (4.78%)	99.93% (0.36%)	99.86% (0.57%)
		1.0	1.44	87.34% (4.67%)	88.41% (6.56%)	99.68% (0.77%)	100.00% (0.00%)
	1.5	0	1.72	88.11% (6.58%)	98.13% (1.82%)	99.13% (1.91%)	99.79% (0.53%)
		0.5	1.72	92.59% (4.16%)	93.29% (4.39%)	98.96% (1.35%)	99.66% (0.67%)
		1.0	1.72	92.65% (6.43%)	92.05% (4.89%)	98.53% (2.17%)	99.73% (0.71%)
	1.2	0	2.30	93.70% (3.07%)	97.56% (1.57%)	89.31% (5.13%)	93.82% (3.41%)
		0.5	2.30	94.50% (3.92%)	94.10% (3.32%)	90.20% (4.50%)	93.11% (3.60%)
		1.0	2.23	92.92% (4.99%)	91.49% (5.17%)	92.85% (5.18%)	96.59% (3.44%)
	1	0	2.52	93.98% (3.93%)	96.60% (2.74%)	86.45% (3.66%)	90.46% (3.04%)
		0.5	2.48	94.42% (3.44%)	93.85% (4.58%)	88.94% (5.51%)	92.84% (4.58%)
		1.0	2.26	94.78% (5.61%)	93.87% (7.50%)	92.17% (4.47%)	97.92% (2.44%)
100	1.8	0	1.44	91.31% (5.93%)	97.12% (2.73%)	99.82% (0.50%)	100.00% (0.00%)
		0.5	1.44	90.80% (5.25%)	90.77% (4.82%)	99.76% (0.74%)	99.97% (0.14%)
		1.0	1.44	89.43% (7.79%)	88.25% (6.35%)	98.95% (1.74%)	99.88% (0.51%)
	1.5	0	1.72	88.32% (6.18%)	97.37% (2.58%)	99.37% (1.38%)	99.82% (0.35%)
		0.5	1.72	92.08% (5.14%)	94.13% (3.53%)	98.97% (2.14%)	99.81% (0.58%)
		1.0	1.72	90.15% (5.75%)	91.89% (4.40%)	97.89% (3.29%)	99.62% (0.66%)
	1.2	0	2.30	92.42% (4.36%)	96.95% (2.87%)	89.12% (5.69%)	92.91% (3.94%)
		0.5	2.30	93.57% (3.36%)	94.88% (2.93%)	89.97% (4.99%)	93.40% (3.83%)
		1.0	2.27	93.78% (3.74%)	93.14% (3.82%)	91.33% (3.00%)	95.29% (3.05%)
	1	0	2.52	95.26% (3.77%)	97.68% (2.87%)	86.23% (5.86%)	89.98% (4.33%)
		0.5	2.50	94.15% (3.18%)	92.45% (3.41%)	87.19% (4.75%)	91.33% (3.75%)
		1.0	2.32	94.38% (4.80%)	92.58% (5.11%)	90.21% (4.12%)	94.25% (4.03%)

Table 1 **Numerical Results (Standard Error relative to the Theoretical Upper-bound in Parentheses)**

7. Conclusion

Personalized sales targets for different product- customer segment pairs are crucial for online retailers and advertising platforms to optimize their operations strategies and maximize market values. This paper has proposed a general modeling framework and the associated efficient algorithms to study assortment personalization in the presence of individualized sales targets. We offer several interesting takeaways.

We propose a general modeling framework to study assortment personalization in the presence of sales target constraints. Our framework is flexible and parsimonious, incorporating different objectives (such as sales revenue maximization and customer traffic procurement cost minimization) and sales targets (personalized and non-personalized). Associated with the proposed framework, we develop a family of simple and effective algorithms, Debt-Weighted Assortment policy. This policy is provably optimal for the assortment personalization problem in the presence of sales target constraints. This policy assigns higher weights on the product-customer pairs that have the largest "debts" from the pre-set desired sales targets, and optimizes the assortment decisions accordingly. The debts serve as a simple but efficient mechanism that synchronizes the first-stage problem to maximize the total value and the second-stage to meet the sales target constraints.

Our modeling framework enables us to further reveal interesting insights across different application contexts. Whereas assortment personalization policies will improve the revenue for an advertising platform, personalized sales targets may require a substantially higher customer traffic if the customer type distribution is highly unbalanced. This potential drawback of sales target personalization can be offset by the multi-sourcing strategy, which efficiently re-balances customer types and personalized sales targets.

Another application of our modeling framework is adaptive assortment personalization with inventory capacity constraints. We reformulate this problem in our modeling framework and develop a simple debt-weighted assortment algorithm that integrates the inventory capacity constraints. The proposed algorithm proves to be asymptotically optimal. Through numerical experiments, we demonstrate that our algorithm outperforms the well-studied CBLP policy and the LP-resolving heuristic especially when the loading factor is high. Furthermore, the proposed algorithms achieve lower revenue variability than the CBLP and re-solving heuristics.

Whereas this paper provides some answers on how firms can optimize the assortment strategy in the presence of sales targets, it raises several intriguing questions as well. First, the proposed framework and algorithm cannot handle the case where the customer arrival process is non-stationary. Developing effective models and algorithms in this scenario would be important for firms whose demand follows non-stable or even unpredictable patterns. Second, our algorithm assumes that the planning horizon length is known apriori. This assumption may not be valid for fashion products whose selling season is uncertain and hard to predict. It is interesting to strengthen our approach to address the problem with random sales horizon length. Third, in our framework, the choice model is assumed to be known to the decision maker. It is an exciting direction to explicitly model how the firm adaptively learns the choice model from the data of sequentially arriving customers. We leave these questions for future research.

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Appendix A. Other Debt-Weighted Assortment Policies

Algorithm 4 RANDOMIZED DEBT-WEIGHTED-ASSORTMENT (RDWA) POLICY

Initialize: $d_i^j(1) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

Random Sample: Randomly generate a customer type sample $\{\xi(1), \xi(2), ..., \xi(T)\}$ from the distribution $\mathcal{P}$, where T is sufficiently large.

For each period $t = 1, 2, ..., T$**:**

1: Observe the customer type $\xi_{(t)} = \hat{j}$.

2: If $\hat{j} = 0$, assign customer choices $y_i^j(t) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

3: If $\hat{j} \in \mathcal{M}$, offer assortment S_t^* to the customer, where

$$S^*(t) \leftarrow \arg\max_S \sum_{i \in S} \left(d_i^{\hat{j}}(t)\right)^+ \phi_i^{\hat{j}}(S) \tag{27}$$

4: Randomly sample customer choices $(y_i^j(t) : i \in \mathcal{N}, j \in \mathcal{M})$, based on customer choice probabilities $\{\phi_i^j(\cdot) : j \in \mathcal{M}, i \in \mathcal{N}\}$.

5: $d_i^j(t+1) \leftarrow d_i^j(t) + \alpha_i^j - y_i^j(t)$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

Output: Uniformly randomly pick up a time index $t \in \{1, 2, 3, ..., T\}$, and offer $S^{j*}(d(t))$ to type j customer based on (8) where $d(t) := (d_i^j(t) : j \in \mathcal{M}, i \in \mathcal{N})$.

We denote the randomized Debt-Weighted-Assortment policy as $\tilde{G}_{RDWA}$. We show that if (λ, α) is feasible, the RDWA policy will ensure the feasibility of sales target constraints.

THEOREM 5. (FEASIBILITY OF RDWA) *If (λ, α) is feasible, i.e., inequality (7) holds, then we have*

$$\mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G}_{RDWA}, \tilde{\xi}))] \geq \alpha_i^j \ \ a.s., \ \text{for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M}.$$

Next, we give the non-personalized debt-weighted assortment policy. As shown by Proposition 7, the following algorithm ensures meeting the sales targets when (33) holds.

Algorithm 5 NON-PERSONALIZED DEBT-WEIGHTED ASSORTMENT (NDWA) POLICY

Initialize: $d_i(1) \leftarrow 0$ for all $i \in \mathcal{N}$.

For each period $t \geq 1$**:**

1: Observe the customer type $\xi_{(t)} = \hat{j}$.

2: If $\hat{j} = 0$, assign customer choices $y_i^j(t) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

3: If $\hat{j} \in \mathcal{M}$, offer assortment S_t^* to the customer, where

$$S^*(t) \leftarrow \arg\max_S \sum_{i \in S} (d_i(t))^+ \phi_i^{\hat{j}}(S) \tag{28}$$

which generates customer choices $(y_i^j(t) : i \in \mathcal{N}, j \in \mathcal{M})$. The revenue, $\sum_i r_i^j y_i^j(t)$, is collected. In the case where product i is stocked out (i.e., $\sum_j \sum_{s \leq t} y_i^j(s) = C_i$), any assortment containing this product will no longer be offered hereafter.

4: $d_i(t+1) \leftarrow d_i(t) + \alpha_i - \sum_{j \in \mathcal{M}} y_i^j(t)$ for all $i \in \mathcal{N}$.

The following algorithm prescribes the dynamic assortment policy with individualized revenue targets.

Algorithm 6 DEBT-WEIGHTED ASSORTMENT WITH REVENUE TARGET (DWA-RT) POLICY $\tilde{G}_{DWA-RT}$

First-stage optimization: Solve (21) to obtain the optimal sales target α^*.

Initialize: $d_i^j(1) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

For each period $t \geq 1$**:**

1: Observe the customer type $\xi_{(t)} = \hat{j}$.

2: If $\hat{j} = 0$, assign customer choices $y_i^j(t) \leftarrow 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

3: If $\hat{j} \in \mathcal{M}$, offer assortment $S^*(t)$ to the customer, where

$$S^*(t) \leftarrow \arg\max_S \sum_{i \in S} \left(d_i^{\hat{j}}(t)\right)^+ \phi_i^{\hat{j}}(S), \tag{29}$$

which generates customer choices $(y_i^j(t) : i \in \mathcal{N}, j \in \mathcal{M})$. The revenue, $\sum_i r_i^j y_i^j(t)$, is collected. In the case where product i is stocked out (i.e., $\sum_j \sum_{s \leq t} y_i^j(s) = C_i$), any assortment containing this product will no longer be offered hereafter.

4: $d_i^j(t+1) \leftarrow d_i^j(t) + r_i^j \alpha_i^{j*} - r_i^j y_i^j(t)$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$.

Appendix B. Concentration Parameter and Reference Sales Targets

In our numerical experiments (Section 6.2), we vary CP to change the uniformness of proportions $p_1, ..., p_m$ of m customer types, which are generated by a m-dimension Dirichlet distribution. The m-dimension Dirichlet distribution has m concentration parameters $\beta_1, ..., \beta_m$. In our setting, let $\beta_1 = \beta_2 = ... = \beta_m = \beta_0$, and

$CP := \beta_0$. Note that, for all j, $\mathbb{E}[p_j] = \frac{\beta_j}{\sum_{k=1}^m \beta_k} = \frac{1}{m}$ and $Var(p_j) = \frac{m-1}{m^2(m\beta_0+1)}$, which is decreasing in β_0. For $j \neq k$, the covariance between p_j and p_k is $-\frac{1}{m^2(m\beta_0+1)}$, which is increasing in β_0. Hence, if β_0 is large, the sampled customer type distribution will be close to the uniform distribution on $\{1,2,...,m\}$. In contrast, if $CP = \beta_0$ is small, the customer type distribution is more likely to be concentrated on a subset of $\{1,2,...,m\}$. In other words, the higher the CP, the more uniform the generated distribution of customer types.

In our numerical experiments, we need to pre-specify a vector of reference sales targets $\alpha(o)$. First, $\alpha(o)$ should be feasible. To avoid the trivial case that the sales target constraints are not binding, we set $\alpha(o)$ to be located on the boundary of the feasible set, characterized by Theorem 2. Without loss of generality, we assume $r_1 \geq r_2 \geq ... \geq r_n$. The reference sales target $\alpha(o)$ is characterized by the following convex optimization:

$$
\begin{aligned}
&\max_{\alpha} \sum_j \alpha_n^j \\
s.t.\ & h(\theta|\lambda,\alpha) \geq 0 \text{ for all } \theta_i^j \geq 0 \text{ for all } i \in \mathcal{N}, j \in \mathcal{M}, \\
&\sum_j \alpha_i^j \leq \frac{C_i}{T} \text{ for all } i \in \mathcal{N}, \\
&0 \leq \alpha_i^j \leq 1 \text{ for all } i \in \mathcal{N}, j \in \mathcal{M}.
\end{aligned}
\tag{30}
$$

Note that, in (30), the objective function is linear, whereas the feasible region is convex and nonempty ($(0,0,...,0)$ is a feasible solution). Therefore, the optimal solution $\alpha(o)$ must lies on the boundary of the feasible region. In our numerical experiments, we set the sales target $\hat{\alpha}_i^j = TR \cdot \alpha_i^j(o)$ for all i and j, where the target ratio $TR \in [0,1]$ captures the tightness of the sales target requirement.

Appendix C. Proofs of Statements

Proof of Theorem 1

The feasibility of (1) is equivalent to that of (4). By weak duality and (6), (4) is feasible only if

$$
\min_{\theta_i^j \geq 0} \left\{ \max_{G \in \hat{G}} \sum_{i \in \mathcal{N}, j \in \mathcal{M}} \lambda p_j \theta_i^j (S^j(G,j)) \theta_i^j - \sum_{i \in \mathcal{N}, j \in \mathcal{M}} \alpha_i^j \theta_i^j \right\} \geq 0,
$$

which is equivalent to (7). This completes the proof of Theorem 1. □

Proof of Theorem 2

First, observe that $h(\theta|\lambda,\alpha)$ is positively homogeneous with respect to θ, i.e., $h(\gamma \cdot \theta|\lambda,\alpha) = \gamma h(\theta|\lambda,\alpha)$ for all $\gamma > 0$. Furthermore, $h(0|\lambda,\alpha) = 0$. It suffices to show that if $h(\theta|\lambda,\alpha) \geq 0$ for all $\theta \in \Theta$, then the sales target constraints (3) can be satisfied, where $\Theta = \{(\theta_i^j : i \in \mathcal{N}, j \in \mathcal{M}) : \sum_{i,j} \theta_i^j = 1\}$.

For any $\theta \in \Theta$, recall from (8) that $S^{j*}(\theta) := \arg\max_S \sum_{i \in S} \lambda p_j \theta_i^j \phi_i^j(S)$. Fix $\epsilon \in (0, \min\{\alpha_i^j\})$, there exists a finite subset of $\Theta_\epsilon \subset \Theta$ such that for any $\theta \in \Theta$, there exists a $\theta(\epsilon) \in \Theta_\epsilon$, such that $||\theta - \theta(\epsilon)||_\infty \leq \frac{\epsilon}{2\lambda}$, $||\cdot||_\infty$

is the $L^\infty-$norm. Denote $mathcalG_\epsilon := \{(S^{j*}(\theta) : j \in \mathcal{M}) : \theta \in \Theta_\epsilon\}$, which is a finite set of deterministic policies. For any $\theta \in \Theta$, define $\theta(\epsilon) \in \Theta_\epsilon$ such that $||\theta - \theta(\epsilon)||_\infty \leq \frac{\epsilon}{2\lambda}$, we have

$$
\begin{aligned}
&\sum_j \max_S \sum_{i\in S} \lambda p_j \theta_i^j \phi_i^j(S) \\
\leq &\sum_j \max_S \sum_{i\in S} \lambda p_j \theta_i^j(\epsilon) \phi_i^j(S) + \sum_j \max_S \sum_{i\in S} \lambda p_j (\theta_i^j - \theta_i^j(\epsilon)) \phi_i^j(S) \\
= &\sum_j \max_{S\in\mathcal{G}_\epsilon} \sum_{i\in S} \lambda p_j \theta_i^j(\epsilon) \phi_i^j(S) + \sum_j \max_S \sum_{i\in S} \lambda p_j (\theta_i^j - \theta_i^j(\epsilon)) \phi_i^j(S) \\
\leq &\sum_j \max_{S\in\mathcal{G}_\epsilon} \sum_{i\in S} \lambda p_j \theta_i^j \phi_i^j(S) + \sum_j \max_{S\in\mathcal{G}_\epsilon} \sum_{i\in S} \lambda p_j (\theta_i^j(\epsilon) - \theta_i^j) \phi_i^j(S) + \sum_j \max_S \sum_{i\in S} \lambda p_j (\theta_i^j - \theta_i^j(\epsilon)) \phi_i^j(S) \\
\leq &\sum_j \max_{S\in\mathcal{G}_\epsilon} \sum_{i\in S} \lambda p_j \theta_i^j \phi_i^j(S) + 2\lambda \cdot \frac{\epsilon}{2\lambda} \\
= &\sum_j \max_{S\in\mathcal{G}_\epsilon} \sum_{i\in S} \lambda p_j \theta_i^j \phi_i^j(S) + \epsilon,
\end{aligned}
$$

where the second inequality follows from the definition of $\mathcal{G}_\epsilon$ and the third from $\sum_{i\in S} \phi_i^j(S) \leq 1$, $\sum_j p_j \leq 1$, and the definition of $\theta(\epsilon)$. Since $h(\theta|\lambda,\alpha) \geq 0$ for any $\theta \in \Theta$, we have

$$
\sum_j \max_{S\in\mathcal{G}_\epsilon} \sum_{i\in S} \lambda p_j \theta_i^j \phi_i^j(S) - \sum_{i,j} \alpha_i^j \theta_i^j \geq h(\theta|\lambda,\alpha) - \epsilon \geq -\epsilon\big(\sum_{i,j} \theta_i^j\big),
$$

where the second inequality follows from $\sum_{i,j} \theta_i^j = 1$. Thus,

$$
\sum_j \max_{S\in\mathcal{G}_\epsilon} \sum_{i\in S} \lambda p_j \theta_i^j \phi_i^j(S) \geq \sum_{i,j} (\alpha_i^j - \epsilon) \theta_i^j
$$

Because $\mathcal{G}_\epsilon$ is a finite set, strong duality holds. Therefore, the following stochastic program with a constant objective function is feasible:

$$
\begin{aligned}
\max_{\tilde{G}}\ & 0 \\
s.t.\ & \mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G},\tilde{\xi}))] \geq \alpha_i^j - \epsilon, \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M} \\
& \tilde{G} \in \mathcal{P}(\mathcal{G}_\epsilon),
\end{aligned}
$$

where $\mathcal{P}(\mathcal{G}_\epsilon)$ is the probability distribution on $\mathcal{G}_\epsilon$. Since ϵ can be arbitrarily small in the interval $(0, \min\{\alpha_i^j\})$ and $\mathcal{P}(\mathcal{G}_\epsilon) \subset \mathcal{G}$ for any $\epsilon > 0$, we have that (3) is feasible as well. This has completed the proof of Theorem 2. $\square$

Proof of Theorem 3

First, by the definition of DWA algorithm, we have that

$$
t\alpha_i^j - \sum_{s=1}^t y_i^j(\tilde{S}^*(s)) = d_i^j(t+1) \leq \big(d_i^j(t+1)\big)^+
$$

Therefore, it suffices to show that, if (7) holds,

$$
\limsup_{t\to+\infty} \frac{1}{t} \cdot \big(d_i^j(t+1)\big)^+ \leq 0 \text{ with probability 1.}
$$

For $t \geq 1$, define $d(t+1) := (d_i^j(t+1) : i \in \mathcal{N}, j \in \mathcal{M})$ and $\tilde{y}(t) := (\tilde{y}_i^j(\tilde{S}^*(t)) : i \in \mathcal{N}, j \in \mathcal{M})$. For a vector $x \in \mathbb{R}^n$, we use x^+ to denote the component-wise positive part of x. Note that, for any A and B, $((A+B)^+)^2 \leq (A^+ + B)^2$. We have

$$\begin{aligned}
\mathbb{E}||\big(d(t+1)\big)^+||_2^2 =& \mathbb{E}||\big(d(t) + \alpha - \tilde{y}(t)\big)^+||_2^2 \\
\leq& \mathbb{E}||(d(t))^+ + \alpha - \tilde{y}(t)||_2^2 \\
=& \mathbb{E}||(d(t))^+||_2^2 + \mathbb{E}||\alpha - \tilde{y}(t)||_2^2 + 2\mathbb{E}[\sum_{i,j}(d_i^j(t))^+ \cdot \alpha_i^j - \sum_{i,j}(d_i^j(t))^+ \cdot \tilde{y}_i^j(\tilde{S}^*(t))],
\end{aligned}$$

where $||\cdot||_2$ denotes the L_2-norm. By (7), since $(d_i^j(t))^+ \geq 0$ for all i and j,

$$\mathbb{E}[\sum_{i,j}(d_i^j(t))^+ \cdot \alpha_i^j - \sum_{i,j}(d_i^j(t))^+ \cdot \tilde{y}_i^j(\tilde{S}^*(t))] \leq 0.$$

Furthermore, $\mathbb{E}||\alpha - \tilde{y}(t)||_2^2 \leq n \cdot m$. Therefore,

$$\mathbb{E}||(d(t+1))^+||_2^2 \leq ||(d(1))^+||_2^2 + tnm \text{ for all } t \geq 1.$$

By Jensen's inequality and that $||\cdot||_2^2$ is convex,

$$||\mathbb{E}[(d(t+1))^+]||_2^2 \leq \mathbb{E}||(d(t+1))^+||_2^2 \leq t \cdot (nm) \text{ for all } t \geq 1$$

Therefore, $\frac{1}{t}||\mathbb{E}[(d(t+1))^+]||_2 \leq \sqrt{\frac{1}{t} \cdot (nm)}$, which implies that $\limsup_{t\to+\infty} \frac{1}{t}||\mathbb{E}[(d(t+1))^+]||_2 = 0$. Hence $\limsup_{t\to+\infty} \frac{1}{t} \cdot (d_i^j(t+1))^+ = 0$ with probability 1, for all $i \in \mathcal{N}$ and $i \in \mathcal{M}$. This completes the proof. □

Proof of Corollary 1

The proof follows immediately from Theorem 1, Theorem 2, and equation (11). □

Proof of Proposition 1

Assume that the assortment offered to customers should satisfy the cardinality constraint, i.e., $\sum_{i\in\mathcal{N}} x_i^j \leq K$ for all $j \in \mathcal{M}$. A standard result in the assortment optimization literature postulates that we can relax the binary constraint $x_i^j \in \{0,1\}$ to $x_i^j \in [0,1]$ in (12), which is therefore equivalent to

$$\max_{x_i^j \in [0,1], \sum_i x_i^j \leq K} \sum_{i\in\mathcal{N}, j\in\mathcal{M}} \frac{\lambda p_j v_i^j \theta_i^j x_i^j}{1 + \sum_{i'\in\mathcal{N}} v_{i'}^j x_{i'}^j} \geq \sum_{i\in\mathcal{N}, j\in\mathcal{M}} \alpha_i^j \theta_i^j \text{ for all } \theta_i^j \geq 0 \ (i \in \mathcal{N}, j \in \mathcal{M}) \tag{31}$$

We change the decision variable and define $z^j := 1/(1 + \sum_{i'} v_{i'}^j x_{i'}^j)$ and $y_i^j := x_i^j z^j = x_i^j/(1 + \sum_{i'} v_{i'}^j x_{i'}^j)$. Then, we can rewrite (31) as

$$\begin{aligned}
&\min_{\theta_i^j \geq 0} \left(\max_{y_i^j, z^j} \sum_{i\in\mathcal{N}, j\in\mathcal{M}} \lambda p_j v_i^j y_i^j \theta_i^j - \sum_{i\in\mathcal{N}, j\in\mathcal{M}} \alpha_i^j \theta_i^j \right) \geq 0 \\
&s.t. \ \sum_{i=1}^n v_i^j y_i^j + z^j = 1 \\
&\qquad \sum_{i\in\mathcal{N}} y_i^j \leq K z^j, \text{ for all } j \in \mathcal{M} \\
&\qquad y_i^j \leq z^j \text{ for all } i \in \mathcal{N}, j \in \mathcal{M}
\end{aligned} \tag{32}$$

By Von Neumann's Minimax Theorem, we exchange the operators:

$$\begin{aligned}\max_{y,z}\min_{\theta\geq 0}\ &\sum_{j=1}^{m}\sum_{i=1}^{n}\theta_i^j(\lambda p_j v_i^j y_i^j-\alpha_i^j)\geq 0,\\ \text{s.t. }&\sum_{i=1}^{n}v_i^j y_i^j+z^j=1,\quad\forall j,\\ &\sum_{i=1}^{n}y_i^j\leq Kz^j,\quad\forall j,\\ &y_i^j\leq z^j,\quad\forall i,j,\\ &y_i^j\geq 0,\quad\forall i,j.\end{aligned}$$

Therefore, we can cancel the operator "max" and the first of the above constraints is equavalent to

$$\begin{aligned}&\sum_{j=1}^{m}\sum_{i=1}^{n}\theta_{ij}(\lambda p_j v_{ij}y_{ij}-\alpha_{ij})\geq 0,\quad\forall\theta\geq 0,\\ \text{or, }&\lambda p_j v_{ij}y_{ij}-\alpha_{ij}\geq 0,\quad\forall i,j.\end{aligned}$$

This completes the proof of Proposition 1. □

Proof of Proposition 2

Part (a). Define $U:=1/\lambda$. Obtaining the minimum traffic λ^* is equivalent to getting the maximum value of U^*. The optimal solution U^* can be obtained through the following:

$$U^*=\max_U\{U|U\leq U_j^*,\forall j,U\geq 1\},$$

where $U_j^*=\max_{U_j,x_i^j}\{U_j|p_j\frac{v_i^jx_i^j}{1+\sum_{i=1}^{n}v_i^jx_i^j}\geq\alpha_i^jU_j,\forall i,\sum_{i=1}^n x_i^j\leq K,0\leq x_i^j\leq 1,\forall i\}$.

That means $U^*=\min\{U_j^*,\forall j\}\geq 1$. Then we argue that, for all i, $U_j^*=\frac{p_j}{\alpha_i^j}\frac{v_i^jx_i^{j*}}{1+\sum_{i=1}^{n}v_i^jx_i^{j*}}$ is correct. If not, there is at least one constraint $p_j\frac{v_{i_j}^jx_{i_j}^{j*}}{1+\sum_{i=1}^{n}v_i^jx_i^{j*}}>\alpha_{i_j}^jU_j^*$ for each j, then we reduce $x_{i_j}^{j*}$ to $\tilde{x}_{i_j}^j$ but keep the constraint not binding at the same time, that is $\frac{p_j}{\alpha_{i_j}^j}\frac{v_{i_j}^j\tilde{x}_{i_j}^j}{1+\sum_{i=1,i\neq i_j}^{n}v_i^jx_i^{j*}+v_{i_j}^j\tilde{x}_{i_j}^j}>U_j^*$. The new solution $\tilde{x}'=\{\{x_i^{j*},\ \forall i\}\setminus\{x_{i_j}^{j*}\}\cup\{\tilde{x}_{i_j}^j\}\}$ is still feasible and $\sum_{i=1,i\neq i_j}^{n}v_i^jx_i^{j*}+v_{i_j}^j\tilde{x}_{i_j}^j$ is less than $\sum_{i=1}^{n}v_i^jx_i^{j*}$ such that $\frac{p_j}{\alpha_i^j}\frac{v_i^jx_i^{j*}}{1+\sum_{i=1,i\neq i_j}^{n}v_i^jx_i^{j*}+v_{i_j}^j\tilde{x}_{i_j}^j}$ for each $i\neq i_j$ is strictly greater than U_j^*. Let $U_j'=\min_i\{\frac{p_j}{\alpha_i^j}\frac{v_i^j\tilde{x}_i^j}{1+\sum_{i=1}^{n}v_i^j\tilde{x}_i^j}\}$, then U_j' is a feasible solution strictly greater than U_j^*, which forms a contradiction. Above all, we have $\frac{x_{i'}^{j*}}{x_{i''}^{j*}}=\frac{\alpha_{i'}^j}{v_{i'}^j}\times\frac{v_{i''}^j}{\alpha_{i''}^j}$ for any i' and i''. Let $x_i^j=\frac{\alpha_i^j}{v_i^j}\kappa_j$, we must have

$$U_j^*=\max_{U_j,\kappa_j}\{U_j|U_j\leq\frac{p_j\kappa_j}{1+\sum_{i=1}^{n}\alpha_i^j\kappa_j},\sum_{i=1}^{n}\frac{\alpha_i^j}{v_i^j}\kappa_j\leq K,\frac{\alpha_i^j}{v_i^j}\kappa_j\leq 1,\forall i,\kappa_j\geq 0,U_j\geq 1\}.$$

We have the optimal solution $\kappa_j^*=\min\{\frac{K}{\sum_{i=1}^n\frac{\alpha_i^j}{v_i^j}},\{\frac{v_i^j}{\alpha_i^j},\forall i\}\}$, $U^*=\min_j\{U_j^*=\frac{p_j\kappa_j^*}{1+\sum_{i=1}^{n}\alpha_i^j\kappa_j^*}\}\geq 1$ and $\lambda^*=1/U^*=\max_j\{\frac{1}{p_j\kappa_j^*}+\frac{1}{p_j}\sum_{i=1}^{n}\alpha_i^j\}$. This proves Part (a).

Part (b). Note that $\lambda \geq \lambda^*(\alpha)$ is equivalent to $\alpha \in \mathcal{A}(\lambda)$, where $\lambda^*(\alpha)$ is defined by (14) and $\mathcal{A}(\lambda)$ is defined by (15). This proves part (b). $\square$

Before proof of Proposition 3, we first provide the following auxiliary proposition, whose proof is given at the end of Appendix C.

PROPOSITION 7. *Given the (required) non-personalized sales targets $(\hat{\alpha}_i : i \in \mathcal{M})$, we have:*

(a) The optimization (17) *is equivalent to the following:*

$$\begin{aligned} \min_{0\leq\lambda\leq 1, G} \quad & c(\lambda) \\ \text{s.t.} \quad & \max_{G\in\hat{G}} \sum_{i,j} \lambda p_j \phi_i^j(S^j(G,j))\theta_i \geq \sum_i \hat{\alpha}_i \theta_i, \ \textit{for all } \theta_i \geq 0 \ (i \in \mathcal{N}) \end{aligned} \tag{33}$$

(b) If (33) *is feasible, the Non-Personalized Debt-Weighted Assortment (NDWA) policy (Algorithm 5 in the Appendix B) meets the non-personalized sales targets $(\hat{\alpha}_i : i \in \mathcal{N})$.*

(c) If customer choices follow the MNL model (11)*, the optimization problem* (33) *can be reformulated as*

$$\begin{aligned} \min_{x,\lambda} \quad & c(\lambda) \\ \text{s.t.} \quad & \sum_{j=1}^{m} \lambda p_j \frac{v_i^j x_i^j}{1 + \sum_{i=1}^{n} v_i^j x_i^j} - \hat{\alpha}_i \geq 0, \quad \forall i \\ & \sum_{i=1}^{n} x_i^j \leq K, \quad \forall j, \\ & 0 \leq x_i^j \leq 1, \quad \forall i,j, \\ & 0 \leq \lambda \leq 1. \end{aligned} \tag{34}$$

Proof of Proposition 3

If the customer choices follow the MNL model, by Proposition 1, (16) can be reformulated as follows:

$$\begin{aligned} \min_{y_i^j, z^j, 0\leq\lambda\leq 1} \quad & c(\lambda) \\ \text{s.t.} \quad & \lambda p_j v_i^j y_i^j \geq \hat{\alpha}_i^j, \quad \forall i,j \\ & \sum_{i=1}^{n} v_i^j y_i^j + z^j = 1, \quad \forall j, \\ & \sum_{i=1}^{n} y_i^j \leq K z^j, \quad \forall j, \\ & y_i^j \leq z^j, \quad \forall i,j, \\ & 0 \leq y_i^j \leq 1, \quad \forall i,j \end{aligned} \tag{35}$$

Because $c(\cdot)$ is strictly increasing in the customer traffic λ, the solution to (35) is the minimum λ with which all the constraints of (35) can be satisfied. By Proposition 2, we know that the minimum traffic to satisfy the required sales targets $(\hat{\alpha}_i^j : i \in \mathcal{N}, j \in \mathcal{M})$ is given by $\lambda^*(\hat{\alpha})$ defined in (14).

We obtain the bounds for $\rho(\alpha)$ by bounding $\lambda_N^*(\alpha)$ from below and above separately.

To get a lower bound of $\lambda_N^*(\alpha)$, define $U_N^*(\alpha) := 1/\lambda_N^*(\alpha)$. A relaxation of the problem with non-personalized targets is constructed as

$$\max_{U_R, x} U_R$$

$$
\begin{aligned}
\text{s.t. } & \sum_{i=1}^{n}\sum_{j=1}^{m} p_j \frac{v_i^j x_i^j}{1+\sum\limits_{i=1}^{n} v_i^j x_i^j} - \sum_{i=1}^{n} \alpha_i U_R \geq 0, \\
& \sum_{i=1}^{n} x_i^j \leq K, \quad \forall j, \\
& x_i^j \leq 1, \quad \forall i,j, \\
& x_i^j \geq 0, \quad \forall i,j, \\
& U_R \geq 1,
\end{aligned}
$$

in which the first constraint is equivalent to $U_R \leq \sum\limits_{j=1}^{m} \frac{p_j}{\sum\limits_{i=1}^{n}\alpha_i} \cdot \frac{\sum\limits_{i=1}^{n} v_i^j x_i^j}{1+\sum\limits_{i=1}^{n} v_i^j x_i^j}$. Let U_R^* be the optimal solution of the optimization problem above. Then, we have $1/\lambda_N^*(\alpha) = U_N^*(\alpha) \leq U_R^* \leq \max\limits_{x_i^j\in\{0,1\}} \sum\limits_{j=1}^{m} \frac{p_j}{\sum\limits_{i=1}^{n}\alpha_i} \cdot \frac{\sum\limits_{i=1}^{n} v_i^j x_i^j}{1+\sum\limits_{i=1}^{n} v_i^j x_i^j}$, and $\max\limits_{x_i^j\in\{0,1\}} \frac{\sum\limits_{i=1}^{n} v_i^j x_i^j}{1+\sum\limits_{i=1}^{n} v_i^j x_i^j} \leq \frac{\sum\limits_{i=1}^{n} v_i^j}{1+\sum\limits_{i=1}^{n} v_i^j}$ for each j, by relaxing the cardinality constraint. Therefore, we have

$$
\lambda_N^*(\alpha) \geq \left(\sum_{j=1}^{m} \frac{p_j}{\sum\limits_{i=1}^{n}\alpha_i} \cdot \frac{\sum\limits_{i=1}^{n} v_i^j x_i^j}{1+\sum\limits_{i=1}^{n} v_i^j x_i^j} \right)^{-1}
$$

To get an upper bound of $\lambda_N^*(\alpha)$, we construct a feasible solution $\bar{x}_i^j = \frac{\alpha_i}{v_i^j}\kappa_N^{j*}$ of the problem with non-personalized targets, such that

$$
\bar{U}_N = \sum_{j=1}^{m} \frac{p_j}{\alpha_i} \cdot \frac{\sum_{i=1}^{n} v_i^j \bar{x}_i^j}{1+\sum\limits_{i=1}^{n} v_i^j \bar{x}_i^j} = \sum_{j=1}^{m} \frac{p_j \kappa_N^{j*}}{1+\sum\limits_{i=1}^{n}\alpha_i \kappa_N^{j*}},
$$

where we define $\kappa_N^{j*} := \min\{\frac{K}{\sum_{i=1}^{n}\frac{\alpha_i}{v_i^j}}, \{\frac{v_i^j}{\alpha_i}, \forall i\}\}$. It is easy to verify that $\bar{x}_i^j \in [0,1]$ for all i and j, and $\sum_{i=1}^{n}\bar{x}_i^j \leq K$ for all j. Therefore the optimal solution $U_N^*(\alpha)$ must satisfy $U_N^*(\alpha) \geq \bar{U}_N = \sum\limits_{j=1}^{m} \frac{p_j \kappa_N^{j*}}{1+\sum\limits_{i=1}^{n}\alpha_i\kappa_N^{j*}}$. Therefore, we have an upper bound for $\lambda_N^*(\alpha)$:

$$
\lambda_N^*(\alpha) \leq \left(\sum_{j=1}^{m} \frac{p_j \kappa_N^{j*}}{1+\sum\limits_{i=1}^{n}\alpha_i \kappa_N^{j*}} \right)^{-1}
$$

Combining the above above two bounds of $\lambda_N^*(\alpha)$ helps us establish those of $\rho_N(\alpha)$, which completes the proof. □

Proof of Proposition 4

Part (a) follows directly from Proposition 1.

Part (b). First, observe that the optimization is equivalent to

$$
\begin{aligned}
\min_{\lambda_l} \quad & \sum_{l=1}^{L} c_l(\lambda_l) \\
\text{s.t.} \quad & \frac{1}{(\sum\limits_{l=1}^{L} p_{l,j}\lambda_l)} \leq V_j^*, \quad \forall j, \\
& 0 \leq \lambda_l \leq 1, \quad \forall l,
\end{aligned}
$$

where $V_j^* = \max_{V_j, x_i^j}\{V_j | \frac{v_i^j x_i^j}{1+\sum_{i=1}^n v_i^j x_i^j} - \alpha_i^j V_j \geq 0, \forall i, \sum_{i=1}^n x_i^j \leq K, 0 \leq x_i^j \leq 1, \forall i\}$.

We solve that $V_j^* = \frac{\kappa_j^*}{1+\sum_{i=1}^n \alpha_i^j \kappa^{j*}}$, which implies that the the constraint in the primal problem can be rewritten as $\sum_{l=1}^L p_{l,j}\lambda_l \geq \frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j$ with $\kappa_j^* = \min\{\frac{K}{\sum_{i=1}^n \frac{\alpha_i^j}{v_i^j}}, \{\frac{v_i^j}{\alpha_i^j}, \forall i\}\}$ for each j.

Assume, in addition, $c_l(\lambda_l) = \lambda_l$. Using the above results, we have an equivalent linear program formulation of the multi-sourcing traffic minimization problem:

$$\begin{aligned} \min_{\lambda_l} \quad & \sum_{l=1}^L \lambda_l \\ \text{s.t.} \quad & \sum_{l=1}^L p_{l,j}\lambda_l \geq \frac{1}{\kappa^{j*}} + \sum_{i=1}^n \alpha_i^j, \quad \forall j, \\ & 0 \leq \lambda_l \leq 1, \quad \forall l, \end{aligned}$$

where $\kappa^{j*} := \min\{\frac{K}{\sum_{i=1}^n \frac{\alpha_i^j}{v_i^j}}, \{\frac{v_i^j}{\alpha_i^j}, \forall i\}\}$. This proves part (b). □

Proof of Proposition 5

By Proposition 4, since $\sum_{j=1}^m p_{l,j} = 1$ for each l, we can have the following lower bound on the minimal value of $\sum_{l=1}^L \lambda_l^{**}$:

$$\sum_{l=1}^L \lambda_l^{**} \geq \sum_{j=1}^m (\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j),$$

which implies that

$$\frac{\sum_{l=1}^L \lambda_l^{**}}{\sum_{l=1}^L \lambda_l^*} \geq \frac{\sum_{j=1}^m (\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j)}{\frac{1}{L}\sum_{l=1}^L \max_j\{\frac{1}{p_{l,j}}(\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j)\}} = \frac{L\sum_{j=1}^m (\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j)}{\sum_{l=1}^L \max_j\{\frac{1}{p_{l,j}}(\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j)\}}.$$

This proves the lower-bound of $\rho_M(\alpha)$.

Then, we construct a feasible solution to get an upper bound of $\sum_{l=1}^L \lambda_l^{**}$. Let $\lambda_l^{feas} = \lambda_l^* h^*$, where h^* is the solution to the following optimization problem:

$$\begin{aligned} \min_{h} \quad & \sum_{l=1}^L \lambda_l^* h \\ \text{s.t.} \quad & \sum_{l=1}^L p_{l,j}\lambda_l^* h \geq \frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j, \quad \forall j, \\ & h \geq 0, \quad \forall l. \end{aligned}$$

We solve that the optimal solution h^* is given by $\max_j \left\{ \frac{\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j}{\sum_{l=1}^L p_{l,j}\lambda_l^*} \right\}$. Because $h = 1$ is feasible, $h^* \leq 1$. Hence, $\lambda_l^{feas} \leq \lambda_l^* \leq 1$. This implies the following inequality:

$$\frac{\sum_{l=1}^L \lambda_l^{**}}{\sum_{l=1}^L \lambda_l^*} \leq \frac{\sum_{l=1}^L \lambda_l^* h^*}{\sum_{l=1}^L \lambda_l^*} = h^* = \max_j \left\{ \frac{\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j}{\frac{1}{L}\sum_{l=1}^L p_{l,j} \max_j\{\frac{1}{p_{l,j}}(\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j)\}} \right\} = \max_j \left\{ \frac{L(\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j)}{\sum_{l=1}^L p_{lj} \max_j\{\frac{1}{p_{l,j}}(\frac{1}{\kappa_j^*} + \sum_{i=1}^n \alpha_i^j)\}} \right\}.$$

This completes the proof. □

Before giving the proof of Theorem 4, we show the following lemma which demonstrates that Algorithm 2 is feasible as the problem scale γ goes to infinity.

LEMMA 1. *For problem $\mathcal{Q}(\gamma)$, Algorithm 2 is feasible as $\gamma \uparrow +\infty$, i.e.,*

$$\liminf_{\gamma\to+\infty} \frac{1}{T(\gamma)} \sum_{t=1}^{T(\gamma)} \tilde{y}_i^j(t) \geq \alpha_i^{j*} \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M},$$

where $\tilde{y}_i^j(t) \in \{0,1\}$ denotes whether a type-j customer chooses product i in period t.

Proof. Let us consider a problem identical to $\mathcal{Q}(\gamma)$ but without inventory constraints (i.e., $C_i(\gamma) = +\infty$ for all i and γ), which we denote as $\mathcal{Q}_*(\gamma)$. Consider one version of Algorithm 2 with $\alpha_i^j = \alpha_i^{j*}$, where α^* is the solution to (23). Let $y_i^j(t) \in \{0,1\}$ denote whether a type-j customer chooses product i in period t under $\mathcal{Q}_*(\gamma)$. It is clear from the construction that the sales process of $\mathcal{Q}_*(\gamma)$ is identical to that of $\mathcal{Q}(\gamma)$ before stock-out occurs at $\mathcal{Q}(\gamma)$, i.e. $y_i^j(t) = \tilde{y}_i^j(t)$ for all t before stock-out occurs at $\mathcal{Q}(\gamma)$. For simplicity, we assume $\gamma \in \mathbb{Z}^+$.

We now show that for the system of $\mathcal{Q}_*(\gamma)$,

$$\limsup_{\gamma\uparrow+\infty} \frac{\sum_j \sum_{t=1}^{T(\gamma)} y_i^j(t)}{T(\gamma)} \leq \frac{C_i(\gamma)}{T(\gamma)} = \frac{C_i}{T} \text{ for all } i.$$

Assume, to the contrary, that,

$$\limsup_{\gamma\uparrow+\infty} \frac{\sum_j \sum_{t=1}^{T(\gamma)} y_{i_0}^j(t)}{T(\gamma)} > \frac{C_{i_0}}{T} \geq \sum_j \alpha_{i_0}^{j*} \text{ for some } i_0, \tag{36}$$

where the last inequality follows from that α^* solves (23). Therefore, by the pigeonhole principle, there exists a j_0 such that

$$\limsup_{\gamma\uparrow+\infty} \frac{\sum_{t=1}^{T(\gamma)} y_{i_0}^{j_0}(t)}{T(\gamma)} > \alpha_{i_0}^{j_0*},$$

i.e., there exists some $\Delta > 0$, such that

$$\frac{\sum_{t=1}^{T(\gamma)} y_{i_0}^{j_0}(t)}{T(\gamma)} > \alpha_{i_0}^{j_0*} + \Delta \text{ for infinitely many } \gamma. \tag{37}$$

Denote the set of γ's that satisfy (37) as Γ. Note that $\frac{1}{T(\gamma)}(\sum_{t=1}^{\tau} y_{i_0}^{j_0}(t))$ increases by at most $1/(T\gamma)$ as τ increases by 1. Hence, for all $\gamma \in \Gamma$ and $\gamma > 3/(T\Delta)$, $\frac{1}{T(\gamma)}(\sum_{t=1}^{\tau} y_{i_0}^{j_0}(t))$ increases by no more than $\Delta/3$ if τ increases by 1. Therefore, for all $\gamma \in \Gamma$ and $\gamma > 3/(T\Delta)$, there exists a $\tau(\gamma) < T(\gamma)$, such that

$$\alpha_{i_0}^{j_0*} + \frac{\Delta}{3} < \frac{\sum_{t=1}^{\tau(\gamma)} y_{i_0}^{j_0}(t)}{T(\gamma)} < \alpha_{i_0}^{j_0*} + \frac{2\Delta}{3} \tag{38}$$

By (38), we have that, for infinitely many γ,

$$\sum_{t=1}^{\tau(\gamma)} y_{i_0}^{j_0}(t) > T(\gamma)\left(\alpha_{i_0}^{j_0*} + \frac{\Delta}{3}\right).$$

Hence, for infinitely many γ,

$$(d_{i_0}^{j_0}(t))^+ = \left((t-1)\alpha_{i_0}^{j_0*} - \sum_{s=1}^{t-1} y_{i_0}^{j_0}(s)\right)^+ = 0 \text{ for all } t \geq \tau(\gamma)+1,$$

where the equality follows from $\sum_{s=1}^{t-1} y_{i_0}^{j_0}(s) \geq \sum_{s=1}^{\tau(\gamma)} y_{i_0}^{j_0}(s) > T(\gamma)\alpha_{i_0}^{j_0*} > (t-1)\alpha_{i_0}^{j_0*}$. Therefore, product i_0 will not be offered to customer type j_0 for all $t \geq \tau(\gamma)+1$. Hence, $y_{i_0}^{j_0}(t) = 0$ for all $t \geq \tau(\gamma)+1$ and $t \leq T(\gamma)$. By (38), we have

$$\frac{\sum_{t=1}^{T(\gamma)} y_{i_0}^{j_0}(t)}{T(\gamma)} = \frac{\sum_{t=1}^{\tau(\gamma)} y_{i_0}^{j_0}(t)}{T(\gamma)} < \alpha_{i_0}^{j_0*} + \frac{2\Delta}{3} \text{ for } \gamma \in \Gamma \text{ and } \gamma > 3/(T\Delta),$$

which contradicts inequality (37). Therefore, for the system of $\mathcal{Q}_*(\gamma)$, we have

$$\limsup_{\gamma\uparrow+\infty} \frac{\sum_j \sum_{t=1}^{T(\gamma)} y_i^j(t)}{T(\gamma)} \leq \frac{C_i(\gamma)}{T(\gamma)} = \frac{C_i}{T} \text{ for all } i. \tag{39}$$

On the other hand, Theorem 3 then implies that, for the system of $\mathcal{Q}_*(\gamma)$, we have

$$\liminf_{\gamma\to+\infty} \frac{1}{T(\gamma)} \sum_{t=1}^{T(\gamma)} y_i^j(t) \geq \alpha_i^{j*} \text{ for each } i \in \mathcal{N} \text{ and } j \in \mathcal{M}. \tag{40}$$

Furthermore, since the sales process of $\mathcal{Q}(\gamma)$ is identical to that of $\mathcal{Q}_*(\gamma)$ before inventory stock-out occurs at $\mathcal{Q}(\gamma)$. Therefore, for the system of $\mathcal{Q}(\gamma)$, a standard coupling argument implies that and (39) and (40) hold as well. This concludes the proof of the lemma. □

Proof of Theorem 4

Because $Rev(\gamma, \tilde{G}_{DWA-I}) \leq Rev^*(\gamma)$, it suffices to show that

$$\liminf_{\gamma\to+\infty} \frac{Rev(\gamma, \tilde{G}_{DWA-I})}{Rev^*(\gamma)} = 1.$$

First, note that as $\gamma \uparrow +\infty$, by Lemma 1,

$$\liminf_{\gamma\to+\infty} \frac{1}{T(\gamma)} \sum_{s=1}^{T(\gamma)} \tilde{y}_i^j(s) \geq \alpha_i^{j*},$$

we must have

$$\liminf_{\gamma\to+\infty} \frac{Rev^*(\gamma)}{T(\gamma)} \geq \liminf_{\gamma\to+\infty} \frac{Rev(\gamma, \tilde{G}_{DWA-I})}{T(\gamma)} \geq \sum_{i,j} r_i^j \alpha_i^{j*}.$$

Hence, it suffices to show that

$$\liminf_{\gamma\to+\infty} \frac{T(\gamma)\sum_{i,j} r_i^j \alpha_i^{j*}}{Rev^*(\gamma)} = 1.$$

Consider the following linear program relaxation of the original revenue management problem:

$$\begin{aligned}
\max\ & \lambda \sum_{i,j,S\in\hat{G}} r_i^j p_j \phi_i^j(S) z^j(S) \\
s.t.\ & T(\gamma)\lambda \sum_{j,S} p_j \phi_i^j(S) z^j(S) \leq C_i(\gamma) \text{ for all } i \in \mathcal{N} \\
& \lambda \sum_{S} p_j \phi_i^j(S) z^j(S) \geq \hat{\alpha}_i^j \text{ for all } i \in \mathcal{N} \text{ and } j \in \mathcal{M} \\
& \sum_{S\in\hat{G}} z^j(S) \leq 1 \text{ for all } j \in \mathcal{M} \\
& z^j(S) \geq 0 \text{ for all } j \in \mathcal{M}, S \in \hat{G}
\end{aligned} \tag{$\mathcal{LP}(\gamma)$}$$

One should note that, for any γ, the optimal value of the linear-program relaxation $\mathcal{LP}(\gamma)$, which we denote as $Rev_{LP}^*(\gamma)$, is an upper bound of $Rev^*(\gamma)$, i.e., $Rev_{LP}^*(\gamma) \geq Rev^*(\gamma)$. Denote the optimal solution to $\mathcal{LP}(\gamma)$

as $z^*(S) := (z^{j*}(S) : j \in \mathcal{M}, S \in \hat{G})$. Clearly, $z^*(S)$ corresponds to a randomized assortment policy that generates sales target $\tilde{\alpha}_i^{j*} = \lambda \sum_S \phi_i^j(S) z^{j*}(S)$. It is easy to observe that $(\tilde{\alpha}_i^{j*} : i \in \mathcal{N}, j \in \mathcal{M})$ is a feasible solution to (21). Thus,

$$\sum_{i,j} r_i^j \alpha_i^{j*} \geq \sum_{i,j} r_i^j \tilde{\alpha}_i^{j*} = \lim_{\gamma \to +\infty} \frac{1}{T(\gamma)} \cdot Rev^*(\gamma),$$

where the last inequality follows from Proposition 2 of Liu and Van Ryzin (2008). It follows immediately that

$$\liminf_{\gamma \to +\infty} \frac{T(\gamma) \sum_{i,j} r_i^j \alpha_i^{j*}}{Rev^*(\gamma)} = 1.$$

This completes the proof of Theorem 4. □

Proof of Theorem 5

Because the algorithm uniformly randomly picks up a time index t,

$$\mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G}_{RDWA}, \tilde{\xi}))] = \frac{1}{T} \sum_{s=2}^{T+1} \mathbb{E}[\tilde{y}_i^j(S^{j*}(s))]$$

Therefore,

$$\mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G}_{RDWA}, \tilde{\xi}))] - \alpha_i^j = \frac{1}{T} \sum_{s=2}^{T+1} (\mathbb{E}[\tilde{y}_i^j(S^{j*}(s))] - \alpha_i^j) = -\frac{1}{T} \mathbb{E}[d_i^j(T+1)].$$

By the proof of Theorem 3,

$$\limsup_{T \to +\infty} \frac{1}{T} \mathbb{E}[d_i^j(T+1)] \leq \limsup_{T \to +\infty} \frac{1}{T} \mathbb{E}[d_i^j(T+1)]^+ = 0 \text{ for all } i \text{ and } j$$

Hence, $\mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G}_{RDWA}, \tilde{\xi}))] \geq \alpha_i^j$ for all i and j as $T \to +\infty$. This completes the proof. □

Proof of Proposition 6

First, we need to show that Algorithm $\tilde{G}_{DWA-ART}$ satisfies that

$$\liminf_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{j \in \mathcal{M}} r_i^j \tilde{y}_i^j(\tilde{S}^*_{\tilde{G}_{DWA-ART}}) \geq \sum_{j \in \mathcal{M}} r_i^j \alpha_i^{j*} \text{ for all } i \in \mathcal{N}. \tag{41}$$

Note that

$$\begin{aligned}
\mathbb{E}||\big(d(t+1)\big)^+||_2^2 =& \mathbb{E}||\big(d(t) + diag(r^T \alpha^*) - diag(r^T \tilde{y}(t))\big)^+||_2^2 \\
\leq& \mathbb{E}||(d^2(t))^+ + diag(r^T \alpha^*) - diag(r^T \tilde{y}(t))||_2^2 \\
=& \mathbb{E}||(d(t))^+||_2^2 + \mathbb{E}||diag(r^T \alpha^*) - diag(r^T \tilde{y}(t))||_2^2 \\
&+ 2\mathbb{E}[\sum_{i,j} (d_i(t))^+ \cdot r_i^j \alpha_i^{j*} - \sum_{i,j} (d_i(t))^+ \cdot r_i^j \tilde{y}_i^j(\tilde{S}^*(t))],
\end{aligned}$$

where r denotes the matrix of $(r_i^j : i \in \mathcal{N}, j \in \mathcal{M})$ and $diag(\cdot)$ denotes the diagonal element vector of a matrix. Since α^* satisfies $h(\theta|\lambda, \alpha) \geq 0$ for all $\theta \geq 0$, we have

$$\mathbb{E}[\sum_{i,j} (d_i(t))^+ \cdot r_i^j \alpha_i^{j*} - \sum_{i,j} (d_i(t))^+ \cdot r_i^j \tilde{y}_i^j(\tilde{S}^*(t)_{\tilde{G}_{DWA-ART}})] \leq 0.$$

Furthermore, we have the following bound $\mathbb{E}||diag(r^T \cdot \alpha^*) - diag(r^T \cdot \tilde{y}(t))||_2^2 \leq (\max_{i,j}\{r_i^j\})^2 \cdot (nm)$. Therefore, by a similar argument to the proof of Theorem 3, we have

$$\frac{1}{t} ||\mathbb{E}[(d(t+1))^+]||_2 \leq \max_{i,j}\{r_i^j\} \sqrt{\frac{1}{t} \cdot (nm)},$$

which implies that

$$\limsup_{t\to+\infty} \frac{1}{t}\cdot (d_i(t+1))^+ = 0 \text{ with probability 1, for all } i\in\mathcal{N}.$$

In other words, we have, for all $i\in\mathcal{N}$,

$$\liminf_{T\to+\infty} \frac{1}{T}\sum_{t=1}^{T}\sum_{j\in\mathcal{M}} r_i^j \tilde{y}_i^j(\tilde{S}^*_{\tilde{G}_{DWA-ART}}) - \sum_j \alpha_i^{j*} = -\limsup_{T\to+\infty}\frac{1}{T} d_i(T+1) \geq -\limsup_{T\to+\infty}\frac{1}{T}(d_i(T+1))^+ = 0,$$

which proves inequality (41).

We are now ready to show that

$$\lim_{\gamma\to+\infty} \frac{Rev(\gamma, \tilde{G}_{DWA-ART})}{Rev^*(\gamma)} = 1.$$

By definition, we have $Rev(\gamma, \tilde{G}_{DWA-ART}) \leq Rev^*(\gamma)$. Thus, it suffices to show that

$$\liminf_{\gamma\to+\infty} \frac{Rev(\gamma, \tilde{G}_{DWA-ART})}{Rev^*(\gamma)} = 1.$$

First, note that as $\gamma\uparrow+\infty$, by (41),

$$\liminf_{\gamma\to+\infty} \frac{1}{T(\gamma)}\sum_{s=1}^{T(\gamma)}\sum_{j\in\mathcal{M}} \tilde{r}_i^j y_i^j(s) \geq \sum_j r_i^j\alpha_i^{j*}, \text{ for all } i\in\mathcal{N}$$

we must have

$$\liminf_{\gamma\to+\infty}\frac{Rev^*(\gamma)}{T(\gamma)} \geq \liminf_{\gamma\to+\infty}\frac{Rev(\gamma,\tilde{G}_{DWA-ART})}{T(\gamma)} = \liminf_{\gamma\to+\infty}\frac{\sum_{t=1}^{T(\gamma)}\sum_{i,j} r_i^j\tilde{y}_i^j}{T(\gamma)} \geq \sum_i(\sum_j r_i^j\alpha_i^{j*}) = \sum_{i,j} r_i^j\alpha_i^{j*}. \quad (42)$$

As we have shown in the Proof of Theorem 4,

$$\liminf_{\gamma\to+\infty}\frac{T(\gamma)\sum_{i,j} r_i^j\alpha_i^{j*}}{Rev^*(\gamma)} = 1.$$

Together with (42), we have

$$\liminf_{\gamma\to+\infty}\frac{Rev(\gamma,\tilde{G}_{DWA-ART})}{Rev^*(\gamma)} \geq \liminf_{\gamma\to+\infty}\frac{T(\gamma)\sum_{i,j} r_i^j\alpha_i^{j*}}{Rev^*(\gamma)} = 1.$$

This completes the proof of Proposition 6. □

Proof of Proposition 7

Part (a). To obtain the feasibility condition, note that (17) is equivalent to the following semi-infinite linear program:

$$\begin{aligned}
&\max_{\mu(\cdot)}\ 0 \\
&s.t.\ \int_{G\in\hat{G}}\sum_j \lambda p_j\phi_i^j(S^j(G,j))\mathrm{d}\mu(G) \geq \alpha_i, \text{ for each } i\in\mathcal{N} \\
&\qquad \int_{G\in\hat{G}}\mathrm{d}\mu(G) = 1 \\
&\qquad \mathrm{d}\mu(G)\geq 0 \text{ for all } G\in\hat{G}.
\end{aligned} \quad (43)$$

The dual of (43) can be written as:

$$\begin{aligned}
&\min_{\theta_0,\theta_i}\ \{\theta_0 - \sum_{i\in\mathcal{N},j\in\mathcal{M}}\alpha_i\theta_i\} \\
&s.t.\ \sum_{i\in\mathcal{N}}\lambda p_j\phi_i^j(S^j(G,j))\theta_i - \theta_0 \leq 0, \text{ for all } G\in\hat{G} \\
&\qquad \theta_i\geq 0 \text{ for all } i\in\mathcal{N}.
\end{aligned} \quad (44)$$

By weak duality, (43) is feasible only if

$$\min_{\theta_i \geq 0} \left\{ \max_{G \in \hat{G}} \sum_{i \in \mathcal{N}, j \in \mathcal{M}} \lambda p_j \phi_i^j (S^j(G,j)) \theta_i - \sum_{i \in \mathcal{N}} \alpha_i \theta_i \right\} \geq 0 \tag{45}$$

Clearly, (45) is equivalent to (33), which proves the necessity of (33).

To show that (33) implies the feasibility of (45), we note that it suffices to show the strong duality between the primal (43) and the dual (43). Following the same argument as the proof of Theorem 2, if (33) holds, for any $\epsilon > 0$ and $\epsilon < \min\{\alpha_i : i \in \mathcal{N}\}$, we can construct a finite subset of $\mathcal{G}$, $\mathcal{G}_\epsilon$, such that

$$\max_{S \in \mathcal{G}_\epsilon} \sum_{i,j} \lambda p_j \theta_i \phi_i^j(S) \geq \sum_i (\alpha_i - \epsilon)\theta_i, \text{ for any } \theta_i \geq 0.$$

Since $\mathcal{G}_\epsilon$ is finite, strong duality holds and therefore, $\sum_{j=1}^m \mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G},\tilde{\xi}))] \geq \alpha_i - \epsilon$ is feasible if $\tilde{G} \in \mathcal{G}$ for any $0 \leq \epsilon < \min\{\alpha_i : i \in \mathcal{N}\}$ and for each $i \in \mathcal{N}$. Since $\epsilon \geq 0$ can be arbitrarily close to 0, (33) implies that $\sum_{j=1}^m \mathbb{E}[\tilde{y}_i^j(\tilde{S}^j(\tilde{G},\tilde{\xi}))] \geq \alpha_i$ holds for each $i \in \mathcal{N}$. This proves part (a).

Part (b). Part (b) of Proposition 7 follows from the same argument as the proof of Theorem 3, so we only sketch the proof. Specifically, it suffices to show that if (33) holds, $d_i(t+1)/t \to 0$ in probability for all i, where $d_i(t)$ is defined in Algorithm 5 for each i and t. To show that $d_i(t+1)/t \to 0$ in probability, we leverage the condition (33) and find that $||\mathbb{E}[(d(t+1))^+]||_2^2 \leq tn$, which implies that

$$\limsup_{t \to +\infty} \frac{1}{t} \cdot (d_i(t+1)) = 0 \text{ with probability 1 for all } i \in \mathcal{N}.$$

This proves Part (b).

Part (c) follows directly from the definition of sales target constraints. □