Optimal Growth of a Two-Sided Platform with Heterogeneous Agents

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We consider the dynamics of a two-sided platform, where the agent population on both sides experiences growth over time with heterogeneous growth rates. The compatibility between buyers and sellers is captured by a bipartite network. The platform sets commissions to optimize its total profit over T periods, considering the trade-off between short-term profit and growth as well as the spatial imbalances in supply and demand. We design an asymptotically optimal policy with the profit loss upper-bounded by a constant independent of T, in contrast with a myopic policy shown to be arbitrarily bad. To derive this policy, we first develop a single-period benchmark problem that captures the platform's optimal steady state. We then identify the agent types with the lowest relative population ratio compared to the benchmark in each period, and adjust the service level of these types to be higher than or equal to their service level in the benchmark problem. A higher service level accelerates growth but requires substantial subsidies during the growth phase. Additionally, we characterize the conditions under which subsidy is necessary.

We further examine the impact of the growth potential and the compatibility network structure on the platform's optimal profit, the agents' payment/income, and the optimal commissions at the optimal steady state. To achieve that, we introduce innovative metrics to quantify the long-run growth potential of each agent type. Using these metrics, we first show that a "balanced" compatibility network, where the relative long-run growth potential between sellers and buyers for all submarkets is the same as that for the entire market, allows the platform to achieve maximum profitability. Our study provides insight into how the growth potential and compatibility network structure jointly influence the commission policy in the growth process and the optimal steady state.

Key words: Two-sided market, platform growth, heterogeneous agent types, commission.

1. Introduction

In recent years, the rapid growth of consumer-to-consumer (C2C) platforms, such as Airbnb and Upwork, has transformed buyer-seller interactions. Their success relies on efficiently growing the agent base on both sides, which drives transaction volume and ultimately enhances platform profitability. Existing literature suggests that a pivotal strategy of the platform involves initially subsidizing agents to stimulate their growth and subsequently implementing charges to ensure long-term profitability (Lian and Van Ryzin 2021). Throughout this process, it is crucial to strike a balance between long-run growth and short-term profitability via a tailored commission structure. However, determining which agent segment to subsidize or charge higher fees becomes challenging, particularly considering the heterogeneity in their growth potentials and preference or popularity on the platform.

In general, the growth of an active agent base over periods hinges on two primary factors: retaining current agents and encouraging new adoptions. The acquisition of new adopters is strongly influenced by word-of-mouth communication between potential adopters and current agents, facilitated through online reviews and comments. Different agent types exhibit varying retention rates and word-of-mouth effects. For example, tourists seeking vacation homes on Airbnb have lower retention rates than regular business travelers due to infrequent revisits (Hamilton et al. 2017). However, they rely more on transaction histories and online reviews from previous guests when selecting properties in unfamiliar destinations (Arndt 1967, Sundaram and Webster 1999). Platforms could tailor their commission structures based on distinct growth patterns across different agent segments. For example, some platforms utilize machine learning algorithms to predict users' churn rate based on their past behaviors, and send coupons to those users with low usage frequency (Yu and Zhu 2021). This targeted promotion campaign can be typically viewed as an indirect way to implement personalized pricing to alleviate backlash from customers.

Furthermore, based on previous works on the cross-side network effect of a two-sided market (Rochet and Tirole 2003, Eisenmann et al. 2006, Chu and Manchanda 2016), the growth on one side of the market has a positive impact on the growth of the other side. However, the value contributed to the opposite side of the market differs across various agent segments, as buyers and sellers are horizontally differentiated in terms of their "popularity" and preferences for agents on the other side of the market. This compatibility difference arises from varying tastes, geographical constraints, or skill mismatches (Birge et al. 2021). For instance, on Airbnb, listings located in popular tourist destinations or with a secure parking space tend to be more popular; on Upwork, freelancers who offer skills that match market demands and have flexible schedules tend to attract more companies. During the platform's growth phase, an increase in the number of "marquee users", typically prominent buyers or high-profile sellers, can attract more users on the other side to join the platform. Therefore, the platform can accelerate its growth by securing the participation of marquee users when setting commissions during the growth phase. For example, Airbnb charges different commissions based on the location of listings, room types, cancellation policies, and so on (Airbnb 2024, Thorn 2024).

With the intricate interplay of *intertemporal factors* marked by heterogeneous growth potentials and *spatial factors* characterized by the compatibility between agent types, it becomes challenging for the platform to find an optimal commission policy to grow the agent base and maximize its long-term profits. Furthermore, gaining insights into how both the intertemporal and spatial factors affect the platform's optimal policy and profit is of utmost importance. These are the two primary focal points of our study.

Results and Contributions. We consider a two-sided platform that charges commissions to sellers and buyers for facilitating their transactions. The compatibility between the buyers and sellers is captured by a bipartite graph, and the transaction quantities and prices between the agents are determined endogenously in a general equilibrium setting. The mass of each agent type in each period depends on the mass and transaction volume in the previous period, capturing the effect of retention and word-of-mouth communication. The platform determines the commissions in each period to maximize the total profit in T periods, taking into account the trade-off between the immediate revenue and the potential for future expansion. Our main findings are summarized as follows.

First, we formulate the platform's problem as a multi-period pricing optimization model, which, however, is challenging to solve due to its high-dimensional state space (determined by the sizes of different agent types) and lack of structural results. To overcome this challenge, we first construct a single-period problem and show that the gap between T times its optimal objective value and that of the original problem is upper bounded by a constant (see Proposition 1). Therefore, the solution to this single-period problem captures the optimal steady state of the system, and we see it as a benchmark. We then develop a heuristic policy that is shown to be asymptotically optimal (see Theorem 1). During the growth phase, the policy identifies the scarcest agent type relative to the benchmark problem in each period and adjusts its service level to be equal to or higher than the corresponding value in the benchmark. The demand and supply quantities of other agent types are matched accordingly to ensure feasibility. We demonstrate that any service level within the proposed range guarantees exponential convergence of population toward the targeted level, but a higher service level accelerates growth and necessitates greater subsidy for users. Additionally, we identify which transactions should be subsidized during the growth stage (see Proposition 2). Once the population of the scarcest type surpasses the targeted level, our policy shifts to align its service level with the service level derived from the benchmark problems, helping to ensure that the population converges to the optimal steady state. We also provide some numerical examples to illustrate possible growth trajectories of the platform and changes in commissions over time applying this policy (see Figure 1).

In comparison, we show that the performance of a myopic policy, which ignores the growth dynamics in the marketplace, can be arbitrarily poor (see Proposition 3). This result further emphasizes that the trade-off between short-term profit and long-term growth must be carefully managed, even for a monopoly platform. Our result provides managerial insights on platform growth: the platform should first identify the optimal state at which it can sustain and maximize profit. Instead of targeting the agent type with the lowest population, the platform should focus on ensuring the service level (e.g., by offering subsidies or lowering commissions) for agent types that lag behind the benchmark in each period during the growth phase.

Second, we focus on the platform's optimal steady state characterized by the single-period benchmark problem. We analyze how the growth potential of agent types (*intertemporal factor*) and compatibility network structure (*spacial factor*) influence (1) the platform's profit, (2) the agents' payments/incomes, and (3) the optimal commissions. Regarding (1), previous literature (see Schrijver et al. 2003, Chou et al. 2011, Birge et al. 2021) considers static settings with exogenous agent bases and shows how the imbalance in supply-demand ratio across the market determines the platform's performance. However, we find that the metric of "balances" in the literature fails in the dynamic setting (see Figure 2). To incorporate the intertemporal factor, we develop a novel metric that captures the long-run growth potential of each agent type. With a more specific growth function, we develop the intuition behind such a metric. We show that a "balanced" compatibility network, where the relative long-run growth potentials of sellers and buyers for all submarkets are the same as that for the entire market, leads to maximum platform profitability (see Theorem 2). In contrast, the extent of the "imbalance" of the compatibility network in terms of the relative long-run growth potentials between the two sides determines the lower bound of optimal profit the platform can achieve.

Regarding (2), we show that the buyer (seller) type with a higher ratio of compatible sellers' (buyers') long-run growth potential to their own long-run growth potential receives lower payments (higher income) at the optimal steady state (see Proposition 4). Based on this result, we conduct a sensitivity analysis to illustrate the impact of each agent type's long-run growth potential on its or others' income/payment (see Corollary 2). For (3), we show that the optimal commission charged from the submarket first decreases in the relative growth potential between sellers and buyers, and then increases (decreases) in it when the value distribution functions of both sides are convex (concave)(see Proposition 5). Our results suggest that the platform should strategically focus its marketing campaigns or loyalty programs on agents who exhibit relatively lower long-run growth potential compared with their compatible agents on the other side of the platform.

Organization of the Paper. The rest of the paper is organized as follows. After reviewing the relevant literature in Section 2, we introduce the model and discuss computational challenges in Section 3. In Section 4, we design a heuristic algorithm with provably good performance. In Section 5, we examine the impact of both the compatibility network structure and growth potential of agents on the platform's profit, agents' payments/incomes, and optimal commissions at the optimal steady state. The concluding remarks are drawn in Section 6.

Throughout the paper, we use "increasing" (and "decreasing") in a weak sense, i.e., meaning "non-decreasing" (and "non-increasing") unless otherwise specified. In addition, we use $\mathbb{R}_+$ to denote the non-negative real number set.

2. Literature Review

Pricing in two-sided platforms has been extensively studied in the field of Economics and Operations Management. Based on Caillaud and Jullien (2003), Rochet and Tirole (2003, 2006), Armstrong (2006), a growing literature has explored the pricing and matching problems in the context of online platforms (e.g., Hagiu 2009, Cachon et al. 2017, Taylor 2018, Bai et al. 2019, Benjaafar et al. 2019, Hu and Zhou 2020, Benjaafar et al. 2022, Cohen and Zhang 2022). Our work features network effects in a potentially incomplete two-sided market that evolves dynamically. Agents on one side of the market can only trade with a subset of agents on the other, and the platform's commissions influence the transactions and the growth of the agent base in the market. Therefore, our work is closely related to two streams of literature: (i) the growth of a marketplace and (ii) pricing in a networked market.

Early literature about the growth of a marketplace mainly focused on product diffusion, which provides a model to forecast the growth of the customer base for a new product, see e.g., Bass (1969), Kalish (1985), Norton and Bass (1987). Based on these papers, more recent literature studies how to leverage discounts or investment incentives to influence the growth of new products (e.g., Bass and Bultez 1982, Shen et al. 2014, Ajorlou et al. 2018) and that of two-sided platforms (e.g., Kabra et al. 2016, Lian and Van Ryzin 2021, He and Goh 2022). Specifically, Lian and Van Ryzin (2021) considered a two-sided market in which the platform can subsidize one or both sides to boost their growth. They show that the optimal policy is to employ a subsidy shock to rapidly steer the market towards its optimal long-term size. He and Goh (2022) studied the dynamics of a hybrid workforce comprising on-demand freelancers and traditional employees, both capable of fulfilling customer demands. They investigated how demand should be allocated between employees and freelancers, and under what conditions the system is sustainable in the long run. Our study differs from this stream of work in that agents have heterogeneous compatibility and growth potentials, which requires us to come up with a customized commission structure for different agent types; in addition, the transaction quantities and prices are both formed endogenously in a general equilibrium in each period.

Our study is also closely related to the literature on networked markets (e.g., Kranton and Minehart 2001, Bimpikis et al. 2019, Baron et al. 2022, Zheng et al. 2023, Chen and Wang 2023).

In this line of work, the edges of the network capture the trading opportunities between agents, and the impacts of network effects on the market outcomes are analyzed. For example, Chen and Chen (2021) explored duopoly competition within a market involving network-connected buyers, and they showed that the existence of symmetric market share equilibrium for two identical sellers depends on the intensity of network effects and the quality of the product. More closely, some recent studies explore how to improve operational efficiency in a two-sided market using centralized price controls (e.g., Banerjee et al. 2015, Ma et al. 2022, Varma et al. 2023) or non-pricing controls (e.g., Kanoria and Saban 2021). For example, Hu and Zhou (2022) considered a platform that strategically matches buyers and sellers, who are categorized into distinct groups based on varying arrival rates and matching values. They provided sufficient conditions under which the optimal matching policy follows a priority hierarchy among matched pairs, determined by factors such as quality and distance. Our work adopts the framework proposed by Birge et al. (2021), in which a platform determines commission structure to maximize the total profit in a two-sided market, and the trades and prices are formed endogenously in a competitive equilibrium given the commissions. Differently, we delve into a dynamic setting and demonstrate that utilizing metrics for network imbalance from static settings in the prior studies to quantify the impact of network structure is inadequate. We introduce a novel metric that incorporates the intertemporal factor (i.e., the growth potentials of agents).

Some recent literature also explores the expansion of the platform's agent base in a network (e.g., Li et al. 2021, Alizamir et al. 2022). These studies assume a uniform retention and growth rate across agents from the same side or all agents in the network, with each agent's payoff determined by an exogenously specified function of the number of participants in the network. In contrast, we account for the heterogeneity of growth potentials among various agent types and introduce a novel metric that incorporates both spatial and intertemporal factors to assess the influence of the network structure on the platform's profitability.

3. Model

Consider a two-sided market in which a platform charges commissions to buyers and sellers for facilitating transactions. The compatibility between buyers and sellers is captured by a bipartite graph $(\mathcal{B}\cup\mathcal{S},E)$, where $\mathcal{B}=\{1,2,...,N_b\}$ denotes the set of buyer type and $\mathcal{S}=\{1,2,...,N_s\}$ denotes the set of seller types; E is the set of edges that captures the potential trading opportunities between them. Specifically, $(i,j)\in E$ if and only if the service or product of type-i sellers can satisfy the demand of type-j buyers for $i\in\mathcal{S}$ and $j\in\mathcal{B}$. This compatibility difference arises from varying tastes, geographical constraints, or skill mismatch. For example, the edge set on Airbnb reflects the preferences of leisure travelers over tourist destinations and business travelers preferring the city center or CBD; the edge set on Upwork/TaskRabbit is determined by the skill set of the freelancer and the demand of firms. This edge set is exogenous and remains stable throughout the decision horizon (see e.g., Hu and Zhou 2022). In the rest of the paper, we will refer to an "incomplete market" as a bipartite network structure where there exists a pair (i,j) such that $i \in \mathcal{S}$ and $j \in \mathcal{B}$, but $(i,j) \notin E$. Otherwise, we call it "complete network".

Note that users may change their preferences over time and thus belong to different user segments at different times. For example, each traveler might occasionally act as a leisure traveler and at other times as a business traveler, and firms may want to outsource different tasks at different times. In this case, our model can treat them as two different users. In practice, some platforms utilize machine learning algorithms to infer user preferences based on their search behavior whenever they interact with listings or make inquiries (Daly 2017, Yu and Zhu 2021).

In each period $t \in \{1,...,T\}$, the populations of type-i sellers and type-j buyers are respectively denoted by $s_i(t)$ for $i \in \mathcal{S}$ and $b_j(t)$ for $j \in \mathcal{B}$. The initial population of each type is finite, i.e., $s_i(1) < \infty$ for $i \in \mathcal{S}$ and $b_j(1) < \infty$ for $j \in \mathcal{B}$. The buyers/sellers are infinitesimal, and each one of them supplies/demands at most one unit of product/service in one period. For $t \in \{1,...,T\}$, we use $q_i^s(t)$ and $q_j^b(t)$ respectively to denote the aggregate supply quantities of type-i sellers and the aggregate demand quantities of type-j buyers, where $q_i^s(t) \in [0, s_i(t)]$ for $i \in \mathcal{S}$ and $q_j^b(t) \in [0, b_j(t)]$ for $j \in \mathcal{B}$. Note that given the commission charged by the platform, the supply/demand vector $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ is endogenously determined in equilibrium, with mechanism details discussed later.

Population Transition. A key feature of our model is that the mass of each agent type evolves dynamically at different rates (see the discussion in Section 1). For any $t \in \{1,...,T-1\}$, we consider the following population transition equations:

$$s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t)), \qquad \forall i \in \mathcal{S} \tag{1a}$$

$$b_j(t+1) = \mathcal{G}_j^b(b_j(t), q_j^b(t)), \qquad \forall j \in \mathcal{B}. \tag{1b}$$

In (1), we assume that the mass of agents for the next period depends on the mass and transaction volume in the current period. A higher mass of agents in the current period contributes to a larger future agent base due to retention (see Lian and Van Ryzin 2021). A higher transaction quantity leads to a higher future agent base due to the word-of-mouth effect or the imitation effect (see Bass 1969, Mahajan and Peterson 1985), i.e., current agents who trade on the platform can share positive information about the platform with potential new adopters, attracting them to join the platform. For each type of agent (e.g., type-i sellers), the equilibrium transactions quantity in each period (e.g., $q_i^s(t)$) also depends on the mass of agents in other categories $(\boldsymbol{s}(t), \boldsymbol{b}(t))$, as discussed later in Definition 1. Therefore, these factors indirectly influence the agent's growth, reflecting the *network effects* mentioned in Section 1.

For the rest of the paper, $\mathcal{G}_i^s(\cdot,\cdot)$ and $\mathcal{G}_j^b(\cdot,\cdot)$ in (1) will be referred to as the *growth functions*. These growth functions can take various forms, such as $\mathcal{G}_i^s(q,s) = sf(q/s)$, with concave $f(\cdot)$ capturing the agent type's average surplus (see Lian and Van Ryzin 2021). Instead of assuming a specific functional form, we impose only basic assumptions for the growth functions. For simplicity of notation, we let $(\mathcal{G}_i^s)_1'(s,q), (\mathcal{G}_i^s)_2'(s,q)$ denote the partial derivatives of $\mathcal{G}_i^s(s,q)$ with respect to $s \geq 0$ and $q \geq 0$; similarly, $(\mathcal{G}_j^b)_1'(b,q), (\mathcal{G}_j^b)_2'(b,q)$ denote the partial derivatives of $\mathcal{G}_j^b(b,q)$ respectively in $b \geq 0$ and $q \geq 0$.

Assumption 1. **(growth functions)** *For any $i \in \mathcal{S}$ and any $j \in \mathcal{B}$,*

(i) $\mathcal{G}_i^s(0,0) = 0$ and $\mathcal{G}_j^b(0,0) = 0$;

(ii) $\mathcal{G}_i^s(s,q)$ is continuously differentiable, increasing and strictly concave in (s,q) for $0 \leq q \leq s$, and moreover, $\lim_{x\to\infty} \left[(\mathcal{G}_i^s)_1'(x,x) + (\mathcal{G}_i^s)_2'(x,x)\right] < 1$ for the seller side; the same properties hold for the buyer side with $\lim_{x\to\infty} \left[(\mathcal{G}_j^b)_1'(x,x) + (\mathcal{G}_j^b)_2'(x,x)\right] < 1$.

Assumption 1(i) implies that if the population mass is zero and there is no transaction from the previous period, then there is no retention or word-of-mouth effect. Assumption 1(ii) requires that the future agent base increases in the current population mass and transaction volume, but the marginal effects of these two factors decrease because the total mass of potential agents in a market is finite. It also requires that the total marginal effects of these two factors are lower than one when the transaction volume and the population mass approach infinity, which implies that the number of agents in the system cannot grow infinitely large. We will delay the discussion about the class of examples under this assumption to Section 5.

We next discuss how the equilibrium supply/demand $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ is formed in an (in)complete market given the commission by the platform in each period $t \in \{1,...,T\}$.

Competitive Equilibrium. In period $t \in \{1,...,T\}$, the platform charges commission $r_i^s(t)$ to type-i sellers and $r_j^b(t)$ to type-j buyers if they trade. The commissions are homogeneous within the same agent type but may vary across different types. When $r_i^s(t) < 0$ or $r_j^b(t) < 0$, the platform subsidizes the sellers or buyers. In practice, platforms could implement heterogeneous prices to different user types through personalized coupon distribution (Park and Hwang 2020). Previous research also showed that revenue loss can be unbounded when using a uniform pricing strategy across types (Birge et al. 2021). Therefore, in our setting, we consider type-dependent, heterogeneous pricing (see e.g., Varma et al. 2023).

Given the commissions, type-i sellers offer their product/service at price $p_i(t)$ and receive $p_i(t) - r_i^s(t)$; type-j buyers pay $p_i(t) + r_j^b(t)$ if they trade with type-i sellers. The market prices $\boldsymbol{p}(t)$ are endogenously formed in equilibrium to match supply and demand, rather than controlled by the platform (see Definition 1 later). For instance, hosts on Airbnb compete on their rental offers, and freelancers on Upwork compete on their hourly rates. We consider the case that a seller cannot charge different prices to different buyers, aligning with the standard practice of many online platforms, where seller prices are openly displayed on the web page. Finally, we assume for a type-j buyer, all compatible sellers (i.e., $i:(i,j)\in E$) provide perfectly substitutable products/services, and the type-j buyer does not have preference over the compatible sellers' products if their prices are the same. Similarly, it is indifferent for a seller to trade with any compatible buyers given that the market price is formed on the seller side. Note that vertical differentiation of sellers can be modeled by adding a quality term for each type of seller in the payoff function of buyers (see Birge et al. 2021), which does not fundamentally change our insights.

We use $F_{b_j}:[0,\overline{v}_{b_j}]\to[0,1]$ and $F_{s_i}:[0,\overline{v}_{s_i}]\to[0,1]$ to denote the cumulative distribution function of the (reservation) values respectively for type-j buyers and type-i sellers, in which $\overline{v}_{b_j}$ and $\overline{v}_{s_i}$ are finite for any $j\in\mathcal{B}$ and $i\in\mathcal{S}$. For simplicity, we refer to a seller by "he" and a buyer by "she". A type-i seller only engages in trades when the amount he receives from the transaction is weakly higher than his reservation value v, i.e., $p_i(t)-r_i^s(t)\geq v$; similarly, a type-j buyer trades when the total payment is weakly lower than her value v, i.e., $p_i(t)+r_j^b(t)\leq v$. To simplify our analysis later, we extend the domains of the value distributions to $\mathbb{R}$: let $F_{b_j}(v)=1$ for $v\geq\overline{v}_{b_j}$ and $F_{b_j}(v)=0$ for $v\leq 0$ for any $j\in\mathcal{B}$; similarly, for the seller side, we let $F_{s_i}(v)=1$ for $v\geq\overline{v}_{s_i}$ and $F_{s_i}(v)=0$ for $v\leq 0$ for any $i\in\mathcal{S}$. In addition, define $f_{b_j}(v)$ and $f_{s_i}(v)$ respectively as the derivative (or the density function) of $F_{b_j}(v)$ for $v\in[0,\overline{v}_{b_j}]$ and $F_{s_i}(v)$ for $v\in[0,\overline{v}_{s_i}]$. We impose the following assumption throughout the paper.

Assumption 2. **(value distribution)** *For any $j\in\mathcal{B}$ and $i\in\mathcal{S}$,*

(i) $F_{b_j}(v)$ and $F_{s_i}(v)$ are strictly increasing in $v\in[0,\overline{v}_{b_j}]$ and $v\in[0,\overline{v}_{s_i}]$;

(ii) $F_{b_j}(v)$ and $F_{s_i}(v)$ are continuously differentiable respectively in $v\in[0,\overline{v}_{b_j}]$ and $v\in[0,\overline{v}_{s_i}]$; $f_{b_j}(v)$ and $f_{s_i}(v)$ are lower bounded by a positive constant.

Under Assumption 2(i), we define the inverse function $F_{b_j}^{-1}:[0,1]\to[0,\overline{v}_{b_j}]$ and $F_{s_i}^{-1}:[0,1]\to[0,\overline{v}_{s_i}]$ such that $F_{b_j}^{-1}(F_{b_j}(v))=v$ for $v\in[0,\overline{v}_{b_j}]$ and $F_{s_i}^{-1}(F_{s_i}(v))=v$ for $v\in[0,\overline{v}_{s_i}]$. Under Assumption 2(ii), $F_{b_j}^{-1}(x)$ and $F_{s_i}^{-1}(x)$ are also continuous and differentiable in $x\in[0,1]$, and their density functions are also bounded. We further impose the following Assumption on $F_{b_j}^{-1}(x)$ and $F_{s_i}^{-1}(x)$.

Assumption 3. **(concavity)** *$F_{b_j}^{-1}(1-a/b)a$ and $-F_{s_i}^{-1}(a/b)a$ are strictly concave in (a,b) for $0\leq a\leq b$.*

Assumptions 2 and 3 hold for many commonly used distributions such as uniform, truncated exponential, and truncated generalized Pareto distribution.

We can finally define the equilibrium in the incomplete market in each period, given the commission vector $(r_i^s(t):i\in\mathcal{S},\ r_j^b(t):j\in\mathcal{B})$ by the platform and the population vector $(\boldsymbol{s}(t),\boldsymbol{b}(t))$. We denote by $x_{ij}(t)$ the amount type-j buyers purchase from type-i sellers.

Definition 1. **(competitive equilibrium)** In period $t \in \{1, \ldots, T\}$, given the platform's commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{N_s} \times \mathbb{R}^{N_b}$ and the population vector of sellers and buyers $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}_+^{N_s} \times \mathbb{R}_+^{N_b}$, a competitive equilibrium is defined as the price-flow vector $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ that satisfies the following conditions:

$$q_i^s(t) = s_i(t) F_{s_i}(p_i(t) - r_i^s(t)), \qquad \forall i \in \mathcal{S}, \tag{2a}$$

$$q_j^b(t) = b_j(t)\Big(1 - F_{b_j}\big(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t)\big)\Big), \qquad \forall j \in \mathcal{B}, \tag{2b}$$

$$q_i^s(t) = \sum_{j':(i,j')\in E} x_{i,j'}(t), \qquad \forall i \in \mathcal{S}, \tag{2c}$$

$$q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B}, \tag{2d}$$

$$x_{ij}(t) \geq 0, \qquad \forall (i,j) \in E, \tag{2e}$$

$$x_{ij}(t) = 0, \qquad \forall i \notin \arg\min_{i':(i',j)\in E}\{p_{i'}\}, \;\; j \in \mathcal{B}. \tag{2f}$$

In Definition 1, Conditions (2a) and (2b) ensure that the total supply/demand quantities of type-i sellers and type-j buyers equal the mass of agents who can obtain nonnegative utilities from the transaction. Specifically, Condition (2b) assumes that type-j buyers only trade with compatible sellers with the lowest market price to maximize their utilities. Conditions (2c) and (2d) characterize the flow conservation conditions in the compatibility network. Finally, Condition (2e) requires that the transaction flow is non-negative, and Condition (2f) requires that the buyers only trade with their compatible sellers with the lowest prices.

The equilibrium concepts similar to Definition 1 have also been adopted in the two-sided market literature by, e.g., Weyl (2010) and Birge et al. (2021). In our setting, the demand/supply quantities only depend on the prices and commissions in the current period, which is commonly seen in the literature about dynamic pricing, e.g., Chen and Gallego (2019), Birge et al. (2023). With Definition 1, given any commission profile and the total mass of agents in each period, we can show that the equilibrium always exists, and the equilibrium supply-demand vector $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ is always unique (see Proposition EC.1 in Appendix EC.1.1).

Platform's Profit Optimization Problem. Given the mass of different types of agents in the first period $(\boldsymbol{s}(1), \boldsymbol{b}(1))$, the platform aims to maximize its total T-period profit by determining the commission for each type in each period. For simplicity of notation, we let $(\boldsymbol{s}, \boldsymbol{b}) := (\boldsymbol{s}(t), \boldsymbol{b}(t))_{t=2}^T$, and $(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q^s}, \boldsymbol{q^b}) := (\boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))_{t=1}^T$, then the platform's T-period profit maximization problem is as follows:

$$\mathcal{R}^*(T) = \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{r},\boldsymbol{p},\boldsymbol{x},\boldsymbol{q^s},\boldsymbol{q^b}} \sum_{t=1}^{T} \left[\sum_{i\in\mathcal{S}} r_i^s(t) q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t) q_j^b(t)\right] \tag{3a}$$

$$\text{s.t.} \quad (\boldsymbol{s}(t), \boldsymbol{b}(t), \boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)) \text{ satisfies (2)}, \qquad \forall t \in \{1, ..., T\}, \tag{3b}$$

$$s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t)), \qquad \forall i \in \mathcal{S}, t \in \{1, ..., T-1\}, \tag{3c}$$

$$b_j(t+1) = \mathcal{G}_j^b(b_j(t), q_j^b(t)), \qquad \forall j \in \mathcal{B}, t \in \{1, ..., T-1\}. \tag{3d}$$

The platform's profit consists of the commissions from the sellers and buyers who trade in the market during the T periods. Constraint (3b) ensures that given the population vector $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ and commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ in period t, vector $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ constitutes a competitive equilibrium; Constraints (3c)-(3d) indicate that the dynamics of populations follow the transition equations given in (1). Given that the equilibrium transaction quantities $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))_{t=1}^T$ are unique under any commission $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))_{t=1}^T$ (see Proposition EC.1(ii) in Appendix EC.1.1), the maximization problem in (3) is well-defined. In the rest of the paper, we refer to Problem (3) as OPT. Since OPT is non-convex (in $(\boldsymbol{r}, \boldsymbol{q})$), we will first reformulate it into a convex optimization problem and then discuss the challenges in solving it.

Reformulation and Challenges. For any period $t \in \{1, \ldots, T\}$, we deduce from (2a) that type-i sellers' incomes per unit are bounded below by the highest reservation value among those who participate in trading, i.e., $p_i(t) - r_i^s(t) \geq F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right)$ for i with $q_i^s(t) > 0$. Similarly, type-j buyers' payments are bounded above by the lowest value among them, i.e., $p_i(t) + r_j^b(t) \leq F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right)$ for j with $q_j^b(t) > 0$. Therefore, the objective value of OPT is upper bounded by $\sum_{t=1}^T \left[\sum_{j \in \mathcal{B}} F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right) q_j^b(t) - \sum_{i \in \mathcal{S}} F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right) q_i^s(t)\right]$, which is concave in $(\boldsymbol{q}, \boldsymbol{s}, \boldsymbol{b})$ under Assumption 3. By further relaxing some constraints of OPT, we can obtain a convex optimization problem where the decision variables only consist of $(\boldsymbol{s}, \boldsymbol{b}, \boldsymbol{q}^s, \boldsymbol{q}^b, \boldsymbol{x})$ but not commission $(\boldsymbol{r}^s, \boldsymbol{r}^b)$. We present the formulation in Problem (EC.2) and show that the relaxation is tight in Proposition EC.2 in Appendix EC.1.1. After obtaining the optimal solution to this convex optimization problem, we can find a feasible commission profile $(\boldsymbol{r}^s, \boldsymbol{r}^b)$ that can induce this equilibrium by solving a system of linear inequalities in each period (see Lemma EC.1 in Appendix EC.1.1). The feasible commissions always exist and are not necessarily unique, but the payments/incomes of agents with positive trades are uniquely determined in any equilibrium.

Even though the non-convexity challenge of OPT can be circumvented by the reformulation, solving Problem (EC.2) remains computationally challenging when the problem's dimensionality is large (i.e., greater than $T \times (2N_s + 2N_b + |E|)$). While Problem (EC.2) can also be reformulated as a deterministic dynamic program (DP) with a high-dimensional state space, the lack of its structural properties makes it difficult to derive clear managerial implications for the growth strategy in the networked market. Therefore, we focus on designing a simpler policy with provable performance guarantees, which offers clear guidance for the platform's growth strategy.

4. Asymptotically Optimal Policy

We define an admissible policy as a sequence of functions $\pi =: \{\pi_t : \mathcal{H}_t \to \boldsymbol{r}^{N_s+N_b}(t)\}_{t=1}^T$ that prescribes the commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ in each period t, where $\mathcal{H}_t$ is the history of population vectors $(\boldsymbol{s}(t'), \boldsymbol{b}(t'))_{t'=1}^t$ and transaction vectors $(\boldsymbol{x}(t'), \boldsymbol{q}^s(t'), \boldsymbol{q}^b(t'))_{t'=1}^{t-1}$. We denote $\mathcal{R}^\pi(T)$ as the platform's total profit in T periods for any admissible policy π. We evaluate the policy's performance by quantifying its profit loss relative to the optimal objective value $\mathcal{R}^*(T)$ of OPT, which is formally defined as

$$\mathcal{L}^\pi(T) \;=\; \mathcal{R}^*(T) - \mathcal{R}^\pi(T). \tag{4}$$

We aim to devise an admissible policy with good performance in the asymptotic setting as $T \to \infty$.

Notice that Flynn (1978) studies heuristic policies for solving infinite-horizon deterministic dynamic programming problems. He proposes a "steady-state policy", which involves solving a static problem to identify the optimal steady state, steering the system to this state, and maintaining it there. Our algorithm shares a similar spirit of "steady-state policy." However, while he provides examples of constructing feasible rules to move the system from the initial state to the target steady state, most of them involve implementing the action at the optimal steady state from the beginning or making straightforward modifications to it (Flynn 1975a, 1981). We will see that those methods cannot be applied to our setting. In other words, it remains unclear how to construct a feasible solution that guides the system toward an optimal steady state in a network problem with endogenous equilibrium constraints. This is particularly challenging due to the flow conservation constraints in the networked market (i.e., Definition 1). To address this issue, we need to identify further which type within the network to prioritize during the growth stage and which metric to target for growth.

In this Section, after establishing the optimal steady state, we introduce a novel approach called the Target-Ratio Policy (TRP). Inspired by the fundamental principle of Economics that "scarcity creates value" (Samuelson and Samuelson 1980), the TRP prioritizes the agent type with the lowest population relative to the optimal steady state, regulating its service level within specified ranges in each period. We prove that the TRP is asymptotically optimal, demonstrating that focusing solely on the scarcest agent type is sufficient for the platform to achieve strong performance. We also identify the conditions under which subsidization should be implemented. Finally, we show that a myopic policy, which maximizes single-period profit while completely ignoring population growth, can perform poorly (see Proposition 3). This finding, again, highlights the critical need for carefully balancing short-term profitability with long-term growth, even for a monopolistic platform.

Long-run Average Value Problem (AVG). We begin by formulating a corresponding steady-state problem for OPT. This step is essential because deriving the exact optimal solution to OPT is challenging, while the optimal objective value of the steady-state problem offers a good approximation of the optimal objective value of OPT and can thus serve as a benchmark for evaluating the policy's profit loss in (4).

For convenience, we define

$$\widetilde{F}_{b_j}(q_j^b, b_j) := \begin{cases} 0, & q_j^b = b_j = 0, \\ F_{b_j}^{-1}\left(1 - \frac{q_j^b}{b_j}\right) q_j^b, & b_j > 0, 0 \le q_j^b \le b_j, \end{cases} \qquad \widetilde{F}_{s_i}(q_i^s, s_i) := \begin{cases} 0, & q_i^s = s_i = 0, \\ F_{s_i}^{-1}\left(\frac{q_i^s}{s_i}\right) q_i^s, & s_i > 0, 0 \le q_i^s \le s_i. \end{cases}$$

Then we consider the following optimization problem which we refer to as AVG:

$$\begin{aligned} \overline{\mathcal{R}} = \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{q^s},\boldsymbol{q^b},\boldsymbol{x}} \quad & \sum_{j\in\mathcal{B}} \widetilde{F}_{b_j}(q_j^b, b_j) - \sum_{i\in\mathcal{S}} \widetilde{F}_{s_i}(q_i^s, s_i), && && \text{(5a)} \\ \text{s.t.} \quad & q_i^s \le s_i, \sum_{j:(i,j)\in E} x_{ij} = q_i^s, && \forall i \in \mathcal{S}, && \text{(5b)} \\ & q_j^b \le b_j, \sum_{i:(i,j)\in E} x_{ij} = q_j^b, && \forall j \in \mathcal{B}, && \text{(5c)} \\ & x_{ij} \ge 0, && \forall (i,j) \in E. && \text{(5d)} \\ & s_i \le \mathcal{G}_i^s(s_i, q_i^s), && \forall i \in \mathcal{S}, && \text{(5e)} \\ & b_j \le \mathcal{G}_j^b(b_j, q_j^b), && \forall j \in \mathcal{B}. && \text{(5f)} \end{aligned}$$

We relax Constraint (3b) of OPT about equilibrium conditions to (5b)-(5d), and relax the population transition equations in Constraint (3c)-(3d) to inequalities in (5e)-(5f). As a result, AVG is a tractable convex optimization problem. We next characterize the properties of its optimal solution denoted by $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}}, \overline{\boldsymbol{x}})$:

Lemma 1. **(optimal solution to AVG)** *The optimal solution to Problem (5) exists, and*

(i) the optimal population $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ and the optimal supply-demand vector $(\overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b)$ are unique;

(ii) the constraints (5e)-(5f) are tight at optimal.

Lemma 1(ii) implies that the population can be sustained at $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ by controlling the supply-demand at the level of $(\overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b)$. We further show that the gap between T times the optimal objective value of AVG from (5) and that of OPT from (3) is upper bounded by a constant for any positive integer T.

Proposition 1. **(alternative benchmark)** *There exists a positive constant C_1 such that for any $T = 1, 2, \ldots,$*

$$|\mathcal{R}^*(T) - T\overline{\mathcal{R}}| \le C_1.$$

Proposition 1 dictates that the difference between $\frac{1}{T}\mathcal{R}^*(T)$ and $\overline{\mathcal{R}}$ converges to zero as T approaches infinity. Therefore, the optimal solution to AVG $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b)$ captures a steady state where the long-run average profit is maximized, and therefore we call it the *optimal steady state (OSS)* in the rest of the paper (see Flynn 1975a, 1992). In addition, as we previously mentioned, in contrast to the high-dimensional problem OPT, AVG is a much more tractable static convex optimization problem. Therefore, we will consider $T\overline{\mathcal{R}}$, instead of $\mathcal{R}^*(T)$ as the benchmark to quantify the policy's profit loss in (4).

We next propose the *Target Ratio Policy (TRP)* that achieves fast convergence to OSS and formally establish its asymptotic optimality.

Target Ratio Policy. For ease of illustration, we refer to $\frac{s_i(t)}{\overline{s}_i}$ for $i \in \mathcal{S}$ and $\frac{b_j(t)}{\overline{b}_j}$ for $j \in \mathcal{B}$ as the *population ratio* of type-i seller and type-j buyer, respectively. In addition, we notice that $\frac{q_j^b(t)}{b_j(t)}$ ($\frac{q_i^s(t)}{s_i(t)}$) is the fraction of type-j buyers (type-i sellers) who trade in period t, and we refer to this fraction as the *service level* of the corresponding agent type. Recall that the service level also determines the payment/income of agents (i.e., $F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})$ and $F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})$).

Inspired by Proposition 1, we design our algorithm to guide the population of all agent types toward the levels at OSS $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ and maintain them there. A straightforward approach is to align the service level (equivalently, the income/payment) of each type with its corresponding level at the OSS, i.e., $\frac{q_i^s(t)}{s_i(t)} \approx \frac{\overline{q}_i^s}{\overline{s}_i}$ for any $i \in \mathcal{S}$ and $\frac{q_j^b(t)}{b_j(t)} \approx \frac{\overline{q}_j^b}{\overline{b}_j}$ for any $j \in \mathcal{B}$ for $t \in \{1, ..., T\}$. However, a significant challenge arises: such a policy is not always feasible in a network. Consider, for instance, a simple scenario with one buyer and one seller type. Given the flow conservation constraint $q^s(1) = q^b(1)$, if we target the service level for the seller side at OSS (i.e., $\frac{q^s(1)}{s(1)} = \frac{\overline{q}^s}{\overline{s}}$), the service level for the buyer side may be different from that of OSS in general (e.g., $\frac{q^b(1)}{b(1)} < \frac{\overline{q}^b}{\overline{b}}$ if $\frac{s(1)}{\overline{s}} < \frac{b(1)}{\overline{b}}$). A critical question remains: which type should be prioritized to foster growth during the early stages, particularly in a market with multiple user segments? If the principle that "scarcity creates value" holds, what is the proper metric to measure "scarcity" in this context?

From the above example, we observe that the type with the lower population ratio constrains the transaction volume of the type with the higher ratio, thereby further hindering its growth. Therefore, in each period, we should prioritize the type with the lowest population ratio and seek to boost its growth (i.e., $q^s(1) = s(1)\frac{\overline{q}^s}{\overline{s}}$ if $\frac{s(1)}{\overline{s}} < \frac{b(1)}{\overline{b}}$), while we match the transaction quantities of other types to guarantee the feasibility of the policy (i.e., $q^b(1) = q^s(1) = s(1)\frac{\overline{q}^b}{\overline{s}}$ if $\frac{s(1)}{\overline{s}} < \frac{b(1)}{\overline{b}}$). As the type with the lowest population ratio may change over time, the focus dynamically shifts to different types in different periods, until the system's state converges to the OSS.

We will demonstrate that the above method guarantees strong performance without requiring the platform to provide subsidies throughout the entire planning horizon. However, for platforms prioritizing rapid growth over short-term profitability, we find that they can further accelerate the growth of all types during the early stages by raising the service level of the most constrained type above its OSS value. One way to implement this is by involving as many agents as possible from the scarcest type in transactions, $q_i^s(1) = q_j^b(1) = \min\{s(1), b(1)\}$, until the system shows signs of overexpansion. As we will show later, this approach may, under certain conditions, require the platform to provide subsidies. Importantly, we will see that any policy falling between these two extremes is asymptotically optimal, but they require different levels of subsidy/commission and lead to varying growth rates. Building on this idea, we formally define the class of Target Ratio Policies parameterized by the acceleration weight w as in Algorithm 1.

Algorithm 1: Target Ratio Policy (TRP)

Input: Optimal solution to AVG $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b, \overline{\boldsymbol{x}})$; initial mass of agents $(\boldsymbol{s}(1), \boldsymbol{b}(1))$; acceleration weight $w \in [0,1]$.
$OverExpansion \leftarrow False$;
for $t = 1$ *to* T **do**
  $m(t) \leftarrow \min\limits_{i,j:\overline{s}_i>0,\overline{b}_j>0} \left\{ \frac{s_i(t)}{\overline{s}_i}, \frac{b_j(t)}{\overline{b}_j} \right\}$; $M(t) \leftarrow \min\limits_{i,j:\overline{q}_i^s, q_j^b>0} \left\{ \frac{s_i(t)}{\overline{q}_i^s}, \frac{b_j(t)}{\overline{q}_j^b} \right\}$;
  $\hat{m}(t) \leftarrow (1-w)m(t) + wM(t)$
  if $\min\limits_{i,j:\overline{s}_i>0,\overline{b}_j>0} \left\{ \frac{\mathcal{G}_i^s(s_i(t), \overline{q}_i^s \hat{m}(t))}{\overline{s}_i}, \frac{\mathcal{G}_j^b(b_j(t), \overline{q}_j^b \hat{m}(t))}{\overline{b}_j} \right\} \le 1$ *and* $OverExpansion = False$ **then**
    for $(i,j) \in E$ **do**
      $q_i^s(t) \leftarrow \overline{q}_i^s \hat{m}(t)$; $q_j^b(t) \leftarrow \overline{q}_j^b \hat{m}(t)$; $x_{ij}(t) \leftarrow \overline{x}_{ij} \hat{m}(t)$;
  else
    $OverExpansion \leftarrow True$;
    for $(i,j) \in E$ **do**
      $q_i^s(t) \leftarrow \overline{q}_i^s m(t)$; $q_j^b(t) \leftarrow \overline{q}_j^b m(t)$; $x_{ij}(t) \leftarrow \overline{x}_{ij} m(t)$;
  Solve (EC.1) in Appendix EC.1.1 to obtain $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$;
  if there are multiple feasible solutions, select one arbitrarily;
  update population profile $(\boldsymbol{s}(t+1), \boldsymbol{b}(t+1))$ by the system dynamics in (1).
Output: $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))_{t=1}^T$.

One key advantage of TRP is its computational efficiency: it only requires solving the single-period optimization problem AVG once. Subsequently, in each period, it identifies two critical values: the lowest population ratio, $m(t)$, and the lowest ratio between population and targeted transaction quantity, $M(t)$. It can be easily seen that $m(t) \le M(t)$. By setting $q_i^s(t) = \overline{q}_i^s m(t)$ and $q_j^b(t) = \overline{q}_j^b m(t)$, the policy sets the service level of the scarcest type at its OSS value. In contrast, by setting $q_i^s(t) = \overline{q}_i^s M(t)$ and $q_j^b(t) = \overline{q}_j^b M(t)$, the policy maximizes the service level of the scarcest type. The platform can select any value between these two extremes $\hat{m}(t)$ by adjusting the acceleration weight $w \in [0,1]$. The parameter w reflects how much the platform can trade off short-term profit for accelerated growth, which we will formally demonstrate after characterizing the performance of the TRP.

In the early stages of growth, if $w > 0$, we continue to raise the service level of the scarcest type above its service level at OSS, as long as this policy does not push the population of the scarcest type above that at OSS (i.e., $\min_{i,j:\bar{s}_i>0,\bar{b}_j>0} \left\{ \frac{\mathcal{G}_i^s(s_i(t),\bar{q}_i^s \hat{m}(t))}{\bar{s}_i}, \frac{\mathcal{G}_j^b(b_j(t),\bar{q}_j^b \hat{m}(t))}{\bar{b}_j} \right\} \leq 1$). However, once this increases its population beyond the targeted population level (i.e. $\min_{i,j:\bar{s}_i>0,\bar{b}_j>0} \left\{ \frac{\mathcal{G}_i^s(s_i(t),\bar{q}_i^s \hat{m}(t))}{\bar{s}_i}, \frac{\mathcal{G}_j^b(b_j(t),\bar{q}_j^b \hat{m}(t))}{\bar{b}_j} \right\} > 1$), our policy shifts to match its service level to that at OSS, i.e., $\frac{q_i^s(t)}{s_i(t)} = \frac{\bar{q}_i^s}{\bar{s}_i}$ or $\frac{q_j^b(t)}{b_j(t)} = \frac{\bar{q}_j^b}{\bar{b}_j}$, which helps ensure that the population converges to the OSS.

Notably, for other types with higher population ratios, their demand/supply quantities are matched correspondingly to guarantee feasibility in the networked market. As a result, their service level will be lower than that at OSS (i.e., $\frac{q_i^s(t)}{s_i(t)} < \frac{\bar{q}_i^s}{\bar{s}_i}$ or $\frac{q_j^b(t)}{b_j(t)} < \frac{\bar{q}_j^b}{\bar{b}_j}$). This may lead to slower population growth or even a decline at the beginning. Consequently, the scarcest agent type may change over time within the market, prompting the platform to focus on enhancing the growth of different types throughout the planning horizon. Perhaps surprisingly, by always guaranteeing the growth of the agent types with the *lowest* population ratio in each period, the entire market could converge to the OSS quickly. Let $\mathcal{L}^{TR}(T)$ denote the profit loss of TRP relative to the optimal objective value $\mathcal{R}^*(T)$, and let $m^w(t)$ denote the lowest population ratio under policy with acceleration weight w, then the following result gives a theoretical performance guarantee for TRP:

THEOREM 1. **(performance of TRP)** *There exists a constant* C_2 *such that for all* $T = 1, 2..$ *and* $w \in [0,1]$,

$$\mathcal{L}^{TR}(T) \leq C_2.$$

Furthermore, for any $w \in (0,1]$ *and* $t = \{1, ..., T\}$, *we have*

$$|m^w(t) - 1| \leq |m^0(t) - 1|.$$

Theorem 1 shows that the profit loss of TRP relative to the optimal policy is uniformly bounded (with respect to T) by a constant, which further suggests that TRP is asymptotically optimal in the networked market. Moreover, it reaffirms the fundamental principle that "scarcity creates value." Notably, our proof technique for showing that the policy is asymptotically optimal differs significantly from the methods used in the literature on "steady-state policy" (Flynn 1975a,b, 1992). We first show that under TRP, even though the type with the lowest ratio may change over time, the lowest ratio $m(t)$ *monotonically* *converges* to one at an exponential rate, i.e., $|m(t+1)-1| \le \gamma|m(t)-1|$ for some $\gamma \in (0,1)$. Therefore, for each type, the transaction quantity $q_i^s(t) = \overline{q}_i^s m(t)$ or $q_j^b(t) = \overline{q}_j^b m(t)$ converges to the optimal level $\overline{q}_i^s$ or $\overline{q}_j^b$ for any $i \in \mathcal{S}$ and $j \in \mathcal{B}$, which ensures that the population profile $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ also converges to OSS $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$. By establishing the fast convergence rate, we observe that there exists a constant C_1' such that $|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)| \le C_1'$. Together with the result in Proposition 1, we conclude that there exists a constant C_2 such that $|\mathcal{R}^*(T) - \mathcal{R}^{TR}(T)| \le C_2$. The detailed proof of Theorem 1 is relegated to Appendix EC.2.

Furthermore, Theorem 1 demonstrates that increasing the service level for the most limited type beyond its value at OSS (by choosing a positive w) accelerates the convergence of its population to the targeted level (will illustrated in Figure 1(b)). Next, we will discuss how the commission or subsidy structure under TRP changes with the parameter w to elucidate the costs associated with achieving this accelerated growth. As mentioned, the commissions used to induce the desired transaction quantities in each period can be obtained by solving a system of linear inequalities (see (EC.1) in Lemma EC.1, Appendix EC.1.1).

Commission Structure: Subsidy and Surcharge. Notice that the commission charged from each type of user is not unique, but the total commission collected from both sides for each transaction is unique (see Proposition EC.1 and Lemma EC.1). Therefore, we focus on the total commission for each transaction with a positive flow x_{ij}. Recall that a negative commission can be interpreted as the platform subsidizing the users.

PROPOSITION 2. **(conditions for subsidy)** *For all (i,j) with $x_{ij} > 0, t \in \{1,...,T\}$,*

(i) under TRP with $w = 0$, $r_i^s(t) + r_j^b(t) \ge 0$;

(ii) under TRP with $w > 0$, there exist positive constants z_{ij} and $\tilde{t}$ such that $r_i^s(t) + r_j^b(t) \le 0$ if $\frac{\max\{s_i(t), b_j(t)\}}{\min\limits_{i' \in \mathcal{S}, j' \in \mathcal{B}} \{s_{i'}(t), b_{j'}(t)\}} < z_{i,j} w$ and $t < \tilde{t}$.

Proposition 2(i) highlights that subsidization during the growth stage is not necessary, particularly in a monopoly environment. Instead, by lowering the commission for appropriate user segments, the platform can already effectively guide the user base toward the OSS. Therefore, a policy with a small w is an ideal option for startup platforms with limited budgets. Proposition 2(ii) identifies the specific user types that should be subsidized during the early growth phase when the platform seeks to accelerate growth and capture market share quickly by choosing a positive w. Subsidies become necessary when the ratio of buyers and sellers involved in transactions to the minimum population of types drops below a certain threshold. This indicates that the number of participants on both sides of a transaction is close to the population of the scarcest type, aligning with real-world practices where platforms often provide coupons or discounts to attract scarce users during the initial growth stages. A higher w is more likely to render subsidization.

To better understand this condition, consider a simple example with one buyer and one seller type. If the buyer type has a lower initial population (i.e., $b(1) < s(1)$), the condition is simplified to $\frac{s(1)}{b(1)} < wz$, suggesting that the population of sellers is moderately larger than that of buyers (see Figure 1(d)-1(c) later). In this case, the platform can stimulate growth on both sides by simultaneously engaging nearly all buyers and sellers in transactions, which requires providing subsidies. On the contrary, if the population of sellers is significantly higher than that of buyers (i.e., $b(1) \ll s(1)$), most buyers naturally participate in transactions at equilibrium without requiring subsidies from the platform. Moreover, the platform can impose higher commissions on sellers due to the limited number of buyers available for transactions.

Finally, we identify how population levels, growth potentials, and network structures influence our policy design. For expositional ease, we denote by $Y_j(t) := \min_{i':(i',j)\in E} \{p_{i'}(t) + r_j^b(t)\}$ the type-j buyer's payment at time t, and denote by $I_i(t) := p_i(t) - r_i^s(t)$ the type-i seller's income at time t.

COROLLARY 1. *Suppose that $F_{s_i} = F_s$ for any $i \in \mathcal{S}$ and $F_{b_j} = F_b$ for any $j \in \mathcal{B}$, then under TRP, for any $t \in \{1, ..., T\}$,*

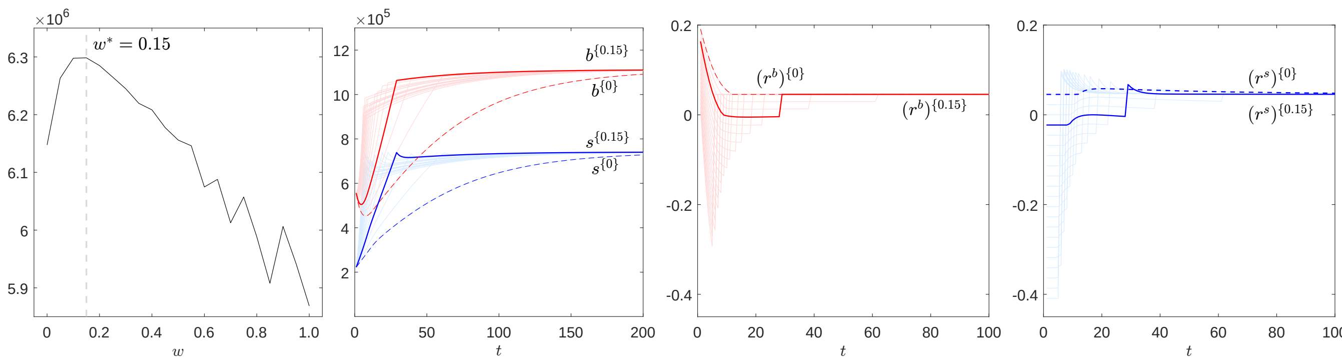
(i) for the buyer side, if $\frac{\bar{q}_j^b}{\bar{b}_j} / \frac{b_j(t)}{\bar{b}_j} < \frac{\bar{q}_l^b}{\bar{b}_l} / \frac{b_l(t)}{\bar{b}_l}$, we have $Y_j(t) > Y_l(t)$, and there exists one feasible commission such that $r_j^b(t) > r_l^b(t)$ for any $j, l \in \mathcal{B}$;

(ii) for the seller side, if $\frac{\bar{q}_i}{\bar{s}_i} / \frac{s_i(t)}{\bar{s}_i} < \frac{\bar{q}_l}{\bar{s}_l} / \frac{s_l(t)}{\bar{s}_l}$, we have $I_i(t) < I_l(t)$, and there exists one feasible commission such that $r_i^s(t) > r_l^s(t)$ for any $i, l \in \mathcal{S}$.

Our findings indicate that the payment or income for each type depends only on two factors: (1) the targeted service level at the OSS, and (2) the current population ratio. In Section 5, we will construct a novel metric to demonstrate how the targeted service level is affected by the network structure and the growth potential of different types (see Proposition 4). Essentially, the targeted service level is lower for a type with higher growth potential but is compatible with types with lower growth potential. Consequently, the platform should charge higher commissions from them. On the other hand, the platform should charge lower commissions from user types with lower population ratios.

We will numerically illustrate the profit (see Figure 1(a)), population growth trajectory (see Figure 1(b)), and feasible commission structure (see Figure 1(c)-Figure 1(d)) for $w \in \{0, 0.05, 0.1, ..., 1\}$ (a total of 21 cases). We use the superscript $\{w\}$ to denote the value under a policy with an acceleration weight w.

Example 1. Consider a one-buyer-one-seller example. The seller side has a lower initial population ratio (i.e., $30\% = \frac{s(1)}{\bar{s}} > \frac{b(1)}{\bar{b}} = 50\%$). The retention rate of the seller side is assumed to be lower than the buyer side: $s(t+1) = 0.7s(t) + 2(q^s(t))^{0.9}$ and $b(t+1) = 0.8b(t) + 2(q^b(t))^{0.9}$.



(a) Profit (b) Population growth (c) Feasible r^b (d) Feasible r^s

Figure 1 performance of FRP.

Figure 1(a) shows that the platform's profit is neither monotonic nor concave with respect to the parameter w, but it can be seen that the profit reaches its maximum value when $w = 0.15$. Therefore, we highlight the population and commission trajectories for $w = 0.15$ using a solid line and use a dashed line to highlight the case when $w = 0$ for comparison. In Figure 1(b), the mass of both types grows over time and eventually stabilizes at OSS under both policies. Specifically, compared to the policy with $w = 0$, the population under the case with $w = 0.15$ approaches the targeted goal more quickly in the early stages. Figures 1(c) and 1(d) illustrate the evolution of feasible commissions for buyers and sellers, respectively, before stabilizing. Under this class of policies, the platform initially charges lower commissions on the seller side to stimulate growth, as sellers are relatively scarce in the early stages. As the seller side's population ratio surpasses that of the buyer side, the platform then increases the commission for sellers while decreasing it for buyers. Specifically, for $w=0$, the platform charges a positive commission throughout the decision horizon, which is consistent with Proposition 2(i). In contrast, for $w=0.15$, the strategy involves subsidizing the type with a lower population ratio rather than merely lowering prices, especially when the population of both sides are close (see the condition in Proposition 2(ii)). This class of policy mirrors Airbnb's growth strategy: the company initially maintained low commission rates for hosts to attract new listings and build a strong inventory. Over time, Airbnb reduced its guest service fees to enhance competitiveness with traditional hotels while increasing host commissions [1].

Myopic Policy (MP). Some prior studies have examined the effectiveness of the myopic pricing policy in the product diffusion process of a monopoly seller, and they draw varying results under different diffusion functions. Robinson and Lakhani (1975) showed that myopic policy results in significant profit loss relative to the optimal policy if a lower current price could stimulate future demand. In contrast, Bass and Bultez (1982) considered the case that the diffusion process does not interact with price and showed by a numerical study that there is only a small difference in the discounted profits between the myopic and optimal policies. Here we examine how MP performs in our model.

Under MP, in each period t, the platform determines the commissions $(\boldsymbol{r}^b(t), \boldsymbol{r}^s(t))$ to maximize its profit in the current period (i.e., $\sum_{i\in\mathcal{S}} r_i^s(t)q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t)q_j^b(t)$) subject to the equilibrium constraints in (2), without considering the population dynamics in (1) and its impact on future profit. The formal definition of the myopic policy is given by Definition EC.1 in Appendix EC.2. We let $\mathcal{R}^{MP}(t)$ denote the platform's profit under MP in period t, and recall that $\overline{\mathcal{R}}$ is the optimal value of AVG and could be achieved under TRP. The following result shows that the performance of MP could be arbitrarily bad.

Proposition 3. **(performance of MP)** *Under MP, for any $\epsilon > 0$, there exists a problem instance such that $\lim_{t\to\infty} \mathcal{R}^{MP}(t) := \overline{\mathcal{R}}^{MP} < \infty$ and $\overline{\mathcal{R}}^{MP} < \epsilon\overline{\mathcal{R}}$. Hence, there exists $C_3 > 0$ such that*

$$\mathcal{L}^{MP}(T) \;\geq\; C_3 T.$$

Proposition 3 suggests that ignoring the commissions' impact on population growth could lead to significant profit loss even if the platform serves as a monopoly intermediary. In

[1] https://www.airbnb.com/e/commissione-semplificata-guida

the proof of Proposition 3, we show that the commissions set by the platform under MP at the steady state are higher than those under TRP. This result, again, highlights that the platform must sacrifice some short-term margin to achieve long-term profitability.

Since the TRP requires controlling the service level (or equivalently, their payment/income) of different agent types at the level of the OSS, we will see in the next section how the service level at OSS is determined by both the compatibility network structure $G(\mathcal{S}\cup\mathcal{B},E)$ and population dynamics from (1).

5. Impact of Population Dynamics and Compatibility Network Structure

In this section, we investigate how the intertemporal factors marked by heterogeneous growth potentials and the spatial factors characterized by the compatibility influence the platform's profit (see Section 5.1) as well as the incomes/payments of agents and optimal commission (see Section 5.2) at OSS. Investigating the impacts of these spatial-temporal factors can provide insights into the platform's revenue management strategy.

The prior studies showed that in a static setting, a network that more efficiently matches supply with demand achieves a better performance from the platform's perspective, and the agent types connected to a larger population on the other side would gain higher surplus (see Schrijver et al. 2003, Chou et al. 2011, Birge et al. 2021). For example, Chou et al. (2011) showed that a bipartite network, in which every subset of nodes is linked to a sufficiently large number of neighboring nodes, is optimal for the system. Similarly, Birge et al. (2021) showed that supply-demand imbalance across the network, measured by the lowest seller-to-buyer population ratio among all submarkets, determines the lower bounds of the platform's achievable profit relative to that with a complete network. Interestingly, we see from the following numerical example that in a dynamic setting, the imbalance in terms of the equilibrium population ratio at OSS can no longer provide a profit guarantee for the platform.

Example 2. Consider a compatibility network shown in Figure 2. Suppose that buyers' and sellers' (reservation) values are uniformly distributed between $[0,1]$. Consider $s_i(t+1)=\alpha s_i(t)+\beta_i^s q_i^s(t)^\xi$ and $b_j(t+1)=\alpha b_j(t)+\beta_j^b q_j^b(t)^\xi$ for $i\in\{1,2\}, j\in\{1,2\}$ with parameters $\alpha=0.5$, $\xi=0.8, \beta_1^s=\beta_2^b=2, \beta_2^s=\beta_1^b=1$. The population ratio in AVG satisfies $\frac{\sum_{i\in N_E(\widetilde{\mathcal{B}})}\bar{s}_i}{\sum_{j\in\widetilde{\mathcal{B}}}\bar{b}_j}\geq 50\%\times\frac{\sum_{i\in\mathcal{S}}\bar{s}_i}{\sum_{j\in\mathcal{B}}\bar{b}_j}$ for any $\widetilde{\mathcal{B}}\subseteq\mathcal{B}$, but the platform's optimal profit at OSS is only about 36 % of that in a complete market, i.e., $\overline{\mathcal{R}}(E,\psi^s,\psi^b)=36\%\times\overline{\mathcal{R}}(\overline{E},\psi^s,\psi^b)$,

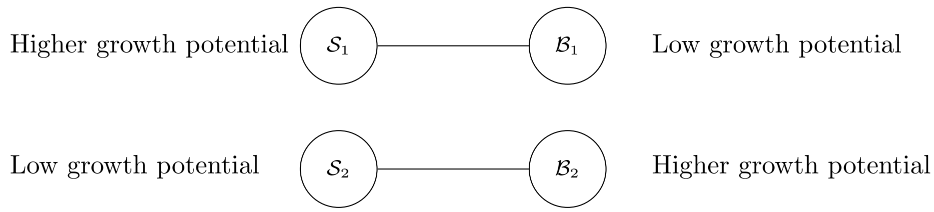


Figure 2 **Bias of Measuring Network Imbalance using Equilibrium Population Ratio**

Hence, it becomes crucial to incorporate temporal factors into the “imbalance” measure for the compatibility network, which requires us to first measure the growth potential of each agent type. As we mentioned in Section 1, the growth of an active agent base consists of retaining previous agents and encouraging word-of-mouth effect to attract new adoption. Therefore, to better quantify these two effects, we consider the following class of growth functions $\mathcal{G}^s(\cdot)$ and $\mathcal{G}^b(\cdot)$ in (1):

$$\mathcal{G}_i^s(s,q) = \alpha_i^s s + \beta_i^s g_s(s,q), \tag{6a}$$

$$\mathcal{G}_j^b(b,q) = \alpha_j^b b + \beta_j^b g_b(b,q). \tag{6b}$$

where we consider homogeneous-degree-ξ_s (and ξ_b) functions $g_s(\cdot,\cdot)$ (and $g_b(\cdot,\cdot)$) with $\xi_s \in (0,1)$ (and $\xi_b \in (0,1)$) for any $q,s,b \geq 0$. A function $g(\cdot,\cdot)$ is homogeneous of degree ξ means that $g(ns,nq) = n^{\xi} g(s,q)$ for any $s \geq q \geq 0$, $n > 0$.

In (6), $\alpha_i^s \in (0,1)$ and $\alpha_j^b \in (0,1)$ respectively represent the retention rate of type-i sellers and type-j buyers (Lian and Van Ryzin 2021, Alizamir et al. 2022, He and Goh 2022). Then $1-\alpha_i^s$ or $1-\alpha_j^b$ can be seen as the attrition rate of the agent type. In practice, platforms can track retention data across different user groups and aim to maintain specific retention rates. For instance, Upwork has an overall retention rate of 58%, while the retention rate for core clients who have spent over 5,000 is 83% [2]. Airbnb's overall retention rate is approximately 30%-40% [3].

The second term captures new agents' adoption. $\beta_i^s \in (0,1)$ and $\beta_j^b \in (0,1)$ measure the type-specific impact of the current user base and transactions on new adoption. Some previous studies on the growth of two-sided platforms assume that the new adoption depends on the transaction volume/price/surplus in the last period and the growth rates are homogeneous for all agents from one side (see Lian and Van Ryzin 2021, He and Goh 2022).

[2] https://www.sec.gov/Archives/edgar/data/1627475/000119312518267594/d575528ds1.htm

[3] https://secondmeasure.com/datapoints/airbnb-sales-surpass-most-hotel-brands/

Instead, we impose a property known as the homogeneous degree of ξ, which measures the elasticity of the future user base with respect to current transactions and user base. $\xi < 1$ suggests a decreasing marginal effect of transaction quantity and user base on the increase in new adoptions For example, in 2022, transaction volume on Airbnb increased by 30% compared to 2021. However, with an estimated retention rate of 40%, the number of new users in 2022 was only 1.03 times greater than in 2021. A similar pattern can be seen from the buyer side [4].

Furthermore, to isolate the impact of compatibility network structure and growth potential, we assume that different types of sellers/buyers have homogeneous value distributions.

ASSUMPTION 4. *$F_{s_i}(v) = F_s(v)$ for any $i \in \mathcal{S}$ and $F_{b_j}(v) = F_b(v)$ for any $j \in \mathcal{B}$.*

We next construct a metric to measure the long-run growth potential of each agent type and then measure the imbalance of the compatibility network using this metric. Before that, we present the main notations used in this section for the reader's reference:

Notations	Definitions	Notes
α_i^s, α_j^b	retention rate	
β_i^s, β_j^b	new adoption rate	exogenous parameters
ξ_i^s, ξ_j^b	elasticity	
ψ_i^s, ψ_j^b	long-run growth potential	depends on (α, β, ξ), see (9)
$\mathcal{S}_\tau, \mathcal{B}_\tau$	seller (buyer) set with ranking τ	depends on (ψ^s, ψ^b), see (10)
ϵ	degree of imbalance of the network	depends on $(\mathcal{S}_\tau, \mathcal{B}_\tau)$, see (11)

Table 1 **Notations**

Agents' long-run growth potential. To obtain an intuitive expression of the long-run growth potential, we use, throughout this section, a simple polynomial term for g_s and g_b as an illustrative example. For $t \in \{1, ..., T-1\}$,

$$s_i(t+1) = \alpha_i^s s_i(t) + \beta_i^s (q_i^s(t))^{\xi_s}, \qquad \text{for } i \in \mathcal{S}, \tag{7a}$$

$$b_j(t+1) = \alpha_j^b b_j(t) + \beta_j^b (q_j^b(t))^{\xi_b}, \qquad \text{for } j \in \mathcal{B}. \tag{7b}$$

[4] https://www.businessofapps.com/data/airbnb-statistics/

Under this form, we can provide a closed-form expression for long-run growth potential, based on which we further deduce all the following results and managerial insights. However, it is worth pointing out that our proofs technique does not rely on the exact expressions of (7). Given type-i sellers' service level $\frac{\overline{q}_i^s}{\overline{s}_i}$ induced by the platform, the population of type-i seller converges to $\overline{s}_i$ that satisfies $\overline{s}_i = \alpha_i^s \overline{s}_i + \beta_i^s (\overline{q}_i^s)^{\xi_s}$. Algebraic manipulations suggest that

$$\overline{s}_i = \left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}} \left(\frac{\overline{q}_i^s}{\overline{s}_i}\right)^{\frac{\xi_s}{1-\xi_s}}, \qquad \overline{q}_i^s = \left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}} \left(\frac{\overline{q}_i^s}{\overline{s}_i}\right)^{\frac{1}{1-\xi_s}} \text{ where } i \in \mathcal{S}, \tag{8}$$

Eqn. (8) reveals that given the service level $\frac{\overline{q}_i^s}{\overline{s}_i}$ for type-i sellers, the population of an agent type and the transaction quantities at OSS are proportional to the coefficients $\left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}}$. The same equations hold for the buyer side. Based on this, we formally define the long-run growth potential as follows:

$$\psi_i^s := \left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}},\ i \in \mathcal{S}, \qquad \psi_j^b := \left(\frac{\beta_j^b}{1-\alpha_j^b}\right)^{\frac{1}{1-\xi_b}},\ j \in \mathcal{B}. \tag{9}$$

We next provide some intuitive explanations for (ψ^s, ψ^b). For simplicity, we omit the superscripts (s, b) and subscripts (i, j). In the population dynamics in (7), β captures the impact of transaction quantities on the population growth, and only a fraction $\alpha < 1$ of agents stays in the system after each period. As $\frac{\beta}{1-\alpha} = \sum_{t=0}^{\infty} \beta \alpha^t$, it captures the net present value for the long-run marginal impact of the transaction quantity q^{ξ}. Similarly, the impact of the population elasticity ξ after t periods can be captured by ξ^t. As $\frac{1}{1-\xi} = \sum_{t=0}^{\infty} \xi^t$, it represents the net present value of the long-term impact of the elasticity ξ. Therefore, we refer to ψ_i^s for $i \in \mathcal{S}$ and ψ_j^b for $j \in \mathcal{B}$ in (9) as the long-run growth potential of each agent type.

Rankings of relative growth potential in the compatibility network. Based on the long-run growth potential, we introduce a ranking of different types of agents. Let $N_E(X)$ denote the set of all neighbors of agent types $X \subseteq \mathcal{B} \cup \mathcal{S}$ in the graph $G(S \cup B, E)$ such that $N_E(X) = \{i \notin X : (i, j) \in E \text{ for } j \in X\}$. Given a compatibility network $G(\mathcal{S} \cup \mathcal{B}, E)$ and the long-run growth potential vector (ψ^s, ψ^b), we first let $\mathcal{B}^0 = B$, $\mathcal{S}^0 = S$ and $E^0 = E$. For $\tau = 0, 1, ...,$ we define $\mathcal{B}_\tau$ and $\mathcal{S}_\tau$ iteratively as follows:

$$\mathcal{B}_{\tau+1} = \underset{\widetilde{\mathcal{B}} \subseteq \mathcal{B}^\tau}{\arg\min} \frac{\sum_{i \in N_{E^\tau}(\widetilde{\mathcal{B}})} \psi_i^s}{\sum_{j \in \widetilde{\mathcal{B}}} \psi_j^b}, \tag{10a}$$

$$\mathcal{S}_{\tau+1} = N_{E^\tau}(\mathcal{B}_{\tau+1}). \tag{10b}$$

where $\mathcal{B}^{\tau} = \mathcal{B}^{\tau-1} \backslash \mathcal{B}_{\tau}$, $\mathcal{S}^{\tau} = \mathcal{S}^{\tau-1} \backslash \mathcal{S}_{\tau}$, $E^{\tau} = \{(i,j) \in E : i \in \mathcal{S}^{\tau} \text{ and } j \in \mathcal{B}^{\tau}\}$ are the remaining buyer set, seller set, and edge set after removing the subgraph labeled in step τ. If multiple sets achieve the minimum, the arg min operator returns the largest one.

In (10a), for each subset of buyer types $\widetilde{\mathcal{B}}$ of $\mathcal{B}^{\tau}$, $\frac{\sum_{i \in N_{E^{\tau}}(\widetilde{\mathcal{B}})} \psi_i^s}{\sum_{j \in \widetilde{\mathcal{B}}} \psi_j^b}$ is the ratio between the total long-run growth potential of its (remaining) compatible sellers and its own. We refer to the ratio as the *relative growth potential* between $N_{E^{\tau}}(\widetilde{\mathcal{B}})$ and $\widetilde{\mathcal{B}}$. This metric, similar to those used for comparing two economies, e.g., in Krugman (1989), captures the relative growth potential of sellers and buyers. In (10), we can iteratively identify a subgraph such that the relative growth potential of sellers is the lowest. Subsequently, we label it and remove this subgraph from the network, and then B^{τ} and S^{τ} are the remaining agent types and E^{τ} is the remaining graph after τ iterations. We repeat the procedure until the remaining subgraph is empty. As a result, the subnetwork with a higher index τ has a higher relative growth potential of sellers against buyers in the graph. This ranking incorporates both intertemporal factors captured by the long-run growth potential ψ and spatial factors captured by the graph structure $G(\mathcal{B} \cup \mathcal{S}, E)$.

We use the example below to illustrate the rankings of relative growth potential. This example illustrates the compatibility between freelance coders and clients in need of IT services on Upwork. Specifically, clients needing AI Services can only be served by coders with AI skills, and clients requiring immediate delivery of work can only choose coders with flexible working hours. By enumeration, we can obtain the index of each type, and the solid (dotted) line represents the lines between sets with the same (different) index. For a large-scale network, we can obtain the ranking by solving a convex optimization problem This procedure borrows the algorithmic idea to characterize the lexicographically optimal bases of polymatroids from Fujishige (1980).

Example 3. Consider a compatibility network as shown in Figure 3.

Suppose that $\psi_i^s = \psi_j^b = 1$ for $i = \{1,2,3\}$ and $j = \{1,2,3\}$. Then by enumeration, we know $\{1,2\} = \arg\min_{\widetilde{\mathcal{B}} \subseteq \mathcal{B}} \frac{\sum_{i \in N_E(\widetilde{\mathcal{B}})} \psi_i^s}{\sum_{j \in \widetilde{\mathcal{B}}} \psi_j^b}$, which means $\mathcal{B}_1 = \{1,2\}$ and $\mathcal{S}_1 = \{1\}$ (blue nodes). After eliminating $\mathcal{B}_1$ and $\mathcal{S}_1$ from the network E, we have $\mathcal{B}^1 = \{3\}, \mathcal{S}^1 = \{2,3\}, E^1 = \{(2,3),(3,3)\}$. Since there is only one buyer type left, we know $\mathcal{B}_2 = \{3\}$ and $\mathcal{S}_2 = \{2,3\}$ (black nodes). Finally, all agent types are labeled with an index.

We will next show the connection between the lowest relative growth potential in the compatibility network and the platform's profit at OSS.

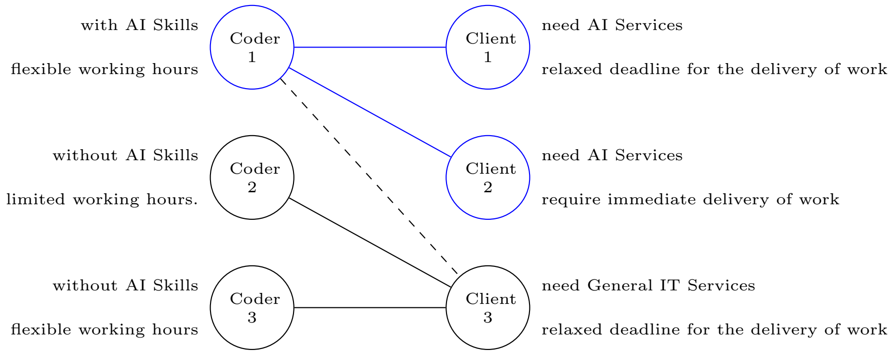


Figure 3 **Compatibility between Freelance Coders and Clients in need of IT Services on Upwork.**

5.1. Optimal Compatibility Network for the Platform's Profit

To signify the dependence on the compatibility network structure $G(\mathcal{S}\cup\mathcal{B},E)$ and long-run growth potential (ψ^s,ψ^b), we let $\overline{\mathcal{R}}(E,\psi^s,\psi^b)$ denote the platform's optimal steady-state profit. Given that the feasible region for a complete graph is the largest in Problem (3), the platform can achieve the maximum optimal profit in a complete graph. Therefore, we let $\overline{E}$ denote the edge set for the complete graph with the set of seller types $\mathcal{S}$ and that of buyer types $\mathcal{B}$, and use $\overline{\mathcal{R}}(\overline{E},\psi^s,\psi^b)$ to benchmark the impact of compatibility network structure E on the platform's profit. The following theorem establishes a connection between the temporal-spatial factors and the platform's optimal profit in network $G(\mathcal{S}\cup\mathcal{B},E)$.

THEOREM 2. **($(1-\epsilon)$-optimal network structure)** *For any $\epsilon\in[0,1]$, if $G(\mathcal{B}\cup\mathcal{S},E)$ satisfies*

$$\frac{\sum_{i\in\mathcal{S}^1}\psi_i^s}{\sum_{j\in\mathcal{B}^1}\psi_j^b} \;\geq\; (1-\epsilon)\frac{\sum_{i\in\mathcal{S}}\psi_i^s}{\sum_{j\in\mathcal{B}}\psi_j^b}, \tag{11a}$$

then

$$\overline{\mathcal{R}}(E,\psi^s,\psi^b) \;\geq\; (1-\epsilon)\overline{\mathcal{R}}(\overline{E},\psi^s,\psi^b). \tag{11b}$$

In Condition (11a), the right-hand-side expression $\frac{\sum_{i\in\mathcal{S}}\psi_i^s}{\sum_{j\in\mathcal{B}}\psi_j^b}$ represents the relative long-run growth potential of all sellers to all buyers within the entire compatibility network $G(\mathcal{B}\cup\mathcal{S},E)$. Likewise, the left-hand-side is the relative growth potential of the compatible sellers to a subset of buyers $\mathcal{B}^1$, whose relative long-run growth potential is the lowest (see (10)). Therefore, ϵ quantifies the degree of imbalance: a positive value of ϵ indicates that there exists no submarket in which the relative growth potential is ϵ lower than that of the entire market. Then (11b) implies that the degree of imbalance ϵ in the compatibility network does not cause more than ϵ optimal profit loss for the platform. When $\epsilon = 0$, the condition in (11a) ensures that the relative growth potential for all submarkets is equal to that for the entire market. In other words, the long-run growth potentials are “balanced” in the compatibility network. In this case, even though the market E may be incomplete, the lower bound in (11b) is tight, and the platform's optimal profit achieves the maximum possible optimal profit, i.e., $\overline{\mathcal{R}}(E, \psi^s, \psi^b) = \overline{\mathcal{R}}(\overline{E}, \psi^s, \psi^b)$.

The managerial insight derived from Theorem 2 suggests that the platform should aim to enhance the balance of the compatibility network in terms of long-run growth potential to maximize its steady-state optimal profit. Specifically, the platform could target its marketing campaign on agent types with relatively low long-run growth potential to increase their retention and attract new users.

Connection to Agent Heterogeneity. The parameter ϵ captures the extent of network imbalance, which is closely associated with the heterogeneity among agents in terms of their growth potentials and compatibility. Specifically, when the growth potentials and the number of compatible types are homogeneous for all agent types from the same side, we have $\epsilon = 0$, which suggests that the network is balanced. To better understand ϵ, we can consider two extreme examples: one with homogeneous growth potential and another with homogeneous compatibility.

(1). If $\psi_i^s = \psi^s$ for any $i \in \mathcal{S}$ and $\psi_j^b = \psi^b$ for any $j \in \mathcal{B}$, then (11a) becomes $\min\limits_{\tilde{\mathcal{B}} \subseteq \mathcal{B}} \frac{|N_E(\tilde{\mathcal{B}})|}{|\tilde{\mathcal{B}}|} \geq (1-\epsilon)\frac{|\mathcal{S}|}{|\mathcal{B}|}$. That is, when the growth potentials are homogeneous for all agent types from the same side, $1-\epsilon$ captures the disparity between the minimum seller-buyer ratio and the average seller-buyer ratio. In other words, it captures the heterogeneity of buyer types' preferences.

(2). If $|\mathcal{B}| = |\mathcal{S}|$ and $(i, j) \in E$ if and only if $i = j$, then (11a) becomes $\min\limits_{l \in \mathcal{B}} \frac{\psi_l^s}{\psi_l^b} \geq (1-\epsilon)\frac{\sum_{i \in \mathcal{S}} \psi_i^s}{\sum_{j \in \mathcal{B}} \psi_j^b}$.When each type of buyer corresponds uniquely to a type of seller and vice versa, the term $1-\epsilon$ reflects only the differences in relative growth potentials in each combination.

5.2. Agent Payments/Incomes and Platform Commissions

In this subsection, we analyze the impact of agents' growth potential on the platform's commission decisions. Recall that the optimal commission $(\overline{\boldsymbol{r}}^{\boldsymbol{s}}, \overline{\boldsymbol{r}}^{\boldsymbol{b}})$ at OSS is not necessarily unique, but any optimal commission profile induces the same (net) payments and incomes for agent types engaged in transactions (see Proposition EC.1 and Lemma EC.1). Furthermore, the total commission generated from a transaction (i.e., $r_i^s + r_j^b$ for $(i,j) \in E$), which represents the difference between buyers' payments and sellers' incomes, is inherently unique. Therefore, in this subsection, we will first study the impact of compatibility network structure and growth potentials on (net) payments and incomes for agent types and then analyze their impact on the total optimal commission.

Buyers' payments and sellers' incomes. We next establish that the ranking of the relative growth potentials of sellers to buyers given in (10) determines the ranking of buyers' payments and sellers' incomes at OSS. We denote by $Y_j = \min_{i':(i',j)\in E} \{\overline{p}_{i'} + \overline{r}_j^b\}$ the payment of any type-j buyers, and denote by $I_i = \overline{p}_i - \overline{r}_i^s$ the income of any type-i sellers at OSS.

Proposition 4. **(ranking of buyers' payments and sellers' incomes)** *In the compatibility network $G(\mathcal{S} \cup \mathcal{B}, E)$, under any platform's optimal commission profile $(\overline{\boldsymbol{r}}^{\boldsymbol{s}}, \overline{\boldsymbol{r}}^{\boldsymbol{b}})$ at the steady state,*

(1) for any $j_1 \in \mathcal{B}_{\tau_1}$ and $j_2 \in \mathcal{B}_{\tau_2}$ with $\tau_1 \leq \tau_2$, $Y_{j_1} \geq Y_{j_2}$ and $\frac{\overline{q}_{j_1}^b}{\overline{b}_{j_1}} \leq \frac{\overline{q}_{j_2}^b}{\overline{b}_{j_2}}$;

(2) for any $i_1 \in \mathcal{S}_{\tau_1}$ and $i_2 \in \mathcal{S}_{\tau_2}$ with $\tau_1 \leq \tau_2$, $I_{i_1} \geq I_{i_2}$ and $\frac{\overline{q}_{i_1}^b}{\overline{b}_{i_1}} \geq \frac{\overline{q}_{i_2}^b}{\overline{b}_{i_2}}$.

Proposition 4 posits that under the platform's optimal commissions at the steady state, with a higher relative long-run growth potential of sellers to buyers (i.e., higher index τ indicates higher $\frac{\sum_{i \in \mathcal{S}_\tau} \psi_i^s}{\sum_{j \in \mathcal{B}_\tau} \psi_j^b}$ in (10)), the buyers pay less and experience a higher service level, while the sellers earn a lower income and experience a lower service level in equilibrium. By using the Example 3 to illustrate, the payments on the buyer (i.e., client) side satisfy $Y_1 = Y_2 > Y_3$ given that $\mathcal{B}_1 = \{1,2\}$ and $\mathcal{B}_2 = \{3\}$; the incomes on the seller (i.e., coder) side satisfy that $I_1 > I_2 = I_3$ given that $\mathcal{S}_1 = \{1\}$ and $\mathcal{S}_2 = \{2,3\}$. The managerial implication from Proposition 4 is that while determining the service level, the platform needs to consider not only the retention rate and growth potentials of the focal agent types but also their trading partners on the other side of the market. Specifically, the platform should incentivize the agents with lower relative growth potential by offering them higher commissions and extract a higher surplus from those with higher relative growth potential.

Proposition 4 suggests that any change in the values of (ψ^s, ψ^b) induces changes in the service level of each agent type, ultimately affecting the equilibrium demand, supply, and population at OSS. Lastly, we examine the influence of the long-run growth potential (ψ^s, ψ^b) to offer guidance for the platform's commission decisions.

Corollary 2. **(impact of the long-run growth potential)** *Given any* $\xi_s \in (0,1)$ *and* $\xi_b \in (0,1)$*, at OSS,*

(1) for the service levels,

(i) given $j \in \mathcal{B}$*,* $\overline{q}_j^b/\overline{b}_j$ *decreases in* $\psi_{j'}^b \geq 0$ *for any* $j' \in \mathcal{B}$ *and increases in* $\psi_{i'}^s \geq 0$ *for* $\forall i' \in \mathcal{S}$*;*

(ii) given $i \in \mathcal{S}$*,* $\overline{q}_i^s/\overline{s}_i$ *decreases in* $\psi_{i'}^s \geq 0$ *for any* $i' \in \mathcal{S}$ *and increases in* $\psi_{j'}^b \geq 0$ *for any* $j' \in \mathcal{B}$*;*

(2) for the transaction quantities and populations,

(i) given $j \in \mathcal{B}$*,* $(\overline{q}_j^b, \overline{b}_j)$ *increases in* $\psi_j^b \geq 0$*, decreases in* $\psi_{j'}^b \geq 0$ *for any* $j' \in \mathcal{B}$ *with* $j' \neq j$*, and increases in* $\psi_{i'}^s \geq 0$ *for any* $i' \in \mathcal{S}$*;*

(ii) given $i \in \mathcal{S}$*,* $(\overline{q}_i^s, \overline{s}_i)$ *increases in* $\psi_i^s \geq 0$*, decreases in* $\psi_{i'}^s \geq 0$ *for* $\forall i' \in \mathcal{S}$ *with* $i' \neq i$ *and increases in* $\psi_{j'}^b \geq 0$ *for any* $j' \in \mathcal{B}$*.*

Note that for any $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, the vectors (ψ^s, ψ^b) are determined by the retention rates $(\boldsymbol{\alpha}^s, \boldsymbol{\alpha}^b)$ and the growth coefficients $(\boldsymbol{\beta}^s, \boldsymbol{\beta}^b)$. Corollary 2(1) suggests that the service level of any agent decreases in the growth potential of all types from the same side but increases in those on the other side of the market. Corollary 2(2) implies that the transaction volume and population of each type are increasing in their own growth potential and those on the other side of the network, but decreasing in those of other types on the same side.

We discuss the intuition using the buyer side as an example. Both a high long-run growth potential and a high service level contribute to an increase in the population of a buyer type at OSS. Consequently, when the long-run growth potential of a buyer type is high, the platform can maintain a high population by inducing a relatively lower service level. However, if other buyer types have higher long-run growth potential, their equilibrium demand will rise, resulting in increased prices for the sellers and a reduced service level for our focal buyer type. Conversely, if the corresponding sellers have higher long-run growth potential, their supply will increase, leading to lower prices and benefiting all buyers.

Platform's optimal commissions. We now focus on the total commission charged by the platform from one transaction, viz., the difference between the buyers' payments and the sellers' incomes. Note that under the optimal commission, type-i sellers with $i \in \mathcal{S}_\tau$ only trade with type-j buyers with $j \in \mathcal{B}_\tau$. Therefore, we will examine how the total commission charged from one transaction between sellers in $\mathcal{S}_\tau$ and buyers in $\mathcal{B}_\tau$ depends on the ranking of the relative growth potential of sellers to buyers τ given in (10). Here, we assume $\xi_s = \xi_b$ to isolate the impact of value distribution.

Proposition 5. **(ranking the platform's optimal commissions)** *Assume that F_s and F_b are twice differentiable in their domains and $\xi_s = \xi_b$. There exists $\widetilde{\tau}$ such that*

(1) $r_i^s + r_j^b$ for $i \in \mathcal{S}_\tau, j \in \mathcal{B}_\tau$ is decreasing in τ for $\tau < \widetilde{\tau}$;

(2) $r_i^s + r_j^b$ for $i \in \mathcal{S}_\tau, j \in \mathcal{B}_\tau$ is decreasing in τ for $\tau \geq \widetilde{\tau}$ if $F_s(v)$ and $F_b(v)$ are concave; whereas it is increasing in τ for $\tau \geq \widetilde{\tau}$ if $F_s(v)$ and $F_b(v)$ are convex.

In Proposition 5(1), when the relative growth potential of sellers to buyers falls below a threshold, the total commission charged from the transaction decreases with the relative growth potential between sellers and buyers. In Proposition 5(2), the concavity of $F_s(v)$ and $F_b(v)$ implies a higher density of agents with lower (reservation) value. In this case, when the relative growth potential of sellers to buyers is higher, the optimal total commission charged by the platform should be lower. Similarly, the convexity of $F_s(v)$ and $F_b(v)$ implies that the number of agents with higher (reservation) value is higher. In this scenario, the platform charges lower (higher) total commissions for transactions involving agents with moderate (high or low) relative growth potentials of sellers to buyers.

Intuitively, when the relative growth potential between sellers and buyers is below a threshold, the number of sellers is significantly smaller than that of buyers. In such cases, the platform uses its commission to keep the sellers' income at a sufficiently high level to ensure the participation of sellers. As the relative growth potential increases, the number of sellers rises, prompting the platform to gradually reduce buyer payments to stimulate demand. As a result, the total commission charged from the transaction, which is the difference between buyer payments and seller incomes, decreases with the relative growth potential between sellers and buyers.

When the relative growth potential between sellers and buyers exceeds the threshold, the number of sellers is already large in the market, and the platform no longer needs to provide high subsidies to ensure their participation. In this case, an increase in the relative growth potential between sellers and buyers suggests that the platform should reduce the service level for sellers and increase the service level for buyers, aimed at achieving a balance between supply and demand. When most agents have a low valuation of the product or service, the platform needs to offer buyers a large price cut to increase their demand, but a slight decrease in sellers' earnings can dampen the supply. As a result, the total commission from the transaction decreases with the relative growth potential. Conversely, when most agents highly value the product or service, providing buyers with a modest price reduction is sufficient to encourage their participation, and the platform can substantially reduce sellers' earnings without significantly impacting their supply. As a result, the total commission charged from the transaction increases with the relative growth potential between sellers and buyers.

6. Conclusion

In this study, we consider a two-sided platform that facilitates transactions between buyers and sellers with heterogeneous growth potentials. The compatibility between buyer and seller types is captured by a bipartite graph, which is not necessarily complete. The platform sets the commissions to maximize its T-period profit. To address the complexity of the platform's profit optimization problem, we consider the long-run average problem (AVG) as a benchmark and propose an algorithm called TRP with a provable performance guarantee. We show that the platform should prioritize boosting the growth of the agent type with the lowest population ratio relative to the long-run average benchmark in each period. Additionally, we demonstrate that providing subsidies to users can accelerate the growth of the user base, highlighting the tradeoff between short-term profit and long-run growth. We also outline the conditions under which subsidies should be implemented.

Furthermore, we delve into the OSS obtained via AVG and explore how the growth potentials of agents and network structure influence the agents' income/payment in the market and the platform's profit. We begin by introducing a set of metrics designed to capture the growth potentials of agents. Based on it, we show that a balanced network, in which sellers with relatively high (low) growth potentials trade with buyers with relatively high (low) growth potentials, results in maximum profitability, while the degree of imbalance in the network establishes a lower bound for the platform's optimal profit (relative to that under the complete graph). We then show that buyer (seller) types compatible with higher sellers' (buyers') growth potentials experience lower payments (higher income). A sensitivity analysis demonstrates the impact of agent type's long-run growth potential on income/payment. Finally, the commission charged by the platform in a submarket depends on the relative growth potentials from the two sides of the market.

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The additional results and proof of Section 3, Section 4 and Section 5 are respectively provided in Appendix EC.1, EC.2 and EC.3.

EC.1. Additional Results and Proof in Section 3

We first present some additional results in Appendix EC.1.1. We provide some Auxiliary Results used to prove the results in Section 3 in Appendix EC.1.2 and we prove the results in Section 3 in Appendix EC.1.3.

EC.1.1. Additional Results in Section 3

Proposition EC.1. **(existence and uniqueness of equilibrium)** *For any $t \in \{1, ..., T\}$, given a commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{N_s} \times \mathbb{R}^{N_b}$ and the total mass of agents $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}_+^{N_s} \times \mathbb{R}_+^{N_b}$,*

(i) a competitive equilibrium $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ always exists;

(ii) all competitive equilibria share the same supply-demand vector $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$, and they share the same prices $p_i(t)$ for $0 < q_i^s(t) < s_i(t)$.

Lemma EC.1. **(commissions for feasible transactions)** *For any $t \in \{1, ..., T\}$, given any positive population vector $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ and non-negative trading vector $(\boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ that satisfy (i) the flow conservation conditions in (2c)-(2e) and (ii) $\boldsymbol{q}^s(t) \le \boldsymbol{s}(t)$ and $\boldsymbol{q}^b(t) \le \boldsymbol{b}(t)$, a commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ supports $(\boldsymbol{s}(t), \boldsymbol{b}(t), \boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ in a competitive equilibrium if there exists a price vector $\boldsymbol{p}(t) \in \mathbb{R}^{N_s}$ that satisfies the following system of linear inequalities:*

$$p_i(t) - r_i^s(t) \;=\; F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big), \qquad \forall i : q_i^s(t) \;>\; 0, \tag{EC.1a}$$

$$p_i(t) - r_i^s(t) \;\le\; F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big), \qquad \forall i : q_i^s(t) \;=\; 0, \tag{EC.1b}$$

$$p_i(t) + r_j^b(t) \;=\; F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big), \qquad \forall (i,j) : x_{ij}(t) \;>\; 0, \tag{EC.1c}$$

$$p_i(t) + r_j^b(t) \;\ge\; F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big), \qquad \forall (i,j) : x_{ij}(t) \;=\; 0. \tag{EC.1d}$$

Consider the following convex optimization problem:

$$\mathcal{R}(T) \;=\; \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{x},\boldsymbol{q^s},\boldsymbol{q^b}} \quad \sum_{t=1}^{T} \Bigg[\sum_{j \in \mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big) q_j^b(t) - \sum_{i \in \mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big) q_i^s(t) \Bigg] \tag{EC.2a}$$

$$\text{s.t.} \quad q_i^s(t) \le s_i(t), \qquad \forall i \in \mathcal{S}, t \in \{1, ..., T\}, \tag{EC.2b}$$

$$q_j^b(t) \le b_j(t), \qquad \forall j \in \mathcal{B},\ t \in \{1, ..., T\}, \tag{EC.2c}$$

$$\sum_{j':(i,j')\in E} x_{i,j'}(t) = q_i^s(t), \qquad \forall i \in \mathcal{S},\ t \in \{1, ..., T\}, \tag{EC.2d}$$

$$q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B},\ t \in \{1, ..., T\}, \tag{EC.2e}$$

$$x_{ij}(t) \ge 0, \qquad \forall (i,j) \in E,\ t \in \{1, ..., T\}, \tag{EC.2f}$$

$$s_i(t+1) \le \mathcal{G}_i^s(s_i(t), q_i^s(t)), \qquad \forall i \in \mathcal{S},\ t \in \{1, ..., T-1\}, \tag{EC.2g}$$

$$b_j(t+1) \le \mathcal{G}_j^b(b_j(t), q_j^b(t)), \qquad \forall j \in \mathcal{B},\ t \in \{1, ..., T-1\}. \tag{EC.2h}$$

From Problem (EC.2), we can establish Proposition EC.2, which enables us to solve a concave maximization problem to obtain the optimal solution $(\boldsymbol{s}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{q^s}, \boldsymbol{q^b})$ to Problem (EC.2), from which we can further establish the optimal commission profile $(\boldsymbol{r}^s, \boldsymbol{r}^b)$ by solving a set of linear inequalities in (EC.1) of Lemma EC.1.

PROPOSITION EC.2. **(tightness of relaxation)** *For any $T \ge 1$, Problem* (EC.2) *is a tight relaxation of Problem* (3)*: $\mathcal{R}^*(T) = \mathcal{R}(T)$ and any optimal solution $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}})$ to Problem* (EC.2) *is also optimal to Problem* (3).

EC.1.2. Auxiliary Results for Section 3

Lemmas EC.2 - EC.4 are needed to prove Proposition EC.1. In Lemma EC.4, we establish the connection between the equilibrium and the optimal solution to an optimization problem in (EC.4). Before that, we establish some properties for the optimization problem in Lemma EC.2. We also establish the existence of the optimal solution to this optimization problem in Lemma EC.3, and show that it is essentially unique. These lemmas enable us to establish the existence and uniqueness of the competitive equilibrium in Definition 1. The proof of Auxiliary Results follows a similar argument as the proof of Proposition EC.1 and Proposition 9 in Birge et al. (2021). Therefore, we omit the detail of the proof of auxiliary results for simplicity.

For simplicity of notation, we first define that

$$W_{b_j}^t(q_j^b(t)) \ := \int_0^{q_j^b(t)} F_{b_j}^{-1}\Big(1 - \frac{z}{b_j(t)}\Big) dz - r_j^b(t) q_j^b(t), \tag{EC.3a}$$

$$W_{s_i}^t(q_i^s(t)) \; := \; -\int_0^{q_i^s(t)} F_{s_i}^{-1}\Big(\frac{z}{s_i(t)}\Big)dz - r_i^s(t)q_i^s(t). \tag{EC.3b}$$

Note that the sum of $W_{b_j}^t(q_j^b(t))$ and $W_{s_i}^t(q_i^s(t))$ can be viewed as the total surplus of buyers and sellers trading in the platform, and is the objective function in Problem (EC.4). Let $W_{b_j}^{t'}(q)$ be the derivative of $W_{b_j}^t(q)$ at $q = q_j^b(t)$ for any $0 < q_j^b(t) < b_j(t)$, and abusing some notation, $W_{b_j}^{t'}(0) = \lim_{q_j^b(t)\downarrow 0} W_{b_j}^t(q_j^b(t))$ and $W_{b_j}^{t'}(b_j(t)) = \lim_{q_j^b(t)\uparrow b_j(t)} W_{b_j}^t(q_j^b(t))$ given Assumption 2(i). Similarly, we let $W_{s_i}^{t'}(q)$ be the derivative of $W_{s_i}^t(q)$ at $q = q_i^s(t)$ for any $0 < q_i^s(t) < s_i(t)$, and we let $W_{s_i}^{t'}(0) = \lim_{q_i^s(t)\downarrow 0} W_{s_i}^t(q_i^s(t))$ and $W_{s_i}^{t'}(s_i(t)) = \lim_{q_i^s(t)\uparrow s_i(t)} W_{s_i}^t(q_i^s(t))$ given Assumption 2(i). We consider the following properties of functions $W_{b_j}^t(q_j^b(t))$ and $W_{s_i}^t(q_i^s(t))$.

LEMMA EC.2. *For any $j \in \mathcal{B}$, $i \in \mathcal{S}$ and $t \in \{1, \dots, T\}$,*

(i) $W_{b_j}^t(q)$ is continuously differentiable and strictly concave in $q \in (0, b_j(t))$; moreover, both $W_{b_j}^t(q)$ and $W_{b_j}^{t'}(q)$ are right continuous at $q = 0$ and left continuous $q = b_j(t)$.

(ii) $W_{s_i}^t(q)$ is continuously differentiable and strictly concave in $q \in (0, s_i(t))$; moreover, both $W_{s_i}^t(q)$ and $W_{s_i}^{t'}(q)$ are right continuous at $q = 0$ and left continuous $q = s_i(t)$.

For any $t \in \{1, ..., T\}$, we proceed to consider the following optimization problem:

$$\begin{aligned} W(t) = \max_{\boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)} \sum_{i \in \mathcal{B}} &\left(\int_0^{q_j^b(t)} F_{b_j}^{-1}\left(1 - \frac{z}{b_j(t)}\right) dz - r_j^b(t) q_j^b(t) \right) \\ &- \sum_{i \in \mathcal{S}} \left(\int_0^{q_i^s(t)} F_{s_i}^{-1}\left(\frac{z}{s_i(t)}\right) dz + r_i^s(t) q_i^s(t) \right) \end{aligned} \tag{EC.4a}$$

$$\text{s.t. } q_j^b(t) = \sum_{i' : (i', j) \in E} x_{i', j}(t), \qquad \forall j \in \mathcal{B}, \tag{EC.4b}$$

$$\sum_{j' : (i, j') \in E} x_{i, j'}(t) = q_i^s(t), \qquad \forall i \in \mathcal{S}, \tag{EC.4c}$$

$$q_j^b(t) \le b_j(t), \qquad \forall j \in \mathcal{B}, \tag{EC.4d}$$

$$q_i^s(t) \le s_i(t), \qquad \forall i \in \mathcal{S}, \tag{EC.4e}$$

$$x_{ij}(t) \ge 0, \qquad \forall (i, j) \in E. \tag{EC.4f}$$

From Problem (EC.4), we establish the result below. Before that, we define the notation "$a \le 0 \perp b \ge 0$" as $a \le 0, b \ge 0, ab = 0$.

LEMMA EC.3. *(i) There exists an optimal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ to Problem (EC.4).*

(ii) Given any optimal primal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$, *there exists a dual multiplier vector* $(\boldsymbol{\theta^b}(t), \boldsymbol{\theta^s}(t), \boldsymbol{\eta}^b(t), \boldsymbol{\eta}^s(t), \boldsymbol{\pi}(t))$ *associated with constraints* (EC.4b)-(EC.4f) *that satisfy the KKT conditions below:*

$$F_{b_j}^{-1}\Big(1-\frac{q_j^b(t)}{b_j(t)}\Big)-r_j^b(t)-\theta_j^b(t)-\eta_j^b(t)=0, \qquad \forall j\in\mathcal{B}, \tag{EC.5a}$$

$$F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)+r_i^s(t)-\theta_i^s(t)+\eta_i^s(t)=0, \qquad \forall i\in\mathcal{S}, \tag{EC.5b}$$

$$\theta_j^b(t)-\theta_i^s(t)+\pi_{ij}(t)=0, \qquad \forall (i,j)\in E, \tag{EC.5c}$$

$$q_j^b(t)-b_j(t)\le 0 \perp \eta_j^b(t)\ge 0, \qquad \forall j\in\mathcal{B}, \tag{EC.5d}$$

$$q_i^s(t)-s_i(t)\le 0 \perp \eta_i^s(t)\ge 0, \qquad \forall i\in\mathcal{S}, \tag{EC.5e}$$

$$x_{ij}(t)\ge 0 \perp \pi_{ij}(t)\ge 0, \qquad \forall (i,j)\in E, \tag{EC.5f}$$

$$q_j^b(t)=\sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j\in\mathcal{B}, \tag{EC.5g}$$

$$q_i^s(t)=\sum_{j':(i,j')\in E} x_{i,j'}(t), \qquad \forall i\in\mathcal{S}. \tag{EC.5h}$$

In addition, these KKT conditions in (EC.5) *are necessary and sufficient conditions for the optimality of solution* $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$.

(iii) All primal optimal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ *share the same vector* $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$;

(iv) The dual solution $\theta_i^s(t)$ *for* $i\in\{i':0<q_{i'}^s<s_{i'}\}$ *that satisfies* (EC.5) *is unique.*

The conditions in Lemma EC.4(i)-(ii) are sufficient and necessary conditions, while those in Lemma EC.4(iii) are only sufficient conditions for equilibrium, as the prices for type $i\in\{i':q_{i'}^s(t)=0$ or $q_{i'}^s(t)=s_{i'}(t)\}$ are not necessarily unique.

Lemma EC.4. *In each period* $t\in\{1,...,T\}$, *given any commission profile* $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))\in\mathbb{R}^{|\mathcal{S}|}\times\mathbb{R}^{|\mathcal{B}|}$ *and population vector* $(\boldsymbol{s}(t), \boldsymbol{b}(t))\in\mathbb{R}^{|\mathcal{S}|}\times\mathbb{R}^{|\mathcal{B}|}$,

(i) $(\boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ *satisfies the equilibrium conditions in Definition 1 if and only if it is an optimal solution to Problem* (EC.4);

(ii) for $i\in\{i':0<q_{i'}^s(t)<s_{i'}(t)\}$, $p_i(t)$ *satisfies the equilibrium conditions in Definition 1 if and only if*

$$p_i(t)=\theta_i^s(t). \tag{EC.6a}$$

(iii) for $i \in \{i' : q^s_{i'}(t) = 0$ or $q^s_{i'}(t) = s_{i'}(t)\}$, $p_i(t)$ satisfies the equilibrium conditions in Definition 1 if

$$p_i(t) = \theta^s_i(t). \tag{EC.6b}$$

Before proceeding, note that functions $F^{-1}_{s_i}(\cdot)$ and $F^{-1}_{b_j}(\cdot)$ have the following properties in an equilibrium:

(1) On the seller side, if $p_i(t) - r^s_i(t) \le 0$, then $q^s_i(t) = 0$ and

$$F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big) \ \ge \ p_i(t) - r^s_i(t), \tag{EC.7a}$$

if $0 < p_i(t) - r^s_i(t) < \overline{v}_{s_i}$, then $0 < q^s_i(t) < s_i(t)$ and

$$F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big) \ = \ p_i(t) - r^s_i(t), \tag{EC.7b}$$

if $\overline{v}_{s_i} \le p_i(t) - r^s_i(t)$, then $q^s_i(t) = s_i(t)$ and

$$F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big) \ \le \ p_i(t) - r^s_i(t). \tag{EC.7c}$$

(2) On the buyer side, if $\min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\} \le 0$, then $q^b_j(t) = b_j(t)$ and

$$F^{-1}_{b_j}\Big(1 - \frac{q^b_j(t)}{b_j(t)}\Big) \ \ge \ \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\}, \tag{EC.8a}$$

if $0 < \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\} < \overline{v}_{b_j}$, then $0 < q^b_j(t) < b_j(t)$ and

$$F^{-1}_{b_j}\Big(1 - \frac{q^b_j(t)}{b_j(t)}\Big) \ = \ \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\}, \tag{EC.8b}$$

if $\min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\} \ \ge \ \overline{v}_{b_j}$, then $q^b_j(t) = 0$ and

$$F^{-1}_{b_j}\Big(1 - \frac{q^b_j(t)}{b_j(t)}\Big) \le \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\}. \tag{EC.8c}$$

EC.1.3. Proof of Results for Section 3

Based on Lemmas EC.2 - EC.4, Proposition EC.1 is proved as below:

Proof of Proposition EC.1. We establish the following two claims of this result.

Claim (i). Lemma EC.3(i) implies that the optimal primal solution to (EC.4) always exists, and Lemma EC.4(i) implies that the $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is the equilibrium if and only if it is the optimal primal solution to (EC.4). Therefore, the equilibrium transaction vector $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ exists.

Lemma EC.3(ii) implies that the optimal dual solution to (EC.4) always exists, and Lemma EC.4(ii) implies that $\boldsymbol{p}$ that satisfies the equality in (EC.6) is the equilibrium price vector. Therefore, there exists a corresponding equilibrium price vector.

Claim (ii). Lemma EC.3(iii) implies that the optimal primal solution $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ to (EC.4) is unique. Lemma EC.4(i) implies that the $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is the equilibrium if and only if it is the optimal primal solution to (EC.4). Therefore, the equilibrium supply-demand vector $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is unique.

Lemma EC.3(iv) implies that the optimal dual solution $\boldsymbol{\theta}^s$ to Problem (EC.4) is unique for $i \in \{i' : 0 < q^s_{i'} < s_{i'}\}$, and Lemma EC.4(ii) implies that $p_i(t) = \theta^s_i(t)$ for i that satisfies $0 < q^s_i(t) < s_i(t)$. Therefore, the equilibrium price is unique for i that satisfies $0 < q^s_i(t) < s_i(t)$. ■

Proof of Lemma EC.1. We establish the sufficiency of (EC.1) in Step 1 and construct a feasible commission profile in Step 2 to show that the feasible commission profile always exists.

Step 1: Sufficiency. We show that for any $(\boldsymbol{q}^b(t), \boldsymbol{q}^s(t), \boldsymbol{x}(t))$ that satisfies (2c)-(2e), if vector $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ satisfies the conditions in (EC.1), then it satisfies the conditions in Definition 1.

We first verify the conditions in Definition 1, in which (2c)-(2e) immediately follow from our conditions.

(2a) We consider the following two cases:

When $q^s_i(t) > 0$, $s_i(t)F_{s_i}(p_i(t) - r^s_i(t)) \overset{(a)}{=} s_i(t)F_{s_i}(F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})) = q^s_i(t)$, (a) follows from (EC.1a).

When $q^s_i(t) = 0$, $0 \le s_i(t)F_{s_i}(p_i(t) - r^s_i(t)) \overset{(b)}{\le} s_i(t)F_{s_i}(F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})) = q^s_i(t) = 0$, (b) follows from (EC.1b).This implies that the inequalities are all tight, then $s_i(t)F_{s_i}(p_i(t) - r^s_i(t)) = q^s_i(t)$.

(2b) We consider the following two cases:

When $q^b_j(t) = 0$, then $x_{ij}(t) = 0$ for any $i : (i,j) \in E$, then $0 \le b_j(t)\Big(1 - F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r^b_j(t))\Big) \overset{(c)}{\le} b_j(t)\Big(1 - F_{b_j}(F^{-1}_{b_j}(1 - \frac{q^b_j(t)}{b_j(t)}))\Big) = q^b_j(t) = 0$, where (c) follows from (EC.1d).This implies that the inequalities are all tight, then $b_j(t)\Big(1 - F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r^b_j(t))\Big) = q^b_j(t)$.

When $q_j^b(t)>0$, pick a i_1 such that $x_{i_1 j}(t)>0$ we have $p_{i_1}(t)=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ based on (EC.1c); if there exists any i_2 such that $x_{i_2 j}(t)=0$, we have $p_{i_2}(t)\geq F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ based on (EC.1d); then $\min_{i':(i',j)\in E}\{p_{i'}(t)\}=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$, then $b_j(t)\Big(1-F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t))\Big)=b_j(t)\Big(1-F_{b_j}(F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)}))\Big)=q_j^b(t)$.

(2f) We consider two cases: When $q_j^b(t)=0$, then $x_{ij}(t)=0$ for any $i:(i,j)\in E$. When $q_j^b(t)>0$, we show in proof of (2b) that $p_i(t)\geq \min_{i':(i',j)\in E}\{p_{i'}\}=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ for $x_{ij}(t)=0$.

Step 2: construct an instance. In each period, given $(\boldsymbol{q}^b(t),\boldsymbol{q}^s(t),\boldsymbol{x}(t))$ that satisfies (2c)-(2e), consider the following one-period problem:

$$\widetilde{R}_t=\max_{\boldsymbol{q^s},\boldsymbol{q^b},\boldsymbol{x}}\Big[\sum_{j\in\mathcal{B}}q_j^b+\sum_{i\in\mathcal{S}}q_i^s\Big]$$

$$\text{s.t. } q_j^b\leq q_j^b(t),\qquad \forall j\in\mathcal{B} \tag{EC.9a}$$

$$q_i^s\leq q_i^s(t),\qquad \forall i\in\mathcal{S} \tag{EC.9b}$$

$$\sum_{j':(i,j')\in E}x_{i,j'}=q_i^s,\qquad \forall i\in\mathcal{S} \tag{EC.9c}$$

$$q_j^b=\sum_{i':(i',j)\in E}x_{i',j},\qquad \forall j\in\mathcal{B} \tag{EC.9d}$$

$$x_{ij}\geq 0,\qquad \forall (i,j)\in E. \tag{EC.9e}$$

Note that the feasible solution set is not empty, as $q_j^b=q_j^b(t)$ for any $j\in\mathcal{B}$, $q_i^s=q_i^s(t)$ for any $i\in\mathcal{S}$ and $x_{ij}=x_{ij}(t)$ for any $(i,j)\in E$ is a feasible solution. Since the constraints are all linear, the KKT conditions are necessary for the optimal solution in (EC.9). Let $(\omega_i^s(t),\omega_j^b(t),\pi_{ij}(t))$ be the Lagrange multipliers corresponding to the constraint in (EC.9c)-(EC.9e), then we can write down the KKT conditions corresponding to $\boldsymbol{x}$:

$$\omega_i^s(t)-\omega_j^b(t)-\pi_{ij}(t)=0,\qquad \forall (i,j)\in E, \tag{EC.10a}$$

$$x_{ij}(t)\geq 0\perp \pi_{ij}(t)\geq 0,\qquad \forall i\in\mathcal{S},\forall (i,j)\in E. \tag{EC.10b}$$

Then we consider the commission and equilibrium price as follows:

$$p_i(t)=\omega_i^s(t),\qquad \forall i\in\mathcal{S}, \tag{EC.11a}$$

$$r_i^s(t)=\omega_i^s(t)-F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right),\qquad \forall i\in\mathcal{S}, \tag{EC.11b}$$

$$r_j^b(t) = F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right) - \omega_j^b(t), \qquad \forall j \in \mathcal{B}. \tag{EC.11c}$$

then the conditions (EC.1a)-(EC.1b) immediately follow. For (EC.1c),

$$p_i(t) + r_j^b(t) = \omega_i^s(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(a)}{=} \omega_j^b(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) = F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}).$$

where (a) follows from (EC.10a) and (EC.10b) that $\pi_{ij}(t) = 0$ when $x_{ij}(t) \geq 0$.

For (EC.1d),

$$p_i(t) + r_j^b(t) = \omega_i^s(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(b)}{=} \omega_j^b(t) + \pi_{ij}(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(c)}{\geq} F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}).$$

where (b) follows from (EC.10a) and (c) follows from (EC.10b). In summary, (EC.1) holds for our construction in (EC.11). ■

Proof of Proposition EC.2 We need to prove that the optimal solutions to (3) exist and that they achieve an objective value of $\mathcal{R}^* = \mathcal{R}$. We first show that $\mathcal{R}^* \leq \mathcal{R}$ in step 1, and construct a solution to (3) whose value equals to $\mathcal{R}$ in step 2, which implies that $\mathcal{R}^* = \mathcal{R}$ and the solution is optimal.

Step 1: Establish that $\mathcal{R}^* \leq \mathcal{R}$. We show that any feasible solution to (3) is feasible in Problem (EC.2) in Step 1.1, and we further show that it leads to a higher objective value in Problem (EC.2) in Step 1.2.

Step 1.1: Any feasible solution in (3) is feasible in (EC.2). To prove the claim, it is sufficient to verify the constraints (EC.2b)-(EC.2c), as other constraints immediately follow from the constraints in (3).

Based on (2a) and (2b), we have $q_i^s(t) = s_i(t)F_{s_i}(p_i(t) - r_i^s(t)) \leq s_i(t)$ as $F_{s_i}(p_i(t) - r_i^s(t)) \in [0,1]$ and $q_j^b(t) = b_j(t)[1 - F_{b_j}(\min_{i:(i,j)\in E}\{p_i(t)\} + r_j^b(t))] \leq b_j(t)$ as $F_{b_j}(\min_{i:(i,j)\in E}\{p_i(t)\} + r_j^b(t)) \in [0,1]$. Therefore, the constraints (EC.2b)-(EC.2c) are satisfied.

Step 1.2: Any feasible solution in (3) results in a higher objective value in (EC.2). We first show that the optimal solution to Problem (3) satisfies the following:

$$\left(F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right)\right) q_i^s(t) \leq \Big(p_i(t) - r_i^s(t)\Big) q_i^s(t), \qquad \forall i \in \mathcal{S}, t \in \{1, ..., T\}, \tag{EC.12a}$$

$$\left(F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right)\right) q_j^b(t) \geq \left(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t)\right) q_j^b(t), \quad \forall j \in \mathcal{B}, t \in \{1, ..., T\}. \tag{EC.12b}$$

For (EC.12a), when $q_i^s(t)=0$, (EC.12a) immediately holds; when $q_i^s(t)>0$, (EC.12a) follows from (EC.7b) and (EC.7c) in the proof of Lemma EC.4. For (EC.12b), when $q_j^b(t)=0$, (EC.12b) immediately holds; when $q_j^b(t)>0$, (EC.12b) follows from (EC.8a) and (EC.8b) in the proof of Lemma EC.4.

Given (EC.12), the objective function in (3a) satisfies the following:

$$
\begin{aligned}
\mathcal{R}^* &= \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} r_j^b(t)q_j^b(t) + \sum_{i\in\mathcal{S}} r_i^s(t)q_i^s(t)\Big]\\
&\overset{(a)}{=} \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} r_j^b(t)\sum_{i':(i',j)\in E} x_{i'j}(t) + \sum_{i\in\mathcal{S}} r_i^s(t)\sum_{j':(i,j')\in E} x_{ij'}(t)\Big]\\
&= \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}}\sum_{i':(i',j)\in E}\Big(p_{i'}(t)+r_j^b(t)\Big)x_{i'j}(t) - \sum_{i\in\mathcal{S}}\Big(p_i(t)-r_i^s(t)\Big)\sum_{j':(i,j')\in E} x_{ij'}(t)\Big]\\
&\overset{(b)}{=} \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t)\Big)\sum_{i':(i',j)\in E} x_{i'j}(t) - \sum_{i\in\mathcal{S}}\Big(p_i(t)-r_i^s(t)\Big)\sum_{j':(i,j')\in E} x_{ij'}(t)\Big]\\
&= \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t)\Big)q_j^b(t) - \sum_{i\in\mathcal{S}}\Big(p_i(t)-r_i^s(t)\Big)q_i^s(t)\Big]\\
&\overset{(c)}{\le} \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}\big(1-\frac{q_j^b(t)}{b_j(t)}\big)q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\big(\frac{q_i^s(t)}{s_i(t)}\big)q_i^s(t)\Big] = \mathcal{R},
\end{aligned}
$$

where (a) follows from (2c)-(2d); (b) follows from (2f) that $x_{ij}=0$ for $i\notin \underset{i':(i',j)\in E}{argmin}\{p_i+r_i^s\}$; (c) follows from (EC.12).

<u>Step 2: Establish that $\mathcal{R}^*=\mathcal{R}$.</u> Given any feasible solution to (EC.2), we construct a feasible solution for (3) in Step 2.1, and we further obtain an objective value that equals $\mathcal{R}$ in Step 2.2.

<u>Step 2.1: Construct a feasible solution for Problem (3).</u>

In each period, given the solution for Problem (EC.2), we consider the construction from (EC.11) as in the proof of Lemma EC.1. We need to verify that all the constraints in (3) hold. Notice that we only need to verify that (2a) (2b) (2f) (3c) and (3d) hold, as other constraints exist in (EC.2) and automatically hold.

(2a) from the construction of $p_i(t)$ and $r_i^s(t)$, we can establish that

$$s_i(t)F_{s_i}(p_i(t)-r_i^s(t)) = s_i(t)F_{s_i}\Big(F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)\Big) = q_i^s(t).$$

(2b) We consider the following two cases:

(i) if $q_j^b > 0$, we pick a i' such that $(i', j) \in E$, then there are two further cases: (1) $x_{i'j} > 0$, then $p_{i'}(t) \overset{(a)}{=} \omega_{i'}^s(t) \overset{(b)}{=} \omega_j^b(t) + \pi_{i'j}(t) \overset{(c)}{=} \omega_j^b(t)$, where (a) follows form the construction of $p_{i'}(t)$; (b) follows from (EC.10a); (c) follows from (EC.10b) for $x_{i'j} > 0$; (2) $x_{i'j} = 0$, then $p_{i'}(t) = \omega_{i'}^s(t) = \omega_j^b(t) + \pi_{i'j}(t) \overset{(d)}{\geq} \omega_j^b(t)$, where (d) follows from (EC.10b) for $x_{i'j} = 0$. In summary, $\min_{i':(i',j)\in E}\{p_{i'}(t)\} = \omega_j^b(t)$, then

$$b_j(t)[1 - F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t))] = b_j(t)[1 - F_{b_j}(\omega_j^b(t) + r_j^b(t))] \overset{(e)}{=} b_j(t)[1 - F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})] = q_j^b(t),$$

where (e) follows from the construction of $r_j^b(t)$;

(ii) if $q_j^b = 0$, we have $p_{i'}(t) = \omega_{i'}^s(t) = \omega_j^b(t) + \pi_{i'j}(t) \geq \omega_j^b(t)$, then $0 \overset{(f)}{\leq} b_j(t)[1 - F_{b_j}(\min\{p_i(t) + r_j^b(t)\})] \leq b_j(t)[1 - F_{b_j}(\omega_j^b(t) + r_j^b(t))] \overset{(g)}{=} b_j(t)[1 - F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})] = q_j^b(t) = 0$,, where (f) follows from $F_{b_j}(\cdot) \leq 1$, (g) follows from the construction of $r_j^b(t)$. This implies that inequality must be tight. Therefore, (2b) holds.

(2f) We have verified in the proof of (2b) that for any $(i, j) \in E$, we have $p_i = \omega_j^b$ for $x_{ij} > 0$ and $p_i \geq \omega_j^b$ for $x_{ij} = 0$. Therefore, $x_{ij} = 0$ for $i \notin \arg\min_{i':(i',j)\in E} p_{i'}$.

(3c) We first prove (EC.2g) holds as equality by contradiction. Suppose that $s_i(t+1) < \mathcal{G}_i^s(s_i(t), q_i^s(t))$ in the optimal solution to (EC.2), then let $s_i'(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t))$, we can obtain higher objective value by replacing the $s_i(t+1)$ in the optimal solution with $s_i'(t+1)$ as (EC.2a) increases in $s_i(t+1)$; in addition, $s_i(t+2) \leq \mathcal{G}_i^s(s_i(t+1), q_i^s(t+1)) < \mathcal{G}_i^s(s_i'(t+1), q_i^s(t+1))$, which implies that the constraint in (EC.2g) still hold. This contradicts to our assumption that $s_i(t+1) < \mathcal{G}_i^s(s_i(t), q_i^s(t))$ is the optimal solution to (EC.2). Therefore, $s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t))$ in the optimal solution to (EC.2), and (3c) immediately holds.

(3d) follows the same argument in (3c).

Step 2.2: Obtain a value that equals $\mathcal{R}$. We can deduce that

$$\begin{aligned}
\mathcal{R}^* &= \sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}} r_i^s(t)q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t)q_j^b(t)\right] \\
&\overset{(a)}{=} \sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}}(\omega_i^s(t) - F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)}))q_i^s(t) + \sum_{j\in\mathcal{B}}(F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t))q_j^b(t)\right] \\
&\overset{(b)}{=} \sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right]
\end{aligned}$$

$$+\sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}}\omega_i^s(t)\sum_{j':(i,j')\in E}x_{ij'}(t)-\sum_{j\in\mathcal{B}}\omega_j^b(t)\sum_{i':(i',j)\in E}x_{i'j}(t)\right]$$
$$=\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}}F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t)-\sum_{i\in\mathcal{S}}F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right]+\sum_{t=1}^{T}\left[\sum_{(i,j)\in E}\Big(\omega_i^s(t)-\omega_j^b(t)\Big)x_{ij}(t)\right]$$
$$\overset{(c)}{\leq}\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}}F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t)-\sum_{i\in\mathcal{S}}F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right]=\mathcal{R},$$

where (a) follows from the construction of $r_i^s(t)$ and $r_j^b(t)$, (b) follows from (EC.2d) and (EC.2e), (c) follows from (EC.10a) and (EC.10b) that when $x_{ij}>0$, $\omega_i^s=\omega_j^b$, while when $x_{ij}=0$, $\omega_i^s\geq\omega_j^b$. ■

EC.2. Proof of Results in Section 4

We provide and prove some auxiliary results in Appendix EC.2.1 and prove the result in Section 4 in Appendix EC.2.2.

EC.2.1. Auxiliary Results for Section 4

Given the definitions of the value functions $\widetilde{F}_{b_j}$ for any $j\in\mathcal{B}$ and $\widetilde{F}_{s_i}$ for any $i\in\mathcal{S}$ from Problem (5), we have the following lemma.

LEMMA EC.5. *$\widetilde{F}_{b_j}(q,b)$ is continuous at $(0,0)$ for $i\in\mathcal{S}$ and $\widetilde{F}_{s_i}(q,s)$ is continuous at $(0,0)$ for $j\in\mathcal{B}$.*

Proof of Lemma EC.5. We need to show that $\lim_{(q,b)\downarrow(0,0)}\widetilde{F}_{b_j}(q,b)=\widetilde{F}_{b_j}(0,0)=0$ and $\lim_{(q,s)\downarrow(0,0)}\widetilde{F}_{s_i}(q,s)=\widetilde{F}_{s_i}(0,0)=0$, which holds because

$$0\;\leq\;\lim_{(q,b)\downarrow(0,0)}\widetilde{F}_{b_j}(q,b)\;=\;\lim_{(q,b)\downarrow(0,0)}F_{b_j}^{-1}\Big(1-\frac{q}{b}\Big)q\;\leq\;\overline{v}_{b_j}\times 0\;=\;0,$$
$$0\;\leq\;\lim_{(q,s)\downarrow(0,0)}\widetilde{F}_{s_i}(q,s)\;=\;\lim_{(q,s)\downarrow(0,0)}F_{s_i}^{-1}\Big(\frac{q}{s}\Big)q\;\leq\;\overline{v}_{s_i}\times 0\;=\;0,$$

where given Assumption 2, all of the inequalities above follow from $F_{b_j}^{-1}(x)\in[0,\overline{v}_{b_j}]$ for $x\in[0,1]$ where $\overline{v}_{b_j}<\infty$ and $F_{s_i}^{-1}(x)\in[0,\overline{v}_{s_i}]$ for $x\in[0,1]$ where $\overline{v}_{s_i}<\infty$. ■

We next develop an auxiliary result about the growth of populations. To simplify the notation, we let $\mathcal{N}:=\{1,...,|\mathcal{S}|,|\mathcal{S}|+1,...,|\mathcal{S}|+|\mathcal{B}|\}$, where the first $|\mathcal{S}|$ nodes represent the types from the seller side and the last $|\mathcal{B}|$ nodes represent the types from the buyer side. In addition, we use $n_i(t)$ and $q_i(t)$ to respectively denote the population and transaction quantity of type $i \in \mathcal{N}$ at time $t \in \{1,\dots,T\}$. We define $\mathcal{G}_i(\cdot,\cdot) := \mathcal{G}_i^s(\cdot,\cdot)$ for $i \in \{1,...,|\mathcal{S}|\}$ and $\mathcal{G}_i(\cdot,\cdot) := \mathcal{G}_{i-|\mathcal{S}|}^b(\cdot,\cdot)$ for $i \in \{|\mathcal{S}|+1,...,|\mathcal{S}|+|\mathcal{B}|\}$. In addition, we define $\mathcal{N}^+ := \{i \in \mathcal{N} : \overline{n}_i > 0\}$.

Recall that

$$m(t) = \min_{i \in \mathcal{N}^+} \frac{n_i(t)}{\overline{n}_i}. \tag{EC.13}$$

Given the minimum population ratio $m(t)$ in (EC.13), we let $l(t)$ be the agent type with the lowest population ratio at time t or "the lowest node at time t" for short:

$$l(t) := \arg\min_{i \in \mathcal{N}^+} \frac{n_i(t)}{\overline{n}_i}. \tag{EC.14}$$

If there is more than one i such that $\frac{n_i(t)}{\overline{n}_i} = m(t)$, we can set $l(t)$ as any node with the minimum population ratio. After the population evolves in period t, it is worth noting that the lowest node can change. Let $\tau_0 := 0$ and $m(\tau_0)$ be a dummy agent type with the minimum ratio in period 0 with $m(\tau_0) \notin \mathcal{S} \cup \mathcal{B}$. Moreover, we let X be the total number of times that the lowest node changes in Algorithm 1 for some $X \in \{1,\dots,T\}$. we let $\tau_x := \min\{t : t > \tau_{x-1}, l(t) \neq l(\tau_{x-1})\}$ for $t \in \{1,\dots,T\}$, in which τ_x is the x^{th} time that the lowest node changes for $x \in \{1,\dots,X\}$. For example, for $x \in \{0,1,...,X\}$, if node i has the lowest ratio at time $\tau_x - 1$, then $n_{l(\tau_x-1)}(\tau_x)$ denotes the population ratio of the node i at time τ_x.

Given the lowest node $l(t) \in \mathcal{S} \cup \mathcal{B}$ we let

$$g_t(n) := \mathcal{G}_{l(t)}\left(n, n\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}}\right), \tag{EC.15}$$

where $n \geq 0$. Then $g_t(n)$ is the transition equation for the lowest node in period t, as by the population transition specified in Algorithm 1 and the definition of $g_t(\cdot)$, we have that

$$n_{l(t)}(t+1) = \mathcal{G}_{l(t)}\left(n_{l(t)}(t), n_{l(t)}(t)\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}}\right) = g_t(n_{l(t)}(t)). \tag{EC.16}$$

We have the following observation about function $g_t(\cdot)$.

Lemma EC.6. *$g_t(n)$ is differentiable, increasing and strictly concave in $n \geq 0$. Moreover, its derivative satisfies $g_t'(\overline{n}_{l(t)}) < 1$ for all $t \in \{1,...,T\}$. Moreover, $g_t(n) - n < 0$ for $n > \overline{n}_{l(t)}$ and $g_t(n) - n > 0$ for $0 < n < \overline{n}_{l(t)}$.*

Proof of Lemma EC.6. We divide the proof arguments into the following components.

Differentiability and monotonicity. From Assumption 1, we have that function $\mathcal{G}_i(n,q)$ is continuously differentiable, increasing and strictly concave in $n \geq 0$, which directly implies that $g_t(n)$ is differentiable, increasing and strictly concave in $n \geq 0$.

$g_t'(\overline{n}_{l(t)}) < 1$ for all $t \in \{1,...,T\}$. By Algorithm 1, we have that $\overline{n}_{l(t)} > 0$. Since $g_t(n)$ is continuous in $n \in [0, \overline{n}_{l(t)}]$ and differentiable $(0, \overline{n}_{l(t)})$, by the mean value theorem, there exists a $\widetilde{n}_{l(t)} \in (0, \overline{n}_{l(t)})$ such that $g_t'(\widetilde{n}_{l(t)}) = \frac{g_t(\overline{n}_{l(t)}) - g_t(0)}{\overline{n}_{l(t)} - 0} \overset{(a)}{=} \frac{\overline{n}_{l(t)} - g_t(0)}{\overline{n}_{l(t)} - 0} \overset{(b)}{=} \frac{\overline{n}_{l(t)} - 0}{\overline{n}_{l(t)} - 0} = 1$, where (a) follows from Lemma 1(ii) and (b) follows from Assumption 1(i). Since $g_t(n)$ is strictly concave in $n \geq 0$, its derivative strictly decreases in $n \geq 0$, which implies that $g_t'(\overline{n}_{l(t)}) < 1$ given that $\widetilde{n}_{l(t)} \in (0, \overline{n}_{l(t)})$.

$g_t(n) - n < 0$ for $n > \overline{n}_{l(t)}$. we define that $y_t(n) := g_t(n) - n$, and it remains to show that $y_t(n) < 0$ for $n > \overline{n}_{l(t)}$. Since $y_t'(n_{l(t)}) = g_t'(n_{l(t)}) - 1 < 0$ for $n_{l(t)} > \overline{n}_{l(t)}$ and $y_t(\overline{n}_{l(t)}) = 0$ based on Lemma 1(ii), $y_t(n_{l(t)}) < 0$ for $n_{l(t)} > \overline{n}_{l(t)}$.

$g_t(n) - n > 0$ for $0 < n < \overline{n}_{l(t)}$. It remains to show that $y_t(n) > 0$ for $0 < n < \overline{n}_{l(t)}$. Note that $y_t(n)$ is concave in n. Since $y_t(0) = g_t(0) - 0 = 0$ and $y_t(\overline{n}_{l(t)}) = g_t(\overline{n}_{l(t)}) - \overline{n}_{l(t)} = 0$, we know $y_t((1-a) \times \overline{n}_{l(t)}) > a y_t(0) + (1-a) y_t(\overline{n}_{l(t)}) = 0 + 0 = 0$ for $a \in (0,1)$, therefore $y_t(n) > 0$ for $0 < n < \overline{n}_{l(t)}$. ∎

Lastly, we formally define the myopic policy and establish its tractability as a supporting result for our proof arguments for Section 4.

Definition EC.1. **(myopic policy)** For $t \in \{1,...,T\}$, given the current population $(\boldsymbol{s}^M(t), \boldsymbol{b}^M(t))$, the myopic policy solves the following optimization problem:

$$\mathcal{R}^{M*}(t) = \max_{\boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)} \sum_{i \in \mathcal{S}} r_i^s(t) q_i^s(t) + \sum_{j \in \mathcal{B}} r_j^b(t) q_j^b(t) \tag{EC.17a}$$

$$\text{s.t.} \quad (\boldsymbol{s^M}(t), \boldsymbol{b^M}(t), \boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)) \text{ satisfies (2)}, \forall t \in \{1,...,T\}. \tag{EC.17b}$$

To solve Problem (EC.17), we consider the following optimization problem:

$$\mathcal{R}^M(t) = \max_{\boldsymbol{q}^s(t), \boldsymbol{q}^b(t), \boldsymbol{x}(t)} \sum_{j \in \mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j^M(t)}\Big) q_j^b(t) - \sum_{i \in \mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i^M(t)}\Big) q_i^s(t) \tag{EC.18a}$$

$$\text{s.t.} \quad q_i^s(t) \leq s_i^M(t), \quad \sum_{j':(i,j') \in E} x_{i,j'}(t) = q_i^s(t), \qquad \forall i \in \mathcal{S}, t \in \{1,...,T\}, \tag{EC.18b}$$

$$q_j^b(t) \le b_j^M(t),\ q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B},\ t \in \{1,...,T\}, \tag{EC.18c}$$

$$x_{ij}(t) \ge 0, \qquad \forall (i,j) \in E,\ t \in \{1,...,T\}. \tag{EC.18d}$$

Recalling the observations about Problem (EC.2), we can apply exactly the same arguments as in the proof of Proposition EC.2 to establish the following result about Problem (EC.18), whose proof will be neglected for avoiding repetition:

Corollary EC.1. *For any $t \in \{1,...,T\}$, Problem (EC.18) is a tight relaxation of Problem (EC.17), i.e., $\mathcal{R}^{M*}(t) = \mathcal{R}^M(t)$ and any optimal solution $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t), \boldsymbol{x}(t))$ to Problem (EC.18) is also optimal to Problem (EC.17).*

EC.2.2. Proof of Results for TRP

Proof of Lemma 1.

Show that AVG's optimal solution and objective value are finite. On the seller side, for any $i \in \mathcal{S}$, we first show that the optimal solution $(\overline{q}_i^s, \overline{s}_i)$ is finite for all $i \in \mathcal{S}$. We first show that $\overline{s}_i$ is finite. The constraint of AVG requires that $s_i \le \mathcal{G}_i^s(s_i, q_i^s) \le \mathcal{G}_i^s(s_i, s_i)$, which requires that $\mathcal{G}_i^s(s_i, s_i) - s_i \ge 0$. Given that $\lim_{x\to\infty}((\mathcal{G}_i^s)_1'(x,x) + (\mathcal{G}_i^s)_2'(x,x)) < 1$ and $\mathcal{G}_i^s(x,x)$ is continuously differentiable in $x \ge 0$ by Assumption 1, there exists a constant $a < 1$ and $\hat{s}_i > 0$ such that $(\mathcal{G}_i^s)_1'(\hat{s}_i, \hat{s}_i) + (\mathcal{G}_i^s)_2'(\hat{s}_i, \hat{s}_i) = a < 1$. Therefore, for any $s_i > \hat{s}_i$, the constraint requires that

$$\begin{aligned}
\mathcal{G}_i^s(s_i, s_i) - s_i \le& \mathcal{G}_i^s(\hat{s}_i, \hat{s}_i) + (\mathcal{G}_i^s)_1'(\hat{s}_i, \hat{s}_i)(s_i - \hat{s}_i) + (\mathcal{G}_i^s)_2'(\hat{s}_i, \hat{s}_i)(s_i - \hat{s}_i) - s_i \\
=& \mathcal{G}_i^s(\hat{s}_i, \hat{s}_i) + a(s_i - \hat{s}_i) - s_i
\end{aligned}$$

which indicates that for any $s_i > \max\{\hat{s}_i, \frac{\mathcal{G}_i^s(\hat{s}_i,\hat{s}_i) - a\hat{s}_i}{1-a}\}$, we have $\mathcal{G}_i^s(s_i, s_i) - s_i < 0$ and therefore is not feasible. Therefore, it is without loss of optimality to focus on the compact set $[0, \hat{s}_i]$ for the optimal solution $\overline{s}_i$. Since $q_i \le s_i$, this suggests that the optimal solution $\overline{q}_i^s \in [0, \hat{s}_i]$, which is also finite. The same arguments hold for the buyer side.

Show that optimal solution $(\overline{\boldsymbol{q}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to AVG exists. For any $u \in [0,1]$, we have that $F_{s_i}^{-1}(u) \le \overline{v}_{s_i} < \infty$ for any $i \in \mathcal{S}$ and $F_{b_j}^{-1}(u) \le \overline{v}_{b_j} < \infty$ for all $j \in \mathcal{B}$. Therefore, the objective value of AVG is also finite. We have already shown that the feasible set of $(\boldsymbol{q}, \boldsymbol{s}, \boldsymbol{b})$ is closed and bounded. The constraints in (5b)-(5c) also ensure that the feasible set of $\boldsymbol{x}$ is closed and bounded. In summary, the feasible set characterized by constraint (5b)-(5f) is compact. In addition, the feasible set is not empty, as solution $\mathbf{0}$ is feasible. Furthermore, the objective function in (5a) is continuous in this compact set based on Assumption 2(i). By the extreme value theorem, an optimal solution $(\overline{\boldsymbol{q}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to AVG exists.

We proceed to prove the lemma.

(i). By the extreme value theorem, the optimal solution to (5) exists. Since the objective function is strictly concave and the feasible region is a convex set, the optimal solution to (5) is unique.

(ii). If there exists a $i \in \mathcal{S}$ such that $\mathcal{G}_i^s(\overline{s}_i, \overline{q}_i^s) - \overline{s}_i > 0$, then given that $\mathcal{G}_i^s(s_i, q_i^s)$ is continuous on s_i, we can always find a $\epsilon > 0$ small enough such that $\mathcal{G}_i^s(\overline{s}_i + \epsilon, \overline{q}_i^s) - (\overline{s}_i + \epsilon) > 0$. In addition, $\overline{s}_i + \epsilon > \overline{s}_i \geq \overline{q}_i^s$. By replacing $\overline{s}_i$ with $\overline{s}_i + \epsilon$, we obtain a higher objective value since the objective function strictly increases in s_i. Therefore, the assumption $\mathcal{G}_i^s(\overline{s}_i, \overline{q}_i^s) - \overline{s}_i > 0$ contradicts the optimality of $(\overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to Problem (5). The same proof arguments can be applied to the buyer side. ■

Proof of Proposition 1. By Proposition EC.2, $\mathcal{R}(T) = \mathcal{R}^*(T)$. So it suffices to show that there exists a constant C_1 such that $|\mathcal{R}(T) - T\overline{\mathcal{R}}| \leq C_1$. To prove the result, we establish the following two claims.

Claim 1: $\mathcal{R}(T) - T\overline{\mathcal{R}} \geq -C_1'$. We delay the proof to the proof of Theorem 1 that there exists a constant C_1' and a policy π such that $\mathcal{R}^\pi(T) - T\overline{\mathcal{R}} \geq -C_1'$, which further implies that $\mathcal{R}(T) - T\overline{\mathcal{R}} \geq \mathcal{R}^\pi(T) - T\overline{\mathcal{R}} \geq -C_1'$ given that $\mathcal{R}(T) \geq \mathcal{R}^\pi(T)$.

Claim 2: $\mathcal{R}(T) - T\overline{\mathcal{R}} \leq C_1''$. Before proving the claim, we first consider the following optimization problem for any $T > 0$:

$$\widetilde{\mathcal{R}} = \max_{\boldsymbol{s}, \boldsymbol{b}, \boldsymbol{q^s}, \boldsymbol{q^b}, \boldsymbol{x}} \quad \sum_{j \in \mathcal{B}} \widetilde{F}_{b_j}(q_j^b, b_j) - \sum_{i \in \mathcal{S}} \widetilde{F}_{s_i}(q_i^s, s_i) \tag{EC.19a}$$

$$\text{s.t.} \quad q_i^s \leq s_i, \qquad \forall i \in \mathcal{S}, \tag{EC.19b}$$

$$q_j^b \leq b_j, \qquad \forall j \in \mathcal{B}, \tag{EC.19c}$$

$$\sum_{j:(i,j) \in E} x_{ij} = q_i^s, \qquad \forall i \in \mathcal{S}, \tag{EC.19d}$$

$$q_j^b = \sum_{i:(i,j) \in E} x_{ij}, \qquad \forall j \in \mathcal{B}, \tag{EC.19e}$$

$$x_{ij} \geq 0, \qquad \forall (i,j) \in E, \tag{EC.19f}$$

$$s_i \leq \mathcal{G}_i^s(s_i, q_i^s) + \frac{s_i(1)}{T}, \qquad \forall i \in \mathcal{S}, \tag{EC.19g}$$

$$b_j \leq \mathcal{G}_j^b(b_j, q_j^b) + \frac{b_j(1)}{T}, \qquad \forall j \in \mathcal{B}. \tag{EC.19h}$$

Note that the only difference between Problem (EC.19) and Problem (5) is the right-hand side of the constraints (EC.19g)-(EC.19h). Given that $s_i(1)>0$ for all $i\in\mathcal{S}$ and $b_j(1)>0$ for all $j\in\mathcal{B}$, Problem (EC.19) could be viewed as a relaxation of Problem (5). We first show that $T\widetilde{\mathcal{R}}\geq\mathcal{R}(T)$ and then show that there exists a positive constant C_1'' such that $T\widetilde{\mathcal{R}}-T\overline{\mathcal{R}}\leq C_1''$ for any $T>0$. Consequently, we can have $\mathcal{R}(T)-T\overline{\mathcal{R}}\leq C_1''$ for any $T>0$.

Step 2.1: Show that $T\widetilde{\mathcal{R}}\geq\mathcal{R}(T)$. For any optimal solution $(\boldsymbol{s}(t),\boldsymbol{b}(t),\boldsymbol{q^s}(t),\boldsymbol{q^b}(t),\boldsymbol{x}(t):t=1,\ldots,T)$ to Problem (EC.2), we construct the following alternative solution vector $(\overline{\boldsymbol{s}},\overline{\boldsymbol{b}},\overline{\boldsymbol{q}}^{\boldsymbol{s}},\overline{\boldsymbol{q}}^{\boldsymbol{b}},\overline{\boldsymbol{x}})$ for Problem (EC.19):

$$\overline{s}_i=\frac{1}{T}\sum_{t=1}^{T}s_i(t)\text{ and }\overline{q}_i^s=\frac{1}{T}\sum_{t=1}^{T}q_i^s(t),\qquad \forall i\in\mathcal{S},$$
$$\overline{b}_j=\frac{1}{T}\sum_{t=1}^{T}b_j(t)\text{ and }\overline{q}_j^b=\frac{1}{T}\sum_{t=1}^{T}q_j^b(t),\qquad \forall j\in\mathcal{B},$$
$$\overline{x}_{ij}=\frac{1}{T}\sum_{t=1}^{T}x_{ij}(t),\qquad \forall(i,j)\in E$$

We establish the feasibility of $(\overline{\boldsymbol{s}},\overline{\boldsymbol{b}},\overline{\boldsymbol{q}}^{\boldsymbol{s}},\overline{\boldsymbol{q}}^{\boldsymbol{b}},\overline{\boldsymbol{x}})$ for Problem (EC.19) in Step 2.1.1 and then show that $T\widetilde{\mathcal{R}}\geq\mathcal{R}(T)$ in Step 2.1.2.

Step 2.1.1: Feasibility. First, from the constraints in Problem (EC.2), we can easily show (EC.19b) - (EC.19f) hold. In particular, $\overline{q}_i^s=\frac{1}{T}\sum_{t=1}^{T}q_i^s(t)\overset{(a)}{\leq}\frac{1}{T}\sum_{t=1}^{T}s_i(t)=\overline{s}_i$. The same argument applies for $\overline{q}_j^b$ and $\overline{b}_j$ on the buyer side. For (EC.19d)-(EC.19e), $\overline{q}_i^s=\frac{1}{T}\sum_{t=1}^{T}q_i^s(t)\overset{(b)}{=}\frac{1}{T}\sum_{j':(i,j')\in E}\sum_{t=1}^{T}x_{ij'}(t)=\sum_{j':(i,j')\in E}\overline{x}_{ij}$. and $\overline{q}_j^b=\frac{1}{T}\sum_{t=1}^{T}q_j^b(t)\overset{(c)}{=}\frac{1}{T}\sum_{i':(i',j)\in E}\sum_{t=1}^{T}x_{i'j}(t)=\sum_{i':(i',j)\in E}\overline{x}_{ij}$. For (EC.19f), $\overline{x}_{ij}=\frac{1}{T}\sum_{t=1}^{T}x_{ij}(t)\overset{(e)}{\geq}0$.

For constraints in (EC.19g)-(EC.19h), we show that

$$\begin{aligned}
\overline{s}_i-\mathcal{G}_i^s(\overline{s}_i,\overline{q}_i^s)-\frac{s_i(1)}{T}&\overset{(a)}{=}\frac{1}{T}\sum_{t=1}^{T}s_i(t)-\mathcal{G}_i^s\Big(\frac{1}{T}\sum_{t=1}^{T}s_i(t),\frac{1}{T}\sum_{t=1}^{T}q_i^s(t)\Big)-\frac{s_i(1)}{T}\\
&\overset{(b)}{\leq}\frac{1}{T}\sum_{t=1}^{T}\Big[s_i(t)-\mathcal{G}_i^s(s_i(t),q_i^s(t))\Big]-\frac{s_i(1)}{T}\\
&=\frac{1}{T}\sum_{t=1}^{T-1}\Big[s_i(t+1)-\mathcal{G}_i^s(s_i(t),q_i^s(t))\Big]+\frac{1}{T}(s_i(1)-\mathcal{G}_i^s(s_i(T),q_i^s(T)))-\frac{s_i(1)}{T}\\
&\leq 0+\frac{1}{T}\Big(-\mathcal{G}_i^s(s_i(T),q_i^s(T))\Big)\;\leq\;0,
\end{aligned}$$

where (a) follows from the construction of $\overline{s}_i$ and $\overline{q}_i^s$ at the beginning of Step 2.1; (b) follows the Assumption 1(ii) that $\mathcal{G}_i^s(\cdot)$ is concave. This proves that Constraint (EC.19g) holds. Following the same argument, we can show that Constraint (EC.19h) holds.

Step 2.1.2: $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$. Given the construction of $\bar{s}_i$ and $\bar{b}_j$, we obtain that $\bar{s}_i > 0$ and $\bar{b}_j > 0$. Given the definitions of $\widetilde{F}_b(\bar{q}^b_j, \bar{b}_j)$ and $\widetilde{F}_s(\bar{q}^s_i, \bar{s}_i)$ in Problem (5), the objective value in (5a) is given by $\sum_{j\in\mathcal{B}} F^{-1}_{b_j}(1-\frac{\bar{q}^b_j}{\bar{b}_j})\bar{q}^b_j - \sum_{i\in\mathcal{S}} F^{-1}_{s_i}(\frac{\bar{q}^s_i}{\bar{s}_i})\bar{q}^s_i$. This allows us to establish that

$$\begin{aligned} T\widetilde{\mathcal{R}} &\overset{(a)}{=} T\Big[\sum_{j\in\mathcal{B}} F^{-1}_{b_j}\Big(1-\frac{\frac{1}{T}\sum_{t=1}^T q^b_j(t)}{\frac{1}{T}\sum_{t=1}^T b_j(t)}\Big)\frac{1}{T}\sum_{t=1}^T q^b_j(t) - \sum_{i\in\mathcal{S}} F^{-1}_{s_i}\Big(\frac{\frac{1}{T}\sum_{t=1}^T q^s_i(t)}{\frac{1}{T}\sum_{t=1}^T s_i(t)}\Big)\frac{1}{T}\sum_{t=1}^T q^s_i(t)\Big] \\ &\overset{(b)}{\geq} T\times\frac{1}{T}\sum_{t=1}^T\Big[\sum_{j\in\mathcal{B}} F^{-1}_{b_j}\left(1-\frac{q^b_j(t)}{b_j(t)}\right)q^b_j(t) - \sum_{i\in\mathcal{S}} F^{-1}_{s_i}\left(\frac{q^s_i(t)}{s_i(t)}\right)q^s_i(t)\Big] = \mathcal{R}(T). \end{aligned}$$

where (a) follows from the construction of $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}}, \bar{\boldsymbol{x}})$ in Step 2-1; (b) follows from the concavity of $F^{-1}_{b_j}(1-\frac{a}{b})a$ and $-F^{-1}_{s_i}(\frac{a}{b})a$ by Assumption 3.

Summarizing the arguments in these two steps, we have $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$.

Step 2.2: Show that $T\widetilde{\mathcal{R}} - T\overline{\mathcal{R}} \leq C''_1$ for some $C''_1 > 0$. Let $(\boldsymbol{\mu}^s, \boldsymbol{\mu}^b)$ be the dual optimal solution corresponding to the constraint $s_i \leq \mathcal{G}^s_i(s_i, q^s_i)$ and $b_j \leq \mathcal{G}^b_j(b_j, q^b_j)$ in Problem (5), then $\mu^s_i \geq 0$ for any $i \in \mathcal{S}$ and $\mu^b_j \geq 0$ for any $j \in \mathcal{B}$ according to duality theory. Note that the only difference between Problem (5) and Problem (EC.19) is the right-hand side of the constraints in (EC.19g)-(EC.19h). Therefore, based on (5.57) in Boyd et al. (2004), we can establish that

$$\widetilde{\mathcal{R}} \leq \overline{\mathcal{R}} + \sum_{i\in\mathcal{S}} \mu^s_i \times \frac{1}{T}s_i(1) + \sum_{j\in\mathcal{B}} \mu^b_j \times \frac{1}{T}b_j(1),$$

which further implies that

$$T(\widetilde{\mathcal{R}} - \overline{\mathcal{R}}) \leq T\Big(\sum_{i\in\mathcal{S}} \mu^s_i \times \frac{1}{T}s_i(1) + \sum_{j\in\mathcal{B}} \mu^b_j \times \frac{1}{T}b_j(1)\Big) = \sum_{i\in\mathcal{S}} \mu^s_i s_i(1) + \sum_{j\in\mathcal{B}} \mu^b_j b_j(1).$$

We let $C''_1 := \sum_{i\in\mathcal{S}} \mu^s_i s_i(1) + \sum_{j\in\mathcal{B}} \mu^b_j b_j(1)$, and obtain the desired result.

In summary, $|\mathcal{R}(T) - T\overline{\mathcal{R}}| \leq C_1$, where $C_1 = \max\{|C'_1|, |C''_1|\}$. ■

Proof of Theorem 1. We divide the proof arguments for the first claim into the following steps: in Step 1, we show that the solution generated by the TRP is feasible to Problem (EC.2); in Step 2, we show when $w = 0$, there exists a constant $\gamma \in (0,1)$ such that $|m(t+1)-1| \leq \gamma|m(t)-1|$ for any $t \in \{1,...,T-1\}$; in Step 3, we show that when $w = 0$, there exists a constant C'_1 such that $T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T) \leq C'_1$ for all w. Then, together with Step 2.1.2 and Step 2.2 of Proposition 1, we conclude that there exists a constant $C_2 := C'_1 + C''_1$ such that $\mathcal{L}^{TR}(T) = \mathcal{R}^*(T) - \mathcal{R}^{TR}(T) \leq T\tilde{\mathcal{R}} - \mathcal{R}^{TR}(T) = \left(T\tilde{\mathcal{R}} - T\overline{\mathcal{R}}\right) + \left(T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\right) \leq C_2$. Finally, in Step 4, we generalize the result to the case when $w > 0$.

The claim $|m^w(t) - 1| \leq |m^0(t) - 1|$ directly follows from Step 4.1 and Step 4.2.1.

Step 1: Show that the solution generated by the TRP is feasible to Problem (EC.2).

(EC.2b)-(EC.2c). $q_i^s(t) \overset{(a)}{=} \overline{q}_i^s \hat{m}(t) \overset{(b)}{\leq} s_i(t)\frac{\overline{q}_i^s}{\overline{q}_i^s} = s_i(t)$, where (a) follows from Algorithm 1; (b) follows directly from the definition of $\hat{m}(t)$. The same argument follows for the buyer side.

(EC.2d)-(EC.2e). $q_i^s(t) = \overline{q}_i^s \hat{m}(t) \overset{(a)}{=} \sum_{j':(i,j')\in E} \overline{x}_{i,j'}\hat{m}(t) \overset{(b)}{=} \sum_{j':(i,j')\in E} x_{i,j'}(t)$, where (a) follows from (5b); (b) follows from Algorithm 1. The same argument follows for the buyer side.

(EC.2f) . $x_{i,j} = \overline{x}_{i,j}\hat{m}(t) \geq 0$ follows from (5d).

(EC.2g)-(EC.2h). Given $s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t))$, the inequality is a relaxation, which directly follows. A similar argument holds for the buyer side.

In summary, the solution generated by the TRP is feasible for Problem (EC.2).

Step 2: Show that when $w = 0$, there exists a constant $\gamma \in (0,1)$ such that $|m(t+1) - 1| \leq \gamma|m(t) - 1|$ for $t \in \{1, ..., T-1\}$. Recall the definition of $l(t)$ and $g_t(n)$ in (EC.14) and (EC.15), respectively. We discuss three cases: (1) $m(1) > 1$, (2) $m(1) < 1$ and (3) $m(1) = 1$. In each case, we will first show that $m(t)$ gets closer to 1 as t increases, and then we show that the convergence rate can be upper bounded by $\gamma < 1$.

Step 2 - Case 1: $m(1) > 1$.

Step 2 - Case 1 - Step 2.1: Show that $m(1) > m(2) > ... > m(T-1) > m(T) > 1$. To prove the claim of this case, we show that for any $t \in \{1, \ldots, T-1\}$, if $m(t) > 1$, then $m(t) > m(t+1) > 1$. Let $X > 0$ denote the number of times the agent type with the lowest ratio changes. We consider the following two cases for any $t \in \{1, \ldots, T\}$: (1) the lowest node does not change in the next period, i.e., $\tau_x \leq t \leq \tau_{x+1} - 2$ for $x \in \{0, ..., X-1\}$; (2) the lowest node changes in next step, i.e., $t = \tau_{x+1} - 1$ for $x \in \{0, ..., X-1\}$.

(1) For any $\tau_x \leq t \leq \tau_{x+1} - 2$ with $x \in \{0, ..., X-1\}$, we show that if $m(t) > 1$, then $m(t) > m(t+1) > 1$.

Recall that $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}$ and $m(t+1) = \frac{n_{l(t+1)}(t+1)}{\overline{n}_{l(t+1)}} \overset{(a)}{=} \frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}$, where (a) holds given that $l(t) = l(t+1)$ for $\tau_x \leq t \leq \tau_{x+1} - 2$ and $x \in \{0, \ldots, X-1\}$. Then, to show that $m(t) > m(t+1) > 1$, it is equivalent to establish that $n_{l(t)}(t) > n_{l(t)}(t+1) > \overline{n}_{l(t)}$. First, we have

$$n_{l(t)}(t+1) - n_{l(t)}(t) \overset{(b)}{=} g_t(n_{l(t)}(t)) - n_{l(t)}(t) \overset{(c)}{<} 0,$$

where (b) follows from (EC.16); (c) follows directly from Lemma EC.6. Second, we deduce that

$$n_{l(t)}(t+1)-\overline{n}_{l(t)} \overset{(d)}{=} g_t(n_{l(t)}(t))-\overline{n}_{l(t)} \overset{(e)}{=} g_t(n_{l(t)}(t))-g_t(\overline{n}_{l(t)}) \overset{(f)}{>} 0,$$

where (d) follows from (EC.16); (e) follows from Lemma 1(ii); (f) follows from $n_{l(t)}(t) > \overline{n}_{l(t)}$ given that $m(t)=\frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}>1$ and that $g_t(n)$ increases in $n \geq 0$ from Lemma EC.6.

In summary, for $\tau_x \leq t \leq \tau_{x+1}-2$, if $m(t)>1$, then $m(t)>m(t+1)>1$.

(2) For $t=\tau_x-1$ with $x\in\{1,\dots,X\}$, we want to show that if $m(\tau_x-1)>1$, then $m(\tau_x-1)>m(\tau_x)>1$. To prove this, we can deduce that

$$m(\tau_x) \;=\; \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} \overset{(a)}{\leq} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}} \overset{(b)}{<} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} \;=\; m(\tau_x-1),$$

where (a) follows directly from the definition that $l(\tau_x)$ in (EC.14); (b) follows from $n_{l(\tau_x-1)}(\tau_x)=g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x-1))<n_{l(\tau_x-1)}(\tau_x-1)$, where the second inequality follows from $n_{l(\tau_x-1)}(\tau_x-1)>\overline{n}_{l(\tau_x-1)}$ given that $m(\tau_x-1)=\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}>1$ and Lemma EC.6. Therefore, $m(\tau_x)<m(\tau_x-1)$.

Next, we show that $m(\tau_x)>1$. Since

$$\begin{aligned} m(\tau_x) \;=\; \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} &\overset{(c)}{=} \frac{\mathcal{G}_{l(\tau_x)}\left(n_{l(\tau_x)}(\tau_x-1), \overline{q}_{l(\tau_x)}\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}\right)}{\overline{n}_{l(\tau_x)}} \\ &\overset{(d)}{\geq} \frac{\mathcal{G}_{l(\tau_x)}(n_{l(\tau_x)}(\tau_x-1),\overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} \overset{(e)}{>} \frac{\mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)},\overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} \;=\; 1, \end{aligned}$$

where (c) follows from Algorithm 1; (d) follows from the condition that $\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}=m(\tau_x-1)>1$ and $\mathcal{G}_{l(\tau_x)}(n,q)$ increases in $q\geq 0$; (e) follows from $\frac{n_{l(\tau_x)}(\tau_x-1))}{\overline{n}_{l(\tau_x)}}\geq m(\tau_x-1)>1$. Therefore, $m(\tau_x)>1$.

Based on the arguments above, if $m(t)>1$, then $m(t)>m(t+1)>1$, which holds for any $t\in\{1,...,T-1\}$. Thus, we can conclude that if $m(1)>1$, then $m(1)>m(2)>...>m(T-1)>m(T)>1$.

<u>Step 2 - Case 1 - Step 2.2: Show that there exists a constant $\gamma_1\in(0,1)$ such that $|m(t+1)-1|\leq\gamma_1|m(t)-1|$ for any $t\in\{1,\dots,T\}$.</u> Again, we consider the following two cases: (1) the lowest node does not change in the next step, i.e., $\tau_x\leq t\leq\tau_{x+1}-2$ for any $x\in\{0,...,X-1\}$; (2) the lowest node changes in next step, i.e., $t=\tau_{x+1}-1$ for any $x\in\{0,...,X-1\}$. For both cases, we first show that $\left|m(t+1)-1\right|\leq g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$. Then we show that there exists a $\gamma_1\in(0,1)$ independent from T such that for any positive integer T, $\max_{t=1,...,T} g_t'(\overline{n}_{l(t)})\leq\gamma_1<1$.

(1) For $\tau_x \leq t \leq \tau_{x+1}-2$, we observe that

$$\begin{aligned}\left|n_{l(t)}(t+1)-\overline{n}_{l(t)}\right| &\overset{(a)}{=} n_{l(t)}(t+1)-\overline{n}_{l(t)} \overset{(b)}{=} g_t(n_{l(t)}(t))-g_t(\overline{n}_{l(t)})\\ &\overset{(c)}{<} (n_{l(t)}(t)-\overline{n}_{l(t)})g_t'(\overline{n}_{l(t)}) \overset{(d)}{=} \left|n_{l(t)}(t)-\overline{n}_{l(t)}\right|g_t'(\overline{n}_{l(t)}),\end{aligned}$$

where (a) follows from $\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}} \geq m(t+1) > 1$ for any $t \in \{1,...,T-1\}$; (b) follows from (EC.16) and Lemma 1(ii); (c) follows from Lemma EC.6 given that $g_t(n)$ is strictly concave in $n \geq 0$; (d) follows from $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}} > 1$ for any $t \in \{1,...,T\}$. Therefore, $\left|m(t+1)-1\right| = \left|\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}-1\right| < g_t'(\overline{n}_{l(t)})\left|\frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}-1\right| = g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$.

(2) For $t = \tau_x - 1$,

$$\begin{aligned}\left|m(\tau_x)-1\right| &\overset{(a)}{=} m(\tau_x)-1 = \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}}-1 \overset{(b)}{\leq} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}-1\\ &\overset{(c)}{=} \frac{g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x-1))-g_{\tau_x-1}(\overline{n}_{l(\tau_x-1)})}{\overline{n}_{l(\tau_x-1)}} \overset{(d)}{<} \left(\frac{n_{l(\tau_x-1)}(\tau_x-1)-\overline{n}_{l(\tau_x-1)}}{\overline{n}_{l(\tau_x-1)}}\right)g'_{\tau_x-1}(\overline{n}_{l(\tau_x-1)})\\ &= (m(\tau_x-1)-1)g'_{\tau_x-1}(\overline{n}_{l(\tau_x-1)}) \overset{(e)}{=} \left|m(\tau_x-1)-1\right|g'_{\tau_x-1}(\overline{n}_{l(\tau_x-1)}),\end{aligned}$$

where (a) follows from $m(t) \geq 1$ for any $t \in \{1,...,T\}$; (b) follows from $\frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = m(\tau_x) \leq \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}$; (c) follows from $g_t(\cdot)$ in (EC.15) and Lemma 1(ii); (d) follows from the strict concavity of $g_t(\cdot)$ in Lemma EC.6; (e) follows from $m(\tau_x-1) = \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} > 1$.

In summary, $\left|m(t+1)-1\right| \leq g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$ for any $t \in \{1,\ldots,T\}$. Define $\gamma_1 := \max\limits_{i\in\mathcal{N}^+} \frac{\partial \mathcal{G}_i}{\partial n}(n, n\frac{\overline{q}_i}{\overline{n}_i})$, then

$$\max_{t=1,\ldots,T} g_t'(\overline{n}_{l(t)}) \overset{(a)}{=} \max_{t=1,\ldots,T} \frac{\partial \mathcal{G}_{l(t)}}{\partial n}(n, n\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}})|_{n=\overline{n}_{l(t)}} \leq \max_{i\in\mathcal{N}^+} \frac{\partial \mathcal{G}_i}{\partial n}(n, n\frac{\overline{q}_i}{\overline{n}_i})|_{n=\overline{n}_i} = \gamma_1 \overset{(b)}{<} 1,$$

where (a) follows from the definition of $g_t(\cdot)$ in (EC.15) and (b) follows from the finite network $G(\mathcal{S}\cup\mathcal{B},E)$ and discussion in Lemma EC.6. This allows us to conclude the contraction arguments for the case of $m(1) > 1$.

Step 2 - Case 2: $m(1) < 1$.

Step 2 - Case 2 - Step 2.1: Show that $m(1) < m(2) < ... < m(T-1) < m(T) < 1$. Similar to the discussions in Step 2 - Case 1, we consider the following two cases: (1) the lowest node does not change in the next step, i.e., $\tau_x \leq t \leq \tau_{x+1}-2$ for any $x \in \{0,...,X-1\}$; (2) the lowest node changes in next step, i.e., $t = \tau_{x+1}-1$ for any $x \in \{0,...,X-1\}$.

(1) For $\tau_x \le t \le \tau_{x+1} - 2$, we want to show that if $m(t) < 1$, then $m(t) < m(t+1) < 1$.

Recall that $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}$ and $m(t+1) = \frac{n_{l(t+1)}(t+1)}{\overline{n}_{l(t+1)}} \overset{(a)}{=} \frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}$, where (a) holds as $l(t) = l(t+1)$ for $\tau_x \le t \le \tau_{x+1} - 2$. Therefore, $m(t) < 1$ implies that $n_{l(t)}(t) < \overline{n}_{l(t)}$. We observe that $m(t) < m(t+1) < 1$ is then equivalent to $n_{l(t)}(t) < n_{l(t)}(t+1) < \overline{n}_{l(t)}$, which holds because

$$n_{l(t)}(t+1) - n_{l(t)}(t) \; = \; g_t(n_{l(t)}(t)) - n_{l(t)}(t) \; > \; 0,$$

where the equality follows from (EC.16) and the inequality follows from the condition that $0 < n_{l(t)}(t) < \overline{n}_{l(t)}$ and Lemma EC.6. In addition,

$$n_{l(t)}(t+1) - \overline{n}_{l(t)} = g_t(n_{l(t)}(t)) - g_t(\overline{n}_{l(t)}) < 0,$$

given that $n_{l(t)}(t) < \overline{n}_{l(t)}$ and that $g_t(n)$ increases in $n \ge 0$ based on Lemma EC.6. The derivations above allow us to establish that $n_{l(t)}(t) < n_{l(t)}(t+1) < \overline{n}_{l(t)}$.

(2) For $t = \tau_x - 1$, we show that $m(\tau_x - 1) < m(\tau_x) < 1$ if $m(\tau_x - 1) < 1$, then

$$\begin{aligned} m(\tau_x) \overset{(a)}{=} \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} &= \frac{\mathcal{G}_{l(\tau_x)}\left(n_{l(\tau_x)}(\tau_x - 1), \overline{q}_{l(\tau_x)} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}\right)}{\overline{n}_{l(\tau_x)}} \overset{(b)}{\ge} \frac{\mathcal{G}_{l(\tau_x)}\left(\overline{n}_{l(\tau_x)} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}, \overline{q}_{l(\tau_x)} \frac{n_{l(\tau_x-1)}(\tau_x-}{\overline{n}_{l(\tau_x-1)}}\right.}{\overline{n}_{l(\tau_x)}} \\ &\overset{(c)}{>} \frac{\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} \mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)}, \overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} \overset{(d)}{=} \frac{n_{l(\tau_x-1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x-1)}} = m(\tau_x - 1), \end{aligned}$$

where (a) follows the definition of $m(\tau_x)$ in (EC.13) and $l(\tau_x)$ in (EC.14); (b) follows from $\frac{n_{l(\tau_x)}(\tau_x-1)}{\overline{n}_{l(\tau_x)}} \ge m(\tau_x - 1) = \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}$ given the definition of $m(\tau_x - 1)$ in (EC.13); (c) follows from

$$\mathcal{G}_i(a\overline{n}_i, a\overline{q}_i) = \mathcal{G}_i(a\overline{n}_i + (1-a)0, a\overline{q}_i + (1-a)0) > a\mathcal{G}_i(\overline{n}_i, \overline{q}_i) + (1-a)\mathcal{G}_i(0,0) = a\mathcal{G}_i(\overline{n}_i, \overline{q}_i), \tag{EC.20}$$

for $0 < a < 1$ given that $\mathcal{G}_i(0,0) = 0$ and $\mathcal{G}_i(n_i, q_i)$ is strictly concave in (n_i, q_i); in addition, (d) follows from $\mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)}, \overline{q}_{l(\tau_x)}) = \overline{n}_{l(\tau_x)}$. In summary, we have $m(\tau_x) > m(\tau_x - 1)$.

To proceed, we further observe that

$$m(\tau_x) \; = \; \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} \overset{(d)}{\le} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}} \overset{(e)}{<} 1,$$

where (d) follows from $\frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = m(\tau_x) \le \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}$ given the definition of $m(\tau_x)$ in (EC.13); (e) follows from Lemma EC.6 that $n_{l(\tau_x-1)}(\tau_x) = g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x - 1)) < \overline{n}_{l(\tau_x-1)}$ for $n_{l(\tau_x-1)}(\tau_x - 1) < \overline{n}_{l(\tau_x-1)}$. Thus, we have that $m(\tau_x) < 1$.

In summary, $m(t) < m(t+1) < 1$ if $m(t) < 1$ for any $t \in \{1, ..., T-1\}$. Since $m(t) < 1$, we obtain that $m(1) < m(2) < ... < m(T-1) < m(T) < 1$.

Step 2 - Case 2 - Step 2.2: Show that there exists a constant $\gamma_2 \in (0,1)$ such that $|m(t+1) - 1| \leq \gamma_2 |m(t) - 1|$ for any $t \in \{1, \ldots, T\}$. Following a similar argument in the previous step, we can obtain the desired results.

Step 2 - Case 3: $m(1) = 1$. When $m(1) = 1$, we want to show that $m(t) = 1$ for any $t \in \{1, ..., T\}$. To establish the claim, we show that inductively, if $m(t) = 1$ then $m(t+1) = 1$ for any $t \in \{1, ..., T-1\}$. We observe that

$$n_{l(t)}(t+1) \overset{(a)}{=} \mathcal{G}_{l(t)}(n_{l(t)}(t), \overline{q}_{l(t)} m(t)) \overset{(b)}{=} \mathcal{G}_{l(t)}(\overline{n}_{l(t)}, \overline{q}_{l(t)}) \overset{(c)}{=} \overline{n}_{l(t)},$$

where (a) follows from the population transition induced by Algorithm 1; (b) holds given that $m(t) = 1$, which further implies that $n_{l(t)}(t) = \overline{n}_{l(t)}$; (c) follows from Lemma 1(ii). Thus, $\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}} = 1$. In addition, for $i \in \mathcal{N}^+$ with $i \neq l(t)$, we can deduce that

$$n_i(t+1) \ = \ \mathcal{G}_i(n_i(t), \overline{q}_i m(t)) \ \overset{(d)}{\geq} \ \mathcal{G}_i(\overline{n}_i, \overline{q}_i) \ = \ \overline{n}_i,$$

where (d) follows from $\frac{n_i(t)}{\overline{n}_i} \geq m(t) = 1$ given the definition of $m(t)$ in (EC.13) and the condition that $i \neq l(t)$. The observation above implies that $\frac{n_i(t+1)}{\overline{n}_i} \geq 1$ for $i \in \mathcal{N}^+$ with $i \neq l(t)$. Therefore, we can establish that

$$m(t+1) \ = \ \min\left\{\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}, \min_{\substack{i \in \mathcal{N}^+, \\ i \neq l(t)}} \left\{\frac{n_i(t+1)}{\overline{n}_i}\right\}\right\} \ = \ 1.$$

Given that $m(1) = 1$, by inductively establishing that $m(t+1) = 1$ for any $t \in \{1, ..., T-1\}$, we have that $m(t) = 1$ for any $t \in \{1, ..., T\}$. Thus, we obtain that $\big|m(t+1) - 1\big| = 0 \leq \gamma_3 \big|m(t) - 1\big| = 0$ for any $\gamma_3 \in (0,1)$.

In summary of the three cases above for $m(t) < 1$, $m(t) > 1$ and $m(t) = 1$, by letting $\gamma = \max\{\gamma_1, \gamma_2, \gamma_3\}$, We have that for some $\gamma \in (0,1)$, $\big|m(t+1) - 1\big| \leq \gamma \big|m(t) - 1\big|$, for any $t = \{1, ..., T-1\}$.

Next, we use the superscript w to denote the value under policy with parameter w.

Step 3: Show that when $w = 0$. there exists a constant C_1' such that $|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)| \leq C_1'$. We prove this by the following steps. Given $\boldsymbol{q}(t)$ and $\boldsymbol{n}(t)$ induced by TRP, we show in Step 3.1 that there exists a positive constant C_{q_i} such that $\lim_{T \to \infty} \sum_{t=1}^{T} |q_i(t) - \overline{q}_i| \leq C_{q_i}$; In Step 3.2, we show that the previous two steps induce a positive constant $C_{\frac{q_i}{n_i}}$ that satisfies $\lim_{T\to\infty} \sum_{t=1}^{T} |\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}| \leq C_{\frac{q_i}{n_i}}$ for any $i \in \mathcal{N}^+$; In Step 3.3, based on Steps 4.1 - 4.2, we conclude that there exists a constant C_1' such that $\left|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\right| \leq C_1'$.

Step 3.1: Show that there exists constants C_{q_i} such that $\lim_{T\to\infty} \sum_{t=1}^{T} |q_i(t) - \overline{q}_i| < C_{q_i}$ for any $i \in \mathcal{N}^+$. Notice that

$$\lim_{T\to\infty} \sum_{t=1}^{T} \left|q_i(t) - \overline{q}_i\right| \stackrel{(a)}{=} \lim_{T\to\infty} \sum_{t=1}^{T} \overline{q}_i \left|m(t) - 1\right| \stackrel{(b)}{\leq} \lim_{T\to\infty} \sum_{t=1}^{T} \overline{q}_i \left|m(1) - 1\right| \gamma^{t-1}$$
$$= \lim_{T\to\infty} \overline{q}_i \left|m(1) - 1\right| \frac{1-\gamma^T}{1-\gamma} \stackrel{(c)}{=} \frac{1}{1-\gamma} \overline{q}_i \left|m(1) - 1\right|,$$

where (a) follows from $q_i(t) = \overline{q}_i m(t)$ in Algorithm 1; (b) follows from the contraction arguments in Step 2; (c) follows from $\gamma < 1$ in Step 2. Let $C_{q_i} = \frac{\overline{q}_i |m(1)-1|}{1-\gamma}$, and then the result follows.

Before proceeding, we provide some supporting results whose proofs will be provided towards the end of this section:

Lemma EC.7. *For any $i \in \mathcal{N}^+$ with $n_i(1) \geq \overline{n}_i$, there exists a positive constant C_{n_i} such that $\lim_{T\to\infty} \sum_{t=1}^{T} |n_i(t) - \overline{n}_i| < C_{n_i}$. Moreover, for any $i \in \mathcal{N}^+$ with $n_i(1) < \overline{n}_i$, if $m(1) < 1$, then $n_i(t) < \overline{n}_i$ for $t \in \{1, ..., T\}$.*

Step 3.2: Show that there exists positive constants $C_{\frac{q_i}{n_i}}$ such that $\lim_{T\to\infty} \sum_{t=1}^{T} |\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}| \leq C_{\frac{q_i}{n_i}}$ for any $i \in \mathcal{N}^+$. To show the claim for this step, we notice that for any $i \in \mathcal{N}_+$,

$$\left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}\right| \stackrel{(a)}{=} \left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{\overline{q}_i m(t)}{n_i(t)}\right| = \frac{\overline{q}_i}{\overline{n}_i} \left|1 - \frac{\overline{n}_i m(t)}{n_i(t)}\right| \stackrel{(b)}{\leq} \frac{\overline{q}_i}{\overline{n}_i} \left(\left|1 - \frac{\overline{n}_i}{n_i(t)}\right| + \frac{\overline{n}_i}{n_i(t)} \left|1 - m(t)\right|\right),$$

where (a) follows from the population transition induced by Algorithm 1, and (b) follows directly from the triangle inequality. Therefore,

$$\begin{aligned}
\lim_{T\to\infty} \sum_{t=1}^{T} \left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}\right| &\leq \lim_{T\to\infty} \sum_{t=1}^{T} \frac{\overline{q}_i}{\overline{n}_i} \left(\left|1 - \frac{\overline{n}_i}{n_i(t)}\right| + \frac{\overline{n}_i}{n_i(t)} \left|1 - m(t)\right|\right) \\
&= \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i} \left(\sum_{t=1}^{T} \frac{\overline{n}_i}{n_i(t)} \left|1 - \frac{n_i(t)}{\overline{n}_i}\right| + \sum_{t=1}^{T} \frac{\overline{n}_i}{n_i(t)} \left|1 - m(t)\right|\right) \\
&\stackrel{(c)}{\leq} \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i} \left(\sum_{t=1}^{T} \frac{1}{m(t)} \left|1 - \frac{n_i(t)}{\overline{n}_i}\right| + \sum_{t=1}^{T} \frac{1}{m(t)} \left|1 - m(t)\right|\right), \quad (*)
\end{aligned} \tag{EC.21}$$

where (c) follow from the definition of $m(t)$ in (EC.13).

Notice that if $m(1)=\min_{i\in\mathcal{N}^+}\frac{n_i(1)}{\overline{n}_i}\geq 1$, then $n(1)\geq\overline{n}_i$ for any $i\in\mathcal{N}^+$. Thus, it is without loss of generality to consider the following three cases for any $i\in\mathcal{N}^+$ to further relax the term in the RHS of (EC.21), which we denote by "(*)".

(1) When $n_i(1)\geq\overline{n}_i$ and $m(1)\geq 1$, we show that

$$(*)\overset{(d)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\left|1-\frac{n_i(t)}{\overline{n}_i}\right|+\sum_{t=1}^{T}\Big|1-m(t)\Big|\right)\overset{(e)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{C_{n_i}}{\overline{n}_i}+\sum_{t=1}^{T}\Big|1-m(1)\Big|\gamma^{t-1}\right)$$
$$=\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{C_{n_i}}{\overline{n}_i}+\Big|1-m(1)\Big|\frac{1}{1-\gamma}\right),$$

where (d) follows from the result in Step 2 - Case 1- Step 2.1 and Step 2 - Case 3 that if $m(1)>1$, then $m(1)\geq m(2)\geq\ldots\geq m(T)\geq 1$; (e) follows from Lemma EC.7 that $\lim_{T\to\infty}\sum_{t=1}^{T}|n_i(t)-\overline{n}_i|\leq C_{n_i}$ given that $n_i(1)\geq\overline{n}_i$, and we also have $|m(t)-1|\leq\gamma|m(t-1)-1|$ for $\gamma<1$ and $t\in\{2,\ldots,T\}$ by Step 2. Therefore, by letting $C_{\frac{q_i}{n_i}}:=\frac{\overline{q}_i}{\overline{n}_i}\big(\frac{C_{n_i}}{\overline{n}_i}+|1-m(1)|\frac{1}{1-\gamma}\big)$, we obtain the desired result.

(2) When $n_i(1)<\overline{n}_i$ and $m(1)<1$, we show that

$$(*)\overset{(f)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\frac{1}{m(t)}\Big|1-m(t)\Big|+\sum_{t=1}^{T}\frac{1}{m(t)}\Big|1-m(t)\Big|\right)$$
$$\overset{(g)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\sum_{t=1}^{T}\Big|1-m(1)\Big|\gamma^{t-1}+\frac{1}{m(1)}\sum_{t=1}^{T}\Big|1-m(1)\Big|\gamma^{t-1}\right)\leq\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{2|1-m(1)|}{m(1)(1-\gamma)}\right),$$

where (f) follows from the observation that $m(t)\leq\frac{n_i(t)}{\overline{n}_i}<1$, where the first inequality follows from the definition of $m(t)$ in (EC.13) and the second inequality follows from Lemma EC.7 that if $n_i(1)<\overline{n}_i$ and $m(1)<1$, then $n_i(t)<\overline{n}_i$ for $t\in\{1,\ldots,T\}$; (g) follows from the observation that $|m(t)-1|\leq\gamma|m(t-1)-1|$ for $\gamma<1$ and $t\in\{2,\ldots,T\}$ by Step 2, and therefore $|m(t)-1|\leq\gamma^{t-1}|m(1)-1|$; in addition, we show in Step 2 - Case 2- Step 2.1 that when $m(1)<1$, we have $m(1)\leq m(t)$ for any $t\in\{1,\ldots,T\}$. Therefore, we can let $C_{\frac{q_i}{n_i}}:=\frac{\overline{q}_i}{\overline{n}_i}\big(\frac{2|1-m(1)|}{m(1)(1-\gamma)}\big)$, and then obtain the desired result.

(3) When $n_i(1)\geq\overline{n}_i$ and $m(1)<1$, we show that

$$(*)\overset{(h)}{<}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i}+\sum_{t=1}^{T}\frac{1}{m(1)}\Big|1-m(t)\Big|\right)$$
$$\overset{(i)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i}+\sum_{t=1}^{T}\frac{1}{m(1)}\Big|1-m(1)\Big|\gamma^{t-1}\right)\overset{(j)}{=}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i}+\left|\frac{1}{m(1)}-1\right|\frac{1}{1-\gamma}\right),$$

where (h) follows from the observation in Step 2 -Case 2- Step 2.1 that $m(1) < m(2) < \ldots < m(T) < 1$ when $m(1) < 1$ and the result in Lemma EC.7 that $\lim_{T\to\infty} \sum_{t=1}^{T} |n_i(t) - \overline{n}_i| \le C_{n_i}$ when $n_i(1) \ge \overline{n}_i$; (i) follows from the results in Step 2 that $|m(t+1) - 1| \le \gamma|m(t) - 1|$; (j) follows from the observation in Step 2 that $\gamma < 1$. Therefore, by letting $C_{\frac{q_i}{n_i}} := \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i} + |\frac{1}{m(1)} - 1|\frac{1}{1-\gamma}\right)$, we can establish the desired result.

In summary, we have that for any $i \in \mathcal{N}^+$, there exists a positive constant $C_{\frac{q_i}{n_i}}$ such that

$$\lim_{T\to\infty} \sum_{t=1}^{T} \left| \frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)} \right| \;\le\; C_{\frac{q_i}{n_i}}.$$

Step 3.3: Show that there exists a constant C_1' such that $|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)| \le C_1'$. Note that for $j \in \mathcal{B}$ with $\overline{b}_j = 0$, we have $\widetilde{F}_{b_j}(\overline{q}_j^b, \overline{b}_j) = 0$ based on the definition of $\widetilde{F}_{b_j}$ before the formulation of (5). Since $\overline{q}_j^b \le \overline{b}_j = 0$, we have $q_j^b(t) = \overline{q}_j^b m(t) = 0$ induced by Algorithm 1, which further implies that $F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})q_j^b(t) = 0$. Therefore,

$$\lim_{T\to\infty} \sum_{t=1}^{T} \sum_{j\in\mathcal{B}:\overline{b}_j=0} \left(\widetilde{F}_{b_j}(\overline{q}_j^b, \overline{b}_j) - F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})q_j^b(t) \right) \;=\; 0.$$

Similarly, we can establish that for any $i \in \mathcal{S}$ with $\overline{s}_i = 0$, we have that $\widetilde{F}_{s_i}(\overline{q}_i^s, \overline{s}_i) = 0$, which further implies that $q_i^s(t) = \overline{q}_i^s m(t) = 0$. Thus, we have that

$$\lim_{T\to\infty} \quad \sum_{t=1}^{T} \sum_{i\in\mathcal{S}:\overline{s}_i=0} \left(\widetilde{F}_{s_i}(\overline{q}_i^s, \overline{s}_i) - F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t) \right) \;=\; 0.$$

Based on the two observations above, with $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t), \boldsymbol{s}(t), \boldsymbol{b}(t) : t = 1, \ldots T)$ induced by the TRP, we can deduce that

$$\begin{aligned}
&\lim_{T\to\infty} \quad \left| T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T) \right| \\
&= \lim_{T\to\infty} \sum_{t=1}^{T} \left[\sum_{j\in\mathcal{B}} \left(\widetilde{F}_{b_j}(\overline{q}_j^b, \overline{b}_j) - F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t) \right) - \sum_{i\in\mathcal{S}} \left(\widetilde{F}_{s_i}(\overline{q}_i^s, \overline{s}_i) - F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t) \right) \right] \\
&= \lim_{T\to\infty} \sum_{t=1}^{T} \left[\sum_{j\in\mathcal{B}:\overline{b}_j>0} \left(F_{b_j}^{-1}\Big(1 - \frac{\overline{q}_j^b}{\overline{b}_j}\Big)\overline{q}_j^b - F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t) \right) - \sum_{i\in\mathcal{S}:\overline{s}_i>0} \left(F_{s_i}^{-1}\Big(\frac{\overline{q}_i^s}{\overline{s}_i}\Big)\overline{q}_i^s - F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t) \right) \right] \\
&\overset{(a)}{\le} \lim_{T\to\infty} \sum_{t=1}^{T} \left[\sum_{j\in\mathcal{B}:\overline{b}_j>0} \left(\left| F_{b_j}^{-1}\Big(1 - \frac{\overline{q}_j^b}{\overline{b}_j}\Big)\overline{q}_j^b - F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)\overline{q}_j^b \right| + \left| F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)\overline{q}_j^b - F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t) \right| \right) \right. \\
&\quad \left. + \sum_{i\in\mathcal{S}:\overline{s}_i>0} \left(\left| F_{s_i}^{-1}\Big(\frac{\overline{q}_i^s}{\overline{s}_i}\Big)\overline{q}_i^s - F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)\overline{q}_i^s \right| + \left| F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)\overline{q}_i^s - F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t) \right| \right) \right] \\
&\overset{(b)}{\le} \lim_{T\to\infty} \sum_{t=1}^{T} \left[\sum_{j\in\mathcal{B}:\overline{b}_j>0} \left(\overline{q}_j^b \frac{1}{d_j^b} \left| \frac{\overline{q}_j^b}{\overline{b}_j} - \frac{q_j^b(t)}{b_j(t)} \right| + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) \left| \overline{q}_j^b - q_j^b(t) \right| \right) \right.
\end{aligned}$$

$$+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}_i^s\frac{1}{d_i^s}\left|\frac{\overline{q}_i^s}{\overline{s}_i}-\frac{q_i^s(t)}{s_i(t)}\right|+F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})\left|\overline{q}_i^s-q_i^s(t)\right|\right)\Bigg]$$

$$\leq\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}_j^b\frac{1}{d_j^b}\lim_{T\to\infty}\sum_{t=1}^T\left|\frac{\overline{q}_j^b}{\overline{b}_j}-\frac{q_j^b(t)}{b_j(t)}\right|+\max_t F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})\lim_{T\to\infty}\sum_{t=1}^T\left|\overline{q}_j^b-q_j^b(t)\right|\right)$$

$$+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}_i^s\frac{1}{d_i^s}\lim_{T\to\infty}\sum_{t=1}^T\left|\frac{\overline{q}_i^s}{\overline{s}_i}-\frac{q_i^s(t)}{s_i(t)}\right|+\max_t F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})\lim_{T\to\infty}\sum_{t=1}^T\left|\overline{q}_i^s-q_i^s(t)\right|\right)$$

$$\overset{(c)}{\leq}\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}_j^b\frac{1}{d_j^b}C_{q_j^b/b_j}+\overline{v}_j^bC_{q_j^b}\right)+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}_i^s\frac{1}{d_i^s}C_{q_i^s/s_i}+\overline{v}_i^sC_{q_i^s}\right):=C_1'.$$

where (a) follows from the triangle inequality; (b) follows from Assumption 2(ii) that the derivative of F_{b_j} (F_{s_i}) is lower bounded by a positive constant d_j^b (d_i^s), and therefore the derivative of $F_{b_j}^{-1}$ ($F_{s_i}^{-1}$) is upper bounded by $\frac{1}{d_j^b}$ ($\frac{1}{d_i^s}$), then $|F_{b_j}^{-1}(x_1)-F_{b_j}^{-1}(x_2)|\leq\frac{1}{d_j^b}|x_1-x_2|$ for any x_1,x_2 in the domain, otherwise $\frac{|F_{b_j}^{-1}(x_1)-F_{b_j}^{-1}(x_2)|}{|x_1-x_2|}>\frac{1}{d_j^b}$ implies that there exists a $x_3\in(x_1,x_2)$ such that $f'(x_3)=\frac{|F_{b_j}^{-1}(x_1)-F_{b_j}^{-1}(x_2)|}{|x_1-x_2|}>\frac{1}{d_j^b}$ by mean value theorem, which contradicts to the fact that the derivative of $F_{b_j}^{-1}$ is upper bounded by $\frac{1}{d_j^b}$; following the same argument, $|F_{s_i}^{-1}(x_1)-F_{s_i}^{-1}(x_2)|\leq\frac{1}{d_i^s}|x_1-x_2|$ for any x_1,x_2 in the domain. (c) follows from the results in Step 3.1- Step 3.2 that $\lim_{T\to\infty}\sum_{t=1}^T|q_i(t)-\overline{q}_i|<C_{q_i}$ and $\lim_{T\to\infty}\sum_{t=1}^T|\frac{\overline{q}_i}{\overline{n}_i}-\frac{q_i(t)}{n_i(t)}|\leq C_{\frac{q_i}{n_i}}$ for any $i\in\mathcal{N}^+$; in addition, $F_{b_j}^{-1}\leq\overline{v}_{b_j}$ and $F_{s_i}^{-1}\leq\overline{v}_{s_i}$. Note that we have $\overline{v}_{b_j}<\infty$ for $j\in\mathcal{B}$ and $\overline{v}_{s_i}<\infty$ for $i\in\mathcal{S}$ and $\frac{1}{d_j^b}<\infty$ for $j\in\mathcal{B}$ and $\frac{1}{d_i^s}<\infty$ for $i\in\mathcal{S}$ given Assumption 2(ii).

Step 4: Show that when $0<w\leq 1$. there exists a constant C_2' such that $|T\overline{\mathcal{R}}-\mathcal{R}^{TR}(T)|\leq C_2'$. We consider the case with $m(1)\geq 1$ and $m(1)<1$ respectively in Step 4.1 and Step 4.2.

Step 4.1. $m(1)\geq 1$. We show that in this case, $Overexpansion=True$ from the beginning.

$$\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(1),\overline{q}_i\hat{m}(1))}{\overline{n}_i}\right\}=\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i\Big(n_i(1),\overline{q}_i((1-w)\min_{i\in\mathcal{N}^+}\{\frac{n_i(1)}{\overline{n}_i}\}+w\min_{i\in\mathcal{N}^+}\{\frac{n_i(1)}{\overline{q}_i}\})\Big)}{\overline{n}_i}\right\}$$

$$\overset{(a)}{>}\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i\Big(n_i(1),\overline{q}_i\min_{i\in\mathcal{N}^+}\{\frac{n_i(1)}{\overline{n}_i}\}\Big)}{\overline{n}_i}\right\}\overset{(b)}{\geq}\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(\overline{n}_i,\overline{q}_i)}{\overline{n}_i}\right\}=\min_{i\in\mathcal{N}^+}\left\{\frac{\overline{n}_i}{\overline{n}_i}\right\}=1,$$

where (a) follows from $0\leq\overline{q}_i\leq n_i$ and $0<w\leq 1$; (b) follows from $m(1)=\min_{i\in\mathcal{N}^+}\left\{\frac{n_i(1)}{\overline{n}_i}\right\}\geq 1$. As a result, $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(1),\overline{q}_i\hat{m}(1))}{\overline{n}_i}\right\}>1$, which means that $OverExpension=True$ from the beginning, and the update rule when $w>0$ the same as that when $w=0$. Therefore, $|T\overline{\mathcal{R}}-\mathcal{R}^{TR}(T)|\leq C_2'$ by Step 3.

Step 4.2. $0<m(1)<1$. We will show in Step 4.2.2 that $OverExpansion$ occurs within a finite period. After that, the policy with $w>0$ becomes identical to the policy with $w=0$.

Prior to this, we must show that the system converges faster under $w>0$ compared to $w=0$ in Step 4.2.1 to facilitate our later proof. We next use the superscript w to denote the value under policy with parameter w.

Step 4.2.1. Show that if $0<m(1)<1$, then $0\le m^0(t)\le m^w(t)\le 1$ for all $w\in(0,1]$ and for $t=\{1,...,T\}$. We already know $0\le m^0(t)\le 1$ from Step 2-Case 2. We will respectively show that $m^0(t)\le m^w(t)$ and $m^w(t)\le 1$ for $t=\{1,...,T\}$.

Step 4.2.1(i). Show that $m^0(t)\le m^w(t)$ for $t=\{1,...,T\}$. Based on the definition that $m^w(t):=\min_{i\in\mathcal{N}}\{\frac{n_i^w(t)}{\overline{n}_i}\}$, it is sufficient to show that $n_i^0(t)\le n_i^w(t)$ for $t\in\{1,...,T\}$ and $i\in\mathcal{N}$. We show it by induction.

We already know that $n_i(1)$ is the same under different w as they are exogenously given. We then show that if $n_i^0(t)\le n_i^w(t)$ for any $i\in\mathcal{N}$, then $n_i^0(t+1)\le n_i^w(t+1)$ for any $i\in\mathcal{N}$. Since the update rule of TRP depends on the state of the system, we need to consider the following two cases:

(1). If $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\}\le 1$ and $OverExpension=False$, then for any $i\in\mathcal{N}$,

$$n_i^0(t+1)\overset{(a)}{=}\mathcal{G}_i^s\Big(n_i^0(t),\overline{q}_i m^0(t)\Big)\overset{(b)}{\le}\mathcal{G}_i^s\Big(n_i^w(t),\overline{q}_i\hat{m}^w(t)\Big)\overset{(c)}{=}n_i^w(t+1).$$

where (a) and (c) follow from the construction of two policies, (b) follows from $n_i^0(t)\le n_i^w(t)$ for any $i\in\mathcal{N}$ and $m^0(t)=\min_{i:\overline{n}_i>0}\left\{\frac{n_i^0(t)}{\overline{n}_i}\right\}\le(1-w)\min_{i:\overline{q}_i>0}\left\{\frac{n_i^w(t)}{\overline{n}_i}\right\}+w\min_{i:\overline{q}_i>0}\left\{\frac{n_i^w(t)}{\overline{q}_i}\right\}=\hat{m}^w(t)$ as $\overline{q}_i\le\overline{n}_i$.

(2). If $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\}>1$ or and $OverExpension=True$, then for any $i\in\mathcal{N}$,

$$n_i^0(t+1)=\mathcal{G}_i^s\Big(n_i^0(t),\overline{q}_i m^0(t)\Big)\le\mathcal{G}_i^s\Big(n_i^w(t),\overline{q}_i m^w(t)\Big)=n_i^w(t+1).$$

where the inequality follows from $n_i^0(t)\le n_i^w(t)$ for any $i\in\mathcal{N}$ and $m^0(t)=\min_{i:\overline{n}_i>0}\left\{\frac{n_i^0(t)}{\overline{n}_i}\right\}\le\min_{i:\overline{n}_i>0}\left\{\frac{n_i^w(t)}{\overline{n}_i}\right\}=m^w(t)$.

Step 4.2.1(ii): Show that $m^w(t)\le 1$ for $t=\{1,...,T\}$. It is equivalent to show that $\min_{i\in\mathcal{N}^+}\left\{\frac{n_i^w(t)}{\overline{n}_i}\right\}<1$ for any $t\in\{1,...,T\}$. We show it by induction. We already know that $m(1)=\min_{i\in\mathcal{N}^+}\left\{\frac{n_i(1)}{\overline{n}_i}\right\}<1$. Then we show that given $m(t)<1$, we have $m(t+1)<1$. Consider the following two cases:

(1). If $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\}\le 1$ and $OverExpension=False$, then $m^w(t+1)=\min_{i\in\mathcal{N}^+}\left\{\frac{n_i^w(t+1)}{\overline{n}_i}\right\}=\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i m^w(t))}{\overline{n}_i}\right\}\le 1;$

(2). If $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i\hat{m}(t))}{\overline{n}_i}\right\} > 1$ or $OverExpension = True$, then the update rule when $w>0$ is the same as that when $w=0$. We need to show that given $m(t)<1$, $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i m(t))}{\overline{n}_i}\right\} < 1$, which is already shown in Step 2-Case 2 of Theorem 1.

Step 4.2.2. Show that there exists a constant $\tilde{t}$ such that if $t>\tilde{t}$, we have $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\} > 1$.

We first show that there exists a constant $\tilde{m}$ such that if $m^w(t) > \tilde{m}$, we have $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\} > 1$. For all $i\in\mathcal{N}^+$, define $\tilde{m}_i = \{0<m<1|\overline{n}_i = \mathcal{G}_i\Big(m\overline{n}_i, \overline{q}_i((1-w)m + wm\min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\}\Big)\}$. Since the RHS of the equation increases in m, and $\mathcal{G}_i(0,0)=0<\overline{n}_i$ and $\mathcal{G}_i\Big(\overline{n}_i, \overline{q}_i((1-w)+w\min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\})\Big)\} > \mathcal{G}_i(\overline{n}_i,\overline{q}_i)\} = \overline{n}_i$, we know $\tilde{m}_i$ is well-defined.

If $m^w(t) > \tilde{m} := \max_{i\in\mathcal{N}^+}\tilde{m}_i$, then for all $i\in\mathcal{N}^+$,

$$\begin{aligned}\mathcal{G}_i\Big(n_i(t),\overline{q}_i\hat{m}^w(t)\Big) &\overset{(a)}{=} \mathcal{G}_i\Big(n_i(t),\overline{q}_i((1-w)m^w(t) + w\min_{i'\in\mathcal{N}}\{\frac{n_{i'}^w(t)}{\overline{q}_{i'}}\})\Big)\\ &\overset{(b)}{\geq} \mathcal{G}_i\Big(m^w(t)\overline{n}_i, \overline{q}_i((1-w)m^w(t) + wm^w(t)\min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\})\Big)\\ &\overset{(c)}{>} \mathcal{G}_i\Big(\tilde{m}_i\overline{n}_i, \overline{q}_i((1-w)\tilde{m}_i + w\tilde{m}_i\min_{i'}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\})\Big) \overset{(d)}{=} \overline{n}_i,\end{aligned}$$

where (a) follows from the definition of $\hat{m}^w(t)$; (b) follows from $m^w(t) \leq \frac{n_i^w(t)}{\overline{n}_i}$ based on its definition, and $\min_{i'\in\mathcal{N}}\{\frac{n_{i'}^w(t)}{\overline{q}_{i'}}\} > \min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\}\min_{i'\in\mathcal{N}}\{\frac{n_{i'}^w(t)}{\overline{n}_{i'}}\} = \min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\}m^w(t)$; (c) follows from $m^w(t) > \tilde{m} := \max_{i\in\mathcal{N}^+}\tilde{m}_i$; (d) follows from the definition of $\tilde{m}_i$. In conclusion, $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\} > 1$.

We then show that there exists a constant $\tilde{t}$ such that if $t>\tilde{t}$, we have $m^w(t)>\tilde{m}$. Define $\tilde{t} = \frac{\log{(1-\tilde{m})/(1-m(1))}}{\log\gamma} + 1$, then when $t>\tilde{t}$, we have

$$1-m^w(t) \overset{(a)}{\leq} 1-m^0(t) \overset{(b)}{\leq} \gamma^{t-1}(1-m^0(1)) \overset{(c)}{<} \gamma^{\tilde{t}-1}(1-m^0(1)) \overset{(d)}{=} 1-\tilde{m},$$

where (a) follows from Step 4.2.1; (b) follows from Step 2; (c) follows from $t>\tilde{t}$ and $0<\gamma<1$; (d) follows from the definition of $\tilde{t}$. Therefore, $m^w(t)>\tilde{m}$ for $t>\tilde{t}$.

In summary of the above two claims, we have $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\} > 1$ for $t>\tilde{t}$, which suggests that the system is in the state of overexpansion after a finite period.

Step 4.3: Conclude the case.

$$\begin{aligned}&\lim_{T\to\infty} \quad \Big|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\Big|\\ =& \sum_{t=1}^{\tilde{t}}\Big[\sum_{j\in\mathcal{B}}\Big(\widetilde{F}_{b_j}(\overline{q}_j^b,\overline{b}_j) - F_{b_j}^{-1}\Big(1-\frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t)\Big) - \sum_{i\in\mathcal{S}}\Big(\widetilde{F}_{s_i}(\overline{q}_i^s,\overline{s}_i) - F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t)\Big)\Big]\end{aligned}$$

$$
\begin{aligned}
&+\lim_{T\to\infty}\sum_{t=\tilde{t}+1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\widetilde{F}_{b_j}(\overline{q}_j^b,\overline{b}_j)-F_{b_j}^{-1}\Big(1-\frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t)\Big)-\sum_{i\in\mathcal{S}}\Big(\widetilde{F}_{s_i}(\overline{q}_i^s,\overline{s}_i)-F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t)\Big)\Big]\\
\le\ &\tilde{t}\Big[\sum_{j\in\mathcal{B}}\widetilde{F}_{b_j}(\overline{q}_j^b,\overline{b}_j)-\sum_{i\in\mathcal{S}}\widetilde{F}_{s_i}(\overline{q}_i^s,\overline{s}_i)\Big]\\
&+\lim_{T\to\infty}\sum_{t=\tilde{t}+1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\widetilde{F}_{b_j}(\overline{q}_j^b,\overline{b}_j)-F_{b_j}^{-1}\Big(1-\frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t)\Big)-\sum_{i\in\mathcal{S}}\Big(\widetilde{F}_{s_i}(\overline{q}_i^s,\overline{s}_i)-F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t)\Big)\Big]\\
\le\ &\tilde{t}\Big[\sum_{j\in\mathcal{B}}\widetilde{F}_{b_j}(\overline{q}_j^b,\overline{b}_j)-\sum_{i\in\mathcal{S}}\widetilde{F}_{s_i}(\overline{q}_i^s,\overline{s}_i)\Big]+C_1':=C_2',
\end{aligned}
$$

where the last inequality follows from Step 3. ■

Summarizing Step 1-4, we conclude the claim of this result.

Proof of Lemma EC.7. We prove the two claims of this result separately. Given that the supporting lemma is located in Step 3 in the proof of Theorem 1, we would borrow some observations from Step 2 in the proof of Theorem 1 in the proof arguments below.

Claim 1. For $i\in\mathcal{N}^+$, when $n_i(1)\ge\overline{n}_i$, we further consider the following two cases: (1) $m(1)\ge 1$; (2) $m(1)<1$.

(1) When $n_i(1)\ge\overline{n}_i$ and $m(1)\ge 1$, we first show that $n_i(t)\ge\overline{n}_i$ for any $t\in\{1,...,T\}$. Given that $n_i(1)\ge\overline{n}_i$ for any $i\in\mathcal{N}^+$, we assume for induction purpose that $n_i(t)\ge\overline{n}_i$, and then we can establish that

$$
n_i(t+1)\overset{(a)}{=}\mathcal{G}_i(n(t),\overline{q}_i m(t))\overset{(b)}{\ge}\mathcal{G}_i(n(t),\overline{q}_i)\ge\mathcal{G}_i(\overline{n}_i,\overline{q}_i)\overset{(c)}{=}\overline{n}_i,
$$

where (a) follows from Algorithm 1; (b) follows from our observations in Step 2 Case 1 in the proof of Theorem 1 that if $m(1)>1$, then we have $m(1)>m(2)>...>m(T)>1$, and in Step 2 Case 3 that if $m(1)=1$, then we have $m(1)=m(2)=...=m(T)=1$; (c) follows directly from Lemma 1(ii). By induction, with $n_i(1)\ge\overline{n}_i$ and $m(1)\ge 1$, we obtain that $n_i(t)\ge\overline{n}_i$ for any $t\in\{1,...,T\}$.

To proceed, we further notice that for any $t\in\{1,...,T\}$,

$$
\begin{aligned}
n_i(t)-\overline{n}_i\overset{(d)}{=}&\mathcal{G}_i(n_i(t-1),\overline{q}_i m(t-1))-\mathcal{G}_i(\overline{n}_i,\overline{q}_i)\\
=&\mathcal{G}_i(n_i(t-1),\overline{q}_i m(t-1))-\mathcal{G}_i(n_i(t-1),\overline{q}_i)+\mathcal{G}_i(n_i(t-1),\overline{q}_i)-\mathcal{G}_i(\overline{n}_i,\overline{q}_i)\\
\overset{(e)}{\le}&\overline{q}_i(m(t-1)-1)(\mathcal{G}_i)_2'(n_i(t-1),\overline{q}_i)+(\overline{n}_i(t-1)-\overline{n}_i)(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i),
\end{aligned}
$$

where (d) follows from Algorithm 1 and Lemma 1(ii); (e) follows from the concavity of $\mathcal{G}_i(\cdot,\cdot)$ by Assumption 1. Since $n_i(t) \geq \overline{n}_i$, the LHS of the inequality for (e) is nonnegative, and we can take the absolute values and obtain the following inequality:

$$
\begin{aligned}
\sum_{t=2}^{T} \left| n_i(t) - \overline{n}_i \right| &\leq \sum_{t=2}^{T} \left[\left| \overline{q}_i (m(t-1)-1)(\mathcal{G}_i)_2'(n_i(t-1), \overline{q}_i) \right| + \left| (\overline{n}_i(t-1) - \overline{n}_i)(\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i) \right| \right] \\
&\overset{(f)}{\leq} \sum_{t=2}^{T} \left| \overline{q}_i (m(t-1)-1) \right| + \sum_{t=2}^{T} \left| (\overline{n}_i(t-1) - \overline{n}_i)(\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i) \right| \\
&\leq \overline{q}_i \sum_{t=2}^{T} \gamma^{t-2} \left| (m(1)-1) \right| + \sum_{t=2}^{T} \left| (\overline{n}_i(t-1) - \overline{n}_i)(\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i) \right|.
\end{aligned}
$$

For (f), we show that $(\mathcal{G}_i)_2'(n_i(t-1), \overline{q}_i) < 1$. Define $y(n) := \mathcal{G}(n, n\frac{\overline{q}_i}{n_i(t-1)})$, by the mean value theorem, there must exist a $\hat{n} \in (0, n_i(t-1))$ such that $y'(\hat{n}) = \frac{y(n_i(t-1)) - y(0)}{n_i(t-1) - 0} = \frac{\mathcal{G}(n_i(t-1), \overline{q}_i)}{n_i(t-1)} < 1$ for $n_i(t-1) > \overline{n}_i$. Therefore, given the concavity of $y(n)$, $y'(n_i(t-1)) < 1$, which suggest that $(\mathcal{G}_i)_1'(n_i(t-1), \overline{q}_i) + (\mathcal{G}_i)_2'(n_i(t-1), \overline{q}_i)\frac{\overline{q}_i}{n_i(t-1)} < 1$, which suggest that $(\mathcal{G}_i)_2'(n_i(t-1), \overline{q}_i) < 1$. Then

$$
\begin{aligned}
\sum_{t=1}^{T} \left| n_i(t) - \overline{n}_i \right| &\leq \frac{\overline{q}_i \sum_{t=2}^{T} \gamma^{t-2} \left| (m(1)-1) \right|}{1 - (\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i)} - \frac{(\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i)}{1 - (\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i)} \times \left| n_i(T) - \overline{n}_i \right| + \frac{\left| n_i(1) - \overline{n}_i \right|}{1 - (\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i)} \\
&\leq \frac{\overline{q}_i \sum_{t=2}^{T} \gamma^{t-2} \left| (m(1)-1) \right|}{1 - (\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i)} + \frac{\left| n_i(1) - \overline{n}_i \right|}{1 - (\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i)}.
\end{aligned}
$$

Therefore, $\lim_{T \to \infty} \sum_{t=1}^{T} \left| n_i(t) - \overline{n}_i \right| \leq \frac{\overline{q}_i \left| (m(1)-1) \right|}{(1 - (\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\left| n_i(1) - \overline{n}_i \right|}{1 - (\mathcal{G}_i)_1'}$. In the end, we define the positive constant

$$
C_{n_i} := \frac{\overline{q}_i \left| (m(1)-1) \right|}{(1 - (\mathcal{G}_i)_1'(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\left| n_i(1) - \overline{n}_i \right|}{1 - (\mathcal{G}_i)_1'},
$$

which allows us to obtain the desired result.

(2) Given that $m(1) < 1$ and that $n_i(1) \geq \overline{n}_i$, we consider two cases. In the first case, we consider the scenario where there exists a $\widetilde{t} \in \{2, ..., T\}$ such that $n_i(\widetilde{t}) \geq \overline{n}_i$. In the second case, we consider the scenario where $n_i(t) \geq \overline{n}_i$ for all $t \in \{1, \ldots, T\}$.

In the first case, given $\widetilde{t} \in \{2, \ldots, T\}$ such that $n_i(\widetilde{t}) < \overline{n}_i$, we want to show that $n_i(t) < \overline{n}_i$ for $t \geq \widetilde{t}$. We prove the claim by induction. Given that $n_i(\widetilde{t}) < \overline{n}_i$, for any $t \geq \widetilde{t}$, suppose towards an induction purpose that $n_i(t) < \overline{n}_i$, and we can establish that

$$
n_i(t+1) \overset{(a)}{=} \mathcal{G}_i(n(t), \overline{q}_i m(t)) \overset{(b)}{<} \mathcal{G}_i(n(t), \overline{q}_i) < \mathcal{G}_i(\overline{n}_i, \overline{q}_i) \overset{(c)}{=} \overline{n}_i, \tag{EC.22}
$$

where (a) follows from Algorithm 1; (b) follows from the condition that $\mathcal{G}_i(q)$ strictly increases in $q \geq 0$ and from the observation in Step 2.1 from the proofs of Theorem 1 that if $m(1) < 1$, then $m(1) < m(2) < ... < m(T) < 1$; (c) follows directly from Lemma 1(ii). Therefore, we obtain that if there exists a $\widetilde{t} \in \{2, \dots, T\}$ such that $n_i(\widetilde{t}) < \overline{n}_i$, we have $n_i(t) < \overline{n}_i$ for $t \geq \widetilde{t}$. We then show that $\widetilde{t}$ is independent of T. Given the definition of $\widetilde{t}$ as the first time that $n_i(t) < \overline{n}_i$, it is equivalent to show that the value of $n_i(t)$ for $0 \leq t \leq \widetilde{t}$ is independent of T. This is true as given $n_i(1)$ and $m(1)$, for $t \in \{1, ..., \widetilde{t}-1\}$, $n_i(t+1) = \mathcal{G}_i(n(t), \overline{q}_i m(t))$, where $m(t) = \min_{i' \in \mathcal{N}^+} \{\frac{n_{i'}(t)}{\overline{n}_{i'}}\}$ is independent of T for $1 \leq t \leq \widetilde{t}-1$.

The observations above allow us to deduce that in the first case,

$$
\begin{aligned}
\lim_{T\to\infty} \sum_{t=1}^{T} \Big| n_i(t) - \overline{n}_i \Big| &= \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \lim_{T\to\infty} \sum_{t=\widetilde{t}}^{T} \Big| n_i(t) - \overline{n}_i \Big| \\
&\overset{(d)}{=} \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \lim_{T\to\infty} \sum_{t=\widetilde{t}}^{T} \Big| m(t) - 1 \Big| \overset{(e)}{\leq} \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \lim_{T\to\infty} \sum_{t=\widetilde{t}}^{T} \Big| m(\widetilde{t}) - \\
&= \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \Big| m(\widetilde{t}) - 1 \Big| \frac{1}{1-\gamma} \overset{(f)}{\leq} \sum_{t=1}^{\widetilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \Big| m(1) - 1 \Big| \frac{1}{1-\gamma} \\
&\overset{(g)}{\leq} \frac{\overline{q}_i \Big| (m(1) - 1) \Big|}{(1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\Big| n_i(1) - \overline{n}_i \Big|}{1 - (\mathcal{G}_i)'_1} + \overline{n}_i \Big| m(1) - 1 \Big| \frac{1}{1-\gamma},
\end{aligned}
$$

where (d) follows from the definition of $m(t)$, (e) follows from Step 2, and (f) follows from $m(1) < m(2) < ... < m(T) < 1$ if $m(1) < 1$ in Step 2.1; (g) follows from the Case (1). Then let $C_{n_i} = \frac{\overline{q}_i \Big| (m(1)-1) \Big|}{(1-(\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\Big| n_i(1) - \overline{n}_i \Big|}{1-(\mathcal{G}_i)'_1} + \overline{n}_i \Big| m(1) - 1 \Big| \frac{1}{1-\gamma}$, we obtain the desired result.

In the second case, if $n_i(t) \geq \overline{n}_i$ for all $t \in \{1, ..., T\}$, we can apply the same upper bound as in Case (1) above under Claim 1.

Claim 2. To establish the second claim of this result, when $n_i(1) \leq \overline{n}_i$ and $m(1) < 1$, by applying the same induction arguments as in (EC.22) from the previous claim, we can establish that $n_i(t) \leq \overline{n}_i$ for any $t \in \{1, \dots, T\}$.

Summarizing the arguments above, we complete the proofs of the two claims in this result. ■

Proof of Proposition 2. Claim (i). Let $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ denote the commission in period $t \in \{1, ..., T\}$ when $w = 0$; in addition, given the optimal solution to AVG in (5) $(\overline{x}, \overline{q}^s, \overline{q}^b, \overline{s}, \overline{b})$, we define $\overline{r}_{ij} := F_{b_j}^{-1}(1-\frac{\overline{q}_j^b}{\overline{b}_j}) - F_{s_i}^{-1}(\frac{\overline{q}_i^s}{\overline{s}_i})$, which can be seen as the total commission for each transaction at AVG in (5). We will respectively show that $\overline{r}_{ij} \geq 0$ and $r_i^s(t) + r_j^b(t) \geq \overline{r}_{ij}$ for any (i,j) with $x_{ij} > 0$ in Step 1.2 and Step 1.3. Before that, we need to establish an auxiliary result in Step 1.1.

Abusing some notations, given any $q > 0$, we use $\overline{s}_i(q)$ to denote the population level at which the transition remains stable, i.e., $\overline{s}_i(q) := \{s > 0 | s = \mathcal{G}_i^s(s,q)\}$. Given that $\mathcal{G}_i^s(s_i,q)$ is increasing and strictly concave in $s \in [q,\infty]$ for any given $q > 0$ and $\mathcal{G}_i^s(0,q) > 0$ (see Assumption 1), it can be easily shown that $\overline{s}_i(q)$ is well-defined. Similarly, we define $\overline{b}_j(q) := \{b > 0 | b = \mathcal{G}_j^b(b,q)\}$.

Step 1.1: show that for any $i \in \mathcal{S}, j \in \mathcal{B}$, if $0 < q_1 < q_2$, then $\frac{q_1}{\overline{s}_i(q_1)} \leq \frac{q_2}{\overline{s}_i(q_2)}$ and $\frac{q_1}{\overline{b}_j(q_1)} \leq \frac{q_2}{\overline{b}_j(q_2)}$. Suppose towards a contradiction that $\frac{q_1}{\overline{s}_i(q_1)} > \frac{q_2}{\overline{s}_i(q_2)}$, then we have

$$\begin{aligned}\overline{s}_i(q_1) &= \frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}\overline{s}_i(q_2) \overset{(a)}{=} \frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}\mathcal{G}_i^s\Big(q_2, \overline{s}_i(q_2)\Big) \\ &\overset{(b)}{<} \mathcal{G}_i^s\Big(\frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}q_2, \frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}\overline{s}_i(q_2)\Big) \overset{(c)}{<} \mathcal{G}_i^s\Big(\frac{q_1}{q_2}q_2, \frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}\overline{s}_i(q_2)\Big) = \mathcal{G}_i^s\Big(q_1, \overline{s}_i(q_1)\Big),\end{aligned}$$

where (a) follows from the definition of $\overline{s}_i(q_2)$; (b) follows from the strict concavity of $\mathcal{G}_i^s$ (see (EC.20)); (c) follows from $\frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)} = \frac{q_1}{q_2}\frac{\frac{q_2}{\overline{s}_2(q_2)}}{\frac{q_1}{\overline{s}_1(q_1)}} < \frac{q_1}{q_2}$ when $\frac{q_1}{\overline{s}_i(q_1)} > \frac{q_2}{\overline{s}_i(q_2)}$, and $\mathcal{G}_i^s(s,q)$ is increasing in (s,q) for $0 \leq q \leq s$ (see Assumption 1). As a result, we have $\overline{s}_i(q_1) < \mathcal{G}_i^s\Big(q_1, \overline{s}_i(q_1)\Big)$, which contradicts to the definition of $\overline{s}_i(q_1)$. Therefore, if $0 < q_1 < q_2$, then $\frac{q_1}{\overline{s}_i(q_1)} \leq \frac{q_2}{\overline{s}_i(q_2)}$. The same argument holds for the buyer side.

Step 1.2: $\overline{r}_{ij} \geq 0$ for (i,j) with $\overline{x}_{ij} > 0$. Suppose towards a contradiction that for the optimal solution to AVG in (5) $\overline{\boldsymbol{x}}$, there exists (i_0,j_0) with $\overline{x}_{i_0j_0} > 0$ such that $\overline{r}_{i_0,j_0} < 0$; based on Lemma 1, we can plug in constraints (5b)(5c)(5d)(5e) and have $\overline{r}_{i_0,j_0} = F_{b_{j_0}}^{-1}(1 - \frac{\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0}}{\overline{b}_{j_0}(\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0})}) - F_{s_{i_0}}^{-1}(\frac{\sum_{j' \in N_E(i_0)} \overline{x}_{i_0j'}}{\overline{s}_{i_0}(\sum_{j' \in N_E(i_0)} \overline{x}_{i_0j'})}) < 0$. Then we construct another feasible solution $\tilde{\boldsymbol{x}}$ in the following way: let $\tilde{x}_{ij} := \overline{x}_{ij}$ for $(i,j) \neq (i_0,j_0)$ and $\tilde{x}_{i_0j_0} := 0$. We can show that $\tilde{\boldsymbol{x}}$ leads to a higher objective value of AVG:

$$\begin{aligned}\overline{\mathcal{R}}(\overline{\boldsymbol{x}}) &\overset{(a)}{=} \sum_{(i,j)\in E} \overline{x}_{ij}\Big(F_{b_j}^{-1}(1 - \frac{\sum_{i' \in N_E(j)} \overline{x}_{i'j}}{\overline{b}_j(\sum_{i' \in N_E(j)} \overline{x}_{i'j})}) - F_{s_i}^{-1}(\frac{\sum_{j' \in N_E(i)} \overline{x}_{ij'}}{\overline{s}_i(\sum_{j' \in N_E(i)} \overline{x}_{ij'})})\Big) \\ &= \overline{x}_{i_0j_0}\Big(F_{b_{j_0}}^{-1}(1 - \frac{\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0}}{\overline{b}_{j_0}(\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0})}) - F_{s_{i_0}}^{-1}(\frac{\sum_{j' \in N_E(i_0)} \overline{x}_{i_0j'}}{\overline{s}_{i_0}(\sum_{j' \in N_E(i_0)} \overline{x}_{i_0j'})})\Big) \\ &\quad + \sum_{i \in N_E(j_0)} \overline{x}_{ij_0}\Big(F_{b_{j_0}}^{-1}(1 - \frac{\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0}}{\overline{b}_{j_0}(\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0})}) - F_{s_i}^{-1}(\frac{\sum_{j' \in N_E(i)} \overline{x}_{ij'}}{\overline{s}_i(\sum_{j' \in N_E(i)} \overline{x}_{ij'})})\Big)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j\in N_E(i_0)}\overline{x}_{i_0j}\Big(F_{b_j}^{-1}(1-\frac{\sum_{i'\in N_E(j)}\overline{x}_{i'j}}{\bar{b}_j(\sum_{i'\in N_E(j)}\overline{x}_{i'j})})-F_{s_{i_0}}^{-1}(\frac{\sum_{j'\in N_E(i_0)}\overline{x}_{i_0j'}}{\bar{s}_{i_0}(\sum_{j'\in N_E(i_0)}\overline{x}_{i_0j'})})\Big)\\
&+\sum_{(i,j)\in E,i\neq i_0,j\neq j_0}\overline{x}_{ij}\Big(F_{b_j}^{-1}(1-\frac{\sum_{i'\in N_E(j)}\overline{x}_{i'j}}{\bar{b}_j(\sum_{i'\in N_E(j)}\overline{x}_{i'j})})-F_{s_i}^{-1}(\frac{\sum_{j'\in N_E(i)}\overline{x}_{ij'}}{\bar{s}_i(\sum_{j'\in N_E(i)}\overline{x}_{ij'})})\Big)\\
&\overset{(b)}{<}\sum_{i\in N_E(j_0)}\tilde{x}_{ij_0}\Big(F_{b_{j_0}}^{-1}(1-\frac{\sum_{i'\in N_E(j_0)}\tilde{x}_{i'j_0}}{\bar{b}_{j_0}(\sum_{i'\in N_E(j_0)}\tilde{x}_{i'j_0})})-F_{s_i}^{-1}(\frac{\sum_{j'\in N_E(i)}\tilde{x}_{ij'}}{\bar{s}_i(\sum_{j'\in N_E(i)}\tilde{x}_{ij'})})\Big)\\
&+\sum_{j\in N_E(i_0)}\tilde{x}_{i_0j}\Big(F_{b_j}^{-1}(1-\frac{\sum_{i'\in N_E(j)}\tilde{x}_{i'j}}{\bar{b}_j(\sum_{i'\in N_E(j)}\tilde{x}_{i'j})})-F_{s_{i_0}}^{-1}(\frac{\sum_{j'\in N_E(i_0)}\tilde{x}_{i_0j'}}{\bar{s}_{i_0}(\sum_{j'\in N_E(i_0)}\tilde{x}_{i_0j'})})\Big)\\
&+\sum_{(i,j)\in E,i\neq i_0,j\neq j_0}\tilde{x}_{ij}\Big(F_{b_j}^{-1}(1-\frac{\sum_{i'\in N_E(j)}\tilde{x}_{i'j}}{\bar{b}_j(\sum_{i'\in N_E(j)}\tilde{x}_{i'j})})-F_{s_i}^{-1}(\frac{\sum_{j'\in N_E(i)}\tilde{x}_{ij'}}{\bar{s}_i(\sum_{j'\in N_E(i)}\tilde{x}_{ij'})})\Big)=\overline{\mathcal{R}}(\tilde{\boldsymbol{x}}),
\end{aligned}
$$

where in (a), we plug in the constraint (5b)(5c)(5d)(5e) into the objective function, where the inequalities in (5d)(5e) hold based on Lemma 1; (b) follows from $\Big(F_{b_{j_0}}^{-1}(1-\frac{\sum_{i'\in N_E(j_0)}\overline{x}_{i'j_0}}{\bar{b}_{j_0}(\sum_{i'\in N_E(j_0)}\overline{x}_{i'j_0})})-F_{s_{i_0}}^{-1}(\frac{\sum_{j'\in N_E(i_0)}\overline{x}_{i_0j'}}{\bar{s}_{i_0}(\sum_{j'\in N_E(i_0)}\overline{x}_{i_0j'})})\Big)<0$ and furthermore, $\sum_{i\in N_E(j)}\overline{x}_{ij}\leq\sum_{i\in N_E(j)}\tilde{x}_{ij}$ for any $j\in\mathcal{B}$ and $\sum_{j\in N_E(i)}\overline{x}_{ij}\leq\sum_{j\in N_E(i)}\tilde{x}_{ij}$ for any $i\in\mathcal{S}$ based on the construction of $\tilde{\boldsymbol{x}}$ and the result in Step 1. As a result, $\overline{\mathcal{R}}(\overline{\boldsymbol{x}})<\overline{\mathcal{R}}(\tilde{\boldsymbol{x}})$, which contradicts to the optimality of $\overline{\boldsymbol{x}}$. Therefore, $\overline{r}_{ij}\geq 0$ for (i,j) with $\overline{x}_{ij}>0$.

<u>Step 1.3: $r_i^s(t)+r_j^b(t)\geq\overline{r}_{ij}$ for (i,j) with $x_{ij}>0$.</u> When $w=0$,

$$
\begin{aligned}
r_i^s(t)+r_j^b(t)=&F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})\\
&\overset{(a)}{=}F_{b_j}^{-1}(1-\frac{\overline{q}_j^b\hat{m}(t)}{b_j(t)})-F_{s_i}^{-1}(\frac{\overline{q}_i^s\hat{m}(t)}{s_i(t)})\overset{(b)}{\geq}F_{b_j}^{-1}(1-\frac{\overline{q}_j^b}{\bar{b}_j})-F_{s_i}^{-1}(\frac{\overline{q}_i^s}{\bar{s}_i})=\overline{r}_{ij}.
\end{aligned}
$$

where (a) follows from the policy rule, (b) holds because when $w=0$, $\hat{m}(t)=m(t)\leq\frac{b_j(t)}{\bar{b}_j}$ for any $j\in\mathcal{B}$ and $\hat{m}(t)=m(t)\leq\frac{s_i(t)}{\bar{s}_i}$ for any $i\in\mathcal{S}$ by definition.

In summary of Step 1.2 and 1.3, $r_i^s(t)+r_j^b(t)\geq\overline{r}_{ij}>0$ for (i,j) with $x_{ij}>0$.

<u>Claim (ii).</u> We define $\tilde{t}$ such that $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}(t))}{\overline{n}_i}\right\}\leq 1$ for $t\in\{1,...,\tilde{t}\}$ and $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(\tilde{t}),\overline{q}_i\hat{m}(\tilde{t}))}{\overline{n}_i}\right\}>1$. Then for $t\in\{\tilde{t}+1,...,T\}$, the update rule when $w>0$ is the same as that when $w=0$, and we have already shown that $r_i^s(t)+r_j^b(t)>0$ for (i,j) with $x_{ij}>0$ (see Step 1). We next establish the result for $t\in\{1,...,\tilde{t}-1\}$.

Define $\kappa_{ij}:=\min\{\frac{\overline{q}_j^b}{\max_{l\in\mathcal{N}}\overline{q}_l},\frac{\overline{q}_i^s}{\max_{l\in\mathcal{N}}\overline{q}_l}\}$ for $(i,j)\in E$ and define $z_{ij}:=\Big\{z\in(\kappa_{ij},+\infty)|F_{b_j}^{-1}(1-\frac{\kappa_{ij}}{z})-F_{s_i}^{-1}(\frac{\kappa_{ij}}{z})=0\Big\}$. Given that $F_{b_j}^{-1}(1-\frac{\kappa_{ij}}{z})-F_{s_i}^{-1}(\frac{\kappa_{ij}}{z})=-\overline{v}_{s_i}<0$ when $z=\kappa_{ij}$ and

$F_{b_j}^{-1}(1-\frac{\kappa_{ij}}{z})-F_{s_i}^{-1}(\frac{\kappa_{ij}}{z})=\overline{v}_{b_j}>0$ when $z=\infty$, and $F_{b_j}^{-1}(1-\frac{\kappa_{ij}}{z})-F_{s_i}^{-1}(\frac{\kappa_{ij}}{z})$ is increasing in z, we know z_{ij} is well-defined. Then

$$\begin{aligned}
r_i^s(t)+r_j^b(t)=&F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})\\
\overset{(a)}{\leq}&F_{b_j}^{-1}(1-w\frac{\overline{q}_j^b}{b_j(t)}\min_{l\in\mathcal{N}^+}\{\frac{n_l(t)}{\overline{q}_l}\})-F_{s_i}^{-1}(w\frac{\overline{q}_i^s}{s_i(t)}\min_{l\in\mathcal{N}^+}\{\frac{n_l(t)}{\overline{q}_l}\})\\
\leq&F_{b_j}^{-1}(1-w\frac{\overline{q}_j^b}{b_j(t)}\min_{l\in\mathcal{N}^+}\{\frac{n_l(t)}{\max_{l'\in\mathcal{N}}\overline{q}_{l'}}\})-F_{s_i}^{-1}(w\frac{\overline{q}_i^s}{s_i(t)}\min_{l\in\mathcal{N}^+}\{\frac{n_l(t)}{\max_{l'\in\mathcal{N}}\overline{q}_{l'}}\})\\
=&F_{b_j}^{-1}(1-w\frac{\overline{q}_j^b}{\max_{l'\in\mathcal{N}}\overline{q}_{l'}}\frac{\min_{l\in\mathcal{N}^+}n_l(t)}{b_j(t)})-F_{s_i}^{-1}(w\frac{\overline{q}_i^s}{\max_{l'\in\mathcal{N}}\overline{q}_{l'}}\frac{\min_{l\in\mathcal{N}^+}n_l(t)}{s_i(t)})\\
\overset{(b)}{<}&F_{b_j}^{-1}(1-\frac{w\kappa_{ij}}{wz_{ij}})-F_{s_i}^{-1}(\frac{w\kappa_{ij}}{wz_{ij}})\overset{(c)}{=}0,
\end{aligned}$$

where (a) follows the definition of $\hat{m}(t)$; (b) follows from the definition of κ_{ij} and the condition that $\max\{s_i(t),b_j(t)\}<wz_{i,j}\min_{i\in\mathcal{N}^+}n_i(t)$; (c) follows from the definition of z_{ij}. ■

Proof of Corollary 1. Under TRP,

$$Y_j^b(t):=r_j^b(t)+\min_{i\in N_E(j)}p_i^s(t)=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})=F_{b_j}^{-1}(1-\frac{\overline{q}_j^b\hat{m}(t)}{b_j(t)})=F_{b_j}^{-1}(1-\hat{m}(t)\frac{\frac{\overline{q}_j^b}{\overline{b}_j}}{\frac{b_j(t)}{\overline{b}_j}}).$$

Since $\hat{m}(t)$ are the same across types and F_{b_j} are assumed to be homogeneous across types, the $Y_j^b(t)$ only differ when $\frac{\frac{\overline{q}_j^b}{\overline{b}_j}}{\frac{b_j(t)}{\overline{b}_j}}$ are different. Similarly, we can show that $I_i^s(t)$ depends only on $\frac{\frac{\overline{q}_i}{\overline{s}_i}}{\frac{s_i(t)}{\overline{s}_i}}$.

For any $t\in\{1,...,T\}$, for any positive constant p, by constructing $p_i(t)=p$ and $r_i^s(t)=p-F_{s_i}^{-1}(1-\frac{q_i^s(t)}{s_i(t)})$ for any $i\in\mathcal{S}$; $r_j^b(t)=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-p$ for any $j\in\mathcal{B}$, we obtain a feasible commission (see Lemma EC.1). For this solution, we can see that $r_j^b(t)$ depends only on and deceases in $\frac{\frac{\overline{q}_j^b}{\overline{b}_j}}{\frac{b_j(t)}{\overline{b}_j}}$, while $r_i^s(t)$ depends only on and decreases in $\frac{\frac{\overline{q}_i}{\overline{s}_i}}{\frac{s_i(t)}{\overline{s}_i}}$. ■

EC.2.3. Proof of Results for MP

Proof of Proposition 3. We denote by $(\boldsymbol{r^{MP}}(t),\boldsymbol{p^{MP}}(t),\boldsymbol{q}^{s,MP}(t),\boldsymbol{q}^{b,MP}(t),\boldsymbol{x}^{MP}(t))$ the optimal solution to the optimization problem for the MP in Definition EC.1. We consider the following problem instance: Consider a simple network in which there is only one buyer type and one seller type with initial population $s(1)=b(1)>0$. Given the commissions $\boldsymbol{r^{MP}}(t)$ induced by the MP, we let the populations for the next period be

$(\boldsymbol{s}^{MP}(t+1), \boldsymbol{b}^{MP}(t+1))$ is updated by $s^{MP}(t+1) = \alpha s^{MP}(t) + \beta(q^{s,MP}(t))^{\xi}$ and $b^{MP}(t+1) = \alpha b^{MP}(t) + \beta(q^{b,MP}(t))^{\xi}$, where we assume $\beta > 0$ and $0 < \xi < 1$ so that the Assumption 1 holds. In addition, we let $F_s(\cdot)$ and $F_b(\cdot)$ be the distribution functions over $[0,1]$ from the uniform distribution.

We establish two claims to complete the proof.

Claim 1: $\lim_{t\to\infty} R^{MP}(t)$ exists . We divide the proof arguments into the following steps. In Step 1.1, we show that if a steady state induced by the MP exists, we characterize the properties of the steady state. In Step 1.2, we show that the populations converge to the steady state under the platform's MP. For simplicity of notations, we let $\mathcal{R}^{MP}(t)$ denote the profit in period t under the MP.

Step 1.1. Characterize the quantity $\overline{q}^{MP}$ and the profit $\overline{R}^{MP}$ in a steady state. We first define a steady state as such that the populations and transaction quantities remain unchanged after the population transition in each period. Given the definition of a steady state, under the platform's myopic policy, the steady-state population vector $(\overline{s}^{MP}, \overline{b}^{MP}, \overline{q}^{MP})$ should satisfy the following three conditions:

$$\overline{q}^{MP} = \operatorname*{arg\,max}_{0 \le q \le \min\{\overline{s}^{MP}, \overline{b}^{MP}\}} \left[\left(1 - \frac{q}{\overline{s}^{MP}} - \frac{q}{\overline{b}^{MP}}\right) q \right], \tag{EC.23a}$$

$$\overline{s}^{MP} = \alpha \overline{s}^{MP} + \beta(\overline{q}^{MP})^{\xi}, \tag{EC.23b}$$

$$\overline{b}^{MP} = \alpha \overline{b}^{MP} + \beta(\overline{q}^{MP})^{\xi}. \tag{EC.23c}$$

Condition (EC.23a) ensures that given the population in each period $(\overline{s}^{MP}, \overline{b}^{MP})$, the platform's commissions r could induce the equilibrium quantity $\overline{q}^{MP}$ to maximize its profit in the current period (see Corollary EC.1 for the formulation of optimization problem); (EC.23b) and (EC.23c) ensure that the population vector $(\overline{s}^{MP}, \overline{b}^{MP})$ remains unchanged after the update in each period.

For Problem (EC.23a), from the first-order-condition $\frac{\partial}{\partial q}\left[\left(1 - \frac{q}{\overline{s}^{MP}} - \frac{q}{\overline{b}^{MP}}\right) q\right] = 0$, we can obtain that $\overline{q}^{MP} = \frac{\overline{s}^{MP}\overline{b}^{MP}}{2\overline{s}^{MP} + 2\overline{b}^{MP}}$, which falls in the region $(0, \min\{\overline{s}^{MP}, \overline{b}^{MP}\})$. Thus, the optimal solution to (EC.23a) is an interior point. Together with the equations in (EC.23b)-(EC.23c), we obtain that

$$\overline{q}^{MP} = \left(\frac{k}{4}\right)^{\frac{1}{1-\xi}}, \overline{b}^{MP} = k\left(\frac{k}{4}\right)^{\frac{\xi}{1-\xi}}, \overline{s}^{MP} = k\left(\frac{k}{4}\right)^{\frac{\xi}{1-\xi}}.$$

where we let $k = \frac{\beta}{1-\alpha}$ for simplicity of notations. This allows us to show that the profit induced by the platform's MP satisfies that

$$\overline{\mathcal{R}}^{MP} \;=\; \Big(1 - \frac{\overline{q}^{MP}}{\overline{s}^{MP}} - \frac{\overline{q}^{MP}}{\overline{b}^{MP}}\Big)\overline{q}^{MP} \;=\; \frac{1}{2}\Big(\frac{k}{4}\Big)^{\frac{1}{1-\xi}}.$$

Step 1.2: For the seller side, show that there exists a $\gamma \in (0,1)$ such that $|\overline{s}^{MP} - s^{MP}(t+1)| \le \gamma|(\overline{s}^{MP} -$

Next, we establish the convergence of the platform's MP. Without loss of generality, we prove the convergence on the seller side, and notice that the same argument would hold for the buyer side as well.

Since we have $s^{MP}(1) = b^{MP}(1)$ in the problem instance, and in each iteration we have $s^{MP}(t+1) = \alpha s^{MP}(t) + \beta(q^{MP}(t))^{\xi}$ and $b^{MP}(t+1) = \alpha b^{MP}(t) + \beta(q^{MP}(t))^{\xi}$, we obtain that $s^{MP}(t) = b^{MP}(t)$ for any $t \in \{1, ..., T\}$. Based on this observation, we can obtain that

$$\begin{aligned} q^{MP}(t) \;&=\; \operatorname*{arg\,max}_{0<q<\min\{s^{MP}(t), b^{MP}(t)\}} \Big\{\Big(1 - \frac{q}{s^{MP}(t)} - \frac{q}{b^{MP}(t)}\Big)q\Big\} \\ &=\; \operatorname*{arg\,max}_{0<q<s^{MP}(t)} \Big\{\Big(1 - \frac{q}{s^{MP}(t)} - \frac{q}{s^{MP}(t)}\Big)q\Big\} \;=\; \frac{s^{MP}(t)}{4}. \end{aligned}$$

From the optimal solution $q^{MP}(t)$ above, we obtain that

$$s^{MP}(t+1) = \alpha s^{MP}(t) + \beta(q^{MP}(t))^{\xi} = \alpha s^{MP}(t) + \beta\Big(\frac{s^{MP}(t)}{4}\Big)^{\xi}.$$

Abusing some notations, we let $g_s(s) := \alpha s + \beta(\frac{s}{4})^{\xi}$ for any $s \ge 0$ such that $g_s(\overline{s}^{MP}) = \overline{s}^{MP}$ based on the condition in (EC.23b). To proceed, we consider the following two cases that $s^{MP}(1) \ge \overline{s}^{MP}$ and $s^{MP}(1) < \overline{s}^{MP}$:

(1) When $s^{MP}(1) \ge \overline{s}^{MP}$, we want to show that $s^{MP}(t) \ge \overline{s}^{MP}$ for $t \in \{1, .., T\}$. By induction, if $s^{MP}(t) \ge \overline{s}^{MP}$, we have $s^{MP}(t+1) = g_s(s^{MP}(t)) \ge g_s(\overline{s}^{MP}) = \overline{s}^{MP}$, where the inequality follows from the fact that $g_s(\cdot)$ is an increasing function. Since $s^{MP}(1) \ge \overline{s}^{MP}$, we obtain that $s^{MP}(t) \ge \overline{s}^{MP}$ for $t \in \{1, .., T\}$.

Based on the observation above, we can establish that

$$\Big|s^{MP}(t+1) - \overline{s}^{MP}\Big| \;=\; \Big|g_s(s^{MP}(t)) - \overline{s}^{MP}\Big| \;\overset{(a)}{=}\; g_s(s^{MP}(t)) - g_s(\overline{s}^{MP}) \;\overset{(b)}{\le}\; \Big|s^{MP}(t) - \overline{s}^{MP}\Big| g_s'(\overline{s}^{MP}), \tag{EC.24}$$

where (a) follows from the observation that $s^{MP}(t) \ge \overline{s}^{MP}$ for $t \in \{1, .., T\}$ in this case; (b) follows from the condion that g_s is concave given that $g_s(s) = \alpha s + \beta(\frac{s}{4})^{\xi}$ with

$a \in (0,1)$. Moreover, we have $g_s'(\overline{s}^{MP}) < 1$ given that $g_s(0) = 0$ and $g_s(\overline{s}^{MP}) = \overline{s}^{MP}$, and so by the mean value theorem, there exists a $\widetilde{s} \in (0, \overline{s}^{MP})$ such that $g_s'(\widetilde{s}) = \frac{g_s(\overline{s}^{MP}) - g_s(0)}{\overline{s}^{MP} - 0} = 1$. Since $g_s(\cdot)$ is concave, we have that $g_s'(\overline{s}^{MP}) < g_s'(\widetilde{s}) = 1$ given that $\overline{s}^{MP} > \widetilde{s}$. By letting $\gamma_1 := g_s'(\overline{s}^{MP})$, we establish that there exists $\gamma_1 \in (0,1)$ such that $|\overline{s}^{MP} - s^{MP}(t+1)| \le \gamma_1 |(\overline{s}^{MP} - s^{MP}(t))|$ for $t \in \{1, ..., T-1\}$ if $s^{MP}(1) \ge \overline{s}^{MP}$. From the definition of $g_s(\cdot)$ and $\overline{s}^{MP}$, we see that γ_1 is independent of T.

(2) When $s^{MP}(1) < \overline{s}^{MP}$, we want to show that $s^{MP}(t) < \overline{s}^{MP}$ for $t \in \{1, .., T\}$. If $s^{MP}(t) < \overline{s}^{MP}$, we have $s^{MP}(t+1) = g_s(s^{MP}(t)) < g_s(\overline{s}^{MP}) = \overline{s}^{MP}$, where the inequality follows from that $g_s(\cdot)$ is an increasing function given that $s^{MP}(t) < \overline{s}^{MP}$. Since $s^{MP}(1) < \overline{s}^{MP}$, by induction we obtain that $s^{MP}(t) < \overline{s}^{MP}$ for any $t \in \{1, .., T\}$.

Then, we can establish that

$$\frac{\overline{s}^{MP} - g_s(s^{MP}(t))}{\overline{s}^{MP} - s^{MP}(t)} \overset{(c)}{<} \frac{\overline{s}^{MP} - g_s(s^{MP}(1))}{\overline{s}^{MP} - s^{MP}(1)} \overset{(d)}{<} 1,$$

where in Step (c), we establish the following set of observations: (c-i) we first establish that $\frac{\overline{s}^{MP} - g_s(s)}{\overline{s}^{MP} - s}$ decreases in $s \ge 0$ by showing that $\frac{\partial}{\partial s}\big(\frac{\overline{s}^{MP} - g_s(s)}{\overline{s}^{MP} - s}\big) = \frac{(s - \overline{s}^{MP}) g_s'(s) - g_s(s) + \overline{s}^{MP}}{(s - \overline{s}^{MP})^2} < 0$, with the inequality following as $g_s(s)$ is strictly concave in $s \ge 0$ such that $\overline{s}^{MP} = g_s(\overline{s}^{MP}) < g_s(s) + (\overline{s}^{MP} - s) g_s'(s)$; (c-ii) we then show that $s^{MP}(t) > s^{MP}(1)$ for $t \in \{2, .., T\}$. Note that $g_s(0) = 0$ and $g_s(\overline{s}^{MP}) = \overline{s}^{MP}$. Since $g_s(s) - s$ is strictly concave in $s \ge 0$, by the Jensen's inequality, we obtain that $g_s(a\overline{s}^{MP}) - a\overline{s}^{MP} > a(g_s(\overline{s}^{MP}) - \overline{s}^{MP}) + (1-a)\big(g_s(0) - 0\big) = 0$ for $0 < a < 1$. Therefore, we have $g_s(a\overline{s}^{MP}) > a\overline{s}^{MP}$ for $0 < a < 1$, which further implies that $s^{MP}(t+1) = g_s(s^{MP}(t)) > s^{MP}(t)$ given that $0 < s^{MP}(t) < \overline{s}^{MP}$. Thus, we can obtain that $s^{MP}(t) < s^{MP}(t+1) < \overline{s}^{MP}$ for $t \in \{1, .., T-1\}$. Combining the observations in (c-i) and (c-ii), since $\frac{\overline{s}^{MP} - g_s(s^{MP}(t))}{\overline{s}^{MP} - s^{MP}(t)}$ decreases in $s^{MP}(t)$ and $s^{MP}(t+1) > s^{MP}(t) > s^{MP}(1)$ for $t \in \{2, .., T-1\}$, we have that Step (c) holds. For Step (d), we have $s^{MP}(1) < s^{MP}(2) = g_s(s^{MP}(1)) < g_s(\overline{s}^{MP}) = \overline{s}^{MP}$, where the first inequality follows from $s^{MP}(t+1) = g_s(s^{MP}(t)) > s^{MP}(t)$ for $0 < s^{MP}(t) < \overline{s}^{MP}$ based on previous discussion; the second inequality follows from the condition that $s^{MP}(1) < \overline{s}^{MP}$ in this case and $g_s(.)$ is a increasing function; the last equation follows directly from the observation in (EC.23b). Therefore, we have that $\frac{\overline{s}^{MP} - g_s(s^{MP}(1))}{\overline{s}^{MP} - s^{MP}(1)} < 1$.

By letting $\gamma_2 = \frac{\overline{s}^{MP} - g_s(s^{MP}(1))}{\overline{s}^{MP} - s^{MP}(1)}$, we obtain that $\frac{\overline{s}^{MP} - g_s(s^{MP}(t))}{\overline{s}^{MP} - s^{MP}(t)} \le \gamma_2$, which implies that

$$\Big|\overline{s}^{MP} - g_s(s^{MP}(t))\Big| \overset{(e)}{=} \overline{s}^{MP} - g_s(s^{MP}(t)) \le \gamma_2 \Big(\overline{s}^{MP} - s^{MP}(t)\Big) \overset{(f)}{=} \gamma_2 \Big|\overline{s}^{MP} - s^{MP}(t)\Big|$$

where (e) and (f) follow from the observations that $s^{MP}(t) < \overline{s}^{MP}$ for $t \in \{1,..,T\}$. In summary, there exists a $\gamma_2 \in (0,1)$ such that $|\overline{s}^{MP} - s^{MP}(t+1)| \le \gamma_2 |(\overline{s}^{MP} - s^{MP}(t))|$ for $t \in \{1,...,T-1\}$ if $s^{MP}(1) < \overline{s}^{MP}$. Again, from the definition of $g_s(\cdot)$, we see that γ_2 is independent of T.

In summary of the two cases above, we let $\gamma := \max\{\gamma_1, \gamma_2\}$, which allows us to obtain the desired result.

Claim 2: For any $\epsilon > 0$, there exists $a \in (0,1)$ for the population transition in this problem instance such that $\overline{\mathcal{R}}^{MP} < \epsilon \overline{\mathcal{R}}$. For the AVG in (5) given the problem instance before Step 1, we have that

$$\begin{aligned}\overline{\mathcal{R}} &= \max_{s,b,q} \quad \Big(1 - \frac{q}{s} - \frac{q}{b}\Big) q \\ \text{s.t. } & 0 \le q \le s, \quad 0 \le q \le b, \quad s \le \alpha s + \beta q^{\xi}, \quad b \le \alpha b + \beta q^{\xi}.\end{aligned}$$

In addition, based on Lemma 1(ii), the inequalities in the last two constraints are both tight. Note that $s = \alpha s + \beta q^{\xi}$ and $b = \alpha b + \beta q^{\xi}$ are equivalent to $s = b = kq^{\xi}$, where $k = \frac{\beta}{1-\alpha}$. By plugging $s = b = kq^{\xi}$ into the objective function we obtain $\overline{\mathcal{R}} = \max_{0 \le q \le kq^{\xi}} (1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}})q$. Since $(1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}})q$ is concave in $q \ge 0$ for $0 < \xi < 1$, from the first-order condition, we have $\overline{q} = \big(\frac{k}{2(2-\xi)}\big)^{\frac{1}{1-\xi}}$, which satisfy $0 < \overline{q} < kq^{\xi}$. Thus, the optimal commission $\overline{r}$ and the optimal profit $\overline{R}$ for the instance of the AVG in (5) satisfies that

$$\begin{aligned}\overline{r} &= 1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}} = \frac{1-\xi}{2-\xi}, \\ \overline{\mathcal{R}} &= \Big(1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}}\Big) q = \frac{1-\xi}{2-\xi}\Big(\frac{k}{2(2-\xi)}\Big)^{\frac{1}{1-\xi}},\end{aligned}$$

which further implies that $\frac{\overline{\mathcal{R}}^{MP}}{\overline{\mathcal{R}}} = (\frac{2-\xi}{2})^{\frac{1}{1-\xi}} \frac{2-\xi}{2(1-\xi)}$. Therefore, we can obtain that

$$\lim_{\xi \to 1} \frac{\overline{\mathcal{R}}^{MP}}{\overline{\mathcal{R}}} = \lim_{\xi \to 1} \Big(\frac{2-\xi}{2}\Big)^{\frac{1}{1-\xi}} \frac{2-\xi}{2(1-\xi)} = 0.$$

∎

EC.3. Proof of Results in Section 5

In this section, we develop some auxiliary results that are needed for the proofs of results in Section 5 in EC.3.1. We then respectively prove the results from Section 5.2 in EC.3.2 and those from 5.1 in EC.3.3.

EC.3.1. Auxiliary Results for Section 5.

In this section, we first develop a simper formulation for Problem (5) in (EC.30). To do that, we first characterize the properties of Problem (5) in Lemma EC.8 and Lemma EC.9. Next, we reformulate it in Lemma EC.10, and will further simplify its formulation into (EC.30) in Lemma EC.11. We then show the connection between the optimal solution to (EC.30) $\boldsymbol{w}^*$ and $(\mathcal{S}_\tau, \mathcal{B}_\tau)$ constructed in (10) in Lemma EC.12. The proof of the auxiliary results follows a similar argument to the proof of Lemma 1, Lemma 2 and Proposition 10 in Birge et al. (2021). Therefore, we omit the detail of the proof of auxiliary results for simplicity.

To develop an equivalent reformulation in $(\boldsymbol{q}, \boldsymbol{x})$ for **AVG**, recall from Lemma 1(ii) that the relaxed population dynamics constraints $s_i \le \alpha_i^s s_i + \mathcal{G}_i^s(q_i^s)$ and $b_j \le \alpha_j^b b_j + \mathcal{G}_j^b(q_j^b)$ with the optimal solutions to **AVG** are tight. Together with (7), on the seller side, we have $s_i = \frac{\beta_i^s (q_i^s)^{\xi_s}}{1-\alpha_i^s}$ for any $i \in \mathcal{S}$. We further let $k_i^s := \frac{\beta_i^s}{1-\alpha_i^s}$, which allows us to obtain that $s_i = k_i^s (q_i^s)^{\xi_s}$ for any $i \in \mathcal{S}$. Similarly, on the buyer side, we have $b_j = k_j^b (q_j^b)^{\xi_b}$ for any $j \in \mathcal{B}$, where $k_j^b = \frac{\beta_j^b}{1-\alpha_j^b}$. Plugging the expressions of $s_i = k_i^s (q_i^s)^{\xi_s}$ and $b_j = k_j^b (q_j^b)^{\xi_b}$ into **AVG**, we obtain the following reformulation of **AVG**:

$$\overline{\mathcal{R}} = \max_{\boldsymbol{q^s}, \boldsymbol{q^b}, \boldsymbol{x}} \left[\sum_{j \in \mathcal{B}} \widetilde{F}_b(q_j^b, k_j^b (q_j^b)^{\xi_b}) - \sum_{i \in \mathcal{S}} \widetilde{F}_s(q_i^s, k_i^s (q_i^s)^{\xi_s}) \right] \tag{EC.25a}$$

$$\text{s.t. } q_i^s \le k_i^s (q_i^s)^{\xi_s}, \qquad \forall i \in \mathcal{S}, \tag{EC.25b}$$

$$q_j^b \le k_j^b (q_j^b)^{\xi_b}, \qquad \forall j \in \mathcal{B}, \tag{EC.25c}$$

$$\sum_{j:(i,j) \in E} x_{ij} = q_i^s, \qquad \forall i \in \mathcal{S}, \tag{EC.25d}$$

$$q_j^b = \sum_{i:(i,j) \in E} x_{ij}, \qquad \forall j \in \mathcal{B}, \tag{EC.25e}$$

$$x_{ij} \ge 0, \qquad \forall (i,j) \in E. \tag{EC.25f}$$

where $\widetilde{F}_b(\cdot)$ and $\widetilde{F}_s(\cdot)$ are defined before Problem (5).

For $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, define $y_b(q) := F_b^{-1}\left(1 - (q)^{1-\xi_b}\right) q$ for $0 \le q \le 1$. Define $y_s(q,u) := -F_s^{-1}\left(\frac{(q)^{1-\xi_s}}{u^{1-\xi_s}}\right) q$ for $0 \le q \le u$ and $u > 0$, $y_s(0,0) := \lim_{(q,u) \to (0,0)} y_s(q,u)$. For simplicity of notations, we let $y_b'(q) := \frac{dy_b(q)}{dq}$ for $0 < q < 1$ and $(y_s)_1'(q,u) := \frac{\partial y_s(q,u)}{\partial q}$ for $0 < q < u$. Furthermore, we let $y_b'(0) := \lim_{q \downarrow 0} y_b'(q)$, $y_b'(1) := \lim_{q \uparrow 1} y_b'(q)$; for $u > 0$, we let $(y_s)_1'(0,u) := \lim_{q \to 0} (y_s)_1'(q,u)$, $(y_s)_1'(u,u) := \lim_{q \to u} (y_s)_1'(q,u)$; for $q > 0$, we let $(y_s)_2'(q,q) := \lim_{u \to q} (y_s)_2'(q,u)$. We show in the following lemma that all of the limiting values are finite.

LEMMA EC.8. *(i)* $y_b(q)$ *is continuously differentiable and strictly concave in* $q \in [0,1]$*;*

(ii) $y_s(q,u)$ *is continuous and strictly concave in* $(q,u) \in \{(q',u') : 0 \le q' \le u'\}$*; moreover,* $y_s(q,u)$ *is continuously differentiable in* $(q,u) \in \{(q',u') : 0 \le q' \le u', u' > 0\}$*;*

(iii) for any $0 < \xi_s < 1$, $-(1-\xi_s)\left[F_s^{-1}\right]'(x)x - F_s^{-1}(x)$ *strictly decreases in* $x \in [0,1]$.

Before the next auxiliary result, we define

$$\rho(u) := \operatorname*{arg\,max}_{0 \le q \le \min\{1,u\}} \Big(y_b(q) + y_s(q,u)\Big), \qquad \text{for } u \ge 0, \tag{EC.26}$$

$$h(u) = \max_{0 \le q \le \min\{1,u\}} \Big(y_b(q) + y_s(q,u)\Big), \qquad \text{for } u \ge 0. \tag{EC.27}$$

Given the definition of $\rho(u)$ and $h(u)$ above, we proceed to consider the following auxiliary result about $(\rho(u), h(u))$ for $u \ge 0$. Notice that $-(y_s)_1'(u,u) = (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s > 0$, which is a constant. To support our proof arguments below, when $u > 0$, if $y_b'(0) > (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s$, we let $\widetilde{u} := (y_b')^{-1}((1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s)$; if $y_b'(0) \le (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s$, we let $\widetilde{u} := 0$.

LEMMA EC.9. *(i)* $\rho(u)$ *is a well-defined and strictly increasing in* $u \ge 0$*; moreover, given* $\widetilde{u} \ge 0$ *defined before the lemma statement,* $\frac{\rho(u)}{u} = 1$ *for* $u \in (0, \widetilde{u}]$ *and* $\frac{\rho(u)}{u}$ *strictly decreases in* $u \ge \widetilde{u}$*;*

(ii) $h(u)$ *is continuous, strictly increasing and strictly concave in* $u \ge 0$.

We next develop an alternative optimization for Problem (EC.25). Consider the following optimization problem:

$$\overline{\mathcal{V}} = \max_{\boldsymbol{w},\boldsymbol{z}} \sum_{j \in \mathcal{B}} \left[(k_j^b)^{\frac{1}{1-\xi_b}} h\left(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} \right) \right] \tag{EC.28a}$$

$$\text{s.t.} \quad (w_j)^{\frac{1}{1-\xi_b}} = \sum_{i:(i,j) \in E} z_{ij}, \qquad j \in \mathcal{B} \tag{EC.28b}$$

$$\sum_{j:(i,j) \in E} z_{ij} = (k_i^s)^{\frac{1}{1-\xi_s}}, \qquad i \in \mathcal{S}, \tag{EC.28c}$$

$$z_{ij} \ge 0, \qquad \forall (i,j) \in E. \tag{EC.28d}$$

where

$$h(u) = \max_{0 \le \widetilde{q}_j \le \min\{1,u\}} F_b^{-1}(1 - (\widetilde{q}_j)^{1-\xi_b})\widetilde{q}_j - F_s^{-1}\left(\frac{(\widetilde{q}_j)^{1-\xi_s}}{u^{1-\xi_s}} \right) \widetilde{q}_j \text{ for any } u > 0 \tag{EC.29}$$

and $h(0) = 0$. We consider the following result:

LEMMA EC.10. *We have the following equivalence properties between Problem (EC.28) and Problem (EC.29):*

(i) let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be the optimal solution to Problem (EC.25), and construct $(\boldsymbol{w}, \boldsymbol{z})$ such that $w_j = (\frac{q_j^b}{q_i^s}(k_i^s)^{\frac{1}{1-\xi_s}})^{1-\xi_b}$ for any $i : x_{ij} > 0$ and $z_{ij} = \frac{x_{ij}}{q_i^s}(k_i^s)^{\frac{1}{1-\xi_s}}$, $\widetilde{q}_j = \frac{q_j^b}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, then $(\boldsymbol{w}, \boldsymbol{z})$ is the optimal solution to Problem (EC.28) and $\widetilde{q}_j$ is the optimal solution to Problem (EC.29) with $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$;

(ii) let $(\boldsymbol{w}, \boldsymbol{z})$ be the optimal solution to Problem (EC.28) and $\widetilde{q}_j$ is the optimal solution to Problem (EC.29) with $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, then construct $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ such that $x_{ij} = \frac{z_{ij}(k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j}{(w_j)^{\frac{1}{1-\xi_b}}}$ and $q_i^s = \frac{(k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j(k_i^s)^{\frac{1}{1-\xi_s}}}{w_j^{\frac{1}{1-\xi_b}}}$ for $j : z_{ij} > 0$, $q_j^b = (k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j$, then $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is the optimal solution to (EC.25);

(iii) Problem (EC.25) and Problem (EC.28) share the same optimal objective value, i.e., $\overline{\mathcal{R}} = \overline{\mathcal{V}}$.

We can further simplify the formulation in (EC.28) in the following Lemma EC.11.

LEMMA EC.11. *Problem (EC.28) and the following problem share the same optimal solution vector $\boldsymbol{w}$,*

$$\overline{\mathcal{Y}} = \max_{\boldsymbol{w}} \sum_{j\in\mathcal{B}} \Big[(k_j^b)^{\frac{1}{1-\xi_b}} h\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big)\Big] \tag{EC.30a}$$

$$s.t. \quad \sum_{j\in\widetilde{\mathcal{B}}} (w_j)^{\frac{1}{1-\xi_b}} \le \sum_{i\in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}}, \qquad \forall \widetilde{\mathcal{B}} \subseteq \mathcal{B}, \tag{EC.30b}$$

$$w_j \ge 0, \qquad \forall j \in \mathcal{B}, \tag{EC.30c}$$

and moreover, $\overline{\mathcal{Y}} = \overline{\mathcal{V}}$ where $\overline{\mathcal{V}}$ is the optimal objective value for Problem (EC.28).

The next lemma establishes the connection between the optimal solution $\boldsymbol{w}^*$ to Problem (EC.30) and the network components $G(\mathcal{S}_\tau \cup \mathcal{B}_\tau, E_\tau)$ constructed in (10). Given the finiteness of the network $G(\mathcal{S} \cup \mathcal{B}, E)$, the iteration in (10) yields a maximum index $\overline{\tau}$.

LEMMA EC.12. *For any $\tau \in \{1, ..., \overline{\tau}\}$ and any $j' \in \mathcal{B}_\tau$, we have $\frac{(w_{j'}^*)^{\frac{1}{1-\xi_b}}}{(k_{j'}^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in\mathcal{S}_\tau}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$.*

EC.3.2. Proof of Results in Section 5.2.

Proof of Proposition 4. Recall that we have established the connection for the optimal solution and the optimal objective value of Problem (EC.25) with those of Problem

(EC.28) and Problem (EC.30) in Lemma EC.10 and Lemma EC.11. Therefore, we focus on characterizing the properties of optimization problems in (EC.28) and (EC.30) instead of (EC.25) in this proof. We have already shown that (EC.28) and (EC.30) share the same optimal solution $\boldsymbol{w}^*$ in Lemma EC.11. To prove the claim, we consider the buyer side in Step 1 and the seller side in Step 2.

Step 1: Establish the ranking of buyers' service levels and payments. Based on Lemma EC.10(ii), we let $(\boldsymbol{w}, \boldsymbol{z})$ be the optimal solution to Problem (EC.28) and $\widetilde{q}_j$ is the optimal solution to Problem (EC.29) with the parameter $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$. We know the optimal solution to Problem (EC.25) satisfies

$$\frac{q_j^b}{b_j} \overset{(a)}{=} \frac{(q_j^b)^{1-\xi_b}}{k_j^b} \overset{(b)}{=} (\widetilde{q}_j)^{1-\xi_b} \overset{(c)}{=} \rho^{1-\xi_b}\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big),$$

where Step (a) follows from the observation that $b_j = k_j^b (q_j^b)^{\xi_b}$ in Problem (EC.25); Step (b) follows from the solution property of $\widetilde{q}_j$ in Problem (EC.29) by Lemma EC.10(ii); Step (c) follows from the definition of the optimal solution ρ to Problem (EC.26). Therefore, the ranking of service levels $(\frac{q_j^b}{b_j})_{j\in\mathcal{B}}$ is the same as that of $\Big(\rho\big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\big)\Big)_{j\in\mathcal{B}}$.

For buyers' payments, we know that

$$\min_{i':(i',j)\in E}\{p_{i'}^s\} + r_j^b = F_b^{-1}\Big(1 - \frac{q_j^b}{b_j}\Big) = F_b^{-1}\Bigg(1 - \rho\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big)\Bigg).$$

Therefore, the ranking of buyers' payments $(\min_{i':(i',j)\in E}\{p_{i'}^s\} + r_j^b)_{j\in\mathcal{B}}$ is the opposite of $\Big(\rho\big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\big)\Big)_{j\in\mathcal{B}}$.

By Lemma EC.9(i), we have that $\rho(u)$ strictly increases in $u>0$. From Lemma EC.12, we know that $\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for $j\in\mathcal{B}_\tau$ and $\tau = 1,\dots\overline{\tau}$. Furthermore, the definition in (10) implies that $\frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ strictly increases in $\tau = 1,\dots,\overline{\tau}$. Therefore, we have

$$\begin{aligned}
&\frac{q_{j_1}^b}{b_{j_1}} = \frac{q_{j_2}^b}{b_{j_2}}, && \text{for } j_1, j_2 \in \mathcal{B}_\tau, \tau\in\{1,...,\overline{\tau}\},\\
&\frac{q_{j_1}^b}{b_{j_1}} < \frac{q_{j_2}^b}{b_{j_2}}, && \text{for } j_1\in\mathcal{B}_{\tau_1}, j_2\in\mathcal{B}_{\tau_2}, \tau_1,\tau_2\in\{1,...,\overline{\tau}\} \text{ and } \tau_1<\tau_2.
\end{aligned}$$

and

$$\min_{i':(i',j_1)\in E}\{p_{i'}^s\} + r_{j_1}^b = \min_{i':(i',j_2)\in E}\{p_{i'}^s\} + r_{j_2}^b, \quad \text{for } j_1, j_2\in\mathcal{B}_\tau, \tau\in\{1,...,\overline{\tau}\},$$

$$\min_{i':(i',j_1)\in E}\{p^s_{i'}\}+r^b_{j_1} > \min_{i':(i',j_2)\in E}\{p^s_{i'}\}+r^b_{j_2}, \quad \text{for } j_1\in\mathcal{B}_{\tau_1}, j_2\in\mathcal{B}_{\tau_2}, \tau_1,\tau_2\in\{1,...,\overline{\tau}\} \text{ and } \tau_1<\tau_2.$$

Step 2: Establish the ranking of sellers' service levels and incomes. To establish the ranking of sellers' service levels, given the optimal solution $\boldsymbol{w}$ to Problem (EC.30) and the optimal solution $\widetilde{q}_j$ to Problem (EC.29) with parameter $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k^b_j)^{\frac{1}{1-\xi_b}}}$, we have that for any $i\in\mathcal{S}$ and $j: x_{ij}>0$,

$$\frac{q^s_i}{s_i} \overset{(a)}{=} \frac{(q^s_i)^{1-\xi_s}}{k^s_i} \overset{(b)}{=} \left(\frac{\rho\big((w_j)^{\frac{1}{1-\xi_b}}/(k^b_j)^{\frac{1}{1-\xi_b}}\big)}{(w_j)^{\frac{1}{1-\xi_b}}/(k^b_j)^{\frac{1}{1-\xi_b}}}\right)^{1-\xi_s}, \tag{EC.31}$$

where (a) follows from our discussion before Problem (EC.25) that $s_i = k^s_i(q^s_i)^{\xi_s}$; (b) follows from Lemma EC.10(ii) for $j: x_{ij}>0$.

We next show that for any $\tau_1\neq\tau_2$, we have $x_{ij}=0$ with $i\in\mathcal{S}_{\tau_1}$ and $j\in B_{\tau_2}$. Based on Lemma EC.10(ii), it is equivalent to show the optimal solution to Problem (EC.28) satisfies that for any $\tau_1\neq\tau_2$, $z_{ij}=0$ for $i\in\mathcal{S}_{\tau_1}$ and $j\in B_{\tau_2}$. We show it by induction. Again, to simplify the notation in Problem (EC.28), we let $W_j := (w_j)^{\frac{1}{1-\xi_b}}$ and $\psi^b_j := (k^b_j)^{\frac{1}{1-\xi_b}}$ for any $j\in\mathcal{B}$ and let $\psi^s_i := (k^s_i)^{\frac{1}{1-\xi_s}}$ for any $i\in\mathcal{S}$. We first consider $\tau=1$. The buyers in $\mathcal{B}_1$ can only trade with the sellers in $\mathcal{S}_1$ given that they are not connected to any other seller types. It remains to show that the sellers in $\mathcal{S}_1$ only trade with the buyers in $\mathcal{B}_1$ at the platform's optimal commissions. Suppose towards a contradiction that there exist $\tau_1\neq 1$ such that $z_{ij}>0$ for some $i\in\mathcal{S}_1$ and $j\in B_{\tau_1}$, then

$$\begin{aligned}\sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E} z_{ij} &= \sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E, j\in\mathcal{B}_1} z_{ij} + \sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E, j\notin\mathcal{B}_1} z_{ij} \\ &\overset{(a)}{>} \sum_{j\in\mathcal{B}_1}\sum_{i:(i,j)\in E, i\in\mathcal{S}_1} z_{ij} \overset{(b)}{=} \sum_{j\in\mathcal{B}_1} W_j \overset{(c)}{=} \sum_{j\in\mathcal{B}_1}\psi^b_j\frac{\sum_{i\in\mathcal{S}_1}\psi^s_i}{\sum_{j\in\mathcal{B}_1}\psi^b_j} = \sum_{i\in\mathcal{S}_1}\psi^s_i\end{aligned} \tag{EC.32}$$

where (a) follows from the assumption that $z_{ij}>0$ for some $i\in\mathcal{S}_1$ and some $j\in B_{\tau_1}$ with $\tau_1\neq 1$; (b) follows from (EC.28b); (c) follows from the observation in Lemma EC.12. In summary, $\sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E} z_{ij} > \sum_{i\in\mathcal{S}_1}\psi^s_i$, which violate Constraint (EC.28c). In summary, we have that $z_{ij}=0$ for all $i\in\mathcal{S}_1$ and $j\in B_{\tau_1}$ if $\tau_1\neq 1$. Assuming that $\mathcal{B}_\tau$ only trade with $\mathcal{S}_\tau$ and vice versa, we proceed to show that $\mathcal{B}_{\tau+1}$ only trade with $\mathcal{S}_{\tau+1}$ and vice versa. First, the buyers in $\mathcal{B}_{\tau+1}$ only trade with the sellers in $\mathcal{S}_{\tau+1}$, because they are not adjacent to the seller types from $\mathcal{S}_{\tau'}$ for any $\tau'\geq\tau+1$; and the seller types with an index lower than $\tau+1$ does not trade with them based on our previous discussion. Second, $\mathcal{S}_{\tau+1}$ only trade with $\mathcal{B}_{\tau+1}$, otherwise we can also obtain $\sum_{i\in\mathcal{S}_{\tau+1}}\sum_{j:(i,j)\in E} z_{ij} > \sum_{i\in\mathcal{S}_{\tau+1}}\psi_i^s$ following the same argument in (EC.32), which violate Constraint (EC.28c) to Problem (EC.28) given that Problem (EC.30) is a reformulation without loss of optimality. In summary, for any $\tau_1 \neq \tau_2$, $x_{ij}=0$ for $i\in\mathcal{S}_{\tau_1}$ and $j\in B_{\tau_2}$. This allows us to show that for any $i\in\mathcal{S}_\tau$ with $\tau=1,\ldots,\overline{\tau}$, we have that if $j: x_{ij}>0$, then we obtain that $j\in\mathcal{B}_\tau$.

Thus, regarding the sellers' incomes, for any $i\in\mathcal{S}_\tau$ with $\tau=1,\ldots,\overline{\tau}$ and any $j:x_{ij}>0$, we have that

$$p_i^s - r_i^s = F_s^{-1}\Big(\frac{q_i^s}{s_i}\Big) = F_s^{-1}\Big(\frac{\rho\Big((w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}\Big)}{(w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}}\Big).$$

Since $\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for $j\in\mathcal{B}_\tau$ with $\tau=1,\ldots,\overline{\tau}$ in Lemma EC.12, we can next focus on the ranking of $\frac{\rho\Big(\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}\Big)}{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for $\tau=1,\ldots,\overline{\tau}$. Recall from Step 1 that $\frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ strictly increases in $\tau=1,\ldots\overline{\tau}$. Based on Lemma EC.9, for some constant $\widetilde{u}\geq 0$, we have that $\frac{\rho(u)}{u}=1$ for $0<u\leq\widetilde{u}$ and $\frac{\rho(u)}{u}$ strictly decreases in u for $u>\widetilde{u}$. Define $\widetilde{\tau} := \max\{\tau | u_j < \widetilde{u} \text{ for } j\in\mathcal{B}_\tau\}$. We observe that (i) for any $\tau\leq\widetilde{\tau}$, we have $\frac{q_i^s}{s_i} = \frac{\rho(u)}{u} = 1$ and $p_i^s - r_i^s = F_s^{-1}\big(\frac{\rho(u)}{u}\big) = F_s^{-1}(1) = \overline{v}_{s_i}$ for $i\in S_\tau$; (ii) for any $\tau>\widetilde{\tau}$, we have $\frac{\rho(\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}})}{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ strictly decreases in τ. Therefore, we can summarize that

$$\begin{aligned}
&\frac{q_{i_1}^s}{s_{i_1}} = \frac{q_{i_2}^s}{s_{i_2}}, && \text{for } i_1,i_2\in\mathcal{S}_\tau, \tau\in\{1,\ldots,\overline{\tau}\},\\
&\frac{q_i^s}{s_i} = 1, && \text{for } i\in\mathcal{S}_\tau, \tau\leq\widetilde{\tau},\\
&\frac{q_{i_1}^s}{s_{i_1}} > \frac{q_{i_2}^s}{s_{i_2}}, && \text{for } i_1\in\mathcal{S}_{\tau_1}, i_2\in\mathcal{S}_{\tau_2}, \tau_1,\tau_2\in\{\widetilde{\tau}+1,\ldots,\overline{\tau}\} \text{ and } \tau_1<\tau_2.
\end{aligned}$$

and

$$\begin{aligned}
&p_{i_1}^s - r_{i_1}^s = p_{i_1}^s - r_{i_1}^s, && \text{for } i_1,i_2\in\mathcal{S}_\tau, \tau\in\{1,\ldots,\overline{\tau}\},\\
&p_i^s - r_i^s = \overline{v}_{s_i}, && \text{for } i\in\mathcal{S}_\tau, \tau\leq\widetilde{\tau},\\
&p_{i_1}^s - r_{i_1}^s > p_{i_2}^s - r_{i_2}^s, && \text{for } i_1\in\mathcal{S}_{\tau_1}, i_2\in\mathcal{S}_{\tau_2}, \tau_1,\tau_2\in\{\widetilde{\tau}+1,\ldots,\overline{\tau}\} \text{ and } \tau_1<\tau_2.
\end{aligned}$$

Summarizing the two steps above, we conclude the claims in this result. ■

Proof of Corollary 2. Given the definition of $(\boldsymbol{k}^s,\boldsymbol{k}^b)$ at the beginning of Appendix EC.3.1, for any $\xi_s\in(0,1)$ and $\xi_b\in(0,1)$, we first let $\psi_i^s = (k_i^s)^{\frac{1}{1-\xi_s}}$ and $\psi_j^b = (k_j^b)^{\frac{1}{1-\xi_b}}$ for simplicity of notations. We consider the equivalent reformulation in Problem (EC.28) with decision variables $(\boldsymbol{w}, \boldsymbol{z})$ by Lemma EC.10 and Problem (EC.30) with the decision variable vector $\boldsymbol{w}$ and Lemma EC.11. We let $W_j = (w_j)^{\frac{1}{1-\xi_b}}$ for all $j \in \mathcal{B}$.

Notice that it is without loss of generality to consider a connected graph $G(\mathcal{S} \cup \mathcal{B}, E)$ for the proof arguments. We prove the impact of ψ^s and ψ^b on the service levels in Step 1, and then the impacts on supply/demand and population in Step 2.

Proof of Claim (1): Establish the impact of ψ^s and ψ^b on the service levels. Recall from Step 1 in the proof arguments of Proposition 4 that for any $j \in \mathcal{B}$, when $\frac{W_j}{\psi_j^b}$ becomes larger under the optimal solution $\boldsymbol{W}$ to Problem (EC.30), $\frac{q_j^b}{b_j}$ becomes larger at the optimal solution as well. As a result, we can focus on the impact of ψ^s and ψ^b on $\frac{W_j}{\psi_j^b}$ for $j \in \mathcal{B}_\tau$.

Step (1-i): Establish the impact of (ψ^s, ψ^b) on the service levels of the buyer side. Let $(\boldsymbol{W}, \boldsymbol{z})$ be the optimal solution to (EC.28) given parameters (ψ^s, ψ^b) and let $\{(\mathcal{S}_\tau, \mathcal{B}_\tau) : \tau = 1, \ldots \overline{\tau}\}$ be the network components obtained from (10) given this parameter set. We define the index set $\tau_i := \{\tau | i \in S_\tau\}$ and $\tau_j := \{\tau | j \in B_\tau\}$. We consider an alternative vector $(\hat{\psi}^s, \hat{\psi}^b)$ in which we pick any $\widetilde{i} \in \mathcal{S}$, and let $\hat{\psi}^s_{\widetilde{i}} > \psi^s_{\widetilde{i}}$; we also let $\hat{\psi}^s_i := \psi^s_i$ for all $i \neq \widetilde{i}$ and let $\hat{\psi}^b_j := \psi^b_j$ for all $j \in \mathcal{B}$. Then we obtain that the parameter vector $(\hat{\psi}^s, \hat{\psi}^b)$ has only one entry on the seller side that is higher than in (ψ^s, ψ^b). Let $(\hat{W}, \hat{z})$ be the optimal solution to (EC.28) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$, and let $\{(\hat{\mathcal{S}}_\tau, \hat{\mathcal{B}}_\tau) : \tau = 1, \ldots \widetilde{\tau}\}$ be the network components obtained from (10) given this parameter set for some positive integer $\widetilde{\tau}$.

To prove the claim of this step, we want to show that $W_j \le \hat{W}_j$ for all $j \in \mathcal{B}$. This leads to the observation that $\frac{W_j}{\psi_j^b} \le \frac{\hat{W}_j}{\hat{\psi}_j^b}$ given our construction that $\hat{\psi}^b_j := \psi^b_j$ for all $j \in \mathcal{B}$. In this way, we can claim that a higher ψ^s_i leads to weakly higher $\frac{W_j}{\psi_j^b}$ for all $j \in \mathcal{B}$.

Suppose towards a contradiction that there exists a $j_1 \in \mathcal{B}$ such that $W_{j_1} > \hat{W}_{j_1}$ at the optimal solution. Based on Constraint (EC.28b), we have that $\sum_{i \in N_E(j_1)} z_{ij_1} = W_{j_1} > \hat{W}_{j_1} = \sum_{i \in N_E(j_1)} \hat{z}_{ij_1}$, which implies that there exists a $i_1 \in N_E(j_1)$ such that $z_{i_1 j_1} > \hat{z}_{i_1 j_1} \ge 0$. Similarly, given $i_1 \in N_E(j_1)$, based on Constraint (EC.28c), we have that $\sum_{j \in N_E(i_1)} z_{i_1 j} = \psi^s_{i_1} \le \hat{\psi}^s_{i_1} = \sum_{j \in N_E(i_1)} \hat{z}_{i_1 j}$ where the inequality follows from the construction of $\hat{\psi}$ above. This implies that there exists $j_2 \in N_E(i_1)$ such that $0 \le z_{i_1 j_2} < \hat{z}_{i_1 j_2}$. Using the same argument as above, there must exist a $i_2 \in N_E(j_2), i_2 \neq i_1$ such that $z_{i_2 j_2} > \hat{z}_{i_2 j_2} \ge 0$ and there exists some $j_3 \in N_E(i_2)$ such that $0 \le z_{i_2 j_3} < \hat{z}_{i_2 j_3}$. In this iteration, given the finiteness of the graph, we have that there exists a finite list $(j_1, i_1, j_2, i_2 ..., j_n)$ such that $W_{j_1} > \hat{W}_{j_1}$ and $W_{j_n} \le \hat{W}_{j_n}$. We let $\mathbb{B}_1 = \{j_1\}$, and $\mathbb{S}_1 = \{i | i \in N_E(j_1), z_{i_1 j_1} > \hat{z}_{i_1 j_1} \ge 0\}$. For $t \in \{2, 3 \ldots\}$, we further let

$\mathbb{B}_t = \{j | j \in N_E(i), 0 \le z_{ij} < \hat{z}_{ij},\ \forall i \in \mathbb{S}_{t-1}\}$, and $\mathbb{S}_t = \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \ge 0,\ \forall j \in \mathbb{B}_{t-1}\}$. We have that $\mathscr{B}_t := \cup_{l\in\{1,\dots,t\}} \mathbb{B}_l$ and $\mathscr{S}_t := \cup_{l\in\{1,\dots,t\}} \mathbb{S}_l$ are the sets of all possible buyer types and seller types accessed within the first $2t$ steps in this iteration. Since $\mathscr{B}_{t-1} \subset \mathscr{B}_t \subset \mathcal{B}$ and $|\mathcal{B}|$ is finite, there exists a finite $\bar{t}$ such that $\mathscr{B}_{\bar{t}} = \mathscr{B}_{\bar{t}-1}$, i.e., the set $\mathscr{B}_t$ stops expanding. Under the assumption that $W_{j_1} > \hat{W}_{j_1}$ at the optimal solution for $j_1 \in \mathbb{B}_1$, we next show that there exists $j \in \mathscr{B}_{\bar{t}}$ such that $W_j < \hat{W}_j$. We further suppose towards a contradiction that $W_j > \hat{W}_j$ for any $j \in \mathscr{B}_{\bar{t}}$. Consider the set of seller types $\widetilde{S} := \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \ge 0,\ \forall j \in \mathscr{B}_{\bar{t}}\}$. We can show that $\widetilde{S} \subseteq \mathscr{S}_{\bar{t}}$ by definition. Moreover, we would obtain that

$$
\begin{aligned}
\sum_{i\in\widetilde{S}} \hat{\psi}^s_i &= \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}<\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} \hat{z}_{ij} \\
&\overset{(a)}{=} \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} \hat{z}_{ij} \\
&< \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} z_{ij} \\
&\le \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} z_{ij} \ = \sum_{i\in\widetilde{S}} \psi^s_i
\end{aligned}
$$

where in Step (a), with $\widetilde{S} \subseteq \mathscr{S}_{\bar{t}} = \cup_{l\in\{1,\dots,\bar{t}\}} \mathbb{S}_l$, in the iterative construction above, given that $\mathbb{B}_t = \{j | j \in N_E(i), 0 \le z_{ij} < \hat{z}_{ij},\ \forall i \in \mathbb{S}_{t-1}\}$ and that $\mathscr{B}_{\bar{t}} = \cup_{l\in\{1,\dots,\bar{t}\}} \mathbb{B}_l$, the subset of buyer types $\{j : z_{ij} < \hat{z}_{ij}$ for some $i \in \widetilde{S}\}$ should be a subset of $\mathscr{B}_{\bar{t}}$; based on the definition $\widetilde{S} = \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \ge 0,\ \forall j \in \mathscr{B}_{\bar{t}}\}$, we have that $z_{ij} > \hat{z}_{ij}$ for any $i \in \widetilde{S}$ and $j \in \mathscr{B}_{\bar{t}}$, which further implies that $\{j : z_{ij} < \hat{z}_{ij},\ \forall i \in \widetilde{S}\} = \emptyset$ and that $\sum_{i\in\widetilde{S}} \sum_{j:z_{ij}<\hat{z}_{ij}} \hat{z}_{ij} = 0$. However, the observation that $\sum_{i\in\widetilde{S}} \hat{\psi}^s_i < \sum_{i\in\widetilde{S}} \psi^s_i$ contradicts with the fact that $\sum_{i\in\widetilde{S}} \hat{\psi}^s_i \ge \sum_{i\in\widetilde{S}} \psi^s_i$ by construction of $(\hat{\psi}^s, \hat{\psi}^b)$ above. Therefore, such a contradiction implies that there exists a $j_l \in \mathbb{B}_l \subset \mathscr{B}_{\bar{t}}$ for some $l \in \mathbb{N}_+$ such that $W_{j_l} \le \hat{W}_{j_l}$. Thus, there must exist a finite path $(j_1, i_1, j_2, i_2 \dots, j_l)$ for $j_t \in \mathbb{B}_t$ and $i_t \in \mathbb{S}_t$ such that $z_{i_t j_t} > 0$ for $t \in \{1, .., l\}$ and $\hat{z}_{i_{t-1} j_t} > 0$ for $t \in \{2, \dots, l\}$ under the assumption that $W_{j_l} \le \hat{W}_{j_l}$. For any $t \in \{1, \dots, l-1\}$, we let τ_{i_t} and τ_{j_t} be the corresponding index for the seller subgroup for $\mathcal{S}_\tau$ and the buyer subgroup $\mathcal{B}_\tau$ by the iterative construction in (10). Since $z_{i_t j_t} > 0$, we know that $\tau_{i_t} = \tau_{j_t}$. With the iterative construction, we have $j_{t+1} \in N_E(i_t)$, which satisfies that $\tau_{i_t} \le \tau_{j_{t+1}}$ given that S_{i_t} is not adjacent to $\mathcal{B}_l$ with $l < \tau_{i_t}$ with the iterative construction in (10). In summary, $\tau_{j_1} = \tau_{i_1} \le \tau_{j_2} = \dots \le \tau_{j_l}$, which implies that $\frac{W_{j_n}}{\psi^b_{j_n}} \ge \frac{W_{j_1}}{\psi^b_{j_1}}$ based on Lemma EC.12. Therefore, $\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}} \ge \frac{W_{j_n}}{\psi^b_{j_n}} \ge \frac{W_{j_1}}{\psi^b_{j_1}} > \frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}$.

We proceed to show that the constructed solution $(\hat{W}, \hat{z})$ cannot be the optimal solution to Problem (EC.28) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$. We first send a flow ϵ along $j_n \to i_{n-1} \to j_{n-1} \to \ldots \to i_1 \to j_1$ to construct a new feasible solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$: since $\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}$ and $\hat{z}_{i_t, j_{t+1}} > 0$ for all $t \in \{1, \ldots, n-1\}$, we can pick any $\epsilon \in (0, \min\{(\hat{W}_{j_n}\hat{\psi}^b_{j_1} - \hat{W}_{j_1}\hat{\psi}^b_{j_n})/(\hat{\psi}^b_{j_1} + \hat{\psi}^b_{j_n}), \min_{t\in\{1,\ldots,n-1\}}\{\hat{z}_{i_t, j_{t+1}}\}\})$; for $t \in \{1, \ldots, n-1\}$, let $\widetilde{z}_{i_t j_t} := \hat{z}_{i_t j_t} + \epsilon$, $\widetilde{z}_{i_t j_{t+1}} := \hat{z}_{i_t j_{t+1}} - \epsilon$, $\widetilde{z}_{ij} := \hat{z}_{ij}$ for all $(i,j) \neq (i_t j_{t+1}), (i,j) \neq (i_t j_t)$. Let $\widetilde{W}_{j_1} := \hat{W}_{j_1} + \epsilon$ and $\widetilde{W}_{j_n} := \hat{W}_{j_n} - \epsilon$, $\widetilde{W}_{j'} := \hat{W}_{j'}$ for all $j' \neq j_1, j' \neq j_n$. We next verify the feasibility of this new solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$ in Problem (EC.28). Since $\epsilon \leq \min_{t\in\{1,\ldots,n-1\}}\{\hat{z}_{i_t, j_{t+1}}\}$, we can obtain that $\widetilde{z}_{i_t j_{t+1}} \geq 0$ such that Constraint (EC.28d) is satisfied. In addition, in our construction of the new feasible solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$, since we only send a flow ϵ along $j_n \to i_{n-1} \to j_{n-1} \to \ldots \to i_1 \to j_1$, Constraints (EC.28b) - (EC.28c) are preserved. Thus, $(\widetilde{W}, \widetilde{z})$ is feasible in Problem (EC.28). We define the super-gradient of $h(u)$ as $\partial h(u) = \{z \in \mathbb{R} | h(t) \leq h(u) + z(t-u), \forall t \geq 0\}$. In addition, we define $\partial_- h(u) := \inf\{\partial h(u)\}$ and $\partial_+ h(u) := \sup\{\partial h(u)\}$. Given the strict concavity of $h(u)$ for $u \geq 0$, we have that if $u_2 > u_1 > 0$, then $\partial_+ h(u_2) < \partial_- h(u_1)$, which implies that

$$
\begin{aligned}
\hat{\psi}^b_{j_1} h\Big(\frac{\widetilde{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\widetilde{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big) &= \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}}\Big) \\
&> \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \epsilon \partial h_-\Big(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big) - \epsilon \partial h_+\Big(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}}\Big) \\
&\geq \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big)
\end{aligned}
$$

where the first inequality follows from the concavity of $h(\cdot)$ in $\mathbb{R}_+$; for the second inequality, since $\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}$ and $\epsilon < \frac{\hat{W}_{j_n}\hat{\psi}_{j_1} + \hat{W}_{j_1}\hat{\psi}_{j_n}}{\hat{\psi}_{j_1} + \hat{\psi}_{j_n}}$, we have $\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}}$, and therefore, $\partial_+ h(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}}) < \partial h_-(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}})$. Since other terms in the objective function remain unchanged, $(\widetilde{W}, \widetilde{z})$ leads to a strictly higher objective value than $(\hat{W}, \hat{z})$, which contradicts with the fact that $(\hat{W}, \hat{z})$ be the optimal solution to (EC.28) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$.

In conclusion, we have that $\frac{W_j}{\psi^b_j} \leq \frac{\hat{W}_j}{\hat{\psi}^b_j}$ for all $j \in \mathcal{B}$. This concludes the claim about the impact of ψ^s_i. For the impact of ψ^b_j, we can apply exactly the same proof-by-contradiction arguments as above to establish that when $\psi^b_{\tilde{j}}$ increases for any $\tilde{j} \in \mathcal{B}$, then we have that the optimal solution $\frac{W_j}{\psi^b_j}$ decreases for any $j \in \mathcal{B}$.

Step (1-ii): Establish the impact of (ψ^s, ψ^b) on the service levels of the seller side. For the impact of ψ^s on the service levels of the seller side, we first recall the construction of

$(\hat{\psi}^s, \hat{\psi}^b)$ based on $(\psi^s, \boldsymbol{\psi}^b)$ in Step (1-i), which satisfies that $\hat{\psi}^s_{\widetilde{i}} > \psi^s_{\widetilde{i}}$, $\hat{\psi}^s_i := \psi^s_i$ for all $i \neq \widetilde{i}$ and $\hat{\psi}^b_j := \psi^b_j$ for all $j \in \mathcal{B}$. Without loss of generality, we suppose that a type-i seller trades with type-j_1 buyer where $i \in \mathcal{S}_{l_1}$ and $j_1 \in \mathcal{B}_{l_1}$ given the parameter set (ψ^s, ψ^b); and given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$, we suppose that the type-i seller trades with type-j_2 buyer for some $j_2 \in \mathcal{B}_{l_2}$. The index satisfies that $l_2 \geq l_1$ given that $\mathcal{S}_{l_1}$ is not connected with $\mathcal{B}_t$ for any $t < l_1$ by the iterative construction of network components in (10). Therefore, we have that $\frac{W_{j_1}}{\psi^b_{j_1}} \leq \frac{W_{j_2}}{\psi^b_{j_2}} \leq \frac{\hat{W}_{j_2}}{\hat{\psi}^b_{j_2}}$, where the first inequality follows from Lemma EC.12 given that $l_2 \geq l_1$, and the second inequality follows from the same arguments in Step (1-i). Since type-i sellers have positive trades with type-j_1 buyers in the optimal solutions given the parameters (ψ^s, ψ^b), and with type-j_2 buyers in the optimal solutions given the parameters $(\hat{\psi}^s, \hat{\psi}^b)$, based on the observation that $\frac{W_{j_1}}{\psi^b_{j_1}} \leq \frac{\hat{W}_{j_2}}{\hat{\psi}^b_{j_2}}$, we can establish that

$$\frac{q^s_i}{s_i} \overset{(a)}{=} \left(\frac{\rho(W_{j_1}/\psi^b_{j_1})}{W_{j_1}/\psi_{j_1}}\right)^{1-\xi_s} \overset{(b)}{\geq} \left(\frac{\rho(\hat{W}_{j_2}/\hat{\psi}^b_{j_2})}{\hat{W}_{j_2}/\hat{\psi}_{j_2}}\right)^{1-\xi_s} \overset{(c)}{=} \frac{\hat{q}^s_i}{\hat{s}_i}, \tag{EC.33}$$

where Step (a) and Step (c) follow from the optimality equation in (EC.31) from the proof arguments in Proposition 4; Step (b) follows from the fact that $\frac{\rho(x)}{x}$ monotonically decreases in $x \geq 0$ (see Lemma EC.9). In summary, when $\psi^s_{\widetilde{i}}$ increases for any $\widetilde{i} \in \mathcal{S}$, we have that $\frac{q^s_i}{s_i}$ becomes weakly lower for all $i \in \mathcal{S}$.

Using the same arguments above, we could establish the impact of ψ^b on the seller side: when $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \in \mathcal{B}$, we have that $\frac{q^s_i}{s_i}$ becomes weakly higher for all $i \in \mathcal{S}$.

<u>Proof of Claim (2): Establish the impact of ψ^s and ψ^b on transaction quantities and populations.</u> Recall from (8) that we have $q^b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$ and $b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{\xi_b}{1-\xi_b}}$ for any $j \in \mathcal{B}$ at the optimal solution to Problem (5) given (7). We establish this claim in the following two substeps.

<u>Step (2-i): Establish the impact of ψ^b on the transaction quantities and populations.</u> For any $j \in \mathcal{B}$, recall from Step (1-i) above that if $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \neq j$, or if $\psi^s_{\widetilde{i}}$ increases for any $\widetilde{i} \in \mathcal{S}$, then $\frac{q^b_j}{b_j}$ weakly decreases at the optimal solution. Given that $q^b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$, we can establish that as $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \neq j$, then q^b_j weakly decreases at the optimal solution for any $j \in \mathcal{B}$. From $q^b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$, we have that $b_j = \psi^b_j(q^b_j)^{\xi_b}$ for any $j \in \mathcal{B}$, which further suggests that b_j weakly decreases at the optimal solution for any $j \in \mathcal{B}$.

For any $j \in \mathcal{B}$, it remains to consider the impact of ψ^b_j on (q^b_j, b_j) at the optimal solution for $j \in \mathcal{B}$. We first show that q^b_j increases in $\psi^b_j \geq 0$ for any $j \in \mathcal{B}$. Recall from Constraints

(EC.25d)-(EC.25e) that $\sum_{i\in\mathcal{S}} q_i^s = \sum_{i\in\mathcal{S}}\sum_{j:(i,j)\in E} x_{ij} = \sum_{j\in\mathcal{B}}\sum_{i:(i,j)\in E} x_{ij} = \sum_{j\in\mathcal{B}} q_j^b$, which means that $q_j^b = \sum_{i\in\mathcal{S}} q_i^s - \sum_{j'\neq j, j'\in\mathcal{B}} q_{j'}^b$. Since higher ψ_j^b leads to weakly higher q_i^s for any $i\in\mathcal{S}$ and weakly lower $q_{j'}^b$ for any $j'\in\mathcal{B}$ with $j'\neq j$, we conclude that higher ψ_j^b leads to weakly higher q_j^b. Similarly, higher ψ_i^s leads to weakly higher q_i^s.

Step (2-ii): Establish the impact of ψ^s on the transaction quantities and populations. By applying the same arguments as in Step (2-i), we can establish that (q_i, s_i) weakly increases in ψ_i^s for all $i\in\mathcal{S}$, and q_i^s and s_i weakly decreases in $\psi_{i'}^s$ for any $i'\neq i$ and weakly increases in ψ_j^b for all $j\in\mathcal{B}$. ■

Proof of Proposition 5. Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be the optimal solution to Problem (EC.25); we let $u_j := (w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}$ for any $j\in\mathcal{B}$ where $(\boldsymbol{w},\boldsymbol{z})$ is the optimal solution to the reformulation into Problem (EC.28) (see Lemma EC.10). Recall that for given $\tau = 1,\dots,\overline{\tau}$ from (10), type-i sellers for $i\in\mathcal{S}_\tau$ trade with type-j buyers for $j\in\mathcal{B}_\tau$. Moreover, for any $i\in\mathcal{S}_\tau$ and $j\in\mathcal{B}_\tau$,

$$r_i^s + r_j^b = F_b^{-1}\Big(1 - \frac{q_j^b}{k_j^b (q_j^b)^{\xi_b}}\Big) - F_s^{-1}\Big(\frac{q_i^s}{k_i^s (q_i^s)^{\xi_s}}\Big) \;=\; F_b^{-1}\Big(1 - \rho^{1-\xi_b}(u_j)\Big) - F_s^{-1}\Big(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\Big),$$

where the first equation follows from the conditions in (EC.1a) and (EC.1c) where the expressions of s_i and b_j are given before Problem (EC.25); the second equation follows from the observations in Lemma EC.10(ii) and the definition of $\rho(u)$ in (EC.26). In addition, at the optimal solution, the value of u_j for any $j\in\mathcal{B}_\tau$ increases in $\tau = 1,\dots,\overline{\tau}$ (see Lemma EC.12 and the definition in (10)). For simplicity of notations, we let $r(u) := F_b^{-1}(1 - \rho^{1-\xi_b}(u)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u)}{u^{1-\xi_s}})$ for any $u > 0$. Recall the definition $\widetilde{u} := (y_b')^{-1}((1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s)$ before Lemma EC.9.

We prove the two claims of this result.

Claim (1). If $u_j \le \widetilde{u}$, we have $\rho(u_j) = u_j$ (see Lemma EC.9(i)). This implies that $F_b^{-1}(1 - \rho^{1-\xi_b}(u_j)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}) = F_b^{-1}(1 - u_j^{1-\xi_b}) - F_s^{-1}(1)$, which is decreasing in $u_j \in [0,1]$ given that $F_b(\cdot)$ is a strictly increasing function in $[0, \overline{v}^b]$ (see Assumption 2). We let $\widetilde{\tau} := \max\{\tau | u_j < \widetilde{u} \text{ for } j\in\mathcal{B}_\tau\}$. Together with the fact that at the optimal solution, the value of u_j for $j\in\mathcal{B}_\tau$ increases in $\tau = 1,\dots,\overline{\tau}$, we obtain that the value $r(u_j)$ increases in $\tau < \widetilde{\tau}$.

Claim (2). If $u_j \ge \widetilde{u}$, we know that $y_b'(\rho(u_j)) + (y_s)_1'(\rho(u_j), u_j) = 0$. Define $Y(\widetilde{q}_j, u_j) := y_b'(\widetilde{q}_j) + (y_s)_1'(\widetilde{q}_j, u_j)$ given the definitions of y_s and y_b before Lemma EC.8: for any $\xi_s \in (0,1)$

and $\xi_b \in (0,1)$, $y_b(q) = F_b^{-1}\left(1-(q)^{1-\xi_b}\right) q$ for $0 \le q \le 1$ and $y_s(q,u) = -F_s^{-1}\left(\frac{(q)^{1-\xi_s}}{u^{1-\xi_s}}\right) q$ for $0 \le q \le u$ and $u > 0$, $y_s(0,0) := \lim_{(q,u)\to(0,0)} y_s(q,u)$. We have that

$$\begin{aligned} Y(\widetilde{q}_j, u_j) &= y_b'(\widetilde{q}_j) + (y_s)_1'(\widetilde{q}_j, u_j) \\ &= \left((\xi_b - 1)\widetilde{q}_j^{1-\xi_b}(F_b^{-1})'\left(1-\widetilde{q}_j^{1-\xi_b}\right) + F_b^{-1}\left(1-\widetilde{q}_j^{1-\xi_b}\right)\right) \\ &\quad + \left((\xi_s - 1)\frac{\widetilde{q}_j}{u_j^{1-\xi_s}}(F_s^{-1})'\left(\frac{\widetilde{q}_j}{u_j^{1-\xi_s}}\right) - F_s^{-1}\left(\frac{\widetilde{q}_j}{u_j^{1-\xi_s}}\right)\right). \end{aligned}$$

Since F_s and F_b are twice differentiable, we know that F_s^{-1} and F_b^{-1} are continuously differentiable, and therefore $Y(\widetilde{q}_j, u_j)$ is continuously differentiable at $(\widetilde{q}_j, u_j)$ for $0 \le \widetilde{q}_j \le \min\{u_j, 1\}$. By the implicit function theorem, there exists a continuously differentiable function $\rho(u_j)$ such that $\widetilde{q}_j = \rho(u_j)$ given $Y(\widetilde{q}_j, u_j) = 0$. By differentiating $Y(\widetilde{\rho}(u_j), u_j) = 0$ with respect to u_j, we obtain

$$\rho'(u_j) = \frac{(\xi_s - 1)u_j^{\xi_s - 3}\rho(u_j)^{1-2\xi_s}\left((\xi_s - 1)\rho(u_j)u_j^{\xi_s}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + (\xi_s - 2)u_j\rho(u_j)^{\xi_s}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\right)}{(\xi_b - 1)\rho(u_j)^{-2\xi_b} f_b + (\xi_s - 1)u_j^{\xi_s - 2}\rho(u_j)^{-2\xi_s} f_s},$$

where

$$\begin{aligned} f_b &:= (\xi_b - 2)\rho(u_j)^{\xi_b}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) - (\xi_b - 1)\rho(u_j)(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right), \\ f_s &:= (\xi_s - 1)\rho(u_j)u_j^{\xi_s}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + (\xi_s - 2)u_j\rho(u_j)^{\xi_s}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right). \end{aligned}$$

We proceed to show that $f_s < 0$ and $f_b < 0$ for later use:

$$\begin{aligned} f_b &:= (1-\xi_b)\rho(u_j)^{\xi_b}\left(\frac{(2-\xi_b)}{(\xi_b - 1)}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) + \rho^{1-\xi_b}(u_j)(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)\right) \\ &\overset{(a)}{<} (1-\xi_b)\rho(u_j)^{\xi_b}\left(-2(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) + \rho^{1-\xi_b}(u_j)(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)\right) \overset{(b)}{<} 0, \\ f_s &:= (\xi_s - 1)u_j\rho^{\xi_s}(u_j)\left(\rho^{1-\xi_s}(u_j)u_j^{\xi_s - 1}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + \frac{\xi_s - 2}{\xi_s - 1}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\right) \\ &\overset{(c)}{<} (\xi_s - 1)u_j\rho^{\xi_s}(u_j)\left(\rho^{1-\xi_s}(u_j)u_j^{\xi_s - 1}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + 2(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\right) \overset{(d)}{<} 0, \end{aligned}$$

where (a) and (c) follow from the facts that $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, which imply that $\frac{2-\xi_b}{\xi_b - 1} < -2$ and $\frac{\xi_s - 2}{\xi_s - 1} > 2$ given that $(F_b^{-1})' > 0$ and $(F_s^{-1})' > 0$ on the domains; (b) and (d) follow from the conditions that $-F_s^{-1}(a/b)a$ and $F_b^{-1}(1-a/b)a$ are concave in (a,b) for

$0 \le a \le b$ and $b > 0$ by Assumption 3, and therefore $\frac{a}{b}(F_s^{-1})''\left(\frac{a}{b}\right) + 2(F_s^{-1})'\left(\frac{a}{b}\right) > 0$ and $\frac{a}{b}(F_b^{-1})''\left(1-\frac{a}{b}\right) - 2(F_b^{-1})'\left(1-\frac{a}{b}\right) < 0$. In summary, we have $f_s < 0$ and $f_b < 0$.

Finally, we want to establish how $r(u_j) = F_b^{-1}(1-\rho^{1-\xi_b}(u_j)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}})$ changes in $u_j > 0$. Again, given the continuity of $r(u)$, we define the sup-derivative

$$\partial r(u) = \{z \in \mathbb{R} \mid r(t) \le r(u) + z(t-u),\ \forall t \ge 0\},$$

which implies that

$$\begin{aligned}\partial r(u) &= (\xi_b - 1)\rho(u_j)^{-\xi_b}\rho'(u_j)(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)\\ &\quad + (\xi_s - 1)u_j^{\xi_s-2}\rho(u_j)^{-\xi_s}\left(u_j\rho'(u_j) - \rho(u_j)\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right).\end{aligned}$$

Plugging in the expression of $\rho'(u_j)$, we obtain that

$$\partial r(u) = \frac{(\xi_b-1)(\xi_s-1)\rho(u_j)\left(f_1+f_2+f_3\right)}{u_j\left((\xi_b-1)s^{2-\xi_s}\rho(u_j)^{2\xi_s}f_b + (\xi_s-1)\rho(u_j)^{2\xi_b}f_s\right)},$$

where

$$\begin{aligned} f_1 &= (\xi_b-1)u_j\rho(u_j)^{\xi_s+1}(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right),\\ f_2 &= (\xi_s-1)u_j^{\xi_s}\rho(u_j)^{\xi_b+1}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right),\\ f_3 &= -u_j(\xi_b-\xi_s)\rho(u_j)^{\xi_b+\xi_s}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right).\end{aligned}$$

Based on the observation above, we discuss the two cases of this claim:

(i) if $F_s(v)$ and $F_b(v)$ are convex in $v \in [0, \overline{v}_s]$ and $v \in [0, \overline{v}_b]$, we have $(F_b^{-1})''(v) < 0$ and $(F_s^{-1})''(v) < 0$ in their domains. Given $(F_b^{-1})'(v) > 0$ and $(F_s^{-1})'(v) > 0$, $\rho(u_j) < 1$ (see (EC.26)) and $\xi_s, \xi_b \in (0,1)$, we know that $f_1 > 0$ and $f_2 > 0$. Since $\xi_s = \xi_b$, $f_3 = 0$. Therefore, the numerator of $\frac{\partial r(u_j)}{\partial u_j}$ is positive. Since $f_s < 0$ and $f_b < 0$, the denominator of $\frac{\partial r(u_j)}{\partial u_j}$ is positive. In summary, $\frac{\partial r(u_j)}{\partial u_j} > 0$ for $u_j \ge \widetilde{u}$;

(ii) if $F_s(v)$ and $F_b(v)$ are concave in $v \in [0, \overline{v}_s]$ and $v \in [0, \overline{v}_b]$ respectively, we have $(F_b^{-1})''(v) > 0$ and $(F_s^{-1})''(v) > 0$, then $f_1 < 0$ and $f_2 < 0$. Therefore, $\frac{\partial r(u_j)}{\partial u_j} < 0$ for $u_j \ge \widetilde{u}$.

■

EC.3.3. Proof of Results in Section 5.1.

Proof of Theorem 2. Recall that $\overline{\mathcal{R}}(E,\psi^s,\psi^b),\overline{\mathcal{V}}(E,\psi^s,\psi^b),\overline{\mathcal{Y}}(E,\psi^s,\psi^b)$ are respectively the optimal objective value to (EC.25), (EC.28) and (EC.30). To simplify the notations, we use $\overline{\mathcal{R}}(E)$, $\overline{\mathcal{V}}(E)$, $\overline{\mathcal{Y}}(E)$ to denote $\overline{\mathcal{R}}(E,\psi^s,\psi^b)$, $\overline{\mathcal{V}}(E,\psi^s,\psi^b)$, $\overline{\mathcal{Y}}(E,\psi^s,\psi^b)$. From Lemma EC.10 and EC.11, we have that $\overline{\mathcal{R}}(E)=\overline{\mathcal{V}}(E)=\overline{\mathcal{Y}}(E)$. Therefore, to prove the claim in this result, it is equivalent to focus on Problem (EC.30) and show that $\overline{\mathcal{Y}}(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$.

We next consider Problem (EC.34) below with an additional constraint $F_b^{-1}(1-q_j^{1-\xi_b})-F_s^{-1}(\frac{q_j^{1-\xi_s}}{u_j^{1-\xi_s}})\geq r$ for some $r\in\mathbb{R}$ in comparison with Problem (EC.30). We then show that even the problem with this constraint can obtain the objective value weakly higher than $(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$, from which we can conclude that $\overline{\mathcal{Y}}(E)\geq\mathcal{Y}^h(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. Given the edge set $\overline{E}$ of the complete graph, for any edge set $E\subset\overline{E}$, we define this auxiliary problem below

$$
\begin{aligned}
\mathcal{Y}^h(E) \;=\; \max_{\boldsymbol{w},r}\ & \sum_{j\in\mathcal{B}}\left[(k_j^b)^{\frac{1}{1-\xi_b}}h\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}},r\Big)\right] && && \text{(EC.34a)}\\
\text{s.t.}\ & \sum_{j\in\widetilde{\mathcal{B}}}(w_j)^{\frac{1}{1-\xi_b}}\leq\sum_{i\in N_E(\widetilde{\mathcal{B}})}(k_i^s)^{\frac{1}{1-\xi_s}}, && \forall\widetilde{\mathcal{B}}\subseteq\mathcal{B}, && \text{(EC.34b)}\\
& w_j\geq 0, && \forall j\in\mathcal{B} && \text{(EC.34c)}\\
& r\leq\overline{v}_b, && && \text{(EC.34d)}
\end{aligned}
$$

where for any $u>0$,

$$
h(u,r)=\max_{\substack{0\leq\widetilde{q}\leq\min\{1,u\},\\ F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u^{1-\xi_s}}\right)\geq r}}\left(F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}\Big(\frac{\widetilde{q}^{1-\xi_s}}{u^{1-\xi_s}}\Big)\right)\widetilde{q}. \tag{EC.34e}
$$

Step 1: Show that $\overline{\mathcal{Y}}(E)\geq\mathcal{Y}^h(E)$. Note that the only difference between (EC.34) and (EC.30) is that one more constraint $F_b^{-1}\big(1-(\widetilde{q}_j)^{1-\xi_b}\big)-F_s^{-1}\big(\frac{\widetilde{q}_j^{1-\xi_s}}{u^{1-\xi_s}}\big)\geq r$ for any $(i,j)\in E$ is added to Problem (EC.34). With $r\leq\overline{v}_b$, we have that the constraint for the maximization problem in (h,r) is non-empty given that solution $\widetilde{q}=0$ is feasible. Therefore, the solution to Problem (EC.34) is also feasible in Problem (EC.30), and two problems share the same objective functions. Thus, we have that

$$
\overline{\mathcal{Y}}(E)\geq\mathcal{Y}^h(E).
$$

Step 2: Show that $\mathcal{Y}^h(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. To establish the claim, we first reformulate the optimization problems for $\mathcal{Y}^h(E)$ and $\overline{\mathcal{Y}}(\overline{E})$.

Step 2.1: Reformulate the problem for $\mathcal{Y}^h(E)$. With $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$ for any $j \in \mathcal{B}$, we define

$$\hat{q}_j(r,u_j) := \max\left\{\widetilde{q}:\ r \le F_b^{-1}(1-(\widetilde{q})^{1-\xi_b}) - F_s^{-1}\Big(\frac{\widetilde{q}^{1-\xi_s}}{u_j^{1-\xi_s}}\Big), 0 \le \widetilde{q} \le \min\{1,u_j\}\right\}. \tag{EC.35}$$

Note that since $F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\Big(\frac{\widetilde{q}^{1-\xi_s}}{u_j^{1-\xi_s}}\Big)$ strictly decreases in $\widetilde{q} \in [0,\min\{1,u_j\}]$, we know $\hat{q}_j(r,u_j)$ is unique given (r,u_j). Given that r is a lower bound of $F_b^{-1}(1-(\widetilde{q})^{1-\xi_b}) - F_s^{-1}\Big(\frac{(\widetilde{q})^{1-\xi_s}}{(u_j)^{1-\xi_s}}\Big)$ and $\hat{q}_j(r,u_j)$ is suboptimal to Problem (EC.34e), the optimal objective value $\mathcal{Y}^h(E)$ from Problem (EC.34e) is weakly higher than the optimal objective value of following optimization problem

$$\begin{aligned}
\max_{\boldsymbol{w},r}\ & \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r\hat{q}_j\Big(r, \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big) && \\
\text{s.t.}\ & \sum_{j\in\widetilde{\mathcal{B}}} w_j^{\frac{1}{1-\xi_b}} \le \sum_{i\in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}}, && \forall \widetilde{\mathcal{B}} \subseteq \mathcal{B}, \\
& w_j \ge 0, && \forall j \in \mathcal{B}, \\
& r \le \overline{v}_b. &&
\end{aligned}$$

For any $r \in (-\infty, \overline{v}^b]$ and $\epsilon \in (0,1)$, we observe that $(w_j)^{\frac{1}{1-\xi_b}} = (k_j^b)^{\frac{1}{1-\xi_b}}(1-\epsilon)\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}$ is feasible in the optimization problem above given that $w_j \ge 0$ for any $j \in \mathcal{B}$ and for any $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$,

$$\sum_{\in\widetilde{\mathcal{B}}} w_j^{\frac{1}{1-\xi_b}} = \sum_{j\in\widetilde{\mathcal{B}}} (k_j^b)^{\frac{1}{1-\xi_b}}(1-\epsilon)\frac{\sum_{i'\in\mathcal{S}}(k_{i'}^s)^{\frac{1}{1-\xi_s}}}{\sum_{j'\in\mathcal{B}}(k_{j'}^b)^{\frac{1}{1-\xi_b}}} \le \sum_{i\in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}},$$

where the inequality follows directly from the condition in the theorem statement. By letting $\overline{u} := \frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}$, we have that

$$\mathcal{Y}^h(E) \ \ge\ \max_{r\le\overline{v}_b} \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r\hat{q}_j\Big(r, (1-\epsilon)\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}\Big) \ =\ \max_{r\le\overline{v}_b} \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r\hat{q}_j\big(r,(1-\epsilon)\overline{u}\big).$$

Step 2.2: Reformulate the problem for $\overline{\mathcal{Y}}(\overline{E})$. We first show that given the graph set to the complete graph $G(\mathcal{S}\cup\mathcal{B},\overline{E})$, the optimal solution to Problem (EC.30) satisfies $(w_{j'}^*)^{\frac{1}{1-\xi_b}} = (k_{j'}^b)^{\frac{1}{1-\xi_b}}\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for any $j' \in \mathcal{B}$. Given the definition of $(\mathcal{S}_\tau,\mathcal{B}_\tau)$ in (10), in a complete graph, we have that $\mathcal{B}_1 = \mathcal{B}$, as for any $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$, we have that

$$\frac{\sum_{i\in N_E(\widetilde{\mathcal{B}})}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\widetilde{\mathcal{B}}}(k_j^b)^{\frac{1}{1-\xi_b}}} \overset{(a)}{=} \frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\widetilde{\mathcal{B}}}(k_j^b)^{\frac{1}{1-\xi_b}}} \overset{(b)}{\ge} \frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_E(\mathcal{B})}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}},$$

where Step (a) follows from the fact that network $G(\mathcal{S}\cup\mathcal{B},\overline{E})$ is complete; Step (b) follows from the condition that $\widetilde{\mathcal{B}}\subseteq\mathcal{B}$. By Lemma EC.12, we have $\frac{(w^*_{j'})^{\frac{1}{1-\xi_b}}}{(k^b_{j'})^{\frac{1}{1-\xi_b}}}=\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}$ for any $j'\in\mathcal{B}$. Therefore, we can obtain that

$$\overline{\mathcal{Y}}(\overline{E}) \;=\; \sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}h\left(\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}\right).$$

Similar to Step 2.1, given definition of $h(.)$ in (EC.29), we could reformulate $h(.)$ by defining that

$$\overline{q}:=\underset{\widetilde{q}\in[0,\min\{1,\overline{u}\}]}{\arg\max}\left(F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}}\right)\right)\widetilde{q}, \tag{EC.36}$$

where we recall that we have set $\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}=\overline{u}$ in Step 2.1 above. By letting $\overline{r}:=F_b^{-1}(1-(\overline{q})^{1-\xi_b})-F_s^{-1}(\frac{(\overline{q})^{1-\xi_s}}{(\overline{u})^{1-\xi_s}})$, given definition of $h(.)$ in (EC.29), we have that

$$\overline{\mathcal{Y}}(\overline{E}) \;=\; \sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}h\left(\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}\right) \;=\; \sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}\overline{rq}.$$

<u>Step 2.3: Establish that $\mathcal{Y}^h(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$.</u> To establish the claim, for any $j\in\mathcal{B}$, we want to show that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})\geq(1-\epsilon)\overline{q}$.

By the definition of $\hat{q}_j(r,u)$ in (EC.35), we have that for any $j\in\mathcal{B}$,

$$\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}):=\max\left\{\widetilde{q}:\overline{r}\leq F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}\right),0\leq\widetilde{q}\leq\min\{1,(1-\epsilon)\overline{u}\}\right\}.$$

For simplicity of notations, we use $\hat{q}_j$ to denote $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})$. Since $F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}(\frac{(\widetilde{q})^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$ decreases in $\widetilde{q}\in[0,\min\{1,(1-\epsilon)\overline{u}\}]$, we have that either $\overline{r}=F_b^{-1}(1-(\hat{q}_j)^{1-\xi_b})-F_s^{-1}(\frac{(\hat{q}_j)^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$ or $\hat{q}_j=\min\{1,(1-\epsilon)\overline{u}\}$.

For any $j\in\mathcal{B}$, to show that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})\geq(1-\epsilon)\overline{q}$, we consider the following two cases:

(1) if $\hat{q}_j=\min\{1,(1-\epsilon)\overline{u}\}$, then $\hat{q}_j=\min\{1,(1-\epsilon)\overline{u}\}\geq(1-\epsilon)\min\{1,\overline{u}\}=(1-\epsilon)\overline{q}$, where the last equality follows from the constraint in Problem (EC.36);

(2) if $\overline{r}=F_b^{-1}(1-\hat{q}_j^{1-\xi_b})-F_s^{-1}(\frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$, then based on the definition that $\overline{r}=F_b^{-1}(1-\overline{q}^{1-\xi_b})-F_s^{-1}(\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}})$ in Step 2.2, we have that

$$F_b^{-1}(1-\overline{q}^{1-\xi_b})-F_s^{-1}\left(\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}}\right)=F_b^{-1}(1-\hat{q}_j^{1-\xi_b})-F_s^{-1}\left(\frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}\right).$$

Note that $F_b^{-1}(1-q^{1-\xi_b}) - F_s^{-1}(\frac{q^{1-\xi_s}}{u^{1-\xi_s}})$ strictly increases in $u \geq q \geq 0$ and strictly decreases in $q \in [0, \min\{1, u\}]$. With the equation above, given that $0 < (1-\epsilon)\overline{u} \leq \overline{u}$, we have that $\overline{q} \geq \hat{q}_j$, which further implies that $\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}} \leq \frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}$. This allows us to establish that $\hat{q}_j^{1-\xi_s} \geq ((1-\epsilon)\overline{u})^{1-\xi_s} \frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}} = (\overline{q})^{1-\xi_s}(1-\epsilon)^{1-\xi_s}$. Therefore, we have $\hat{q}_j \geq (1-\epsilon)\overline{q}$.

Summarizing the two cases above, we can establish that

$$\mathcal{Y}^h(E) \overset{(a)}{\geq} \sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} \overline{r} \hat{q}_j(\overline{r}, (1-\epsilon)\overline{u}) \overset{(b)}{=} \sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} \overline{r}(1-\epsilon)\overline{q} \overset{(c)}{=} (1-\epsilon)\overline{\mathcal{Y}}(\overline{E}),$$

where (a) follows from Step 2.1 and $\overline{r} = F_b^{-1}(1-(\overline{q})^{1-\xi_b}) - F_s^{-1}(\frac{(\overline{q})^{1-\xi_s}}{(\overline{u})^{1-\xi_s}}) \leq F_b^{-1}(1) = \overline{v}_b$; (b) follows from the observation that $\hat{q}_j(\overline{r}, (1-\epsilon)\overline{u}) \geq (1-\epsilon)\overline{q}$ for any $j \in \mathcal{B}$; (c) follows directly from the reformulation in Step 2.2. ■