Carpool Services for Ride-sharing Platforms: Price and Welfare Implications

Xuan Wang
Department of Information Systems, Business Statistics and Operations Management,
Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, xuanwang@ust.hk

Renyu Zhang
New York University Shanghai,
1555 Century Avenue, Shanghai, China, 200122, renyu.zhang@nyu.edu

There has been rapid growth in on-demand ride-hailing platforms that serve as an intermediary to match individual service providers (drivers) with consumer demand (riders). Several major players of this market have introduced carpool services that allow passengers heading towards the same direction to share a ride at a discounted fare. In this paper, we develop an analytical model to study the pricing issues of ride-sharing platforms in the presence of carpool services, and their economical and social implications. We show that the carpool service should be provided when its quality and/or the pooling efficiency is high. Adopting carpool services enables the platform to achieve a larger market coverage and allows the customers to enjoy more affordable rides without any sacrifice in service quality. Our analysis reveals that the provision of carpool services benefits the platform and the riders in general, but may hurt the drivers. Our extensive numerical studies suggest that the carpool service and surge pricing, which are two operational levers to match supply with demand, are strategic complements when the demand-supply imbalance is severe, and they become strategic substitutes with balanced demand and supply.

Key words: Sharing economy, on-demand two-sided platforms, carpool services, surge pricing, welfare analysis

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1. Introduction

Recent years have witnessed a phenomenal growth in on-demand ride-hailing platforms (e.g., Uber, Lyft, Grab, Ola and Didi Chuxing) that serve as an intermediary to match individual service providers (drivers) with consumer demand (riders). Instead of planning the provider resources (e.g., drivers, cars, etc) in advance, ride-hailing platforms operate by matching self-scheduling and earning-sensitive drivers with price-sensitive customers in a real time. The rapid growth in popularity and success of these on-demand ride-sharing platforms have been phenomenal. Uber, for instance, is now operating in more than 900 cities globally (Uber 2020) and has achieved 111 million active users by the end of 2019 (DMR 2020a). As another example, Didi Chuxing, the largest on-demand ride-hailing platform in China, processes 10 billion trips per year for more than 550 million users as of 2019 (DMR 2020b).

Several major players of the on-demand ride-hailing market have introduced carpool services, such as UberPool by Uber, Lyft Line by Lyft and Didi Pinche by Didi Chuxing. The carpool service option enables drivers to pick up multiple passengers travelling along similar routes and the riders need to share the ride with each other. The platform typically offers a discounted price for carpool rides. However, this discounted fare often comes at the expense of a lower-quality service because of the lack of privacy and longer trip duration if detours are made to pick up or drop off other passengers. Even before the era of on-demand ride-hailing platforms, carpool has been widely applauded and promoted for their value in reducing the number of vehicles on the road, which helps curtail exhaust pollution and alleviate traffic congestions (see, e.g., Chan and Shaheen 2012). Uber has also promoted UberPool by highlighting the value of their carpool services from the social and environmental perspectives (e.g., UberBlog 2016). However, it is worth noting that the carpool services have also received controversial responses since their introduction. Some riders favor the carpool service because it is less expensive and more affordable than the normal service, and sometimes sharing a ride with others even facilitates new connections (e.g., Jess 2015). On the other hand, one major critique of the carpool service is the potential compromises on privacy, security, and inconvenience involved with riding with a stranger in the closed and confined environment within a car. Moreover, many drivers complain that they work more for UberPool or Lyft Line but get paid less, and receive lower ratings because carpool riders usually receive lower-quality services. Despite the prevalence and controversial perceptions of carpool services in the ride-sharing market, research in the extant literature that rigorously studies their operational, economical and social implications is scarce.

In this paper, our primary goal is to model the *carpool* service for the on-demand ride-hailing platforms and investigate the price and welfare implications it bears. More specifically, we consider a monopoly ride-sharing platform who offers both normal (non-pool) and carpool services with vertically differentiated qualities. Riders have heterogeneous values over service quality and would choose the travel option with the highest surplus. Drivers are self-scheduling with heterogeneous reservation wages, and would work for the platform only if the wages distributed by the platform dominate their outside options. To maximize its expected profit, the platform designs its pricing and wage schemes, and matches drivers with riders for both normal and carpool services. We characterize the optimal policy of the platform, study the impact of carpool services on different stake holders of the market (the platform, riders, and drivers), and draw insights on the conditions under which carpool services are most valuable. Our theoretical analysis is also complemented with computational studies which help strengthen the practical relevance of this work.

1.1. Main Contributions

We next summarize our main results and contributions below.

1.1.1. Pricing Implications of Carpool Services. Our analysis reveals that it is optimal for the platform to offer carpool services as long as it is provided with high quality and/or high efficiency. We show that the provision of carpool services enables the platform to better utilize the driver capacity and expands its market coverage. We find that, as expected, the optimal price for the carpool services is lower than that of the normal services, which is consistent with the business practice. More interestingly, the presence of carpool services also prompts the platform to charge a lower price for its normal service when both service modes are offered compared with the benchmark system where the carpool option is not offered. One may intuit that, by offering an additional service mode, the platform should be in a better position to discriminate between customers with different valuations and can charge a higher price for the high-quality service. However, our results suggest that the opposite is true. The additional leverage of the carpool services helps the platform expand both the rider coverage and the supply base, thus thickening the market and allowing for lower prices of both service modes. As a consequence, carpool services enable riders to enjoy more affordable rides without any sacrifice in service quality.

1.1.2. Social Implications of Carpool Services. In addition to the implications of the carpool services for the platform, we also study what societal influence the carpool services would lead to. As for the rider surplus, we find that offering carpool services benefits the riders in general. This is intuitive given that riders are provided with an additional service option when the platform offers carpool services. However, we show that the provision of carpool services acts as a double-edged sword for the drivers as it may improve or hurt the driver surplus. On one hand, the presence of carpool services expands market coverage and hence brings in more riders to be matched with drivers. On the other hand, carpool services also enlarge the capacity per driver and, as a result, decrease the need and potential earning for the drivers. The overall impact of the above two contrasting effects may benefit or hurt the drivers. We show that when both service modes are offered, the latter effect always dominates and the total driver surplus is lower in the presence of carpool services. From the perspective of the entire society, we demonstrate that carpool services increase the total social welfare in general through our numerical studies. In sum, the platform and the riders benefit from carpool services at the cost of the drivers. Hence, it is advised that, while advocating carpool services, policy makers and platforms may consider devising some appropriate compensation schemes (e.g., coupons and bonuses) to re-balance each party's interests.

1.1.3. Carpool Services and Surge Pricing. We characterize the interplay between carpool services and surge pricing as two levers to match supply with demand. In the presence of carpool services, it is optimal for the platform to adopt the surge pricing strategy for both services. More specifically, the platform should raise the prices for both the normal services and the carpool services when the rider arrival rate increases. We show that the price increase for the carpool services is lower than that for the normal services in the face of a demand spike. Intuitively, the carpool services enable the platform to better utilize driver capacity and therefore could partially alleviate surge pricing. We further numerically examine the strategic relationship between the provision of carpool services and surge pricing. We find that the strategic relationship between these two operational strategies exhibits an interesting pattern that depends on the demand-supply ratio of the platform. In particular, carpool services and surge pricing are strategic substitutes if the total number of registered drivers is neither too large nor too small compared with the total demand rate. When there exists extreme imbalance between the total supply and demand, these two operations levers complement each other instead.

1.2. Related Literature

The emergence and phenomenal growth of the sharing economy in recent years have attracted considerable academic interest from the operations research/operations management community. Our work is most closely related to the stream of research that examines the problem of capacity management via dynamic pricing, and in particular, how on-demand service platforms can adjust service prices and agent wages to effectively coordinate supply with demand. Banerjee et al. (2015) model the ride-hailing problem as a queueing network where customers arrival and the drivers work hours depend on the real-time dynamic service price, and they demonstrate that dynamic pricing with prices reacting instantaneously to demand-supply imbalances does not provide more benefit than the optimal static pricing. In a similar vein, Hu and Zhou (2019b) show that a flat-commission contract can be optimal or near-optimal for the platform compared with the benchmark where the platform is allowed to freely determine the price and wage under various market conditions. Chen and Hu (2019) consider the dynamic pricing decisions of a ride-sharing platform in a strategic environment where the customers and suppliers may wait strategically for better prices. They show that under a thick market with large transaction volume, a waiting-adjusted fixed pricing heuristic is close to optimal. The above papers demonstrate the near-optimality of static pricing, whereas Cachon et al. (2017) and Guda and Subramanian (2019) demonstrate the merit of the surge pricing policy for on-demand service platforms with self-scheduling capacity. Our work in this paper also considers the price and wage optimization problems faced by the on-demand platforms in order to effectively coordinate supply with demand, but with a specific focus on the carpool service, and its operational, economic, and societal implications. We show that the provision of carpool services enables the platform to expand market coverage and allows customers to pay less without any sacrifice of the service quality. Our analysis reveals that the surge pricing strategy is optimal for the platform in the presence of carpool services. However, as two operations levers to match supply with demand, surge pricing and carpool services may be either strategic complements or substitutes, depending on the level of imbalance between supply and demand.

In addition to studying the price and wage optimization problems and examining the impact of dynamic pricing, researchers have also explored various operational issues that arise from on-demand service platforms. Bimpikis et al. (2019) consider the spatial transition of a ride-sharing network and characterize the value of spacial price discrimination for a ride-sharing platform that serves a network of locations with deterministic demand patterns, and show that the pricing policy that uses a fixed commission rate could result in significant profit loss in case of heterogeneous demand patterns across different locations. Bai et al. (2019) use the steady-state equilibrium to characterize the optimal price and wage for a monopoly on-demand platform where an $M/M/k$ queuing model is used to get the approximated waiting time for passengers. They show that the price and wage policy with a fixed payout ratio could capture most of the profit from an optimal policy. Gurvich et al. (2019) use a newsvendor model to study the capacity management problem in sharing marketplaces and find that workers' flexibility to choose their own work schedules reduces worker participation and increases price levels. Taylor (2018) studies how two defining features of an on-demand service platform — congestion-driven delay sensitivity and agent independence — affect the platform's optimal per-service price and wage. Chen and Sheldon (2016) empirically examine the impact of surge pricing (dynamic wages) on the duration for which a driver works on Uber's platform, and find that drivers are more likely to continue working if surge pricing is in effect upon they finish a trip. Hu and Zhou (2019a) consider an intermediary's problem of dynamically matching demand and supply of heterogeneous types in a periodic-review fashion, and they provide sufficient and robustly necessary conditions on matching rewards such that the optimal matching policy follows a priority hierarchy.

Our work is also related to the stream of research on carpool services in the operations and transportation literature. Alonso-Mora et al. (2017) present a general model of large-scale ride-sharing systems with carpool services, and develop an algorithm that dynamically generates optimal routes with respect to online demand and vehicle locations. Gopalakrishnan et al. (2017) introduce the new notions of sequential individual rationality (SIR) and sequential fairness (SF) in a cost sharing framework for carpool services. The authors characterize the routes and cost sharing schemes that satisfy SIR and SF. Wen et al. (2017) use reinforce learning to address the fleet rebalancing needs for carpool services. Under a multi-nomial logit (MNL) model, Cohen and Zhang (2017) show that, with a well-designed profit-sharing contract, it would benefit two competing ride-sharing platforms to partner with each other and jointly offer a new carpool service. A recent paper by Jacob and Roet-Green (2018) develops a queueing-theoretic model and designs the incentive-compatible price-service menu that maximizes the ride sharing platform's revenue at equilibrium. They find that offering both solo and pooled rides is optimal only when the distribution of high- and low-type passengers is not skewed and the congestion (the ratio of passenger-demand to driver-supply) is not very high.

In addition to on-demand ride-hailing platforms, other types of online platforms have also been studied in the operations literature, such as e-commerce marketplaces (e.g., Cui et al. 2019b, Zhang et al. 2020, Qi et al. 2020), vacation rental platforms (e.g., Cui et al. 2019a), short-video sharing platforms (e.g., Chen et al. 2020a,b), peer-to-peer product sharing and rental markets (e.g., Benjaafar et al. 2019, Fraiberger and Sundararajan 2017, Jiang and Tian 2018, Li et al. 2017), peer-to-peer service platforms (e.g., Cullen and Farronato 2018), moderating service platforms (e.g., Allon et al. 2012), bike-sharing systems (e.g., Shu et al. 2013, Kabra et al. 2016), and electric car sharing system (e.g., He et al. 2020, 2017).

The remainder of the paper is organized as follows. We formally introduce our model in Section 2. In Sections 3-5, we analyze the operational implications of carpool services, the interplay between carpool services and surge pricing, and the societal impact of carpool services, respectively. Finally, we summarize and conclude the paper in Section 6 with directions for future research. All proofs are relegated to the Appendix.

2. Model

We consider a ride-sharing platform that offers both *normal services* (i.e., rides with a single destination without carpool) with quality v_n and *carpool services* (i.e., rides with multiple destinations shared by several passengers) with quality v_p. Let $\Delta := v_n - v_p \geq 0$ denote the quality difference between the two services, which reflects that carpool services have longer waiting time, lower privacy, and less comfortableness. Riders arrive randomly at the platform and request at most one ride service. In the same spirit as in Bai et al. (2019), we assume that each ride request consists of a certain amount of *service units* to be served by the driver, where a service unit (e.g., travel distance in kilometers/miles, trip time in minutes, or a combination of the two) is the unit of measure that riders get charged and drivers get paid. The platform charges a price rate p_n per service unit for the normal service and a price rate p_p per service unit for the carpool service, and pays a wage rate w per service unit to the drivers. We assume that the price rates (p_n, p_p) and wage rate w are respectively public information known to the riders and drivers. Therefore, the "supply" of participating drivers and the "demand" of rider requests are endogenously determined by the platform's pricing decisions — customers either choose a normal service, a carpool service, or leave the platform without requesting any service based on whichever option results in the highest utility, and each driver registered on the platform decides whether or not to work for the platform based on the expected earning compared with his outside option. Hereafter, we call customers and riders interchangeably.

2.1. Customers Ride Request Choice and Effective Arrival Rates

Suppose that for a certain time period (e.g., one hour), the maximum potential demand rate for ride service during this time period is $\bar{\lambda}$. To model the heterogeneity among riders without losing tractability, we assume that there is a continuum of customer types and the *type* of each customer represents her valuation for quality. Moreover, a rider's *type*, denoted by θ, is independently and uniformly distributed on the interval $[0,1]$, and a type-θ customer's valuation from taking a normal ride is $v_n\theta$ whereas that from taking a carpool ride is $v_p\theta$.

Let d_n denote the average service units of a normal ride, and let d_p represent the average service units of each rider in a carpool ride. It then follows that the utility of a type-θ customer to request a normal service is $\theta v_n - p_n d_n$, and the utility of a type-θ customer to request a carpool service is $\theta v_p - p_p d_p$. We assume that customers are rational and make service request choices based on whichever alternative gives them the highest utility. Let θ_p be the threshold such that a customer would request a ride service if and only if $\theta \geq \theta_p$ and $\theta_n > \theta_p$ be the threshold such that a rider would request a normal service if and only if $\theta \geq \theta_n$. Therefore, a customer with type θ would choose a normal service if $\theta \in [\theta_n, 1]$, a carpool service if $\theta \in [\theta_p, \theta_n)$, and would leave the platform without requesting any service if $\theta \in [0, \theta_p)$, where θ_n and θ_p are given by the following conditions:

$$\begin{aligned} &\theta_n v_n - p_n d_n = \theta_n v_p - p_p d_p, \\ &\theta_p v_p - p_p d_p = 0. \end{aligned} \tag{1}$$

Let s_n be the fraction of customers who choose a normal ride and s_p be the fraction of customers who choose a carpool ride in equilibrium. It is easy to see that s_n and s_p satisfy the following relationship with θ_n and θ_p:

$$\begin{aligned} &\theta_n = 1 - s_n, \\ &\theta_p = 1 - s_p - s_n. \end{aligned} \tag{2}$$

It is worth noticing that s_n and s_p represent the market share of the normal and carpool services, respectively. Assuming that on average each carpool ride is shared by m passengers, the effective demand arrival rate for normal services, λ_n, and that for carpool services λ_p are given by

$$\begin{aligned} &\lambda_n = \bar{\lambda} s_n, \\ &\lambda_p = \frac{1}{m}\bar{\lambda} s_p. \end{aligned} \tag{3}$$

In view of the one-to-one correspondence between the market shares of the two service modes and their effective demand rates, we shall focus our analysis on (s_n, s_p) instead of (λ_n, λ_p) for mathematical convenience. Moreover, from (1) and (2), the price rates (p_n, p_p) satisfy the following equations:

$$\begin{aligned} &p_n = \frac{(1-s_n)\Delta + (1-s_n-s_p)(v_n-\Delta)}{d_n}, \\ &p_p = \frac{(1-s_p-s_n)(v_n-\Delta)}{d_p}. \end{aligned} \tag{4}$$

2.2. Drivers' Decision and the Number of Active Drivers

Now we consider the self-scheduling drivers' decision on whether or not to work for the platform, depending on wage they can get. Assume that a continuum of drivers with total mass K are registered on the platform. In other words, K represents the maximum number of drivers potentially available to offer a ride service for the platform. Given p_n, p_p and w, let $k \in [0, K]$ be the actual number of drivers who opt to work on the platform, and we assume that the drivers would accept all the ride requests that the platform assigns to them.

The drivers are earnings-sensitive and they would opt to work on the platform if the expected per-unit-time wage is higher than what his outside option would offer. We consider heterogeneous drivers and let $G(\cdot)$ be the cumulative distribution of a driver's reservation earning rate for his outside option. The total per-unit-time wage offered by the platform is

$$w\left(\lambda_n d_n + \lambda_p d_p'\right) = w\left(\bar{\lambda} s_n d_n + \frac{\bar{\lambda} s_p d_p'}{m}\right),$$

where d_p' denotes the average service units that a driver provides in a carpool ride.[1] With k active drivers, the expected per-unit-time wage for each driver who participates to work is

$$\frac{w}{k}\left(\bar{\lambda} s_n d_n + \frac{\bar{\lambda} s_p d_p'}{m}\right).$$

A driver would participate to offer service if and only if the reservation earning rate of his outside option does not exceed the expected per-unit time wage. Since the drivers are infinitesimal, the total number of active drivers should satisfy that $k = KG\left(\frac{w}{k}\left(\bar{\lambda} s_n d_n + \frac{\bar{\lambda} s_p d_p'}{m}\right)\right)$. Therefore, in equilibrium, we have

$$G\left(\frac{w}{k}\left(\bar{\lambda} s_n d_n + \frac{\bar{\lambda} s_p d_p'}{m}\right)\right) = \frac{k}{K}.$$

Equivalently,

$$w = \frac{k}{\bar{\lambda}\left(s_n d_n + \frac{s_p d_p'}{m}\right)} G^{-1}\left(\frac{k}{K}\right). \tag{5}$$

2.3. Platform's Optimization Problem

We now consider the platform's optimal price and wage decisions. The platform earns an average profit of $(p_n - w)d_n$ for each normal ride, and the profit margin of a carpool ride with an average of m passengers is $mp_p d_p - wd_p'$. Therefore, the platform's expected per-unit time profit is equal to

$$\lambda_n (p_n - w) d_n + \lambda_p (m p_p d_p - w d_p'). \tag{6}$$

[1] Here we would like to remark that in general, we have $d_p' \le m d_p$ since the riders in a carpool ride share a proportion of the ride.

By substituting (4) and (5) into (6), the expected per-unit time profit of the platform as a function of (s_n, s_p, k) is given by

$$\Pi_p(s_n, s_p, k) = \bar{\lambda}\left[((1-s_n-s_p)(v_n-\Delta)+(1-s_n)\Delta)s_n + (1-s_n-s_p)s_p(v_n-\Delta)\right] - kG^{-1}\left(\frac{k}{K}\right). \tag{7}$$

Throughout the paper, we assume that $G(\cdot)$ satisfies the log-concave property[2]. It then can be shown that $C(y) := yG^{-1}(y)$ is convexly increasing in $0 \le y \le 1$.

To better hedge against demand uncertainty and achieve a satisfactory service experience such that customers do not wait too long after submitting a request for ride service, we require that the average driver utilization (i.e., the ratio between the number of drivers in service and the number of drivers that opt to work) cannot exceed a pre-specified threshold $\rho_{\max}$. Let T_n and T_p be the average service time (i.e., average trip length) of a normal ride and a carpool ride, respectively. Since a carpool ride requires additional pick-ups and drop-offs and may necessitate detours, we have $T_n < T_p$. Moreover, we assume that $T_p \le mT_n$ since a carpool ride is usually shared among passengers heading in similar directions and hence its trip duration should not exceed the summation of the service times when each one of the m passengers take a normal ride separately. By Little's law, the average number of drivers in service should be $\lambda_p T_p + \lambda_n T_n = \bar{\lambda}(\frac{1}{m}s_p T_p + s_n T_n)$. Therefore, the service requirement on the average utilization is given by $\frac{\bar{\lambda}(\frac{1}{m}s_p T_p + s_n T_n)}{k} \le \rho_{\max}$, and hence the platform's optimization problem reads

$$\begin{aligned} \max_{s_n, s_p, k} \quad & \Pi_p(s_n, s_p, k) \\ \text{s.t.} \quad & \frac{\bar{\lambda}(\frac{1}{m}s_p T_p + s_n T_n)}{k} \le \rho_{\max}, \\ & s_n + s_p \le 1, s_p \ge 0, s_n \ge 0, \\ & 0 \le k \le K. \end{aligned} \tag{8}$$

The optimal price rates (p_p^*, p_n^*) and wage rate w^* can be then obtained from the optimal solutions (s_n^*, s_p^*, k^*) to the above problem (8) through the identities (4) and (5). For reference, Table 1 below summarizes the relevant notations used in the model.

2.4. Benchmark Model without Carpools

In this subsection, we consider a benchmark model without carpools, i.e., $s_p \equiv 0$. As we shall show later, the comparison between our focal model and this benchmark would lead to interesting insights on the implications of providing carpool services.

[2] Note that many common probability distributions are log-concave, such as normal distribution, exponential distribution, logistic distribution, chi distribution, and uniform distribution over any convex set.

Table 1 Summary of Notation

v_n :	quality of normal service
v_p :	quality of carpool service
Δ :	quality difference between two service modes ($\Delta = v_n - v_p$)
θ :	customer valuation of quality, $\theta \sim U[0,1]$
d_n :	average service units of a normal ride
d_p :	average service units of each rider in a carpool ride
d'_p :	average service units that a driver provides in a carpool ride
p_n :	price rate per service unit for normal service
p_p :	price rate per service unit for carpool service
w :	wage per service unit for drivers
$\bar{\lambda}$:	maximum rider arrival rate
s_n :	market share of normal service
s_p :	market share of carpool service
m :	average number of riders per ride for carpool service
K :	number of registered drivers on the platform
k :	number of active drivers
r :	reservation wage of drivers in the outside option
$G(\cdot)$:	CDF of r which is assumed to satisfy the log-concave condition
$\rho_{\max}$:	maximum driver utilization on the platform

In the benchmark, the platform only provides normal services with quality v_n, average service units d_n, and average service time T_n. The price rate charged to the riders and wage rate paid to the drivers are respectively $\tilde{p}_n$ and $\tilde{w}_n$. In this case, a type-θ customer would request a normal ride if $\theta \geq \tilde{\theta}_n$, where the threshold $\tilde{\theta}_n$ satisfies $\tilde{\theta}_n v_n - \tilde{p}_n d_n = 0$. Let $\tilde{s}_n$ be the proportion of customers who decide to request a normal service, and the effective arrival rate is $\tilde{\lambda}_n = \bar{\lambda}\tilde{s}_n$. It then follows that $\tilde{\theta}_n = 1 - \tilde{s}_n$ and $\tilde{p}_n = (1-\tilde{s}_n)v_n/d_n$. With $\tilde{k}_n$ active drivers in equilibrium, the per-unit time wage $\tilde{w}_n$ satisfies the following equation:

$$\tilde{w}_n = \frac{\tilde{k}_n}{\bar{\lambda}\tilde{s}_n d_n} G^{-1}\left(\frac{\tilde{k}_n}{K}\right).$$

The service level constraint requires that the average driver utilization shall not exceed $\rho_{\max}$, i.e., $\bar{\lambda}\tilde{s}_n T_n/\tilde{k}_n \leq \rho_{\max}$. The platform's expected per-unit time profit is given by

$$\tilde{\lambda}_n(\tilde{p}_n - \tilde{w}_n)d_n = \bar{\lambda}\tilde{s}_n(1-\tilde{s}_n)v_n - \tilde{\lambda}_n\tilde{w}_n d_n = \bar{\lambda}\tilde{s}_n(1-\tilde{s}_n)v_n - \tilde{k}_n G^{-1}\left(\frac{\tilde{k}_n}{K}\right).$$

Therefore, the platform's optimization problem when only offering normal services is given by

$$\begin{aligned}
\max_{\tilde{s}_n,\tilde{k}_n} \quad & \bar{\lambda}\tilde{s}_n(1-\tilde{s}_n)v_n - \tilde{k}_n G^{-1}\left(\frac{\tilde{k}_n}{K}\right) \\
\text{s.t.} \quad & \frac{\bar{\lambda}\tilde{s}_n T_n}{\tilde{k}_n} \leq \rho_{\max}, \\
& 0 \leq \tilde{s}_n \leq 1, \\
& 0 \leq \tilde{k}_n \leq K.
\end{aligned} \tag{9}$$

Let $(\tilde{s}_n^*, \tilde{k}_n^*)$ be the optimal solutions to the above problem. Then the optimal price rate and wage rate can be computed as

$$\tilde{p}_n^* = \frac{(1-\tilde{s}_n^*)v_n}{d_n} \quad \text{and} \quad \tilde{w}_n^* = \frac{\tilde{k}_n^*}{\bar{\lambda}\tilde{s}_n^* d_n} G^{-1}\left(\frac{\tilde{k}_n^*}{K}\right).$$

The following proposition characterizes the impact of model primitives on the market outcome when the platform only offers normal services.

PROPOSITION 1. *(1) If $\bar{\lambda}$ increases, then (a) $\tilde{s}_n^*$ decreases, (b) $\bar{\lambda}\tilde{s}_n^*$ increases, (c) $\tilde{p}_n^*$ increases, (d) $\tilde{k}_n^*$ increases, (e) $\tilde{w}_n^*$ increases; and (f) the optimal profit of the platform π_b^* increases;*
(2) If K increases, then (a) $\tilde{s}_n^*$ increases, (b) $\tilde{p}_n^*$ decreases, (c) $\tilde{k}_n^*$ increases; (d) $\tilde{k}_n^*/K$ decreases; (e) $\tilde{w}_n^*$ decreases; and (f) π_b^* increases.

Proposition 1 delivers insights on how the demand (i.e., the rider arrival rate $\bar{\lambda}$) and the supply (i.e., the driver capacity K) would affect the optimal strategy and the optimal profit of the platform. An important implication of Proposition 1 is that, if the platform does not offer carpool services, the surge pricing strategy is optimal, i.e., the optimal price $\tilde{p}_n^*$ is increasing in the rider arrival rate $\bar{\lambda}$ (see part (1.c) of Proposition 1). As a consequence, the platform should also distribute a higher wage $\tilde{w}_n^*$ to ensure enough drivers to offer rides. The influence of driver capacity is opposite to that of rider arrival rate in the sense that, if there are more registered drivers on the platform, the price of normal service and the wage for drivers will both decrease. As for the profit, the platform could earn a higher profit either when the rider arrival rate or the driver capacity is higher.

3. Market Coverage and Pricing

In this section, we analyze the platform's optimization problem (8) to characterize its optimal price and wage decisions. On top of this, we further investigate the operational implications that the carpool services may lead to. Recall that the platform's objective function is given by

$$\Pi_p(s_n, s_p, k) = \bar{\lambda}\left[((1-s_n-s_p)(v_n-\Delta)+(1-s_n)\Delta)s_n + (1-s_n-s_p)s_p(v_n-\Delta)\right] - kG^{-1}\left(\frac{k}{K}\right).$$

Let (s_n^*, s_p^*, k^*) be the optimal solutions to problem (8), and denote Π_p^* as the corresponding optimal profit achieved by the platform. It is worth noticing that the optimization problem (8) reduces to the benchmark model (9) by letting $s_p \equiv 0$, which immediately implies that the provision of carpool services can help the platform achieve a higher profit.

LEMMA 1. $\Pi_p^* \geq \tilde{\Pi}_n^*$.

Now we focus on the analysis of (8). It's clear that $\Pi_p(s_n, s_p, k)$ is decreasing in k, so at optimality we must have $\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n) = \rho_{\max} k^*$. It then follows that the platform's optimization problem reduces to

$$\begin{aligned}(s_n^*, s_p^*) = \arg\max \quad & f_p(s_n, s_p)\\ \text{s.t.} \quad & s_n + s_p \le 1,\\ & \frac{\bar{\lambda}(\frac{1}{m}s_p T_p + s_n T_n)}{\rho_{\max}} \le K,\\ & s_n, s_p \ge 0,\end{aligned}$$

where

$$f_p(s_n, s_p) := \bar{\lambda}\left[((1 - s_n - s_p)(v_n - \Delta) + (1 - s_n)\Delta)s_n + (1 - s_n - s_p)s_p(v_n - \Delta)\right] - KC\left(\frac{\bar{\lambda}(\frac{1}{m}s_p T_p + s_n T_n)}{\rho_{\max} K}\right).$$

We begin our analysis by establishing the joint concavity of $f_p(\cdot)$ in (s_n, s_p).

LEMMA 2. *$f_p(s_n, s_p)$ is jointly concave in (s_n, s_p).*

Our next result shows how the quality difference Δ would affect the structure of the optimal policy for the platform. We show that the proportion of riders who request a normal service s_n^* is increasing in the quality difference, whereas the fraction of riders who request a carpool service decreases as the quality difference becomes more significant.

PROPOSITION 2. *There exist two thresholds $\underline{\Delta}$ and $\bar{\Delta}$ $(0 < \underline{\Delta} < \bar{\Delta} < v_n)$, such that $s_n^* \begin{cases} = 0, & \text{if } \Delta \in [0, \underline{\Delta}], \\ > 0, & \text{if } \Delta \in (\underline{\Delta}, v_n]; \end{cases}$ and $s_p^* \begin{cases} > 0, & \text{if } \Delta \in [0, \bar{\Delta}], \\ = 0, & \text{if } \Delta \in (\bar{\Delta}, v_n]. \end{cases}$ In particular, $\bar{\Delta} = v_n(1 - \frac{T_p}{mT_n})$. Moreover, Π_p^* is decreasing in Δ, s_n^* is increasing in Δ, and s_p^* is decreasing in Δ.*

In view of Proposition 2, the optimal service provision strategy of the platform bears an interesting threshold structure. Specifically, if the quality difference Δ is small $(\Delta < \underline{\Delta})$, the platform should offer the carpool service alone ($s_p^* > 0$ and $s_n^* = 0$). If the quality difference is moderate $(\Delta \in [\underline{\Delta}, \bar{\Delta}])$, it is optimal for the platform to provide both normal and carpool services ($s_p^* > 0$ and $s_n^* > 0$). If the quality difference is large $(\Delta > \bar{\Delta})$, only the normal service should be provided. Furthermore, as the dis-utility of riders to take a carpool ride increases, the platform should adjust the prices so that the number of riders for normal services increases whereas the number of riders for carpool services decreases. Note that, it is optimal for the platform to offer carpool services if and only if $\frac{\Delta}{v_n} < 1 - \frac{T_p}{mT_n}$, or equivalently, $\frac{v_p}{v_n} > \frac{T_p}{mT_n}$. The ratio $\gamma := m/T_p$ can be viewed as the pooling efficiency of the platform. A higher pooling efficiency means that the platform is able to pool more riders together in a single trip (i.e., m is large) without increasing the trip duration too much (i.e., T_p is not too long). The condition $\frac{v_p}{v_n} > \frac{1}{\gamma T_n}$ highlights a clear insight that the platform should offer carpool services when the carpool service quality v_p and/or the pooling efficiency γ is high.

Theorem 1. *The total market coverage $s^* := s_n^* + s_p^*$ is decreasing in Δ. Therefore, the provision of carpool services expands market coverage of the platform.*

Theorem 1 proves that the total market coverage of the platform s^* will shrink if the quality difference between the two services is larger. According to Proposition 2, we remark that our benchmark model, (9), is a special case of the focal model with $\Delta \geq \bar{\Delta}$. Therefore, Theorem 1 further demonstrates that providing carpool services enables the platform to better leverage its driver capacity and achieve a larger total market coverage ($s^*(\Delta) \geq s^*(\bar{\Delta})$ for all $\Delta \leq \bar{\Delta}$).

In addition to the market coverage, we next examine the implications of carpool services on the platform's optimal pricing decisions. In particular, we compare the optimal price rates (p_n^*, p_p^*) in the focal model where both service modes are offered, with the optimal price rate $\tilde{p}_n^*$ in the model where the platform does not offer carpool services. Intuitively, it is expected that the optimal price rate for carpool services p_p^* should be lower than the optimal price rate for normal services in the benchmark model $\tilde{p}_n^*$, due to the quality difference between the two services. More interestingly, we show in the following theorem that not only do we have $p_p^* \leq \tilde{p}_n^*$, but the optimal price rate charged for the normal services p_n^* in the presence of carpool services is also dominated by its counterpart in the benchmark model $\tilde{p}_n^*$.

Theorem 2. *(a) For all $\Delta \in [0, \bar{\Delta}]$, $p_p^* \leq \tilde{p}_n^*$. (b) For all $\Delta \in [0, \bar{\Delta}]$, $p_n^* \leq \tilde{p}_n^*$.*

As shown in Theorem 2(a), customers experience a lower quality ride from the carpool service in exchange for a discounted fare p_p^*. Interestingly, if the carpool service is offered, customers who take a normal ride and enjoy the same level of service quality also pay a lower fare p_n^* than the case where only normal services are offered. This result is somewhat intriguing as one may intuit that, by offering an additional service mode, the platform should be able to discriminate between customers with different preferences over quality and hence can charge a higher price for the high-quality normal service. However, Theorem 2(b) suggests that the opposite is true — the provision of carpool services not only enables the platform to expand market coverage, but also allows riders to enjoy less expensive (normal and pooled) rides. With the additional leverage of the carpool services, the platform can expand both the rider coverage and the supply base and thus thicken the market, which allows for lower prices of both service modes. As a consequence, carpool services enable riders to enjoy more affordable rides without any sacrifice in service quality. We also remark that Theorem 2 is consistent with our subsequent analysis in Section 5 that carpool services enhance the rider surplus.

Recall that $\gamma = m/T_p$ can be viewed as the pooling efficiency of the platform. Our next result investigates how the pooling efficiency affects the equilibrium outcome.

Proposition 3. *Assume that* $\Delta \in [0, \bar{\Delta}]$*. If* γ *increases, then (a)* Π_p^* *increases, (b)* s_p^* *increases, (c)* $s^* = s_n^* + s_p^*$ *increases, (d)* p_n^* *decreases, and (e)* p_p^* *decreases.*

Proposition 3 characterizes the impact of pooling efficiency on the optimal market coverage and pricing policy of the platform. More specifically, as the pooling efficiency of the platform increases, the platform would achieve a more expanded market coverage and enjoy a higher profit. Intuitively, a more efficient carpool system that routes drivers to match with riders prompts the platform to increase the usage of carpool services, which in turn also expands the total market coverage. To match demand with supply, the platform decreases the prices p_n^* and p_p^* under a higher pooling efficiency, and therefore customers can enjoy a ride at more affordable prices for both service modes.

3.1. Numerical Experiments

To further illustrate the implications of the carpool services on the platform's pricing decisions and profits, we complement our theoretical analysis with computational studies. We first provide the setup of our numerical experiments. In view of Lemma 1, the provision of carpool service option enables the platform to achieve a higher profit. We next numerically evaluate this profit gain, and investigate when the benefit of providing carpool services will be most significant.

We now describe the setup of our numerical experiments. For simplicity, we use travel distance as a proxy for the service units (d_n, d_p) and assume that the average travel distance is the same at different hours of a day. In our numerical experiments, we fix the quality of the normal service as $v_n = 2$, the average service time of normal rides and carpool rides as $T_n = 1$ and $T_p = 1.5$, and the average service units (travel distance) of normal rides and carpool rides as $d_n = 1$ and $d_p = 1.2$. The distribution of the drivers' reservation wage rate for outside option is uniformly distributed on $[0, 1]$. The average number of passengers per ride for the carpool services is $m = 2$. Notice that with the above parameters, we have $\bar{\Delta} = v_n(1 - T_P/(mT_n)) = 0.5$. Therefore, we vary the quality difference Δ between the normal service and the carpool service in the range $[0, 0.5]$. When $\Delta = 0.5$, our model reduces to the benchmark model that only offers the normal service, i.e., $s_p^* = 0$ and $s_n^* = \tilde{s}_n^*$. For the demand and supply parameters, i.e., the maximum rider arrival rates $\bar{\lambda}$ and the total number of registered drivers K, we calibrate the model parameters based on real Didi's ride data from Bai et al. (2019), which records rides that took place in Hangzhou, China during the time periods between September 7-13 and November 1-30 in 2015.[3] According to Bai et al. (2019), Didi had about 7,800 registered Express/Private drivers in Hangzhou. In our numerical studies, we use the range between 3,000 and 10,000 for the number of registered drivers to cover a large parameter space. For the rider arrival rate, the data from Bai et al. (2019) suggests that the demand rate is relatively stable across the day except during two peak periods in the morning and afternoon rush hours. Similar to Bai

[3] We refer interested readers to Bai et al. (2019) for the detailed description of the data set.

et al. (2019), we set the average customer demand rate during peak time periods 7:00-10:00 and 17:00-20:00 as $\bar{\lambda} = 4,000$, and set $\bar{\lambda} = 2,000$ during off-peak periods 10:00-17:00 and 20:00-23:00. The arrival rates from midnight to early morning were omitted Bai et al. (2019) due to incomplete data in the database. In our numerical studies, we set the average arrival rate as $\bar{\lambda} = 500$ between 23:00 and 7:00 for completeness. Table 2a summarizes the above customer arrival pattern.

Time period	$\bar{\lambda}$
7:00-10:00	4,000
10:00-17:00	2,000
17:00-20:00	4,000
20:00-23:00	2,000
23:00-7:00	500

(a) Arrival Pattern I

Time period	$\bar{\lambda}$
7:00-10:00	6,000
10:00-17:00	1,000
17:00-20:00	6,000
20:00-23:00	1,000
23:00-7:00	250

(b) Arrival Pattern II

Table 2 Distribution of Rider Arrival Rate $\bar{\lambda}$

We next examine when carpool services will be the most beneficial to the platform. For various values of the quality difference Δ and the total number of registered drivers K, we evaluate the (relative) daily platform profit improvement of adopting carpool services compared with only providing the normal services, i.e., $(\Pi_p^* - \tilde{\Pi}_n^*)/\tilde{\Pi}_n^* \times 100\%$, where Π_p^* is the total daily profit of the platform to offer both normal and carpool services whereas $\tilde{\Pi}_n^*$ is the total daily profit of the platform if only the normal service is provided.

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	20.68%	17.02%	14.48%	12.60%	11.16%	10.01%	9.08%	8.31%
$\Delta = 0.1$	13.21%	10.12%	7.96%	6.41%	5.27%	4.39%	3.70%	3.13%
$\Delta = 0.2$	6.61%	4.50%	3.21%	2.41%	1.89%	1.52%	1.24%	1.04%
$\Delta = 0.3$	2.30%	1.50%	1.07%	0.80%	0.62%	0.50%	0.41%	0.34%
$\Delta = 0.4$	0.47%	0.31%	0.22%	0.16%	0.13%	0.10%	0.08%	0.07%

Table 3 Daily Profit Improvement of Adopting Carpool Services (%) ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d'_p = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ **is distributed according to Table 2a)**

Table 3 summarizes the relative profit improvement of the platform when the carpool service is offered compared with the benchmark model with normal service only where $\rho_{\max}$ is set to 0.8 and $\bar{\lambda}$ follows the distribution in Table 2a throughout the day. We observe from Table 3 that the profit increase becomes more significant and hence the provision of carpool services becomes more valuable when the number of total registered drivers decreases. Intuitively, carpool services can help the platform to enlarge the capacity of the registered drivers, and such benefit is most significant when the driver capacity is limited (i.e., when K is small). Moreover, the numerical results in Table 3 suggest that the profit improvement is more prominent when the quality difference between the two service modes becomes smaller. This observation echoes our analytical results in Proposition 2, and suggests that it is to the benefit of the platform to provide high quality carpool services (by, e.g., improving route design to reduce detours during multiple pick-ups and drop-offs).

Notice that the mean arrival rate when $\bar{\lambda}$ follows the distribution in Table 2a is 2,000 rides per hour, which is even less than the smallest K that we have tested, and the improvement ranges between 10% and 25% for small to medium values of quality difference. When the supply K is far more than the mean demand rate, the platform is still able to achieve a considerable profit gain. These observations suggest that offering the carpool service can provide the platform an operations lever to hedge against the demand variability. Table 4, which summarizes the profit gain when the demand arrival rate $\bar{\lambda}$ has a higher variability throughout the day, further confirms this intuition. More specifically, Table 4 evaluates the relative profit improvement when $\bar{\lambda}$ is distributed according to Table 2b, which has the same mean arrival rate 2,000 rides per hour but with a higher variance than that in Table 2a. Comparing Table 4 with Table 3, we observe that the benefit of offering the carpool service option is more significant when the rider arrival pattern is more volatile.

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	24.94%	21.62%	19.09%	17.09%	15.46%	14.12%	12.99%	12.02%
$\Delta = 0.1$	16.88%	14.21%	12.18%	10.57%	9.25%	8.16%	7.23%	6.44%
$\Delta = 0.2$	10.25%	8.16%	6.50%	5.14%	4.12%	3.39%	2.84%	2.41%
$\Delta = 0.3$	4.34%	3.02%	2.23%	1.73%	1.38%	1.13%	0.94%	0.80%
$\Delta = 0.4$	0.90%	0.62%	0.46%	0.35%	0.28%	0.23%	0.19%	0.16%

Table 4 **Daily Profit Improvement of Adopting Carpool Services (%)** ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d_p' = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ **is distributed according to Table 2b)**

We have conducted further numerical studies to examine how the maximum utilization $\rho_{\max}$ would affect the profit gain achieved by the provision of carpool services. Table 5 and Table 6 summarize the profit gain of offering the carpool service when $\rho_{\max}$ is equal to 0.7 and 0.9, respectively. As expected, a lower $\rho_{\max}$ decreases the effective supply capacity of each driver and hence the provision of the carpool service becomes more valuable. Under equilibrium, the realized driver utilization is equal the maximum utilization ρ_{max}, i.e., $\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n) = \rho_{max} k^*$. As is well-established in the queueing and ride-sharing literature, the expected waiting time of the customers/riders grows exponentially as the server/driver utilization increases to 1 (see, e.g., Bai et al. 2019, Taylor 2018). Therefore, the maximum driver utilization has a material impact on the waiting time of the riders, which is crucial for the rider experience of the platform. In this paper, we assume ρ_{max} is exogenous and it would be an interesting future research direction to endogenize and optimize the maximum driver utilization.

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	24.41%	20.14%	17.55%	15.41%	13.74%	12.40%	11.30%	10.38%
$\Delta = 0.1$	16.34%	12.98%	10.57%	8.76%	7.34%	6.25%	5.38%	4.67%
$\Delta = 0.2$	8.91%	6.45%	4.80%	3.65%	2.88%	2.34%	1.93%	1.63%
$\Delta = 0.3$	3.35%	2.23%	1.61%	1.22%	0.95%	0.77%	0.64%	0.53%
$\Delta = 0.4$	0.69%	0.46%	0.33%	0.25%	0.19%	0.16%	0.13%	0.11%

Table 5 **Daily Profit Improvement of Adopting Carpool Services (%)** ($v_n = 2$, $\rho_{\max} = 0.7$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d'_p = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ **is distributed according to Table 2a)**

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	17.65%	14.34%	12.09%	10.45%	9.20%	8.22%	7.43%	6.78%
$\Delta = 0.1$	10.66%	7.85%	6.00%	4.72%	3.79%	3.07%	2.53%	2.12%
$\Delta = 0.2$	4.85%	3.15%	2.22%	1.65%	1.28%	1.02%	0.83%	0.69%
$\Delta = 0.3$	1.63%	1.04%	0.73%	0.54%	0.42%	0.33%	0.27%	0.22%
$\Delta = 0.4$	0.33%	0.21%	0.15%	0.11%	0.08%	0.07%	0.05%	0.05%

Table 6 **Daily Profit Improvement of Adopting Carpool Services (%)** ($v_n = 2$, $\rho_{\max} = 0.9$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d'_p = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ **is distributed according to Table 2a)**

4. Carpool Services and Surge Pricing

We have shown in Section 3 that the carpool service serves as an operations lever to better match supply with demand and enables the platform to achieve a larger market share and higher profit. In this section, we consider another popular operations strategy to coordinate supply with demand for ride-sharing platforms, *surge pricing*, and examine the strategic interplay between carpool services and surge pricing under demand fluctuation. The idea behind surge pricing is to dynamically adjust the prices of rides to match driver supply with rider demand, and the term "surge" refers to the practice that the platform increases ride fares during periods of excessive demand when the driver supply is not sufficient and customer waiting times are long.

We first characterize the impact of demand rate $\bar{\lambda}$ on the market outcome. We show in the following result that the optimal price rates p_n^* and p_p^* for normal and carpool services are both increasing in $\bar{\lambda}$. In other words, surge pricing is optimal in the presence of carpool services.

Proposition 4. *Assume that $\Delta < \bar{\Delta}$. Then (a) s_n^* is decreasing in $\bar{\lambda}$. (b) There exists some threshold λ_0 such that $s_n^* = 0$ for $\bar{\lambda} \geq \lambda_0$. (c) s_p^* is increasing (resp. decreasing) in $\bar{\lambda}$ for $\bar{\lambda} < \lambda_0$ (resp. $\bar{\lambda} > \lambda_0$). (d) p_n^* and p_p^* are increasing in $\bar{\lambda}$, and p_n^* increases faster than p_p^* does.*

An interesting implication from Proposition 4 is that the proportion of customers who choose the carpool service s_p^* is increasing in the demand arrival rate $\bar{\lambda}$ when both services are provided (i.e., $s_n^* > 0$), whereas the number of customers who choose the normal service $\lambda_n^* = \bar{\lambda} s_n^*$ decreases to 0 as the rider arrival rate $\bar{\lambda}$ increases. As the rider arrival increases, the platform is gradually incentivizing the riders to shift from normal services to carpool services (s_n^* decreases whereas s_p^* increases in $\bar{\lambda}$

in the range $s_n^* > 0$). Proposition 4 also suggests that it is beneficial to the platform to adopt surge pricing when both normal and carpool services are offered (p_n^* and p_p^* are both increasing in $\bar{\lambda}$). Furthermore, in the face of a demand spike, the price increase for the carpool service is not as sharp as that for the normal service. As another leverage to match supply with demand, the carpool service partially alleviates surge pricing. Therefore, as shown in Proposition 4(a,c), riders will gradually switch from normal services to carpool services when the demand traffic increases. This is consistent with DidiChuxing's strategy to deal with demand spikes in peak hours. The platform encourages the riders to take the carpool service by putting those requesting carpool services at the front of their rider waiting queue.

In the remainder of this section, we aim to study the strategic relationship between the carpool service option and the practice of surge pricing. More specifically, are these two strategies complements or substitutes? Complementarity means that one strategy becomes more beneficial when the other strategy is also adopted. On the contrary, substitutability means that the benefit of using one strategy becomes weaker when the other strategy is also used.

We now formally define the strategic relationship between surge pricing and carpool services. Denote Π_p^s as the platform's optimal expected profit when both carpool service and surge pricing are used, and Π_p^{ns} as the platform's optimal expected profit when carpool service is offered but surge pricing is not allowed (the superscript "ns" represents "no surge"). Therefore, $\Pi_p^s - \Pi_p^{ns}$ is the incremental profit from adopting surge pricing when the carpool service is offered. Analogously, $\Pi_n^s - \Pi_n^{ns}$ represents the incremental profit from adopting surge pricing when only the normal service is available, where Π_n^s and Π_n^{ns} are defined in a similar fashion as Π_p^s and Π_p^{ns}, respectively. In Section 4.1, we present the mathematical formulation to calculate Π_n^s, Π_n^{ns}, Π_p^s and Π_p^{ns}. Then in Section 4.2, we evaluate the cross difference between the two incremental profits:

$$\Gamma := (\Pi_p^s - \Pi_p^{ns}) - (\Pi_n^s - \Pi_n^{ns}). \tag{10}$$

Note that $\Gamma > 0$ implies that $\Pi_p^s - \Pi_p^{ns} > \Pi_n^s - \Pi_n^{ns}$, i.e., surge pricing is more profitable in the presence of carpool services. Alternately, $\Gamma > 0$ is also equivalent to $\Pi_p^s - \Pi_n^s > \Pi_p^{ns} - \Pi_n^{ns}$, i.e., the incremental profit from offering carpool services is higher when surge pricing is adopted. Therefore, $\Gamma > 0$ implies that surge pricing and carpool services are strategic complements. On the other hand, $\Gamma < 0$ implies that the incremental benefit of using one strategy becomes weaker when the other strategy is also adopted, and hence surge pricing and carpool services are strategic substitutes.

4.1. Surge Pricing vs. No Surge Pricing

We first consider the case where the platform is allowed to adjust the price rates in response to the changes in rider arrival rate $\bar{\lambda}$, i.e., the surge pricing strategy is adopted. For a given $\bar{\lambda}$, let $f_n(\tilde{s}_n|\bar{\lambda})$ denote the platform's profit when only the normal service is available, and let $f_p(s_n, s_p|\bar{\lambda})$ denote the platform's profit when both normal and carpool services are offered. It then follows from (7) and (9) that

$$f_n(\tilde{s}_n|\bar{\lambda}) = \bar{\lambda} v_n \tilde{s}_n(1-\tilde{s}_n) - KC\left(\frac{\bar{\lambda}\tilde{s}_n T_n}{\rho_{\max} K}\right) \text{ and}$$

$$f_p(s_n, s_p|\bar{\lambda}) = \bar{\lambda}[((v_n - \Delta)(1 - s_n - s_p) + \Delta(1 - s_n))s_n + (1 - s_n - s_p)(v_n - \Delta)s_p] - KC\left(\frac{\bar{\lambda}(s_n T_n + \frac{1}{2}s_p T_p)}{\rho_{\max} K}\right).$$

Therefore, the expected optimal profit of the platform with surge pricing when only the carpool service is available is given by

$$\Pi_n^s = \mathbb{E} \max_{0 \le \tilde{s}_n \le 1} \{f_n(\tilde{s}_n|\bar{\lambda})\}, \tag{11}$$

where the expectation is taken over the arrival rate $\bar{\lambda}$. Similarly, the platform's expected optimal profit with surge pricing when both normal and carpool services are offered is given by

$$\Pi_p^s = \mathbb{E} \max_{0 \le s_n + s_p \le 1} \{f_p(s_n, s_p|\bar{\lambda})\}. \tag{12}$$

Next we consider the case where no surge pricing is allowed and the price rates are set independent of the arrival rate $\bar{\lambda}$. The sequence of events is as follows. First, the platform determines the price rate $\tilde{p}_n$ (or the price rates (p_n, p_p) when carpool service is also offered). Then, the rider arrival rate $\bar{\lambda}$ is realized and the platform determines the wage rate w for its drivers. Depending on the price rates, each rider decides which service option to choose or leaves the platform without requesting a ride. The drivers observe w and decide whether opt to work. Notice that we allow the wage rate w to depend on the arrival rate $\bar{\lambda}$ (e.g., the platform may adjust w in real-time by offering promotions to incentivize the drivers to work in case of high demand). This treatment can help us single out the effect of surge pricing.

We now derive the expected optimal profit of the platform when surge pricing is not allowed. If the platform only offers the normal service, the platform charges the price rate $\tilde{p}_n$ independent of $\bar{\lambda}$. Equivalently, the platform sets the probability that a rider will request a ride, $\tilde{s}_n = 1 - (\tilde{p}_n d_n)/v_n$. After observing $\bar{\lambda}$ (which can be estimated with historical data, or through the per-unit-time sessions that riders open the app in an online fashion), the platform determines the wage rate $\tilde{w}_n$. Let $\tilde{k}_n$ be the number of active drivers, and in equilibrium we have $\tilde{w}_n = \frac{\tilde{k}_n}{\bar{\lambda}\tilde{s}_n d_n} G^{-1}\left(\frac{\tilde{k}_n}{K}\right)$. By Little's Law, the maximum rider arrival rate that the active drivers could serve is $\frac{\rho_{\max}\tilde{k}_n}{T_n}$. For the ease of exposition, we use $\tilde{k}_n$ (instead of $\tilde{w}_n$) as the decision variable of the platform. Then, for a given $\tilde{s}_n$, the platform's wage optimization problem in response to $\bar{\lambda}$ is given by

$$\max_{\tilde{k}_n \in [0,K]} \left\{ \min\left\{\bar{\lambda}\tilde{s}_n, \frac{\rho_{\max}\tilde{k}_n}{T_n}\right\}\tilde{p}_n d_n - \tilde{k}_n G^{-1}\left(\frac{\tilde{k}_n}{K}\right) \right\},$$

where $\tilde{p}_n = v_n(1-\tilde{s}_n)/d_n$. The above maximization problem is a one-dimensional convex program and can be solved efficiently. It then follows that the platform's optimal expected profit is given by

$$\Pi_n^{ns} = \max_{\tilde{s}_n \in [0,1]} \mathbb{E}\left[\max_{\tilde{k}_n \in [0,K]} \left\{ \min\left\{\bar{\lambda}\tilde{s}_n, \frac{\rho_{\max}\tilde{k}_n}{T_n}\right\}\tilde{p}_n d_n - \tilde{k}_n G^{-1}\left(\frac{\tilde{k}_n}{K}\right) \right\}\right], \tag{13}$$

where $\tilde{p}_n = v_n(1-\tilde{s}_n)/d_n$.

We now consider the case where the platform offers both normal and carpool services, and it does not adopt the surge pricing strategy. In this case, the platform determines the price rate p_n for the normal service, and p_p for the carpool service, both of which are independent of the rider arrival rate $\bar{\lambda}$. Equivalently, the platform sets the probability s_n that a rider will request a normal ride and the probability s_p that a rider will request a carpool ride. After setting (s_n, s_p), the platform observes $\bar{\lambda}$ and determines the wage rate w. Let k_n and k_p respectively denote the average number of active drivers that provide normal and carpool services in equilibrium, where we have

$$w = \frac{k_n + k_p}{\bar{\lambda}\left(s_n d_n + \frac{s_p d_p'}{m}\right)} G^{-1}\left(\frac{k_n + k_p}{K}\right).$$

Similar to the case without offering carpool services, we use k_n and k_p as the decision variables. Note that the platform can dynamically determine k_n and k_p in response to $\bar{\lambda}$ through the routing algorithm that assigns drivers to riders. Notice that the maximum rider arrival rate that active normal service drivers can serve is $\frac{\rho_{\max} k_n}{T_n}$, and the maximum demand that active carpool service drivers can serve is $\frac{m\rho_{\max} k_p}{T_p}$. For given (s_n, s_p), the platform's wage optimization problem in response to $\bar{\lambda}$ reduces to the following optimization:

$$\max_{0\le k_n+k_p\le K} \left\{ \min\left\{\bar{\lambda}s_n, \frac{\rho_{\max}k_n}{T_n}\right\} p_n d_n + \min\left\{\bar{\lambda}s_p, \frac{m\rho_{\max}k_p}{T_p}\right\} p_p d_p - (k_n+k_p)G^{-1}\left(\frac{k_n+k_p}{K}\right)\right\},$$

where $p_n = ((1-s_n-s_p)(v_n-\Delta) + (1-s_n)\Delta)/d_n$ and $p_p = (1-s_n-s_p)(v_n-\Delta)/d_p$. This maximization problem is a two-dimensional convex program that can be solved efficiently. It then follows that the platform's optimal expected profit is given by

$$\Pi_p^{ns} := \max_{0\le s_n+s_p\le 1} \mathbb{E}\left\{ \max_{0\le k_n+k_p\le K} \left\{ \min\left\{\bar{\lambda}s_n, \frac{\rho_{\max}k_n}{T_n}\right\} p_n d_n + \min\left\{\bar{\lambda}s_p, \frac{m\rho_{\max}k_p}{T_p}\right\} p_p d_p - (k_n+k_p)G^{-1}\left(\frac{k_n+k_p}{K}\right)\right\}\right\}, \tag{14}$$

where $p_n = ((1-s_n-s_p)(v_n-\Delta)+(1-s_n)\Delta)/d_n$ and $p_p = (1-s_n-s_p)(v_n-\Delta)/d_p$. With (11)-(14) at hand, we are able to evaluate the cross difference (10) given a rider arrival distribution.

4.2. Strategic Relationship Between Surge Pricing and Carpool Service

In this section, we numerically examine the strategic relationship between surge pricing and carpool service by evaluating the cross difference Γ. For various combinations of parameter values of Δ and K, we have computed the corresponding cross difference Γ, and the results are summarized in Table 7.

An interesting observation from Table 7 is that, the strategic relationship between surge pricing and carpool services depends on the driver capacity K. Specifically, regardless of the quality difference between normal and carpool services, the two strategies are strategic complements ($\Gamma > 0$) when the number of registered drivers K is very small or very large compared with the rider arrival rate.

K	500	1,000	2,000	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0.1$	101.5	−17.5	−230.0	−193.2	−146.9	−106.9	−65.6	−22.1	10.6	25.3	28.6
$\Delta = 0.2$	55.8	−14.4	−159.8	−53.2	−3.6	6.2	10.6	12.5	13.0	12.9	12.4
$\Delta = 0.3$	41.9	−16.9	−21.8	−6.6	0.1	3.0	4.3	4.7	4.7	4.6	4.3
$\Delta = 0.4$	4.9	4.7	−4.3	−1.2	0.1	0.7	0.9	1.0	1.0	1.0	0.9

Table 7 Cross Differences ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d'_p = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ is distributed according to Table 2a)

However, if the number of registered drivers K on the platform is moderate relative to the rider arrival rate, surge pricing and carpool services become strategic substitutes ($\Gamma < 0$). Therefore, these two operations levers complement each other in case of extreme imbalance between the total supply and demand, whereas they become strategic substitutes when the total capacity is neither too large nor too small compared with the total rider arrival rate. Table 8 reports cross difference Γ when the arrival rate $\bar{\lambda}$ follows the distribution in Table 2b with a higher demand variability. We observe a similar pattern in Table 8 as that in Table 7. The magnitude of Γ is larger when the demand variability is higher, which suggests a stronger strategic relationship between the two operations levers in a more volatile environment.

K	500	1,000	2,000	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0.1$	321.4	263.7	−413.9	−415.5	−353.2	−280.8	−213.8	−155.7	−106.2	−64.2	−28.6
$\Delta = 0.2$	245.6	149.7	−312.6	−225.8	−120.1	−30.1	2.9	13.0	18.7	21.8	23.2
$\Delta = 0.3$	142.7	70.5	−103.6	−30.8	−12.5	−2.5	3.0	6.0	7.6	8.4	8.6
$\Delta = 0.4$	39.9	36.3	−13.1	−6.1	−2.3	−0.3	0.8	1.4	1.7	1.8	1.8

Table 8 Cross Differences ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d'_p = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ is distributed according to Table 2b)

To understand this interesting pattern for the strategic relationship between surge pricing and carpool services revealed in Table 7, we note that if the driver supply is abundant (i.e., K is large), providing carpool services offers an additional operations lever that facilitates the platform to segment riders who have different preferences of quality over price. This in turn enables the platform to extract higher rider surplus by responding to rider arrival fluctuations under the surge pricing policy. On the other hand, if the driver supply is scarce (i.e., K is small), providing carpool services expands supply capacity and hence alleviates the supply shortage of the platform. In the case of limited supply, surge pricing can improve the profit the platform generates from each unit of supply capacity. Therefore, surge pricing will be more beneficial if the platform expands the driver capacity by offering carpool services. If the number of drivers K is moderate relative to the rider arrival rate, surge pricing could serve as an important lever to match supply with demand under rider arrival fluctuations. On the other hand, with moderate driver capacity, the main value of providing carpool services also comes from reducing the supply-demand mismatch. Thus, the presence of carpool services would devalue surge pricing as they bear similar roles with moderate driver capacity. Therefore, these two operational levers become strategic substitutes when driver capacity is moderate compared with rider arrival rate.

Our results on the strategic relationship between surge pricing and carpool services delivers actionable insights for the operations of ride-sharing platforms. In a market where rider arrival variability is prominent and the platform has a moderate amount of registered drivers, the platform could provide carpool services in place of surge pricing without too much sacrificing its profit. It is worth noticing that surge pricing has been long criticized by riders, government, and media since it creates unfairness and uncertainty in the prices charged to the riders, because the price surges when it is most difficult to get a ride such as during a terror attack (Riley 2017). In fact, the government caps the rate of surge pricing in a lot of cities (Willingham 2017). Offering carpool services not only effectively matches supply with demand, but also alleviates the controversial practice of surge pricing. This is also consistent with Didi Chuxing's strategy to deal with the surged demand in peak hours — the platform does not increase the price, but prioritizes riders who request a carpool service by putting them at the front of their rider waiting queue. On the other hand, if the driver capacity of the platform is far from enough or far exceeding the average demand, however, integrating surge pricing and carpool services is highly recommended as the two strategies reinforce each other to extract the highest profit from the bottleneck.

5. Rider Surplus, Driver Surplus, and Social Welfare

Besides the on-demand platform's own profit and optimal operational decisions, we are also interested in investigating the social implications of offering the carpool services. In this section, we examine how would the provision of the carpool services affect the welfare of various parties in the system. In what follows, we first investigate the impact of carpool services on riders' welfare. Given the platform's optimal price rates p_n^* and p_p^*, the total rider surplus is given by

$$\begin{aligned}
RS_p^* =&\bar{\lambda}\left(\mathbb{E}[\theta v_n - p_n^* d_n|\theta\in[1-s_n^*,1]] + \mathbb{E}[\theta(v_n-\Delta) - p_p^* d_p|\theta\in[1-s_p^*-s_n^*,1-s_n^*]]\right)\\
=&\bar{\lambda}\left(v_n\mathbb{E}\left[\theta-(1-s_n^*)+\frac{v_n-\Delta}{v_n}s_p^*|\theta\in[1-s_n^*,1]\right] + (v_n-\Delta)\mathbb{E}[\theta-(1-s_n^*-s_p^*)|\theta\in[1-s_p^*-s_n^*,1-s_n^*]]\right)\\
=&\bar{\lambda}\left(\frac{1}{2}v_n(s_n^*)^2+\frac{1}{2}(v_n-\Delta)(s_p^*)^2\right)+\bar{\lambda}(v_n-\Delta)s_n^*s_p^*,
\end{aligned} \tag{15}$$

where the second equality follows from substituting (4) into the first equation. In view of (15), we use the notation $RS_p^*(\Delta)$ to capture the dependence of the rider surplus on the quality difference Δ when both service modes are offered. We use $\tilde{RS}_n^* = RS_p^*(\bar{\Delta})$ to denote the rider surplus in the benchmark model where only the normal service is available. The following result characterizes the comparison between $RS_p^*(\Delta)$ and $\tilde{RS}_n^*$.

PROPOSITION 5. *There exists a threshold $0\le\underline{\Delta}_r\le\bar{\Delta}$ such that $RS_p^*(\Delta) > \tilde{RS}_n^*$ for $\Delta<\underline{\Delta}_r$. Moreover, if $G(r)=r$, $RS_p^*(\Delta)>\tilde{RS}_n^*$ for all $\Delta<\bar{\Delta}$.*

In view of Proposition 5, the provision of carpool services benefits the riders when the quality difference is not too large. If, in addition, the driver reservation wage is uniformly distributed (i.e., $G(r)=r$), the total rider surplus is higher if the platform provides carpool services. Figure 1 illustrates the relationship between the total rider surplus RS_p^* and the quality difference Δ for various values of rider arrival rate $\bar{\lambda}$, where the drivers' reservation rate for the outside option is assumed to be uniformly distributed on $[0,1]$. From Figure 1, we observe that the rider surplus is decreasing in Δ for the entire range $\Delta \in [0,\bar{\Delta}]$. Notice that by Proposition 2, in equilibrium the platform should offer carpool services only when $\Delta < \bar{\Delta}$, and therefore only normal services will be available in the regime where $\Delta \geq \bar{\Delta}$.

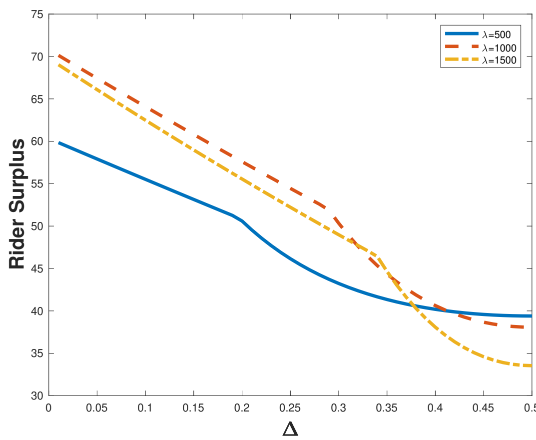

Figure 1 **Rider Surplus ($v_n=2$, $\rho_{\max}=0.8$, $T_p=1.5$, $T_n=1$, $m=2$, $d_p=1.2$, $d_n=1$, $d_p'=2$, $G(\cdot)\sim U[0,1]$, $\bar{\Delta}=v_n(1-T_p/(mT_n))=0.5$, $K=500$)**

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta=0$	42.44%	34.84%	29.56%	25.68%	22.70%	20.34%	18.43%	16.85%
$\Delta=0.1$	31.51%	25.22%	20.76%	16.94%	14.23%	12.20%	10.62%	9.30%
$\Delta=0.2$	19.53%	14.39%	10.11%	7.53%	5.84%	4.67%	3.82%	3.19%
$\Delta=0.3$	7.68%	4.89%	3.40%	2.52%	1.94%	1.54%	1.26%	1.05%
$\Delta=0.4$	1.58%	1.00%	0.70%	0.51%	0.40%	0.31%	0.26%	0.21%

Table 9 **Change in Daily Rider Surplus when Adopting Carpool Services (%) ($v_n=2$, $\rho_{\max}=0.8$, $T_p=1.5$, $T_n=1$, $m=2$, $d_p=1.2$, $d_n=1$, $d_p'=2$, $G(\cdot)\sim U[0,1]$, $\bar{\lambda}$ is distributed according to Table 2a)**

The provision of carpool services has the following two contrasting effects on riders: (a) the (positive) market expansion effect and (b) the (negative) quality downgrade effect. First, providing carpool services expands the market coverage of the platform (cf. Theorem 1), which allows more riders to obtain a ride. On the other hand, the quality of carpool services is lower than that of the normal service ($v_n \geq v_p$). As a result, the average service quality a rider obtains from the platform is lower in the presence of carpool service. Our results suggest that the market expansion effect dominates the quality downgrade effect and offering carpool services benefit the riders. To check the robustness of this insight, we have conducted extensive numerical experiments, and our numerical results in Table 9 suggest that the total rider surplus is monotonically decreasing in Δ and the provision of carpool services is beneficial to the riders.

Since the provision of carpool services expands the market coverage of the platform, one may conjecture that drivers will benefit from the carpool services as well. However, as we elaborate below, the offering of carpool services may actually hurt the drivers' welfare. Note that when both normal and carpool services are offered, the total driver surplus DS_p^* is given by

$$DS_p^* = K\mathbb{E}\left[\frac{w(\lambda_n d_n + \lambda_p d_p')}{k_p^*} - r\right]^+ = K\mathbb{E}\left[G^{-1}\left(\frac{k_p^*}{K}\right) - r\right]^+, \tag{16}$$

where $r \sim G(\cdot)$ is the reservation rate of the driver's outside option and $k_p^* = \frac{\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n)}{\rho_{\max}}$ is the number of active drivers. Figure 2 illustrates how the driver surplus DS_p^* changes with respect to

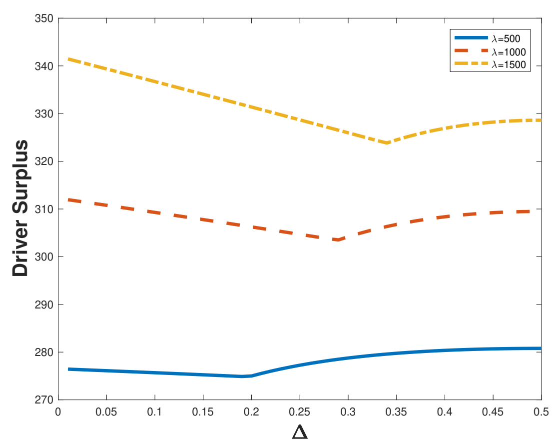

Figure 2 **Driver Surplus (**$v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d_p' = 2$, $G(\cdot) \sim U[0,1]$, $K = 500$**)**

Δ for various values of rider arrival rate $\bar{\lambda}$ when the drivers' reservation rate for the outside option is uniformly distributed on $[0,1]$. In view of Figure 2, we observe that the driver surplus is not necessarily monotone in Δ, and the provision of the carpool services may turn out to be detrimental to the drivers. In what follows, we formalize the observation from Figure 2 in the model where the drivers' reservation wage rate for the outside option is uniformly distributed. We show in the next proposition that the provision of carpool services leads to lower driver surplus compared with the benchmark where the platform only offers the normal service.

Proposition 6. *Assume that (i) r follows the uniform distribution on $[0,1]$, i.e., $G(r) = r$; and (ii) $\Delta \in (\underline{\Delta}, \bar{\Delta})$, i.e., the platform would offer both normal and carpool services. We have $DS_p^* < \tilde{D}S_n^*$.*

In light of Proposition 6, when the drivers' reservation wage rate is uniformly distributed and the quality difference is in the regime where both normal and carpool services have a positive market share in equilibrium, the drivers would be worse off if they are to provide carpool services. Offering carpool services have the following two opposing effects on drivers: (a) the (positive) market expansion effect and (b) the (negative) demand pooling effect. On one hand, as shown in Theorem 1 and Proposition 3, offering carpool services thickens the market by inducing more riders to hail a ride using the platform ($s_n^* + s_p^* \geq \tilde{s}_n^*$) and this effect is enhanced by higher carpool efficiency. Therefore, carpool services bring more demand to the drivers and may increase their earnings and surplus. On the other hand, carpool services enlarge the capacity of each driver, thus facilitating a driver to serve multiple riders simultaneously, which decreases the total need and earning potentials for the drivers. Proposition 6 suggests that, if both normal and carpool services are offered, the demand pooling effect actually dominates the market expansion effect and the overall impact is harmful to the drivers.

To check the robustness of the above insight, we have performed extensive numerical experiments. Table 10 reports how the total driver surplus changes with respect to the service quality difference and driver capacity, when the drivers' reservation wage rate is uniformly distributed on $[0,1]$ and the rider arrival rate $\bar{\lambda}$ is distributed according to Table 2a. Our numerical results in Table 10 suggest that the total driver surplus is not necessarily monotone in Δ and the drivers may be worse off in the presence of carpool service. In particular, offering carpool services would make the drivers worse off for all the parameter combinations in our numerical experiments. This result leads to important practical implications for the operations of the platform. Offering carpool services is likely to benefit the platform and the riders, but at the cost of the drivers. Therefore, an important actionable insight from our study is that, the platform may need to carefully redistribute the additional profit from the provision of carpool services to the drivers (by, e.g., distributing coupons or bonuses) so as to protect their welfare and retain a large enough supply base. This idea has been adopted in practice to re-balance the interests of ride-sharing platforms and their drivers (see, e.g., Cohen and Zhang 2017).

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	-1.01%	-1.00%	-0.90%	-0.79%	-0.68%	-0.59%	-0.52%	-0.45%
$\Delta = 0.1$	-1.14%	-1.07%	-0.93%	-0.75%	-0.61%	-0.51%	-0.43%	-0.37%
$\Delta = 0.2$	-1.00%	-0.81%	-0.54%	-0.38%	-0.28%	-0.21%	-0.16%	-0.13%
$\Delta = 0.3$	-0.45%	-0.28%	-0.19%	-0.13%	-0.09%	-0.07%	-0.05%	-0.04%
$\Delta = 0.4$	-0.09%	-0.06%	-0.04%	-0.03%	-0.02%	-0.01%	-0.01%	-0.01%

Table 10 Change in Daily Driver Surplus when Adopting Carpool Services (%) ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d_p' = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ **is distributed according to Table 2a)**

Besides the rider and driver surplus, we further investigate the impact of the provision of carpool services on the social welfare. The social welfare equals the sum of the rider surplus RS_p^*, the platform's profit Π_p^*, and the driver surplus DS_p^*. It then follows that the social welfare can be computed as follows:

$$\begin{aligned} SW_p^* =& RS_p^* + \Pi_p^* + DS_p^* \\ =& \bar{\lambda}\left(\frac{1}{2}v_n(s_n^*)^2 + \frac{1}{2}(v_n - \Delta)(s_p^*)^2\right) + \bar{\lambda}(v_n - \Delta)s_n^* s_p^* + \bar{\lambda}(p_n^* d_n s_n^* + p_p^* d_p s_p^*) - k_p^* G^{-1}\left(\frac{k_p^*}{K}\right) \\ &+ K\mathbb{E}\left[G^{-1}\left(\frac{k_p^*}{K}\right) - r\right]^+ . \end{aligned}$$

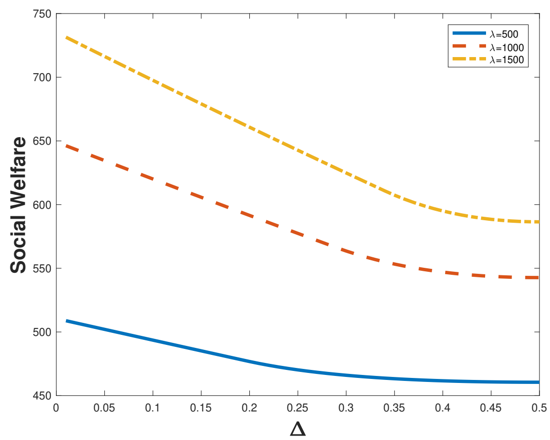

Figure 3 **Social Welfare ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $d_p = 1.2$, $d_n = 1$, $m = 2$, $G(\cdot) \sim U[0,1]$, $K = 500$)**

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	7.70%	5.61%	4.29%	3.39%	2.76%	2.28%	1.92%	1.64%
$\Delta = 0.1$	4.92%	3.34%	2.36%	1.73%	1.30%	1.00%	0.78%	0.62%
$\Delta = 0.2$	2.46%	1.48%	0.95%	0.65%	0.47%	0.35%	0.26%	0.21%
$\Delta = 0.3$	0.86%	0.50%	0.32%	0.21%	0.15%	0.11%	0.09%	0.07%
$\Delta = 0.4$	0.18%	0.10%	0.06%	0.04%	0.03%	0.02%	0.02%	0.01%

Table 11 **Change in Daily Social Welfare when Adopting Carpool Services (%) ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d_p' = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ is distributed according to Table 2a)**

Figure 3 illustrates how the social welfare SW_p^* changes with respect to Δ for various values of rider arrival rate $\bar{\lambda}$ when the drivers' reservation wage for the outside option is uniformly distributed. As shown in Figure 3, the social welfare is decreasing in Δ and it is to the benefit of the entire society for the platform to offer carpool services. We have further conducted extensive numerical experiments to check the robustness of our results. Table 11 suggest that the total social welfare increases as the quality difference becomes smaller, and the provision of carpool services would achieve a higher social welfare. Our analysis and results in this section may prove helpful in providing guidelines for policy makers. The entire society would benefit from the carpool service of ride-sharing platforms from the total social-welfare perspective, so the policy maker is advised to advocate and encourage such services. However, not everyone equally benefits from the carpool service, so care must also be taken and appropriate compensation schemes may be needed for platform drivers as they may be worse off with the introduction of the service.

6. Conclusion

Motivated by the increasing popularity of on-demand service platforms with self-scheduling and earning-sensitive service providers and price-sensitive customers, we develop an analytical framework to examine the operational, economical, and social implications of carpool services on ride-hailing platforms. We characterize the optimal price and wage strategy of the platform, and find that it is optimal for the platform to offer the carpool service if its quality is sufficiently high and/or the pooling efficiency is high. Providing carpool services enables the platform to achieve a larger market coverage and allows the passengers to pay less for the same service quality. If the pooling efficiency improves, the platform can further enlarge its market coverage, and decrease the prices. In the presence of carpool services, surge pricing is still optimal, and, as the demand increases, the platform will gradually encourage customers to switch from normal services to carpool services. We show that the provision of carpool services benefits the riders if the quality difference between the normal and carpool service modes is not too large. However, drivers may be worse off in the presence of carpool services. In particular, we find that the provision of carpool services will result in a lower driver surplus when the drivers' reservation wage is uniformly distributed. From our extensive numerical analysis, we find that the carpool service and surge pricing, both of which are operations strategies to coordinate supply with demand, are strategic substitutes when the supply-demand imbalance is moderate, and they complement each other in the case of severe imbalance between supply and demand.

There are several modeling limitations in our paper which we hope to address in future research. First, we assume that the customer's valuation type is uniformly distributed between zero and one. It would be interesting to see whether the results and insights are robust under more general distributions. Second, we have focused on the equilibrium behavior of the system and didn't consider the spatial heterogeneities of riders and drivers for a ride-hailing platform. An interesting future direction is to introduce the spatial dimension into the joint price and wage optimization problem under demand and supply uncertainty with carpool services. Finally, this paper considers a monopoly platform and ignored competition among platforms. It is not uncommon in practice that there may exist multiple platforms competing for both riders and drivers in the market. Another potential future research direction is to study platform competition in the presence of carpool services, and characterize the optimal price and wage strategies in a competitive setting.

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Appendix to "Carpool Services for Ride-sharing Platforms: Price and Welfare Implications".

LEMMA 3. *Assume that $G(\cdot)$ satisfies the log-concave property. Then $C(y) := yG^{-1}(y)$ is convexly increasing in $0 \le y \le 1$.*

Proof. Let $h(x) := \log G(x)$. Since $h(x)$ is concave, we have $h''(x) = \frac{G''(x)\cdot G(x) - (G'(x))^2}{(G(x))^2} \le 0$, which implies

$$G''(x) \cdot G(x) \le (G'(x))^2 \tag{17}$$

To show $C(y)$ is convexly increasing in y, it suffices to show that $C'(y) \ge 0$ and $C''(y) \ge 0$. Since $G(\cdot)$ is non-decreasing and by the inverse function theorem, we have $C'(y) = G^{-1}(y) + y \cdot (G^{-1})'(y) = G^{-1}(y) + \frac{y}{G'(G^{-1}(y))} \ge 0$. It then follows that

$$\begin{aligned} C''(y) &= (G^{-1})'(y) + \frac{G'(G^{-1}(y)) - y \cdot [G''(G^{-1}(y)) \cdot (G^{-1})'(y)]}{(G'(G^{-1}(y)))^2} \\ &= \frac{1}{G'(G^{-1}(y))} + \frac{G'(G^{-1}(y)) - y \cdot \frac{G''(G^{-1}(y))}{G'(G^{-1}(y))}}{(G'(G^{-1}(y)))^2} = \frac{2(G'(G^{-1}(y)))^2 - y \cdot G''(G^{-1}(y))}{(G'(G^{-1}(y)))^3} \ge 0 \end{aligned}$$

where the last inequality follows from $y = G(x)$ and (17). *Q.E.D.*

Proof of Proposition 1. We write $\Pi_n(s,k) = \bar{\lambda} v_n(1-s_n)s_n - KC(k/K)$. Clearly, $\Pi_n(s,k)$ is decreasing in k, so $\bar{\lambda} T_n \tilde{s}_n^* = \rho_{\max} \tilde{k}_n^*$. Plugging this into $\Pi_n(s,k)$, we have that it suffices to solve the optimization problem:

$$\tilde{s}_n^* = \arg\max_s f(s) := \bar{\lambda} v_n (1-s)s - KC\left(\frac{\bar{\lambda} T_n s}{\rho_{\max} K}\right)$$

subject to the constraints $s \in [0,1]$ and $\frac{\bar{\lambda} T_n s}{\rho_{\max}} \le K$.

When $\bar{\lambda}$ increases: <u>1(a) $\tilde{s}_n^*$ is decreasing in $\bar{\lambda}$:</u> We have $f'(s) = \bar{\lambda} v_n(1-2s) - \frac{\bar{\lambda} T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda} T_n s}{\rho_{\max} K}\right)$. Let s^* satisfy $f'(s^*) = 0$, which is unique. We have $\tilde{s}_n^* = \min\{s^*, (K\rho_{\max})/(\bar{\lambda} T_n)\}$. It is easy to check that s^* and $(K\rho_{\max})/(\bar{\lambda} T_n)$ are both decreasing in $\bar{\lambda}$. Hence, $\tilde{s}_n^*$ is decreasing in $\bar{\lambda}$.

<u>1(b) $\bar{\lambda}\tilde{s}_n^*$ is increasing in $\bar{\lambda}$:</u> Let $\bar{\lambda} s := \lambda$. We have $f(s) = g(\lambda) = v_n\left(1 - \frac{\lambda}{\bar{\lambda}}\right)\lambda - KC\left(\frac{\lambda T_n}{\rho_{\max} K}\right)$. Thus, we have $g'(\lambda) = v_n\left(1 - \frac{2\lambda}{\bar{\lambda}}\right) - \frac{T_n}{\rho_{\max}} C'\left(\frac{T_n \lambda}{\rho_{\max} K}\right)$. Let λ^* satisfies $g'(\lambda^*) = 0$, so $\bar{\lambda}\tilde{s}_n^* = \min\{\lambda^*, (K\rho_{\max})/T_n\}$. Since $g'(\lambda^*) = 0$ implies that $\lambda^* \le 0.5\bar{\lambda}$, λ^* is increasing in $\bar{\lambda}$. Thus, $\bar{\lambda}\tilde{s}_n^*$ is also increasing in $\bar{\lambda}$.

<u>1(c) $\tilde{p}_n^*$ is increasing in $\bar{\lambda}$:</u> It follows immediately that $\tilde{p}_n^* = (1-\tilde{s}_n^*)v_n/d_n$ is increasing in $\bar{\lambda}$.

<u>1(d) $\tilde{k}_n^*$ is increasing in $\bar{\lambda}$:</u> Note that $\tilde{k}_n^* = \bar{\lambda}\tilde{s}_n^* T_n/\rho_{\max}$. By (b), $\tilde{k}_n^*$ is increasing in $\bar{\lambda}$.

<u>1(e) $\tilde{w}_n^*$ is increasing in $\bar{\lambda}$:</u> Note that $\tilde{w}_n^* = \tilde{k}_n^*/(\bar{\lambda}\tilde{s}_n^* d_n) G^{-1}(\frac{\tilde{k}_n^*}{K}) = T_n/(\rho_{\max} d_n) G^{-1}(\tilde{k}_n^*/K)$. Since $\tilde{k}_n^*$ is increasing in $\bar{\lambda}$, $\tilde{w}_n^*$ is also increasing in $\bar{\lambda}$.

<u>1(f) π_b^* is increasing in $\bar{\lambda}$:</u> By the envelope theorem, $\pi_b^* = \max \Pi_b(s,k)$ is continuously differentiable in $\bar{\lambda}$ with $\frac{\partial \pi_b^*}{\partial \bar{\lambda}} = v_n(1-s_b^*)s_b^* > 0$. Thus, π_b^* is increasing in $\bar{\lambda}$.

When K increases: <u>2(a) $\tilde{s}_n^*$ is increasing in K:</u> As shown in part 1(a), $\tilde{s}_n^* = \min\{s^*, (K\rho_{\max})/(\bar{\lambda} T_n)\}$, where s^* satisfies $f'(s^*) = 0$. It is easy to check that s^* and $(K\rho_{\max})/(\bar{\lambda} T_n)$ are both increasing in K. Hence, $\tilde{s}_n^*$ is also increasing in K.

<u>2(b) $\tilde{p}_n^*$ is decreasing in K:</u> By part 2(a), it follows immediately that $\tilde{p}_n^* = (1-\tilde{s}_n^*)v_n/d_n$ decreases in K.

<u>2(c) $\tilde{k}_n^*$ is increasing in K:</u> Note that $\tilde{k}_n^* = \bar{\lambda}\tilde{s}_n^* T_n/\rho_{\max}$. By part 2(a), $\tilde{k}_n^*$ is increasing in K.

2(d) $\tilde{k}_n^*/K$ is decreasing in K: Let $z := k/K$. We have $f(s) = h(z) = \bar{\lambda} v_n \left(1 - \frac{\rho_{\max} K z}{\bar{\lambda} T_n}\right) \frac{\rho_{\max} K z}{\bar{\lambda} T_n} - KC(z)$. Thus, we have $h'(z) = \bar{\lambda} v_n \left(\frac{\rho_{\max} K}{\bar{\lambda} T_n} - 2\left(\frac{\rho_{\max} K}{\bar{\lambda} T_n}\right)^2 z\right) - KC'(z)$. Let z^* satisfies $h'(z^*) = 0$. By $\frac{\tilde{k}_n^*}{K} = \frac{\bar{\lambda} T_n \tilde{s}_n^*}{\rho_{\max} K}$ and $\tilde{s}_n^* \le 1$, we then have $\tilde{k}_n^*/K = \min\{z^*, 1, \frac{\bar{\lambda} T_n}{\rho_{\max} K}\}$. It is easy to check that if K increases, z^* will decrease. Since $\frac{\bar{\lambda} T_n}{\rho_{\max} K}$ is also decreasing in K, $\tilde{k}_n^*/K$ is decreasing in K.

2(e) $\tilde{w}_n^*$ is decreasing in K. Note that $\tilde{w}_n^* = \tilde{k}_n^*/(\bar{\lambda} \tilde{s}_n^* d_n) G^{-1}(\frac{\tilde{k}_n^*}{K}) = T_n/(\rho_{\max} d_n) G^{-1}(\tilde{k}_n^*/K)$. Since $\tilde{k}_n^*/K$ is decreasing in K, $\tilde{w}_n^*$ is also decreasing in K.

2(f) π_b^* is increasing in K. Since $G^{-1}\left(\frac{k}{K}\right)$ is decreasing K, $\Pi_b(s,k) = \bar{\lambda} v_n (1-s)s - kG^{-1}\left(\frac{k}{K}\right)$ is increasing in K. Furthermore, the constraints $k \le K$ is less tight as K increases. Thus, $\pi_b^* = \max \Pi_b(s,k)$ is increasing in K as well. *Q.E.D.*

Proof of Lemma 2. We prove joint concavity by showing that the Hessian matrix of $f_p(\cdot)$ is negative semidefinite, or alternatively, its leading principal minors have alternate signs. Taking derivatives and by $v_n = v_p + \Delta$, we have

$$\frac{\partial f_p(s_n, s_p)}{\partial s_n} = \bar{\lambda}[-2v_n s_n - 2s_p v_p + v_n] - \frac{\bar{\lambda} T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m} s_p T_p + s_n T_n)}{\rho_{\max} K}\right),$$
$$\frac{\partial f_p(s_n, s_p)}{\partial s_p} = \bar{\lambda}[-2v_p s_n - 2s_p v_p + v_p] - \frac{\bar{\lambda} T_p}{m\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m} s_p T_p + s_n T_n)}{\rho_{\max} K}\right).$$

It then follows that $\frac{\partial^2 f_p(s_n,s_p)}{\partial s_n^2} = -2v_n\bar{\lambda} - \frac{\bar{\lambda}^2 T_n^2}{\rho_{\max}^2 K} C''\left(\frac{\bar{\lambda}(\frac{1}{m} s_p T_p + s_n T_n)}{\rho_{\max} K}\right) \le 0$ because $C(\cdot)$ is convexly increasing. Similarly, we have $\frac{\partial^2 f_p(s_n,s_p)}{\partial s_p^2} = -2v_p\bar{\lambda} - \frac{\bar{\lambda}^2 T_p^2}{m^2 \rho_{\max}^2 K} C''\left(\frac{\bar{\lambda}(\frac{1}{m} s_p T_p + s_n T_n)}{\rho_{\max} K}\right) \le 0$. It remains to show

$$\frac{\partial^2 f_p(s_n, s_p)}{\partial s_n^2} \cdot \frac{\partial^2 f_p(s_n, s_p)}{\partial s_p^2} \ge \left(\frac{\partial^2 f_p(s_n, s_p)}{\partial s_p \partial s_n}\right)^2. \tag{18}$$

It is straightforward to check that (18) holds if and only if

$$2\bar{\lambda}^2 v_p v_n + \frac{\bar{\lambda}^3 T_p^2 \alpha v_n}{m^2 \rho_{\max}^2 K} + \frac{\bar{\lambda}^3 T_n^2 \alpha v_p}{\rho_{\max}^2 K} \ge 2\bar{\lambda}^2 v_p^2 + \frac{2\bar{\lambda}^3 T_p T_n \alpha v_p}{m \rho_{\max}^2 K}, \tag{19}$$

where $\alpha := C''\left(\frac{\bar{\lambda}(\frac{1}{m} s_p T_p + s_n T_n)}{\rho_{\max} K}\right)$. Since $v_n \ge v_p$ and $\alpha \ge 0$, a sufficient condition for (19) to hold is $T_p^2 v_n + m^2 T_n^2 v_p \ge 2m T_p T_n v_p$, which is clearly true since $v_n \ge v_p$ and $(T_p - mT_n)^2 \ge 0$. *Q.E.D.*

Proof of Proposition 2. We first show that if $\Delta = 0$, $s_n^* = 0$. If $\Delta = 0$,

$$f_p(s_n, s_p) = \bar{\lambda}\left[(1 - s_n - s_p)(s_n + s_p) v_n\right] - KC\left(\frac{\bar{\lambda}(\frac{1}{m} s_p T_p + s_n T_n)}{\rho_{\max} K}\right).$$

Assume to the contrary that $s_n^* > 0$. Let $\epsilon > 0$ be small enough such that $s_n' = s_n^* - \epsilon \ge 0$, $s_p' = s_p^* + \epsilon$. Since $T_n > \frac{T_p}{m}$, we have

$$\frac{\bar{\lambda}(\frac{1}{m} s_p' T_p + s_n' T_n)}{\rho_{\max} K} = \frac{\bar{\lambda}(\frac{1}{m}(s_p^* + \epsilon) T_p + (s_n^* - \epsilon) T_n)}{\rho_{\max} K} = \frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K} + \frac{\bar{\lambda}(\frac{1}{m} T_p - T_n)\epsilon}{\rho_{\max} K} < \frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}.$$

Thus,

$$C\left(\frac{\bar{\lambda}(\frac{1}{m} s_p' T_p + s_n' T_n)}{\rho_{\max} K}\right) < C\left(\frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right).$$

In addition, $(1-s'_n-s'_p)(s'_n+s'_p)v_n=(1-s^*_n-s^*_p)(s^*_n+s^*_p)v_n$. Hence,

$$\begin{aligned}f_p(s'_n,s'_p)=&\bar{\lambda}\left[(1-s'_n-s'_p)(s'_n+s'_p)v_n\right]-KC\left(\frac{\bar{\lambda}(\frac{1}{m}s'_pT_p+s'_nT_n)}{\rho_{\max}K}\right)\\>&\bar{\lambda}\left[(1-s^*_n-s^*_p)(s^*_n+s^*_p)v_n\right]-KC\left(\frac{\bar{\lambda}(\frac{1}{m}s^*_pT_p+s^*_nT_n)}{\rho_{\max}K}\right)=f_p(s^*_n,s^*_p).\end{aligned}$$

Therefore, $s^*_n=0$ if $\Delta=0$.

We now show that $\underline{\text{if } \Delta=v_n,\ s^*_p=0}$. If $\Delta=v_n$, we have $f_p(s_n,s_p):=\bar{\lambda}\left[(1-s_n)v_ns_n\right]-KC\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right)$. Since $C(\cdot)$ is convexly increasing, $f_p(s_n,s_p)$ is decreasing in s_p for all s_n. Therefore, $s^*_p=0$ if $\Delta=v_n$.

Next, we show that $\underline{s^*_n \text{ is increasing in } \Delta}$. Assume $\hat{\Delta}>\Delta$, $\hat{f}_p(\cdot,\cdot)$ is the profit function associated with $\hat{\Delta}$, and $(\hat{s}^*_n,\hat{s}^*_p)$ is the maximizer of $\hat{f}_p(\cdot,\cdot)$. Assume to the contrary that $\hat{s}^*_n<s^*_n$. Then we have $\partial_{s_n}\hat{f}_p(\hat{s}^*_n,\hat{s}^*_p)\le 0\le\partial_{s_n}f_p(s^*_n,s^*_p)$. Therefore,

$$-2\bar{\lambda}v_ns^*_n-2\bar{\lambda}(v_n-\Delta)s^*_p+\bar{\lambda}v_n-\frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s^*_pT_p+s^*_nT_n)}{\rho_{\max}K}\right)\ge-2\bar{\lambda}v_n\hat{s}^*_n-2\bar{\lambda}(v_n-\hat{\Delta})\hat{s}^*_p+\bar{\lambda}v_n-\frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}\hat{s}^*_pT_p+\hat{s}^*_nT_n)}{\rho_{\max}K}\right),$$

which implies that

$$y^*-\hat{y}^*\le 2v_n(\hat{s}^*_n-s^*_n)+2(v_n-\hat{\Delta})\hat{s}^*_p-2(v_n-\Delta)s^*_p, \tag{20}$$

where

$$y^*:=\frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s^*_pT_p+s^*_nT_n)}{\rho_{\max}K}\right)\text{ and }\hat{y}^*:=\frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}\hat{s}^*_pT_p+\hat{s}^*_nT_n)}{\rho_{\max}K}\right).$$

If $\hat{s}^*_p\le s^*_p$, the convexity of $C(\cdot)$ suggests that $y^*-\hat{y}^*>0$. Since $\hat{s}^*_n<s^*_n$, $\hat{\Delta}>\Delta$, and $\hat{s}^*_p\le s^*_p$, we have $2v_n(\hat{s}^*_n-s^*_n)+2(v_n-\hat{\Delta})\hat{s}^*_p-2(v_n-\Delta)s^*_p<0$. This forms a contradiction. Thus, we have $\hat{s}^*_p>s^*_p$. It then follows that $\partial_{s_p}\hat{f}_p(\hat{s}^*_n,\hat{s}^*_p)\ge 0\ge\partial_{s_p}f_p(s^*_n,s^*_p)$. Therefore, we have $(v_n-\hat{\Delta})(1-2\hat{s}^*_n-2\hat{s}^*_p)-\frac{T_p}{mT_n}\hat{y}^*\ge(v_n-\Delta)(1-2s^*_n-2s^*_p)-\frac{T_p}{mT_n}y^*$. It then follows that

$$\begin{aligned}y^*-\hat{y}^*\ge&\frac{mT_n}{T_p}((v_n-\Delta)(1-2s^*_n-2s^*_p)-(v_n-\hat{\Delta})(1-2\hat{s}^*_n-2\hat{s}^*_p))\\\ge&(v_n-\Delta)(1-2s^*_n-2s^*_p)-(v_n-\hat{\Delta})(1-2\hat{s}^*_n-2\hat{s}^*_p)\\=&(\hat{\Delta}-\Delta)+2(v_n-\hat{\Delta})\hat{s}^*_n-2(v_n-\Delta)s^*_n+2(v_n-\hat{\Delta})\hat{s}^*_p-2(v_n-\Delta)s^*_p\\>&2(v_n-\Delta)(\hat{s}^*_n-s^*_n)+2(v_n-\hat{\Delta})\hat{s}^*_p-2(v_n-\Delta)s^*_p\\>&2v_n(\hat{s}^*_n-s^*_n)+2(v_n-\hat{\Delta})\hat{s}^*_p-2(v_n-\Delta)s^*_p,\end{aligned}\tag{21}$$

where the second inequality follows from $T_n>\frac{1}{m}T_p$ and $s^*_n+s^*_p\le 0.5$ (which will be shown later in (23)), the third inequality follows from $\hat{s}^*_n<s^*_n$, and the last inequality follows from the assumption that $\hat{s}^*_n<s^*_n$. Inequality (20) contradicts with inequality (21). Therefore, $\hat{s}^*_n\ge s^*_n$ if $\hat{\Delta}>\Delta$.

Next, we show that $\underline{\hat{s}^*_p\le s^*_p \text{ if } \hat{\Delta}>\Delta}$. Assume to the contrary that $\hat{s}^*_p>s^*_p$. Then we have $\partial_{s_p}\hat{f}_p(\hat{s}^*_n,\hat{s}^*_p)\ge 0\ge\partial_{s_p}f_p(s^*_n,s^*_p)$, and therefore

$$(v_n-\hat{\Delta})(1-2\hat{s}^*_n-2\hat{s}^*_p)-\frac{T_p}{mT_n}\hat{y}^*\ge(v_n-\Delta)(1-2s^*_n-2s^*_p)-\frac{T_p}{mT_n}y^*. \tag{22}$$

We have shown that $\hat{s}^*_n\ge s^*_n$. Thus, $\hat{s}^*_n+\hat{s}^*_p>s^*_n+s^*_p$, $(v_n-\hat{\Delta})(1-2\hat{s}^*_n-2\hat{s}^*_p)<(v_n-\Delta)(1-2s^*_n-2s^*_p)$, and

$$\hat{y}^*=\frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}\hat{s}^*_pT_p+\hat{s}^*_nT_n)}{\rho_{\max}K}\right)>\frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s^*_pT_p+s^*_nT_n)}{\rho_{\max}K}\right)=y^*.$$

Therefore,

$$(v_n - \hat{\Delta})(1 - 2\hat{s}_n^* - 2\hat{s}_p^*) - \frac{T_p}{mT_n}\hat{y}^* < (v_n - \Delta)(1 - 2s_n^* - 2s_p^*) - \frac{T_p}{mT_n}y^*.$$

The above inequality contradicts with (22) and hence implies that $\hat{s}_p^* \le s_p^*$ if $\hat{\Delta} > \Delta$.

Next, we show the existence of $\underline{\Delta}$ and $\bar{\Delta}$. Note that if $\Delta = 0$ we have $s_p^* > 0$, and if $\Delta = v_n$ we have $s_n^* > 0$. Since $f_p(s_n, s_p|\Delta)$ is continuously differentiable with respect to (s_n, s_p, Δ), by the maximum theorem, the maximizer $(s_n^*(\Delta), s_p^*(\Delta))$ is continuous in Δ. Therefore, the monotonicity and continuity of s_n^* and s_p^* with respect to Δ yields that there exists $\underline{\Delta}$ and $\bar{\Delta}$ such that

$$s_n^* \begin{cases} = 0, & \text{if } \Delta \in [0, \underline{\Delta}], \\ > 0, & \text{if } \Delta \in (\underline{\Delta}, v_n]; \end{cases} \quad \text{and} \quad s_p^* \begin{cases} > 0, & \text{if } \Delta \in [0, \bar{\Delta}), \\ = 0, & \text{if } \Delta \in [\bar{\Delta}, v_n]. \end{cases}$$

To show $\bar{\Delta} > \underline{\Delta}$, observe that $s_n^* = s_p^* = 0$ is never optimal for any $\Delta \in [0, v_n]$, which immediately implies that $\underline{\Delta} < \bar{\Delta}$. In the remainder of the proof, we show that

$$s_p^* + s_n^* \le 0.5, \text{ and} \tag{23}$$

$$\bar{\Delta} = v_n(1 - \frac{T_p}{mT_n}). \tag{24}$$

We first show (23). Assume to the contrary that $s_p^* + s_n^* > 0.5$. We have

$$\partial_{s_p} f_p(s_n^*, s_p^*) = \bar{\lambda}\left[(v_n - \Delta)(1 - 2s_n^* - s_p^*) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)\right] < 0,$$

so we must have $s_p^* = 0$, and thus $s_n^* > 0.5$. Therefore,

$$\partial_{s_n} f_p(s_n^*, s_p^*) = \bar{\lambda}\left[v_n(1 - 2s_n^*) - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)\right] < 0,$$

which implies that $s_n^* = 0$, contradicting with $s_n^* > 0.5$. We next show (24). It suffices to show that if $\Delta > v_n(1 - \frac{T_p}{mT_n})$ (resp. $\Delta < v_n(1 - \frac{T_p}{mT_n})$), $s_p^* = 0$ (resp. $s_p^* > 0$). If $\Delta > v_n(1 - \frac{T_p}{mT_n})$ and $s_p^* > 0$, the First Order Condition (FOC) with respect to s_p implies that

$$(v_n - \Delta)(1 - 2s_n^* - 2s_p^*) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right) = \frac{T_p}{m}\mu_2^*,$$

where μ_2^* is the Lagrangian multiplier with respect to the constraint $\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n) \le \rho_{\max}K$. By $\Delta > v_n(1 - \frac{T_p}{mT_n})$, we have $\frac{v_n - \Delta}{v_n} < \frac{T_p}{mT_n}$. It then follows that

$$\begin{aligned} \partial_{s_n} f_p(s_n^*, s_p^*) =& \bar{\lambda}\left(-2v_n s_n^* - 2(v_n - \Delta)s_p^* + v_n - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)\right) \\ =& \bar{\lambda}\left(-2v_n s_n^* - 2(v_n - \Delta)s_p^* + v_n - \frac{m(v_n - \Delta)(1 - 2s_n^* - 2s_p^*)T_n}{T_p} + T_n\mu_2^*\right) \\ >& \bar{\lambda}\left(-2v_n s_n^* - 2(v_n - \Delta)s_p^* + v_n - v_n(1 - 2s_n^* - 2s_p^*) + T_n\mu_2^*\right) = 2\bar{\lambda}\Delta s_p^* + \bar{\lambda}T_n\mu_2^* > \bar{\lambda}T_n\mu_2^*, \end{aligned}$$

where the first inequality follows from $\frac{v_n - \Delta}{v_n} < \frac{T_p}{mT_n}$. Therefore we have $\partial_{s_n} f_p(s_n^*, s_p^*) - \bar{\lambda}T_n\mu_2^* > 0$, which contradicts the FOC that $\partial_{s_n} f_p(s_n^*, s_p^*) - \bar{\lambda}T_n\mu_2^* = 0$. If then follows that $s_p^* = 0$ if $\Delta > v_n(1 - \frac{T_p}{mT_n})$.

If $\Delta < v_n(1 - \frac{T_p}{mT_n})$ and $s_p^* = 0$, we have that $s_n^* > 0$ since both of s_n^* and s_p^* being equal to zero is clearly suboptimal. The FOC with respect to s_n implies that

$$v_n - 2v_n s_n^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}s_n^*T_n}{\rho_{\max}K}\right) = T_n\mu_2^*,$$

and by $\Delta < v_n(1-\frac{T_p}{mT_n})$ we have $\frac{v_n-\Delta}{v_n} > \frac{T_p}{mT_n}$. It then follows that

$$\begin{aligned}
\partial_{s_p} f_p(s_n^*, s_p^*) =& \bar{\lambda}\left((v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{m\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)\right) \\
>& \bar{\lambda}\left(v_n(1-2s_n^*)\frac{T_p}{mT_n} - \frac{T_p}{m\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)\right) \\
=& \frac{\bar{\lambda}T_p}{mT_n}\left(v_n(1-2s_n^*) - \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{2}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)\right) = \frac{\bar{\lambda}T_p}{m}\mu_2^*,
\end{aligned}$$

where the inequality follows from $\frac{v_n-\Delta}{v_n} > \frac{T_p}{mT_n}$ and the assumption $s_p^* = 0$. Thus, $\partial_{s_p} f_p(s_n^*, s_p^*) - \frac{\bar{\lambda}T_p}{m}\mu_2^* > 0$, which contradicts with $\partial_{s_p} f_p(s_n^*, s_p^*) - \frac{\bar{\lambda}T_p}{m}\mu_2^* = 0$. Therefore, we have $s_p^* > 0$ if $\Delta < v_n(1-\frac{T_p}{mT_n})$. *Q.E.D.*

Proof of Theorem 1. Let $\hat{\Delta} > \Delta$. We need to show that $\hat{s}^* = \hat{s}_n^* + \hat{s}_p^* \le s^* = s_n^* + s_p^*$. Notice that $\hat{s}_n^* \ge s_n^*$ by Proposition 2. If $\hat{s}_n^* = s_n^*$, then we have $\hat{s}^* = \hat{s}_n^* + \hat{s}_p^* \le s^* = s_n^* + s_p^*$ since $\hat{s}_p^* \le s_p^*$ by Proposition 2. Therefore, it remains to consider the case where $\hat{s}_n^* > s_n^*$.

If $\hat{s}_n^* > s_n^*$, we have $\partial_{s_n} \hat{f}_p(\hat{s}_n^*, \hat{s}_p^*) \ge 0 \ge \partial_{s_n} f_p(s_n^*, s_p^*)$, i.e.,

$$-2v_n\hat{s}_n^* - 2(v_n-\hat{\Delta})\hat{s}_p^* - \hat{y}^* \ge -2v_n s_n^* - 2(v_n-\Delta)s_p^* - y^*,$$

where $y^* = \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)$ and $\hat{y}^* = \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}\hat{s}_p^*T_p + \hat{s}_n^*T_n)}{\rho_{\max}K}\right)$. It then follows that

$$2(v_n-\hat{\Delta})(\hat{s}^* - s^*) \le 2\hat{\Delta}(s_n^* - \hat{s}_n^*) + 2(\Delta-\hat{\Delta})s_p^* + y^* - \hat{y}^*.$$

If $y^* \le \hat{y}^*$, then $s^* > \hat{s}^*$ immediately follows from $s_n^* < \hat{s}_n^*$ and $\Delta < \hat{\Delta}$. If $y^* > \hat{y}^*$, the convexity of $C(\cdot)$ implies that $\frac{1}{m}s_p^*T_p + s_n^*T_n > \frac{1}{m}\hat{s}_p^*T_p + \hat{s}_n^*T_n$. Since $(T_p/m) < T_n$, it then follows that $s_p^* - \hat{s}_p^* > \hat{s}_n^* - s_n^*$, or equivalently, $s^* = s_n^* + s_p^* > \hat{s}_n^* + \hat{s}_p^* = \hat{s}^*$. *Q.E.D.*

Proof of Theorem 2. We first show $\underline{p_p^* \le \tilde{p}_n^* \text{ for all } \Delta \in [0, v_n]}$. Note that $p_p^* = (1-s_p^*-s_n^*)(v_n-\Delta)/d_p$ and $\tilde{p}_n^* = (1-\tilde{s}_n^*)v_n/d_n$. By Theorem 1, we have $\tilde{s}_n^* \le s_p^* + s_n^*$ ($\tilde{s}_n^*$ corresponds to $s_n^* + s_p^*$ in the case with $\Delta = v_n$), and $p_p^* \le \tilde{p}_n^*$ follows immediately from $\Delta \ge 0$ and $d_p \ge d_n$. Next, we show that $\underline{p_n^* \le \tilde{p}_n^* \text{ for all } \Delta \in [0, \bar{\Delta})}$. We proceed in two steps. First, we show that $p_n^* \le \tilde{p}_n^*$ when $\Delta \in [\underline{\Delta}, \bar{\Delta})$. Then we show that p_n^* is increasing in Δ on $\Delta \in [0, \underline{\Delta}]$, which would complete the proof.

First, consider the case where $\Delta \in (\underline{\Delta}, \bar{\Delta})$ (i.e., $s_n^* > 0$ and $s_p^* > 0$). Assume, to the contrary, that $p_n^* > \tilde{p}_n^*$, i.e., $(1-s_n^*)v_n - s_p^*(v_n-\Delta) > (1-\tilde{s}_n^*)v_n$. Rearranging terms, we get

$$\tilde{s}_n^* > s_n^* + \frac{v_n-\Delta}{v_n}s_p^* > s_n^* + \frac{T_p}{mT_n}s_p^*,$$

where the second inequality holds because $\Delta < \bar{\Delta}$ implies $\frac{v_n-\Delta}{v_n} > \frac{T_p}{mT_n}$. Note that

$$\begin{aligned}
\partial_{s_n} f_p(s_n^*, s_p^*) =& \bar{\lambda}\left(v_n - 2v_n s_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)\right) \\
>& \bar{\lambda}\left(v_n - 2v_n s_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}\tilde{s}_n^*T_n}{\rho_{\max}K}\right)\right) \\
>& \bar{\lambda}\left(v_n - 2v_n s_n^* - 2v_n(\tilde{s}_n^* - s_n^*) - \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}\tilde{s}_n^*T_n}{\rho_{\max}K}\right)\right) \\
=& \bar{\lambda}\left(v_n - 2v_n\tilde{s}_n^* - \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}\tilde{s}_n^*T_n}{\rho_{\max}K}\right)\right) = f_b'(\tilde{s}_n^*) \ge 0,
\end{aligned} \tag{25}$$

where $f_b(s_b) := \bar{\lambda} v_n(1-s_b)s_b - KC\left(\frac{\bar{\lambda} s_b T_n}{\rho_{\max} K}\right)$ is the profit of the platform which only offers the normal service. In (25), the first inequality follows from $\tilde{s}_n^* > s_n^* + \frac{T_p}{mT_n} s_p^*$, the second inequality follows from $\tilde{s}_n^* > s_n^* + \frac{v_n - \Delta}{v_n} s_p^*$, and the last inequality follows from $\tilde{s}_n^* > 0$. In addition, it is straightforward to check that $\tilde{s}_n^* > s_n^* + \frac{T_p}{mT_n} s_p^*$ implies $\frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K} < 1$. It then follows from (25) that $\partial_{s_n} f_p(s_n^*, s_p^*) > 0$, which contradicts with (s_n^*, s_p^*) being the optimal solution. Therefore, we have $p_n^* \le \tilde{p}_n^*$ when $\Delta \in (\underline{\Delta}, \bar{\Delta})$. Finally, we show $\underline{p_n^* \text{ is increasing in } \Delta \text{ on } \Delta \in [0, \underline{\Delta}]}$. When $\Delta \in [0, \underline{\Delta}]$, we have $s_n^* = 0$ and

$$p_n^* = ((1-s_n^*)\Delta + (1-s_n^*-s_p^*)(v_n-\Delta))/d_n = (\Delta + (1-s_p^*)(v_n-\Delta))/d_n = (v_n - (v_n-\Delta)s_p^*)/d_n.$$

By Proposition 2, s_p^* is decreasing in Δ. Therefore, $(v_n - \Delta)s_p^*$ is decreasing in Δ and it follows that p_n^* is increasing in Δ on $\Delta \in [0, \underline{\Delta}]$.

Proof of Proposition 3. First, it follows immediately from

$$\Pi_p^* = \max\left\{\bar{\lambda}[((1-s_n-s_p)(v_n-\Delta) + (1-s_n)\Delta)s_n + (1-s_n-s_p)(v_n-\Delta)s_p] - KC\left(\frac{\bar{\lambda}(\frac{s_p}{\gamma} + s_n T_n)}{\rho_{\max} K}\right)\right\}$$

that Π_p^* is increasing in γ ($C(\cdot)$ is decreasing in y). If $\Delta \le \underline{\Delta}$, as shown in Proposition 2, $s_n^* = 0$. It can be easily checked that $\partial_{s_p} f_p(0, s_p) = (v_n - \Delta)(1-2s_p) - \frac{\bar{\lambda}}{\rho_{\max}\gamma} C'\left(\frac{\bar{\lambda} s_p}{\gamma}\right)$ is increasing in s_p, so $f_p(0, s_p)$ is supermodular in (s_p, γ). Hence, s_p^* is increasing in γ. Since $s_n^* = 0$, $s^* = s_n^* + s_p^* = s_p^*$ is increasing in γ, whereas $p_n^* = ((1-s^*)v_p + (1-s_n^*)\Delta)/d_n = ((1-s^*)v_p + \Delta)/d_n$ and $p_p^* = (1-s^*)v_p/d_p$ are decreasing in s^* and thus in γ as well.

We now consider the case $\Delta > \underline{\Delta}$, in which case $s_p^* > 0$ and $s_n^* > 0$. Assume that $\hat{\gamma} > \gamma$, $\hat{f}_p(\cdot,\cdot)$ is the profit function associated with $\hat{\gamma}$, and $(\hat{s}_n^*, \hat{s}_n^*)$ is the optimizer of $\hat{f}_p(\cdot,\cdot)$. We first show $\underline{\hat{s}_p^* \ge s_p^*}$. Assume to the contrary that $\hat{s}_p^* < s_p^*$. Then we have $\partial_{s_p}\hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \le 0 \le \partial_{s_p} f_p(s_n^*, s_p^*)$, or alternatively, $(v_n - \Delta)(1 - 2\hat{s}_n^* - 2\hat{s}_p^*) - \frac{\hat{y}^*}{\hat{\gamma} T_n} \le (v_n - \Delta)(1 - 2s_n^* - 2s_p^*) - \frac{y^*}{\gamma T_n}$, where $\hat{y}^* := \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{\hat{s}_p^*}{\hat{\gamma}} + \hat{s}_n^* T_n)}{\rho_{\max} K}\right)$ and $y^* := \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{s_p^*}{\gamma} + s_n^* T_n)}{\rho_{\max} K}\right)$. Equivalently,

$$\frac{\hat{y}^*}{\hat{\gamma} T_n} - \frac{y^*}{\gamma T_n} \ge 2(v_n - \Delta)(s_n^* - \hat{s}_n^*) + 2(v_n - \Delta)(s_p^* - \hat{s}_p^*). \tag{26}$$

If in addition we have $\hat{s}_n^* \le s_n^*$, the convexity of $C(\cdot)$ suggests that $\hat{y}^* < y^*$. However, (26) implies that $\hat{y}^* > y^*$, which forms a contradiction. Hence, we must have $\hat{s}_n^* > s_n^*$. Thus, $\partial_{s_n}\hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \ge 0 \ge \partial_{s_n} f_p(s_n^*, s_p^*)$, or alternatively, $-2(v_n - \Delta)\hat{s}_p^* + v_n(1 - 2\hat{s}_n^*) - \hat{y}^* \ge -2(v_n - \Delta)s_p^* + v_n(1 - 2s_n^*) - y^*$. Equivalently,

$$\hat{y}^* - y^* \le 2(v_n - \Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*). \tag{27}$$

By (26) and $\hat{\gamma} T_n > \gamma T_n > 1$, $\hat{y}^* - y^* > 2(v_n - \Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*)$, which contradicts (27). We have thus shown that $\hat{s}_p^* \ge s_p^*$.

Next, we show that $\underline{\hat{s}_p^* + \hat{s}_n^* \ge s_p^* + s_n^*}$. If $\hat{s}_p^* = s_p^*$, then $\frac{\hat{s}_p^*}{\hat{\gamma}} \le \frac{s_p^*}{\gamma}$. We have

$$\begin{aligned}
\partial_{s_n}\hat{f}_p(s_n^*, \hat{s}_p^*) &= \bar{\lambda}(v_n - 2v_n s_n^* - 2(v_n - \Delta)\hat{s}_p^*) - \frac{\bar{\lambda} T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{\hat{s}_p^*}{\hat{\gamma}} + s_n^* T_n)}{\rho_{\max} K}\right) \\
&\ge \bar{\lambda}(v_n - 2v_n s_n^* - 2(v_n - \Delta)\hat{s}_p^*) - \frac{\bar{\lambda} T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{s_p^*}{\gamma} + s_n^* T_n)}{\rho_{\max} K}\right) = \partial_{s_n} f_p(s_n^*, s_p^*) \ge 0.
\end{aligned}$$

Therefore, we have $\hat{s}_n^* \geq s_n^*$ and hence, $\hat{s}_p^* + \hat{s}_n^* \geq s_p^* + s_n^*$.

Now we consider the case $\hat{s}_p^* > s_p^*$. If $\hat{s}_n^* + \hat{s}_p^* \leq s_n^* + s_p^*$, we must have $\hat{s}_n^* < s_n^*$. Thus, $\partial_{s_n} \hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \leq 0 \leq \partial_{s_n} f_p(s_n^*, s_p^*)$, i.e., $-2(v_n - \Delta)\hat{s}_p^* + v_n(1 - 2\hat{s}_n^*) - \hat{y}^* \leq -2(v_n - \Delta)s_p^* + v_n(1 - 2s_n^*) - y^*$. Equivalently,

$$\hat{y}^* - y^* \geq 2(v_n - \Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*) > 0, \tag{28}$$

where the last inequality follows from $\hat{s}_n^* + \hat{s}_p^* \leq s_n^* + s_p^*$. Since $C(\cdot)$ is convex, (28) implies that $\frac{\hat{s}_p^*}{\hat{\gamma}} + T_n \hat{s}_n^* > \frac{s_p^*}{\gamma} + T_n s_n^*$, which is equivalent to that $s_n^* - \hat{s}_n^* < \frac{\hat{s}_p^*}{\hat{\gamma} T_n} - \frac{s_p^*}{\gamma T_n} < \hat{s}_p^* - s_p^*$, where the inequality follows from that $\hat{\gamma} T_n > \gamma T_n > 1$. Thus, $\hat{s}_n^* + \hat{s}_p^* > s_n^* + s_p^*$, contradicting with $\hat{s}_n^* + \hat{s}_p^* \leq s_n^* + s_p^*$. Therefore, we must have $\hat{s}_n^* + \hat{s}_p^* \geq s_n^* + s_p^*$.

Next, we show that $\underline{\hat{p}_p^* \leq p_p^*}$. Note that $\hat{p}_p^* = (v_n - \Delta)(1 - \hat{s}_n^* - \hat{s}_p^*)/d_p$ and $p_p^* = (v_n - \Delta)(1 - \hat{s}_n^* - \hat{s}_p^*)/d_p$, $\hat{p}_p^* \leq p_p^*$ follows immediately from $\hat{s}_n^* + \hat{s}_p^* \geq s_n^* + s_p^*$.

Finally, we show that $\underline{\hat{p}_n^* \leq p_n^*}$. Assume to the contrary that $\hat{p}_n^* > p_n^*$, i.e., $(v_n - \Delta)(1 - \hat{s}_n^* - \hat{s}_p^*) + \Delta(1 - \hat{s}_n^*) > (v_n - \Delta)(1 - s_n^* - s_p^*) + \Delta(1 - s_n^*)$. Hence, $v_n(s_n^* - \hat{s}_n^*) > (v_n - \Delta)(\hat{s}_p^* - s_p^*) > 0$, where the second inequality follows from that $\hat{s}_p^* > s_p^*$. The inequality $\hat{s}_n^* < s_n^*$ implies that $\partial_{s_n} \hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \leq 0 \leq \partial_{s_n} f_p(s_n^*, s_p^*)$, i.e., $-2(v_n - \Delta)\hat{s}_p^* + v_n(1 - 2\hat{s}_n^*) - \hat{y}^* \leq -2(v_n - \Delta)s_p^* + v_n(1 - 2s_n^*) - y^*$. Equivalently,

$$\hat{y}^* - y^* \geq 2(v_n - \Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*) > 0, \tag{29}$$

where the last inequality follows from $v_n(s_n^* - \hat{s}_n^*) > (v_n - \Delta)(\hat{s}_p^* - s_p^*) > 0$. Since $C(\cdot)$ is convex, (29) implies that $\frac{\hat{s}_p^*}{\hat{\gamma}} + T_n \hat{s}_n^* > \frac{s_p^*}{\gamma} + T_n s_n^*$, which is equivalent to that $s_n^* - \hat{s}_n^* < \frac{\hat{s}_p^*}{\hat{\gamma} T_n} - \frac{s_p^*}{\gamma T_n} < \frac{\hat{s}_p^* - s_p^*}{\hat{\gamma} T_n}$, where the inequality follows from that $\hat{\gamma} T_n > \gamma T_n$. Since $\Delta < \bar{\Delta}$, $(v_n - \Delta)/v_n > 1/(\hat{\gamma} T_n)$. So we have $\frac{(v_n - \Delta)(\hat{s}_p^* - s_p^*)}{v_n} > \frac{\hat{s}_p^* - s_p^*}{\hat{\gamma} T_n} > s_n^* - \hat{s}_n^*$. This inequality contradicts that $v_n(s_n^* - \hat{s}_n^*) > (v_n - \Delta)(\hat{s}_p^* - s_p^*)$. Therefore, we must have $\hat{p}_n^* \leq p_n^*$. *Q.E.D.*

Proof of Proposition 4. We use $\lambda_p^* := \bar{\lambda} s_p^*$ and $\lambda_n^* := \bar{\lambda} s_n^*$. Notice that when $(v_n - \Delta)/v_n > T_p/(mT_n)$, we have $\Delta < \bar{\Delta}$ and hence $s_p^* > 0$. By the KKT condition (which is both necessary and sufficient for optimality by the joint concavity of $f_p(\cdot)$ and compactness of the feasible region of (s_n, s_p)), we have

$$\bar{\lambda}\left[-2v_p s_n^* - 2\Delta s_n^* - 2s_p^* v_p + v_p + \Delta\right] - \frac{\bar{\lambda} T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right) = \mu_1^* + \bar{\lambda} T_n \mu_2^* - \eta_1^*, \tag{30}$$

$$\bar{\lambda}\left[-2v_p s_n^* - 2s_p^* v_p + v_p\right] - \frac{\bar{\lambda} T_p}{m\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right) = \mu_1^* + \frac{1}{m}\bar{\lambda} T_p \mu_2^* - \eta_2^*, \tag{31}$$

$$\mu_1^*(1 - s_n^* - s_p^*) = 0, \tag{32}$$

$$\mu_2^*\left(\rho_{\max} K - \bar{\lambda}\left(\frac{1}{m} s_p^* T_p + s_n^* T_n\right)\right) = 0, \tag{33}$$

$$\eta_1^* s_n^* = 0, \quad \eta_2^* s_p^* = 0, \tag{34}$$

$$\mu_1^*, \mu_2^*, \eta_1^*, \eta_2^* \geq 0, \tag{35}$$

where $\mu_1^*, \mu_2^*, \eta_1^*$, and η_2^* are the Lagrangian multipliers with respect to the constraints $s_n^* + s_p^* \leq 1$, $\bar{\lambda}(\frac{1}{m} s_n^* T_n + s_p^* T_p) \leq \rho_{\max} K$, $s_n^* \geq 0$ and $s_p^* \geq 0$, respectively. Notice that by (23), $s_n^* + s_p^* < 1$ and hence by complementary slackness condition, we have $\mu_1^* = 0$.

(a) s_n^* is decreasing in $\bar{\lambda}$. Consider $\hat{\bar{\lambda}}$ and $\bar{\lambda}$ with $\hat{\bar{\lambda}} > \bar{\lambda}$. Notice that $(v_n - \Delta)/v_n > T_p/(mT_n)$ and hence $\Delta < \bar{\Delta}$, we have $s_p^* > 0$ and $\hat{s}_p^* > 0$. By (34), $\eta_2^* = \hat{\eta}_2 = 0$. We first consider the case where $\hat{s}_n^*, s_n^* > 0$, and therefore $\eta_1^* = \hat{\eta}_1 = 0$. Then the KKT conditions (30) and (31) imply that:

$$\begin{aligned}
&v_n - 2v_n s_n^* - 2(v_n-\Delta)s_p^* - y^* - \mu_2^* T_n = 0,\\
&(v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{mT_n}y^* - \mu_2^*\frac{T_p}{m} = 0,\\
&v_n - 2v_n \hat{s}_n^* - 2(v_n-\Delta)\hat{s}_p^* - \hat{y}^* - \hat{\mu}_2^* T_n = 0,\\
&(v_n-\Delta)(1-2\hat{s}_n^*-2\hat{s}_p^*) - \frac{T_p}{mT_n}\hat{y}^* - \hat{\mu}_2^*\frac{T_p}{m} = 0,
\end{aligned} \tag{36}$$

where $y^* := \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)$ and $\hat{y}^* := \frac{T_n}{\rho_{\max}} C'\left(\frac{\hat{\bar{\lambda}}(\frac{1}{m}\hat{s}_p^*T_p + \hat{s}_n^*T_n)}{\rho_{\max}K}\right)$. Observe that both (s_n^*, s_p^*) and $(\hat{s}_n^*, \hat{s}_p^*)$ are located on the line

$$\frac{v_n - 2v_n s_n - 2(v_n-\Delta)s_p}{(v_n-\Delta)(1-2s_n-2s_p)} = \frac{mT_n}{T_p}. \tag{37}$$

If $\hat{s}_n^* - s_n^* = \delta > 0$, then it is easy to check by (37) that $s_p^* > \hat{s}_p^*$ and $s_p^* - \hat{s}_p^* < \delta$. Thus, we have $\hat{s}_n^* + \hat{s}_p^* > s_n^* + s_p^*$ and $\hat{\bar{\lambda}}(\hat{s}_n^* T_n + \frac{1}{m}T_p\hat{s}_p^*) > \bar{\lambda}(s_n^*T_n + \frac{1}{m}s_p^*T_p)$. Hence, $\hat{y}^* > y^*$. Moreover, by the complementary slackness condition (33), $\hat{\mu}_2^* \geq \mu_2^*$. Therefore,

$$(v_n-\Delta)(1-2\hat{s}_n^*-2\hat{s}_p^*) - \frac{T_p}{mT_n}\hat{y}^* - \hat{\mu}_2^*\frac{T_p}{m} < (v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{mT_n}y^* - \mu_2^*\frac{T_p}{m},$$

which contradicts with (36). Hence, in the range of $s_n^* > 0$, s_n^* is decreasing in $\bar{\lambda}$. By the continuity of s_n^*, it is clear that s_n^* is decreasing in $\bar{\lambda}$ for all $\bar{\lambda}$.

(b) There exists a λ_0 such that $s_n^* = 0$ for $\bar{\lambda} \geq \lambda_0$. Note that $\Delta < \bar{\Delta}$ is equivalent to $\frac{v_n - \Delta}{v_n} > \frac{T_p}{mT_n}$. We use $\lambda_p := \bar{\lambda}s_p$ and $\lambda_n := \bar{\lambda}s_n$ as the decision variables. The platform is then to maximize

$$f_p(\lambda_n,\lambda_p) = \left(\left(1-\frac{\lambda_p}{\bar{\lambda}}-\frac{\lambda_n}{\bar{\lambda}}\right)(v_n-\Delta) + \left(1-\frac{\lambda_n}{\bar{\lambda}}\right)\Delta\right)\lambda_n + \left(1-\frac{\lambda_n}{\bar{\lambda}}-\frac{\lambda_p}{\bar{\lambda}}\right)(v_n-\Delta)\lambda_p - KC\left(\frac{\frac{1}{m}\lambda_pT_p + \lambda_nT_n}{\rho_{\max}K}\right), \tag{38}$$

subject to the constraint $0 \leq \lambda_n + \lambda_p \leq \bar{\lambda}$ and $\lambda_nT_n + \lambda_p\frac{T_p}{m} \leq \rho_{\max}K$. We have

$$\begin{aligned}
\partial_{\lambda_n} f_p(\lambda_n^*,\lambda_p^*) =& v_n - 2v_n\frac{\lambda_n^*}{\bar{\lambda}} - 2(v_n-\Delta)\frac{\lambda_p^*}{\bar{\lambda}} - \frac{T_n}{\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right)\\
=& v_n - 2v_ns_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right),
\end{aligned}$$

and

$$\begin{aligned}
\partial_{\lambda_p} f_p(\lambda_n^*,\lambda_p^*) =& (v_n-\Delta)\left(1-\frac{2\lambda_n^*}{\bar{\lambda}}-\frac{2\lambda_p^*}{\bar{\lambda}}\right) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right)\\
=& (v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right).
\end{aligned}$$

Since $\lambda_n^*T_n + \lambda_p^*\frac{T_p}{m} \leq \rho_{\max}K$, it follows that $s_n^* = \frac{\lambda_n^*}{\bar{\lambda}} \leq \frac{\rho_{\max}K}{T_n\bar{\lambda}}$ and $s_p^* = \frac{\lambda_p^*}{\bar{\lambda}} \leq \frac{m\rho_{\max}K}{T_n\bar{\lambda}}$. Therefore, we have $s_n^* \to 0$ and $s_p^* \to 0$ as $\bar{\lambda} \to +\infty$. Because $\Delta < v_n\left(1-\frac{T_p}{mT_n}\right)$, we have

$$v_n - \Delta - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right) > \frac{T_p}{mT_n}\left(v_n - \frac{T_n}{\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right)\right).$$

Therefore, when $\bar{\lambda}$ is sufficiently large (where $s_n^* \to 0$ and $s_p^* \to 0$), we have

$$\begin{aligned}
\partial_{\lambda_p} f_p(\lambda_n^*,\lambda_p^*) - \frac{T_p}{m}\mu_2^* =& (v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right) - \frac{T_p}{m}\mu_2^*\\
>& \frac{T_p}{mT_n}\left(v_n - 2v_ns_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right) - T_n\mu_2^*\right)\\
=& \frac{T_p}{mT_n}(\partial_{\lambda_n} f_p(\lambda_n^*,\lambda_p^*) - T_n\mu_2^*),
\end{aligned} \tag{39}$$

where μ_2^* is the Lagrangian multiplier with respect to the constraint $\lambda_n T_n + \lambda_p \frac{T_p}{m} \leq \rho_{\max} K$. Since $s_p^* > 0$ and thus $\lambda_p^* > 0$, the first-order condition $\partial_{\lambda_p} f_p(\lambda_n^*, \lambda_p^*) - \frac{T_p}{m}\mu_2^* = 0$ when $\bar{\lambda}$ is sufficiently large. In this case, (39) implies that $\partial_{\lambda_n} f_n(\lambda_n^*, \lambda_p^*) - T_n \mu_2^* < \frac{mT_n}{T_p}\left(\partial_{\lambda_p} f_p(\lambda_n^*, \lambda_p^*) - \frac{T_p}{m}\right) = 0$. It is straightforward to check that by the KKT condition of optimization problem (38), $\partial_{\lambda_n} f_n(\lambda_n^*, \lambda_p^*) - T_n \mu_2^* < 0$ implies that $\lambda_n^* = 0$. It then follows that $s_n^* = 0$ when $\bar{\lambda}$ is sufficiently large, or, there exists a threshold λ_0, such that $s_n^* = 0$ for $\bar{\lambda} \geq \lambda_0$.

(c) s_p^* is increasing (resp. decreasing) in $\bar{\lambda}$ for $\bar{\lambda} < \lambda_0$ (resp. $\bar{\lambda} > \lambda_0$). Recall that $\lambda_0 := \min\{\bar{\lambda} : s_n^* = 0\}$. If $\bar{\lambda} < \lambda_0$, (s_n^*, s_p^*) satisfies (37). Since s_n^* is decreasing in $\bar{\lambda}$, it is straightforward to check that s_p^* is decreasing in s_n^*, thus increasing in $\bar{\lambda}$ as well. If $\bar{\lambda} > \lambda_0$, then we have $s_n^* = 0$. By Proposition 1, s_p^* is decreasing in $\bar{\lambda}$.

(d) p_n^* and p_p^* are increasing in $\bar{\lambda}$, and $p_n^* d_n - p_p^* d_p$ is increasing in $\bar{\lambda}$. Note that $p_p^* = (v_n - \Delta)(1 - s_n^* - s_p^*)/d_p$. If $\bar{\lambda} < \lambda_0$, $s_n^* > 0$ and (s_n^*, s_p^*) satisfies (37). Since s_n^* is decreasing in $\bar{\lambda}$, it is easy to check, by (37), that $s_n^* + s_p^*$ is decreasing in $\bar{\lambda}$. Thus, $p_p^* = (v_n - \Delta)(1 - s_n^* - s_p^*)/d_p$ is increasing in $\bar{\lambda}$. Furthermore, $p_n^* d_n - p_p^* d_p = (1 - s_n^*)\Delta$ is decreasing in s_n^*, thus increasing $\bar{\lambda}$. Hence, $p_n^* = (p_p^* d_p + (1 - s_n^*)\Delta)/d_n$ is also increasing in $\bar{\lambda}$. *Q.E.D.*

Proof of Proposition 5. It follows from (15) that if $\Delta = 0$, $RS_p^* = \frac{1}{2}\bar{\lambda}(s_p^*)^2$. $\tilde{RS}_n^* = RS_p^*(\bar{\Delta}) = \frac{1}{2}\bar{\lambda}(s_n^*)^2$. We now show that $s_p^*(0) > s_n^*(\bar{\Delta})$. By Theorem 1, $s_p^*(0) + s_n^*(0) > s_p^*(\bar{\Delta}) + s_n^*(\bar{\Delta})$. By Proposition 2, $s_n^*(0) = s_p^*(\bar{\Delta}) = 0$, we have $s_p^*(0) > s_n^*(\bar{\Delta})$, which implies that $RS_p^*(0) > RS_p^*(\bar{\Delta})$. The existence of $\underline{\Delta}_r$ then follows directly from $RS_p^*(\Delta)$ being continuous in Δ.

For the ease of exposition, we normalize $K = 1$, $T_n = 1$, and $v_n = 1$. We also define $\gamma = m/T_p$ and $\eta = v_n - \Delta$. Then, we have the constraints $\gamma > 1$, $\eta < 1$, and $\eta\gamma > 1$. If $G(r) = r$, we first compare $RS_p^*(\Delta)$ with $\tilde{RS}_n^*$ for $\Delta \in (\underline{\Delta}, \bar{\Delta})$. In this case, $s_n^*(\Delta) > 0$. Then, It is straightforward to calculate that

$$\begin{cases} s_n^*(\Delta) = & \frac{1}{2}\left(1 - \frac{\eta\bar{\lambda}(1/\gamma - 1)}{-\eta\bar{\lambda} + (\eta-1)\eta\rho_{\max}^2 + 2\eta\bar{\lambda}/\gamma - \lambda/\gamma^2}\right) \\ s_p^*(\Delta) = & \frac{\bar{\lambda}(1/\gamma - \eta)}{-2\eta\bar{\lambda} + 2(\eta-1)\eta\rho_{\max}^2 + 4\eta\bar{\lambda}/\gamma - 2\bar{\lambda}/\gamma^2} \\ \tilde{s}_n^* = & \frac{\rho_{\max}^2}{2(\bar{\lambda} + \rho_{\max}^2)} \end{cases}$$

Then, we can calculate the difference between the setting with carpool services and that without:

$$RS_p^*(\Delta) - \tilde{RS}_n^* = -\frac{\bar{\lambda}^2(\eta - 1/\gamma)^2(\eta(-\bar{\lambda}^2 + 2(\eta-2)\bar{\lambda}\rho_{\max}^2 + 3(\eta-1)\rho_{\max}^4) + 2\eta\bar{\lambda}(\bar{\lambda} + 2\rho_{\max}^2)/\gamma - \lambda(\lambda + 2\rho_{\max}^2)/\gamma^2)}{4(\lambda + \rho_{\max}^2)^2(\eta\lambda - (\eta-1)\eta\rho_{\max}^2 - 2\eta\bar{\lambda}/\gamma + \bar{\lambda}/\gamma^2)^2}$$

Hence, it suffices to show that

$$\eta(-\bar{\lambda}^2 + 2(\eta-2)\bar{\lambda}\rho_{\max}^2 + 3(\eta-1)\rho_{\max}^4) + 2\eta\bar{\lambda}(\bar{\lambda} + 2\rho_{\max}^2)/\gamma - \lambda(\lambda + 2\rho_{\max}^2)/\gamma^2 < 0.$$

Rearranging the terms, it suffices to show that

$$\bar{\lambda}^2(\eta - 2\eta/\gamma + 1/\gamma^2) > 0, \tag{40}$$

$$2\bar{\lambda}\rho_{\max}^2((2-\eta)\eta - 2\eta/\gamma + 1/\gamma^2) > 0, \tag{41}$$

$$3(1-\eta)\eta\rho_{\max}^4 > 0. \tag{42}$$

To show (40), observe that $\bar{\lambda}^2(\eta - 2\eta/\gamma + 1/\gamma^2) > \bar{\lambda}^2(\eta^2 - 2\eta/\gamma + 1/\gamma^2) = \bar{\lambda}^2(\eta - 1/\gamma)^2 > 0$, where the first inequality follows from $\eta < 1$ and the second from $\eta\gamma > 1$. To show (41), observe that $2\bar{\lambda}\rho_{\max}^2((2-\eta)\eta - 2\eta/\gamma + 1/\gamma^2) > 2\bar{\lambda}\rho_{\max}^2(\eta^2 - 2\eta/\gamma + 1/\gamma^2) = 2\bar{\lambda}\rho_{\max}^2(\eta - 1/\gamma)^2 > 0$, where the first inequality follows from

$(2-\eta)\eta > \eta^2$ for $\eta \in (0,1)$, and the second from $\eta > 1/\gamma$. This proves that if $\Delta \in (\underline{\Delta}, \bar{\Delta})$, $RS_p^*(\Delta) > \tilde{RS}_n^*$. Inequality (42) follows immediately from $0 < \eta < 1$. Putting everything together, we have that $RS_p^*(\Delta) > \tilde{RS}_p^*$ for $\Delta \in [\underline{\Delta}, \bar{\Delta})$.

Finally we show that for the case $\Delta \le \underline{\Delta}$, $RS_p^*(\Delta) > \tilde{RS}_n^*$. By continuity, if $\Delta = \underline{\Delta}$, $RS_p^*(\Delta) > \tilde{RS}_n^*$. Furthermore, $s_p^*(\Delta)$ is decreasing in Δ (by Proposition 2). Therefore, $RS_p^*(\Delta) = \frac{1}{2}(v_n - \Delta)(s_p^*(\Delta))^2$ is decreasing in Δ. Hence, $RS_p^*(\Delta) > RS_p^*(\underline{\Delta})$ for all $\Delta < \underline{\Delta}$. This concludes the proof of Proposition 5. *Q.E.D.*

Proof of Proposition 6. It is clear from (16) that DS_p^* is strictly increasing in k_p^*, and hence it boils down to analyzing the impact of carpool services on the number of active drivers k_p^* in equilibrium. When $\Delta \in (\underline{\Delta}, \bar{\Delta})$, it follows from Proposition 2 that $s_n^* > 0$ and $s_p^* > 0$. Then by first order conditions $\partial_{s_n} f_p(s_n^*, s_p^*) = 0$ and $\partial_{s_p} f_p(s_n^*, s_p^*) = 0$, it is straightforward to derive that

$$\begin{cases} s_n^* = \dfrac{(\Delta^2 K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)T_p v_n - \Delta m(\bar{\lambda} T_n T_p + K m \rho_{\max}^2 v_n))}{(2\Delta m(\Delta K m \rho_{\max}^2 + \bar{\lambda} T_n(mT_n - 2T_p)) - 2(\Delta K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)^2) v_n)}, \\ s_p^* = \dfrac{\bar{\lambda} m T_n(\Delta m T_n - (mT_n - T_p)v_n)}{2\Delta m(\Delta K m \rho_{\max}^2 + \bar{\lambda} T_n(mT_n - 2T_p)) - 2(\Delta K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)^2) v_n}. \end{cases}$$

Similarly, the first order condition $\partial_{\tilde{s}_n} f_b(\tilde{s}_n^* | \bar{\lambda}) = 0$ implies that

$$\tilde{s}_n^* = \frac{K \rho_{\max}^2 v_n}{2\bar{\lambda} T_n^2 + 2K \rho_{\max}^2 v_n}.$$

Note that

$$\tilde{k}_n^* - k_p^* = \frac{\bar{\lambda} T_n(\tilde{s}_n^* - s_n^* - (T_p s_p^*)/(mT_n))}{\rho_{\max}}.$$

Therefore, $\tilde{k}_n^* > k_p^*$ is equivalent to $\tilde{s}_n^* > s_n^* + \frac{T_p s_p^*}{mT_n}$. We next compute $\tilde{s}_n^* - \left(s_n^* + \frac{T_p s_p^*}{mT_n}\right)$ as follows:

$$\begin{aligned} \tilde{s}_n^* - \left(s_n^* + \frac{T_p s_p^*}{mT_n}\right) &= \frac{K \bar{\lambda} \rho_{\max}^2 (\Delta m T_n - (mT_n - T_p)v_n)^2}{2(\bar{\lambda} T_n^2 + K \rho_{\max}^2 v_n)(-\Delta m(\Delta K m \rho_{\max}^2 + \bar{\lambda} T_n(mT_n - 2T_p)) + (\Delta K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)^2) v_n)} \\ &= \frac{K \bar{\lambda} \rho_{\max}^2 (\Delta m T_n - (mT_n - T_p)v_n)^2}{2(\bar{\lambda} T_n^2 + K \rho_{\max}^2 v_n)[\Delta(v_n - \Delta) K m^2 \rho_{\max}^2 + \bar{\lambda}((mT_n - T_p)^2(v_n - \Delta) + T_p^2 \Delta))]} \\ &> 0, \end{aligned}$$

where the inequality follows from $v_n > \Delta \ge 0$. Therefore, $\tilde{s}_n^* > s_n^* + \frac{T_p s_p^*}{mT_n}$ and hence $\tilde{k}_n^* > k_p^*$. *Q.E.D.*