Competition and Coopetition for Two-Sided Platforms

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Two-sided platforms have become omnipresent. In this context, firms compete not only for customers but also for flexible self-scheduling workers who can work for multiple platforms. We consider a setting where two-sided platforms simultaneously choose prices and wages to compete on both sides of the market. We assume that customers and workers each follow an endogenous generalized attraction model that accounts for network effects. In our model, the behavior of an agent depends not only on the price or wage set by the platforms, but also on the strategic interactions among agents on both sides of the market. We show that a unique equilibrium exists and that it can be computed using a *tatônnement* scheme. The proof technique for the competition between two-sided platforms is not a simple extension of the traditional (one-sided) setting and involves different arguments. Armed with this result, we study the impact of *coopetition* between two-sided platforms, that is, the business strategy of cooperating with competitors. Motivated by recent practices in the ride-sharing industry, we analyze a setting where two competing platforms engage in a profit sharing contract by introducing a new joint service. We show that a well-designed profit sharing contract (e.g., under Nash bargaining) will benefit every party in the market (platforms, riders, and drivers), especially when the platforms are facing intensive competition on the demand side. However, if the platforms are facing intensive competition on the supply side, the coopetition partnership may hurt the profit of at least one platform.

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1. Introduction

The service industry has significantly evolved in recent years. Thanks to the emergence of online platforms, several types of services are now offered on-demand. Specifically, customers can use their smartphones to request services from anywhere at any time. These services include ride-sharing, food delivery, cleaning, and repair works, just to name a few. According to a survey by the National Technology Readiness Survey, in October 2015, the on-demand economy was attracting more than 22.4 million consumers annually and \$57.6 billion in spending.[1] This new

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[1] `https://hbr.org/2016/04/the-on-demand-economy-is-growing-and-not-just-for-the-young-and-wealthy`

trend has also made the market increasingly competitive. In each sector, several competing firms offer the same type of service (a list of companies that offer on-demand services can be found at `https://theondemandeconomy.org/participants/`). For example, in the U.S. ride-sharing market, one can find several competitors including Uber, Lyft, Via, Gett, and Curb. These platforms compete not only for customers (riders) but also for workers (drivers). They often send enticing monetary incentives to attract both sides of the market. Traditionally, firms were competing only for customers while hiring permanent workers. In two-sided markets, platforms also compete for workers who can work for multiple platforms and seemingly switch back and forth between companies. As of 2017, 70% of on-demand U.S. drivers work for both Uber and Lyft, and 25% drive for more than just those two, according to a survey by The Rideshare Guy.[2]

The first part of this paper (Sections 2 and 3) studies the competition between two-sided platforms that compete for both customers and workers (e.g., Uber). We model this problem using an endogenous general attraction model (Gallego et al. 2006, Luce 2012) that accounts for network effects in both sides of the market. In our model, the utilities of customers and workers are endogenously determined by the total demand and supply of each platform. Consequently, the behavior of an agent depends not only on the price or wage set by the platform, but also on the strategic interactions among agents on both sides of the market. The customers belong to different segments with heterogeneous preferences over the platforms and different price sensitivities, whereas workers are endorsed with different types with heterogeneous preferences and wage sensitivities. Given that firms compete for customers and workers, the standard equilibrium analysis from the choice model literature should be revisited. The two-sidedness nature of our setting makes the objective function non-differentiable, so that traditional arguments from the literature are not applicable. Instead, we use an approach based on analyzing the best-response strategy to characterize the equilibrium. We ultimately show the existence and uniqueness of the equilibrium market outcome.

Within the ride-sharing industry, a recent trend of partnerships has emerged. One such example is the partnership between Curb and Via in NYC. Curb[3] is an online platform that allows taxi rides to be ordered from a smartphone application and the payment can be completed either via the app or in person. Via[4] is a ride-sharing platform that allows riders heading in the same direction to carpool and share a ride. One can definitely view these two platforms as competitors. Yet, they decided to collaborate and engage in a joint partnership. Specifically, on June 6, 2017, they started offering a joint service through a *profit sharing contract*, under which Curb and Via each earn a portion of the net profit from the joint service. This type of partnership is often referred

[2] `https://docs.google.com/document/d/1QSUFSqasfjM9b9UsqBwZlpa8EgqNj6EBfWybFBSHj3o/edit`

[3] `https://gocurb.com/`

[4] `https://ridewithvia.com/`

to as *coopetition*, a term coined to describe cooperative competition (see, e.g., Brandenburger and Nalebuff 2011). The new service introduced by Curb and Via allows users to book a shared taxi from either platform (this is called Shared Taxi). Shared Taxi fares are calculated using the meter price and paid directly to the driver. If the matching algorithm finds another rider heading in the same direction, the two riders will carpool and save 40% on any shared portion of the trip.[5]

The recent partnership between Curb and Via is not an exception. Below, we report four additional similar examples:

1. In December 2016, Uber partnered with Indonesia's second largest taxi operator PT Express Transindo Utama Tbk. This partnership gave Uber access to Express fleet of taxis and drivers. Express drivers who participate in the program can serve requests from Uber.
2. In October 2014, Uber partnered with For Hire taxis to expand pick-up availability in Seattle. In this partnership, riders can select multiple options directly from the Uber app (UberX, UberXL, Black Car, SUV, and For Hire).
3. In March 2017, Grab partnered with SMRT Taxis with the goal of building the largest car fleet (taxi and private-hire) in Southeast Asia. In this partnership, all SMRT drivers will use only Grab's application for third-party bookings (to complement street-hail pickups).
4. On January 31, 2017, Go-Jek partnered with PT Blue Bird Tbk in Indonesia. In this partnership, riders will simply be served by the closest driver.[6]

It is clear that both platforms have their incentives to engage in such partnerships. For example, it allows ride-sharing platforms to expand their fleet of drivers and increase their market share. Platforms can also benefit from technological advances developed by other firms (e.g., efficient matching algorithms). At the same time, such partnerships can cannibalize the original market shares (customers who were riding with one of the platforms may now switch to the new service).

The second part of this paper (Sections 4 and 5) is motivated by the type of partnerships described above. We study the implications of introducing a new joint service between two competing platforms via a profit sharing contract. Our goal is to examine the impact of the new service on both platforms, riders, and drivers. In conformance with recent partnerships (e.g., Curb and Via), we assume that the workers are coming from two separate labor pools. Although it is not a-priori obvious whether coopetition will benefit the platforms, our analysis shows that—under the Nash bargaining framework— a well-designed contract is beneficial for both platforms, riders, and drivers (i.e., yields a Pareto improvement), in particular when the platforms are facing intensive

[5] The partnership between Curb and Via in NYC was the topic of extensive media coverage. See for example: `https://www.nytimes.com/2017/06/06/nyregion/new-york-yellow-taxis-ride-sharing.html`, `https://techcrunch.com/2017/06/06/curb-and-via-bring-ride-sharing-to-nycs-yellow-taxis/` and `https://qz.com/999132/can-shared-rides-save-the-iconic-new-york-city-yellow-cab/`

[6] `https://www.techinasia.com/go-jek-launches-blue-bird-partnership-now-on-iphone`

competition on the demand side. However, if the platforms are facing intensive competition on the supply side, the coopetition partnership may hurt the profit of at least one platform.

1.1. Contributions

Given the recent popularity of two-sided platforms, this paper extends our understanding on competition and coopetition models in this context. We next summarize our main contributions.

Equilibrium in a two-sided general attraction model with network effects. This paper is among the first to study the (price and wage) competition between two-sided platforms. We use an endogenous general attraction model with network effects to capture the decision process of potential customers and workers. Our model explicitly captures the potential heterogeneity in customers and workers. We prove the existence and uniqueness of equilibrium under general price and wage (Theorem 1) and under a fixed-commission rate (Theorem 2). We also convey that the equilibrium outcome can be computed efficiently using a *tatônnement scheme*. Interestingly, the proof technique for two-sided markets is not a simple extension of the traditional (one-sided) setting. Instead, we show that the best-response strategy is a monotone contraction mapping, allowing us to prove the existence and uniqueness of equilibrium.

Win-win coopetition using a profit sharing contract. Motivated by recent practices in ride-sharing, we study how introducing a new joint service affects the competing platforms. We first show that there exists a unique equilibrium even after introducing the coopetition partnership. We also capture the strategic interactions between the platforms using the Nash bargaining framework. We show that the platforms will agree on a profit sharing contract that increases the profit of each platform. We then identify conditions under which the coopetition partnership is *strictly* beneficial for both participating platforms, namely, engaging in a coopetition is a win-win strategy when the platforms are facing intensive competition on the demand side. However, if the platforms are facing intensive competition on the supply side, coopetition may hurt the profit of at least one platform. Finally, we identify three main effects induced by the coopetition partnership: new market share, cannibalization, and wage variation.

Pareto improvement under coopetition. As expected, riders will also benefit from coopetition. Interestingly, we show that one can design a profit sharing contract that will also benefit drivers. Consequently, when the coopetition terms are carefully designed, it will benefit every party (both participating platforms, riders, and drivers).

1.2. Related Literature

This paper is related to three streams of literature: price competition under choice models, economics of ride-sharing platforms, and coopetition models.

Price competition under choice models: The first relevant stream of literature is related to choice models (for a review on this topic, see Train 2009, and the references therein), and in particular price competition under the MNL model and its extensions (see, e.g., Anderson et al. 1992, Gallego et al. 2006, Konovalov and Sándor 2010, Li and Huh 2011, Aksoy-Pierson et al. 2013, Gallego and Wang 2014). Talluri and Van Ryzin (2004) consider a revenue management setting under a general discrete choice model. The authors propose an exact solution and derive several structural properties. Using the MNL model, Gallego et al. (2006) show that a unique equilibrium exists when costs are increasing and convex in sales. In Li and Huh (2011), the authors consider the problem of pricing multiple products under the nested-MNL model and show that characterizing the equilibrium is analytically tractable. In this literature, the main focus is on showing the existence and/or uniqueness of the equilibrium outcome. In this paper, we extend the results of Gallego et al. (2006) and Li and Huh (2011) to show that a unique equilibrium exists in a two-sided market where firms compete for both customers and workers. As mentioned, the proof technique for two-sided markets is not a simple extension of the traditional (one-sided) setting and involves different arguments. To capture the fact that the decision of an agent depends on other agents' decisions, we consider a general attraction model that accounts for network effects. Specifically, our choice model is constructed in a similar fashion as the MNL-type models with endogenous network effects. Wang and Wang (2016) and Du et al. (2016) incorporate demand-side network effects into the standard MNL model and study the optimal pricing and assortment strategies. We extend this framework by endogenizing the network effects on both the demand and supply sides to examine the two-sided competition between platforms. The network effects considered in our model are threshold-based, which generalize the traditional linear network effects in the platform competition literature (e.g., Rochet and Tirole 2003, Armstrong 2006, Rochet and Tirole 2006).

Economics of ride-sharing platforms: The popularity of ride-sharing platforms triggered a great interest in studying pricing decisions in this context. Several papers consider the problem of designing incentives on prices and wages to coordinate supply with demand for on-demand service platforms (see, e.g., Chen and Hu 2020, Bai et al. 2019, Taylor 2018, Hu and Zhou 2017, Bimpikis et al. 2019, Yu et al. 2020, Benjaafar et al. 2021). Our work has a similar motivation but is among the first to explicitly capture the competition between platforms using a general attraction choice model for each side of the market. The recent work by Nikzad (2017) also analyzes the competition between ride-sharing platforms but with a different focus. The author shows that the effect of competition on prices and wages crucially depends on market thickness (i.e., the number of potential workers). The author identifies an underlying mechanism which is quite similar to the one we advocate in our work: monopoly may soften competition (which may hurt workers and customers), but given the resource pooling on the supply side, it may actually benefit all parties. In Hu and Zhou (2017), the authors study the pricing decisions of an on-demand platform and demonstrate the good performance of a flat-commission contract. We will also consider the special case of flat-commission contracts.

Coopetition models: As mentioned, when two competitors cooperate, this is often referred to as *coopetition* (see, e.g., Brandenburger and Nalebuff 2011). Closer to our work, there are several papers on coopetition in operations management. For example, Nagarajan and Sošić (2007) propose a model for coalition formation among competitors and characterize the equilibrium behavior of the resulting strategic alliances. Casadesus-Masanell and Yoffie (2007) study the simultaneously competitive and cooperative relationship between two manufacturers of complementary products, such as Intel and Microsoft, on their R&D investment, pricing, and timing of new product releases. In a strategic alliance setting with capacity sharing, Roels and Tang (2017) show that an ex-ante capacity reservation contract will benefit both firms. In the revenue management literature, several papers have studied a common form of coopetition among airline companies, called airline alliances (see, e.g., Netessine and Shumsky 2005, Wright et al. 2010). Coopetition and its related contractual issues have also been studied in the context of service operations (as opposed to manufacturing and supply chain). For example, Roels et al. (2010) analyze the contracting issues that arise in collaborative services and identify the optimal contracts. In a recent work, Yuan et al. (2021) show that as price competition increases, the service providers may surprisingly charge higher prices under coopetition. Our contribution with respect to this literature lies in the fact that, motivated by recent partnerships, we are the first to study coopetition in ride-sharing.

Finally, our work is related to the economics literature on competition between two-sided platforms (see Rochet and Tirole 2003, Armstrong 2006, and the references therein). Our paper differs from this literature in two important ways. First, we explicitly consider a supply-constrained setting where the sales (i.e., matches between customers and workers) are truncated by both demand and supply. Second, we focus on a setting where each side of the market (customers and workers) follow a choice model to decide which platform to use and work. As a result, our model is especially applicable to the increasingly competitive environment between on-demand service platforms.

Structure of the paper. Section 2 presents our model of competition between two-sided platforms (in the absence of coopetition), and Section 3 reports our equilibrium analysis for this model. We next consider introducing the coopetition partnership: Section 4 presents our coopetition model, and Section 5 studies the impact of coopetition. We present computational experiments in Section 6. Finally, we consider an extension of our model with endogenous waiting times in Section 7, and we report our conclusions in Section 8. All the proofs of the technical results are relegated to the appendix.

2. Competition Between Two-sided Platforms: Model

In this section, we present our model of competition between two-sided platforms. We consider a general model of n competing online platforms denoted $P_1, P_2, ..., P_n$. Each platform P_i ($i = 1, 2, ..., n$) offers a service via its mobile or online application. Customers belong to m segments, where segment j has a total mass of $\Lambda_j > 0$. Workers belong to l types, where type k has a total mass of $\Gamma_k > 0$. Platform P_i charges a price of p_i to its customers and offers a wage of w_i to its workers. A summary of the notation can be found in Appendix A.

Demand side: We assume that customers follow an endogenous general attraction model (e.g., Gallego et al. 2006, Luce 2012) that accounts for network effects. We denote the utility of the outside option of a customer of any segment by u_0. A customer in segment j can choose between $n+1$ alternatives: $P_1, P_2, ..., P_n$, and the outside option. The utility derived by a segment-j customer from the service offered by P_i *endogenously* depends on the aggregate behavior of all customers (captured by P_i's total demand d_i) and on the aggregate behavior of all workers (captured by P_i's total supply s_i). If P_i's demand is dominated by its supply (i.e., $d_i \leq s_i$), then every customer who opts for P_i will be served. In this case, a segment-j customer who selects P_i earns a utility of $(q_{ij} - \kappa_j p_i)$, where q_{ij} is the perceived quality of platform P_i for segment j and κ_j is the price sensitivity of segment j. If $d_i > s_i$, then the supply capacity has to be rationed. In this case, we assume that the platform randomly allocates its supply s_i to the customers who choose its service. If a segment-j customer chooses P_i's service and successfully receives it, its utility is $(q_{ij} - \kappa_j p_i)$. A customer who opts for P_i but does not get served will be forced to select the outside option, whose utility is ν_j. Without loss of generality, we assume that $\nu_j \leq u_0$ to capture the inconvenience of requesting but not receiving service from P_i. Thus, if $d_i > s_i$, a customer will have a probability of s_i/d_i to be served and a probability of $1 - s_i/d_i$ to opt for the outside option. Consequently, the expected utility of a segment-j customer who opts for P_i is $u_{ij} = \min\{1, s_i/d_i\}(q_{ij} - \kappa_j p_i) + [1 - \min\{1, s_i/d_i\}]\nu_j$. Recall that the expected utility of directly taking the outside option is u_0. Based on the general attraction model, the total demand for P_i from segment-j customers is given by:

$$\begin{aligned} d_{ij} &= \frac{\Lambda_j \exp(u_{ij})}{\exp(u_0) + \sum_{i'=1}^{n} \exp(u_{i'j})} \\ &= \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i/d_i\}(q_{ij} - \kappa_j p_i - \nu_j)]}{\exp(u_0) + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]}. \end{aligned}$$

Thus, the total demand for P_i is given by:

$$d_i = \sum_{j=1}^{m} d_{ij} = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i/d_i\}(q_{ij} - \kappa_j p_i - \nu_j)]}{\exp(u_0) + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]}.$$

To ease the exposition, we normalize $u_0 = 0$ and hence the demand functions become

$$\begin{aligned} d_{ij} &= \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i/d_i\}(q_{ij} - \kappa_j p_i - \nu_j)]}{1 + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]} \text{ for all } 1 \le i \le n, 1 \le j \le m, \\ d_i &= \sum_{j=1}^{m} d_{ij} = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i/d_i\}(q_{ij} - \kappa_j p_i - \nu_j)]}{1 + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]} \text{ for all } 1 \le i \le n. \end{aligned} \tag{1}$$

Supply side: Workers also follow an endogenous general attraction model with network effects to select the platform to work for. Similar to the demand side, the expected utility that a worker derives from working for P_i depends endogenously on the aggregate behavior of all customers (captured by the total demand d_i) and on the aggregate behavior of all workers (captured by the total supply s_i). Each worker chooses one of $n+1$ alternatives: $P_1, P_2, ..., P_n$, and the outside option. We denote the attractiveness of P_i for a type-k worker by a_{ik}, while a_0 represents the attractiveness of the outside option for any worker. If demand dominates supply ($d_i \ge s_i$), then any worker of type-k who opts for P_i can serve a customer and will receive the wage w_i, so the utility is given by $a_{ik} + \eta_k w_i$, where η_k is the wage sensitivity of type-k workers. If $s_i > d_i$, demand will be randomly rationed to the workers who opt for P_i. In this case, a worker will have a probability of d_i/s_i to be matched with a demand request (while earning a utility of $a_{ik} + \eta_k w_i$) and a probability of $1 - d_i/s_i$ to not be matched with any customer (while earning a utility of ω_k). Without loss of generality, we assume that $\omega_k \le a_0$ to capture the inconvenience associated with not being forced to work for the outside option. Thus, the expected utility of a type-k worker choosing P_i is $v_{ik} = \min\{1, d_i/s_i\}(a_{ik} + \eta_k w_i) + (1 - \min\{1, d_i/s_i\})\omega_k = \omega_k + \min\{1, d_i/s_i\}(a_{ik} + \eta_k w_i - \omega_k)$. Based on the general attraction model, the total supply for P_i from type k customers is given by:

$$s_{ik} = \frac{\Gamma_k \exp(v_{ik})}{\exp(v_0) + \sum_{i'=1}^{n} \exp(v_{i'})} = \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i/s_i\}(a_{ik} + \eta_{ik} w_i - \omega_k)]}{\exp(a_0) + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_{i'k} w_{i'} - \omega_k)]}.$$

The total supply for P_i is given by:

$$s_i = \sum_{k=1}^{l} s_{ik} = \sum_{k=1}^{l} \frac{\Gamma_k \exp(v_{ik})}{\exp(v_0) + \sum_{i'=1}^{n} \exp(v_{i'})} = \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i/s_i\}(a_{ik} + \eta_k w_i - \omega_k)]}{\exp(a_0) + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]}.$$

To ease the exposition, we normalize $a_0 = 0$ and hence the supply functions become

$$\begin{aligned} s_{ik} &= \frac{\Gamma_k \exp(\omega_k + \min\{1, d_i/s_i\}(a_{ik} + \eta_{ik} w_i - \omega_k)}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]} \text{ for all } 1 \le i \le n, 1 \le k \le l, \\ s_i &= \sum_{k=1}^{l} s_{ik} = \sum_{k=1}^{l} \frac{\Gamma_k \exp(\omega_k + \min\{1, d_i/s_i\}(a_{ik} + \eta_k w_i - \omega_k)}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]} \text{ for all } 1 \le i \le n. \end{aligned} \tag{2}$$

We note that our model naturally incorporates the heterogeneity in customers' and workers' perceptions of quality, attractiveness, and monetary rewards for the different platforms. Such heterogeneity factors are parameterized by the heterogeneous qualities and price sensitivities of different customer segments $(q_{ij}, \kappa_j, \nu_j : 1 \le i \le n, 1 \le j \le m)$ and by the heterogeneous attractiveness and wage sensitivities of different worker types $(a_{ik}, \eta_k, \omega_k : 1 \le i \le n, 1 \le k \le l)$.

Our analysis begins by formally showing the validity of the above endogenous attraction model. Specifically, we show that given a price and wage vector $(p_i, w_i : 1 \le i \le n)$, the demand and supply functions in Eqs. (1) and (2) uniquely determine a demand and supply vector $(d_{ij}, s_{ik} : 1 \le i \le n, 1 \le j \le m, 1 \le k \le l)$.

LEMMA 1. *Given the price and wage vector $(p_i, w_i : 1 \le i \le n)$, there exists a unique demand and supply vector $(d, s) = (d_i, s_i : 1 \le i \le n)$ that satisfies Eqs.* (1) *and* (2)*.*

Several remarks with respect to our endogenous attraction model are in order. As discussed, in our model, the utilities of both customers and workers are endogenously determined by the total demand and supply of each platform. Consequently, customers and workers account for the strategic interactions among themselves, which give rise to endogenous market outcomes. Our endogenous two-sided choice model is constructed in a similar fashion as the MNL model with endogenous network effects (see Wang and Wang 2016, Du et al. 2016). In these papers, the utility (and the purchase probability) of choosing one product is endogenously determined by the demand of each product through network effects. In our model, the utility of a customer will increase if his/her chance of being served increases. Analogously, the utility of a worker will increase if his/her chance of working for the platform increases. Ultimately, the choice behavior of a customer (or a worker) depends not only on the price (or wage) set by the platform, but also on the strategic interactions among agents on both sides on the market.

In equilibrium, each two-sided platform exhibits *positive* cross-side network effects and *negative* same-side network effects. Namely, if we add one agent (a customer or a worker) to either platform, the utility of other agents from the same side of the market will decrease; conversely, the utility of agents from the opposite side of the market will increase. This extends the framework proposed in Wang and Wang (2016) and Du et al. (2016), which only considers (positive or negative) network effects on the demand side.

We also highlight that the positive and negative effects captured by our model correspond to threshold-type network effects, which will occur only if a certain threshold is attained (i.e., $s_i/d_i > 1$ for the demand side and $d_i/s_i > 1$ for the supply side). As a result, our model generalizes the standard linear network effect model from the traditional two-sided platforms literature (e.g., Rochet and Tirole 2003, Armstrong 2006, Rochet and Tirole 2006). In fact, the extension of our model to endogenous waiting times (see Section 7) can be viewed as a generalization of the network effects to a (quasi-)linear type, because the waiting time of customers who select platform P_i, $\kappa(s_i - d_i)$, is quasi-linear in both the total supply s_i and the total demand d_i of platform P_i. Thus, we are able to extend our results to other models with network effects.

The total sales of P_i are truncated by both demand and supply, that is, $\min\{d_i, s_i\}$. Thus, the profit earned by P_i is given by:

$$\pi_i(p,w) = (p_i - w_i)\min\{d_i, s_i\},$$

where $(p,w) = (p_1, p_2, ..., p_n, w_1, w_2, ..., w_n)$,

$$d_i = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i/d_i\}(q_{ij} - \kappa_j p_i - \nu_j)]}{1 + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]},$$

and

$$s_i = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i/s_i\}(a_{ik} + \eta_k w_i - \omega_k)]}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]}.$$

In the special case of a fixed-commission rate, each platform allocates a fixed proportion $0 < \beta_i < 1$ of the price paid by customers to its workers, that is, $w_i = \beta_i p_i$. In this case, the profit earned by P_i for each $i = 1, 2, ..., n$ can be calculated as

$$\pi_i^c(p) = (p_i - \beta_i p_i)\min\{d_i, s_i\} = (1 - \beta_i) p_i \min\{d_i, s_i\},$$

where $p = (p_1, p_2, ..., p_n)$,

$$d_i = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i/d_i\}(q_{ij} - \kappa_j p_i - \nu_j)]}{1 + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]},$$

and

$$s_i = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i/s_i\}(a_{ik} + \beta_i \eta_k w_i - \omega_k)]}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \beta_{i'} \eta_k w_{i'} - \omega_k)]}.$$

3. Competition Between Two-sided Platforms: Equilibrium Analysis

Recall that the platforms $P_1, P_2, ..., P_n$ compete on both price and wage. More specifically, they engage in a simultaneous game in which P_i sets p_i and w_i to maximize $\pi_i(p,w)$. After observing the platforms' price and wage decisions, customers and workers engage in a game where every agent selects an action (i.e., decide between one of the platforms and the outside option) that best responds to the aggregate decisions of all other players.

In this section, we characterize the equilibrium outcome of this game, which we call the *two-sided competition game*. A strategy profile of both platforms is an equilibrium, if each platform maximizes its own profit given the competitors' strategy, that is,

$$(p_i^*, w_i^*) \in \arg\max_{(p_i, w_i)} \pi_i(p_i, w_i, p_{-i}^*, w_{-i}^*),$$

where (p_{-i}^*, w_{-i}^*) is the equilibrium price and wage vector of all other platforms. We also denote the equilibrium demand and supply of P_i by d_i^* and s_i^*, respectively, where

$$d_i^* = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i^*/d_i^*\}(q_{ij} - \kappa_j p_i^* - \nu_j)]}{1 + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}^*/d_{i'}^*\}(q_{i'j} - \kappa_j p_{i'}^* - \nu_j)]}$$

and

$$s_i^* = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i^*/s_i^*\}(a_{ik} + \eta_k w_i^* - \omega_k)]}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}^*/s_{i'}^*\}(a_{i'k} + \eta_k w_{i'}^* - \omega_k)]}.$$

The following theorem shows that a unique equilibrium exists and that at equilibrium, supply matches with demand.

THEOREM 1. *Consider the above two-sided competition game. Then, the following holds:*

1. *Under equilibrium, supply matches with demand, that is,* $s_i^* = d_i^*$ *for* $i = 1, 2, ..., n$.
2. *The two-sided competition game admits a unique equilibrium* $(p^*, w^*) = (p_1^*, p_2^*, ..., p_n^*, w_1^*, w_2^*, ..., w_n^*)$. *Further, the equilibrium can be computed using a tatônnement scheme.*

In the two-sided competition game, if supply does not match with demand, one can always find a profitable unilateral deviation by increasing the price (when demand exceeds supply) or by decreasing the wage (when supply exceeds demand). See more details in the proof of Theorem 1 in Appendix B. In addition, based on the second part of Theorem 1, the equilibrium can be computed using a *tatônnement* scheme, namely, if each platform uses the best-response strategy based on the price and wage of its competitor in the previous iteration, the sequence of price and wage strategies converge to the unique equilibrium $(p_1^*, p_2^*, ..., p_n^*, w_1^*, w_2^*, ..., w_n^*)$.

When establishing the existence and uniqueness of equilibrium in one-sided competition with logit-type demand models without network effects (e.g., MNL, nested-MNL, and mixed-MNL), existing results in the literature typically leverage the first-order optimality condition (FOC) of the profit function. A common approach is to show that the system of equations that characterizes the FOC has a unique solution (see, e.g., Gallego et al. 2006, Li and Huh 2011, Aksoy-Pierson et al. 2013, Gallego and Wang 2014). In the two-sided competition game with the endogenous choice model considered in this paper, the FOC turns out to be difficult to analyze. This is driven by the fact that the platforms have more flexibility in decisions (price and wage). Furthermore, the total sales of each platform is endogenously determined by the choice behavior of all customers and workers in the market, and is truncated by both demand and supply. As a result, the objective function becomes non-differentiable, making traditional arguments not applicable to our model. To overcome this technical challenge, we exploit the structural properties of the best-response mapping of each platform and prove that it is a contraction mapping. Consequently, a *tatônnement* scheme converges to the unique equilibrium. An important insight of price competition under MNL or nested-MNL models from the literature is the optimality of the so-called equal or adjusted markup policy (see, Li and Huh 2011, Gallego and Wang 2014). This property, however, no longer holds in our two-sided competition setting where demand and supply are endogenous and the platforms have the flexibility to adjust both price and wage.

The proof of Theorem 1 also helps us inductively characterize the desired properties of the *tatônnement* scheme in each iteration, implying that the same properties hold under equilibrium by taking the limit. We next exploit this technique to (i) compare the equilibrium outcome of our two-sided competition game to a monopoly market (i.e., both platforms are owned by a single entity) and (ii) characterize how the equilibrium strategy reacts to real-time demand changes. We denote the prices and wages under the monopolistic setting by $(p^{m*}, w^{m*}) = (p_1^{m*}, p_2^{m*}, ..., p_n^{m*}, w_1^{m*}, w_2^{m*}, ..., w_n^{m*})$, where $(p^{m*}, w^{m*}) = \arg\max_{(p,w)} \sum_{i=1}^{n} \pi_i(p_i, w_i)$.

Proposition 1. *The following comparative statics results hold:*

(a) $p_i^* < p_i^{m*}$ *and* $w_i^* > w_i^{m*}$ *for* $i = 1, 2, ..., n$.

(b) p_i^* *and* w_i^* *are increasing in* Λ_j *for* $i = 1, 2, .., n$ *and* $j = 1, 2, ..., m$.

As stated in Proposition 1(a), in a competitive market, each platform will decrease (resp. increase) its price (resp. wage) to attract customers (resp. workers). Traditionally, it was shown that price competition decreases the price of each firm relative to a monopoly (see, e.g., Li and Huh 2011). Proposition 1(a) generalizes this result to a two-sided market by showing that competition not only decreases the price, but also raises the wage offered by each platform. Once again, the method we use to prove Proposition 1 is different from the typical argument used in the literature. In the literature, the main argument relies on analyzing the FOC (see, e.g., Li and Huh 2011), whereas in our model, we directly exploit the properties of the best response in each iteration of the *tatônnement* scheme. Specifically, we input the monopoly prices and wages as the initial variables of the *tatônnement* scheme and show that the prices (resp. wages) will be lower (resp. higher) relative to the monopolistic setting, for each iteration of the scheme. At the limit of the *tatônnement* scheme, the equilibrium prices and wages under two-sided platform competition will be higher relative to the monopoly.

Consistent with the ride-sharing business practice, Proposition 1(b) suggests that both platforms adopt a surge pricing strategy under equilibrium, that is, they react to real-time peak demand levels by increasing both their price and wage. This result generalizes the well-known optimality of surge pricing for a monopoly (see, e.g., Bai et al. 2019) to a competitive two-sided setting with endogenous supply and demand.

3.1. Fixed-Commission Rate

Platforms often use a fixed-commission rate to pay their workers. Namely, P_i allocates a fixed share $0 < \beta_i < 1$ of the price paid by customers to workers, that is, $w_i = \beta p_i$ (see, e.g., Hu and Zhou 2017), where β is a pre-specified parameter that does not change with the state of the market. For example, for Lyft drivers who applied before 12AM on January 1, 2016, they earn 80% of the passenger's time, distance, and base rates in each trip.[7] In the model with a fixed-commission rate, the equilibrium $(p_1^{c*}, p_2^{c*}, ..., p_n^{c*})$ is defined as follows:

$$p_i^{c*} \in \arg\max_{p_i} \pi_i^c(p_i, p_{-i}^{c*}),$$

where p_{-i}^{c*} is the equilibrium price vector of all other platforms (except P_i) under a fixed-commission rate. We also denote the equilibrium demand and supply of P_i by

$$d_i^{c*} = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i^{c*}/d_i^{c*}\}(q_{ij} - \kappa_j p_i^{c*} - \nu_j)]}{1 + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}^{c*}/d_{i'}^{c*}\}(q_{i'j} - \kappa_j p_{i'}^{c*} - \nu_j)]}$$

and

$$s_i^{c*} = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i^{c*}/s_i^{c*}\}(a_{ik} + \eta_k \beta_i p_i^{c*} - \omega_k)]}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}^{c*}/s_{i'}^{c*}\}(a_{i'k} + \eta_k \beta_{i'} p_{i'}^{c*} - \omega_k)]}.$$

THEOREM 2. *Consider the two-sided competition game under a fixed-commission rate. Then, the following holds:*

(a) Under equilibrium, supply exceeds demand, that is, $s_i^{c*} \geq d_i^{c*}$ for $i = 1, 2, ..., n$.

(b) The game under admits a unique equilibrium $(p_1^{c*}, p_2^{c*}, ..., p_n^{c*})$. Further, the equilibrium can be computed using a tatônnement scheme.

Note that the first part of Theorem 2 is different from Theorem 1. Indeed, under a fixed-commission rate, the platforms have less flexibility in the decision-making process since the wage is tied to the price. Consequently, the argument of finding a profitable unilateral deviation does not hold anymore when $d_i < s_i$. When $d_i > s_i$, by increasing p_i (and thus also $w_i = \beta_i p_i$ and the resulting profit margin $(1-\beta_i)p_i$), it will raise d_i and s_i so that P_i's profit increases. However, when $d_i < s_i$, such an approach does not necessarily increase the platform's profit: by decreasing p_i (and thus w_i and the margin), d_i will increase while s_i will decrease, so that the impact on P_i's profit is not clear. Consequently, as shown in the proof of Theorem 2, the model with a fixed-commission rate requires a different equilibrium analysis to carefully examine the case when $d_i < s_i$.

We next compare the equilibrium outcomes of the base model and those of the model with a fixed-commission rate. For technical tractability, we assume that the model is symmetric, namely, the model primitives are identical for all platforms, all demand segments, and worker types. In particular, the fixed-commission rate is the same across different platforms, denoted as $\beta = \beta_1 = \beta_2 = \cdots = \beta_n$.

PROPOSITION 2. *Assume that Λ_j, κ_j, and ν_j are the same for all j; Γ_k, η_k, and ω_k are the same for all k; q_{ij} are the same for all i and j; a_{ik} are the same for all i and k; and $\beta = \beta_1 = \beta_2 = \cdots = \beta_n$. We then have*

[7] `https://thehub.lyft.com/pay-breakdown/`

(a) For the base model, the equilibrium prices and wages are identical for all platforms, i.e., $p_1^* = p_2^* = \cdots = p_n^*$ *and* $w_1^* = w_2^* = \cdots = w_n^*$*. For the model with a fixed-commission rate, the equilibrium prices are identical for all platforms, i.e.,* $p_1^{c*} = p_2^{c*} = \cdots = p_n^{c*}$.

(b) The equilibrium profit of each platform in the base model is higher relative to the model with a fixed-commission rate, i.e., $\pi_i^* = (p_i^* - w_i^*)d_i^* \geq (1-\beta)p_i^{c*}d_i^{c*} = \pi_i^{c*}$ *for each* i*, where the inequality holds as equality if* $\beta = w_i^*/p_i^*$.

(c) If $\beta < w_i^*/p_i^*$*, it follows that* $p_i^{c*} > p_i^*$, $w_i^{c*} < w_i^*$*, and* $d_i^{c*} < d_i^*$ *for each* i.

(d) If $\beta > w_i^*/p_i^*$*, it follows that* $p_i^{c*} < p_i^*$, $w_i^{c*} > w_i^*$*, and* $d_i^{c*} > d_i^*$ *for each* i.

Proposition 2 shows that, in the symmetric setting, the equilibrium outcomes are also symmetric. Furthermore, due to the additional constraint, the equilibrium profit of each platform without a fixed-commission rate is lower relative to the base model. Depending on the magnitude of the commission rate, adopting a fixed-commission rate may have a different impact on the equilibrium price, wage, and demand of each platform. Specifically, if the commission rate is lower (resp. higher) than the ratio between the equilibrium wage and price from the base model, adopting a fixed-commission rate will result in a higher (resp. lower) price, a lower (resp. higher) wage, and a lower (resp. higher) demand under equilibrium.

3.2. Separate Pools of Workers

To conclude this section, we consider a two-sided competition model where workers come from different labor pools (e.g., Curb's taxi drivers and Via's self-employed drivers). This is different from our base model in which all workers belong to the same pool. In this case, a worker faces only two choices: working for the focal platform or selecting the outside option. As before, both customers and workers follow an endogenous general attraction model. Thus, the total demand and supply for P_i are given by:

$$d_i^s = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i^s/d_i^s\}(q_{ij} - \kappa_j p_i - \nu_j)]}{1 + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}^s/d_{i'}^s\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]}, \quad i = 1, 2, ..., n$$

and

$$s_i^s = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i^s/s_i^s\}(a_{ik} + \eta_k w_i - \omega_k)]}{1 + \exp[\omega_k + \min\{1, d_i^s/s_i^s\}(a_{ik} + \eta_k w_i - \omega_k)]}, \quad i = 1, 2, ..., n.$$

As a corollary of Theorem 1, we show that a unique equilibrium exists for the model with separate pools of workers.[8]

[8] By conducting computational tests, we find that the market equilibrium under separate pools of workers has the following properties relative to a joint pool of workers: (i) The equilibrium volume of transactions and profit of each platform are higher, (ii) the equilibrium price of each platform is lower, and (iii) the equilibrium wage of each platform is lower. These properties are due to the softened competition on the supply side and the cross-side network effects (the computational results are omitted for conciseness).

Corollary 1. *Consider the two-sided competition game with separate pools of workers. Then, the following holds:*

(a) Under equilibrium, supply matches with demand, that is, $s_i^{s*} = d_i^{s*}$ *for* $i = 1, 2, ..., n$.

(b) The game admits a unique equilibrium $(p_1^{s*}, p_2^{s*}, ..., p_n^{s*}, w_1^{s*}, w_2^{s*}, ..., w_n^{s*})$. *Further, the equilibrium can be computed efficiently using a tatônnement scheme.*

4. Coopetition Between Two-sided Platforms: Model

Inspired by recent practices in ride-sharing, we model the setting where a coopetition partnership is introduced via a profit sharing contract between two platforms. In particular, the two competing platforms P_1 and P_2 collaborate and offer a new joint service, which is available to riders from either platform. As mentioned before, one such recent example is the partnership between Curb and Via with the introduction of a taxi sharing service in NYC on June 6, 2017. For the rest of this paper, we use the terms "new joint service," "new service," and "coopetition" interchangeably. Since the coopetition partnership is mainly adopted in the ride-sharing market, we refer to customers as riders and to workers as drivers in the model with coopetition. As is clear from this model setup, P_1 and P_2 are also competing with the other $n-2$ competitors, where we assume $n \geq 3$. The intensity of competition is parameterized by the total number of platforms on the market n.

In this section, our main inspiration is the coopetition partnership between Curb and Via. We use the superscript $\tilde{}$ to denote the different variables in the presence of coopetition. To be consistent with the business practice of Curb (P_1) and Via (P_2), we assume that P_1 and P_2 have separate pools of drivers. We denote the prices of the original services offered by P_1 and P_2, after introducing the new service by $\tilde{p}_1$ and $\tilde{p}_2$. The quality and price of the new service are denoted by $\tilde{q}_x$ and $\tilde{p}_x$, respectively. In addition, we propose to capture the pooling effect of the new service by the parameter $\tilde{n}$. This parameter corresponds to the (average) number of customers per service (i.e., riders per ride) for the new service. If the new joint service does not offer a pooling option (i.e., only private rides), $\tilde{n} = 1$ and otherwise, $\tilde{n} > 1$ (which is the case for the Curb-Via partnership). Since the new service is a combination of the original taxi-hailing (Curb) and carpooling (Via) services, its quality for rider-segment j, q_{xj}, can be interpreted as a convex combination of the qualities of the original services, that is, $q_{xj} = \sigma q_{1j} + (1-\sigma) q_{2j}$ for all $j = 1, 2, ..., m$ (where $\sigma \in [0, 1]$). Furthermore, as in the competition model, the utility earned by a customer or by a worker depends on the decisions of all customers and workers. Namely, in the presence of coopetition, the demand and supply of each service are also endogenous market outcomes.

Inspired by the coopetition partnership between Curb and Via, we assume that the new service is solely provided by P_1's drivers. Nevertheless, our results and insights extend to the situation where the new service is provided by workers from both platforms. Motivated by practice, we assume that Curb's drivers have no choice whether or not to accept requests for the new service. More generally, in most carpooling platforms, drivers need to serve all incoming requests. Moreover, both Curb and Via prioritize the new service in the same as their original services. In the presence of coopetition, the driver pool for P_i $(3 \le i \le n)$ will remain the same as without coopetition.

We use $\tilde{\lambda}_1$ to denote the total demand for P_1's drivers, which is the sum of the demand of its original service and the new service, namely $\tilde{\lambda}_1 := \tilde{d}_1 + \tilde{d}_x/\tilde{n}$, where $\tilde{d}_x$ is the total number of demand requests for the new joint service. We also denote the total demand for P_i's drivers as $\tilde{\lambda}_i = \tilde{d}_i$ for $i = 2, 3, ..., n$. We assume that P_1's drivers are randomly assigned either to the original service or to the new service. More specifically, if $\tilde{s}_1 > \tilde{\lambda}_1$, where $\tilde{s}_1$ is the total number of workers who choose to work for P_1, then P_1 has enough drivers to fulfill all demand requests. Otherwise, we assume that drivers are proportionally allocated to the original service and to the new service (considering random arrivals and a first-come-first-serve allocation rule).[9] Recall that the demand of P_1's original service (resp. new service) is $\tilde{d}_1$ (resp. $\frac{\tilde{d}_x}{\tilde{n}}$). Thus, P_1 will allocate $\tilde{s}_1 \cdot \frac{\tilde{d}_1}{\tilde{\lambda}_1}$ drivers to its original service and $\tilde{s}_x = \tilde{s}_1 \cdot \frac{\tilde{d}_x/\tilde{n}}{\tilde{\lambda}_1}$ drivers to the new service. For notational convenience, we denote $\tilde{N} := \{1, 2, ..., n, x\}$ as the set of services in the presence of coopetition.

As in the competition model, the utility earned by a consumer from choosing one of the services is endogenously determined by the aggregate demand and supply. The expected utility that a rider from segment j derives from P_1's original service is $\tilde{u}_{1j} = \nu_j + \min\{1, \tilde{s}_1 \cdot \frac{\tilde{d}_1}{\tilde{\lambda}_1}/\tilde{d}_1\}(q_{1j} - \kappa_j \tilde{p}_1 - \nu_j) = \nu_j + \min\{1, \tilde{s}_1/\tilde{\lambda}_1\}(q_{1j} - \kappa_j \tilde{p}_1 - \nu_j)$, from P_i's original service is $\tilde{u}_{ij} = \nu_j + \min\{1, \tilde{s}_i/\tilde{d}_i\}(q_{ij} - \kappa_j \tilde{p}_i - \nu_j)$ $(i = 2, ..., n)$, from the new service is $\tilde{u}_{xj} = \nu_j + \min\{1, \tilde{s}_x \tilde{n}/\tilde{d}_x\}(q_{xj} - \kappa_j \tilde{p}_x - \nu_j)$, and from the outside option is $\tilde{u}_0 = 0$. Thus, the demand for each service is given by:

$$\tilde{d}_i = \sum_{j=1}^{m} \tilde{d}_{ij} = \sum_{j=1}^{m} \frac{\Lambda_j \exp(\tilde{u}_{ij})}{\exp(\tilde{u}_0) + \sum_{i' \in \tilde{N}} \exp(\tilde{u}_{i'j})}, \text{ for } i \in \tilde{N}.$$

Consequently, in the presence of coopetition, the demand for P_i's service $(i \in \tilde{N})$ is

$$\tilde{d}_i = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, \tilde{s}_i/\tilde{\lambda}_i\}(q_{ij} - \kappa_j \tilde{p}_i - \nu_j)]}{1 + \sum_{i' \in \tilde{N}} \exp[\nu_j + \min\{1, \tilde{s}_{i'}/\tilde{\lambda}_{i'}\}(q_{i'j} - \kappa_j \tilde{p}_{i'} - \nu_j)]}.$$

Analogously, a type-k driver working for P_i will earn an expected utility of $\tilde{v}_{ik} = \omega_k + \min\{1, \tilde{\lambda}_i/\tilde{s}_i\}(a_{ik} + \eta_k w_i - \omega_k)$ $(i = 1, 2, 3, ..., n,\ k = 1, 2, ..., l)$, whereas the expected utility of the outside option is $\tilde{v}_0 = 0$. Thus, the total supply of P_i's drivers is $(i = 1, 2, ..., n)$

$$\tilde{s}_i = \sum_{k=1}^{l} \frac{\Gamma_k \exp(\tilde{v}_i)}{\exp(\tilde{v}_0) + \exp(\tilde{v}_i)} = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, \tilde{\lambda}_i/\tilde{s}_i\}(a_{ik} + \eta_k w_i - \omega_k)]}{1 + \sum_{i' \in \tilde{N}} \exp[\omega_k + \min\{1, \tilde{\lambda}_{i'}/\tilde{s}_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]}.$$

[9] For analytical tractability and to isolate the effect of interest (i.e., studying the impact of competition and coopetition in two-sided platforms), we do not consider the spatial heterogeneity of riders and drivers. This extension is an interesting avenue for future research.

As before, the utility earned by a worker is endogenously determined by the aggregate demand and supply. We note that the introduction of the new service affects the realized utility of the platforms as well as the perceived wage of the workers. Moreover, the value of $\tilde{n}$ affects the equilibrium price, wage, demand, and supply. We also note that the wage for the new service is the same as the original service. In our motivating example, this follows from the fact that Curb's drivers (who fulfill the new service) are compensated according to the meter price.

We consider a *profit sharing contract* under which P_1 and P_2 split the net profit generated by the new service. More precisely, P_1 receives a fraction $\gamma_1 = \gamma \in (0,1)$ of the profit generated by the new service and P_2 receives $\gamma_2 = 1-\gamma$. For notational consistency, we denote $\gamma_i = 0$ for $i = 3, 4, ..., n$.

Under coopetition, P_i's profit comprises two parts: the profit from its original service and the profit from the new service allocated to P_i. Specifically, P_i's profit from its original service amounts to $(\tilde{p}_i - \tilde{w}_i)\min\{\tilde{d}_i, \frac{\tilde{s}_i\tilde{d}_i}{\tilde{\lambda}_i}\} = (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i \min\left\{1, \frac{\tilde{s}_i}{\tilde{\lambda}_i}\right\}$, and P_i's profit from the new service is $\gamma_i(\tilde{n}\tilde{p}_x - \tilde{w}_1)\min\left\{\frac{\tilde{d}_x}{\tilde{n}}, \tilde{s}_1 \cdot \frac{\tilde{d}_x}{\tilde{n}\tilde{\lambda}_1}\right\} = \gamma_i\left(\tilde{p}_x - \frac{\tilde{w}_1}{\tilde{n}}\right)\tilde{d}_x \min\left\{1, \frac{\tilde{s}_1}{\tilde{\lambda}_1}\right\}$. Putting everything together, the expression for the total profit earned by P_i is given by:

$$\tilde{\pi}_i(\tilde{p}_i, \tilde{w}_i, \tilde{p}_{-i}, \tilde{w}_{-i}) = (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i \min\left\{1, \frac{\tilde{s}_i}{\tilde{\lambda}_i}\right\} + \gamma_i\left(\tilde{p}_x - \frac{\tilde{w}_1}{\tilde{n}}\right)\tilde{d}_x \min\left\{1, \frac{\tilde{s}_1}{\tilde{\lambda}_1}\right\}, \text{ for } i = 1, 2, ..., n.$$

5. Impact of Coopetition

In this section, we analyze the impact of coopetition (i.e., introducing the new joint service) based on the model proposed in Section 4. We first consider the profit implications on both platforms, and then examine the impact on riders and on drivers.

We first show that even in the presence of coopetition, there still exists a unique equilibrium. The sequence of events unfolds as follows:

1. The platforms agree upon the price of the new service $\tilde{p}_x$ and the profit-sharing parameter γ (see more details below).

2. Given $(\tilde{p}_x, \gamma)$, each of the n platforms simultaneously decides the price and wage of its original service $\tilde{p}_i$ and $\tilde{w}_i$ to maximize its own profit ($i = 1, 2, ..., n$).

Using backward induction, we start by characterizing the equilibrium of the second step. Given $(\tilde{p}_x, \gamma)$, the platforms engage in a price and wage competition using the model presented in Section 4. As before, the equilibrium outcome $(\tilde{p}^*, \tilde{w}^*) = (\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*, ..., \tilde{p}_n^*, \tilde{w}_n^*)$ should satisfy $(\tilde{p}_i^*, \tilde{w}_i^*) \in \arg\max_{(p_i, w_i)} \tilde{\pi}_i(p_i, w_i, p_{-i}^*, w_{-i}^*)$ for each $i = 1, 2, ..., n$. We next extend the result on existence and uniqueness of equilibrium in the presence of coopetition. Recall that the supply and demand of P_i are denoted by $\tilde{s}_i$ and $\tilde{\lambda}_i$, respectively.

THEOREM 3. *Consider the two-sided competition game in the presence of coopetition. Then, the following holds:*

1. *Under equilibrium, supply matches with demand for each platform, that is,* $\tilde{s}_i^* = \tilde{\lambda}_i^*$ *for* $i = 1, 2, ..., n$.

2. *For any* $(\tilde{p}_x, \gamma)$*, there exists a unique equilibrium* $(\tilde{p}^*, \tilde{w}^*)$ *that can be computed using a tatônnement scheme.*

Note that when $\tilde{p}_x \uparrow +\infty$, the model with coopetition converges to the original model (without coopetition) for any $\gamma \in (0,1)$, because the demand for the new joint service shrinks to 0 in this case. We also remark that one can construct examples of coopetition partnerships that are detrimental to both partnering platforms. In other words, if the platforms do not carefully decide $\tilde{p}_x$ and γ, introducing the new service may lead to an undesirable lose-lose outcome for the two platforms that provide shared services.

5.1. Impact on Platforms' Profits

At a high level, the coopetition will induce three effects: (i) a new market share effect (i.e., capturing new riders who were previously choosing the outside option), (ii) a cannibalization effect (i.e., losing some existing market share to the new service), and (iii) a wage variation (i.e., adapting the wage to match supply with demand). Our goal is to study how the two focal platforms (which engage in a coopetition partnership) could use well-designed profit sharing contracts to balance these effects and ultimately benefit from coopetition. The other competing platforms, as expected, will be worse off in the presence of the new joint service. We first show that when the price of the new service $\tilde{p}_x$ and the profit-sharing parameter γ are carefully chosen, coopetition will increase the profits of both focal platforms (i.e., P_1 and P_2). To unlock the fullest potential of coopetition, we consider the case where $\tilde{p}_x$ is jointly set by both platforms to maximize the total profit, that is,

$$\tilde{p}_x^* \in \arg\max_{\tilde{p}_x} \{\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*) + \tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*)\}.$$

Note that, since the equilibrium outcome $(\tilde{p}^*, \tilde{w}^*)$ depends on γ, then $\tilde{p}_x^*$ will also depend on γ.

Theorem 4. *If* $\tilde{p}_x$ *is set to maximize the total profit of* P_1 *and* P_2*, then there exists an interval* $(\underline{\gamma}, \bar{\gamma}) \subset (0,1)$ *such that if* $\gamma \in (\underline{\gamma}, \bar{\gamma})$, $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) > \pi_i(p^*, w^*)$ *for* $i = 1, 2$. *In addition, for any* $(\tilde{p}_x, \gamma)$*, we have* $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) < \pi_i(p^*, w^*)$ *for* $i = 3, 4, ..., n$.

Theorem 4 implies that in the presence of coopetition, P_1 and P_2 can set $\tilde{p}_x$ and γ so that introducing the new service will lead to a profit increase for each platform. As mentioned before, when the terms of the coopetition (i.e., $\tilde{p}_x$ and γ) are not carefully designed, introducing the new service can yield lower profits for each platform. Recall that the coopetition induces three effects: (i) a new market share, (ii) an adverse cannibalization, and (iii) a wage variation. Theorem 4 shows that under a well-designed profit sharing contract, the new market share effect dominates the cannibalization and wage variation effects for each platform. We will discuss in greater detail the implications of these effects later in this section. The other competing platforms P_i $(i = 3, 4, ..., n)$ will always be worse off with the introduction of the new service jointly offered by P_1 and P_2, since they do not benefit from the new market share effect but suffer from the cannibalization induced by the new joint service.

We next elaborate on how $(\tilde{p}_x, \gamma)$ can be determined. In practice (e.g., the Curb-Via partnership), P_1 and P_2 negotiate to decide the values of $\tilde{p}_x$ and γ. To model the negotiation process, we use the Nash bargaining framework (see, e.g., Nash Jr 1950, Osborne and Rubinstein 1990). We define $\theta_i \in (0,1)$ as the bargaining power of P_i $(i = 1, 2)$ with $\theta_1 + \theta_2 = 1$. Then, the equilibrium price and profit-sharing parameter $(\tilde{p}_x^{**}, \gamma^{**})$ satisfy the following:

$$\begin{aligned} (\tilde{p}_x^{**}, \gamma^{**}) \in & \operatorname*{arg\,max}_{\tilde{p}_x, \gamma \in (0,1)} [\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*) - \pi_1(p^*, w^*)]^{\theta_1} \cdot [\tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*) - \pi_2(p^*, w^*)]^{\theta_2} \\ & s.t.\ \tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) \geq \pi_i(p^*, w^*) \text{ for } i = 1, 2. \end{aligned} \tag{3}$$

We know from Theorem 4 that there exists $(\tilde{p}_x, \gamma)$ such that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) > \pi_i(p^*, w^*)$ for $i = 1, 2$. Thus, for any (θ_1, θ_2), the parameters $(\tilde{p}_x^{**}, \gamma^{**})$ are well defined. We next show that the equilibrium profit of each platform under Nash bargaining increases under coopetition.

Proposition 3. *Under Nash bargaining, that is, when the platforms set* $(\tilde{p}_x^{**}, \gamma^{**})$, *we have* $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) > \pi_i(p^*, w^*)$ *for* $i = 1, 2$. *Furthermore,* $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) < \pi_i(p^*, w^*)$ *for* $i = 3, 4, ..., n$.

So far, we have shown that the focal platforms can set the price of the new service and the profit-sharing parameter to ensure that coopetition is beneficial. Nevertheless, we are interested in avoiding extreme cases and in identifying conditions under which coopetition yields a strict benefit for both platforms in the presence of a price constraint for the new service. For instance, it is always possible to set $\tilde{p}_x$ to a large value, so that no customer will opt for the new service, and we are back to the original setting. We next assume that the value of $\tilde{p}_x$ is bounded (i.e., the platforms cannot set the price of the new service to an arbitrarily high value). The following proposition shows that when the value of $\tilde{p}_x$ is bounded, both platforms will be strictly better off only when the demand-supply ratio is not too high. Specifically, we denote $\Lambda_j = \Lambda \cdot \tau_j$ for each customer segment j, where $\tau_j > 0$ is the market proportion of segment j with $\sum_{j=1}^{m} \tau_j = 1$. Analogously, we denote $\Gamma_k = \Gamma \cdot \xi_k$ for each driver segment k, where $\xi_k > 0$ is the proportion of drivers from segment k with $\sum_{k=1}^{l} \xi_k = 1$. We then define $r := \Lambda / \Gamma$ as the demand-supply ratio of the market.

Proposition 4. *The following statements hold:*

1. *If* $r \uparrow +\infty$, *then* $\tilde{p}_x^* \uparrow +\infty$ *and* $\tilde{p}_x^{**} \uparrow +\infty$.

2. *Assume that the price of the new service is bounded, that is,* $\tilde{p}_x \leq \bar{p}$ *for some* $\bar{p} < +\infty$. *Then, there exists a threshold* $\bar{r}(\bar{p})$ *such that (i) if* $\bar{p}$ *is sufficiently large and* $r < \bar{r}(\bar{p})$, *then, for each* $i = 1, 2$, $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) > \pi_i(p^*, w^*)$ *under some* $(\tilde{p}_x, \gamma)$ *with* $\tilde{p}_x \leq \bar{p}$, *and (ii) if* $r > \bar{r}(\bar{p})$, *then* $\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*) < \pi_1(p^*, w^*)$ *or* $\tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*) < \pi_2(p^*, w^*)$ *under any* $(\tilde{p}_x, \gamma)$ *with* $\tilde{p}_x \leq \bar{p}$.

Proposition 4 shows that, when $\tilde{p}_x$ is bounded, the demand-supply ratio has a critical implication on the impact of coopetition on the platforms' profits. Specifically, if the demand-supply ratio is not too high (i.e., $r < \bar{r}(\bar{p})$), the platforms can design a profit sharing contract (by setting $\tilde{p}_x$ and γ) that will make the coopetition partnership strictly beneficial for both platforms (i.e., a Pareto improvement in the profits of both partnering platforms). However, if the demand-supply ratio becomes too high, at least one platform will be hurt by coopetition (assuming $\tilde{p}_x$ is bounded). In this case, introducing the new service will make (at least) one of the platforms over-demanded. This will in turn induce the platform(s) to increase their wage, and hence reduce profit. On the other hand, when the demand-supply ratio is not too high, introducing the new service expands the market share of both platforms, thus increasing revenues without imposing high additional wages. Proposition 4 provides evidence on why dominating ride-sharing platforms in the U.S. market, such as Uber and Lyft, do not engage in a coopetition partnership to offer joint services. Indeed, these platforms typically have sufficient customers but insufficient drivers,[10] and thus by Proposition 4(b), coopetition would decrease the profit of at least one platform.

We next study how the market competition structure affects the value of coopetition. We show that both focal platforms will strictly benefit from coopetition if they are facing intensive competition on the demand side (which is quantified by the attractiveness of the platforms to the riders). On the other hand, if the competition is intensive on the supply side (i.e., the platforms are highly attractive to the drivers), then either P_1 or P_2 will be worse off with the new joint service. For simplicity, we assume that $n=3$ to derive Proposition 5 (i.e., there is only one additional platform P_3 that competes with the partnering platforms P_1 and P_2). Furthermore, we denote $q_{3j} = q_3 \iota_j$ for all rider segments $j = 1,2,...,m$, and $a_{3k} = a_3\psi_k$ for all driver types $k = 1,2,...,l$. This facilitates us to parameterize the demand-side competition intensity with q_3, and the supply-side competition intensity with a_3.

PROPOSITION 5. *Assume that $n=3$ and that the price of the new service is bounded (i.e., $\tilde{p}_x \le \bar{p}$ for some $\bar{p} < +\infty$). Then, there exist a threshold $\bar{q}_3(\bar{p})$ for q_3 and a threshold $\bar{a}_3(\bar{p})$ for a_3 such that*

1. *If $q_3 > \bar{q}_3(\bar{p})$, then $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) > \pi_i(p^*, w^*)$ for $i = 1,2$ under some properly chosen $(\tilde{p}_x, \gamma)$.*
2. *If $a_3 > \bar{a}_3(\bar{p})$, then either $\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*) < \pi_1(p^*, w^*)$ or $\tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*) < \pi_2(p^*, w^*)$ under any $(\tilde{p}_x, \gamma)$.*

Proposition 5 shows that, when the price of the new service is bounded, the market competition intensity plays a critical role on the impact of coopetition on the focal platforms' profits. Specifically, if the market competition on the demand side is intensive (i.e., $q_3 > \bar{q}_3(\bar{p})$), then the focal platforms P_1 and P_2 can design a profit sharing contract (by setting $\tilde{p}_x$ and γ) that will make the coopetition partnership strictly beneficial for both platforms (i.e., a Pareto improvement in both profits). In

[10] `https://www.businessinsider.com/uber-lyft-fares-price-driver-shortage-travel-ride-hailing-app-2021-6`.

this case, introducing the new service would expand the market share of both platforms, thus increasing revenues without imposing high additional wages. However, if the market competition is intensive on the supply side, then at least one platform will be hurt by the coopetition. In this case, introducing the new service will make (at least) one of the platforms over-demanded. This will in turn induce the platform(s) to increase their wage, and hence reduce the profit. This insight has also been confirmed and illustrated in our numerical study in Section 6 (Figures 3 and 4).

We next revisit the three effects induced by the coopetition on all three platforms and discuss how a well-designed profit sharing contract can help balance these effects to benefit the focal platforms.

New market share. The new service may attract customers who would otherwise leave the market. Mathematically, the new market share effect for P_i $(i=1,2)$ can be quantified as the profit portion generated by the new service that is allocated to P_i, namely, $\frac{\gamma_i \tilde{n} \tilde{p}_x \tilde{d}_x^*}{\tilde{n}} = \gamma_i \tilde{p}_x \tilde{d}_x^*$. It can be shown that, under equilibrium, the new market share effect for P_i is increasing in γ_i for $i=1,2$.

Cannibalization. Introducing the new service will cannibalize demand since customers may switch from the original services to the new one. The cannibalization effect for P_i is captured by $\tilde{p}_i^* \tilde{d}_i^* - p_i^* d_i^*$ $(i=1,2,3)$, which is always negative (i.e., decreasing the profits from the original services) unless $\tilde{p}_x = +\infty$. We can show that the cannibalization effect will gradually shrink to 0 as $\tilde{p}_x \uparrow +\infty$, as expected.

Wage variation. To match supply with demand in the presence of coopetition (which is the equilibrium condition), the platforms will adjust their wages. While the new market share effect is beneficial and the cannibalization effect is harmful to the platforms, the wage variation effect may go either way. Specifically, the wage variation for P_i amounts to $\tilde{w}_i^* \tilde{d}_i^* + \gamma_i \tilde{w}_1^* \frac{\tilde{d}_x^*}{\tilde{n}} - w_i^* d_i^*$. We can show that the wage variation effect shrinks to 0 as $\tilde{p}_x \uparrow +\infty$ and is strengthened as γ_i increases.

To summarize, our results show that a well-designed profit sharing contract can successfully balance these effects and lead to an overall positive benefit for the focal platforms, regardless of whether the coopetition parameters (p_x, γ) are jointly set by both platforms or determined through bargaining. The other competing platform P_3, however, does not capture any benefit from the new market share of the joint service, but its original service gets cannibalized. As a result, P_3 is always worse off in the presence of a coopetition partnership between its competitors. We next turn our attention to the impact of coopetition on riders and on drivers.

5.2. Surpluses of Riders and Drivers

We investigate the impact of coopetition on riders and drivers. Note that the surpluses of riders and drivers are not (explicitly) dependent on the profit-sharing parameter γ. We use RS to denote the expected rider surplus of the benchmark setting (i.e., without coopetition):

$$RS = \sum_{j=1}^{m} \frac{\Lambda_j}{\kappa_j} \log\left(1 + \sum_{i=1}^{n} \exp(\nu_j + \min\{1, s_i/d_i\}[q_{ij} - \kappa_j p_i - \nu_j)]\right).$$

Analogously, we let $\tilde{RS}$ denote the expected rider surplus after introducing the new service:

$$\tilde{RS}=\sum_{j=1}^{m}\frac{\Lambda_j}{\kappa_j}\log\Big(1+\exp[\nu_j+\min\{1,\tilde{s}_x\tilde{n}/\tilde{d}_x\}(q_{xj}-\kappa_j\tilde{p}_x-\nu_j)]+\sum_{i=1}^{n}\exp[\nu_j+\min\{1,\tilde{s}_i/\tilde{\lambda}_i\}(q_{ij}-\kappa_j\tilde{p}_i-\nu_j)]\Big).$$

For more details on consumer surplus under MNL-type models and on the derivation of the above expressions, see, e.g., Chapter 3.5 of Train (2009). We use $\tilde{RS}^*$ and RS^* to denote the equilibrium rider surplus with and without coopetition, respectively.

Proposition 6. *For any $(\tilde{p}_x,\gamma)$, $\tilde{RS}^*>RS^*$.*

Proposition 6 shows that introducing the new service will increase the expected rider surplus, regardless of the price of the new service and of the profit-sharing parameter. This result is expected given that riders can now enjoy an additional alternative for service.

We next examine the impact of coopetition on drivers, which appears to be more subtle. Since P_1 and P_2 have separate pools of drivers, we need to evaluate the effect of coopetition separately. Let DS_i denote the surplus of P_i's drivers before the coopetition partnership $(i=1,2,...,n)$:

$$DS_i=\sum_{k=1}^{l}\frac{\Gamma_k}{\eta_k}\log\Big(1+\exp[\omega_k+\min\{1,d_i/s_i\}(a_{ik}+\eta_k w_i-\omega_k)]\Big),\ i=1,2,...,n.$$

Analogously, the surplus of P_i's drivers with coopetition is given by:

$$\tilde{DS}_i=\sum_{k=1}^{l}\frac{\Gamma_k}{\eta_k}\log\Big(1+\exp[\omega_k+\min\{1,\tilde{\lambda}_i/\tilde{s}_i\}(a_{ik}+\eta_k\tilde{w}_i-\omega_k)]\Big),\ i=1,2,...,n.$$

Finally, $\tilde{DS}_i^*$ and DS_i^* denote the equilibrium surplus of P_i's drivers with and without coopetition, respectively.

Proposition 7. *For any $(\tilde{p}_x,\gamma)$, the following holds:*

1. *There exists a threshold $\bar{n}_d>1$ such that $\tilde{DS}_1^*>DS_1^*$ if and only if $\tilde{n}<\bar{n}_d$.*
2. *$\tilde{DS}_i^*<DS_i^*$ for all $i=2,3,...,n$.*

As shown in Proposition 7, P_1's drivers may not necessarily benefit from coopetition. When the average number of riders per trip for the new service is not too high (i.e., $\tilde{n}<\bar{n}_d$), there exist profit-sharing contracts that will strictly benefit P_1's drivers. Indeed, when $\tilde{n}$ is small, the platform needs to increase its wage to attract additional P_1's drivers to satisfy the demand for the new service. When $\tilde{n}$ is large, however, P_1's drivers will be worse off in the presence of coopetition. In this case, fewer drivers are needed, so that the platform can reduce its wage. This finding explains partially why several coopetition partnerships either have no carpooling option for the new service (i.e., $\tilde{n}=1$) or impose a restriction on the number of riders per trip. For example, in the case of Curb and Via, the platforms imposed a limit of at most two riders who can share a ride for the new taxi-sharing service (i.e., $\tilde{n}\le 2$). Note that when $\tilde{n}=1$, P_1's drivers will always benefit from coopetition. Proposition 7 also shows that introducing the new service will always decrease the surplus of the drivers from the platforms that do not offer the new joint service (i.e., P_2, P_3,..., P_n). This follows from the fact that the drivers from any platform except P_1 are directly affected by the market share reduction ($\tilde{s}_i^* < s_i^*$ for $i \neq 1$) induced by the cannibalization effect.

We next propose a simple and realistic way to address the issue that some drivers working for P_1 and P_2 may be hurt by coopetition. In particular, the platforms P_1 and P_2 can reallocate some of their profit gains to their drivers. In practice, the incentives can be provided to the drivers through promotions, bonuses, or other monetary compensations. For example, the platform Grab started subsidizing its trip fares on June 19, 2018, to boost driver earnings.[11] In fact, bonuses are widely used in practice as a competitive lever in two-sided platforms (see, e.g., Allon et al. 2018, Chen et al. 2020). We denote the total surplus (i.e., platform and drivers) of P_i with and without coopetition as $\tilde{\pi}_i + \tilde{DS}_i$ and $\pi_i + DS_i$, respectively. We next show that the platforms can reach an agreement on $(\tilde{p}_x, \gamma)$ that will guarantee a strict total surplus gain for each platform.

PROPOSITION 8. *For any $\tilde{n}$, there exist $(\tilde{p}_x, \gamma)$ such that under equilibrium, $\tilde{\pi}_i^* + \tilde{DS}_i^* > \pi_i^* + DS_i^*$ $(i = 1, 2)$. However, for any $(\tilde{p}_x, \gamma)$, $\tilde{\pi}_i^* + \tilde{DS}_i^* < \pi_i^* + DS_i^*$ $(i = 3, 4, ..., n)$.*

Proposition 8 shows that under a well-designed profit sharing contract, the coopetition partnership between P_1 and P_2 can strictly increase the total surplus (i.e., the sum of the platform's profit and the driver surplus) of each platform. Consequently, if each platform redistributes a portion of its profit gain to its drivers (e.g., by offering bonuses), then both platforms and all their drivers will be better off. This will also result in more drivers joining each platform. Ultimately, the platforms can design a profit sharing contract $(\tilde{p}_x, \gamma)$ that will benefit all the stakeholders of P_1 and P_2 (i.e., both platforms, riders, and drivers). We note that since both the platform's profit and the driver surplus for P_i $(i = 3, 4, ..., n)$ are lower in the presence of the coopetition partnership between P_1 and P_2, a similar strategy to redistribute the total surplus to its drivers cannot be used by the other platforms.

6. Computational Experiments

In this section, we investigate computationally how the competition intensity on either side of the market impacts the platforms' equilibrium profits under competition, as well as the benefit of coopetition. As in Proposition 5, we consider a market composed of three competing platforms. We parameterize the competition intensity faced by the focal platforms (i.e., P_1 and P_2) on the demand side using the quality of the third platform P_3 (i.e., q_3). Similarly, we parameterize the competition intensity on the supply side using the attractiveness of P_3 (i.e., a_3). By conducting

[11] `https://www.rappler.com/business/205322-grab-philippines-drivers-subsidy-fares-earnings`

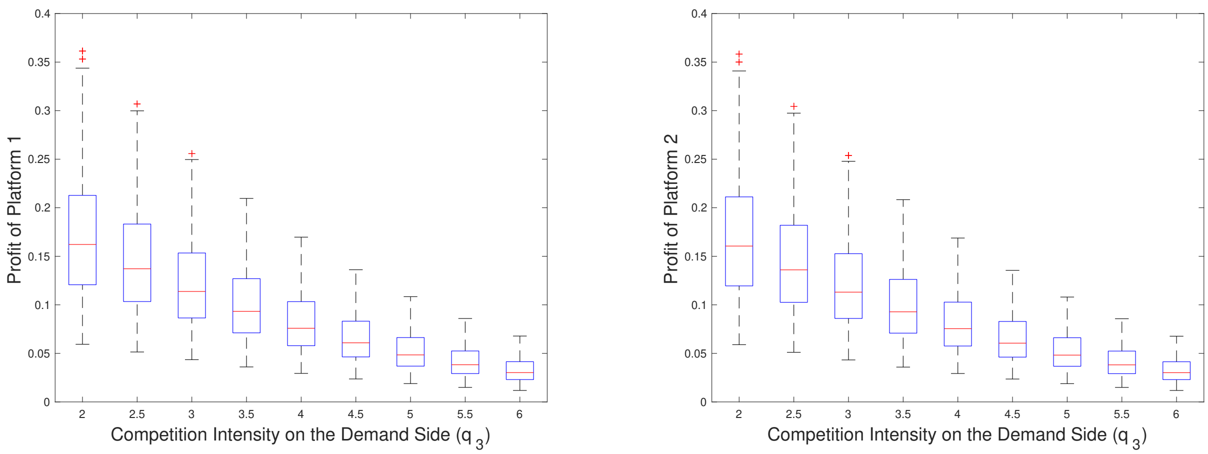
a sensitivity analysis with respect to q_3 and a_3, we can isolate the effect of demand- and supply-side competition intensities on the equilibrium profit as well as on the profit improvement from coopetition for the focal platforms.

Our numerical experiments cover a wide range of model primitives, hence ensuring that our results and insights are robust. Specifically, we vary the platform quality parameters q_1 and q_2 in the set $\{0.5, 1, 1.5\}$, and the platform attractiveness parameters a_1 and a_2 in $\{-1.5, -1, -0.5\}$. Thus, the quality and attractiveness of P_1 may be either higher or lower than P_2. We normalize the total mass of workers to $\Gamma = 1$ and vary the total mass of customers Λ in $\{0.5, 1, 1.5, 2\}$. The quality of the new joint service is given by $q_x = (q_1 + q_2)/2$ for each instance, whereas the average number of customers per service for the new service under coopetition, $\tilde{n}$, takes a value in $\{1.2, 1.5, 1.8\}$. Following the analysis in Propositions 4 and 5, we set the upper bound for the price of the new service as $\bar{p} = 3$. Therefore, there are a total of $3 \times 3 \times 3 \times 3 \times 4 \times 3 = 972$ parameter combinations examined in our numerical experiments.

We base our numerical experiments on a setting with two customer segments and two worker types. For the first customer segment, their platform quality sensitivity is $\iota_1 = 1$ (i.e., the perceived quality of the first customer segment for P_i is $q_{i1} = q_i \iota_1 = q_i$ for $i = 1, 2, 3$) and their price sensitivity is $\kappa_1 = 1$. We set the market size of the first segment as $\Lambda_1 = 0.25\Lambda$, which depends on the total mass of customers Λ. For the second customer segment, their quality sensitivity is $\iota_2 = 0.8$ (i.e., the perceived quality of the second customer segment for P_i is $q_{i2} = q_i \iota_2 = 0.8 q_i$ for $i = 1, 2, 3$) and their price sensitivity is $\kappa_2 = 0.75$. The market size of the second segment is $\Lambda_2 = \Lambda - \Lambda_1 = 0.75\Lambda$. For the first worker type, their platform attractiveness sensitivity is $\psi_1 = 1$ (i.e., the perceived attractiveness of the first worker type for P_i is $a_{i1} = a_i \psi_1 = a_i$ for $i = 1, 2, 3$) and their wage sensitivity is $\eta_1 = 1$. The total supply of the first worker type is $\Gamma_1 = 0.375\Gamma = 0.375$. For the second worker type, their platform attractiveness sensitivity is $\psi_2 = 0.85$ (i.e., the perceived attractiveness of the second worker type for P_i is $a_{i2} = a_i \psi_2 = 0.85 a_i$ for $i = 1, 2, 3$) and their wage sensitivity is $\eta_2 = 1.3$. The total mass of the second worker type is $\Gamma_2 = \Gamma - \Gamma_2 = 0.625$. We highlight that the set of parameters we consider in this section encompasses a wide range of realistic instances and hence, it allows us to quantify the practical impact of the coopetition partnership.

For all problem instances, we first evaluate the profits of P_1 and P_2 under equilibrium without the coopetition partnership. For the setting with coopetition, we assume that the price of the new service is set to maximize the total profit of P_1 and P_2 (i.e., $\tilde{p}_x^*$). Without loss of generality, we assume that P_1 and P_2 equally share the profit of the new joint service (i.e., $\gamma = 0.5$).

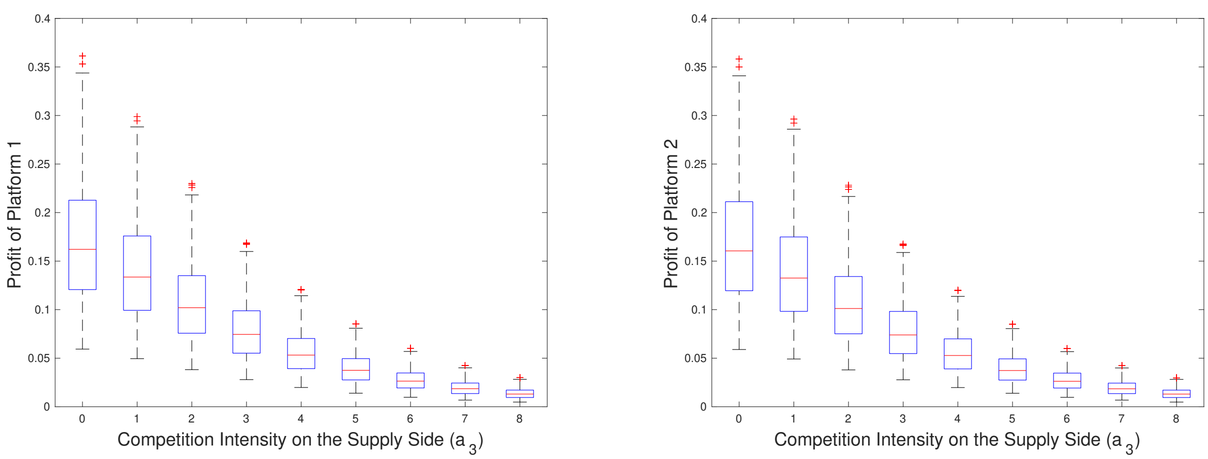
Our first set of results quantify the impact of competition intensities from both sides of the market on the equilibrium profits of the focal platforms (P_1 and P_2). Specifically, Figures 1(a) and 1(b) present the box plots of the equilibrium profits of P_1 and P_2 under different values of q_3



(a) Equilibrium profit of P_1 without coopetition (b) Equilibrium profit of P_2 without coopetition

Figure 1 Impact of demand-side competition intensity on equilibrium profits ($a_3 = 0$).

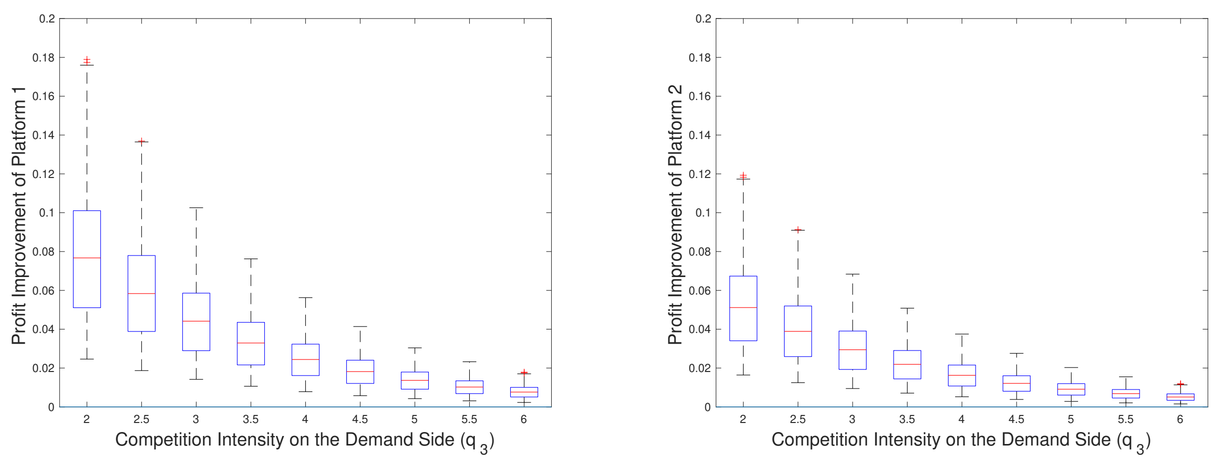
for all 972 problem instances. It is clear from Figure 1 that when the competition on the demand side becomes more intense, the equilibrium profits of P_1 and P_2 decrease. In fact, for each problem instance, the profit of both P_1 and P_2 decrease with q_3. A similar insight also applies to the supply-side competition. Figures 2(a) and 1(b) illustrate the box plots of the equilibrium profits of P_1 and P_2 under different worker attractions of their competing platform P_3. As for demand-side competition, we find that a more intensive competition on the supply side will decrease the equilibrium profits of both P_1 and P_2.



(a) Equilibrium profit of P_1 without coopetition (b) Equilibrium profit of P_2 without coopetition

Figure 2 Impact of supply-side competition intensity on equilibrium profits ($q_3 = 2$).

We next examine the benefit of coopetition for the focal platforms. Our numerical results complement our theoretical analysis (Proposition 5) and confirm the finding that the coopetition benefits the focal platforms under intensive competition on the demand side, but may be detrimental when the competition on the supply side is intensive. Figure 3 (resp. Figure 4) shows the box plots of the profit improvements for P_1 and P_2 under different competition intensities on the demand side (resp. supply side). Comparing Figures 3 and 4 reveals a contrasting effect of the competition intensity on both sides of the market. More precisely, coopetition always improves the profits of the focal platforms regardless of the competition intensity on the demand side, whereas coopetition may be detrimental to the platforms when the competition on the supply side is too intense. As discussed after Proposition 5, when the competition on the demand side is intensive, introducing a new joint service will help the focal platforms expand their market shares. In this case, the new market share can be covered by P_1's workers without offering a high additional wage. Ultimately, the coopetition partnership will benefit both platforms. When the competition on the supply side is intensive, however, P_1 has to offer a high additional wage to attract enough workers to satisfy the additional demand. If the price of the new service is upper bounded, the coopetition partnership may lead to a negative profit margin, thus making both platforms worse-off. In closing, we remark that the above insight holds when the price of the new joint service is upper bounded by $\bar{p} < +\infty$. Otherwise, as we have shown in Theorem 4, the focal platforms that are engaged in coopetition can always set the price of the new service high enough to avoid the negative effect of coopetition.



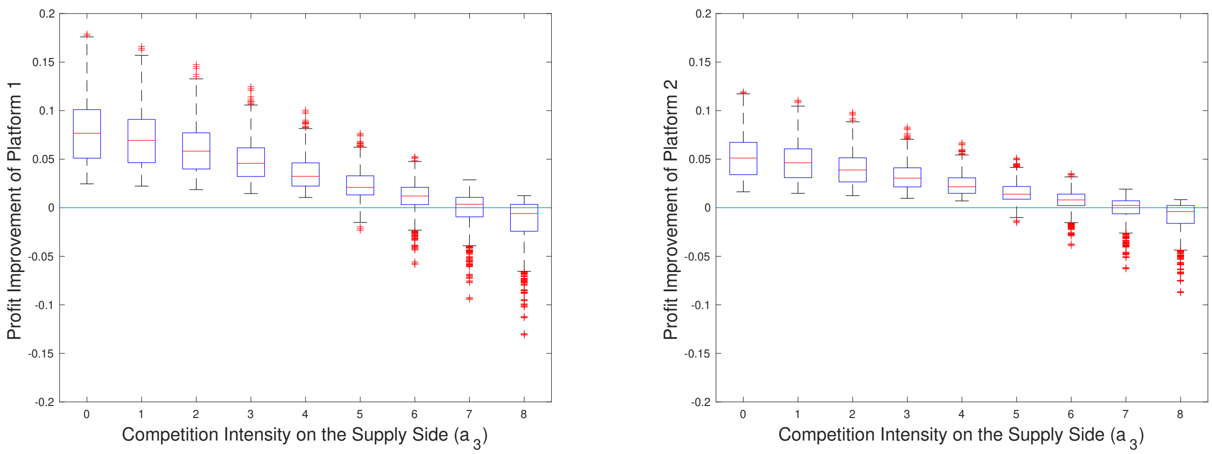
(a) Profit improvement of coopetition for P_1 (b) Profit improvement of coopetition for P_2

Figure 3 Impact of demand-side competition intensity on the profit improvement of coopetition ($a_3 = 0$).

7. Extension: Endogenous Waiting Times

We extend our model by explicitly considering a key feature of ride-sharing platforms: the waiting time experienced by riders. We assume that the expected waiting time depends on the number of available drivers. Specifically, the expected waiting time for P_i (without coopetition) is given by:

$$T_i = \kappa(s_i - d_i),$$



(a) Profit improvement of coopetition for P_1 (b) Profit improvement of coopetition for P_2

Figure 4 Impact of supply-side competition intensity on equilibrium profit ($q_3 = 2$).

where $s_i - d_i$ is the number of available (or idle) drivers and $\kappa(\cdot) > 0$ is a strictly decreasing and convex function on $(0, +\infty)$ with $\lim_{x \downarrow 0} \kappa(x) = +\infty$ and $\lim_{x \uparrow +\infty} \kappa(x) = 0$.[12] Note that this includes as special cases the $M/M/k$ queuing system and the situation where idle drivers are uniformly distributed on a circle so that the expected travel time to pick up a new rider is $c/(s_i - d_i)$ for some constant $c > 0$. Similar modeling approaches have been used in the literature on ride-sharing platforms (see, e.g., Nikzad 2017, Bai et al. 2019, Benjaafar et al. 2021).

Following a similar approach as Cachon and Harker (2002), we assume that the platforms compete on the *total price*, $f_i = p_i + g_i$, where g_i is the operational performance of P_i, which we define as $g_i = T_i$. Hence, the actual price charged by P_i to its riders is $p_i = f_i - g_i = f_i - \kappa(s_i - d_i)$.

Since $\kappa(\cdot)$ satisfies $\lim_{x \downarrow 0} \kappa(x) = +\infty$, we must have $s_i > d_i$ under equilibrium, that is, $\min\{s_i, d_i\} = d_i$. Thus, the profit earned by P_i when waiting times are endogenous is

$$\pi_i^e(f, w) = [f_i - \kappa(s_i - d_i) - w_i] d_i, \text{ for } i = 1, 2, ..., n,$$

where $d_i = \sum_{j=1}^m \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i/d_i\}(q_i - \kappa_j f_i - \nu_j)]}{1 + \sum_{i'=1}^n \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'} - \kappa_j f_{i'} - \nu_j)]} = \sum_{j=1}^m \frac{\Lambda_j \exp(q_i - \kappa_j f_i)}{1 + \sum_{i'=1}^n \exp(q_{i'} - \kappa_j f_{i'})}$, $s_i = \sum_{k=1}^l \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i/s_i\}(a_{ik} + \eta_k w_i - \omega_k)]}{1 + \sum_{i'=1}^n \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]}$, and (s_i, d_i) is such that $s_i > d_i$. An equilibrium (f^{e*}, w^{e*}) should then satisfy

$$(f_i^{e*}, w_i^{e*}) \in \arg\max_{(f_i, w_i)} \pi_i^e(f_i, w_i, f_{-i}^{e*}, w_{-i}^{e*}).$$

We next extend Theorem 1 to the setting with endogenous waiting times.

Theorem 5. *The two-sided competition game with endogenous waiting times admits a unique equilibrium (f^{e*}, w^{e*}) that can be computed using a tatônnement scheme.*

[12] The expected waiting time is defined only when $s_i > d_i$, as otherwise, the system is not stable.

As in the original setting, we can study the impact of coopetition for the model with endogenous waiting times. We denote the total prices of the original services offered by P_1 and P_2 after introducing the new service by $\tilde{f}_1$ and $\tilde{f}_2$, respectively. We also denote by $\tilde{f}_x$ the total price of the new service. Under coopetition, P_i's demand $(i=1,2)$ is $\tilde{d}_i = \sum_{j=1}^m \tilde{d}'_{ij}$, where (since $\tilde{d}_i < \tilde{s}_i$ for each i)

$$\tilde{d}'_{ij} = \frac{\Lambda_j \exp(q_i - \kappa_j \tilde{f}_i)}{1 + \sum_{i'=1}^n \exp(q_{i'} - \kappa_j p_{i'}) + \exp(q_x - \kappa_j \tilde{f}_x)}.$$

Recall that the total number of requests for P_1 and P_2 drivers are $\tilde{\lambda}_1 = \tilde{d}_1 + \tilde{d}_x/\tilde{n}$ and $\tilde{\lambda}_2 = \tilde{d}_2$, respectively. Since the number of idle drivers in P_i is $\tilde{s}_i - \tilde{\lambda}_i > 0$, the expected waiting time on this platform is $\kappa(\tilde{s}_i - \tilde{\lambda}_i)$. Consequently, the actual price charged by P_i for its original service under coopetition is $\tilde{p}_i = \tilde{f}_i - \kappa(\tilde{s}_i - \tilde{\lambda}_i)$. We note that the expected waiting time of a customer who requests the new service is the same as in the original P_1's service, that is, $\kappa(\tilde{s}_1 - \tilde{\lambda}_1)$. As before, the actual price of the new service is the difference between the full price and the expected waiting time: $\tilde{p}_x = \tilde{f}_x - \kappa(\tilde{s}_1 - \tilde{\lambda}_1)$. We can now write P_i's profit as

$$\begin{aligned}\tilde{\pi}_i^e(\tilde{f}, \tilde{w}) =& (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i + \gamma_i(\tilde{p}_x - \tilde{w}_1)\tilde{d}_x \\ =& [\tilde{f}_i - \kappa(\tilde{s}_i - \tilde{\lambda}_i) - \tilde{w}_i]\tilde{d}_i + \gamma_i\left[\tilde{f}_x - \kappa(\tilde{s}_1 - \tilde{\lambda}_1) - \frac{\tilde{w}_1}{\tilde{n}}\right]\tilde{d}_x,\end{aligned}$$

where $\tilde{d}_i = \sum_{j=1}^m \frac{\Lambda_j \exp(q_i - \kappa_j \tilde{f}_i)}{1+\sum_{i'=1}^n \exp(q_{i'} - \kappa_j p_{i'}) + \exp(q_x - \kappa_j \tilde{f}_x)}$, $\tilde{d}_x = \sum_{j=1}^m \frac{\Lambda_j \exp(q_x - \kappa_j \tilde{f}_x)}{1+\sum_{i'=1}^n \exp(q_{i'} - \kappa_j p_{i'}) + \exp(q_x - \kappa_j \tilde{f}_x)}$, $\tilde{s}_i = \sum_{k=1}^l \frac{\Gamma_k \exp[\omega_k + \min\{1, \tilde{\lambda}_i/\tilde{s}_i\}(a_{ik} + \eta_k w_i - \omega_k)]}{1+\exp[\omega_k + \min\{1, \tilde{\lambda}_i/\tilde{s}_i\}(a_{ik} + \eta_k w_i - \omega_k)]}$, $\tilde{\lambda}_1 = \tilde{d}_1 + \tilde{d}_x/\tilde{n}$, $\tilde{\lambda}_i = \tilde{d}_i$ $(i \neq 2)$, and $(\tilde{s}_i, \tilde{\lambda}_i)$ is such that $\tilde{s}_i > \tilde{\lambda}_i$ for all $i = 1, 2, ..., n$.

As in the model without coopetition, the platforms first jointly decide $\tilde{f}_x$ and γ. They then engage in a competition game to maximize their profits by setting the equilibrium $(\tilde{f}^{e*}, \tilde{w}^{e*})$, which satisfies $(\tilde{f}_i^{e*}, \tilde{w}_i^{e*}) \in \arg\max_{(f_i, w_i)} \tilde{\pi}_i^e(f_i, w_i, \tilde{f}_{-i}^{e*}, \tilde{w}_{-i}^{e*})$. We can show the existence and uniqueness of equilibrium in the model under coopetition with endogenous waiting times. Furthermore, all the results of Section 5 also extend to this model (the proofs are omitted for conciseness).

8. Conclusions

The ubiquity of two-sided platforms has increased significantly over the past few years. These platforms compete not only for customers but also for flexible workers. In the first part of this paper, we study the problem of competition between two-sided platforms. We propose to model this problem using an endogenous general attraction choice model that accounts for network effects across both sides of the market. In our model, the behavior of a customer or a worker depends not only on the price or wage set by the platform, but also on the strategic interactions among agents on both sides of the market. The two-sidedness nature of our setting makes the objective function non-differentiable, and hence traditional arguments from the literature are not applicable. Instead, we use an approach based on analyzing the best-response strategy to characterize the equilibrium. We ultimately show the existence and uniqueness of equilibrium.

Recently, several coopetition partnerships emerged in the ride-sharing industry. Examples include Curb and Via in NYC and Uber and PT Express in Indonesia. The second part of this paper is motivated by such partnerships that can be implemented via a profit sharing contract. It is not clear a-priori whether the competing platforms will benefit from coopetition. We present a rigorous analysis to show that—when properly designed (e.g., using the Nash bargaining framework)—such coopetition partnerships are beneficial for both platforms, especially when the platforms are facing intensive competition on the demand side. We convey that riders and drivers can also benefit from coopetition. In summary, our results suggest that when the coopetition terms are carefully designed, it will benefit every party (both participating platforms, riders, and drivers).

This paper is among the first to propose a tractable model to study competition and partnerships in the ride-sharing industry. It allows us to draw practical insights on the impact of some recent partnerships observed in practice. Several interesting extensions are left for future research. For example, what is the long-term impact of such partnerships? Shall the platforms consider more complicated contracts such as two-part piecewise linear agreements (i.e., allowing two different profit portions depending on the scale of the new service)? A second direction for future research is to study an alternative form of coopetition, known as joint ownership of a subsidiary. For example, Uber and the Russian taxi-sharing platform Yandex.Taxi merged their businesses in Russia under a new company.[13] It could be interesting to compare the two different forms of coopetition.

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Appendix A: Summary of Notation

Table 1 Summary of Notation

P_i :	Platform i
q_{ij} :	Perceived quality of Platform i $(i=1,2,...,n)$ for customer segment j $(j=1,2,...,m)$
κ_j :	Price sensitivity of segment j
q_{xj} :	Perceived quality of the new joint service for customer segment j
p_i :	Price of P_i without the new joint service
$\tilde{p}_i$:	Price of P_i with the new joint service
$\tilde{p}_x$:	Price of the new joint service
Λ_j :	Total arrival rate of customer segment j
d_{ij} :	Arrival rate of customer segment j to platform P_i without the new joint service
$\tilde{d}_{ij}$:	Arrival rate of customer segment j to platform P_i with the new joint service
$\tilde{d}_{xj}$:	Arrival rate of customer segment j to the new joint service
a_{ik} :	Attractiveness of Platform i for worker type k $(k=1,2,...,l)$
η_k :	Wage sensitivity of worker type k
a_{xk} :	Attractiveness of the new joint service for worker type k
w_i :	Wage of P_i's workers without the new joint service
$\tilde{w}_i$:	Wage of P_i's workers with the new joint service
Γ_k:	Total number of workers of type k
s_{ik} :	Number of workers of type k working for P_i without the new joint service
$\tilde{s}_{ik}$:	Number of workers of type k working for P_i with the new joint service
γ :	Fraction of profit generated by the new joint service allocated to P_1
$\tilde{\lambda}_i$:	Total number of workers needed by P_i (with coopetition)
β_i :	Fixed share of the price allocated to workers under a fixed-commission rate at P_i
$\tilde{n}$:	Number of customers per service for the new joint service

Appendix B: Proof of Statements

Auxiliary Lemma

Before presenting the proofs of our results, we state and prove an auxiliary lemma which is extensively used throughout this appendix.

LEMMA 2. *Define*

$$\bar{d}_{ij} := \frac{\Lambda_j \exp(q_{ij} - \kappa_j p_i)}{1 + \sum_{i'=1}^{n} \exp(q_{i'j} - \kappa_j p_{i'})} \text{ for all } i,j,$$
$$\text{and } \bar{d}_i := \sum_{j=1}^{m} \bar{d}_{ij} = \sum_{j=1}^{m} \frac{\Lambda_j \exp(q_{ij} - \kappa_j p_i)}{1 + \sum_{i'=1}^{n} \exp(q_{i'j} - \kappa_j p_{i'})} \text{ for all } i.$$

Then, we have for $i = 1,2,...,n$ *and* $i' \neq i$, $\partial_{p_i} \bar{d}_{ij} = -\kappa_j (1 - \bar{d}_{ij}/\Lambda_j)\bar{d}_{ij}$, $\partial_{p_i} \bar{d}_i = \sum_{j=1}^{m} \partial_{p_i} \bar{d}_{ij} = -\sum_{j=1}^{m} \kappa_j (1 - \bar{d}_{ij}/\Lambda_j)\bar{d}_{ij}$, $\partial_{p_{i'}} \bar{d}_{ij} = \kappa_j \bar{d}_{ij} \bar{d}_{i'j}/\Lambda_j$, *and* $\partial_{p_{i'}} \bar{d}_i = \sum_{j=1}^{m} \partial_{p_{i'}} \bar{d}_{ij} = \sum_{j=1}^{m} \kappa_j \bar{d}_{ij} \bar{d}_{i'j}/\Lambda_j$.

Proof. Since $\bar{d}_{ij} = \frac{\Lambda_j \exp(q_{ij} - \kappa_{ij} p_i)}{1 + \sum_{i'=1}^{n} \exp(q_{i'j} - \kappa_{i'j} p_{i'})}$, we have

$$\begin{aligned}
\partial_{p_i} \bar{d}_{ij} =& \Lambda_j \frac{-\kappa_j \exp(q_{ij} - \kappa_j p_i)[1 + \sum_{i'=1}^{n} \exp(q_{i'j} - \kappa_j p_{i'})] + \kappa_j [\exp(q_{ij} - \kappa_j p_i)]^2}{[1 + \sum_{i'=1}^{n} \exp(q_{i'j} - \kappa_j p_{i'})]^2} \\
=& -\frac{\Lambda_j \kappa_j \exp(q_{ij} - \kappa_j p_i)}{1 + \sum_{i'=1}^{n} \exp(q_{i'j} - \kappa_j p_{i'})} + \frac{\kappa_j}{\Lambda_j} \left(\frac{\Lambda_j \exp(q_{ij} - p_i)}{1 + \sum_{i'=1}^{n} \exp(q_{i'j} - \kappa_j p_{i'})} \right)^2 \\
=& -\kappa_j \bar{d}_{ij} + \kappa_j/\Lambda_j (\bar{d}_{ij})^2 = -\kappa_j (1 - \bar{d}_{ij}/\Lambda_j)\bar{d}_{ij}.
\end{aligned}$$

Hence,

$$\partial_{p_i} \bar{d}_i = \sum_{j=1}^{m} \partial_{p_i} \bar{d}_{ij} = -\sum_{j=1}^{m} \kappa_j (1 - \bar{d}_{ij}/\Lambda_j)\bar{d}_{ij}.$$

Analogously,

$$\begin{aligned}\partial_{p_{i'}}\bar{d}_{ij} =&\frac{\kappa_j\Lambda_j\exp(q_{i'j}-\kappa_j p_{i'})\exp(q_{ij}-\kappa_j p_i)}{[1+\sum_{i''=1}^n\exp(q_{i''j}-\kappa_j p_{i''})]^2}\\ =&\frac{\kappa_j}{\Lambda_j}\cdot\frac{\Lambda_j\exp(q_{ij}-\kappa_j p_i)}{1+\sum_{i''=1}^n\exp(q_{i''j}-\kappa_j p_{i''})}\cdot\frac{\exp(q_{i'j}-\kappa_j p_{i'})}{1+\sum_{i''=1}^n\exp(q_{i''j}-\kappa_j p_{i''})}\\ =&\kappa_j\bar{d}_{ij}\bar{d}_{i'j}/\Lambda_j.\end{aligned}$$

Thus, for $i\neq i'$,

$$\partial_{p_{i'}}\bar{d}_i=\sum_{j=1}^m\partial_{p_{i'}}\bar{d}_{ij}=\sum_{j=1}^m\kappa_j\bar{d}_{ij}\bar{d}_{i'j}/\Lambda_j. \quad \square$$

Proof of Lemma 1

For each $i\in 1,2,...,n$, we define the following:

$$f_i(d_i,s_i)=\sum_{j=1}^m\frac{\Lambda_j\exp[\nu_j+\min\{1,s_i/d_i\}(q_{ij}-\kappa_j p_i-\nu_j)]}{1+\sum_{i'=1}^n\exp[\nu_j+\min\{1,s_{i'}/d_{i'}\}(q_{i'j}-\kappa_j p_{i'}-\nu_j)]},$$

and

$$g_i(d_i,s_i)=\sum_{k=1}^l\frac{\Gamma_k\exp[\omega_k+\min\{1,d_i/s_i\}(a_{ik}+\eta_k w_i-\omega_k)]}{1+\sum_{i'=1}^n\exp[\omega_k+\min\{1,d_{i'}/s_{i'}\}(a_{i'k}+\eta_k w_{i'}-\omega_k)]}.$$

It suffices to show that, for each i, there exists (d_i,s_i) such that

$$\begin{cases}d_i=f_i(d_i,s_i)\\ s_i=g_i(d_i,s_i).\end{cases}$$

We next show that, given s_i, there exists a unique $d_i(s_i)$ increasing in s_i, such that $d_i(s_i)=f_i(d_i(s_i),s_i)$. One should note that $\exp[\nu_j+\min\{1,s_i/d_i\}(q_{ij}-\kappa_j p_i-\nu_j)]$ is continuously decreasing in d_i for any s_i. Hence, $f_i(d_i,s_i)$ is also continuously decreasing in d_i. Furthermore, $f_i(0+,s_i)>0$ and $f_i(+\infty,s_i)=0$. Therefore, there exists a unique $d_i(s_i)$ such that $d_i(s_i)=f_i(d_i(s_i),s_i)$. To show that $d_i(s_i)$ is increasing in s_i, we observe that $f_i(d_i,s_i)$ is increasing in s_i for any d_i. For $\hat{s}_i>s_i$, $d_i(s_i)=f_i(d_i(s_i),s_i)\le f_i(d_i(s_i),\hat{s}_i)$, which implies that $d_i(\hat{s}_i)\ge d_i(s_i)$, i.e., $d_i(s_i)$ increasing in s_i. The exact same argument implies that, given d_i, there exists a unique $s_i(d_i)$ increasing in d_i, such that $s_i(d_i)=g_i(d_i,s_i(d_i))$. Tarski's Fixed Point Theorem (see, e.g., Milgrom and Roberts 1990) suggests that there exists (d_i,s_i) such that $d_i=f_i(d_i,s_i)$ and $s_i=g_i(d_i,s_i)$.

We now show that, for $\hat{s}_i>s_i$, $d_i(\hat{s}_i)-d_i(s_i)<\hat{s}_i-s_i$. Denote $\delta:=\hat{s}_i-s_i$. It is straightforward to check that $f_i(d_i(s_i)+\delta,s_i+\delta)<d(s_i)+\delta$. Thus, $d(\hat{s}_i)=d(s_i+\delta)<d_i(s_i)+\delta$, i.e., $d_i(\hat{s}_i)-d_i(s_i)<\hat{s}_i-s_i$. Analogously, we have for $\hat{d}_i>d_i$, $s_i(\hat{d}_i)-s_i(d_i)<\hat{d}_i-d_i$.

Finally, we show the uniqueness of (d_i,s_i), such that $d_i=f_i(d_i,s_i)$ and $s_i=g_i(d_i,s_i)$. If there exist distinct (d_i^1,s_i^1) and (d_i^2,s_i^2) such that $d_i^j=f_i(d_i^j,s_i^j)$ and $s_i^j=g_i(d_i^j,s_i^j)$ for $j=1,2$, then we have $d_i^j=d_i(s_i^j)$ and $s_i^j=s_i(d_i^j)$ for $j=1,2$. Therefore,

$$|d_i^1-d_i^2|+|s_i^1-s_i^2|=|d_i(s_i^1)-d_i(s_i^2)|+|s_i(d_i^1)-s_i(d_i^2)|<|s_i^1-s_i^2|+|d_i^1-d_i^2|,$$

which leads to a contradiction. Thus, we must have $(d_i^1,s_i^1)=(d_i^2,s_i^2)$, so that there exists a unique (d_i,s_i) such that $d_i=f_i(d_i,s_i)$ and $s_i=g_i(d_i,s_i)$. This completes the proof. $\square$

Proof of Theorem 1

We first introduce some notation that will prove useful in our analysis. Given the competitors' strategy (p_{-i}, w_{-i}), we define $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ as P_i's best price and wage responses. We also define the best-response mapping of the two-sided competition game as

$$T(p,w) := \Big(p_i(p_{-i}, w_{-i}), w_i(p_{-i}, w_{-i}) : 1 \le i \le n\Big).$$

We then iteratively define the k-fold best-response mapping ($k \ge 2$) as

$$T^{(k)}(p,w) = \Big(p_i^{(k)}(p_{-i}, w_{-i}), w_i^{(k)}(p_{-i}, w_{-i}) : 1 \le i \le n\Big),$$

where for $i = 1, 2, ..., n$

$$p_i^{(k)}(p_{-i}, w_{-i}) = p_i\Big(p_1^{(k-1)}(p_{-1}, w_{-1}), w_1^{(k-1)}(p_{-1}, w_{-1}), ..., p_n^{(k-1)}(p_{-n}, w_{-n}), w_n^{(k-1)}(p_{-n}, w_{-n})\Big),$$
$$w_i^{(k)}(p_{-i}, w_{-i}) = p_i\Big(p_1^{(k-1)}(p_{-1}, w_{-1}), w_1^{(k-1)}(p_{-1}, w_{-1}), ..., p_n^{(k-1)}(p_{-n}, w_{-n}), w_n^{(k-1)}(p_{-n}, w_{-n})\Big).$$

We use $||\cdot||_1$ to represent the ℓ_1 norm, that is, $||x||_1 = \sum_{i=1}^n |x_i|$ for $x \in \mathbb{R}^n$. The proof of Theorem 1 is based on the following four steps:

- Step I. Under equilibrium, $s_i^* = d_i^*$ for $i = 1, 2, ..., n$.
- Step II. The best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are continuously increasing in p_{-i} and w_{-i}. This will imply that an equilibrium exists.
- Step III. There exists a k^*, such that the k^*-fold best response is a contraction mapping under the ℓ_1 norm, i.e., there exists a constant $\theta \in (0,1)$, such that

$$||T^{(k^*)}(p,w) - T^{(k^*)}(p',w')||_1 \le \theta ||(p,w) - (p',w')||_1.$$

This will imply that the equilibrium is unique.

- Step IV. For any (p,w), the sequence $\{T^{(k)}(p,w) : k = 1, 2, ...\}$ converges to the unique equilibrium (p^*, w^*) as $k \uparrow +\infty$. This will imply that the equilibrium can be computed using a *tatônnement* scheme.

Step I is proved by contradiction (see Lemma 3 below). We show that if $s_i^* > d_i^*$, then P_i can unilaterally decrease w_i to increase its profit; and if $s_i^* < d_i^*$, then P_i can unilaterally increase p_i to increase its profit. This implies that we must have $s_i^* = d_i^*$ under equilibrium.

Step II is proved by exploiting structural properties of the best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$, and by using the fact that $d_i^* = s_i^*$ under equilibrium (see Lemma 4 below). Since the feasible region of (p_{-i}, w_{-i}) is a lattice, Step II immediately implies that an equilibrium exists by Tarski's Fixed Point Theorem.

Step III is proved by bounding the ℓ_1 norm of $T(p,w)$. We note that $T(\cdot)$ is not necessarily a contraction mapping, but $T^{(k*)}(\cdot)$ for some $k^* > 1$ is (see Lemma 5 below). Using the result of Step III, a standard contradiction argument will show that the equilibrium is unique.

Step IV is proved by exploiting the contraction mapping property of $T^{(k)}(\cdot)$ (see Lemma 6 below). Putting Steps I–IV together concludes the proof of Theorem 1. □

The following lemma proves Step I in the proof of Theorem 1.

LEMMA 3. *Under equilibrium, $d_i^* = s_i^*$ for $i = 1, 2$.*

Proof. Assume by contradiction that $s_i^* < d_i^*$. This implies that $d_i^* > \min\{d_i^*, s_i^*\} = s_i^*$,

$$d_i^* = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i^*/d_i^*\}(q_{ij} - \kappa_j p_i^* - \nu_j)]}{1 + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}^*/d_{i'}^*\}(q_{i'j} - \kappa_j p_{i'}^* - \nu_j)]},$$

and

$$s_i^* = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i^*/s_i^*\}(a_{ik} + \eta_k w_i^* - \omega_k)]}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}^*/s_{i'}^*\}(a_{i'k} + \eta_k w_{i'}^* - \omega_k)]}.$$

Consequently, P_i can increase its price to $p_i^*(\epsilon) = p_i^* + \epsilon$ (for a sufficiently small $\epsilon > 0$) and $(w_i^*, p_{-i}^*, w_{-i}^*)$ remain unchanged, with the induced market outcome $(d_i^*(\epsilon), s_i^*(\epsilon), d_{-i}^*(\epsilon), s_{-i}^*(\epsilon))$, which satisfies

$$d_i^*(\epsilon) = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i^*(\epsilon)/d_i^*(\epsilon)\}(q_{ij} - \kappa_{ij}(p_i^* + \epsilon) - \nu_j)]}{1 + \exp[\nu_j + \min\{1, s_i^*(\epsilon)/d_i^*(\epsilon)\}(q_{ij} - \kappa_j(p_i^* + \epsilon) - \nu_j)] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, s_{i'}^*(\epsilon)/d_{i'}^*(\epsilon)\}(q_{i'j} - \kappa_j p_{i'}^* - \nu_j)]}$$

and

$$s_i^*(\epsilon) = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i^*(\epsilon)/s_i^*(\epsilon)\}(a_{ik} + \eta_k w_i^* - \omega_k)]}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}^*(\epsilon)/s_{i'}^*(\epsilon)\}(a_{i'k} + \eta_k w_{i'}^* - \omega_k)]}.$$

One can check that, for a sufficiently small $\epsilon > 0$, $s_i^*(\epsilon) < d_i^*(\epsilon) < d_i^*$, $s_i^*(\epsilon) \geq s_i^*$, and hence $\min\{d_i^*(\epsilon), s_i^*(\epsilon)\} = s_i^*(\epsilon)$, where the inequality follows from the fact that $d_i(\epsilon)$ and $s_i(\epsilon)$ are continuous in ϵ. Thus, $\pi_i(\epsilon) = (p_i^* + \epsilon - w_i^*) \min\{d_i^*(\epsilon), s_i^*(\epsilon)\} > (p_i^* - w_i^*) s_i^* = \pi_i^*$, which contradicts the fact that $(p_i^*, w_i^*, p_{-i}^*, w_{-i}^*)$ is an equilibrium. Therefore, we must have $s_i^* \geq d_i^*$.

Assume by contradiction that $s_i^* > d_i^*$. This implies that $s_i^* > \min\{d_i^*, s_i^*\} = d_i^*$. Consequently, P_i can decrease its wage to $w_i^*(\epsilon) = w_i^* - \epsilon$ (for a sufficiently small $\epsilon > 0$) and (p_i^*, w_i^*, p_{-i}^*) remain unchanged, with the induced market outcome $(d_i^*(\epsilon), s_i^*(\epsilon), d_{-i}^*(\epsilon), s_{-i}^*(\epsilon))$, which satisfies

$$d_i^*(\epsilon) = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i^*(\epsilon)/d_i^*(\epsilon)\}(q_{ij} - \kappa_j p_i^* - \nu_j)]}{1 + \exp[\nu_j + \min\{1, s_i^*(\epsilon)/d_i^*(\epsilon)\}(q_{ij} - \kappa_j p_i^* - \nu_j)] + \sum_{i' \neq i}^{n} \exp[\nu_j + \min\{1, s_{i'}^*(\epsilon)/d_{i'}^*(\epsilon)\}(q_{i'j} - \kappa_j p_{i'}^* - \nu_j)]},$$

and

$$s_i^*(\epsilon) = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i^*(\epsilon)/s_i^*(\epsilon)\}(a_{ik} + \eta_k(w_i^* - \epsilon) - \omega_k)]}{1 + \exp[\omega_k + \min\{1, d_i^*(\epsilon)/s_i^*(\epsilon)\}(a_{i'k} + \eta_k(w_i^* - \epsilon) - \omega_k)] + \sum_{i' \neq i} \exp[\omega_k + \min\{1, d_{i'}^*(\epsilon)/s_{i'}^*(\epsilon)\}(a_{i'k} + \eta_k w_{i'}^* - \omega_k)]}.$$

One check that, for a sufficiently small $\epsilon > 0$, $s_i^* > s_i^*(\epsilon) > d_i^*(\epsilon) > d_i^*$, and hence $\min\{d_i^*(\epsilon), s_i^*(\epsilon)\} = d_i^*(\epsilon) > d_i^*$, where the inequality follows from the fact that $d_i(\epsilon)$ and $s_i(\epsilon)$ are continuous in ϵ. Thus, $\pi_i(\epsilon) = (p_i^* - w_i^* + \epsilon) \min\{d_i^*(\epsilon), s_i^*(\epsilon)\} > (p_i^* - w_i^*) d_i^* = \pi_i^*$, contradicting that $(p_i^*, w_i^*, p_{-i}^*, w_{-i}^*)$ is an equilibrium. Therefore, we have $s_i^* \leq d_i^*$. Since $s_i^* \geq d_i^*$ and $s_i^* \leq d_i^*$, we conclude that $s_i^* = d_i^*$. $\square$

The following lemma establishes Step II in the proof of Theorem 1.

LEMMA 4. *$p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are continuously increasing in p_{-i} and w_{-i}. Hence, an equilibrium exists in the two-sided competition model.*

Proof. Since $s_i^* = d_i^*$, we denote $s = s_i = d_i$ as the demand/supply of P_i. Given (p_{-i}, w_{-i}, s), we can formulate the price and wage optimization of P_i as follows:

$$
\begin{aligned}
&\max_{(p_i,w_i,s)} \ \pi_i(p_i,w_i,s|p_{-i},w_{-i})\\
&\text{where } \pi_i(p_i,w_i,s|p_{-i},w_{-i}) = (p_i - w_i)s\\
&\qquad \sum_{j=1}^{m} d_{ij} = s\\
&\qquad p_i = \frac{q_{ij}}{\kappa_j} - \frac{1}{\kappa_j}\log\left(\frac{d_{ij}/\Lambda_j}{1-d_{ij}/\Lambda_j}\right) - \frac{1}{\kappa_j}\log\left(1+\sum_{i'\neq i}\exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]\right) \ \forall j\\
&\qquad \sum_{k=1}^{l} s_{ik} = s\\
&\qquad w_i = -\frac{a_{ik}}{\eta_k} + \frac{1}{\eta_{ik}}\log\left(\frac{s_{ik}/\Gamma_k}{1-s_{ik}/\Gamma_k}\right) + \frac{1}{\eta_k}\log\left(1+\sum_{i'\neq i}\exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]\right) \ \forall k.
\end{aligned} \tag{4}
$$

Since $p_i = \frac{q_{ij}}{\kappa_j} - \frac{1}{\kappa_j}\log\left(\frac{d_{ij}/\Lambda_j}{1-d_{ij}/\Lambda_j}\right) - \frac{1}{\kappa_j}\log\left(1+\sum_{i'\neq i}\exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]\right)$ for all j, then d_{ij} is strictly decreasing in p_i for all j. Together with $\sum_{j=1}^{m} d_{ij} = s$, it implies that given s, there exists a unique p_i and a unique associated vector $(d_{i1}, d_{i2}, ..., d_{im})$ that satisfy the constraints $p_i = \frac{q_{ij}}{\kappa_j} - \frac{1}{\kappa_j}\log\left(\frac{d_{ij}/\Lambda_j}{1-d_{ij}/\Lambda_j}\right) - \frac{1}{\kappa_j}\log\left(1+\sum_{i'\neq i}\exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]\right)$ for all j. Thus, given (p_{-i}, w_{-i}) and s, there exists a unique price $p_i(s, p_{-i}, w_{-i})$ that satisfies all the constraints in (4). Analogously, there exists a unique wage $w_i(s, p_{-i}, w_{-i})$ that satisfies all the constraints in (4). The corresponding demand for P_i from each segment j, d_{ij}, and the corresponding supply for P_i from each worker type k, s_{ik}, are also uniquely determined. It is clear by (4) that given s, $p_i(s, p_{-i}, w_{-i})$ is strictly increasing in $p_{i'}$ and that $w_i(s, p_{-i}, w_{-i})$ is strictly increasing in $w_{i'}$ for all $i' \neq i$. In addition, given (p_{-i}, w_{-i}), $p_i(s, p_{-i}, w_{-i})$ is strictly decreasing in s, whereas $w_i(s, p_{-i}, w_{-i})$ is strictly increasing in s. By calculating the cross derivative, we can show that $\pi_i(s|p_{-i}, w_{-i}) := (p_i(s, p_{-i}, w_{-i}) - w_i(s, p_{-i}, w_{-i}))s$ is supermodular in $(p_{i'}, s)$ for any $i' \neq i$. Therefore, $s^* := \arg\max_s \pi_i(s|p_{-i}, w_{-i})$ is increasing in $p_{i'}$, which implies that $w_i(p_{-i}, w_{-i}) = w_i(s^*, p_{-i}, w_{-i})$ is also increasing in $p_{i'}$ for any $i' \neq i$.

We next show that $p_i(p_{-i}, w_{-i}) = p_i(s^*, p_{-i}, w_{-i})$ is also strictly increasing in $p_{i'}$ for $i' \neq i$. We define $m(s, p_{-i}, w_{-i}) := p_i(s, p_{-i}, w_{-i}) - w_i(s, p_{-i}, w_{-i})$ as P_i's profit margin given (s, p_{-i}, w_{-i}). Thus,

$$\pi_i'(s|p_{-i}, w_{-i}) = \partial_s m(s, p_{-i}, w_{-i})s + m(s, p_{-i}, w_{-i}).$$

Since $\pi_i'(s^*|p_{-i}, w_{-i}) = 0$, we have $\partial_s m(p_{-i}, w_{-i}, s^*)s^* + m(p_{-i}, w_{-i}, s^*) = 0$. One should note by (4) that $\partial_s m(p_{-i}, w_{-i}, s)s$ is strictly decreasing in s and independent of (p_{-i}, w_{-i}). Assume that $\hat{p}_{i'} > p_i$ $(i' \neq i)$, so we have $\hat{s}^* > s^*$. Thus, $\partial_s m(\hat{s}^*, \hat{p}_{-i}, w_{-i})\hat{s}^* < \partial_s m(\hat{s}^*, p_{-i}, w_{-i})s^*$. By the first-order condition (FOC), $\pi_i'(\hat{s}^*|\hat{p}_{-i}, w_{-i}) = \pi_i'(s^*|p_{-i}, w_{-i}) = 0$, that is, $\partial_s m(\hat{s}^*, \hat{p}_{-i}, w_{-i})\hat{s}^* + m(\hat{s}^*, \hat{p}_{-i}, w_{-i}) = \partial_s m(s^*, p_{-i}, w_{-i})s^* + m(s^*, p_{-i}, w_{-i}) = 0$. Hence, we must have $m(\hat{s}^*, \hat{p}_{-i}, w_{-i}) > m(s^*, p_{-i}, w_{-i})$. Therefore

$$p_i(\hat{s}^*, \hat{p}_{-i}, w_{-i}) = w_i(\hat{s}^*, \hat{p}_{-i}, w_{-i}) + m_i(\hat{s}^*, \hat{p}_{-i}, w_{-i}) > w_i(s^*, p_{-i}, w_{-i}) + m_i(s^*, p_{-i}, w_{-i}) = p_i(s^*, p_{-i}, w_{-i}).$$

Thus, both $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in $p_{i'}$ $(i' \neq i)$. By using a similar argument, we can show that s^* is decreasing in $w_{i'}$ $(i' \neq i)$, which further implies that $p_i(p_{-i}, w_{-i}) = p_i(s^*, p_{-i}, w_{-i})$ is increasing in $w_{i'}$ for $i' \neq i$. Moreover, a similar first-order argument suggests that the profit margin $m(s^*, p_{-i}, w_{-i})$ is decreasing in $w_{i'}$ for $i' \neq i$. We then conclude that

$$w_i(p_{-i}, w_{-i}) = w_i(s^*, p_{-i}, w_{-i}) = p_i(s^*, p_{-i}, w_{-i}) - m_i(s^*, p_{-i}, w_{-i})$$

is increasing in $w_{i'}$ for $i' \neq i$. Thus, we have shown that both $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in $p_{i'}$ and in $w_{i'}$ for $i' \neq i$. The continuity of $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ follows from the fact that $\pi_i(s|p_{-i}, w_{-i})$ is continuous. This completes the proof of Step II. By Tarski's Fixed Point Theorem (see, e.g., Milgrom and Roberts 1990), the continuity and monotonicity of $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$, together with the fact that the feasible sets of $p_i(\cdot,\cdot)$ and $w_i(\cdot,\cdot)$ are lattices, imply that an equilibrium exists. □

We next show that the best response mapping is a contraction mapping, so that a unique equilibrium exists.

LEMMA 5. *There exists a k^*, such that the k^*-fold best response is a contraction mapping under the ℓ_1 norm, that is, there exists a constant $\theta \in (0,1)$, such that*

$$||T^{(k^*)}(p,w) - T^{(k^*)}(p',w')||_1 \leq \theta ||(p,w) - (p',w')||_1.$$

Furthermore, the equilibrium is unique.

Proof. We assume that (p,w) and $(\hat{p},\hat{w})$ are identical except that $\hat{p}_{i'} = p_{i'} + \delta$ for some i'. We observe that, for any $i \neq i'$ and any j

$$\partial_{p_{i'}}\Big\{-\frac{1}{\kappa_j}\log\Big[1+\sum_{i''\neq i}\exp[\nu_j+\min\{1, s_{i''}/d_{i''}\}(q_{i''j}-\kappa_j p_{i'}-\nu_j)]\Big]\Big\} \leq \frac{\exp(q_{i'j}-\kappa_j p_{i'})}{1+\sum_{i''\neq i}\exp(q_{i''j}-\kappa_j p_{i'})} < \frac{\exp(q_{i'j})}{1+\exp(q_{ij})}.$$

By the mean-value theorem, for $\delta > 0$ and any j,

$$0 < \frac{1}{\kappa_j}\log\Big[1+\sum_{i''\neq i}\exp[\nu_j+\min\{1,\hat{s}_{i''}/\hat{d}_{i''}\}(q_{i''j}-\kappa_j\hat{p}_{i''}-\nu_j)]\Big] - \frac{1}{\kappa_j}\log\Big[1+\sum_{i''\neq i}\exp[\nu_j+\min\{1,s_{i''}/d_{i''}\}(q_{i''j}-\kappa_j p_{i''}-\nu_j)]\Big] < C_{i'j}\delta,$$

where $C_{i'j} := \frac{\exp(q_{i'j})}{1+\exp(q_{i'j})} < 1$. Similarly, we have, for $\delta > 0$ and $i \neq i'$ and any k,

$$0 < \frac{1}{\eta_k}\log\Big[1+\sum_{i''\neq i}\exp[\omega_k+\min\{1,\hat{d}_{i''}/\hat{s}_{i''}\}(a_{i''k}+\eta_k\hat{p}_{i''}-\omega_k)]\Big] - \frac{1}{\eta_k}\log\Big[1+\sum_{i''\neq i}\exp[\omega_k+\min\{1,d_{i''}/s_{i''}\}(a_{i''k}+\eta_k p_{i''}-\omega_k)]\Big] < D_{i'k}\delta,$$

where $D_{i'k} := \frac{\exp(a_{i'k})}{1+\exp(a_{i'k})} < 1$. Define $s_i^* := \arg\max_s \pi_i(s|p_{-i}, w_{-i})$ and $\hat{s}_i^* := \arg\max_s \pi_i(s|\hat{p}_{-i}, w_{-i})$ for $i \neq i'$. We denote the demand from each customer segment j for P_i associated with price vector $\hat{p}_{-i}$ (resp. p_{-i}) as $\hat{d}_{ij}^*$ (resp. d_{ij}^*). The supply of worker type k for P_i associated with price vector $\hat{p}_{-i}$ (resp. p_{-i}) is denoted as $\hat{s}_{ik}^*$ (resp. s_{ik}^*). Thus, we have $\sum_{j=1}^m \hat{d}_{ij}^* = \sum_{k=1}^l \hat{s}_{ik}^* = \hat{s}_i^*$ and $\sum_{j=1}^m d_{ij}^* = \sum_{k=1}^l s_{ik}^* = s_i^*$.

We denote $\delta_i^2 := \max_j \Big[\log\Big(\frac{\hat{d}_{ij}^*/\Lambda_j}{1-\hat{d}_{ij}^*/\Lambda_j}\Big) - \log\Big(\frac{d_{ij}^*/\Lambda_j}{1-s_{ij}^*/\Lambda_j}\Big)\Big] > 0$ and $\delta_i^3 := \max_k \Big[\log\Big(\frac{\hat{s}_{ik}^*/\Gamma_k}{1-\hat{s}_{ik}^*/\Gamma_k}\Big) - \log\Big(\frac{s_{ij}^*/\Gamma_k}{1-s_{ik}^*/\Gamma_k}\Big)\Big] > 0$. As shown in the proof of Step II of Theorem 1, $\hat{d}_{ij}^* > d_{ij}^*$ for all j and $\hat{s}_{ik}^* > s_{ik}^*$ for all k, and $m_i(\hat{s}_i^*, \hat{p}_{-i}, w_{-i}) > m_i(s_i^*, p_{-i}, w_{-i})$, that is, for any j,

$$\begin{aligned} 0 &< [p_i(\hat{p}_{-i}, w_{-i}) - w_i(\hat{p}_{-i}, w_{-i})] - [p_i(p_{-i}, w_{-i}) - w_i(p_{-i}, w_{-i})] < \frac{1}{\kappa_j}\log\Big[1+\sum_{i''\neq i}\exp[\nu_j+\min\{1,\hat{s}_{i''}/\hat{d}_{i''}\}(q_{i''j}-\kappa_j\hat{p}_{i''}-\nu_j)]\Big] \\ &-\frac{1}{\kappa_j}\log\Big[1+\sum_{i''\neq i}\exp[\nu_j+\min\{1,s_{i''}/d_{i''}\}(q_{i''j}-\kappa_j p_{i''}-\nu_j)]\Big] < C_{i'j}\delta. \end{aligned} \quad (5)$$

Inequality (5) implies that $\delta_i^2+\delta_i^3<C_{i'j}\delta$ for any j. Therefore, we obtain

$$\begin{aligned}&p_i(\hat{p}_{-i},w_{-i})-p_i(p_{-i},w_{-i})=-\log\Big(\frac{\hat{d}^*_{ij}/\Lambda_j}{1-\hat{d}^*_{ij}/\Lambda_j}\Big)+\log\Big(\frac{d^*_{ij}/\Lambda_j}{1-d^*_{ij}/\Lambda}\Big)+\\&\frac{1}{\kappa_j}\log\Big[1+\sum_{i''\neq i}\exp[\nu_j+\min\{1,\hat{s}_{i''}/\hat{d}_{i''}\}(q_{i''j}-\kappa_j\hat{p}_{i''}-\nu_j)]\Big]-\frac{1}{\kappa_j}\log\Big[1+\sum_{i''\neq i}\exp[\nu_j+\min\{1,s_{i''}/d_{i''}\}(q_{i''j}-\kappa_jp_{i''}-\nu_j)]\Big]\\&<C_{i'j}\delta-\delta_i^2.\end{aligned}$$

Analogously, for all k, $w_i(\hat{p}_{-i},w_{-i})-w_i(p_{-i},w_{-i})=\log\Big(\frac{\hat{s}^*_{ik}/\Gamma_k}{1-\hat{s}^*_{ik}/\Gamma_k}\Big)-\log\Big(\frac{s^*_{ik}/\Gamma_k}{1-s^*_{ik}/\Gamma_k}\Big)=\delta_i^3<D_{i'k}\delta-\delta_i^2$. As a result, for all $i\neq i'$ and any j and k,

$$|p_i(\hat{p}_{-i},w_{-i})-p_i(p_{-i},w_{-i})|<C_{i'j}\delta \text{ and } |w_i(\hat{p}_{-i},w_{-i})-w_i(p_{-i},w_{-i})|<D_{i'k}\delta.$$

We define $p_i^{(k)}$ (resp. $w_i^{(k)}$) as the value of p_i (resp. w_i) for the k-th iteration of T operated on (p,w). Analogously, $\hat{p}_i^{(k)}$ (resp. $\hat{w}_i^{(k)}$) as the value of p_i (resp. w_i) for the k-th iteration of T operated on $(\hat{p},w)$. Repeating the argument above, we have that, for any i and any $k\geq 1$,

$$|\hat{p}_i^{(k)}-p_i^{(k)}|<C^{k-1}C_{i'}\delta \text{ and } |\hat{w}_i^{(k)}-w_i^{(k)}|<D^{k-1}D_{i'}\delta,$$

where

$$C:=\max\Big\{\frac{\exp(q_{ij})}{1+\exp(q_{ij})}:1\leq i\leq n,1\leq j\leq m\Big\}<1 \text{ and } D:=\max\Big\{\frac{\exp(a_{ik})}{1+\exp(a_{ik})}:1\leq i\leq n,1\leq k\leq l\Big\}<1.$$

Define $(\hat{p}^{(k)},\hat{w}^{(k)}):=(\hat{p}_i^{(k)},\hat{w}_i^{(k)}:1\leq i\leq n)$ and $(p^{(k)},w^{(k)}):=(p_i^{(k)},w_i^{(k)}:1\leq i\leq n)$. We have, for any $k\geq 1$,

$$||(\hat{p}^{(k)},\hat{w}^{(k)})-(p^{(k)},w^{(k)})||_1\leq(C^{k-1}C_{i'}+D^{k-1}D_{i'})\delta<2E^k\delta,$$

where $E:=\max\{C,D\}<1$. By using the triangular inequality, we have, for any $k\geq 1$,

$$||T^{(k)}(p,w)-T^{(k)}(p',w')||_1\leq 2E^k||(p,w)-(p',w')||_1.$$

We define k^* as the smallest integer k such that $2E^k<1$ (i.e., the smallest integer k such that $k>-\log(2)/\log(E)$). Therefore, we obtain

$$||T^{(k^*)}(p,w)-T^{(k^*)}(p',w')||_1\leq 2E^{k^*}||(p,w)-(p',w')||_1<\theta||(p,w)-(p',w')||_1,$$

where $\theta:=2E^{(k^*)}<1$. We conclude that $T^{(k^*)}(\cdot,\cdot)$ is a contraction mapping under the ℓ_1 norm.

We next show that the equilibrium is unique. Assume by contradiction that there are two distinct equilibria (p^*,w^*) and $(\bar{p}^*,\bar{w}^*)$. Then, by the equilibrium definition, we have

$$T(p^*,w^*)=(p^*,w^*) \text{ and } T(\bar{p}^*,\bar{w}^*)=(\bar{p}^*,\bar{w}^*).$$

Therefore,

$$T^{(k^*)}(p^*,w^*)=(p^*,w^*) \text{ and } T^{(k^*)}(\bar{p}^*,\bar{w}^*)=(\bar{p}^*,\bar{w}^*).$$

Hence, we have

$$||T^{(k^*)}(p^*,w^*)-T^{(k^*)}(\bar{p}^*,\bar{w}^*)||_1=||(p^*,w^*)-(\bar{p}^*,\bar{w}^*)||_1. \tag{6}$$

Since $T^{(k^*)}(\cdot,\cdot)$ is a contraction mapping, we have

$$||T^{(k^*)}(p^*,w^*)-T^{(k^*)}(\bar{p}^*,\bar{w}^*)||_1<\theta||(p^*,w^*)-(\bar{p}^*,\bar{w}^*)||_1,$$

contradicting Equation (6) if $(p^*,w^*)\neq(\bar{p}^*,\bar{w}^*)$. Thus, a unique equilibrium exists. □

The following lemma establishes Step IV in the proof of Theorem 1.

LEMMA 6. *$T^{(k)}(p,w)$ converges to the unique equilibrium as $k\uparrow+\infty$.*

Proof. It suffices to show that $T^{(k)}(p,w)$ converges to the equilibrium (p^*,w^*) in the (p,w) space as $k\uparrow+\infty$. As shown in Step III, $||T^{(k)}(p,w)-T^{(k)}(p',w')||_1\le 2E^k||(p,w)-(p',w')||_1$ for any (p,w) and (p',w'). We define $(p^{(k)},w^{(k)}):=T^{(k)}(p,w)$ for $k\ge 1$. For any k and $l>0$,

$$||(p^{(k)},w^{(k)})-(p^{(k+l)},w^{(k+l)})||_1\le\sum_{i=1}^{l}||(p^{(k+i)},w^{(k+i)})-(p^{(k+i-1)},w^{(k+i-1)})||_1$$

$$\le\sum_{i=1}^{l}2E^{(k+i-1)}||(p^{(1)},w^{(1)})-(p,w)||_1\le\sum_{i=1}^{+\infty}2E^{(k+i-1)}||(p^{(1)},w^{(1)})-(p,w)||_1=\frac{2||(p^{(1)},w^{(1)})-(p,w)||_1E^k}{1-E},$$

where the first inequality follows from the triangle inequality. Thus, $||(p^{(k)},w^{(k)})-(p^{(k+l)},w^{(k+l)})||_1\to 0$ uniformly with respect to l as $k\uparrow+\infty$, that is, $\{(p^{(k)},w^{(k)}):k\ge 1\}$ is a Cauchy sequence, and hence $(p^{(k)},w^{(k)})$ converges to (p^*,w^*), which is a fixed point of $T(\cdot,\cdot)$, namely, $T(p^*,w^*)=(p^*,w^*)$ so that (p^*,w^*) is the unique equilibrium. Hence, the unique equilibrium can be obtained using a *tatônnement* scheme, and this concludes the proof of Theorem 1. □

Proof of Proposition 1

Part (a). As shown in the proof of Theorem 1, the sequence $\{T^{(k)}(p^{m*},w^{m*}):k\ge 1\}$ converges to the equilibrium (p^*,w^*). In the proof of Proposition 1, we have defined:

$$(p^{(k)},w^{(k)}):=T^{(k)}(p^{m*},w^{m*})\text{ for }k\ge 1,$$

and $(p^{(0)},w^{(0)}):=(p^{m*},w^{m*})$. We have also defined $s_i^{(k)}$ as the optimal demand/supply of P_i in the k-th iteration of the *tatônnement* scheme. Then, it suffices to show that $p_i^{(k)}<p_i^{(m*)}$ and $w_i^{(k)}>w_i^{(m*)}$ for $k\ge 1$ and $i=1,2,...,n$.

Note that for a monopoly (i.e., a setting where a centralized decision maker seeks to maximize the total profit from all n platforms), we have $d_i^{m*}=s_i^{m*}$ for $i=1,2$. Indeed, following the same argument as in the proof of Step I of Theorem 1, if $d_i^{m*}>s_i^{m*}$, we can increase p_i and strictly increase the profit of each platform. Analogously, if $d_i^{m*}<s_i^{m*}$, we can increase w_i and strictly increase the profit of each platform. As a result, under the optimal price and wage policies, $d_i^{m*}=s_i^{m*}$ for $i=1,2,...,n$.

We next show that $p_i^{(1)}<p_i^{(0)}$ and $w_i^{(1)}>w_i^{(0)}$ for all i. As shown in the proof of Theorem 1, $(p_i^{(1)},w_i^{(1)})$ can be represented by $\big(p_i(s_i^{(1)},p_{-i}^{(0)},w_{-i}^{(0)}),w_i(s_i^{(1)},p_{-i}^{(0)},w_{-i}^{(0)}\big)$, where $p_i(\cdot,\cdot,\cdot)$ (resp. $w_i(\cdot,\cdot,\cdot)$) is the price (resp. wage) policy of P_i given (s,p_{-i},w_{-i}) and $s_i^{(1)}$ is the optimal supply (which is equal to demand) obtained by solving the following optimization problem:

$$\max_s\ \pi_i(s|p_{-i}^{(0)},w_{-i}^{(0)})$$

where $\pi_i(s|p_{-i},w_{-i})=(p_i-w_i)s$

$$\sum_{j=1}^{m}d_{ij}=s$$

$$p_i=\frac{q_{ij}}{\kappa_j}-\frac{1}{\kappa_j}\log\Big(\frac{d_{ij}/\Lambda_j}{1-d_{ij}/\Lambda_j}\Big)-\frac{1}{\kappa_j}\log\Big(1+\sum_{i'\neq i}\exp[\nu_j+\min\{1,s_{i'}/d_{i'}\}(q_{i'j}-\kappa_jp_{i'}-\nu_j)]\Big)\text{ for all }j$$

$$\sum_{k=1}^{l}s_{ik}=s$$

$$w_i=-\frac{a_{ik}}{\eta_k}-\frac{1}{\eta_{ik}}\log\Big(\frac{s_{ik}/\Gamma_k}{1-s_{ik}/\Gamma_k}\Big)+\frac{1}{\eta_k}\log\Big(1+\sum_{i'\neq i}\exp[\omega_k+\min\{1,d_{i'}/s_{i'}\}(a_{i'k}+\eta_kw_{i'}-\omega_k)]\Big)\text{ for all }k.$$

Under the optimal policy, we have $s_i^{m*} = d_i^{m*}$, so the optimal price and wage of a monopoly (p_i^{m*}, w_i^{m*}) can be obtained by $\big(p_i(s_i^{m*}, p_{-i}^{(0)}, w_{-i}^{(0)}), w_i(s_i^{m*}, p_{-i}^{(0)}, w_{-i}^{(0)})\big)$, where s_i^{m*} is the solution to the following optimization problem:

$$\begin{aligned}
&\max_s \Big[\pi_i(s|p_{-i}^{(0)}, w_{-i}^{(0)}) + \sum_{i'\neq i}\pi_{i'}(s)\Big]\\
&\text{where } \pi_{i'}(s) = (p_{i'}^{(0)} - w_{i'}^{(0)})\min\{d_{i'}, s_{i'}\},\ i'\neq i\\
&\text{with } d_{i'} = \sum_{j=1}^{m}\frac{\Lambda_j\exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]}{1 + \sum_{i''=1}^{n}\exp[\nu_j + \min\{1, s_{i''}/d_{i''}\}(q_{i''j} - \kappa_j p_{i''} - \nu_j)]}\\
&s_{i'} = \sum_{k=1}^{l}\frac{\Gamma_k\exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]}{1 + \sum_{i''=1}^{n}\exp[\omega_k + \min\{1, d_{i''}/s_{i''}\}(a_{i''k} + \eta_k w_{i''} - \omega_k)]}.
\end{aligned}$$

One can easily check that, for all $i' \neq i$, $d_{i'}$, $s_{i'}$, and $\pi_{i'}(\cdot)$ are all strictly decreasing in s. Since $s_i^{(1)}$ is the maximizer of $\pi_i(s)$, we must have $s_i^{m*} < s_i^{(1)}$. Since, by the Proof of Lemma 4, $p_i(s, p_{-i}^{(0)}, w_{-i}^{(0)})$ is strictly decreasing in s, whereas $w_i(s, p_{-i}^{(0)}, w_{-i}^{(0)})$ is strictly increasing in s, we have $p_i^{(1)} = p_i(s_i^{(1)}, p_{-i}^{(0)}, w_{-i}^{(0)}) < p_i(s_i^{(m*)}, p_{-i}^{(0)}, w_{-i}^{(0)})$ and $w_i^{(1)} = w_i(s_i^{(1)}, p_{-i}^{(0)}, w_{-i}^{(0)}) > w_i(s_i^{(m*)}, p_{-i}^{(0)}, w_{-i}^{(0)})$. Then, we have shown that $p_i^{(1)} < p_i^{(0)}$ and $w_i^{(1)} > w_i^{(0)}$ for all $i = 1, 2, ..., n$.

We next show that if $p_{i'}^{(k)} < p_{i'}^{(m*)}$ and $w_{i'}^{(k)} > w_{i'}^{(m*)}$ for any $i' \neq i$, then $p_i^{(k+1)} < p_i^{(m*)}$ and $w_i^{(k+1)} > w_i^{(m*)}$. Assume by contradiction that either $p_i^{(k+1)} \geq p_i^{(m*)}$ or $w_i^{(k+1)} \leq w_i^{(m*)}$. Then, we have $s_i^{(k+1)} < s_i^{m*}$ and $m_i^{(k+1)} := p_i^{(k+1)} - w_i^{(k+1)} > m_i^{m*} := p_i^{(m*)} - w_i^{(m*)}$. As shown in the proof of Theorem 1, $\partial_s m(s, p_{-i}, w_{-i})s$ is independent of (p_{-i}, w_{-i}) and decreasing in s. Thus, we have:

$$\partial_s\pi_i(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) = \partial_s m_i^{(k+1)} s_i^{(k+1)} + m_i^{(k+1)} > \partial_s m_i^{(m*)} s_i^{(m*)} + m_i^{(m*)} = \partial_s\pi_i(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}),$$

where the inequality follows from $s_i^{(k+1)} < s_i^{m*}$ and $m_i^{(k+1)} > m_i^{m*}$. By the FOC of the monopoly model,

$$\partial_s\pi_i(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) + \sum_{i'\neq i}\partial_s\pi_{i'}(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) = 0,$$

so, we have that

$$\partial_s\pi_i(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) = -\sum_{i'\neq i}\partial_s\pi_{i'}(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) > 0,$$

where the inequality follows from the fact that $\pi_{i'}(\cdot|p_{-i}^{(m*)}, w_{-i}^{(m*)})$ is strictly decreasing in s (for $i' \neq i$). This implies that $\partial_s\pi_i(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) > 0$, which contradicts the FOC $\partial_s\pi_i(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) = 0$. Thus, we must have $p_i^{(k+1)} < p_i^{(m*)}$ and $w_i^{(k+1)} < w_i^{(m*)}$ for all i. Proposition 1(a) then follows from taking the limit $p_i^* = \lim_{k\to+\infty} p_i^{(k)} < p_i^{(0)} = p_i^{m*}$ and $w_i^* = \lim_{k\to+\infty} w_i^{(k)} > w_i^{(0)} = w_i^{m*}$ for $i = 1, 2, ..., n$.

Part (b). We first show that the best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in Λ_j for any $j = 1, 2, ..., m$. Recall from the proof of Theorem 1 that $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ can be characterized as the solution to the following optimization problem:

$$\max_s \ \pi_i(s|p_{-i}, w_{-i}, \Lambda_j)$$

where $\pi_i(s|p_{-i}, w_{-i}) = (p_i - w_i)s$

$$\sum_{j'=1}^{m} d_{ij'} = s$$

$$p_i = \frac{q_{ij'}}{\kappa_{j'}} - \frac{1}{\kappa_{j'}} \log\left(\frac{d_{ij'}/\Lambda_{j'}}{1 - d_{ij'}/\Lambda_{j'}}\right) - \frac{1}{\kappa_{j'}} \log\left(1 + \sum_{i' \neq i} \exp[\nu_{j'} + \min\{1, s_{i'}/d_{i'}\}(q_{i'j'} - \kappa_{j'} p_{i'} - \nu_{j'})]\right) \text{ for all } j'$$

$$\sum_{k=1}^{l} s_{ik} = s$$

$$w_i = -\frac{a_{ik}}{\eta_k} - \frac{1}{\eta_k} \log\left(\frac{s_{ik}/\Gamma_k}{1 - s_{ik}/\Gamma_k}\right) - \frac{1}{\eta_k} \log\left(1 + \sum_{i' \neq i} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]\right) \text{ for all } k.$$

By computing the cross derivative, one can see that $\pi_i(s|p_{-i}, w_{-i}, \Lambda_j)$ is supermodular in (s, Λ_j) for any j. Therefore, s^* and

$$w_i(s, p_{-i}, w_{-i}) = \frac{a_{ik}}{\eta_k} - \frac{1}{\eta_k} \log\left(\frac{s_{ik}/\Gamma_k}{1 - s_{ik}/\Gamma_k}\right) - \frac{1}{\eta_k} \log\left(1 + \sum_{i' \neq i} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]\right) \text{ for all } k$$

are increasing in Λ_j for any i and j.

We define $t_{ij'} := \frac{d_{ij'}/\Lambda_{j'}}{1 - d_{ij'}/\Lambda_{ij'}} = \frac{d_{ij'}}{\Lambda_{j'} - d_{ij'}}$. We then have $d_{ij'} = \frac{\Lambda_{j'} t_{ij'}}{1 - t_{ij'}}$ and can write the following:

$$s^* = \max_s \ \pi_i(s|p_{-i}, w_{-i}, \Lambda_j)$$

where $\pi_i(s|p_{-i}, w_{-i}) = (p_i - w_i)s$

$$\sum_{j'=1}^{m} \frac{\Lambda_{j'} t_{ij'}}{1 - t_{ij'}} = s$$

$$p_i = \frac{q_{ij'}}{\kappa_{j'}} - \frac{1}{\kappa_{j'}} \log\left(t_{ij'}\right) - \frac{1}{\kappa_{j'}} \log\left(1 + \sum_{i' \neq i} \exp[\nu_{j'} + \min\{1, s_{i'}/d_{i'}\}(q_{i'j'} - \kappa_{j'} p_{i'} - \nu_{j'})]\right) \text{ for all } j'$$

$$\sum_{k=1}^{l} s_{ik} = s$$

$$w_i = -\frac{a_{ik}}{\eta_k} - \frac{1}{\eta_{ik}} \log\left(\frac{s_{ik}/\Gamma_k}{1 - s_{ik}/\Gamma_k}\right) + \frac{1}{\eta_k} \log\left(1 + \sum_{i' \neq i} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]\right) \text{ for all } k.$$

If Λ_j increases, it follows that $t^*_{ij'}$, which solves the above optimization problem decreases for all j'. Thus,

$$p_i(p_{-i}, w_{-i}) = \frac{q_{ij'}}{\kappa_{j'}} - \frac{1}{\kappa_{j'}} \log\left(t^*_{ij'}\right) - \frac{1}{\kappa_{j'}} \log\left(1 + \sum_{i' \neq i} \exp[\nu_{j'} + \min\{1, s_{i'}/d_{i'}\}(q_{i'j'} - \kappa_{j'} p_{i'} - \nu_{j'})]\right)$$

is increasing in Λ_j. We then have proved that both $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in Λ_j. Since $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are both increasing in p_{-i} and w_{-i}, then $p_i^{(k)}$ and $w_i^{(k)}$ are increasing in Λ for any $k \geq 1$. By Theorem 1, $(p^*, w^*) = \lim_{k \uparrow +\infty}(p^{(k)}, w^{(k)})$. Thus, $p_i^* = \lim_{k \uparrow +\infty} p_i^{(k)}$ and $w_i^* = \lim_{k \uparrow +\infty} w_i^{(k)}$ for $i = 1, 2, ..., n$ are increasing in Λ_j for any j. This concludes the proof of Proposition 1(b). □

Proof of Theorem 2

As in the proof of Theorem 1, we prove Theorem 2 using the following three steps:

- Under equilibrium, $s_i^{c*} \geq d_i^{c*}$, that is, supply dominates demand.

- The best-response price $p_i^c(p_{-i})$ is continuously increasing in p_j for any $j \neq i$. By Tarski's Fixed Point Theorem, this monotonicity implies that an equilibrium exists.
- The best-response mapping $T^c(p) = (p_i^c(p_{-i}) : i = 1, 2, ..., n)$ satisfies

$$||T^c(p) - T^c(p')||_1 \leq q_c ||p - p'||_1 \text{ for some } q_c \in (0, 1).$$

This will imply that the equilibrium is unique and can be computed using a *tatônnement* scheme.

Step I. $s_i^{c*} \geq d_i^{c*}$

If $s_i^{c*} < d_i^{c*}$, then P_i can increase its price from p_i^{c*} to $\hat{p}_i^{c*} = p_i^{c*} + \epsilon$ (for a small $\epsilon > 0$), and accordingly its wage from $\beta_i p_i^{c*}$ to $\beta_i p_i^{c*} + \beta_i \epsilon$, where ϵ is small enough so that $\hat{s}_i^{c*} \leq \hat{d}_i$. With this price adjustment, P_i's profit increases by at least $(1 - \beta_i)\epsilon s_i^{c*} > 0$, hence contradicting that (p_i^{c*}, p_{-i}^{c*}) is an equilibrium. Therefore, we must have $s_i^{c*} \geq d_i^{c*}$ for $i = 1, 2, ..., n$.

Step II. $p_i^c(p_{-i})$ *is continuously increasing in* p_j *for all* $j \neq i$

Since $s_i^{c*} \geq d_i^{c*}$, the price/wage optimization of P_i can be formulated as follows:

$$\begin{aligned} &\max_{p_i} (1 - \beta_i) p_i d_i \\ &\text{s.t. } d_i = \sum_{j=1}^{m} \frac{\Lambda_j \exp(q_{ij} - \kappa_j p_i)}{1 + \exp(q_{ij} - \kappa_j p_i) + \sum_{i' \neq i} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]} \\ &\qquad s_i = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i p_i - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i p_i - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_j \beta_{i'} p_{i'} - \omega_j)]} \\ &\qquad s_i \geq d_i. \end{aligned}$$

Note that the objective function is supermodular in $(p_1, p_2, ..., p_n)$ and that the feasible set is a lattice. Thus, the best-response price $p_i^c(p_{-i})$ is continuously increasing in $p_{i'}$ for all $i' \neq i$. By Tarski's Fixed Point Theorem, an equilibrium p^{c*} exists.

Step III. $T^c(\cdot)$ *is a contraction mapping under the* ℓ_1 *norm*

As shown in the proof of Step II above,

$$\begin{aligned} p_i^c(p_{-i}) = &\arg\max_{p_i} (1 - \beta_i) p_i d_i \\ &\text{s.t. } d_i = \sum_{j=1}^{m} \frac{\Lambda_j \exp(q_{ij} - \kappa_j p_i)}{1 + \exp(q_{ij} - \kappa_j p_i) + \sum_{i' \neq i} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]} \\ &\qquad s_i = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + (a_{ik} + \eta_k \beta_i p_i - \omega_k) \cdot d_i/s_i]}{1 + \exp[\omega_k + (a_{ik} + \eta_k \beta_i p_i - \omega_k) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k \beta_{i'} p_{i'} - \omega_k)]} \\ &\qquad s_i \geq d_i. \end{aligned}$$

We define $\underline{p}_i(p_{-i})$ as the unconstrained optimizer of $p_i d_i$ (without the constraint $s_i \geq d_i$), which is increasing in $p_{i'}$ for each $i' \neq i$, as shown in Step II. We also define $\bar{p}_i(p_{-i})$ as the unique p_i such that $s_i = d_i$, which is also increasing in $p_{i'}$ $(i' \neq i)$. We have $p_i^c(p_{-i}) = \max\{\underline{p}_i(p_{-i}), \bar{p}_i(p_{-i})\}$. It suffices to show that both $\underline{p}(\cdot) := (\underline{p}_1(\cdot), \underline{p}_2(\cdot), ..., \underline{p}_n(\cdot))$ and $\bar{p}(\cdot) := (\bar{p}_1(\cdot), \bar{p}_2(\cdot), ..., \bar{p}_n(\cdot))$ are contraction mappings under the ℓ_1 norm. We next show that there exists a constant $C \in (0, 1)$, such that for any $p, p' \in \mathbb{R}_+^n$,

$$||\underline{p}(p) - \underline{p}(p')||_1 \leq C||p - p'||_1 \text{ and } ||\bar{p}(p) - \bar{p}(p')||_1 \leq C||p - p'||_1.$$

Since the MNL demand model satisfies the diagonal dominance condition, that is, for any j, $\frac{\partial^2 d_{ij}}{\partial p_i \partial p_{i'}} > 0$ for any $i' \neq i$, and

$$\frac{\partial^2 d_{ij}}{\partial (p_i)^2} < -\sum_{i' \neq i} \frac{\partial^2 d_{ij}}{\partial p_i \partial p_{i'}} < 0,$$

we have that, the ℓ_1 matrix norm for the Jacobian of $\underline{p}(\cdot)$, denoted by $\underline{C}$, is strictly below 1 (i.e., $\underline{C} < 1$). Thus,

$$||\underline{p}(p) - \underline{p}(p')||_1 \leq \underline{C}||p - p'||_1. \tag{7}$$

We also note that $\bar{p}_i(p_{-i})$ satisfies the following equation:

$$\begin{aligned}
&\sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]} \\
&= \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_j \beta_{i'} p_{i'} - \omega_j)]} := s.
\end{aligned}$$

If $\hat{p}_{i'} = p_{i'} + \delta$ for some $i' \neq i$ and $\delta > 0$ and $\hat{p}_{i''} = p_{i''}$ for all other $i'' \neq i$ and $i'' \neq i'$, we have

$$\sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]} > s, \text{ whereas}$$

$$\sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]} < s.$$

We denote the induced supply and demand under the price vector $\hat{p}$ as $\hat{s}$. Furthermore, we have

$$\begin{aligned}
&\sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(\hat{p}_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(\hat{p}_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]} \\
&< \sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]} := \bar{s}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(\hat{p}_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(\hat{p}_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]} \\
&> \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]} =: \underline{s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{s} &= \sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(\hat{p}_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(\hat{p}_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]} \\
&> \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(\hat{p}_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(\hat{p}_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]} \in (\underline{s}, \bar{s}).
\end{aligned}$$

If $\hat{s} < s$, define p' as the solution to $\sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i p' - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]} = s > \hat{s}$. Hence, $\bar{p}_i(\hat{p}_{-i}) < p'$. By the diagonal dominance property of the MNL demand model, we have $0 < \bar{p}(\hat{p}_{-i}) - \bar{p}(p_{-i}) < p' - \bar{p}(p_{-i}) < q_s \delta$, where $q_s := \max\{\frac{\exp(a_{ik})\beta_i}{1 + \exp(a_{ik})} : 1 \leq i \leq n, 1 \leq k \leq l\} < 1$.

Analogously, if $\hat{s} > s$, assume that p'' satisfies $\sum_{j=1}^{m} \frac{\Lambda_j \exp(q_{ij} - \kappa_j p'')}{1 + \exp(q_{ij} - \kappa_j p'') + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]} = s < \hat{s}$. Since $\hat{s} > s$, we have $\bar{p}(\hat{p}_{-i}) < p''$. By the diagonal dominance condition of the MNL model, we have $0 < \bar{p}(\hat{p}_{-i}) - \bar{p}(p_{-i}) < p'' - \bar{p}(p_{-i}) < q_d \delta$, where $q_d := \max\{\frac{\exp(q_{ij})\beta_i}{1 + \exp(a_{ij})} : 1 \leq i \leq n, 1 \leq j \leq m\} < 1$.

We define $q_c := \max\{q_d, q_s\} < 1$. The above analysis implies that

$$||\bar{p}(p) - \bar{p}(p')||_1 \le q_c ||p - p'||_1 \tag{8}$$

By combining Equations (7) and (8), we obtain $||\bar{p}(p) - \bar{p}(p')||_1 \le C||p - p'||_1$, where $C := \max\{\underline{C}, q_c\} < 1$. We have established that under a fixed commission, the best-response is a contraction mapping over the strategy space. Then, by using Banach's Fixed Point Theorem, a unique equilibrium exists and can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 2. □

B.1. Proof of Proposition 2

In the the proof of Proposition 2, we define:

$$(p^{(k)}, w^{(k)}) := T^{(k)}(p^{c*}, \beta p^{c*}) \text{ for } k \ge 1,$$

and $(p^{(0)}, w^{(0)}) = (p^{c*}, \beta p^{c*})$. By Theorem 1, we have $(p^{(k)}, w^{(k)})$ converges to (p^*, w^*), as $k \uparrow +\infty$. Furthermore, by the symmetry of the model primitives, we have, if p^{c*} is symmetric, $p_1^{(k)} = p_2^{(k)} = ... = p_n^{(k)}$ and $w_1^{(k)} = w_2^{(k)} = ... = w_n^{(k)}$ for each $k \ge 0$.

Part (a). By Theorems 1 and 2, there exist a unique equilibrium (p^*, w^*) in the base model and a unique equilibrium p^{c*} in the model with a fixed-commission rate. If (p^*, w^*) is not symmetric, since all the model parameters are symmetric with respect to different platforms, customer segments, and worker types, we can find a permutation of (p^*, w^*), which is not identical to (p^*, w^*), but still an equilibrium, thus contradicting the uniqueness of (p^*, w^*). Therefore, (p^*, w^*) must be symmetric, i.e. $p_1^* = p_2^* = ... = p_n^*$ and $w_1^* = w_2^* = ... = w_n^*$. Similarly, we have p^{c*} is symmetric for the model with a fixed-commission rate, i.e., $p_1^{c*} = p_2^{c*} = ... = p_n^{c*}$. This proves **part (a)**.

Part (b). If $\beta = w_i^*/p_i^*$, it is straightforward to check that $p_i^* = p_i^{c*}$ and $w_i^* = \beta p_i^* = \beta p_i^{c*} = w_i^{c*}$ for all i. Therefore, $d_i^* = d_i^{c*}$ for all i as well. $\pi_i^* = (p_i^* - w_i^*)d_i^* = (p_i^{c*} - w_i^{c*})d_i^{c*} = \pi_i^{c*}$. If $\beta \ne w_i^*/p_i^*$, by the definition of the best-response operator $T(\cdot, \cdot)$, we have that, for each $k \ge 0$,

$$(p_i^{(k+1)} - w_i^{(k+1)})d_i^{(k+1)} > (p_i^{(k)} - w_i^{(k)})d_i^{(k)}.$$

Therefore,

$$\pi_i^* = (p_i^* - w_i^*)d_i^* = \lim_{k\uparrow+\infty} (p_i^{(k)} - w_i^{(k)})d_i^{(k)} > (p_i^{(0)} - w_i^{(0)})d_i^{(0)} = (p_i^{c*} - w_i^{c*})d_i^{c*} = \pi_i^{c*}.$$

This proves **part (b)**.

Part (c). Because $w_i^{(0)}/p_i^{(0)} = \beta < w_i^*/p_i^*$, exactly the same argument as the proof of Proposition 1(a) demonstrates that $p_i^{(1)} < p_i^{(0)}$ and $w_i^{(1)} > w_i^{(0)}$. Furthermore, by an induction argument similar to the proof of Proposition 1(a), we have if $p_i^{(k)} < p_i^{(0)}$ and $w_i^{(k)} > w_i^{(0)}$ then $p_i^{(k+1)} < p_i^{(0)}$ and $w_i^{(k+1)} > w_i^{(0)}$ for all $k \ge 1$. Putting these inequalities together and taking k to limit, we have $p_i^* = \lim_{k\uparrow+\infty} p_i^{(k)} < p_i^{(0)} = p_i^{c*}$ and $w_i^* = \lim_{k\uparrow+\infty} w_i^{(k)} > w_i^{(0)} = w_i^{c*}$ for all i. Finally, $d_i^* > d_i^{c*}$ follows directly from $p_i^* < p_i^{c*}$. This proves **part (c)**.

Part (c). Because $w_i^{(0)}/p_i^{(0)} = \beta > w_i^*/p_i^*$, exactly the same argument as the proof of Proposition 1(a) demonstrates that $p_i^{(1)} > p_i^{(0)}$ and $w_i^{(1)} < w_i^{(0)}$. Furthermore, by an induction argument similar to the proof of Proposition 1(a), we have if $p_i^{(k)} > p_i^{(0)}$ and $w_i^{(k)} < w_i^{(0)}$ then $p_i^{(k+1)} > p_i^{(0)}$ and $w_i^{(k+1)} < w_i^{(0)}$ for all $k \ge 1$. Putting these inequalities together and taking k to limit, we have $p_i^* = \lim_{k\uparrow+\infty} p_i^{(k)} > p_i^{(0)} = p_i^{c*}$ and $w_i^* = \lim_{k\uparrow+\infty} w_i^{(k)} < w_i^{(0)} = w_i^{c*}$ for all i. Finally, $d_i^* < d_i^{c*}$ follows directly from $p_i^* > p_i^{c*}$. This proves **part (d)**. □

Proof of Corollary 1

The first part follows from the same argument as in the proof of Theorem 1. If $s_i^{s*} < d_i^{s*}$, then P_i can increase its price and strictly increase its profit. If $s_i^{s*} > d_i^{s*}$, then P_i can decrease its price and strictly increase its profit. As a result, under equilibrium, we must have $s_i^{s*} = d_i^{s*}$ for $i = 1, 2, ..., n$. Similarly, the equilibrium existence and uniqueness follow from the same argument as in the proof of Theorem 1. □

Proof of Theorem 3

We first observe that the same argument as in the proof of Step I of Theorem 1 implies that, in equilibrium, the supply and demand of each platform should match. More specifically, if $\tilde{s}_i^* > \tilde{\lambda}_i^*$ (resp. $\tilde{s}_i^* < \tilde{\lambda}_i^*$), P_i can decrease (resp. increase) its wage $\tilde{w}_i$ (resp. price $\tilde{p}_i$) by a sufficiently small amount to strictly increase its profit. Here, $\tilde{s}_i^*$ is the equilibrium supply of P_i, $\tilde{\lambda}_1^* = \tilde{d}_1^* + \tilde{d}_x^*/\tilde{n}$ is the total equilibrium demand for P_1's workers, and $\tilde{\lambda}_i^* = \tilde{d}_i^*$ is the total equilibrium demand for P_i's workers ($i = 2, 3, ..., n$). Using $\tilde{s}_i^* = \tilde{\lambda}_i^*$, we can write P_i's profit function as follows:

$$\tilde{\pi}_i(\tilde{p}, \tilde{w}) = (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i + \gamma_i\Big(\tilde{p}_x - \frac{\tilde{w}_1}{\tilde{n}}\Big)\tilde{d}_x.$$

Given P_{-i}'s strategy, $(\tilde{p}_{-i}, \tilde{w}_{-i})$, we use $\tilde{p}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ and $\tilde{w}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ to denote the best-response price and wage of P_i under coopetition. Given $(\tilde{p}_{-i}, \tilde{w}_{-i}, \tilde{p}_x)$, the price and wage optimization of P_1 can be formulated as follows:

$$\begin{aligned}
\max_{(\tilde{p}_1, \tilde{w}_1, \tilde{d}_1, \tilde{d}_x)} \quad & (\tilde{p}_1 - \tilde{w}_1)\tilde{d}_1 + \gamma_1\Big(\tilde{p}_x - \frac{\tilde{w}_1}{\tilde{n}}\Big)\tilde{d}_x \\
\text{where} \quad & \sum_{j=1}^{m} \tilde{d}_{1j} = \tilde{d}_1 \\
& \tilde{p}_1 = \frac{q_{1j}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{\tilde{d}_{1j}/\Lambda_j}{1 - \tilde{d}_{1j}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1 + \sum_{i' \neq 1}\exp[\nu_j + \min\{1, \tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j} - \kappa_j\tilde{p}_{i'} - \nu_j)]\Big) \text{ for all } j \\
& \sum_{j=1}^{m} \tilde{d}_{xj} = \tilde{d}_x \\
& \tilde{p}_x = \frac{q_{xj}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{\tilde{d}_{xj}/\Lambda_j}{1 - \tilde{d}_{xj}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1 + \sum_{i' \neq x}\exp[\nu_j + \min\{1, \tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j} - \kappa_j\tilde{p}_{i'} - \nu_j)]\Big) \text{ for all } j \\
& \sum_{k=1}^{l} \tilde{s}_{1k} = \tilde{d}_1 + \frac{\tilde{d}_x}{\tilde{n}} \\
& \tilde{w}_1 = -\frac{a_{1k}}{\eta_k} - \frac{1}{\eta_{1k}}\log\Big(\frac{\tilde{s}_{1k}/\Gamma_k}{1 - \tilde{s}_{1k}/\Gamma_k}\Big) + \frac{1}{\eta_k}\log\Big(1 + \sum_{i' \neq 1}\exp[\omega_k + \min\{1, \tilde{d}_{i'}/\tilde{s}_{i'}\}(a_{i'k} + \eta_k\tilde{w}_{i'} - \omega_k)]\Big) \text{ for all } k.
\end{aligned} \tag{9}$$

In addition, the price and wage optimization for P_i $(i=2,3,...,n)$ can be formulated as follows (we use $\gamma_2 = 1-\gamma_1$ and $\gamma_i = 0$ for $i=3,4,...,n$):

$$\begin{aligned}
&\max_{(\tilde{p}_i,\tilde{w}_i,\tilde{d}_i)} \ (\tilde{p}_i-\tilde{w}_i)\tilde{d}_i + \gamma_i\Big(\tilde{p}_x - \frac{\tilde{w}_1}{\tilde{n}}\Big)\tilde{d}_x \\
&\text{where } \sum_{j=1}^m \tilde{d}_{ij} = \tilde{d}_i \\
&\qquad \tilde{p}_i = \frac{q_{ij}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{\tilde{d}_{ij}/\Lambda_j}{1-\tilde{d}_{ij}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1+\sum_{i'\neq i}\exp[\nu_j + \min\{1,\tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j}-\kappa_j\tilde{p}_{i'}-\nu_j)]\Big) \text{ for all } j \\
&\qquad \sum_{j=1}^m \tilde{d}_{xj} = \tilde{d}_x \\
&\qquad \tilde{p}_x = \frac{q_{xj}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{\tilde{d}_{xj}/\Lambda_j}{1-\tilde{d}_{xj}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1+\sum_{i'\neq x}\exp[\nu_j + \min\{1,\tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j}-\kappa_j\tilde{p}_{i'}-\nu_j)]\Big) \text{ for all } j \\
&\qquad \sum_{k=1}^l \tilde{s}_{ik} = \tilde{d}_i \\
&\qquad \tilde{w}_i = -\frac{a_{ik}}{\eta_k} - \frac{1}{\eta_{ik}}\log\Big(\frac{\tilde{s}_{ik}/\Gamma_k}{1-\tilde{s}_{ik}/\Gamma_k}\Big) + \frac{1}{\eta_k}\log\Big(1+\sum_{i'\neq i}\exp[\omega_k + \min\{1,\tilde{d}_{i'}/\tilde{s}_{i'}\}(a_{i'k}+\eta_k\tilde{w}_{i'}-\omega_k)]\Big) \text{ for all } k.
\end{aligned} \tag{10}$$

Following the same argument as in the proof of Step II of Theorem 1, we have that both $\tilde{p}_i(\tilde{p}_{-i},\tilde{w}_{-i})$ and $\tilde{w}_i(\tilde{p}_{-i},\tilde{w}_{-i})$ are continuously increasing in $\tilde{p}_{-i}$ and $\tilde{w}_{-i}$ for $i=1,2,...,n$. Therefore, by Tarski's Fixed Point Theorem, an equilibrium exists for the model with coopetition.

To show that the equilibrium is unique, we follow the same argument as in the proof of Lemma 5. It suffices to show that for some k, the k-fold best-response mapping, $\tilde{T}^{(k)}(\tilde{p},\tilde{w})$, (defined in a similar fashion as $T^{(k)}(\cdot,\cdot)$, but for the model with coopetition) is a contraction mapping. The same argument as in the proof of Lemma 5 implies that for any $(\tilde{p},\tilde{w})$ and $(\tilde{p}',\tilde{w}')$, we have

$$||\tilde{T}^{(k)}(\tilde{p},\tilde{w}) - \tilde{T}^{(k)}(\tilde{p}',\tilde{w}')||_1 \leq 2E^k||(p,w)-(\tilde{p}',\tilde{w}')||_1,$$

where $E<1$ is defined in the proof of Lemma 5. Consequently, $\tilde{T}^{(k^*)}$ is a contraction mapping under the ℓ_1 norm, where k^* is the smallest integer satisfying $2C^{(k^*)}<1$ (i.e., $k^* > -\log(2)/\log(E)$). The contraction mapping property of $T^{(k^*)}(\cdot,\cdot)$, as shown in the proof of Theorem 1, implies that the equilibrium is unique in the presence of coopetition, and that it can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 3. □

Proof of Theorem 4

We first show that if $\tilde{p}_n \uparrow +\infty$, then $(\tilde{p}_i^*,\tilde{w}_i^*)$ converges to (p_i^*,w_i^*) for $i=1,2...,n$. For given $(\tilde{p},\tilde{w}) = (\tilde{p}_1,\tilde{w}_1,\tilde{p}_2,\tilde{w}_2,...,\tilde{p}_3,\tilde{w}_3)$, we define the two-dimensional sequence $\{\big(\tilde{p}_i(k,j),\tilde{w}_i(k,j)\big) : 1\leq i\leq n, k\geq 1, j\geq 1\}$, where $\big(\tilde{p}(k,j),\tilde{w}(k,j)\big) = \tilde{T}^{(k)}(\tilde{p},\tilde{w})$ with $\tilde{p}_x = j$. From the proof of Lemma 5, we know that $\lim_{j\uparrow+\infty}\big(\tilde{p}(k,j),\tilde{w}(k,j)\big) = T^{(k)}(\tilde{p},\tilde{w})$.

Therefore, as shown in the proof of Theorem 3, the equilibrium strategies with $\tilde{p}_x = j$ satisfy $\big(\tilde{p}^*(j),\tilde{w}^*(j)\big) = \lim_{k\uparrow+\infty}\big(\tilde{p}(k,j),\tilde{w}(k,j)\big)$. Using the proof of Theorem 3, we have $||T^{(k)}(\tilde{p},\tilde{w}) - T^{(k)}(\tilde{p}',\tilde{w}')||_1 \leq 2E^k||(\tilde{p},\tilde{w})-(\tilde{p}',\tilde{w}')||_1$ for $k\geq 1$. Thus,

$$|\tilde{p}_i(k+1,j) - \tilde{p}_i(k,j)| \leq 2E^k||(\tilde{p}(1,j),\tilde{w}(1,j)) - (\tilde{p},\tilde{w})||_1,$$

$$|\tilde{w}_i(k+1,j)-\tilde{w}_i(k,j)|\leq 2E^k||(\tilde{p}(1,j),\tilde{w}(1,j))-(\tilde{p},\tilde{w})||_1.$$

As a result, $\sum_{k=1}^{+\infty}|\tilde{p}_i(k+1,j)-\tilde{p}_i(k,j)|<+\infty$ and $\sum_{k=1}^{+\infty}|\tilde{w}_i(k+1,j)-\tilde{w}_i(k,j)|<+\infty$ for $i=1,2,...,n$. Using the dominated convergence theorem, we obtain, for all i,

$$\lim_{j\uparrow+\infty}\lim_{k\uparrow+\infty}(\tilde{p}_i(k,j),\tilde{w}_i(k,j))=\lim_{k\uparrow+\infty}\lim_{j\uparrow+\infty}(\tilde{p}_i(k,j),\tilde{w}_i(k,j))$$

that is,

$$\lim_{j\uparrow+\infty}(\tilde{p}^*(j),\tilde{w}^*(j))=\lim_{j\uparrow+\infty}\lim_{k\uparrow+\infty}(\tilde{p}(k,j),\tilde{w}(k,j))=\lim_{k\uparrow+\infty}\lim_{j\uparrow+\infty}(\tilde{p}(k,j),\tilde{w}(k,j))=\lim_{k\uparrow+\infty}T^{(k)}(\tilde{p},\tilde{w})=(p^*,w^*),$$

which states that if $\tilde{p}_x\uparrow+\infty$, then $(\tilde{p}_i^*,\tilde{w}_i^*)$ converges to (p_i^*,w_i^*) for $i=1,2,...,n$.

We next show that $\tilde{\pi}(\tilde{p}_x):=\tilde{\pi}_1(\tilde{p}^*(\tilde{p}_x),\tilde{w}^*(\tilde{p}_x))+\tilde{\pi}_2(\tilde{p}^*(\tilde{p}_x),\tilde{w}^*(\tilde{p}_x))$ is decreasing in $\tilde{p}_x$ for sufficiently large $\tilde{p}_x$, where $(\tilde{p}_i^*(\tilde{p}_x),\tilde{w}_i^*(\tilde{p}_x))$ is the equilibrium outcome of P_i under coopetition when the price of the new service is $\tilde{p}_x$.

We first show that, under a given equilibrium price and wage vector $(\tilde{p}^*,\tilde{w}^*)$ associated with $\tilde{p}_x$, the total profit of P_1 and P_2, $\tilde{\pi}(\tilde{p}_x|\tilde{p}^*,\tilde{w}^*)$ is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_x$, where

$$\tilde{\pi}(\tilde{p}_x|\tilde{p}^*,\tilde{w}^*)=(\tilde{p}_1^*-\tilde{w}_1^*)\tilde{d}_1+(\tilde{p}_2^*-\tilde{w}_2^*)\tilde{d}_2+\Big(\tilde{p}_x-\frac{\tilde{w}_1^*}{\tilde{n}}\Big)\tilde{d}_x.$$

By Lemma 2, we have

$$\begin{aligned}\partial_{\tilde{p}_x}\tilde{\pi}(\tilde{p}_x|\tilde{p}^*,\tilde{w}^*)=&(\tilde{p}_1^*-\tilde{w}_1^*)\partial_{\tilde{p}_x}\tilde{d}_1+(\tilde{p}_2^*-\tilde{w}_2^*)\partial_{\tilde{p}_x}\tilde{d}_2+\tilde{d}_x+\Big(\tilde{p}_x-\frac{\tilde{w}_1^*}{\tilde{n}}\Big)\partial_{\tilde{p}_x}\tilde{d}_x\\ =&(\tilde{p}_1^*-\tilde{w}_1^*)\sum_{j=1}^m\kappa_j\bar{d}_{1j}\bar{d}_{xj}/\Lambda_j+(\tilde{p}_2^*-\tilde{w}_2^*)\sum_{j=1}^m\kappa_j\bar{d}_{2j}\bar{d}_{xj}/\Lambda_j+\tilde{d}_x\\ &-\Big(\tilde{p}_x-\frac{\tilde{w}_1^*}{\tilde{n}}\Big)\sum_{j=1}^m\kappa_j(1-\bar{d}_{xj}/\Lambda_j)\bar{d}_{xj}.\end{aligned}$$

Hence, $\partial_{\tilde{p}_x}\tilde{\pi}(\tilde{p}_x^*|\tilde{p}^*,\tilde{w}^*)=0$ implies that

$$\tilde{p}_x^*=(\tilde{p}_1^*-\tilde{w}_1^*)\frac{\sum_{j=1}^m\kappa_j\bar{d}_{1j}^*\bar{d}_{xj}^*/\Lambda_j}{\sum_{j=1}^m\kappa_j(1-\bar{d}_{xj}^*/\Lambda_j)\bar{d}_{xj}^*}+(\tilde{p}_2^*-\tilde{w}_2^*)\frac{\sum_{j=1}^m\kappa_j\bar{d}_{2j}^*\bar{d}_{xj}^*/\Lambda_j}{\sum_{j=1}^m\kappa_j(1-\bar{d}_{xj}^*/\Lambda_j)\bar{d}_{xj}^*}+\frac{\tilde{w}_1^*}{\tilde{n}},\tag{11}$$

where $\bar{d}_{ij}^*$ is the equilibrium demand of P_i's service, when $\tilde{p}_x=\tilde{p}_x^*$ satisfies Equation (11). We observe that the right-hand side of Equation (11) is decreasing with respect to $\tilde{p}_x$. Therefore, there exists a unique $\tilde{p}_x^*$ such that Equation (11) holds. Furthermore, one can check that $\partial_{\tilde{p}_x}\tilde{\pi}(\tilde{p}_x|\tilde{p}^*,\tilde{w}^*)>0$ (resp. <0) if $\tilde{p}_x<\tilde{p}_x^*$ (resp. $\tilde{p}_x>\tilde{p}_x^*$). As a result, $\tilde{\pi}(\cdot|\tilde{p}^*,\tilde{w}^*)$ is decreasing in $\tilde{p}_x$ for $\tilde{p}_x\geq\tilde{p}_x^*$. Note that $\tilde{p}_x^*$ is uniformly bounded from above by an upper bound on the right-hand side of Equation (11), say $\bar{p}^*:=(p_1^*-w_1^*+p_2^*-w_2^*)+w_1^*+\frac{1}{1-\bar{d}_0'/(\sum_j\Lambda_j)}$, where $\bar{d}_0'$ is the market share of the new joint service with $\tilde{p}_x=0$. It then follows that, when $\tilde{p}_x\geq\bar{p}^*$, $\tilde{\pi}(\tilde{p}_x)=\tilde{\pi}(\tilde{p}_x|\tilde{p}^*(\tilde{p}_n),\tilde{w}^*(\tilde{p}_x))$ is strictly decreasing in $\tilde{p}_x$.

We observe that as $\tilde{p}_x\uparrow+\infty$, $\tilde{d}_x\downarrow 0$. Since $(\tilde{p}^*(\tilde{p}_x),\tilde{w}^*(\tilde{p}_x))$ approaches (p^*,w^*) when $\tilde{p}_x\uparrow+\infty$, then $\tilde{\pi}(\tilde{p}_x)=\tilde{\pi}(\tilde{p}_x|\tilde{p}^*(\tilde{p}_x),\tilde{w}^*(\tilde{p}_x))$ approaches the equilibrium total profit of P_1 and P_2 without coopetition, that is, $\pi^*:=\pi_1(p^*,w^*)+\pi_2(p^*,w^*)$. Since we have shown that $\tilde{\pi}(\cdot)$ is strictly decreasing in $\tilde{p}_x\geq\bar{p}^*$ and $\lim_{\tilde{p}_x\to+\infty}\tilde{\pi}(\tilde{p}_x)=\pi^*$, then $\tilde{\pi}^*:=\max_{\tilde{p}_x}\tilde{\pi}(\tilde{p}_x)>\pi^*$, that is, the maximum total profit of P_1 and P_2 with coopetition dominates the maximum total profit without coopetition for any $\gamma \in (0,1)$. In other words, for $\tilde{p}_x \geq \bar{p}^*$, $\tilde{\pi}_1(\tilde{p}_x|\tilde{p}^*(\tilde{p}_x), \tilde{w}^*(\tilde{p}_x)) + \tilde{\pi}_2(\tilde{p}_x|\tilde{p}^*(\tilde{p}_x), \tilde{w}^*(\tilde{p}_x)) > \pi_1(p^*, w^*) + \pi_2(p^*, w^*)$. Thus, there exist a range of profit sharing parameters $(\underline{\gamma}, \bar{\gamma}) \subset (0,1)$, such that when $\gamma \in (\underline{\gamma}, \bar{\gamma})$, $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) > \pi_i(p^*, w^*)$ for $i = 1, 2$.

We next show that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) < \pi_i(p^*, w^*)$ for $i = 3, 4, ..., n$. Since $\lim_{\tilde{p}_x \uparrow +\infty} \tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) = \pi_i(p^*, w^*)$ for $i = 3, 4, ..., n$, it suffices to show that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*)$ is increasing in $\tilde{p}_x$. To show this monotonicity result, we prove that, for any k, $(\tilde{p}_i(k,j) - \tilde{w}_i(k,j))\tilde{d}_i(k,j)$ is increasing in j for $i = 3, 4, ..., n$, where $\tilde{p}_i(k,j)$ and $\tilde{w}_i(k,j)$ are defined above, and $\tilde{d}_i(k,j)$ is the associated demand (and supply) for P_i in round k of the *tatônnement* scheme. By the proof of Lemma 4, for each k, both the profit margin $\tilde{m}_i(k,j) := \tilde{p}_i(k,j) - \tilde{w}_i(k,j)$ and the demand $\tilde{d}_i(k,j)$ are increasing in the price of the new service $\tilde{p}_x = j$, and so is $(\tilde{p}_i(k,j) - \tilde{w}_i(k,j))\tilde{d}_i(k,j)$. Taking k to infinity, we obtain that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) = \lim_{k \to \infty} (\tilde{p}_i(k,j) - \tilde{w}_i(k,j))\tilde{d}_i(k,j)$ is increasing in $\tilde{p}_x = j$. This concludes the proof of Theorem 4. □

Proof of Proposition 3

By Theorem 4, we can select $\gamma_0 \in (\underline{\gamma}, \bar{\gamma})$ and $\tilde{p}_x^* = \arg\max_{\tilde{p}_x} \{\pi_1(\tilde{p}^*, \tilde{w}^*) + \pi_2(\tilde{p}^*, \tilde{w}^*)\}$ that maximize the total profit of both platforms, so that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^*, \gamma_0) > \pi_i(p^*, w^*)$, for $i = 1, 2$. Thus, for any $\theta_1 + \theta_2 = 1$ $(\theta_i > 0)$, $(\tilde{p}_x^*, \gamma_0)$ is a feasible solution to the optimization problem in (3). Therefore, an optimal solution to (3), $(\tilde{p}_x^{**}, \gamma^{**})$, exists and satisfies the following:

$$\begin{aligned}
&(\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^{**}, \gamma^{**}) - \pi_1(p^*, w^*))^{\theta_1} \cdot (\tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^{**}, \gamma^{**}) - \pi_2(p^*, w^*))^{\theta_2} \\
\geq &(\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*|\tilde{p}_n^*, \gamma_0) - \pi_1(p^*, w^*))^{\theta_1} \cdot (\tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^*, \gamma_0) - \pi_2(p^*, w^*))^{\theta_2} > 0.
\end{aligned}$$

As a result, we have $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^{**}, \gamma^{**}) > \pi_i(p^*, w^*)$ for $i = 1, 2$. Finally, by the proof of Theorem 4, we have that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*)$ $(i = 3, 4, ..., n)$ is increasing in $\tilde{p}_x$, which, together with $\lim_{\tilde{p}_x \uparrow +\infty} \tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) = \pi_i(p^*, w^*)$, implies that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) < \pi_i(p^*, w^*)$ for all $i = 3, 4, ..., n$. This concludes the proof of Proposition 3. □

Proof of Proposition 4

Part (a). Denote $(\tilde{p}_i(k,j), \tilde{w}_i(k,j) : 1 \leq i \leq n)$ as the price and wage of each platform's original service under the price of the new service $\tilde{p}_x = j$. By Equations (9) and (10), as $r \uparrow +\infty$, $\tilde{w}_i(k,j) \uparrow +\infty$ for all $i = 1, 2, ..., n$, $k = 1, 2, ...$, and $j > 0$. Then, by taking $k \uparrow +\infty$, we have that, for any $\tilde{p}_x$, the equilibrium wage of P_1, $\tilde{w}_1^* \uparrow +\infty$. By (11), we must have $\lim_{r \uparrow +\infty} \tilde{p}_x^* = +\infty$. To show that $\lim_{r \uparrow +\infty} \tilde{p}_x^{**} \uparrow +\infty$, we note that $\lim_{r \uparrow +\infty} \tilde{w}_1^*/\tilde{n} = +\infty$. Under the Nash Bargaining equilibrium, we must have $\tilde{p}_x > \tilde{w}_1/\tilde{n}$, which together with $\lim_{r \uparrow +\infty} \tilde{w}_1^*/\tilde{n} = +\infty$ leads to $\lim_{r \uparrow +\infty} \tilde{p}_x^{**} \uparrow +\infty$. This concludes the proof of Part (a).

Part (b). We next show that the total profit under coopetition increases when $\tilde{p}_x = \bar{p}$ and r is sufficiently small. Note that, as $r \downarrow 0$, by Equations (9) and (10), $\tilde{w}_i(k,j) \downarrow 0$ for all $i = 1, 2, ..., n$, $k = 1, 2, ...$, and $j > 0$. Then, by taking $k \uparrow +\infty$, we have that, for any $\tilde{p}_x$, the equilibrium wage of P_1, $\tilde{w}_1^* \downarrow 0$. Therefore, for $\tilde{p}_x = \bar{p}$, the equilibrium profit from the new service $(\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x^* > 0$. This implies that the total profit under coopetition increases when $\tilde{p}_x = \bar{p}$ and r is sufficiently small. Consequently, we can find a profit sharing parameter γ such that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*|\bar{p}, \gamma) > \pi_i(p^*, w^*)$ for $i = 1, 2$. This concludes the proof of Part (b-i).

Finally, we show Part (b-ii). Specifically, we prove that if there is a finite upper bound on the price of the new service set by the platforms, i.e., $\tilde{p}_x \leq \bar{p}$, at least one platform would be worse off under coopetition, namely, either $\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*) < \pi_1(p^*, w^*)$ or $\tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*) < \pi_2(p^*, w^*)$, when r is sufficiently large. By the proof of

Part (a), as $r \uparrow +\infty$, we have $\tilde{w}_1^* \uparrow +\infty$ for any $\tilde{p}_x$. Since $\tilde{p}_x \le \bar{p}$, the profit from the new joint service is such that $(\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x < 0$.

Furthermore, in the presence of coopetition, P_i needs to charge a lower price relative to the setting without coopetition in order to induce the same demand assuming that its competitor offers the same price. Thus, for any (p_{-i}, w_{-i}), P_i's optimal profit from its original service is lower under coopetition. By taking the index of the best-response mapping k to infinity, we have that P_i's equilibrium profit from its original service is lower under coopetition for $i = 1,2$. Since we have shown that for a sufficiently large r, the total profit from the new service $(\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x$ is negative, then the total profit of P_1 and P_2 is lower under coopetition:

$$\tilde{\pi}_1^* + \tilde{\pi}_2^* = (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1^* + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2^* + (\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x < (p_1^* - w_1^*)d_1^* + (p_2^* - w_2^*)d_2^* = \pi_1^* + \pi_2^*.$$

Consequently, when r is sufficiently large, at least one of the platforms is worse off for any γ, and this concludes the proof of Proposition 4. □

Proof of Proposition 5

We first show that the total profit under coopetition increases when $\tilde{p}_x = \bar{p}$ and q_3 is sufficiently large. Recall that given $(\tilde{p}_i, \tilde{w}_i)$ for $i = 1,2$ and $(\tilde{p}_x, \gamma)$, the price and wage optimization of P_3 can be formulated as follows:

$$\begin{aligned}
\max_{(\tilde{p}_3, \tilde{w}_3, \tilde{d}_3)} \quad & (\tilde{p}_3 - \tilde{w}_3)\tilde{d}_3 \\
\text{where} \quad & \sum_{j=1}^{m} \tilde{d}_{3j} = \tilde{d}_3 \\
& \tilde{p}_3 = \frac{q_3 \iota_j}{\kappa_j} - \frac{1}{\kappa_j}\log\left(\frac{\tilde{d}_{3j}/\Lambda_j}{1 - \tilde{d}_{3j}/\Lambda_j}\right) - \frac{1}{\kappa_j}\log\left(1 + \sum_{i' \neq 3} \exp[\nu_j + \min\{1, \tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j} - \kappa_j \tilde{p}_{i'} - \nu_j)]\right) \ \forall j \\
& \sum_{j=1}^{m} \tilde{d}_{xj} = \tilde{d}_x \\
& \tilde{p}_x = \frac{q_{xj}}{\kappa_j} - \frac{1}{\kappa_j}\log\left(\frac{\tilde{d}_{xj}/\Lambda_j}{1 - \tilde{d}_{xj}/\Lambda_j}\right) - \frac{1}{\kappa_j}\log\left(1 + \sum_{i' \neq x} \exp[\nu_j + \min\{1, \tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j} - \kappa_j \tilde{p}_{i'} - \nu_j)]\right) \ \forall j \\
& \sum_{k=1}^{l} \tilde{s}_{3k} = \tilde{d}_3 \\
& \tilde{w}_3 = -\frac{a_3 \psi_k}{\eta_k} + \frac{1}{\eta_{3k}}\log\left(\frac{\tilde{s}_{3k}/\Gamma_k}{1 - \tilde{s}_{3k}/\Gamma_k}\right) + \frac{1}{\eta_k}\log\left(1 + \sum_{i' \neq 3} \exp[\omega_k + \min\{1, \tilde{d}_{i'}/\tilde{s}_{i'}\}(a_{i'k} + \eta_k \tilde{w}_{i'} - \omega_k)]\right) \ \forall k.
\end{aligned} \tag{12}$$

It follows from the optimization problem in (12) that, given $(\tilde{p}_i, \tilde{w}_i)$ for $i = 1,2$ and $(\tilde{p}_x, \tilde{n})$, if we take $q_3 \uparrow +\infty$, the best responses of P_3 will satisfy $\tilde{p}_3 \uparrow +\infty$ and $\tilde{d}_{3j} \uparrow \Lambda_j$ for all j. Consequently, as $q_3 \uparrow +\infty$, $\tilde{d}_{1j} \downarrow 0$ and $\tilde{d}_{2j} \downarrow 0$ for all j. Since supply equals demand, we have $\tilde{s}_{1k} \downarrow 0$ and $\tilde{s}_{2k} \downarrow 0$ for all k, which imply that $w_i^* \downarrow 0$ for $i = 1,2$. Therefore, for $\tilde{p}_x = \bar{p}$, the equilibrium profit from the new service is such that $(\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x^* > 0$. Similarly, for the model without coopetition, as $q_3 \uparrow +\infty$, $d_{1j}^* \downarrow 0$ and $d_{2j}^* \downarrow 0$ for all j under equilibrium. So $d_1^* = \sum_j d_{1j}^*$ and $d_2^* = \sum_j d_{2j}^*$ will both decrease to 0 as $q_3 \downarrow 0$. Therefore, the profit of P_i without coopetition, $(p_i^* - w_i^*)d_i^*$ will decrease to 0 as $q_3 \uparrow +\infty$. This implies that the total profit of P_1 and P_2 under coopetition will increase when $\tilde{p}_x = \bar{p}$ and q_3 is sufficiently large. Consequently, we can find a profit sharing parameter γ and a price for the joint new service $\tilde{p}_x \le \bar{p}$, such that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^* | \bar{p}, \gamma) > \pi_i(p^*, w^*)$ for $i = 1,2$. This concludes the proof of the first part.

We next show that the total profit under coopetition will decrease for all $\tilde{p}_x \le \bar{p}$ and when a_3 is sufficiently large. It follows from (12) that, given $(\tilde{p}_i, \tilde{w}_i)$ for $i=1,2$ and $(\tilde{p}_x, \tilde{n})$, if we take $a_3 \uparrow +\infty$, $\tilde{w}_3 \downarrow 0$ and $\tilde{s}_{3k} \uparrow \Gamma_k$ for all k. Then, for P_i $(i=1,2)$, the wage $\tilde{w}_i$ satisfies $\tilde{w}_i = -\frac{a_{ik}}{\eta_k} + \frac{1}{\eta_{ik}} \log\left(\frac{\tilde{s}_{ik}/\Gamma_k}{1-\tilde{s}_{ik}/\Gamma_k}\right) + \frac{1}{\eta_k}\log\Big(1+\sum_{i'\neq i}\exp[\omega_k + \min\{1, \tilde{d}_{i'}/\tilde{s}_{i'}\}(a_{i'k}+\eta_k\tilde{w}_{i'} - \omega_k)]\Big)$ for all k. Since $a_{3k} = a_3\psi_k$ increases to $+\infty$ as $a_3 \uparrow \infty$, then $\tilde{w}_i$ will increase to $+\infty$ as $a_3 \uparrow +\infty$. Thus, since $\tilde{p}_x \le \bar{p} < +\infty$, the profit margin of the new joint service is negative when a_3 is sufficiently large, that is, $\tilde{p}_x - \tilde{w}_1^*/\tilde{n} < 0$.

In the presence of coopetition, P_i needs to charge a lower price relative to the setting without coopetition in order to induce the same demand assuming that its competitor offers the same price. As a result, for any (p_{-i}, w_{-i}), P_i's optimal profit from its original service is lower under coopetition. In particular, under equilibrium, P_i's profit from its original service is lower in the presence of coopetition relative to the setting without coopetition for $i=1,2$. Since we have shown that for a sufficiently large a_3, the total profit from the new service is negative, then the total profit of P_1 and P_2 is lower under coopetition, that is,

$$\tilde{\pi}_1^* + \tilde{\pi}_2^* = (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1^* + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2^* + (\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x < (p_1^* - w_1^*)d_1^* + (p_2^* - w_2^*)d_2^* = \pi_1^* + \pi_2^*.$$

Consequently, at least one of the platforms is worse off for any γ, when a_3 is sufficiently large, and this concludes the proof of Proposition 5. □

Proof of Proposition 6

First, since $d_i^* = s_i^*$ without coopetition and $\tilde{\lambda}_i^* = \tilde{s}_i^*$ with coopetition for $i=1,2,...,n$, we have

$$RS^* = \sum_{j=1}^m \frac{\Lambda_j}{\kappa_j} \log\Big(1 + \sum_{i=1}^n \exp(q_{ij} - \kappa_j p_i^*)\Big)$$

and

$$\tilde{RS}^* = \sum_{j=1}^m \frac{\Lambda_j}{\kappa_j} \log\Big(1 + \exp(q_{xj} - \kappa_j \tilde{p}_x) + \sum_{i=1}^n \exp(q_{ij} - \kappa_j \tilde{p}_i^*)\Big).$$

We observe that if $\tilde{p}_i^* \le p_i^*$ for $i=1,2,...,n$, then we have

$$\begin{aligned}\tilde{RS}^* &= \sum_{j=1}^m \frac{\Lambda_j}{\kappa_j} \log\Big(1 + \exp(q_{xj} - \kappa_j \tilde{p}_x) + \sum_{i=1}^n \exp(q_{ij} - \kappa_j \tilde{p}_i^*)\Big) \\ &> \sum_{j=1}^m \frac{\Lambda_j}{\kappa_j} \log\Big(1 + \sum_{i=1}^n \exp(q_{ij} - \kappa_j \tilde{p}_i^*)\Big) \\ &\ge \sum_{j=1}^m \frac{\Lambda_j}{\kappa_j} \log\Big(1 + \sum_{i=1}^n \exp(q_{ij} - \kappa_j p_i^*)\Big) = RS^*.\end{aligned}$$

Consequently, it suffices to show that $\tilde{p}_i^* \le p_i^*$ for $i=1,2,...,n$.

We define $(\tilde{p}^*(k,\tilde{p}_x), \tilde{w}^*(k,\tilde{p}_x)) := \tilde{T}^{(k)}(p^*, w^*)$, where $\tilde{T}^{(k)}(\cdot,\cdot)$ is the k-fold best-response mapping when the price of the new service is $\tilde{p}_x$. Then, the corresponding price and wage for P_i are given by $(\tilde{p}_i^*(k,\tilde{p}_x), \tilde{w}_i^*(k,\tilde{p}_x))$. On the other hand, we know that $(p^*, w^*) = T^{(k)}(p^*, w^*)$ for any $k \ge 1$, where $T^{(k)}(\cdot,\cdot)$ is the k-fold best-response mapping of the model without coopetition, which can also be viewed as a special case of $\tilde{T}^{(k)}(\cdot,\cdot)$ with $\tilde{p}_x = +\infty$. Comparing the best-response formulations of $\tilde{T}^{(1)}$ and $T^{(1)}$ (see the proof of Theorems 1 and 3), one can show that given (p_{-i}^*, w_{-i}^*), the best-response price $\tilde{p}_i^*(1,\tilde{p}_x)$ is increasing in $\tilde{p}_x$. Since the model without coopetition can be viewed as a special case of the model with coopetition when $\tilde{p}_x = +\infty$, we have $\tilde{p}_i^*(1,\tilde{p}_x) < \tilde{p}_i^*(1,+\infty) = p_i^*$ for all $i = 1,2,...,n$. Then, by following the same argument as in the proof of Theorem 3, we conclude that $\tilde{p}_i^*(k+1,\tilde{p}_x)$ is strictly increasing in both $\tilde{p}_x$ and $\tilde{p}_{i'}^*(k)$ for $i = 1,2,...,n$, $i' \neq i$, and for any k. Using a standard induction argument, we obtain $\tilde{p}_i^*(k,\tilde{p}_x) < \tilde{p}_i^*(k,+\infty) = p_i^*$ for $k \geq 1$ and $i = 1,2,...,n$. Thus, $\tilde{p}_i^* = \lim_{k\uparrow+\infty} \tilde{p}_i^*(k,\tilde{p}_x) < p_i^*$ for $i = 1,2,...,n$, and this concludes the proof of Proposition 6. □

Proof of Proposition 7

First, we highlight that for the model without coopetition $s_i^* = d_i^*$ for $i = 1,2,...,n$, whereas for the model with coopetition $\tilde{s}_i^* = \tilde{\lambda}_i^*$ for $i = 1,2,...,n$. We have

$$DS_i^* = \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1 + \exp(a_{ik} + \eta_k w_i^*)\right],\ i = 1,2,...,n,$$

and

$$\tilde{DS}_i^* = \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1 + \exp(a_{ik} + \eta_k \tilde{w}_i^*)\right],\ i = 1,2,...,n.$$

We next show the first part. Specifically, we show the following three claims: (a) if $\tilde{n} = 1$, then $\tilde{w}_1^* > w_1^*$; (b) if $\tilde{n}$ is sufficiently large, then $\tilde{w}_1^* < w_1^*$; and (c) $\tilde{w}_1^*$ is continuously decreasing in $\tilde{n}$. Then, Claims (a), (b), and (c) would imply the first part of Proposition 7.

Claim (a): If $\tilde{n} = 1$, from the proof of Theorem 3, we have $\tilde{s}_1^* = \tilde{\lambda}_1^* = \tilde{d}_1^* + \tilde{d}_x^*/\tilde{n} = \tilde{d}_1^* + \tilde{d}_x^*$. As shown in the proof of Proposition 6, $\tilde{p}_1^* < p_1^*$, and hence $\tilde{s}_1^* = \tilde{d}_1^* + \tilde{d}_x^* > d_1^* = s_1^*$. This implies that $\tilde{w}_1^* > w_1^*$ and concludes the proof of Claim (a).

Claim (b): As $\tilde{n} \uparrow +\infty$, we have $\tilde{s}_1^* = \tilde{\lambda}_1^* = \tilde{d}_1^* + \tilde{d}_x^*/\tilde{n} = \tilde{d}_1^*$. We next show that $\tilde{d}_1^* < d_1^*$. As in the proof of Proposition 6, for any $(\tilde{p}_x,\gamma)$, we define $(\tilde{p}^*(k,\tilde{p}_x), \tilde{w}^*(k,\tilde{p}_x)) := \tilde{T}^{(k)}(p^*,w^*)$, where $\tilde{T}^{(k)}(\cdot,\cdot)$ is the k-fold best-response mapping when the price of the new service is $\tilde{p}_x$. Then, the corresponding price and wage for P_i are given by $(\tilde{p}_i^*(k,\tilde{p}_x), \tilde{w}_i^*(k,\tilde{p}_x))$. On the other hand, we know that $(p^*,w^*) = T^{(k)}(p^*,w^*)$ for any $k \geq 1$, where $T^{(k)}(\cdot,\cdot)$ is the k-fold best-response mapping of the model without coopetition, which can also be viewed as a special case of $\tilde{T}^{(k)}(\cdot,\cdot)$ with $\tilde{p}_x = +\infty$. By comparing the best-response formulations of $\tilde{T}^{(1)}$ and $T^{(1)}$ (see the proof of Theorems 1 and 3), one can show that given (p_{-1}^*, w_{-1}^*), the best-response demand $\tilde{d}_1^*(1,\tilde{p}_x)$ is increasing in $\tilde{p}_x$. Since the model without coopetition can be viewed as a special case of the model with coopetition with $\tilde{p}_x = +\infty$, we have $\tilde{d}_1^*(1,\tilde{p}_x) < \tilde{d}_1^*(1,+\infty) = d_1^*$. Then, by following the same argument as in the proof of Theorem 3, we conclude that $\tilde{d}_1^*(k+1,\tilde{p}_x)$ is strictly increasing in $\tilde{p}_x$ for $k \geq 1$. Using an induction argument, we obtain $\tilde{d}_1^* = \lim_{k\uparrow+\infty} \tilde{d}_1^*(k,\tilde{p}_x) < d_1^*$. Thus, $\tilde{s}_1^* = \tilde{d}_1^* < d_1^* = s_1^*$. This implies that $\tilde{w}_1^* < w_1^*$ and concludes the proof of Claim (b).

Claim (c): We show that $\tilde{w}_i^*$ is decreasing in $\tilde{n}$ for any $i = 1,2,...,n$. We define $(\tilde{p}^*(k,\tilde{n}), \tilde{w}^*(k,\tilde{n})) := \tilde{T}^{(k)}(p^*,w^*)$, where $\tilde{T}^{(k)}(\cdot,\cdot)$ is the k-fold best-response mapping when the price of the new service is $\tilde{p}_x$ and the pooling parameter is $\tilde{n}$. By examining the best-response mapping $\tilde{T}^{(1)}$ (see the proof of Theorem 4), we obtain that given (p_{-i}^*, w_{-i}^*), $\tilde{w}_i^*(1,\tilde{n})$ is decreasing in $\tilde{n}$ for $i = 1,2,...,n$. Furthermore, the best-response mapping is increasing in $\tilde{w}_{-i}^*$ (see the proof of Theorem 1). Using an induction argument, we obtain that $\tilde{w}_i^*(k,\tilde{n})$ is increasing in $\tilde{w}_{-i}^*(k-1,\tilde{n})$, which is decreasing in $\tilde{n}$. Thus, $\tilde{w}_i^*(k,\tilde{n})$ is decreasing in $\tilde{n}$ for $k \geq 1$ and for $i = 1,2,...,n$. As a result, the equilibrium wage under coopetition $\tilde{w}_i^* = \lim_{k\uparrow+\infty} \tilde{w}_i^*(k,\tilde{n})$ is decreasing in $\tilde{n}$ for $i = 1, 2, ..., n$. This concludes the proof of Claim (c). Claims (a), (b), and (c) together imply that Proposition 7(a) holds.

We next show the second part of the proposition. The same argument as the proof of Claim (b) above implies that $\tilde{w}_i^* < w_i^*$ for $i = 2, 3, ..., n$, so we must have

$$\tilde{DS}_i^* = \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1 + \exp(a_{ik} + \eta_k \tilde{w}_i^*)\right] < \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1 + \exp(a_{ik} + \eta_k w_i^*)\right] = DS_i^* \text{ for all } i = 2, 3, ..., n.$$

This concludes the proof of part (b). $\square$

Proof of Proposition 8

Following the same argument as in the proof of Theorem 4, we know that if $\tilde{p}_x \to +\infty$, then $\lim_{\tilde{p}_x \uparrow +\infty}(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = (p_1^*, w_1^*, p_2^*, w_2^*)$, $\lim_{\tilde{p}_x \uparrow +\infty}(\tilde{d}_1^*, \tilde{d}_2^*) = (d_1^*, d_2^*)$, and $\lim_{\tilde{p}_x \uparrow +\infty}(\tilde{s}_1^*, \tilde{s}_2^*) = (s_1^*, s_2^*)$. Therefore, we have $\lim_{\tilde{p}_n \uparrow +\infty} \tilde{\pi}_i^* + \tilde{DS}_i^* = \pi_i^* + DS_i^*$ for $i = 1, 2$.

We next show that $\tilde{R}_i(\tilde{p}_x) := \tilde{\pi}_i^* + \tilde{DS}_i^*$ $(i = 1, 2)$ is decreasing in $\tilde{p}_x$ for a sufficiently large $\tilde{p}_n$, where $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, w_2^*)$ is the equilibrium under coopetition with $\tilde{p}_x$. Given the equilibrium price and wage vector $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$, we define the total platform and driver surplus of both platforms as follows:

$$\begin{aligned}
\tilde{R}(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) =& \tilde{R}_1(\tilde{p}_x) + \tilde{R}_2(\tilde{p}_x) \\
=& (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1^* + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2^* + (\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x \\
&+ \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1 + \exp(a_{1k} + \eta_k \tilde{w}_1^*)\right] + \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1 + \exp(a_{2k} + \eta_k \tilde{w}_2^*)\right],
\end{aligned}$$

where $\tilde{s}_1^* = \tilde{d}_1^* + \tilde{d}_x/\tilde{n}$ and $\tilde{s}_2^* = \tilde{d}_2^*$. Following the same argument as in the proof of Theorem 4, we have $\partial_{\tilde{p}_x} R(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < 0$ for a sufficiently large $\tilde{p}_x$. This also shows that $R(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ is strictly decreasing in $\tilde{p}_x$ for a sufficiently large $\tilde{p}_x$. We have also shown that $\lim_{\tilde{p}_x \uparrow +\infty} \tilde{R}(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = \lim_{\tilde{p}_x \uparrow +\infty}(\tilde{\pi}_1^* + \tilde{DS}_1^* + \tilde{\pi}_2^* + \tilde{DS}_2^*) = \pi_1^* + DS_1^* + \pi_2^* + DS_2^*$. Since $\tilde{R}(\cdot | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ is strictly decreasing in $\tilde{p}_x$ for a sufficiently large $\tilde{p}_x$, one can find a value of $\tilde{p}_x$ such that $\tilde{R}(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_1^* + DS_1^* + \pi_2^* + DS_2^*$. Since $\tilde{R}(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = \tilde{\pi}_1^* + \tilde{DS}_1^* + \tilde{\pi}_2^* + \tilde{DS}_2^*$, one can find a value of γ such that, under the price of the new service $\tilde{p}_x$, $\tilde{\pi}_i^* + \tilde{DS}_i^* > \pi_i^* + DS_i^*$ for $i = 1, 2$. By Theorem 4 and Proposition 7, for any $(\tilde{p}_x, \gamma)$, $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) < \pi_i(p^*, w^*)$ and $\tilde{DS}_i^* < DS_i^*$ for all $i = 3, 4, ..., n$. Hence, $\tilde{\pi}_i^* + \tilde{DS}_i^* < \pi_i^* + DS_i^*$ for all $i = 3, 4, ..., n$. This concludes the proof of Proposition 8. $\square$

Proof of Theorem 5

Since $\kappa(0+) = +\infty$, we must have $s_i^{e*} > d_i^{e*}$ for $i = 1, 2$ under equilibrium. Hence, P_i's profit under equilibrium can be written as $[f_i - \kappa(s_i - d_i) - w_i]d_i$. Given (p_{-i}, w_{-i}), we rewrite P_i's profit as a function of d_i and s_i:

$$
\begin{aligned}
&\max_{(f_i,w_i,d_i,s_i)} \pi_i^e(f_i, w_i, s_i, d_i | f_{-i}, w_{-i}) \\
&\quad \text{where } \pi_i^e(f_i, w_i, s_i, d_i | f_{-i}, w_{-i}) = (f_i - \kappa(s_i - d_i) - w_i)d_i \\
&\qquad\qquad \sum_{j=1}^{m} d_{ij} = d_i \\
&\qquad\qquad p_i = \frac{q_{ij}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{d_{ij}/\Lambda_j}{1 - d_{ij}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1 + \sum_{i'\neq i}\exp(q_{i'j} - \kappa_j p_{i'})\Big)\ \forall j \\
&\qquad\qquad \sum_{k=1}^{l} s_{ik} = s_i \\
&\qquad\qquad w_i = -\frac{a_{ik}}{\eta_k} + \frac{1}{\eta_{ik}}\log\Big(\frac{s_{ik}/\Gamma_k}{1 - s_{ik}/\Gamma_k}\Big) + \frac{1}{\eta_k}\log\Big(1 + \sum_{i'\neq i}\exp[\omega_k + \frac{d_{i'}}{s_{i'}}(a_{i'k} + \eta_k w_{i'} - \omega_k)]\Big)\ \forall k.
\end{aligned}
\tag{13}
$$

Hence, given d_i, there exists a unique price f_i such that all the constraints in (13) hold, which we denote as $f_i(d_i)$. Analogously, given s_i, there exists a unique wage w_i such that all the constraints in (13) hold, which we denote as $w_i(s_i)$. Thus, given (f_{-i}, w_{-i}), P_i's best response can be characterized as follows:

$$
\begin{aligned}
&\max_{(d_i,s_i)} \pi_i^e(f_i(d_i), w_i(s_i), d_i, s_i | f_{-i}, w_{-i}) \\
&\ s.t.\ d_i < s_i.
\end{aligned}
$$

Given P_i's demand, d_i, the best-response supply of P_i should be $\arg\max_{s>d_i}[-w_i(s) + \kappa(s - d_i)]$. As a result, we can reduce $\pi_i^e(f_i(d_i), w_i(s_i), d_i, s_i | f_{-i}, w_{-i})$ to the single-variable function $\pi_i^e(d_i | f_{-i}, w_{-i}) = (f_i(d_i) - h_i(d_i))d_i$, where $h(d_i) := \max_{s>d_i}[-w_i(s) + \kappa(s - d_i)]$.

We denote by $(f_i^e(f_{-i}, w_{-i}), w_i^e(f_{-i}, w_{-i}))$ P_i's best-response price and wage functions given (f_{-i}, w_{-i}). Following the same argument as in Step II of the proof of Theorem 1, we can show that $(f_i^e(f_{-i}, w_{-i}), w_i^e(f_{-i}, w_{-i}))$ is continuously increasing in f_{-i} and w_{-i}. Thus, an equilibrium (f^{e*}, w^{e*}) exists.

To show that the equilibrium is unique, we denote by $T_e(\cdot,\cdot)$ the best-response mapping of the model with endogenous waiting times, that is, $T_e(f, w) = \big(f_i^e(f_{-i}, w_{-i}), w_i^e(f_{-i}, w_{-i}) : 1 \le i \le n\big)$. Using the same argument as in the proof of Lemma 5, we obtain that there exists a constant $C = \max\big\{\frac{\exp(q_i)}{1+\exp(q_i)}, \frac{\exp(a_i)}{1+\exp(a_i)} : i = 1, 2, ..., n\big\} \in (0, 1)$, such that

$$||T_e^{(k)}(f, w) - T_e^{(k)}(f', w')||_1 \le 2C^{(k)}||(f, w) - (f', w')||_1,$$

and hence the k^*-fold best-response mapping, $T_e^{(k^*)}(\cdot,\cdot)$, is a contraction mapping, where $k^* > -\log(2)/\log(C)$. Consequently, using the same argument as in the proof of Lemma 5, the equilibrium is unique and can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 5. □