

# Dynamic Pricing and Inventory Management under Network Externalities

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We study the impact of *network externalities* upon a firm's pricing and inventory policy under demand uncertainty. The firm sells a product associated with an online service or communication network, which is formed by (part of) the customers who have purchased the product. The product exhibits network externalities, i.e., a customer's willingness-to-pay and, thus, the potential demand are increasing in the size of the associated network. We show that a network-size-dependent base-stock/list-price policy is optimal. Interestingly, the inventory dynamics of the firm do not influence the optimal policy as long as the initial inventory is below the initial base-stock level. Hence, we can reduce the dynamic program that characterizes the optimal policy to one with a single-dimensional state-space (the network size). Network externalities give rise to the trade-off between generating current profits and inducing future demands, thus having several important implications upon the firm's operations decisions. First, network externalities drive the firm to deliver a better service and attract more customers into the network. Hence, the safety-stock and base-stock levels are higher in the presence of network externalities. Second, the network size evolution follows a mean-reverting pattern: When the current network size is small (resp. large), it has an increasing (resp. decreasing) trend in expectation. Third, although myopic profit optimization leads to significant optimality losses under network externalities, balancing the current profits and near-future demands suffices to exploit network externalities. We propose a dynamic look-ahead heuristic policy that well leverages this idea and achieves small optimality gaps which decay exponentially in the length of the look-ahead time-window. Finally, we develop a general network expansion strategy, which facilitates the firm to (partially) separate generating current profits and inducing future demands via network externalities.

*Key words:* dynamic pricing and inventory management; network externalities

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## 1. Introduction

*Network externalities* (also called the network effect) refer to the general phenomenon that a customer's utility of purchasing a product is increasing in the number of other customers who buy the same product (see, e.g., Economides 1996). With the fast development of information technology, network externalities have become a key driver of profitability for a high-tech firm.

Take Apple as an example. Around year 2000, Apple computers were better, by all accounts, than the PCs with the Windows system. However, the vast majority of desktop and laptop computers ran Windows as their operating systems because of network externalities (see, e.g., Krugman 2013). Due to Windows' dominating role in the operating system market, software developers made only one sixth as many applications for Macintosh as they did for Windows by the time of Microsoft's antitrust trial. This, in turn, made Apple computers unattractive to new consumers, despite its functional advantages (see Eisenmann et al. 2006). In the era of smartphones, however, Apple becomes the winning side of the network externalities game. Since the launch of App Store in 2008, there have been more than 1.4 million mobile apps with more than 75 billion downloads on this digital distribution platform. The App Store not only generates huge revenues (Apple takes 30% of all revenues generated through apps), but also creates large availability of apps for iPhones, thus enabling Apple to exploit network externalities to a large extent. As a consequence, iPhones have a market share of 47.4% among all smartphones in November 2014 (see, e.g., Jones 2015). The example of Apple clearly demonstrates the importance of network externalities upon a firm's success in the market. In particular, the online mobile software distributing platform App Store plays an important role in strengthening the network externalities of Apple products and in boosting the sales of iPhones.

As an analogous example, Xbox Live, the online multiplayer gaming network for Xbox game consoles, significantly intensifies network externalities. This is because the value of an Xbox to a user increases if she has more opportunities to play games with her friends on Xbox Live (see, e.g., Mirrokni et al. 2012). Thus, the size of the online gaming network Xbox Live is crucial to Microsoft's game console business, and the firm has employed various strategies to expand the size of Xbox Live. For example, Microsoft offered a discount of \$50 for Xbox One customers who guaranteed to sign up for Xbox Live Gold membership for at least one year (Geddes 2015). In another promotion, the 12-month Xbox Live Gold membership was discounted by 33% in February 2015 (Soper 2015).

Firms like Apple and Microsoft naturally face the question of how to optimally coordinate the price and inventory policy of their products (iPhone and Xbox One) under strong network externalities. To address this question, we study a periodic-review single-item dynamic pricing and inventory management model in the presence of network externalities. The firm may launch an online service network associated with the product (e.g., App Store and Xbox Live). With the recent trends of online social media, the associated network can also be in the form of a social communication network (e.g., Facebook and Twitter), where customers share their experiences and excitements in purchasing and consuming their favorite products. These sharings can prompt potential customers to purchase products of higher popularity, thus creating network externalities.

To model network externalities, we assume that a customer's willingness-to-pay is increasing in the size of the associated network. In each period, a fraction of the customers who make a purchase would join the network, depending on the customers' preferences and inventory availability. The probability that a customer will join the network is higher if the product she requests is available, since she receives a better service in this case. The firm may generate revenue from the network via, e.g., service fees. In this model, we characterize the optimal pricing and inventory policy of a profit-maximizing firm under network externalities. Our analysis highlights the significant impact of network externalities on the firm's optimal price and inventory policy, and identifies effective strategies to exploit network externalities.

To the best of our knowledge, we are the first in the literature to operationalize network externalities in an inventory model and study its impact upon the joint pricing and inventory policy of a firm. We show that a network-size-dependent base-stock/list-price policy is optimal. Moreover, we make an interesting technical contribution in this paper: Although the inventory stocking decision affects potential network sizes via stock-outs, the inventory dynamics of the firm have no impact on the optimal policy in the probabilistic sense. As a consequence, the optimal policy can be characterized by a dynamic program with a single-dimensional state space (the network size). We perform a sample path analysis of the inventory system and show that, if the firm adopts the optimal policy and the initial inventory is below the initial base-stock level, the inventory level of the firm stays below the optimal base-stock level in each period throughout the planning horizon with probability one. Under the base-stock/list-price policy, the inventory level does not affect the optimal policy as long as it is below the base-stock level. Therefore, while the price and inventory *decisions* cannot be decoupled, the *state space* of the dynamic program is reduced to a single-dimensional one (network size). Such dimensionality reduction significantly simplifies the analysis and computation of the optimal policy, and thus helps deliver sharper insights on the managerial implications of network externalities.

Our analysis reveals that network externalities drive the firm to balance the trade-off between generating current profits and inducing future demands and, thus, have quite a few important and interesting managerial implications. First, network externalities lead to the service effect and network-size-dependent pricing: The firm provides better services to customers and charges different prices with different network sizes. Customers who are satisfied immediately (i.e., not being put onto the wait-list) are more likely to join the network and attract future customers. Therefore, if benchmarked against the case without network externalities, the firm under network externalities provides a better service to customers. Hence, the safety-stock and base-stock levels are both higher in the presence of network externalities. Network externalities also have significant impact upon

the firm’s pricing policy. If the current network size is small, the presence of network externalities decreases the sales price to induce higher future demands. Otherwise, if the current network size is large, the firm increases the sales price to exploit the better market condition. From an intertemporal perspective, the firm should put more weight on inducing future demands at the early stage of a sales season. Thus, when the market is stationary, the firm charges lower prices at the beginning of the planning horizon. Hence, the widely-adopted introductory price strategy (offering price discounts at the beginning of the sales season of a product) may stem from network externalities.

Second, the aforementioned trade-off between generating current profits and inducing future demands drives the stochastic network size process to follow an interesting mean-reverting pattern. As long as the firm adopts the optimal joint pricing and inventory policy, the network size will increase (resp. decrease) in expectation when it is below (resp. above) the “mean” currently. With a small current network size, the firm underscores inducing future demands, so the network size has an increasing trend. On the other hand, with a large current network size, the firm simply extracts high profits from the good market condition, so the network size will fall in expectation.

Third, although myopic profit optimization leads to significant losses in the presence of network externalities, balancing the current profits and near-future demands suffices to exploit network externalities. Our extensive numerical studies demonstrate that the myopic policy ignores the (inter-temporal) demand-inducing effect under network externalities, so it substantially erodes the firm’s profits. We propose a look-head heuristic policy that dynamically maximizes the profit in a (short) moving time-window. Interestingly, this heuristic policy yields small optimality gaps (less than 2% in our extensive numerical experiments) with exponential decay in the length of the moving time-window. Thus, with the dynamic look-ahead heuristic, the firm can effectively leverage network externalities by balancing the current profits and the *near future* demands.

Finally, we develop a general network expansion strategy (with a cost) that effectively exploits network externalities and improves the profit. This strategy is commonly used in practice. For example, Microsoft offers discounts for the customers who commit to subscribe to the Xbox live service (Geddes 2015). It is also documented in the literature (e.g., Fainmesser and Galeotti 2016) that firms can now use an online platform called SponsoredTweets to pay customers to tweet about their products. The key idea underlying these network expansion strategies is that, the firm employs an additional leverage (i.e., network expansion investment) to partially separate generating current profits and inducing future demands through network externalities. With sufficiently strong network externalities, it is optimal for the firm to invest in network expansions, regardless of its inventory level. The optimal price is higher, whereas the safety-stock and base-stock levels are lower with network expansion investment than without. In other words, the firm which invests in

network expansions to induce future demands via network externalities is able to charge a premium product price and maintain a low inventory level to generate higher current profits.

The rest of this paper is organized as follows. In Section 2, we position this paper in the related literature. Section 3 presents the basic formulation, notations and assumptions of our model. Section 4 simplifies the model to a single-dimensional dynamic program. We investigate the key trade-off under network externalities in Section 5. The network expansion strategy to exploit network externalities is analyzed in Section 6. We discuss the extension with high initial inventory level in Section 7. Section 8 concludes this paper by summarizing our main findings. All proofs are relegated to the Appendix. We will use  $\mathbb{E}[\cdot]$  to denote the expectation operation. In addition, for any  $x, y \in \mathbb{R}$ ,  $x \wedge y := \min\{x, y\}$ ,  $x^+ := \max\{x, 0\}$ , and  $x^- := \max\{-x, 0\}$ .

## 2. Literature Review

This paper is built upon two streams of research in the literature: (a) network externalities and (b) joint pricing and inventory management.

Network externalities have been extensively studied in the economics literature. In their seminal papers, Katz and Shapiro (1985, 1986) characterize the impact of network externalities upon market competition, product compatibility, and technology adoption. Diamond (1982) and Economides and Siow (1988) study the impact of network externalities in financial markets. Several papers also study dynamic pricing under network externalities. For example, Dhebar and Oren (1986) characterize the optimal nonlinear pricing strategy for a network product with heterogeneous customers. Bensaid and Lesne (1996) consider the optimal dynamic monopoly pricing under network externalities and show that the equilibrium prices increase as time passes. Cabral et al. (1999) show that, for a monopolist, the introductory price strategy is optimal under demand information incompleteness or asymmetry. Recently, the operations management (OM) literature starts to take into account the impact of network externalities upon a firm's operations strategy. For example, Ifrach et al. (2014) study the monopoly pricing problem in which network externalities arise from the fact that customers learn the quality of the product from their peers. Papanastasiou and Savva (2015) and Yu et al. (2015) study the impact of social learning (on product quality) when the product quality is uncertain and the customers are strategic. Hu and Wang (2014) study whether a firm should reveal the sales information of a network product under demand uncertainty. Wang and Wang (2015) propose and analyze the consumer choice models that endogenize network externalities. Under a newsvendor framework, Hu et al. (2016) propose efficient solutions for a firm to better cope with and benefit from the social influences between customers on online social media. In all aforementioned papers, network externalities depend on the total number of consumptions and have a uniform effect for all potential customers. We also adopt this approach to study the operations impact of network externalities.

There is another line of research that examines the local network effect by explicitly modeling the topology of the underlying network. Ballester et al. (2006) analyze the consumption game between players on a network with a given network structure, and characterize the key player therein. Zhou and Chen (2015) extend this work to the setting with sequential moves of the players. Candogan et al. (2012) and Bloch and Qu  rou (2013) study the optimal pricing strategy in a network with given network structure and characterize the relationship between optimal prices and consumers' centrality. Cohen and Harsha (2013) derive efficient and scalable methods to determine the optimal pricing strategies when customers are embedded in a social network. When the network structure is random, Campbell (2013) studies the optimal pricing strategy when a consumer makes a purchase only after s/he is informed of the product through a word-of-mouth percolation process on the network.

The literature on the joint pricing and inventory management problem under stochastic demand is rich. Petruzzi and Dada (1999) give a comprehensive review on the single period joint pricing and inventory control problem, and extend the results in the newsvendor problem with pricing. Federgruen and Heching (1999) show that a list-price/order-up-to policy is optimal for a general periodic-review joint pricing and inventory management model. When the demand distribution is unknown, Petruzzi and Dada (2002) address the joint pricing and inventory management problem under demand learning. Chen and Simchi-Levi (2004a, b, 2006) analyze the joint pricing and inventory control problem with fixed ordering cost. They show that  $(s, S, p)$  policy is optimal for finite horizon, infinite horizon and continuous review models. Chen et al. (2006) and Huh and Janakiraman (2008), among others, study the joint pricing and inventory control problem under lost sales. In the case of a single unreliable supplier, Li and Zheng (2006) and Feng (2010) show that supply uncertainty drives the firm to charge a higher price in the settings with random yield and uncertain capacity, respectively. Gong et al. (2014) and Chao et al. (2014) characterize the joint dynamic pricing and dual-sourcing policy of an inventory system facing the random yield risk and the disruption risk, respectively. When the replenishment leadtime is positive, the joint pricing and inventory control problem under periodic review is extremely difficult. For this problem, Pang et al. (2012) partially characterize the structure of the optimal policy, whereas Bernstein et al. (2015) develop a simple heuristic that resolves the computational complexity. Chen et al. (2014) characterize the optimal joint pricing and inventory control policy with positive procurement leadtime and perishable inventory. Several papers in this stream of literature also integrate consumer behaviors. Huh et al. (2010) characterize the optimal pricing and production policy under customer subscription and retention/attrition. Li and Huh (2011) establish the concavity of the objective function in the nested logit model, and apply this model to analyze the joint pricing and inventory management problem with multiple products. Yang and Zhang (2014) characterize the optimal

price and inventory policies under the scarcity effect of inventory, i.e., demand negatively correlates with inventory. Chen et al. (2007) and Yang (2012) adopt the idea of constant absolute risk aversion and time-consistent coherent and Markov risk measure, respectively, to characterize the optimal joint price and inventory control policy under risk aversion. When the firm adopts supply diversification to complement its pricing strategy, Zhou and Chao (2014) characterize the optimal dynamic pricing/dual-sourcing strategy, whereas Xiao et al. (2015) demonstrate how a firm should coordinate its pricing and sourcing strategies to address procurement cost fluctuation. We refer interested readers to Chen and Simchi-Levi (2012) for a comprehensive survey on joint pricing and inventory control models.

This paper contributes to the above two streams of research by operationalizing network externalities in the standard joint pricing and inventory management model, studying the impact of network externalities upon a firm's pricing and inventory policy, and identifying effective strategies and heuristics to exploit network externalities.

Finally, from the modeling perspective, this paper is related to the literature on inventory systems with positive inter-temporal demand correlations (see, e.g., Johnson and Thompson 1975, Graves 1997, Aviv 2002). We differentiate our paper from this line of research from the following two perspectives: First, we endogenize the pricing decision in our model and, thus, the firm can partially control the demand process via network externalities; Second, the inventory decision (i.e., service level) influences the future network size and, thus, the potential demand. As a consequence, our focus is on the trade-off between generating current profits and inducing future demands, whereas that literature focuses on the demand learning and inventory control issues with inter-temporally correlated demands. The new perspective and focus of our paper facilitate us to deliver new insights on the managerial implications of network externalities to the literature on inventory management with inter-temporal demand correlations.

### 3. Model Formulation

Consider a periodic-review stochastic joint pricing and inventory management model with full backlog. The firm sells a network product (e.g., a smartphone or a video game console) over a  $T$ -period planning horizon, labeled backwards as  $\{T, T-1, \dots, 1\}$ . For most of our analysis,  $T$  is finite. We will also discuss the case with  $T = +\infty$  when necessary. We assume that there is an online network associated with the product, which is either a service network (e.g., the App Store or the Xbox Live) or a social communication network (e.g. Facebook). Customers who purchase the product may join the network and exert network externalities onto potential customers in the future. We use  $N_t$  to denote the size of the associated network at the beginning of period  $t$ .

In each period  $t$ , a continuum of infinitesimal customers arrive at the market. Each customer has a type  $V$  and requests at most one product. Following Katz and Shapiro (1985), we assume

that the willingness-to-pay of a new customer in period  $t$  is given by  $V + \gamma(N_t)$ . The customer type  $V$  captures her intrinsic valuation of the product, which is independent of network externalities. The customer type  $V$  is uniformly positioned on the interval  $(-\infty, \bar{V}_t]$  with density 1. Clearly, there is an infinite mass of potential customers in the market, but, as we will show shortly, the actual demand in each period  $t$  is finite. We take this modeling approach to rule out potential corner solutions and thus simplify expositions (see also Katz and Shapiro 1985). It is possible that a customer derives no intrinsic value from the product ( $V \leq 0$ ). Such customers will not make a purchase without network externalities. The function  $\gamma(\cdot)$  captures network externalities, and is concavely increasing and twice continuously differentiable in  $N_t$ . Thus, the larger the associated network, the greater utilities customers could derive from purchasing the product. We normalize  $\gamma(0) = 0$ , and assume that the intensity of network externalities diminishes as the network size goes to infinity (i.e.,  $\lim_{N_t \rightarrow +\infty} \gamma'(N_t) = 0$ ). For technical tractability, we assume that customers are bounded rational so that they base their purchasing decisions on the current sales price and network size, instead of rational expectations on future prices and network sizes. Therefore, a type- $V$  customer would make a purchase in period  $t$  if and only if  $V + \gamma(N_t) \geq p_t$ , where  $p_t \in [\underline{p}, \bar{p}]$  is the product price in period  $t$  and  $\underline{p}$  (resp.  $\bar{p}$ ) is the minimum (resp. maximum) allowable price. In each period  $t$ , there exists a random additive demand shock,  $\xi_t$ , which captures other uncertainties not explicitly modeled (e.g., the macro-economic condition of period  $t$ ). Hence, the actual demand in period  $t$  is given by:

$$D_t(p_t, N_t) := \int_{-\infty}^{\bar{V}_t} 1_{\{V + \gamma(N_t) \geq p_t\}} dV + \xi_t = \bar{V}_t + \gamma(N_t) - p_t + \xi_t,$$

where  $\xi_t$  is independent of the price  $p_t$  and the network size  $N_t$  with  $\mathbb{E}[\xi_t] = 0$ . Moreover,  $\{\xi_t : t = T, T-1, \dots, 1\}$  are *i.i.d.* continuously distributed random variables. Without loss of generality, we assume that  $D_t(p_t, N_t) \geq 0$  with probability 1, for all  $p_t \in [\underline{p}, \bar{p}]$  and  $N_t \geq 0$ .

We now introduce the dynamics of the network sizes  $\{N_t : t = T, T-1, \dots, 1\}$ . Given the current network size  $N_t$ , the network size of the next period,  $N_{t-1}$ , is determined by two effects. First, some customers who are already in the network will continue to stay there, but the rest may leave the network. For example, an Xbox player may remain enthusiastic about online gaming on Xbox Live for the first couple of years after purchasing an Xbox console. Analogously, a customer may keep posting her amazing experiences with her new car on Facebook for a few months. In period  $t$ , the percentage of customers who will stay in the network in the next period is  $\eta$ , the carry-over rate of network size. Thus, given  $N_t$ , there will be  $\eta N_t$  customers staying in the network in period  $t-1$ . Second, a fraction of new customers who purchase the product in period  $t$  are willing to join the network. For instance, some Xbox buyers also subscribe to the Xbox Live Gold Membership



and, thus, become part of the Xbox Live network. When demand  $D_t(p_t, N_t)$  exceeds the available inventory level  $x_t$ , some customers are backlogged and put onto a wait-list. Customers who get the products immediately have better consumer experiences and, thus, have a higher probability to join the network than those wait-listed. For each customer receiving the product, the probability that she will join the network is  $\theta \in (0, 1]$ . If a customer is wait-listed, the probability that she will join the network is  $\theta - \sigma$ , where  $\sigma \in [0, \theta]$ . Whether a customer will join the network is independent of her type  $V$ , the network size  $N_t$  and the attribute of other customers. Therefore, given  $(x_t, p_t, N_t)$ , there will be  $\theta(D_t(p_t, N_t) \wedge x_t) + (\theta - \sigma)(D_t(p_t, N_t) - x_t)^+$  customers joining the network in period  $t$ . We remark that the parameter  $\sigma$  measures the impact of service-level/inventory-availability on the customers' network-joining behavior. The larger the  $\sigma$ , the more significant the impact. When  $\sigma = 0$ , future demand depends on cumulative past *demand*; when  $\sigma = \theta$ , future demand depends on cumulative past *sales*. In addition, there exists a random shock  $\epsilon_t$  in the network size dynamics, capturing any uncertainty not explicitly modeled. Hence, the network size at the beginning of period  $t - 1$  is given by:

$$N_{t-1} = \eta N_t + \theta(x_t \wedge D_t(p_t, N_t)) + (\theta - \sigma)(D_t(p_t, N_t) - x_t)^+ + \epsilon_t = \eta N_t + \theta D_t(p_t, N_t) - \sigma(D_t(p_t, N_t) - x_t)^+ + \epsilon_t, \quad (1)$$

where  $\epsilon_t$  is independent of the price  $p_t$ , the network size  $N_t$ , the available inventory  $x_t$ , and the demand perturbations  $\{\xi_t : t = T, T - 1, \dots, 1\}$  with  $\mathbb{E}[\epsilon_t] = 0$ . Moreover,  $\{\epsilon_t : t = T, T - 1, \dots, 1\}$  are *i.i.d.* continuously distributed random variables.

If the associated network is a service network, the firm can generate profits via this network by charging service/subscription fees. For example, Microsoft charges an annual subscription fee of \$59.99 for the Xbox Live Gold membership, whereas Apple takes 30% of all revenues generated through apps in the App Store. For any network size  $N_t \geq 0$ , let  $r_n(N_t) \geq 0$  denote the net profit from the network in period  $t$ . Without loss of generality, we assume that  $r_n(\cdot)$  is a concavely increasing and continuously differentiable function with  $r_n(0) = 0$ . To focus on the firm's pricing and inventory policy of its *product*, we do not explicitly model the firm's pricing decision of its network service. Hence, the profit function of the network,  $r_n(\cdot)$ , is exogenously given. If the associated network is a social communication network where the customers share their purchasing and consumption experiences (e.g., Facebook), the firm earns no profit from the network, i.e.,  $r_n(\cdot) \equiv 0$ .

The state of the inventory system is given by  $(I_t, N_t) \in \mathbb{R} \times \mathbb{R}_+$ , where

$I_t$  = the starting inventory level before replenishment in period  $t$ ,  $t = T, T - 1, \dots, 1$ ;

$N_t$  = the starting network size of the product in period  $t$ ,  $t = T, T - 1, \dots, 1$ .

The decisions of the firm is given by  $(x_t, p_t) \in \hat{\mathcal{F}}(I_t) := [I_t, +\infty) \times [\underline{p}, \bar{p}]$ , where

$x_t$  = the available inventory level after replenishment in period  $t$ ,  $t = T, T - 1, \dots, 1$ ;

$p_t$  = the sales price charged in period  $t$ ,  $t = T, T - 1, \dots, 1$ .

In each period, the sequence of events unfolds as follows: At the beginning of period  $t$ , after observing the inventory level  $I_t$  and the network size  $N_t$ , the firm simultaneously chooses the inventory stocking level  $x_t \geq I_t$  and the sales price  $p_t$ , and pays the ordering cost  $c(x_t - I_t)$ . The inventory procurement leadtime is assumed to be zero, so that the replenished inventory is received immediately. The demand  $D_t(p_t, N_t)$  then realizes. The revenue from selling the product,  $p_t \mathbb{E}[D_t(p_t, N_t)]$ , and the profit from the associated network,  $r_n(N_t)$ , are collected. Unmet demand is fully backlogged. At the end of period  $t$ , the holding and backlogging costs are paid, the net inventory is carried over to the next period, and the network size is updated according to the network size dynamics (1).

We introduce the following model primitives:

$\alpha$  = discount factor of revenues and costs in future periods,  $0 < \alpha < 1$ ;

$c$  = inventory purchasing cost per unit ordered;

$b$  = backlogging cost per unit backlogged at the end of a period;

$h$  = holding cost per unit stocked at the end of a period.

Without loss of generality, we make the following assumptions on the model primitives:

$b > (1 - \alpha)c$ : The backlogging penalty is higher than the saving from delaying an order to the next period, so that the firm will not backlog all of its demand;

$\underline{p} > b + \alpha c$ : The margin for backlogged demand is positive.

For technical tractability, we make the following assumption throughout our theoretical analysis.

ASSUMPTION 1. For each period  $t$ ,  $R_t(\cdot, \cdot)$  is jointly concave in  $(p_t, N_t) \in [\underline{p}, \bar{p}] \times [0, +\infty)$ , where

$$R_t(p_t, N_t) := (p_t - b - \alpha c)(\bar{V}_t - p_t + \gamma(N_t)). \quad (2)$$

Given the sales price,  $p_t$ , and network size,  $N_t$ , of period  $t$ ,  $R_t(p_t, N_t)$  is the expected difference between the revenue and the total cost, which consists of ordering and backlogging costs, to satisfy the current demand in the next period. Hence, the joint concavity of  $R_t(\cdot, \cdot)$  implies that such difference has decreasing marginal values with respect to the current sales price and network size. While the concavity of revenue with respect to price is a common assumption in the pricing literature, the joint concavity of  $R_t(\cdot, \cdot)$  is a slightly stronger assumption, as it also captures the impact of network externalities upon revenue, procurement cost, and backlogging cost. We remark that  $R_t(\cdot, N_t)$  is strictly concave in  $p_t$  for any given  $N_t$ . Moreover, the monotonicity of  $\gamma(\cdot)$  suggests that  $R_t(\cdot, \cdot)$  is supermodular in  $(p_t, N_t)$ .

Assumption 1 is essential to show the analytical results in this paper, because it ensures the concavity of the objective function in each period (see Lemma EC.2). Thus, we discuss the conditions

under which this assumption holds in the Appendix. As shown by Lemma EC.4, the necessary and sufficient condition for Assumption 1 is that  $\underline{p} \geq \alpha c + b + \frac{M}{2}$ , where  $M := \sup\{-(\gamma'(N_t))^2/\gamma''(N_t) : N_t \geq 0\}$ . Since the sensitivity of demand with respect to price  $\frac{\partial \mathbb{E}[D_t(p_t, N_t)]}{\partial p_t} = -1$  is a constant, the condition  $\underline{p} \geq \alpha c + b + \frac{M}{2}$  is equivalent to that the price elasticity of demand,  $|\frac{d\mathbb{E}[D_t(p_t, N_t)]/\mathbb{E}[D_t(p_t, N_t)]}{dp_t/p_t}|$ , is sufficiently high relative to the network size elasticity of demand,  $|\frac{d\mathbb{E}[D_t(p_t, N_t)]/\mathbb{E}[D_t(p_t, N_t)]}{dN_t/N_t}|$ . Therefore, Assumption 1 has a clear and nonrestrictive economic interpretation: Compared with the primary demand leverage (i.e., sales price), network externalities have relatively less impact upon demand in general.

In Appendix EC.2.2, we demonstrate that Assumption 1 can be satisfied for a wide variety of network externalities functions, by giving some concrete examples of network externalities functions and deriving necessary and sufficient conditions for the concavity of  $R_t(\cdot, \cdot)$ . We also conduct extensive numerical experiments which verify the robustness of our analytical results when Assumption 1 does not hold. These numerical verifications are reported in Appendix EC.2.3.

## 4. Model Analysis

The purpose of this section is to simplify our model by demonstrating that the state space dimension of the dynamic program for the joint pricing and inventory replenishment problem can be reduced to 1. To this end, we first characterize the structure of the optimal joint pricing and inventory policy under network externalities.

### 4.1. Optimal Policy

We now formulate the planning problem as a dynamic program. Define

$v_t(I_t, N_t) :=$  the maximum expected discounted profits in periods  $t, t-1, \dots, 1$ , when starting period  $t$  with an inventory level  $I_t$  and network size  $N_t$ .

Without loss of generality, we assume that, in the last period (period 1), the excess inventory is salvaged with unit value  $c$ , and the backlogged demand is filled with ordering cost  $c$ , i.e.,  $v_0(I_0, N_0) = cI_0$  for any  $(I_0, N_0)$ . We define  $y_t(p_t, N_t) := \bar{V}_t - p_t + \gamma(N_t)$  as the expected demand with price  $p_t$  and network size  $N_t$ . The optimal value function  $v_t(I_t, N_t)$  satisfies the following recursive scheme:

$$v_t(I_t, N_t) = cI_t + \max_{(x_t, p_t) \in \hat{\mathcal{F}}(I_t)} J_t(x_t, p_t, N_t), \quad (3)$$

where  $\hat{\mathcal{F}}(I_t) := [I_t, +\infty) \times [\underline{p}, \bar{p}]$  denotes the set of feasible decisions and,

$$\begin{aligned} J_t(x_t, p_t, N_t) = & R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - y_t(p_t, N_t)) + r_n(N_t) \\ & + \mathbb{E}[\Psi_t(x_t - y_t(p_t, N_t) - \xi_t, \eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+)], \end{aligned}$$

with  $\Psi_t(x, y) := \alpha \mathbb{E}\{[v_{t-1}(x, y + \epsilon_t) - cx]\}$ ,

$$\Lambda(x) := \mathbb{E}\{-(b+h)(x - \xi_t)^+\},$$

$\beta := b - (1 - \alpha)c$  = the effective monetary benefit of ordering one unit of inventory.

The detailed derivation of  $J_t(x_t, p_t, N_t)$  is given by (EC.1) in the Appendix. Hence, for each period  $t$ , the firm selects

$$(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) := \arg \max_{(x_t, p_t) \in \hat{\mathcal{F}}(I_t)} J_t(x_t, p_t, N_t) \quad (4)$$

as the optimal inventory and price policy contingent on the state variable  $(I_t, N_t)$ .

As a stepping stone for our subsequent analysis, Lemma EC.2 in the Appendix shows that the value and objective functions  $v_t(\cdot, \cdot)$ ,  $J_t(\cdot, \cdot, \cdot)$ , and  $\Psi_t(\cdot, \cdot)$  are all jointly concave and continuously differentiable. In particular, the concavity and continuous differentiability of  $J_t(\cdot, \cdot, \cdot)$  ensure that, the optimal price and inventory policy,  $(x_t^*(I_t, N_t), p_t^*(I_t, N_t))$ , is well-defined and can be obtained via first-order conditions. Moreover, we can define the inventory-independent optimizer in each period  $t$  as follows:

$$(x_t(N_t), p_t(N_t)) := \arg \max_{x_t \in \mathbb{R}, p_t \in [\underline{p}, \bar{p}]} J_t(x_t, p_t, N_t). \quad (5)$$

In case of multiple optimizers, we select the lexicographically smallest one.

**THEOREM 1.** *For any  $t$ , the following statements hold:*

- (a) *If  $I_t \leq x_t(N_t)$ ,  $(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) = (x_t(N_t), p_t(N_t))$ .*
- (b) *If  $I_t > x_t(N_t)$ ,  $x_t^*(I_t, N_t) = I_t$  and  $p_t^*(I_t, N_t) = \arg \max_{p_t \in [\underline{p}, \bar{p}]} J_t(I_t, p_t, N_t)$ .*
- (c) *For any  $I_t \in \mathbb{R}$  and  $N_t \geq 0$ ,  $x_t^*(I_t, N_t) > 0$ .*

Theorem 1 characterizes the optimal policy as a network-size-dependent base-stock/list-price policy. If the starting inventory level  $I_t$  is below the network-size-dependent base-stock level  $x_t(N_t)$ , it is optimal to order up to this base-stock level, and charge a network-size-dependent list-price  $p_t(N_t)$ . If the starting inventory level is above the base-stock level, it is optimal to order nothing and charge an inventory-dependent sales price  $p_t^*(I_t, N_t)$ . Moreover, as shown in Theorem 1(c), the optimal period- $t$  order-up-to level  $x_t^*(I_t, N_t)$  is always positive for any inventory level  $I_t$  and network size  $N_t$ .

## 4.2. State Space Dimension Reduction

The original dynamic program to characterize the optimal joint pricing and inventory policy has a state space of two dimensions (inventory level  $I_t$  and network size  $N_t$ , see Section 4.1). Hence, it is analytically challenging and computationally complex to directly work with the recursive Bellman equation (3). In this subsection, we demonstrate that the dynamic program can actually be reduced to a much simpler one with a single-dimensional state space (network size  $N_t$ ). Moreover, as long

as the initial inventory level  $I_T$  is below the period- $T$  optimal base-stock level  $x_T(N_T)$ , the optimal policy in each period  $t$ ,  $(x_t^*(I_t, N_t), p_t^*(I_t, N_t))$ , is independent of the (stochastic and endogenous) inventory dynamics till this period  $\{I_s : s = T, T-1, \dots, t\}$  with probability 1. As we will show in Section 5, the state space dimension reduction serves as our stepping stone to deliver sharper insights on the managerial implications of network externalities.

We first simplify the objective function  $J_t(\cdot, \cdot, \cdot)$ . Let  $\Delta_t := x_t - y_t(p_t, N_t)$  be the safety stock level in period  $t$ , given that the firm sets inventory stocking level  $x_t$  and price  $p_t$ , and the current network size is  $N_t$ . It is straightforward to show that, in period  $t$ , maximizing  $(x_t, p_t)$  over the feasible set  $\hat{\mathcal{F}}(I_t)$  is equivalent to maximizing  $(\Delta_t, p_t)$  over the feasible set  $\mathcal{F}(I_t) := \{(\Delta_t, p_t) \in \mathbb{R} \times [\underline{p}, \bar{p}] : \Delta_t + y_t(p_t, N_t) \geq I_t\}$ . Therefore, the objective function in period  $t$  can be written as:

$$O_t(\Delta_t, p_t, N_t) = Q_t(p_t, N_t) + \beta\Delta_t + \Lambda(\Delta_t) + r_n(N_t) + \mathbb{E}[\Psi_t(\Delta_t - \xi_t, \eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(-\Delta_t + \xi_t)^+)],$$

where  $Q_t(p_t, N_t) := R_t(p_t, N_t) + \beta y_t(p_t, N_t) = (p_t - c)(\bar{V}_t - p_t + \gamma(N_t))$  is jointly concave in  $(p_t, N_t)$ . The detailed derivation of  $O_t(\cdot, \cdot, \cdot)$  is given by (EC.4) in the Appendix. For each network size  $N_t$ , we define  $(\Delta_t(N_t), p_t(N_t))$  as the inventory-independent optimizer of  $O_t(\cdot, \cdot, N_t)$ , i.e.,

$$(\Delta_t(N_t), p_t(N_t)) := \arg \max_{\Delta_t \in \mathbb{R}, p_t \in [\underline{p}, \bar{p}]} O_t(\Delta_t, p_t, N_t). \quad (6)$$

Thus,  $\Delta_t(N_t) = x_t(N_t) - y_t(p_t(N_t), N_t)$  is the optimal inventory-independent safety-stock level with network size  $N_t$ .

We now employ sample path analysis to characterize the inventory dynamics under the optimal joint pricing and inventory policy.

LEMMA 1. *For each period  $t$  and any network size  $N_t$ , we have*

$$\mathbb{P}[x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1}) | N_t] = 1. \quad (7)$$

Lemma 1 is a key technical result of our paper. We show that, if the firm adopts the optimal policy and period- $t$  starts with an inventory level below the optimal base-stock level (i.e.,  $I_t \leq x_t(N_t)$ ), the starting inventory level in period- $(t-1)$ ,  $I_{t-1} = x_t(N_t) - D_t(p_t(N_t), N_t)$ , will stay below the period- $(t-1)$  optimal base-stock level,  $x_{t-1}(N_{t-1})$ , with probability 1. The sample-path property (7) is a version of Condition 3(b) in Veinott (1965) in our joint pricing and inventory management model with network externalities. It has been widely shown in the inventory literature that this property (or its corresponding version in a different model) is essential in establishing the structure of an inventory system (e.g., Veinott 1965 and Section 6.3 of Porteus 2002). What makes Lemma 1 particularly interesting is that (7) holds even under a highly non-stationary demand process with dynamic pricing. The non-stationarity of the demand process under network externalities stems

from the fact that the network size process  $\{N_t : t = T, \dots, 1\}$  and customer type distribution (equivalently, the highest customer type  $\{\bar{V}_t : t = T, \dots, 1\}$ ) could be highly non-stationary. Such non-stationarity is also magnified by the intertwined pricing and inventory decisions. The key to the proof of Lemma 1 is to iteratively leverage the following two facts: (a) If being put on the wait-list does not affect the probability that a customer joins the network (i.e.,  $\sigma = 0$ ), the optimal safety-stock should be invariant throughout the planning horizon; and (b) if stock-out occurs in period  $t$  (i.e.,  $x_t(N_t) < D_t(p_t(N_t), N_t)$ ), the starting inventory is always below the optimal base-stock level in the next period, (i.e.,  $I_{t-1} < 0 < x_{t-1}(N_{t-1})$ ). The details of the proof are given in the Appendix.

An important implication of Lemma 1 is that once the starting inventory level falls below the optimal base-stock level in *some* period, it is optimal for the firm to replenish in *each* period thereafter throughout the planning horizon with probability 1. Since our model best fits the network product that is either a new product (e.g., the first-generation iPhone) or a new generation of an existing product (e.g., Xbox One), zero inventory is stocked at the beginning of the sales season, i.e.,  $I_T = 0$ . Therefore, Theorem 1(c) and Lemma 1 together imply that, with probability 1,  $I_t \leq x_t(N_t)$  for each period  $t$ .

Based on Lemma 1, we now show that the bivariate profit-to-go functions,  $\{v_t(I_t, N_t) : t = T, T-1, \dots, 1\}$ , can be transformed into univariate ones of network size  $N_t$  by normalizing the value of inventory  $cI_t$ . Let  $\pi_t(N_t) := \max\{O_t(\Delta_t, p_t, N_t) : \Delta_t \in \mathbb{R}, p_t \in [\underline{p}, \bar{p}]\}$  for any  $N_t \geq 0$ .

LEMMA 2. *The sequence of functions  $\{\pi_t(\cdot) : t = T, T-1, \dots, 1\}$  satisfy that: (i)  $\pi_t(\cdot)$  is concavely increasing and continuously differentiable in  $N_t$ ; (ii)  $v_t(I_t, N_t) = cI_t + \pi_t(N_t)$  for all  $N_t \geq 0$  and  $I_t \leq x_t(N_t)$ ; (iii) for all  $N_t \geq 0$  and  $\Delta_t \leq \Delta_t(N_t)$ ,*

$$O_t(\Delta_t, p_t, N_t) = Q_t(p_t, N_t) + \beta\Delta_t + \Lambda(\Delta_t) + r_n(N_t) + \mathbb{E}[G_t(\eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(-\Delta_t + \xi_t)^+)], \quad (8)$$

where  $G_t(y) := \alpha \mathbb{E}[\pi_{t-1}(y + \epsilon_t)]$ ; and (v)  $(\Delta_t(N_t), p_t(N_t))$  maximizes the right-hand side of equation (8) over the feasible set  $\mathbb{R} \times [\underline{p}, \bar{p}]$ .

Lemma 2 paves our way to reduce the original dynamic program (3), which has a two-dimension state-space, to one with a single-dimension state space. More specifically, it follows immediately from Lemma 2 that the optimal network-size-dependent safety-stock level and list-price in each period  $t$ ,  $(\Delta_t(N_t), p_t(N_t))$ , can be recursively determined by solving the following dynamic program with a single dimensional state-space of network size  $N_t$ :

$$\begin{aligned} \pi_t(N_t) &= \max_{\Delta_t \in \mathbb{R}, p_t \in [\underline{p}, \bar{p}]} O_t(\Delta_t, p_t, N_t), \text{ where} \\ O_t(\Delta_t, p_t, N_t) &= Q_t(p_t, N_t) + \beta\Delta_t + \Lambda(\Delta_t) + r_n(N_t) + \mathbb{E}[G_t(\eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(-\Delta_t + \xi_t)^+)], \\ \text{with } G_t(y) &:= \alpha \mathbb{E}[\pi_{t-1}(y + \epsilon_t)], \text{ and } \pi_0(\cdot) \equiv 0. \end{aligned} \quad (9)$$

Through reducing the original dynamic program (3) to the new one (9), we have essentially decoupled inventory and network size in the *state space*. This is interesting since the optimal inventory and pricing *decisions* have some intertwined impact upon the future network size and, thus, cannot be decoupled (the objective function  $O_t(\Delta_t, p_t, N_t)$  is non-separable in  $\Delta_t$  and  $p_t$  when  $t \geq 2$ ). To conclude this subsection, we give the following sharper characterization of the optimal joint pricing and inventory policy based on Theorem 1, Lemma 1, and Lemma 2.

**THEOREM 2.** *Assume that  $I_T \leq x_T(N_T)$ . In each period  $t$ ,  $(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) = (x_t(N_t), p_t(N_t))$  with probability 1, where  $x_t(N_t) = \Delta_t(N_t) + y_t(p_t(N_t), N_t)$ . Moreover,  $\{(\Delta_t(N_t), p_t(N_t)) : t = T, T-1, \dots, 1\}$  is the solution to the Bellman equation (9).*

Theorem 2 shows that, as long as the planning horizon starts with an inventory level below the period- $T$  optimal base-stock level (i.e.,  $I_T \leq x_T(N_T)$ ), the optimal pricing and inventory policy in each period  $t$ ,  $(x_t^*(I_t, N_t), p_t^*(I_t, N_t))$ , is identical to the optimal base-stock level and list-price,  $(x_t(N_t), p_t(N_t))$ , with probability 1. Although the firm holds inventory throughout the sales horizon, the optimal policy is independent of the inventory dynamics if the initial inventory level  $I_T$  is sufficiently low. As discussed above, in most applications, the firm holds zero initial inventory at the beginning of the sales season, i.e.,  $I_T = 0$ . Hence,  $(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) = (x_t(N_t), p_t(N_t))$  for all  $(I_t, N_t)$  with probability 1. The state-space dimensionality reduction of the dynamic program also helps alleviate the complexity of numerically computing the optimal policy  $\{(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) : t = T, T-1, \dots, 1\}$ . As shown in Theorem 2, it suffices to compute the inventory-independent policy  $\{(x_t(N_t), p_t(N_t)) : t = T, T-1, \dots, 1\}$ , which is the solution to a dynamic program with a single dimensional state space, (9). Based on Theorem 2, unless otherwise specified, we will confine our analysis to the properties of the optimal base-stock level and list-price  $(x_t(N_t), p_t(N_t))$  for the rest of this paper.

## 5. Trade-off between Current Profits and Future Demands

This section strives to answer the following two questions: (a) How does the presence of network externalities impact the operations decisions of the firm? (b) What strategies can the firm employ to exploit network externalities? The answers to these questions shed lights on the managerial implications of network externalities. We show that the firm under network externalities is facing the key trade-off between generating current profits and inducing future demands. This trade-off yields several interesting new insights on the operational implications of network externalities.

### 5.1. Impact on Joint Pricing and Inventory Policy

We start our analysis with a comparison between our joint pricing and inventory management model with network externalities and the benchmark model without (i.e., Federgruen and Heching 1999). The benchmark model corresponds to a special case of our model with  $\gamma(\cdot) \equiv 0$  and  $r_n(\cdot) \equiv 0$ .

**THEOREM 3.** *Assume that two inventory systems are identical except that one with network externalities function  $\gamma(\cdot)$  and the other with  $\hat{\gamma}(\cdot)$ , where  $\gamma(0) = \hat{\gamma}(0) = 0$  and  $\hat{\gamma}(N_t) \geq \gamma(N_t) \equiv 0$  for all  $N_t \geq 0$ , i.e., the inventory system with  $\gamma(\cdot)$  exhibits no network externalities. Moreover, let  $\hat{r}_n(\cdot) = r_n(\cdot) \equiv 0$ . For each period  $t$  and any network size  $N_t \geq 0$ , the following statements hold: (a)  $\hat{\Delta}_t(N_t) \geq \Delta_t(N_t) \equiv \Delta_*$ , where  $\Delta_* := \arg \max_{\Delta} \{\beta\Delta + \Lambda(\Delta)\}$  and the inequality is strict if  $\sigma > 0$  and  $\hat{\gamma}'(\cdot) > 0$ ; (b)  $\hat{x}_t(N_t) \geq x_t(N_t) \equiv x_*(t)$ , where the inequality is strict if  $\sigma > 0$  and  $\hat{\gamma}'(\cdot) > 0$ ; (c) there exists a threshold  $\mathfrak{N}_t \geq 0$ , such that  $\hat{p}_t(N_t) \leq p_t(N_t) \equiv p_*(t)$  for  $N_t \leq \mathfrak{N}_t$ , whereas  $\hat{p}_t(N_t) \geq p_t(N_t) \equiv p_*(t)$  for  $N_t \geq \mathfrak{N}_t$ .*

Under network externalities, a service effect emerges: The firm should keep a higher safety-stock to mitigate the potential reductions of network size due to unsatisfied customers when stock-out occurs. As shown in Theorem 3(a,b), the presence of network externalities drives the firm to increase both the safety-stock and the base-stock levels in each period  $t$  (i.e.,  $\hat{\Delta}_t(N_t) \geq \Delta_*$  and  $\hat{x}_t(N_t) \geq x_*(t)$ ). Unlike the standard setting without network externalities where the price is independent of the network size, the firm under network externalities should charge differentiated prices contingent on different network sizes. Theorem 3(c) demonstrates the interesting effect of network externalities upon the firm's pricing policy: The optimal price with network externalities,  $\hat{p}_t(\cdot)$ , may be either higher or lower than that without,  $p_*(t)$ . More specifically, if the network size  $N_t$  is sufficiently small (i.e., below the threshold  $\mathfrak{N}_t$ ), the optimal price in the presence of network externalities,  $\hat{p}_t(\cdot)$ , is lower than that without,  $p_*(t)$ . On the other hand, if the network size is sufficiently large (i.e., above the threshold  $\mathfrak{N}_t$ ), the firm should increase the price in the presence of network externalities, i.e.,  $\hat{p}_t(N_t) \geq p_*(t)$ . Under network externalities, the firm faces the trade-off between decreasing the price to induce high future demands and increasing the price to exploit the current market condition. When the current network size is small ( $N_t \leq \mathfrak{N}_t$ ), the firm should put higher weight on inducing future demands, so the optimal price is lower with network externalities. Otherwise,  $N_t \geq \mathfrak{N}_t$ , generating current profits outweighs inducing future demands, and, hence, the optimal price is higher with network externalities. In short, Theorem 3 reveals that, because of the trade-off between generating current profits and inducing future demands, network externalities give rise to the service effect and network-size-depended pricing. We now proceed to investigate how the firm's optimal pricing and inventory policy responds to different network sizes.

**THEOREM 4.** *For period  $t$ , assume that  $\hat{N}_t > N_t$ . We have: (a)  $p_t(\hat{N}_t) \geq p_t(N_t)$ ; (b)  $\Delta_t(\hat{N}_t) \leq \Delta_t(N_t)$ ; and (c) if  $\gamma(\hat{N}_t) = \gamma(N_t)$ , then  $x_t(\hat{N}_t) \leq x_t(N_t)$ .*

Theorem 4 sharpens our understanding of how the trade-off between generating current profits and inducing future demands impacts the price and inventory policy of the firm. More specifically, we show that the optimal price  $p_t(N_t)$  is increasing in the current network size  $N_t$ , whereas the



optimal safety-stock  $\Delta_t(N_t)$  is decreasing in  $N_t$ . As the network size increases (resp. decreases), the potential demand becomes larger (resp. smaller), and thus the firm is prompted to focus more on generating current profits (resp. inducing future demands) by increasing (resp. decreasing) the price. Analogously, with a larger network size, the safety-stock level should be decreased. In this case, the service effect is weakened and the firm sets a lower safety-stock to save the procurement and holding costs. In summary, the firm puts more weight on generating current profits when the network size is large, whereas it focuses more on inducing future demands when the network size is small.

Contrary to our intuition, Theorem 4(c) shows that the optimal base-stock level  $x_t(N_t)$  may not necessarily be increasing in  $N_t$ . In the region without network externalities (i.e.,  $\gamma'(\cdot) = 0$ ), the current market condition is not better with a larger network size, but an increased sales price (Theorem 4(a)) and a lower safety-stock level (Theorem 4(b)) drive the resulting optimal base-stock level lower as the network size increases.

The trade-off between generating current profits and inducing future demands gives rise to the service effect and network-size-dependent pricing. Our next step is to study how these two phenomena evolve throughout the planning horizon. As shown in the following theorem, when the market is stationary, network externalities motivate the firm to set lower sales prices and higher safety-stock and base-stock levels at the beginning of the sales horizon.

**THEOREM 5.** *Assume that  $\bar{V}_\tau = \bar{V}$  for all  $\tau$ . For each  $t \geq 2$  and any network size  $N \geq 0$ , we have (a)  $\Delta_t(N) \geq \Delta_{t-1}(N)$ , (b)  $x_t(N) \geq x_{t-1}(N)$ , and (c)  $p_t(N) \leq p_{t-1}(N)$ .*

Theorem 5 shows that the service effect becomes less intensive as the time approaches the end of the sales horizon. Specifically, with stationary customer type distribution (i.e.,  $\bar{V}_t$  is independent of time  $t$ ) and the same network size  $N$  (thus, the same potential demand), the optimal safety-stock  $\Delta_t(N)$  and the optimal base-stock level  $x_t(N)$  are decreasing, whereas the optimal sales price  $p_t(N_t)$  is increasing over the sales horizon. Under network externalities, the firm should put more weight on inducing future demands at the beginning of the planning horizon and turn to generating current profits as it approaches the end of the sales season. Hence, the firm offers discounts and increases the safety-stock (and, thus, base-stock) level to attract more customers to purchase the product and join the network at the early stage of a sales season. On the other hand, the firm charges a higher price and sets a lower safety-stock to exploit the current market towards the end of the planning horizon.

Theorem 5 is consistent with the commonly used introductory price strategy under which price discounts are offered at the beginning of the sales season. For example, when Microsoft introduced the 500 GB Xbox 360 to the India video game market, it charged a surprisingly low introductory

price of \$313.9 (see, e.g., Kumar 2015). When the customer valuation is not stationary (i.e.,  $\bar{V}_t$  is time-dependent), the introductory price strategy may not necessarily be optimal. This is because, if the customer valuation is higher at the beginning of the sales season, the firm may charge a higher price to exploit the customer preference over the product, as opposed to offering discounts.

To conclude this subsection, we analyze the impact of discount factor  $\alpha$  on the optimal joint pricing and inventory policy. The discount factor measures how the firm values future profits relative to the current profits. Studying its impact helps us better understand the trade-off between generating current profits and inducing future demands.

**THEOREM 6.** *Assume that two inventory systems are identical except that one with discount factor  $\hat{\alpha}$ , and the other with discount factor  $\alpha$ , where  $\hat{\alpha} > \alpha$ . For each period  $t$  and any network size  $N_t$ , we have (a)  $\hat{\Delta}_t(N_t) \geq \Delta_t(N_t)$ ; (b)  $\hat{x}_t(N_t) \geq x_t(N_t)$ ; and (c)  $\hat{p}_t(N_t) \leq p_t(N_t)$ .*

Future profits are more valuable to the firm with a larger discount factor, so the firm puts more weight on inducing future demands relative to generating current profits. Therefore, as the discount factor  $\alpha$  increases, the service effect gets strengthened (the safety-stock level  $\Delta_t(N_t)$  and the base-stock level  $x_t(N_t)$  both increase), and price discounts are offered (the price  $p_t(N_t)$  decreases). Both changes facilitate the firm to attract more customers into the network, and better balance the trade-off between current profits and future demands.

## 5.2. Mean-Reverting Network Sizes

We now proceed to study the stochastic network size evolution. Interestingly, under the optimal pricing and inventory policy, the network size process  $\{N_t : t = T, T-1, \dots, 1\}$  follows a mean-reverting pattern.

**THEOREM 7.** *Assume that  $\eta < 1$  and  $I_T \leq x_T(N_T)$ .*

(a) *There exists a threshold value  $\bar{N}_t \in (0, +\infty)$ , such that  $\mathbb{E}[N_{t-1}|N_t] > N_t$  for  $N_t < \bar{N}_t$ , and  $\mathbb{E}[N_{t-1}|N_t] < N_t$  for  $N_t > \bar{N}_t$ .*

(b) *Assume that  $\bar{V}_\tau \equiv \bar{V}$  for all  $\tau$  and  $T = +\infty$  (i.e., the infinite-horizon discounted reward criterion). We have  $N_t$  is ergodic and has a stationary distribution  $\nu(\cdot)$ , i.e.,  $\mathbb{P}(N_t \leq z) = \nu(z)$  for all  $t$  and all  $z$ . Moreover, there exists a threshold  $\bar{N} = \lim_{t \rightarrow +\infty} \bar{N}_t \in (0, +\infty)$ , such that,  $\mathbb{E}[N_{t-1}|N_t] > N_t$  for  $N_t < \bar{N}$ , and  $\mathbb{E}[N_{t-1}|N_t] < N_t$  for  $N_t > \bar{N}$ .*

(c)  *$\bar{N}_t$  and, thus,  $\bar{N}$  are increasing in  $\eta$ . In particular, as  $\eta \uparrow 1$ ,  $\bar{N}_t \uparrow +\infty$  and, thus,  $\bar{N} \uparrow +\infty$ .*

As long as some customers leave the network in each period (i.e.,  $\eta < 1$ ) and the initial inventory is below the initial base-stock level (i.e.,  $I_T \leq x_T(N_T)$ ), the network size dynamics exhibit a mean-reverting pattern: The future network size has an increasing trend ( $\mathbb{E}[N_{t-1}|N_t] > N_t$ ) if the current network size  $N_t$  is below the “mean”  $\bar{N}_t$ , but would in expectation decrease ( $\mathbb{E}[N_{t-1}|N_t] < N_t$ ) if

$N_t$  is above  $\bar{N}_t$ . Curiously, such mean-reversion persists even when there are only a tiny fraction of customers leaving the network (i.e.,  $\eta$  is very close to 1). Theorem 7(b) further shows that, if the customer preference of the product is stationary (i.e., the maximum intrinsic customer valuation  $\bar{V}_t$  is time-invariant), the network size  $N_t$  has a stationary distribution  $\nu(\cdot)$ . Therefore, the “mean”  $\bar{N}$  is time-invariant in this case. In Theorem 7(c), we demonstrate that if the network size carry-over rate  $\eta$  is higher, the “mean”  $\bar{N}_t$  (also  $\bar{N}$ ) increases, and, thus, the network size is more likely to grow. The mean-reverting pattern of the network size process clearly reflects the trade-off between generating current profits and inducing future demands via network externalities. With a small network size, the firm cares more about inducing future demands and, thus, strives to expand the network size through its joint pricing and inventory decisions. On the other hand, with a large network size, the firm focuses on exploiting the current market and, hence, the future network size would fall in expectation. In particular, if the network size carry-over rate  $\eta$  is larger, current customers are more influential on future ones through network externalities. Hence, the trade-off between current profits and future demands is more intensive and the firm adopts the joint pricing and inventory policy that drives the network size process to grow. In the limiting case where no existing customer in the network leaves (i.e.,  $\eta = 1$ ), the mean-reverting pattern vanishes and the network size grows throughout the planning horizon with probability 1.

### 5.3. Dynamic Look-Ahead Heuristic

The goal of this subsection is to propose an easy-to-implement dynamic look-ahead heuristic policy, and quantitatively justify its effectiveness in exploiting network externalities. This heuristic prescribes the joint pricing and inventory decisions that maximize the total profit of a (short) moving time-window throughout the planning horizon. We theoretically show that the profit gap between the optimal policy and the proposed heuristic decays exponentially as the length of the moving time-window increases. Therefore, our heuristic can achieve small optimality gaps even with a short moving time-window. This is also verified by our numerical examples. The key insight from our analysis is that the computationally efficient dynamic look-ahead heuristic policy could effectively exploit network externalities by balancing generating current profits and inducing demands in the *near future*.

To begin, we first numerically examine the profit losses of the benchmark heuristic, the *myopic policy*. The myopic policy is the simplest heuristic policy and it completely ignores the demand-inducing opportunities in the presence of network externalities. Adopting the myopic policy, the firm adjusts its price and inventory decisions to maximize the expected current-period profit, so the joint price and inventory decision under the myopic policy is the solution to a newsvendor problem with endogenous pricing.

Throughout the numerical studies in this section, we assume that the maximum intrinsic valuation  $\bar{V}_t = 30$  is stationary for each period  $t$ . The planning horizon length is  $T = 20$ . The network externalities function is  $\gamma(N_t) = kN_t$  ( $k \geq 0$ ). The parameter  $k$  measures the network externalities intensity. The larger the  $k$ , the more intensive network externalities the firm faces. Hence, the demand in each period  $t$  is  $D_t(p_t, N_t) = 30 + kN_t - p_t + \xi_t$ , where  $\{\xi_t\}_{t=1}^T$  follow *i.i.d.* normal distributions with mean 0 and standard deviation 2 truncated so that  $D_t(p_t, N_t) \geq 0$  with probability 1 for any  $(p_t, N_t)$ . We set the discount factor  $\alpha = 0.99$ , the unit procurement cost  $c = 8$ , the unit holding cost  $h = 1$ , the unit backlogging cost  $b = 10$ , and the feasible price range  $[p, \bar{p}] = [0, 25]$ . For simplicity, we assume that the random perturbation in the market size dynamics  $\epsilon_t$  is degenerate, i.e.,  $\epsilon_t = 0$  with probability 1. We also assume that a customer who gets the product immediately has the same network-joining probability as that of a wait-listed customer, i.e.,  $\sigma = 0$ . In the evaluation of the expected profits, we take  $I_t = 0$  as the reference initial inventory level and  $N_t = 0$  as the reference initial network size.

Let  $v_t^0(I_t, N_t)$  be the expected total profits in periods  $t, t-1, \dots, 0$  under the myopic policy, if period  $t$  starts with inventory  $I_t$  and network size  $N_t$ . The metric of interest is

$$\lambda_m := \frac{v_t(\cdot, \cdot) - v_t^0(\cdot, \cdot)}{v_t(\cdot, \cdot)} \times 100\%, \text{ which evaluates the (relative) profit loss of the myopic policy.}$$

We report the numerical results with the parameters  $t = 5, 10, 15, 20$ ,  $k = 0.2, 0.5, 0.8$ ,  $\theta = 0.2, 0.5, 0.8$ , and  $\eta = 0.2, 0.5, 0.8$ .

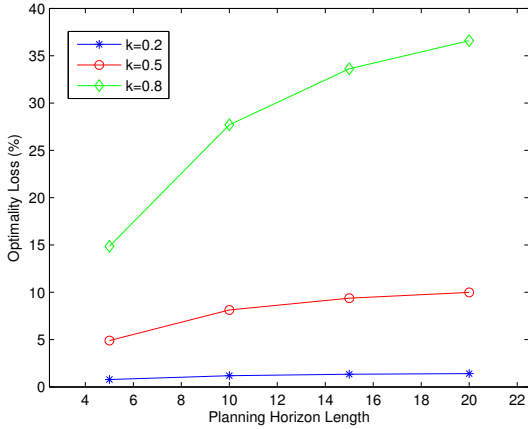


Figure 1 Value of  $\lambda_m$ :  $\theta = 0.5$ ,  $\eta = 0.5$

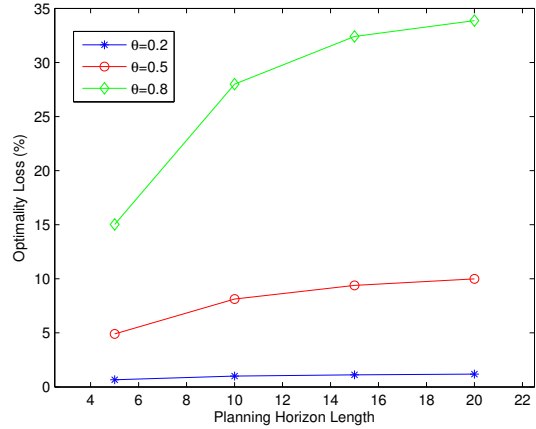


Figure 2 Value of  $\lambda_m$ :  $k = 0.5$ ,  $\eta = 0.5$

Figures 1-3 summarize the results of our numerical study on the profit performance of the myopic policy. As long as the network externalities intensity  $k$ , the network-joining probability of customers  $\theta$ , and the network size carry-over rate  $\eta$ , are not too low (greater than 0.2 in our numerical cases),

the myopic policy leads to a significant profit loss, which is at least 4.90% and can be as high as 36.60%. If  $k$ ,  $\theta$ , or  $\eta$  is large, the current operations decisions have great impact upon future network sizes, thus leading to intensive trade-off between generating current profits and inducing future demands. Therefore, the myopic policy results in significant profit losses if  $k$ ,  $\theta$ , and  $\eta$  are not too low. Another important implication of Figures 1-3 is that, if  $k$ ,  $\theta$ , and  $\eta$  are not too low, the profit loss of ignoring network externalities may be significant even when the planning horizon is short (i.e.,  $t = 5$ ). This calls for caution that the firm under network externalities should not overlook the trade-off between generating current profits and inducing future demands even for a short sales horizon.

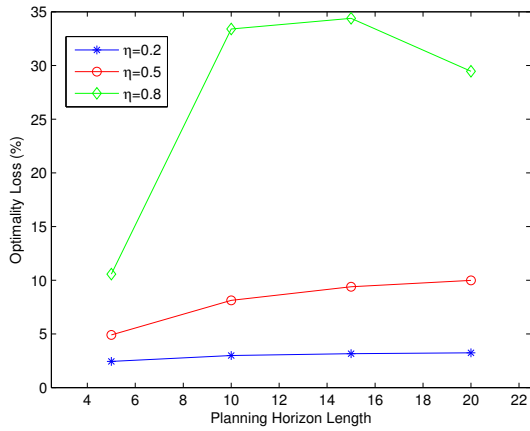


Figure 3 Value of  $\lambda_m$ :  $k = 0.5$ ,  $\theta = 0.5$

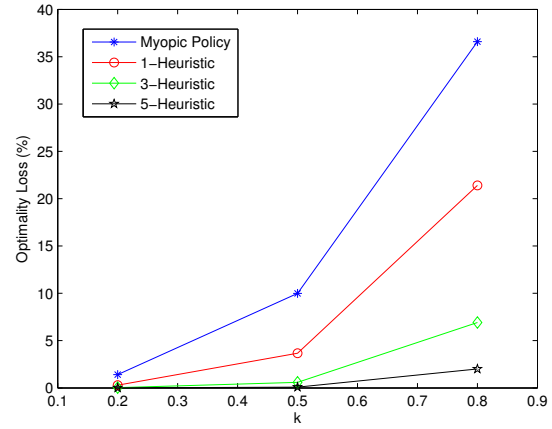


Figure 4 Value of  $\lambda_m$  and  $\lambda_h^i$ :  $\theta = 0.5$ ,  $\eta = 0.5$

The myopic policy completely ignores the demand-inducing opportunities under network externalities, and, therefore, gives rise to substantial profit losses. We now propose the dynamic look-ahead heuristic policy and study its value in exploiting network externalities. The key idea of the dynamic look-ahead heuristic is to (slightly) leverage the demand-inducing opportunities while keeping the computational simplicity of the myopic policy. This heuristic policy balances generating current profits and inducing demands in the *near future* through network externalities. More specifically, in each period  $t$ , the firm adopts the joint pricing and inventory management policy that maximizes the expected total discounted profits in the moving time-window of  $w + 1$  periods: The firm looks forward for  $w$  periods into the future, and maximizes the total profit from period  $t$  to period  $\min\{t - w, 1\}$ . Similar dynamic look-ahead heuristics (also called the rolling-horizon procedures) are widely used in the literature to (approximately) solve complex dynamic programming problems with a long planning horizon (see, e.g., Powell 2011). We refer to the dynamic look-ahead heuristic with the moving time-window of  $w + 1$  periods as the  $w$ -heuristic hereafter. Note that the

0–heuristic corresponds to the myopic policy, whereas the  $T$ –heuristic corresponds to the optimal policy. Obtaining the  $w$ –heuristic involves solving a dynamic program with planning horizon length  $w + 1$ , so it is computationally efficient and easy to implement if  $w$  is small.

We first theoretically justify the effectiveness of the  $w$ –heuristic policy in leveraging network externalities. More specifically, we show that, if the customer preference of the product is stationary (i.e.,  $\bar{V}_\tau \equiv \bar{V}$  for all  $\tau$ ), the gap between the optimal total profit and the total profit associated with the  $w$ –heuristic decays exponentially in the length of the moving time-window  $w$ . Formally, in the finite-horizon model (i.e.,  $T < +\infty$ ), let  $v_t^w(I_t, N_t)$  be the expected profits in periods  $t, t-1, \dots, 0$  when the firm adopts the  $w$ –heuristic and period  $t$  starts with inventory  $I_t$  and network size  $N_t$ . In the infinite-horizon discounted reward criterion model (i.e.,  $T = +\infty$ ), let  $v^w(I, N)$  (resp.  $v(I, N)$ ) be the expected discounted total profit when the firm adopts the  $w$ –heuristic (resp. optimal policy) and the planning horizon starts with inventory  $I$  and network size  $N$ .

**THEOREM 8.** *Assume that  $\bar{V}_\tau \equiv \bar{V}$  for all  $\tau$ ,  $\eta < 1$  and  $I_T \leq x_T(N_T)$ .*

(a) *If  $T < +\infty$ , we have  $v_t^w(\cdot, \cdot) \leq v_t^{w+1}(\cdot, \cdot) \leq v_t(\cdot, \cdot)$  for all  $w \geq 0$ . Moreover,  $v_t^w(\cdot, \cdot) = v_t(\cdot, \cdot)$  for  $w \geq t - 1$ .*

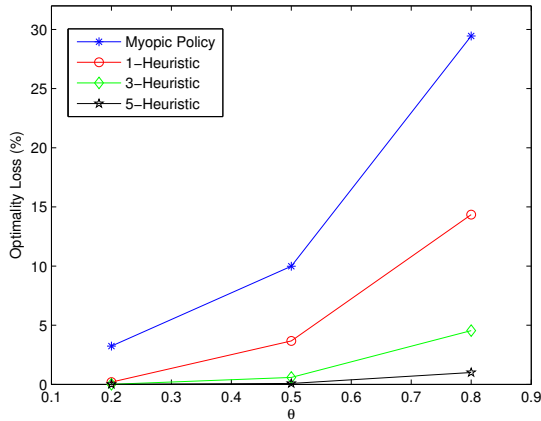
(b) *If  $T = +\infty$ , we have  $v^w(\cdot, \cdot) \leq v^{w+1}(\cdot, \cdot) \leq v(\cdot, \cdot)$  for all  $w \geq 0$ . There exist two constants  $C > 0$  and  $\delta > 0$ , such that  $\sup |v^w(\cdot, \cdot) - v(\cdot, \cdot)| \leq Ce^{-\delta w}$ . Thus,  $\lim_{w \rightarrow +\infty} v^w(\cdot, \cdot) = v(\cdot, \cdot)$ .*

As shown in Theorem 8, the  $w$ –heuristic is sub-optimal, but its performance improves as the moving time-window length  $w$  increases. Thus, if the firm looks ahead more into the future, it can better balance the current profits and future demands. Technically, this property results from the finding in Theorem 5 that, for a stationary market, the optimal price  $p_t(\cdot)$  is decreasing, whereas the optimal safety-stock level  $\Delta_t(\cdot)$  and base-stock level  $x_t(\cdot)$  are increasing in the time index  $t$ . The choice of the moving time-window length  $w$  highlights the trade-off between computational efficiency and profitability. A longer (resp. shorter) moving time-window yields more (resp. fewer) profits, but requires more (resp. fewer) computational efforts as well. Interestingly, this trade-off is not very intensive in the sense that, the optimality gap  $\sup |v^w(\cdot, \cdot) - v(\cdot, \cdot)|$  decays exponentially in the forward-looking length  $w$ . Therefore, even with a short moving time-window and, thus, light computational efforts, the  $w$ –heuristic could effectively exploit network externalities and achieve excellent performance in profitability. Specifically, to achieve an optimality gap of  $\epsilon$  (i.e.,  $\sup |v^w(\cdot, \cdot) - v(\cdot, \cdot)| < \epsilon$ ), it suffices to employ the  $w$ –heuristic with moving time-window length  $w \sim O(\log(1/\epsilon))$ .

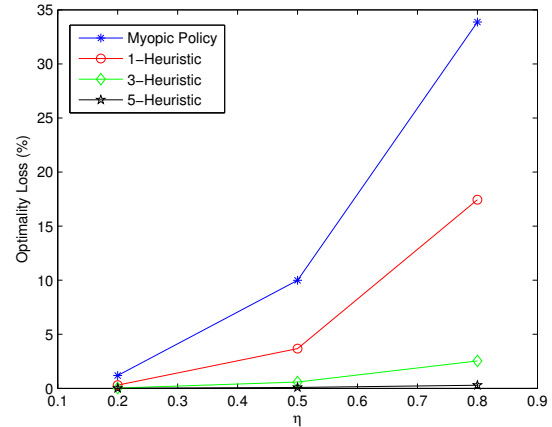
We now proceed to numerically demonstrate the effectiveness of the  $w$ –heuristic in exploiting network externalities even for a short moving time-window. The metric of interest is

$$\lambda_h^w := \frac{v_t(\cdot, \cdot) - v_t^w(\cdot, \cdot)}{v_t(\cdot, \cdot)} \times 100\%, \text{ which measures the optimality gap of the } w\text{–heuristic.}$$

The numerical experiments are under the parameters  $t = 20$ ,  $k = 0.2, 0.5, 0.8$ ,  $\theta = 0.2, 0.5, 0.8$ ,  $\eta = 0.2, 0.5, 0.8$ , and  $w = 1, 3, 5$ .



**Figure 5** Value of  $\lambda_m$  and  $\lambda_h^i$ :  $k = 0.5$ ,  $\eta = 0.5$



**Figure 6** Value of  $\lambda_m$  and  $\lambda_h^i$ :  $k = 0.5$ ,  $\theta = 0.5$

Figures 4-6 summarize the results of our numerical study on the performance of  $w$ -heuristics. We find that, compared with the myopic policy that completely ignores the demand-inducing opportunities, the  $w$ -heuristics significantly improve the profits in the presence of network externalities. In particular, the 5-heuristic leads to substantially lower profit losses than those of the myopic policy (below 2%, in contrast to the above 30% optimality gap of the myopic policy). This confirms our theoretical prediction that, even for a small  $w$  ( $w = 1, 3, 5$ ), the  $w$ -heuristic can achieve a very good profit performance. Therefore, the firm can effectively exploit network externalities by looking ahead into the near future and balancing the trade-off between generating current profits and inducing demands in the *near future*. Moreover, Figures 4-6 show that, as  $k$ ,  $\theta$ , or  $\eta$  increases, the trade-off between current profits and future demands becomes more intensive, and, thus, the look-ahead  $w$ -heuristics can deliver higher values to the firm when benchmarked against the myopic policy. We have also performed numerical analysis for the  $w$ -heuristics with longer moving time-windows (i.e.,  $w > 5$ ), which do not yield a significantly better performance over those with  $w = 5$ . This further demonstrates that, to exploit network externalities, it suffices to balance generating current profits and inducing demands in the *near future*. Finally, we remark that our numerical results are robust and continue to hold in the settings where the planning horizon length is greater than 20, and/or the market is non-stationary (i.e., the maximum intrinsic customer valuation  $\bar{V}_t$  is time-dependent), and/or wait-listed customers have a lower network-joining probability (i.e.,  $\sigma > 0$ ). For concision, we only present the results for the case where  $T = 20$ ,  $\bar{V}_t$  is time-invariant, and  $\sigma = 0$  in this paper.

In summary, network externalities have several important operational implications upon the joint pricing and inventory policy of the firm. Most notably, network externalities create another layer of complexity in balancing the trade-off between generating current profits and inducing future demands. Therefore, network externalities result in the service effect, network-size-dependent pricing, and the mean-reverting pattern of the network size process. Although completely ignoring the trade-off between current profits and future demands leads to substantial profit losses, it suffices to adopt the dynamic look-head heuristic policies that balance current profits and near-future demands. This family of heuristics are easy to implement, and achieve low optimality gaps with exponential decay in the length of the look-ahead time-window.

## 6. Network Expansion Strategy

In this section, we study an effective strategy to exploit network externalities: the network expansion strategy (with a cost). Since the willingness-to-pay of the customers is increasing in the size of the associated network, the firm may invest in network expansions to attract more customers into the network and, hence, increase its profitability.

The network expansion strategy is widely used to leverage network externalities, and it takes several different forms in practice. For example, in 2015, Microsoft offered price discounts for Xbox One buyers who commit to signing up for the Xbox Live Gold membership for at least one year (see, e.g., Godes 2015). In the case where the associated network is an online communication network (i.e.,  $r_n(\cdot) \equiv 0$ ), network expansion is the effort and investment the firm makes in social media marketing to attract customers to create and share the messages about the product on the social network (i.e., through the electronic word-of-mouth). In October 2014, Apple bought Twitter's Promoted Trend at a daily cost of \$200,000 to engage Twitter users for the new iPad Air 2 launch (Heine, 2014). Analogously, an online platform SponsoredTweets enables firms to pay customers to tweet about their products so as to leverage network externalities on social media (Fainmesser and Galeotti, 2016).

Let  $n_t$  be the number of additional customers who join the associated service or communication network in period  $t$ , which results from the firm's network expansion investment in this period. The total cost of attracting  $n_t$  additional customers into the network is  $c_n(n_t)$ , where  $c_n(\cdot)$  is a continuously differentiable and convexly increasing function of  $n_t$  with  $c_n(0) = 0$ . The convexity of  $c_n(\cdot)$  captures the decreasing marginal value of investing in network expansion. Note that although the network expansion investment does not change the inventory dynamics of the firm, it does have some impact on the network size dynamics. More specifically, by (1), the network size at the beginning of period  $t - 1$  with network expansion is given by:

$$N_{t-1} = \eta N_t + \theta D_t(p_t, N_t) - \sigma(D_t(p_t, N_t) - x_t)^+ + n_t + \epsilon_t.$$



We now formulate the dynamic program for the planning problem with the network expansion strategy. Define

$v_t^e(I_t, N_t) :=$  the maximum expected discounted profits with the network expansion strategy in periods

$t, t-1, \dots, 1, 0$ , when starting period  $t$  with an inventory level  $I_t$  and network size  $N_t$ ;

and  $(x_t^{e*}(I_t, N_t), p_t^{e*}(I_t, N_t), n_t^*(I_t, N_t))$  as the optimal pricing and inventory policy. As in the base model, we assume that, in the last period, the excess inventory is salvaged with unit value  $c$ , and the backlogged demand is filled with ordering cost  $c$ , i.e.,  $v_0^e(I_0, N_0) = cI_0$  for any  $(I_0, N_0)$ . Employing similar dynamic programming and sample path analysis methods, we characterize the optimal policy in the model with network expansion investment in the following lemma.

LEMMA 3. *Iteratively define a sequence of functions  $\{\pi_t^e(N_t) : t = T, T-1, \dots, 1\}$  and a sequence of pricing and inventory policies  $\{(x_t^e(N_t), p_t^e(N_t), n_t(N_t)) : t = T, T-1, \dots, 1\}$  as follows:*

$$\pi_t^e(N_t) = \max_{(\Delta_t, p_t, n_t) \in \mathcal{F}_e} O_t^e(\Delta_t, p_t, n_t, N_t), \quad (10)$$

$$\text{where } O_t^e(x_t, p_t, n_t, N_t) = Q_t(p_t, N_t) + \beta \Delta_t + \Lambda(\Delta_t) + r_n(N_t) - c_n(n_t) \\ + \mathbb{E}[G_t^e(\eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(-\Delta_t + \xi_t)^+ + n_t)],$$

$$\text{with } G_t^e(y) := \alpha \mathbb{E}[\pi_{t-1}^e(y + \epsilon_t)], \quad \pi_0^e(\cdot) \equiv 0,$$

$$(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t)) := \arg \max_{(\Delta_t, p_t, n_t) \in \mathcal{F}_e} O_t^e(x_t, p_t, n_t, N_t), \text{ and } \mathcal{F}_e := \mathbb{R} \times [\underline{p}, \bar{p}] \times \mathbb{R}_+.$$

Also define  $x_t^e(N_t) := \Delta_t^e(N_t) + y_t(p_t^e(N_t), N_t)$ .

(a) If  $I_t \leq x_t^e(N_t)$ ,  $(x_t^{e*}(I_t, N_t), p_t^{e*}(I_t, N_t), n_t^*(I_t, N_t)) = (x_t^e(N_t), p_t^e(N_t), n_t(N_t))$  and  $v_t^e(I_t, N_t) = cI_t + \pi_t^e(N_t)$ . Otherwise,  $I_t > x_t^e(N_t)$ ,  $x_t^{e*}(I_t, N_t) = I_t$ .

(b) If  $I_T \leq x_T^e(N_T)$ ,  $(x_t^{e*}(I_t, N_t), p_t^{e*}(I_t, N_t), n_t^*(I_t, N_t)) = (x_t^e(N_t), p_t^e(N_t), n_t(N_t))$  for all  $t$  with probability 1.

Lemma 3 demonstrates that a network-size-dependent base-stock/list-price/expansion-investment policy is optimal. Employing the same sample path analysis technique as in the base model, we can reduce the state space dimension of the dynamic program to 1: As long as the initial inventory level  $I_T$  is below the optimal period- $T$  base-stock level  $x_T^e(N_T)$ , the optimal policy is independent of the starting inventory level in each period with probability 1.

We remark that Theorems 3-8 are readily generalizable to the model with network expansion. For brevity, these results are not presented in the paper, but available from the authors upon request. We now demonstrate the impact of network externalities intensity upon the effectiveness of the network expansion strategy.

THEOREM 9. (a) Let  $0 < \iota < 1$ , and  $\bar{S}(N) := \sup\{z : \mathbb{P}(N_{t-1} \geq z | N_t = N) \geq \iota\}$ . If

$$\alpha(1 - \iota)[r'_n(\bar{S}(N)) + (\underline{p} - c)\gamma'(\bar{S}(N))] > c'_n(0), \quad (11)$$

then  $n_t^*(I_t, N) > 0$  for all  $I_t$ . Moreover,  $\bar{S}(N)$  is continuously increasing in  $N$  and, for each  $0 < \iota < 1$ , there exists an  $N_*(\iota) \geq 0$ , such that (11) holds for all  $N < N_*(\iota)$ .

(b) If  $\alpha(\sum_{\tau=1}^{t-1} (\alpha\eta)^\tau)(r'_n(0) + (\bar{p} - c)\gamma'(0)) \leq c'_n(0)$ ,  $n_t^*(I_t, N_t) \equiv 0$  for all  $I_t$  and  $N_t \geq 0$ .

Theorem 9 characterizes the dichotomy on when the firm should invest in network expansions. Theorem 9(a) shows that, when either (i) the intensity of network externalities is sufficiently strong or (ii) the associated service network is sufficiently profitable (as characterized by inequality (11)), it is optimal for the firm to invest in network expansions. In particular, the firm should adopt the network expansion strategy for a sufficiently low current network size (i.e.,  $n_t^*(I_t, N_t) > 0$  if  $N_t \leq N_*(\iota)$ ). The intuition behind Theorem 9(a) is that, if a lower bound of the marginal value of network expansion investment,  $\alpha(1 - \iota)[r'_n(\bar{S}(N)) + (\underline{p} - c)\gamma'(\bar{S}(N))]$ , dominates its marginal cost  $c'_n(0)$ , the firm should invest in network expansions. Here,  $\bar{S}(N)$  can be interpreted as the threshold such that, conditioned on  $N_t = N$ , the probability that the network size in period  $t - 1$  exceeds  $\bar{S}(N)$  is smaller than  $\iota$ , regardless of the joint pricing and inventory policy the firm employs. On the other hand, Theorem 9(b) shows that if network externalities are not sufficiently strong or the associated service network is not sufficiently profitable (i.e.,  $\alpha(\sum_{\tau=1}^{t-1} (\alpha\eta)^\tau)(r'_n(0) + (\bar{p} - c)\gamma'(0)) \leq c'_n(0)$ ), it is optimal for the firm not to invest in network expansions.

We now characterize the impact of the network expansion strategy upon the firm's optimal policy.

THEOREM 10. Assume that two inventory systems are identical except that one with the network expansion strategy and the other without. For each period  $t$  and any network size  $N_t \geq 0$ , the following statements hold: (a)  $\Delta_t^e(N_t) \leq \Delta_t(N_t)$ ; (b)  $x_t^e(N_t) \leq x_t(N_t)$ ; (c)  $p_t^e(N_t) \geq p_t(N_t)$ ; and (d)  $\pi_t^e(N_t) \geq \pi_t(N_t)$ , where the inequality is strict if  $n_t(N_t) > 0$ .

Theorem 10 characterizes how the firm should adjust its joint pricing and inventory policy under the network expansion strategy. Since the sales price, the safety-stock, and the network expansion investment all help induce future demands via network externalities, the adoption of the network expansion strategy allows the firm to set a lower safety-stock and increase the sales price to generate higher profit in the current period. As a result, the optimal base-stock level is also lower with market expansion investment. Theorem 10(d) further shows that the network expansion strategy can improve the profit in the presence of network externalities.

The network expansion strategy helps the firm exploit network externalities by attracting more customers into the network (with a cost) in each period. In particular, this strategy allows the firm

to induce future demands with network expansion investments, while generating higher current profit with a lower safety-stock and a higher sales price. The firm should invest in network expansions when the intensity of network externalities is sufficiently strong or the associated service network is sufficiently profitable.

## 7. Extension with Excessive Starting Inventory

The main focus of our analysis is on the scenario with the starting inventory level below the optimal base-stock level ( $I_t \leq x_t(N_t)$ ), because this scenario occurs with probability 1 as long as  $I_T \leq x_T(N_T)$  (see Theorem 2). This section partially characterizes the structural properties of the optimal policy when the starting inventory exceeds the optimal base-stock level (i.e.,  $I_t > x_t(N_t)$ ).

**THEOREM 11.** *Assume that  $\eta = 0$  and  $\sigma = 0$ . For each period  $t$ , the following statements hold:*

- (a)  $v_t(I_t, N_t)$  is supermodular in  $(I_t, N_t)$ .
- (b)  $x_t^*(I_t, N_t)$  is continuously increasing in  $I_t$  and  $N_t$ .
- (c)  $p_t^*(I_t, N_t)$  is continuously decreasing in  $I_t$ , and continuously increasing in  $N_t$ .
- (d) The optimal expected demand  $y_t^*(I_t, N_t) := \bar{V}_t - p_t^*(I_t, N_t) + \gamma(N_t)$  is continuously increasing in  $I_t$  and  $N_t$ . Hence,  $\mathbb{E}[N_{t-1}|N_t] = \theta y_t^*(I_t, N_t)$  is continuously increasing in  $I_t$  and  $N_t$ .
- (e) The optimal safety-stock  $\Delta_t^*(I_t, N_t) := x_t^*(I_t, N_t) - y_t^*(I_t, N_t)$  is continuously increasing in  $I_t$  and continuously decreasing in  $N_t$ .

In the special case of our general model where customers in the network will leave in the next period and the network-joining probability is irrelevant to whether a customer is wait-listed (i.e.,  $\eta = 0$  and  $\sigma = 0$ ), we are able to characterize some properties of the optimal policy for any starting inventory  $I_t$ . More specifically, Theorem 11(a) shows that the value function in each period  $t$ ,  $v_t(I_t, N_t)$  is supermodular in  $(I_t, N_t)$ . This is because, a larger network size leads to a larger potential demand and, thus, a higher marginal value of inventory. Analogously, the optimal expected demand  $y_t^*(I_t, N_t)$  and the optimal expected network size in the next period are both increasing in the network size  $N_t$ . As a consequence, if the network size is larger, the firm increases the order-up-to level  $x_t^*(I_t, N_t)$  to match demand with supply, and charges a higher sales price  $p_t^*(I_t, N_t)$  to exploit the better market condition. Theorem 11 also shows how the starting inventory level  $I_t$  influences the optimal policy: A higher starting inventory level prompts the firm to increase the safety stock and to charge a lower sales price.

## 8. Concluding Remarks

This is the first paper in the literature to study the joint pricing and inventory management problem under network externalities. To model network externalities, we assume that there is an online

service or communication network associated with the product, and the customers' willingness-to-pay is increasing in the size of this network. Moreover, in each period, a fraction of the customers who purchase the product would join the network and exert network externalities over potential customers in the future. The firm may directly generate profits from the network via, e.g., service subscription fees. The firm therefore faces an important trade-off between generating current profits and inducing future demands via network externalities.

The optimal policy is a network-size-dependent base-stock/list-price policy. Moreover, we demonstrate that, as long as the initial inventory level is below the initial optimal base-stock level, the inventory dynamics do not influence the optimal policy of the firm with probability 1. As a consequence, the state space dimension of the dynamic program can be reduced to one (network size) by normalizing the current inventory value. Such state space dimension reduction greatly facilitates the analysis and computation of the optimal policy, and paves our way to deliver sharper insights from our model.

Our analysis highlights the central role of the trade-off between generating current profits and inducing future demands through network externalities in the firm's operations strategy. The presence of network externalities gives rise to the service effect and network-size-dependent pricing, which are absent without network externalities. With network externalities, the firm should decrease the sales price to exploit the demand-inducing opportunity through network externalities when the network size is small, and increase the sales price to leverage the better market condition when the network size is large. From the inter-temporal perspective, the firm should put more weight on inducing future demands at early stages of a sales season than at later stages. Thus, when the market is stationary, the firm employs the introductory price strategy that offers early purchase discounts to induce high future demands at the beginning of the sale season. As a consequence of the firm's effort to balance the trade-off between current profits and future demands via network externalities, the network size process follows an interesting mean-reverting pattern: If the network size is small (resp. large), it has an increasing (resp. decreasing) trend in expectation. We also find that the dynamic look-ahead heuristic that maximizes the total profits in a (short) moving time-window can achieve small optimality gaps which decay exponentially in the length of the moving time-window. Therefore, it suffices to balance generating current profits and inducing demands in the near future.

Our analysis demonstrates the effectiveness of the widely adopted network expansion strategy in exploiting network externalities. This strategy facilitates the firm to partially separate generating current profits and inducing future demands through network externalities with an additional leverage (network expansion investment). The network expansion strategy should be employed when the intensity of network externalities is sufficiently strong or the associated service network

is sufficiently profitable. The firm invests in network expansions to induce future demands via network externalities, while generating a higher current profit with a decreased safety-stock level and an increased price.

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# Electronic Companions to “Dynamic Pricing and Inventory Management under Network Externalities”

## EC.1. Proofs of Statements

We use  $\partial$  to denote the derivative operator of a single variable function,  $\partial_x$  to denote the partial derivative operator of a multi-variable function with respect to variable  $x$ , and  $1_{\{\cdot\}}$  to denote the indicator function. For any multivariate continuously differentiable function  $f(x_1, x_2, \dots, x_n)$  and  $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$  in  $f(\cdot)$ 's domain,  $\forall i$ , we use  $\partial_{x_i} f(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$  to denote  $\partial_{x_i} f(x_1, x_2, \dots, x_n)|_{x=\tilde{x}}$ . We use  $\epsilon_1 \stackrel{d}{=} \epsilon_2$  to denote that two random variables  $\epsilon_1$  and  $\epsilon_2$  follow the same distribution.

The following lemma is used throughout our proofs.

**LEMMA EC.1.** *Let  $F_i(z, Z)$  be a continuously differentiable and jointly concave function in  $(z, Z)$  for  $i = 1, 2$ , where  $z \in [\underline{z}, \bar{z}]$  ( $\underline{z}$  and  $\bar{z}$  might be infinite) and  $Z \in \mathbb{R}^n$ . For  $i = 1, 2$ , let  $(z_i, Z_i) := \arg \max_{(z, Z)} F_i(z, Z)$  be the optimizers of  $F_i(\cdot, \cdot)$ . If  $z_1 < z_2$ , we have:  $\partial_z F_1(z_1, Z_1) \leq \partial_z F_2(z_2, Z_2)$ .*

**Proof:**  $z_1 < z_2$ , so  $\underline{z} \leq z_1 < z_2 \leq \bar{z}$ . Hence,  $\partial_z F_1(z_1, Z_1) \begin{cases} = 0 & \text{if } z_1 > \underline{z}, \\ \leq 0 & \text{if } z_1 = \underline{z}; \end{cases}$  and  $\partial_z F_2(z_2, Z_2) \begin{cases} = 0 & \text{if } z_2 < \bar{z}, \\ \geq 0 & \text{if } z_2 = \bar{z}, \end{cases}$  i.e.,  $\partial_z F_1(z_1, Z_1) \leq 0 \leq \partial_z F_2(z_2, Z_2)$ . *Q.E.D.*

**Derivation of  $J_t(x_t, p_t, N_t)$ :**

$$\begin{aligned}
 J_t(x_t, p_t, N_t) &= -cI_t + \mathbb{E}\{p_t D_t(p_t, N_t) - c(x_t - I_t) - h(x_t - D_t(p_t, N_t))^+ - b(x_t - D_t(p_t, N_t))^- \\
 &\quad + r_n(N_t) + \alpha v_{t-1}(x_t - D_t(p_t, N_t), \eta N_t + \theta D_t(p_t, N_t) - \sigma(D_t(p_t, N_t) - x_t)^+ + \epsilon_t) | N_t\}, \\
 &= (p_t - \alpha c - b)y_t(p_t, N_t) + (b - (1 - \alpha)c)x_t + r_n(N_t) + \mathbb{E}\{-(h + b)(x_t - y_t(p_t, N_t) - \xi_t)^+ \\
 &\quad + \alpha[v_{t-1}(x_t - y_t(p_t, N_t) - \xi_t, \theta(y_t(p_t, N_t) + \xi_t) + \eta N_t - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+ + \epsilon_t) \\
 &\quad - c(x_t - y_t(p_t, N_t) - \xi_t)] | N_t\} \\
 &= R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - y_t(p_t, N_t)) + r_n(N_t) \\
 &\quad + \mathbb{E}[\Psi_t(x_t - y_t(p_t, N_t) - \xi_t, \eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+)]. \quad (\text{EC.1})
 \end{aligned}$$

We begin with characterizing the preliminary concavity and differentiability properties of the value and objective functions in the following lemma, which serves as a stepping stone for our subsequent analysis.

**LEMMA EC.2.** *For each  $t = T, T - 1, \dots, 1$ , the following statements hold:*

- (a)  $\Psi_t(\cdot, \cdot)$  is jointly concave and continuously differentiable in  $(x, y)$ . Moreover,  $\Psi_t(x, y)$  is decreasing in  $x$  and increasing in  $y$ .
- (b)  $J_t(\cdot, \cdot, \cdot)$  is jointly concave and continuously differentiable in  $(x_t, p_t, N_t)$ .
- (b)  $v_t(\cdot, \cdot)$  is jointly concave and continuously differentiable in  $(I_t, N_t)$ . Moreover,  $v_t(I_t, N_t)$  is increasing in  $N_t$ , and  $v_t(I_t, N_t) - cI_t$  is decreasing in  $I_t$ .

Lemma EC.2 proves that, in each period  $t$ , the objective and value functions are concave and continuously differentiable. Moreover, after normalized with the value of inventory, the profit-to-go,  $v_t(I_t, N_t) - cI_t$ , is decreasing in the inventory level  $I_t$  and increasing in the network size  $N_t$ . Results similar to lemma EC.2 have also been established in other joint pricing and inventory management settings (see, e.g., Theorem 1 in Federgruen and Heching 1999).

**Proof of Lemma EC.2:** We prove parts (a) - (c) together by backward induction.

We first show, by backward induction that if  $v_{t-1}(I_{t-1}, N_{t-1}) - cI_{t-1}$  is jointly concave in  $(I_{t-1}, N_{t-1})$ , decreasing in  $I_{t-1}$ , and increasing in  $N_{t-1}$ , (i)  $\Psi_t(\cdot, \cdot)$  is jointly concave in  $(x, y)$ , decreasing in  $x$ , and increasing in  $y$ ; (ii)  $J_t(\cdot, \cdot, \cdot)$  is jointly concave in  $(x_t, p_t, N_t)$ ; and (iii)  $v_t(I_t, N_t) - cI_t$  is jointly concave in  $(I_t, N_t)$ , decreasing in  $I_t$ , and increasing in  $N_t$ . It is clear that  $v_0(I_0, N_0) - cI_0 = 0$  is jointly concave, decreasing in  $I_0$ , and increasing in  $N_0$ . Hence, the initial condition holds.

Assume that  $v_{t-1}(I_{t-1}, N_{t-1}) - cI_{t-1}$  is jointly concave in  $(I_{t-1}, N_{t-1})$ , decreasing in  $I_{t-1}$ , and increasing in  $N_{t-1}$ . Since concavity and monotonicity are preserved under expectation,  $\Psi_t(\cdot, \cdot)$  is jointly concave in  $(x, y)$ , decreasing in  $x$ , and increasing in  $y$ . Analogously,  $\Lambda(x)$  is concavely decreasing in  $x$ . We now verify that

$$\Phi_t(x_t, p_t, N_t) := \mathbb{E}[\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t, \eta N_t + \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) - \sigma(\bar{V}_t - p_t + \gamma(N_t) + \xi_t - x_t)^+)]$$

is jointly concave in  $(x_t, p_t, N_t)$  and increasing in  $N_t$ . Since  $\gamma(\cdot)$  is increasing in  $N_t$ ,  $\sigma \leq \theta$ , and  $\Psi_t(x, y)$  is decreasing in  $x$  and increasing in  $y$ ,  $\Phi_t(x_t, p_t, N_t)$  is increasing in  $N_t$ . Let  $\lambda \in [0, 1]$ ,  $x_* = \lambda x_t + (1 - \lambda)\hat{x}_t$ ,  $p_* = \lambda p_t + (1 - \lambda)\hat{p}_t$ , and  $N_* = \lambda N_t + (1 - \lambda)\hat{N}_t$ , we have:

$$\begin{aligned} & \lambda \Phi_t(x_t, p_t, N_t) + (1 - \lambda) \Phi_t(\hat{x}_t, \hat{p}_t, \hat{N}_t) \\ &= \lambda \mathbb{E}[\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t, \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) - \sigma(\bar{V}_t - p_t + \gamma(N_t) + \xi_t - x_t)^+ + \eta N_t)] \\ & \quad + (1 - \lambda) \mathbb{E}[\Psi_t(\hat{x}_t - \bar{V}_t + \hat{p}_t - \gamma(\hat{N}_t) - \xi_t, \theta(\bar{V}_t - \hat{p}_t + \gamma(\hat{N}_t) + \xi_t) - \sigma(\bar{V}_t - \hat{p}_t + \gamma(\hat{N}_t) + \xi_t - \hat{x}_t)^+ + \eta \hat{N}_t)] \\ &\leq \mathbb{E}[\Psi_t(A_t - \xi_t, \eta N_* + \theta B_t + \sigma C_t)] \end{aligned}$$

where

$$\begin{aligned} A_t &= x_* - \bar{V}_t + p_* - \lambda \gamma(N_t) - (1 - \lambda) \gamma(\hat{N}_t), \\ B_t &= \lambda(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) + (1 - \lambda)(\bar{V}_t - \hat{p}_t + \gamma(\hat{N}_t) + \xi_t), \\ C_t &= -\lambda(\bar{V}_t - p_t + \gamma(N_t) + \xi_t - x_t)^+ - (1 - \lambda)(\bar{V}_t - \hat{p}_t + \gamma(\hat{N}_t) + \xi_t - \hat{x}_t)^+, \end{aligned}$$

and the inequality follows from the joint concavity of  $\Psi_t(\cdot, \cdot)$ . Since  $\gamma(\cdot)$  is concave,  $A_t \geq x_* - \bar{V}_t + p_* - \gamma(N_*)$ . Since  $\gamma(\cdot)$  is concave,  $\theta \geq \sigma$ , and  $-(\cdot)^+$  is concavely decreasing,  $\theta B_t + \sigma C_t \leq \theta(\bar{V}_t - p_* + \gamma(N_*) + \xi_t) - \sigma(\bar{V}_t - p_* + \gamma(N_*) + \xi_t - x_*)^+$ . Therefore, since  $\Psi_t(x, y)$  is decreasing in  $x$  and increasing in  $y$ ,

$$\begin{aligned} & \mathbb{E}[\Psi_t(A_t - \xi_t, \eta N_* + \theta B_t + \sigma C_t)] \\ &\leq \mathbb{E}[\Psi_t(x_* - \bar{V}_t + p_* - \gamma(N_*) - \xi_t, \eta N_* + \theta(\bar{V}_t - p_* + \gamma(N_*) + \xi_t) - \sigma(\bar{V}_t - p_* + \gamma(N_*) + \xi_t - x_*)^+)], \end{aligned}$$

i.e.,  $\Psi_t(\cdot, \cdot, \cdot)$  is jointly concave in  $(x_t, p_t, N_t)$ .

Since  $\Lambda(x) = \mathbb{E}\{-(h+b)(x - \xi_t)^+\}$  is concavely decreasing in  $x$ , similar argument to the case of  $\Phi_t(x_t, p_t, N_t)$  implies that  $\Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t))$  is jointly concave in  $(x_t, p_t, N_t)$  and increasing in  $N_t$ . By Assumption 1,

$R_t(p_t, N_t)$  is jointly concave in  $(p_t, N_t)$ . Moreover, since  $\gamma(\cdot)$  is increasing in  $N_t$ ,  $R_t(p_t, N_t)$  is increasing in  $N_t$  as well. Finally, by definition,  $r_n(N_t)$  is concavely increasing in  $N_t$ . Hence,

$$J_t(x_t, p_t, N_t) = R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t)) + r_n(N_t) \\ + \mathbb{E}[\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t, \eta N_t + \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) - \sigma(\bar{V}_t - p_t + \gamma(N_t) + \xi_t - x_t)^+)]$$

is jointly concave in  $(x_t, p_t, N_t)$  and increasing in  $N_t$ .

Since concavity is preserved under maximization (e.g., Boyd and Vanderberghe 2004), the joint concavity of  $v_t(\cdot, \cdot)$  follows directly from that of  $J_t(\cdot, \cdot, \cdot)$ . Note that for any  $\hat{I}_t > I_t$ ,  $\hat{\mathcal{F}}(\hat{I}_t) \subseteq \hat{\mathcal{F}}(I_t)$ . Thus,

$$v_t(\hat{I}_t, N_t) - c\hat{I}_t = \max_{(x_t, p_t) \in \hat{\mathcal{F}}(\hat{I}_t)} J_t(x_t, p_t, N_t) \\ \leq \max_{(x_t, p_t) \in \hat{\mathcal{F}}(I_t)} J_t(x_t, p_t, N_t) \\ = v_t(I_t, N_t) - cI_t.$$

Hence,  $v_t(I_t, N_t) - cI_t$  is decreasing in  $I_t$ . Since  $J_t(x_t, p_t, N_t)$  is increasing in  $N_t$  for any  $(x_t, p_t, N_t)$ , for any  $\hat{N}_t > N_t$ ,

$$v_t(I_t, \hat{N}_t) - cI_t = \max_{(x_t, p_t) \in \hat{\mathcal{F}}(I_t)} J_t(x_t, p_t, \hat{N}_t) \\ \geq \max_{(x_t, p_t) \in \hat{\mathcal{F}}(I_t)} J_t(x_t, p_t, N_t) \\ = v_t(I_t, N_t) - cI_t.$$

Thus,  $v_t(I_t, N_t) - cI_t$  is increasing in  $N_t$ .

Second, we show, by backward induction, that if  $v_{t-1}(\cdot, \cdot)$  is continuously differentiable,  $\Psi_t(\cdot, \cdot)$ ,  $J_t(\cdot, \cdot, \cdot)$ , and  $v_t(\cdot, \cdot)$  are continuously differentiable as well. For  $t = 0$ ,  $v_0(I_0, N_0) = cI_0$  is clearly continuously differentiable. Thus, the initial condition holds.

If  $v_{t-1}(\cdot, \cdot)$  is continuously differentiable,  $\Psi_t(\cdot, \cdot)$  is continuously differentiable with partial derivatives given by

$$\partial_x \Psi_t(x, y) = \mathbb{E}\{\alpha[\partial_{I_t} v_{t-1}(x, y + \epsilon_t) - c]\}, \quad (\text{EC.2})$$

$$\partial_y \Psi_t(x, y) = \alpha \mathbb{E}\{\partial_{N_{t-1}} v_{t-1}(x, y + \epsilon_t)\}, \quad (\text{EC.3})$$

where the exchangeability of differentiation and expectation is easily justified using the canonical argument (see, e.g., Theorem A.5.1 in Durrett 2010, the condition of which can be easily verified observing the continuity of the partial derivatives of  $v_{t-1}(\cdot, \cdot)$ , and that the distributions of  $\xi_t$  and  $\epsilon_t$  are continuous.). Moreover, since  $\xi_t$  is continuously distributed,  $\Lambda(\cdot)$  and  $\Phi_t(x_t, p_t, N_t)$  is continuously differentiable with

$$\partial_{x_t} \Phi_t(x_t, p_t, N_t) = \mathbb{E}[\partial_x \Psi_t(x_t - y_t(p_t, N_t) - \xi_t, \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+ + \eta N_t)] \\ + \sigma \mathbb{E}[\partial_y \Psi_t(x_t - y_t(p_t, N_t) - \xi_t, \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+ + \eta N_t) 1_{\{\xi_t \geq x_t - y_t(p_t, N_t)\}}] \\ \partial_{p_t} \Phi_t(x_t, p_t, N_t) = -\mathbb{E}[\partial_x \Psi_t(x_t - y_t(p_t, N_t) - \xi_t, \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+ + \eta N_t)] \\ - \theta \mathbb{E}[\partial_y \Psi_t(x_t - y_t(p_t, N_t) - \xi_t, \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+ + \eta N_t)] \\ + \sigma \mathbb{E}[\partial_y \Psi_t(x_t - y_t(p_t, N_t) - \xi_t, \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+ + \eta N_t) 1_{\{\xi_t \geq x_t - y_t(p_t, N_t)\}}]$$

By assumption,  $R_t(\cdot, \cdot)$  and  $r_n(\cdot)$  are continuously differentiable. Therefore,  $J_t(\cdot, \cdot, \cdot)$  is continuously differentiable in  $(x_t, p_t, N_t)$ . If  $I_t \neq x_t(N_t)$ , the continuous differentiability of  $v_t(\cdot, \cdot)$  follows immediately from that of  $J_t(\cdot, \cdot, \cdot)$  and the envelope theorem. To complete the proof, it suffices to check that, for all  $N_t \geq 0$ , the left and right partial derivatives of the first variable at  $(x_t(N_t), N_t)$ ,  $\partial_{I_t} v_t(x_t(N_t)-, N_t)$  and  $\partial_{I_t} v_t(x_t(N_t)+, N_t)$  are equal. By the envelope theorem,

$$\begin{cases} \partial_{I_t} v_t(x_t(N_t)-, N_t) = c, \\ \partial_{I_t} v_t(x_t(N_t)+, N_t) = c + \beta + \partial_x \Lambda(x_t(N_t) - y_t(p_t(N_t), N_t)) + \partial_x \Phi_t(x_t(N_t), p_t(N_t), N_t). \end{cases}$$

The first-order condition with respect to  $x_t$  implies that

$$\beta + \partial_x \Lambda(x_t(N_t) - y_t(p_t(N_t), N_t)) + \partial_x \Phi_t(x_t(N_t), p_t(N_t), N_t) = 0.$$

Therefore,  $\partial_{I_t} v_t(x_t(N_t)-, N_t) = \partial_{I_t} v_t(x_t(N_t)+, N_t) = c$ . This completes the induction and, thus, the proof of Lemma EC.2. *Q.E.D.*

**Proof of Theorem 1: Parts (a)-(b)** follow immediately from the joint concavity of  $J_t(\cdot, \cdot, N_t)$  in  $(x_t, p_t)$  for any  $N_t \geq 0$ .

We now show **part (c)** by backward induction. More specifically, we prove that if  $x_{t-1}(N_{t-1}) > 0$  for all  $N_{t-1} \geq 0$ ,  $x_t(N_t) > 0$  for all  $N_t \geq 0$ . Since  $v_0(I_0, N_0) = cI_0$ ,  $\Psi_1(x, y) \equiv 0$ . Since  $D_1 \geq 0$  with probability 1,  $\partial_x \Lambda(-\bar{V}_1 + p_1 - \gamma(N_1)) = 0$  for all  $p_1 \in [\underline{p}, \bar{p}]$  and  $N_1 \geq 0$ . Hence, for any  $p_1 \in [\underline{p}, \bar{p}]$  and  $N_1 \geq 0$ ,

$$\partial_{x_1} J_1(0, p_1, N_1) = \beta - \partial_x \Lambda(-\bar{V}_1 + p_1 - \gamma(N_1)) = \beta > 0.$$

Hence,  $x_1(N_1) > 0$  for any  $N_1 \geq 0$ . Thus, the initial condition is satisfied.

Now we assume that  $x_{t-1}(N_{t-1}) > 0$  for all  $N_{t-1} \geq 0$  and  $x_t(\tilde{N}_t) \leq 0$  for some  $\tilde{N}_t \geq 0$ . Thus,

$$I_{t-1} = x_t(\tilde{N}_t) - D_t(p_t(\tilde{N}_t), \tilde{N}_t) \leq 0 < x_{t-1}(\tilde{N}_{t-1})$$

almost surely, where

$$\tilde{N}_{t-1} = \eta \tilde{N}_t + \theta D_t(p_t(\tilde{N}_t), \tilde{N}_t) - \sigma(D_t(p_t(\tilde{N}_t), N_t) - x_t(\tilde{N}_t))^+ + \epsilon_t.$$

Thus, by **part (a)**,  $\partial_{I_{t-1}} v_{t-1}(I_{t-1}, \tilde{N}_{t-1}) = c$  almost surely, when conditioned on  $N_t = \tilde{N}_t$ . Hence, conditioned on  $N_t = \tilde{N}_t$ ,  $\partial_x \Psi_t(x, y) = \alpha \mathbb{E}\{\partial_{I_{t-1}} v_{t-1}(I_{t-1}, \tilde{N}_{t-1}) - c | N_t = \tilde{N}_t\} = c - c = 0$ , when  $(x_t, p_t)$  lies in the neighborhood of  $(x_t(\tilde{N}_t), p_t(\tilde{N}_t))$ . Since  $x_t(\tilde{N}_t) \leq 0$ ,  $\partial_x \Lambda(x_t(\tilde{N}_t) - \bar{V}_t + p_t - \gamma(\tilde{N}_t)) = 0$  for all  $p_t \in [\underline{p}, \bar{p}]$ . Hence, for any  $p_t \in [\underline{p}, \bar{p}]$ ,

$$\partial_{x_t} J_t(x_t(\tilde{N}_t), p_t, \tilde{N}_t) = \beta - \partial_x \Lambda(x_t(\tilde{N}_t) - \bar{V}_t + p_t - \gamma(\tilde{N}_t)) = \beta > 0.$$

Hence,  $x_t(\tilde{N}_t) > 0$ , which contradicts the assumption that  $x_t(\tilde{N}_t) \leq 0$  is the optimizer of (5) when  $N_t = \tilde{N}_t$ . Therefore,  $x_t(N_t) > 0$  for all  $N_t \geq 0$ , whenever  $x_t(\tilde{N}_t) > 0$  for all  $\tilde{N}_t$ . This completes the induction and, thus, the proof of **part (c)**. *Q.E.D.*

**Derivation of  $O_t(\cdot, \cdot, \cdot)$ .**

$$\begin{aligned}
O_t(\Delta_t, p_t, N_t) &:= J_t(\Delta_t + y_t(p_t, N_t), p_t, N_t) \\
&= R_t(p_t, N_t) + \beta(\Delta_t + y_t(p_t, N_t)) + \Lambda(\Delta_t) + r_n(N_t) \\
&\quad + \mathbb{E}[\Psi_t(\Delta_t - \xi_t, \eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(-\Delta_t + \xi_t)^+)] \\
&= Q_t(p_t, N_t) + \beta\Delta_t + \Lambda(\Delta_t) + r_n(N_t) + \mathbb{E}[\Psi_t(\Delta_t - \xi_t, \eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(-\Delta_t + \xi_t)^+)].
\end{aligned} \tag{EC.4}$$

**Proof of Lemma 1:** We first show that (7) holds for the case  $\sigma = 0$ . Specifically, with backward induction, we show that, if  $\sigma = 0$ , (i) (7) holds for each period  $t$  and (ii)  $\Delta_t(N_t) = \Delta_*$  for all  $t$  and  $N_t$ , where  $\Delta_* = \arg \max_{\Delta} \{\beta\Delta + \Lambda(\Delta)\}$ . When  $t = 1$ ,  $\Psi_t(\cdot, \cdot) \equiv 0$ , optimizing the objective function in period 1, (EC.4), indicates that  $\Delta_1(N_1) = \Delta_*$  for all  $N_1$  and (7) automatically holds.

We now show that if (i) and (ii) hold for period  $t - 1$ , they also hold for period  $t$ . First, we prove that  $\Delta_t(N_t) \leq \Delta_*$ . If, to the contrary,  $\Delta_t(N_t) > \Delta_*$ , Lemma EC.1 yields that

$$\begin{aligned}
&\partial_{\Delta_t} [Q_t(p_t(N_t), N_t) + \beta\Delta_t(N_t) + \Lambda(\Delta_t(N_t)) + \mathbb{E}[\Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta \xi_t)]] \\
&\geq \partial_{\Delta} [\beta\Delta + \Lambda(\Delta)],
\end{aligned}$$

i.e.,

$$\beta + \Lambda'(\Delta_t(N_t)) + \mathbb{E}[\partial_x \Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta \xi_t)] \geq \beta + \Lambda'(\Delta_*).$$

The concavity of  $\Lambda(\cdot)$  implies that  $\Lambda'(\Delta_t(N_t)) \leq \Lambda'(\Delta_*)$ . Moreover, since  $\Psi_t(x, y)$  is decreasing in  $x$ ,  $\mathbb{E}[\partial_x \Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta \xi_t)] \leq 0$ . Therefore,  $\Lambda'(\Delta_t(N_t)) = \Lambda'(\Delta_*)$  and  $\mathbb{E}[\partial_x \Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta \xi_t)] = 0$ . Thus, by the first-order condition with respect to  $\Delta_t$ ,  $(p_t(N_t), \Delta_*)$  is also the optimal price and safety-stock level, which is strictly lexicographically smaller than  $(p_t(N_t), \Delta_t(N_t))$ . This contradicts the assumption that we select the lexicographically smallest optimizer in each period. Hence,  $\Delta_t(N_t) \leq \Delta_*$  for all  $N_t \geq 0$ .

We now show that (7) holds. Note that, conditioned on  $N_t$ ,

$$\begin{aligned}
x_t(N_t) - D_t(p_t(N_t), N_t) &= \Delta_t(N_t) - \xi_t \leq \Delta_* - \xi_t = x_{t-1}(N_{t-1}) - y_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) - \xi_t \\
&= x_{t-1}(N_{t-1}) - D_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) \text{ with probability 1,}
\end{aligned}$$

where the inequality follows from  $\Delta_t(N_t) \leq \Delta_*$ , the second equality from the hypothesis induction that  $x_{t-1}(N_{t-1}) = y_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) + \Delta_*$  for all  $N_{t-1} \geq 0$ , and the last equality from  $\xi_{t-1} \stackrel{d}{=} \xi_t$ . Because  $D_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) \geq 0$  with probability 1, conditioned on  $N_t$ ,

$$x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1}) - D_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) \leq x_{t-1}(N_{t-1})$$

with probability 1, i.e., (7) holds for period  $t$ .

To complete the induction, we show that  $\Delta_t(N_t) = \Delta_*$ . If, to the contrary  $\Delta_t(N_t) < \Delta_*$ , Lemma EC.1 implies that

$$\begin{aligned} & \partial_{\Delta_t} [Q_t(p_t(N_t), N_t) + \beta \Delta_t(N_t) + \Lambda(\Delta_t(N_t)) + \mathbb{E}[\Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta \xi_t)]] \\ & \leq \partial_{\Delta} [\beta \Delta + \Lambda(\Delta)]. \end{aligned}$$

On the other hand, (7) implies that  $\mathbb{E}[\partial_x \Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta \xi_t)] = 0$ . Thus,

$$\beta + \Lambda'(\Delta_t(N_t)) \leq \beta + \Lambda'(\Delta_*).$$

The concavity of  $\Lambda(\cdot)$  indicates that  $\Lambda'(\Delta_t(N_t)) = \Lambda'(\Delta_*)$ . Since  $\Delta_*$  is the smallest minimizer of  $[\beta \Delta + \Lambda(\Delta)]$ , we have  $\Delta_t(N_t) = \Delta_*$ . This completes the induction and, thus, the proof of the sample path property (7) for the case with  $\sigma = 0$ .

To complete the proof, it suffices to show (7) holds for the case  $0 < \sigma \leq \theta$ . Observe that the above argument continues to hold if  $\xi_t < \Delta_t(N_t)$ , since, in this case, the inventory stocking level does not affect future network size evolutions. By Theorem 1(c), if  $\xi_t \geq \Delta_t(N_t)$ , when conditioned on  $N_t$ ,

$$x_t(N_t) - D_t(p_t(N_t), N_t) = \Delta_t(N_t) - \xi_t \leq 0 < x_{t-1}(N_{t-1}) \text{ with probability 1,}$$

i.e., (7) holds for this case. Therefore, for any  $\sigma \in [0, \theta]$ , the sample-path property (7) holds. This completes the proof of Lemma 1. *Q.E.D.*

**Proof of Lemma 2:** By parts (a) and (b) of Theorem 1, if  $I_t \leq x_t(N_t)$ ,

$$v_t(I_t, N_t) = cI_t + \pi_t(N_t),$$

where

$$\pi_t(N_t) := \max\{J_t(x_t, p_t, N_t) : x_t \geq 0, p_t \in [\underline{p}, \bar{p}]\}.$$

By Lemma EC.2,  $\pi_t(\cdot)$  is concavely increasing and continuously differentiable in  $N_t$ .

By Lemma 1, for each  $N_t \geq 0$ ,  $x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1})$  with probability 1. Since  $v_{t-1}(I_{t-1}, N_{t-1}) = cI_{t-1} + \pi_{t-1}(N_{t-1})$  for all  $I_{t-1} \leq x_{t-1}(N_{t-1})$ ,

$$\begin{aligned} & v_{t-1}(x_t(N_t) - D_t(p_t(N_t), N_t), \theta D_t(p_t(N_t), N_t) + \eta N_t - \sigma(D_t(p_t(N_t), N_t) - x_t(N_t))^+ + \epsilon_t) \\ & = c[x_t(N_t) - D_t(p_t(N_t), N_t)] + \pi_{t-1}(\theta D_t(p_t(N_t), N_t) + \eta N_t - \sigma(D_t(p_t(N_t), N_t) - x_t(N_t))^+ + \epsilon_t) \end{aligned}$$

with probability 1. Taking expectation with respect to  $\epsilon_t$ , we have, for all  $N_t \geq 0$  and  $x_t \leq x_t(N_t)$ ,

$$\begin{aligned} & \Psi_t(\Delta_t(N_t) - \xi_t, \theta(y_t(p_t(N_t), N_t) + \xi_t) + \eta N_t - \sigma(\xi_t - \Delta_t(N_t))^+) \\ & = \alpha \mathbb{E}[\pi_{t-1}(\theta(y_t(p_t(N_t), N_t) + \xi_t) + \eta N_t - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Therefore, for all  $N_t \geq 0$ , if  $\Delta_t \leq \Delta_t(N_t)$ ,

$$O_t(\Delta_t, p_t, N_t) = Q_t(p_t, N_t) + \beta \Delta_t + \Lambda(\Delta_t) + r_n(N_t) + \mathbb{E}[G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

where  $G_t(y) := \alpha \mathbb{E}[\pi_{t-1}(y + \epsilon_t)]$ .

Finally, it remains to show that  $(\Delta_t(N_t), p_t(N_t))$  maximizes the right-hand side of (8). Note that Theorem 1(c) and Lemma 1 imply that, if  $I_t \leq x_t(N_t)$ , with probability 1,  $I_\tau \leq x_\tau(N_\tau)$  for all  $\tau = t, t-1, \dots, 1$

and, hence,  $\{(x_\tau(N_\tau), p_\tau(N_\tau))\}_{\tau=t, t-1, \dots, 1}$  is the optimal policy in periods  $t, t-1, \dots, 1$ . In particular,  $(x_t(N_t), p_t(N_t))$  maximizes the total expected discounted profit given that the firm adopts  $\{(x_\tau(N_\tau), p_\tau(N_\tau))\}$  for  $\tau = t-1, \dots, 1$ . It's straightforward to check that if the firm adopts the policy  $\{(x_\tau(N_\tau), p_\tau(N_\tau))\}$  for  $\tau = t-1, \dots, 1$ ; and sets the safety-stock level  $\Delta_t$  and charges  $p_t$  in period  $t$ , the (normalized) total expected discounted profit of the firm in period  $t$  is given by the right-hand side of (8). Since  $(\Delta_t(N_t), p_t(N_t))$  maximizes the (normalized) total expected discounted profit in period  $t$ , it also maximizes the right-hand side of (8) for each  $t$ . This proves Lemma 2. *Q.E.D.*

**Proof of Theorem 2:** By Theorem 1(c) and Lemma 1, if  $I_T \leq x_T(N_T)$ ,  $I_t \leq x_t(N_t)$  for all  $t = T, T-1, \dots, 1$  with probability 1. Therefore, by Theorem 1(a),  $(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) = (x_t(N_t), p_t(N_t))$  with probability 1 if  $I_T \leq x_T(N_T)$ . The characterization of  $(\Delta_t(N_t), p_t(N_t))$  follows immediately from Lemma 2 and its discussions. *Q.E.D.*

Before giving the proof of Theorem 3, we first show Theorem 2.

**Proof of Theorem 4: Part (a).** If  $\sigma = 0$ ,  $O_t(\Delta_t, p_t, N_t) = f_1(\Delta_t) + f_2(p_t, N_t)$ , where

$$f_1(\Delta_t) := \beta \Delta_t + \Lambda(\Delta_t) \text{ and}$$

$$f_2(p_t, N_t) := Q_t(p_t, N_t) + r_n(N_t) + \mathbb{E}[G_t(\eta N_t + \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t))].$$

Since  $G_t(\cdot)$  is concave,  $f_2(\cdot, \cdot)$  is supermodular in  $(p_t, N_t)$ . Thus,  $p_t(\hat{N}_t) \geq p_t(N_t)$  follows immediately (see Topkis 1998). Now we only consider the case  $\sigma > 0$ .

Assume, to the contrary,  $p_t(\hat{N}_t) < p_t(N_t)$ , Lemma EC.1 implies that  $\partial_{p_t} O_t(\Delta_t(\hat{N}_t), p_t(\hat{N}_t), \hat{N}_t) \leq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \partial_{p_t} Q_t(p_t(\hat{N}_t), \hat{N}_t) - \theta \mathbb{E}[G'_t(\eta \hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)] \\ & \leq \partial_{p_t} Q_t(p_t(N_t), N_t) - \theta \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Since  $Q_t(\cdot, \cdot)$  is supermodular in  $(p_t, N_t)$  and strictly concave in  $p_t$ ,  $\partial_{p_t} Q_t(p_t(\hat{N}_t), \hat{N}_t) > \partial_{p_t} Q_t(p_t(N_t), N_t)$ . Hence,

$$\mathbb{E}[G'_t(\eta \hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)] > \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \quad (\text{EC.5})$$

Since  $\hat{N}_t > N_t$  and  $p_t(\hat{N}_t) < p_t(N_t)$ ,  $y_t(p_t(\hat{N}_t), \hat{N}_t) > y_t(p_t(N_t), N_t)$ . The concavity of  $G_t(\cdot)$  and (EC.5) imply that  $\Delta_t(\hat{N}_t) < \Delta_t(N_t)$ . Thus, invoking Lemma EC.1, we have  $\partial_{\Delta_t} O_t(\Delta_t(\hat{N}_t), p_t(\hat{N}_t), \hat{N}_t) \leq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \beta + \Lambda'(\Delta_t(\hat{N}_t)) + \sigma \mathbb{E}[G'_t(\eta \hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+) 1_{\{\xi_t \geq \Delta_t(\hat{N}_t)\}}] \\ & \leq \beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\Delta_t(\hat{N}_t)) \geq \Lambda'(\Delta_t(N_t))$  and, thus,

$$\begin{aligned} & \mathbb{E}[G'_t(\eta \hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+) 1_{\{\xi_t \geq \Delta_t(\hat{N}_t)\}}] \\ & \leq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.6})$$

Since  $\Delta_t(\hat{N}_t) < \Delta_t(N_t)$ , it follows immediately that, for any realization of  $\xi_t$ ,

$$\begin{aligned} & G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+) 1_{\{\xi_t < \Delta_t(\hat{N}_t)\}} \\ & \leq G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and we have

$$\begin{aligned} & \mathbb{E}[G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+) 1_{\{\xi_t < \Delta_t(\hat{N}_t)\}}] \\ & \leq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.7})$$

Sum up (EC.6) and (EC.7) and we have:

$$\mathbb{E}[G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)] \leq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts (EC.5). Therefore,  $p_t(\hat{N}_t) \geq p_t(N_t)$  for all  $\hat{N}_t > N_t$ . This proves part (a).

**Part (b).** If  $\sigma = 0$ ,  $O_t(\Delta_t, p_t, N_t) = f_1(\Delta_t) + f_2(p_t, N_t)$ , so  $\Delta_t(\hat{N}_t) = \Delta_t(N_t)$ . Now we restrict ourselves to the case  $\sigma > 0$ .

Assume, to the contrary, that  $\Delta_t(\hat{N}_t) > \Delta_t(N_t)$ . Lemma EC.1 implies that  $\partial_{\Delta_t} O_t(\Delta_t(\hat{N}_t), p_t(\hat{N}_t), \hat{N}_t) \geq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \beta + \Lambda'(\Delta_t(\hat{N}_t)) + \sigma \mathbb{E}[G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+) 1_{\{\xi_t \geq \Delta_t(\hat{N}_t)\}}] \\ & \geq \beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\Delta_t(\hat{N}_t)) \leq \Lambda'(\Delta_t(N_t))$  and, thus,

$$\begin{aligned} & \mathbb{E}[G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+) 1_{\{\xi_t \geq \Delta_t(\hat{N}_t)\}}] \\ & \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.8})$$

The concavity of  $G_t(\cdot)$  implies that  $\eta\hat{N}_t + \theta y_t(p_t(\hat{N}_t), \hat{N}_t) < \eta N_t + \theta y_t(p_t(N_t), N_t)$  and, thus,  $p_t(\hat{N}_t) > p_t(N_t)$ . Since  $\Delta_t(\hat{N}_t) > \Delta_t(N_t)$ , it follows immediately that, for any realization of  $\xi_t$ ,

$$\begin{aligned} & G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+) 1_{\{\xi_t < \Delta_t(\hat{N}_t)\}} \\ & \geq G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and we have

$$\begin{aligned} & \mathbb{E}[G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+) 1_{\{\xi_t < \Delta_t(\hat{N}_t)\}}] \\ & \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.9})$$

Sum up (EC.8) and (EC.9) and we have:

$$\mathbb{E}[G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)] \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \quad (\text{EC.10})$$

Since  $p_t(\hat{N}_t) > p_t(N_t)$ , Lemma EC.1 implies that  $\partial_{p_t} O_t(\Delta_t(\hat{N}_t), p_t(\hat{N}_t), \hat{N}_t) \geq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \partial_{p_t} Q_t(p_t(\hat{N}_t), \hat{N}_t) - \theta \mathbb{E}[G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)] \\ & \geq \partial_{p_t} Q_t(p_t(N_t), N_t) - \theta \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Since  $y_t(p_t(\hat{N}_t), \hat{N}_t) < y_t(p_t(N_t), N_t)$ ,

$$\partial_{p_t} Q_t(p_t(\hat{N}_t), \hat{N}_t) = y_t(p_t(\hat{N}_t), \hat{N}_t) - p_t(\hat{N}_t) + c < y_t(p_t(N_t), N_t) - p_t(N_t) + c = \partial_{p_t} Q_t(p_t(N_t), N_t).$$



Thus,

$$\mathbb{E}[G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)] < \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts inequality (EC.10). Therefore,  $\Delta_t(\hat{N}_t) \leq \Delta_t(N_t)$  for any  $\hat{N}_t > N_t$ . This proves part (b).

**Part (c).** Since  $\gamma(\hat{N}_t) = \gamma(N_t)$ ,  $p_t(\hat{N}_t) \geq p_t(N_t)$  implies that  $y_t(p_t(\hat{N}_t), \hat{N}_t) \leq y_t(p_t(N_t), N_t)$ . Moreover, by part (b),  $\Delta_t(\hat{N}_t) \leq \Delta_t(N_t)$ . Therefore,

$$x_t(\hat{N}_t) = \Delta_t(\hat{N}_t) + y_t(p_t(\hat{N}_t), \hat{N}_t) \leq \Delta_t(N_t) + y_t(p_t(N_t), N_t) = x_t(N_t).$$

This proves part (c). *Q.E.D.*

**Proof of Theorem 3: Part (a).** Since  $\gamma(\cdot) \equiv \gamma_0$  and  $r_n(\cdot) \equiv 0$ ,  $\partial_{N_t} v_t(\cdot, \cdot) \equiv 0$ ,  $\pi'_t(\cdot) \equiv 0$ , and thus  $G'_t(\cdot) \equiv 0$ . Therefore, optimizing (9) yields that  $\Delta_t(N_t) \equiv \Delta_*$  for any  $t$  and  $N_t$ . To show  $\hat{\Delta}_t(N_t) \geq \Delta_*$ , we assume, to the contrary, that  $\hat{\Delta}_t(N_t) < \Delta_*$ . Lemma EC.1 yields that  $\partial_{\Delta_t} \hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \leq \beta + \Lambda'(\Delta_*)$ , i.e.,

$$\beta + \Lambda'(\hat{\Delta}_t(N_t)) + \sigma \mathbb{E}[G'_t(\eta N_t + \theta(y_t(\hat{p}_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \leq \beta + \Lambda'(\Delta_*).$$

Since  $\sigma \mathbb{E}[G'_t(\eta N_t + \theta(y_t(\hat{p}_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \geq 0$ , we have  $\Lambda'(\hat{\Delta}_t(N_t)) \leq \Lambda'(\Delta_*)$ . The concavity of  $\Lambda(\cdot)$  indicates that  $\Lambda'(\hat{\Delta}_t(N_t)) = \Lambda'(\Delta_*)$  and thus, by our assumption that  $\Delta_*$  is the lexicographically smallest optimizer,  $\hat{\Delta}_t(N_t) \geq \Delta_*$ . This contradicts with  $\hat{\Delta}_t(N_t) < \Delta_*$ . Thus,  $\hat{\Delta}_t(N_t) \geq \Delta_*$ .

If  $\hat{\gamma}'(\cdot) > 0$ ,  $\hat{G}'_t(\cdot) > 0$ . Thus, for any  $p_t$ ,

$$\begin{aligned} \partial_{\Delta_t} \hat{O}_t(\Delta_*, p_t, N_t) &= \beta + \Lambda'(\Delta_*) + \sigma \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(\xi_t - \Delta_*)^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ &= \sigma \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(\xi_t - \Delta_*)^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ &> 0, \end{aligned}$$

where the second equality follows from the first-order condition  $\beta + \Lambda'(\Delta_*) = 0$  and the inequality from  $\sigma > 0$  and  $\hat{G}'(\cdot) > 0$ . Hence,  $\hat{\Delta}_t(N_t) > \Delta_*$ . This proves part (a).

**Part (b).** We rewrite the objective function (9) in  $(\Delta_t, y_t, N_t)$ , where  $y_t = y_t(p_t, N_t) = \bar{V}_t - p_t + \gamma(N_t)$  is the expected demand in period  $t$ . It is clear that, given the network size  $N_t$ , price  $p_t$  and expected demand  $y_t$  have a one-to-one correspondence. Hence, optimizing over  $(\Delta_t, p_t, N_t)$  is equivalent to optimizing over  $(\Delta_t, y_t, N_t)$ . We transform the objective function  $O_t(\cdot, \cdot, \cdot)$  into

$$K_t(\Delta_t, y_t, N_t) = (\bar{V}_t + \gamma(N_t) - y_t - c)y_t + \beta \Delta_t + \Lambda(\Delta_t) + r_n(N_t) + \mathbb{E}[G_t(\eta N_t + \theta(y_t + \xi_t) - \sigma(\xi_t - \Delta_t))^+].$$

Let  $y_t(N_t)$  be the optimal expected demand in period  $t$  with network size  $N_t$ . We have  $y_t(N_t) = y_t(p_t(N_t), N_t) = \bar{V}_t - p_t(N_t) + \gamma(N_t)$ .

We now show that  $\hat{y}_t(N_t) \geq y_t(N_t)$  for all  $N_t$ . Assume, to the contrary, that  $\hat{y}_t(N_t) < y_t(N_t)$ . Lemma EC.1 yields that  $\partial_{y_t} \hat{K}_t(\hat{\Delta}_t(N_t), \hat{y}_t(N_t), N_t) \leq \partial_{y_t} K_t(\Delta_t(N_t), y_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} &\bar{V}_t - c - 2\hat{y}_t(N_t) + \hat{\gamma}(N_t) + \theta \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ &\leq \bar{V}_t - c - 2y_t(N_t) + \gamma(N_t) + \theta \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Because  $\hat{G}'_t(\cdot) \geq G'_t(\cdot) \equiv 0$  and  $\hat{y}_t(N_t) < y_t(N_t)$ , the concavity of  $\hat{G}_t(\cdot)$  implies that

$$\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].$$

Since  $\hat{\gamma}(N_t) \geq \gamma(N_t) \equiv 0$ , we have  $-2\hat{y}_t(N_t) \leq -2y_t(N_t)$ , which contradicts the assumption that  $\hat{y}_t(N_t) < y_t(N_t)$ . Hence,  $\hat{y}_t(N_t) \geq y_t(N_t)$ .

We now show  $y_t(N_t) \equiv y_*(t)$ . Observe that, since  $\gamma(\cdot) \equiv 0$  and  $G'_t(\cdot) \equiv 0$ , optimizing (9) yields that

$$p_t(N_t) = \arg \max_{p_t \in [\underline{p}, \bar{p}]} Q_t(p_t, N_t) = \min\{\max\{\frac{\bar{V}_t + c}{2}, \underline{p}\}, \bar{p}\} =: p_*(t).$$

Clearly,  $p_*(t)$  is independent of the network size  $N_t$ . Hence,  $y_t(N_t) = \bar{V}_t - p_t(N_t) + \gamma(N_t) = \bar{V}_t - p_*(t) =: y_*(t)$ , which is independent of the network size  $N_t$  as well.

Putting everything together, we have

$$\hat{x}_t(N_t) = \hat{y}_t(N_t) + \hat{\Delta}_t(N_t) \geq y_t(N_t) + \Delta_t(N_t) = y_*(t) + \Delta_* =: x_*(t).$$

Here,  $x_*(t)$  is independent of the network size  $N_t$ . By part (a), the inequality is strict if  $\sigma > 0$  and  $\gamma'(\cdot) > 0$ . This proves part (b).

**Part (c).** The equality  $p_t(N_t) \equiv p_*(t)$  has been shown in part (b). To show the existence of the threshold  $\mathfrak{N}_t$ , we first prove that  $\hat{p}_t(0) \leq p_t(0)$ . Observe that  $\hat{p}_t(0) = \bar{V}_t + \hat{\gamma}(0) - \hat{y}_t(0)$  and  $p_t(0) = \bar{V}_t + \gamma(0) - y_t(0)$ . By the proof of part (b),  $\hat{y}_t(0) \geq y_t(0)$ . Moreover, since  $\hat{\gamma}(0) = \gamma(0) = 0$ ,  $\hat{p}_t(0) \leq p_t(0)$ . By part (b),  $p_t(N_t) \equiv p_t(0)$  for all  $N_t$ . Recall from Theorem 4(a) that  $\hat{p}_t(N_t)$  is increasing in  $N_t$ . The joint concavity of  $\hat{O}_t(\cdot, \cdot, \cdot)$  implies that  $\hat{p}_t(N_t)$  is continuously increasing in  $N_t$ . Thus, let  $\mathfrak{N}_t$  be the smallest  $N_t$  such that  $\hat{p}_t(N_t) \geq p_t(N_t) = p_*(t)$ . The monotonicity of  $\hat{p}_t(\cdot)$  then suggests that  $\hat{p}_t(N_t) \leq p_t(N_t) \equiv p_*(t)$  if  $N_t \leq \mathfrak{N}_t$ , and  $\hat{p}_t(N_t) \geq p_t(N_t) \equiv p_*(t)$  if  $N_t \geq \mathfrak{N}_t$ . This proves part (c). *Q.E.D.*

**Proof of Theorem 5:** We show Theorem 5 by backward induction. More specifically, we show that if  $\bar{V}_\tau \equiv \bar{V}$  for all  $\tau$  and  $\pi'_{t-1}(N) \geq \pi'_{t-2}(N)$  for all  $N \geq 0$ , (i)  $p_t(N) \leq p_{t-1}(N)$  for all  $N \geq 0$ , (ii)  $\Delta_t(N) \geq \Delta_{t-1}(N)$  for all  $N \geq 0$ , (iii)  $x_t(N) \geq x_{t-1}(N)$  for all  $N \geq 0$ , and (iv)  $\pi'_t(N) \geq \pi'_{t-1}(N)$  for all  $N \geq 0$ . Since  $\pi'_1(N) \geq \pi'_0(N) \equiv 0$  for all  $N$ , the initial condition is satisfied.

Note that  $\pi'_{t-1}(N) \geq \pi'_{t-2}(N)$  for all  $N \geq 0$  implies that

$$G'_t(y) = \alpha \mathbb{E}[\pi'_{t-1}(y + \epsilon_t)] \geq \alpha \mathbb{E}[\pi'_{t-2}(y + \epsilon_t)] = G'_{t-1}(y),$$

for all  $y$ . Since  $\bar{V}_t \equiv \bar{V}$ ,  $Q_t(p_t, N_t) = (p_t - c)(\bar{V} - p_t + \gamma(N_t)) =: Q(p_t, N_t)$  for all  $t$ . We use  $y_t(N) := y_t(p_t(N), N)$  to denote the expected demand in period  $t$  with network size  $N$  under the optimal policy.

We first prove that  $p_t(N) \leq p_{t-1}(N)$  for all  $N$ . Assume, to the contrary, that  $p_t(N) > p_{t-1}(N)$  for some  $N$ . Lemma EC.1 implies that  $\partial_{p_t} O_t(\Delta_t(N), p_t(N), N) \geq \partial_{p_{t-1}} O_{t-1}(\Delta_{t-1}(N), p_{t-1}(N), N)$ , i.e.,

$$\begin{aligned} & \partial_p Q(p_t(N), N) - \theta \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \\ & \geq \partial_p Q(p_{t-1}(N), N) - \theta \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)]. \end{aligned}$$

Since  $Q(\cdot, N)$  is strictly concave in  $p$  and  $p_t(N) > p_{t-1}(N)$ ,  $\partial_p Q(p_t(N), N) < \partial_p Q(p_{t-1}(N), N)$ . Thus,

$$\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] < \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)]. \quad (\text{EC.11})$$

Note that  $G'_t(\cdot) \geq G'_{t-1}(\cdot)$  for all  $y$ ,  $p_t(N) > p_{t-1}(N)$ , and  $G_t(\cdot)$  and  $G_{t-1}(\cdot)$  are concave. We have (EC.11) implies that  $\sigma > 0$  and  $\Delta_t(N) > \Delta_{t-1}(N)$ . Thus, Lemma EC.1 implies that  $\partial_{\Delta_t} O_t(\Delta_t(N), p_t(N), N) \geq \partial_{\Delta_{t-1}} O_{t-1}(\Delta_{t-1}(N), p_{t-1}(N), N)$ , i.e.,

$$\begin{aligned} & \beta + \Lambda'(\Delta_t(N)) + \sigma \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t \geq \Delta_t(N)\}}] \\ & \geq \beta + \Lambda'(\Delta_{t-1}(N)) + \sigma \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\Delta_t(N)) \leq \Lambda'(\Delta_{t-1}(N))$  and, thus,

$$\begin{aligned} & \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t \geq \Delta_t(N)\}}] \\ & \geq \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}]. \end{aligned} \tag{EC.12}$$

Since  $\Delta_t(N) > \Delta_{t-1}(N)$  and  $G'_t(\cdot) \geq G'_{t-1}(\cdot) \geq 0$ , it follows immediately that, for any realization of  $\xi_t = \xi_{t-1}$ ,

$$\begin{aligned} & G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t < \Delta_t(N)\}} \\ & \geq G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and  $\xi_{t-1}$  and we have

$$\begin{aligned} & \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t < \Delta_t(N)\}}] \\ & \geq \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}]. \end{aligned} \tag{EC.13}$$

Sum up (EC.12) and (EC.13) and we have:

$$\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \geq \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)],$$

which contradicts (EC.11). Therefore,  $p_t(N) \leq p_{t-1}(N)$  for all  $N$ .

Next, we show that  $\Delta_t(N) \geq \Delta_{t-1}(N)$ . If  $\sigma = 0$ , it is straightforward to show that  $\Delta_t(N) = \Delta_{t-1}(N) = \Delta_*$ . Hence, we confine ourselves to the interesting case of  $\sigma > 0$ .

Assume, to the contrary, that  $\Delta_t(N) < \Delta_{t-1}(N)$ . Lemma EC.1 implies that  $\partial_{\Delta_t} O_t(\Delta_t(N), p_t(N), N) \leq \partial_{\Delta_{t-1}} O_{t-1}(\Delta_{t-1}(N), p_{t-1}(N), N)$ , i.e.,

$$\begin{aligned} & \beta + \Lambda'(\Delta_t(N)) + \sigma \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t \geq \Delta_t(N)\}}] \\ & \leq \beta + \Lambda'(\Delta_{t-1}(N)) + \sigma \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\Delta_t(N)) \geq \Lambda'(\Delta_{t-1}(N))$  and, thus,

$$\begin{aligned} & \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t \geq \Delta_t(N)\}}] \\ & \leq \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}]. \end{aligned} \tag{EC.14}$$

The concavity of  $G_t(\cdot)$  and  $G_{t-1}(\cdot)$  and that  $G'_t(\cdot) \geq G'_{t-1}(\cdot)$  imply that  $y_t(N) > y_{t-1}(N)$  and, thus,  $p_t(N) < p_{t-1}(N)$ . Since  $\Delta_t(N) < \Delta_{t-1}(N)$ , it follows immediately that, for any realization of  $\xi_t = \xi_{t-1}$ ,

$$\begin{aligned} & G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t < \Delta_t(N)\}} \\ & \leq G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and  $\xi_{t-1}$  and we have

$$\begin{aligned} & \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t < \Delta_t(N)\}}] \\ & \leq \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}]. \end{aligned} \tag{EC.15}$$

Sum up (EC.14) and (EC.15) and we have:

$$\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \leq \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)]. \quad (\text{EC.16})$$

By Lemma EC.1,  $p_t(N) < p_{t-1}(N)$  yields that  $\partial_{p_t} O_t(\Delta_t(N), p_t(N), N) \leq \partial_{p_{t-1}} O_{t-1}(\Delta_{t-1}(N), p_{t-1}(N), N)$ , i.e.,

$$\begin{aligned} & \partial_p Q(p_t(N), N) - \theta \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \\ & \leq \partial_p Q(p_{t-1}(N), N) - \theta \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)]. \end{aligned}$$

Since  $Q(\cdot, N)$  is strictly concave in  $p$ ,  $\partial_p Q(p_t(N), N) > \partial_p Q(p_{t-1}(N), N)$ . Thus,

$$\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] > \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)],$$

which contradicts inequality (EC.16). Therefore,  $\Delta_t(N) \geq \Delta_{t-1}(N)$  for any  $N$ .

Next, we show  $x_t(N) \geq x_{t-1}(N)$ . Note that  $p_t(N) \leq p_{t-1}(N)$  implies that  $y_t(N) \geq y_{t-1}(N)$ . Thus,

$$x_t(N) = y_t(N) + \Delta_t(N) \geq y_{t-1}(N) + \Delta_{t-1}(N) = x_{t-1}(N).$$

Finally, to complete the induction, we show that  $\pi'_t(N) \geq \pi'_{t-1}(N)$  for all  $N$ . By the envelope theorem,

$$\pi'_t(N) = r'_n(N) + (p_t(N) - c)\gamma'(N) + (\eta + \theta\gamma'(N))\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)],$$

and

$$\pi'_{t-1}(N) = r'_n(N) + (p_{t-1}(N) - c)\gamma'(N) + (\eta + \theta\gamma'(N))\mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)],$$

If  $p_t(N) = p_{t-1}(N)$  and  $\Delta_t(N) = \Delta_{t-1}(N)$ ,  $\pi'_t(N) \geq \pi'_{t-1}(N)$  follows immediately from  $\gamma'(N) \geq 0$  and  $G'_t(\cdot) \geq G'_{t-1}(\cdot)$ .

If  $p_t(N) = p_{t-1}(N)$  and  $\Delta_t(N) > \Delta_{t-1}(N)$ , Lemma EC.1 yields that  $\partial_{\Delta_t} O_t(\Delta_t(N), p_t(N), N) \geq \partial_{\Delta_{t-1}} O_{t-1}(\Delta_{t-1}(N), p_{t-1}(N), N)$ , i.e.,

$$\begin{aligned} & \beta + \Lambda'(\Delta_t(N)) + \sigma \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t \geq \Delta_t(N)\}}] \\ & \geq \beta + \Lambda'(\Delta_{t-1}(N)) + \sigma \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\Delta_t(N)) \leq \Lambda'(\Delta_{t-1}(N))$  and, thus,

$$\begin{aligned} & \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t \geq \Delta_t(N)\}}] \\ & \geq \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}]. \end{aligned} \quad (\text{EC.17})$$

Since  $\Delta_t(N) > \Delta_{t-1}(N)$ , it follows immediately that, for any realization of  $\xi_t = \xi_{t-1}$ ,

$$\begin{aligned} & G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t < \Delta_t(N)\}} \\ & \geq G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and  $\xi_{t-1}$  and we have

$$\begin{aligned} & \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+) 1_{\{\xi_t < \Delta_t(N)\}}] \\ & \geq \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+) 1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}]. \end{aligned} \quad (\text{EC.18})$$

Sum up (EC.17) and (EC.18) and we have:

$$\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \geq \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)]. \quad (\text{EC.19})$$

Plugging (EC.19) into the formulas of  $\pi'_t(\cdot)$  and  $\pi'_{t-1}(\cdot)$ , we have that the inequality  $\pi'_t(N) \geq \pi'_{t-1}(N)$  follows immediately from  $p_t(N) = p_{t-1}(N)$ .

If  $p_t(N) < p_{t-1}(N)$ , Lemma EC.1 yields that  $\partial_{p_t} O_t(\Delta_t(N), p_t(N), N) \leq \partial_{p_{t-1}} O_{t-1}(\Delta_{t-1}(N), p_{t-1}(N), N)$ , i.e.,

$$\begin{aligned} & \partial_p Q(p_t(N), N) - \theta \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \\ & \leq \partial_p Q(p_{t-1}(N), N) - \theta \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)], \end{aligned}$$

i.e.,

$$\begin{aligned} & \bar{V} + c - 2p_t(N) + \gamma(N) - \theta \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \\ & \leq \bar{V} + c - 2p_{t-1}(N) + \gamma(N) - \theta \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)]. \end{aligned}$$

Thus,

$$\begin{aligned} & (p_t(N) - p_{t-1}(N)) + \theta(\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \\ & \quad - \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)] \\ & \geq p_{t-1}(N) - p_t(N) \\ & > 0. \end{aligned}$$

Moreover,

$$\begin{aligned} & \mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \\ & \quad - \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)] \\ & \geq \frac{2}{\theta}(p_{t-1}(N) - p_t(N)) \\ & > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \pi'_t(N) - \pi'_{t-1}(N) &= ((p_t(N) - p_{t-1}(N)) + \theta(\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \\ & \quad - \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)])\gamma'(N) \\ & \quad + \eta(\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \\ & \quad - \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)])) \\ & \geq 0. \end{aligned}$$

Hence,  $\pi'_t(N) \geq \pi'_{t-1}(N)$  for all  $N$ . This completes the induction and, thus, the proof of Theorem 5. *Q.E.D.*

**Proof of Theorem 6:** We show Theorem 6 by backward induction. More specifically, we show that if  $\hat{\alpha} > \alpha$  and  $\hat{\pi}'_{t-1}(N_{t-1}) \geq \pi'_{t-1}(N_{t-1})$  for all  $N_{t-1} \geq 0$ , (i)  $\hat{p}_t(N_t) \leq p_t(N_t)$  for all  $N_t \geq 0$ , (ii)  $\hat{\Delta}_t(N_t) \geq \Delta_t(N_t)$  for all  $N_t \geq 0$ , (iii)  $\hat{x}_t(N_t) \geq x_t(N_t)$  for all  $N_t \geq 0$ , and (iv)  $\hat{\pi}'_t(N_t) \geq \pi'_t(N_t)$  for all  $N_t \geq 0$ . Since  $\hat{\pi}'_0(N_0) = \pi'_0(N_0) \equiv 0$  for all  $N$ , the initial condition is satisfied.

Note that  $\hat{\pi}'_{t-1}(N_{t-1}) \geq \pi'_{t-1}(N_{t-1})$  for all  $N_{t-1} \geq 0$  implies that

$$\hat{G}'_t(y) = \hat{\alpha} \mathbb{E}[\hat{\pi}'_{t-1}(y + \epsilon_t)] \geq \alpha \mathbb{E}[\pi'_{t-1}(y + \epsilon_t)] = G'_t(y),$$

for all  $y$ . We use  $\hat{y}_t(N_t) := y_t(\hat{p}_t(N_t), N_t)$  and  $y_t(N_t) := y_t(p_t(N_t), N_t)$  to denote the expected demand in period  $t$  under the optimal policy with discount factor  $\hat{\alpha}$  and  $\alpha$ , respectively. Since  $\hat{\alpha} > \alpha$ ,  $\hat{\beta} = b - (1 - \hat{\alpha})c > b - (1 - \alpha)c = \beta$ .

We first prove that  $\hat{p}_t(N_t) \leq p_t(N_t)$  for all  $N_t$ . Assume, to the contrary, that  $\hat{p}_t(N_t) > p_t(N_t)$  for some  $N_t$ . Lemma EC.1 implies that  $\partial_{p_t} O_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \geq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \partial_{p_t} Q_t(\hat{p}_t(N_t), N_t) - \theta \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \geq \partial_{p_t} Q_t(p_t(N_t), N_t) - \theta \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Since  $Q_t(\cdot, N_t)$  is strictly concave in  $p_t$  and  $\hat{p}_t(N_t) > p_t(N_t)$ ,  $\partial_{p_t} \hat{Q}_t(p_t(N_t), N_t) < \partial_{p_t} Q_t(p_t(N_t), N_t)$ . Thus,

$$\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] < \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \quad (\text{EC.20})$$

Note that  $\hat{G}'_t(\cdot) \geq G'_t(\cdot)$  for all  $y$ ,  $\hat{p}_t(N_t) > p_t(N_t)$ , and  $\hat{G}_t(\cdot)$  and  $G_t(\cdot)$  are concave. We have (EC.20) implies that  $\sigma > 0$  and  $\hat{\Delta}_t(N_t) > \Delta_t(N_t)$ . Thus, Lemma EC.1 implies that  $\partial_{\Delta_t} \hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \geq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \hat{\beta} + \Lambda'(\hat{\Delta}_t(N_t)) + \sigma \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ & \geq \beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\hat{\Delta}_t(N_t)) \leq \Lambda'(\Delta_t(N_t))$ . In addition,  $\hat{\beta} > \beta$ , thus,

$$\begin{aligned} & \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ & \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.21})$$

Since  $\hat{\Delta}_t(N_t) > \Delta_t(N_t)$  and  $\hat{G}'_t(\cdot) \geq G'_t(\cdot) \geq 0$ , it follows immediately that, for any realization of  $\xi_t$ ,

$$\begin{aligned} & \hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t < \hat{\Delta}_t(N_t)\}} \\ & \geq G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and we have

$$\begin{aligned} & \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t < \hat{\Delta}_t(N_t)\}}] \\ & \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.22})$$

Sum up (EC.21) and (EC.22) and we have:

$$\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts (EC.20). Therefore,  $\hat{p}_t(N_t) \leq p_t(N_t)$  for all  $N_t$ .

Next, we show that  $\hat{\Delta}_t(N_t) \geq \Delta_t(N_t)$ . If  $\sigma = 0$ ,  $\hat{\Delta}_t(N_t) = \arg \max_{\Delta_t} [\hat{\beta} \Delta_t + L(\Delta_t)]$ , whereas  $\Delta_t(N_t) = \arg \max_{\Delta_t} [\beta \Delta_t + L(\Delta_t)]$ . Since  $\hat{\beta} > \beta$ , it follows immediately that  $\hat{\Delta}_t(N_t) \geq \Delta_t(N_t)$ . Hence, we confine ourselves to the interesting case of  $\sigma > 0$ .

Assume, to the contrary, that  $\hat{\Delta}_t(N_t) < \Delta_t(N_t)$ . Lemma EC.1 implies that  $\partial_{\Delta_t} \hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \leq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \hat{\beta} + \Lambda'(\hat{\Delta}_t(N_t)) + \sigma \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ & \leq \beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\hat{\Delta}_t(N_t)) \geq \Lambda'(\Delta_t(N_t))$ . Since  $\hat{\beta} > \beta$ , we have

$$\begin{aligned} & \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ & \leq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.23})$$

The concavity of  $\hat{G}_t(\cdot)$  and  $G_t(\cdot)$  and that  $\hat{G}'_t(\cdot) \geq G'_t(\cdot)$  imply that  $\hat{y}_t(N_t) > y_t(N_t)$  and, thus,  $\hat{p}_t(N) < p_t(N)$ . Since  $\hat{\Delta}_t(N_t) < \Delta_t(N_t)$ , it follows immediately that, for any realization of  $\xi_t$ ,

$$\begin{aligned} & \hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t < \hat{\Delta}_t(N_t)\}} \\ & \leq G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and we have

$$\begin{aligned} & \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t < \hat{\Delta}_t(N_t)\}}] \\ & \leq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.24})$$

Sum up (EC.23) and (EC.24) and we have:

$$\begin{aligned} & \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \leq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned} \quad (\text{EC.25})$$

By Lemma EC.1,  $\hat{p}_t(N_t) < p_t(N_t)$  yields that  $\partial_{p_t} \hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \leq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \partial_{p_t} \hat{Q}(\hat{p}_t(N_t), N_t) - \theta \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \leq \partial_{p_t} Q(p_t(N_t), N_t) - \theta \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Since  $Q_t(\cdot, N_t)$  is strictly concave in  $p_t$ ,  $\partial_{p_t} Q_t(\hat{p}_t(N_t), N_t) > \partial_{p_t} Q_t(p_t(N_t), N_t)$ . Thus,

$$\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] > \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts inequality (EC.25). Therefore,  $\hat{\Delta}_t(N_t) \geq \Delta_t(N_t)$  for any  $N_t$ .

Next, we show  $\hat{x}_t(N_t) \geq x_t(N_t)$ . Note that  $\hat{p}_t(N_t) \leq p_t(N_t)$  implies that  $\hat{y}_t(N_t) \geq y_t(N_t)$ . Thus,

$$\hat{x}_t(N_t) = \hat{y}_t(N_t) + \hat{\Delta}_t(N_t) \geq y_t(N_t) + \Delta_t(N_t) = x_t(N_t).$$

Finally, to complete the induction, we show that  $\hat{\pi}'_t(N_t) \geq \pi'_t(N_t)$  for all  $N_t$ . By the envelope theorem,

$$\hat{\pi}'_t(N_t) = r'_n(N_t) + (\hat{p}_t(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)],$$

and

$$\pi'_t(N_t) = r'_n(N_t) + (p_t(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

If  $\hat{p}_t(N_t) = p_t(N_t)$  and  $\hat{\Delta}_t(N_t) = \Delta_t(N_t)$ ,  $\hat{\pi}'_t(N_t) \geq \pi'_t(N_t)$  follows immediately from  $\gamma'(N_t) \geq 0$  and  $\hat{G}'_t(\cdot) \geq G'_t(\cdot)$ .

If  $\hat{p}_t(N_t) = p_t(N_t)$  and  $\hat{\Delta}_t(N_t) > \Delta_t(N_t)$ , Lemma EC.1 yields that  $\partial_{\Delta_t} \hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \geq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \beta + \Lambda'(\hat{\Delta}_t(N_t)) + \sigma \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ & \geq \beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\hat{\Delta}_t(N_t)) \leq \Lambda'(\Delta_t(N_t))$  and, thus,

$$\begin{aligned} & \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ & \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.26})$$

Since  $\hat{\Delta}_t(N_t) > \Delta_t(N_t)$ , it follows immediately that, for any realization of  $\xi_t$ ,

$$\begin{aligned} & \hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t < \hat{\Delta}_t(N_t)\}} \\ & \geq G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and we have

$$\begin{aligned} & \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t < \hat{\Delta}_t(N_t)\}}] \\ & \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.27})$$

Sum up (EC.26) and (EC.27) and we have:

$$\begin{aligned} & \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned} \quad (\text{EC.28})$$

Plugging (EC.28) into the formulas of  $\hat{\pi}'_t(\cdot)$  and  $\pi'_t(\cdot)$ , we have that the inequality  $\hat{\pi}'_t(N_t) \geq \pi'_t(N_t)$  follows immediately from  $\hat{p}_t(N_t) = p_t(N_t)$ .

If  $\hat{p}_t(N_t) < p_t(N_t)$ , Lemma EC.1 yields that  $\partial_{p_t} \hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \leq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \partial_{p_t} Q_t(\hat{p}_t(N_t), N_t) - \theta \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \leq \partial_{p_t} Q_t(p_t(N_t), N_t) - \theta \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)], \end{aligned}$$

i.e.,

$$\begin{aligned} & \bar{V}_t + c - 2\hat{p}_t(N_t) + \gamma(N_t) - \theta \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \leq \bar{V}_t + c - 2p_t(N_t) + \gamma(N_t) - \theta \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Thus,

$$\begin{aligned} & (\hat{p}_t(N_t) - p_t(N_t)) + \theta(\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \quad - \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)] \\ & \geq \hat{p}_t(N_t) - p_t(N_t) \\ & > 0. \end{aligned}$$

Moreover,

$$\begin{aligned} & \mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \quad - \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)] \\ & \geq \frac{2}{\theta}(p_t(N_t) - \hat{p}_t(N_t)) \\ & > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{\pi}'_t(N_t) - \pi'_t(N_t) &= ((\hat{p}_t(N_t) - p_t(N_t)) + \theta(\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \quad - \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]]) \gamma'(N_t) \\ & \quad + \eta(\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ & \quad - \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]]) \\ & \geq 0. \end{aligned}$$



Hence,  $\hat{\pi}'_t(N_t) \geq \pi'_t(N_t)$  for all  $N_t$ . This completes the induction and, thus, the proof of Theorem 6. *Q.E.D.*

**Proof of Theorem 7: Part (a).** For any  $\hat{N}_t > N_t$ , since  $p_t(\hat{N}_t) \geq p_t(N_t)$  (Theorem 4(a)),

$$y_t(\hat{N}_t) - y_t(N_t) = \gamma(\hat{N}_t) - \gamma(N_t) - (p_t(\hat{N}_t) - p_t(N_t)) \leq \gamma(\hat{N}_t) - \gamma(N_t) \leq (\hat{N}_t - N_t)\gamma'(N_t), \quad (\text{EC.29})$$

where the last inequality follows from the concavity of  $\gamma(\cdot)$ .

Let  $\mathcal{N} := \min\{N_t \geq 0 : \theta\gamma'(N_t) \leq 1 - \eta\}$ . Since  $\lim_{N_t \rightarrow +\infty} \gamma'(N_t) = 0$ ,  $\mathcal{N} < +\infty$ . Since  $I_T \leq x_T(N_T)$ ,  $I_t \leq x_t(N_t)$  for all  $t$  with probability 1. Thus,

$$\mathbb{E}[N_{t-1}|N_t] = \eta N_t + \theta y_t(N_t) - \sigma \mathbb{E}(-\Delta_t(N_t) + \xi_t)^+$$

We now show that, if  $N_t > \mathcal{N}$ ,  $\mathbb{E}[N_{t-1}|N_t] - N_t$  is decreasing in  $N_t$ . For any  $\hat{N}_t > N_t > \mathcal{N}$ , we have

$$\begin{aligned} \mathbb{E}[N_{t-1}|\hat{N}_t] - \hat{N}_t - (\mathbb{E}[N_{t-1}|N_t] - N_t) &= \eta \hat{N}_t + \theta y_t(\hat{N}_t) - \sigma \mathbb{E}(\xi_t - \Delta_t(\hat{N}_t))^+ - \hat{N}_t - (\eta N_t + \theta y_t(N_t) - \sigma \mathbb{E}(\xi_t - \Delta_t(N_t))^+ - N_t) \\ &= -(1 - \eta)(\hat{N}_t - N_t) + \theta(y_t(\hat{N}_t) - y_t(N_t)) - \sigma(\mathbb{E}(\xi_t - \Delta_t(\hat{N}_t))^+ - \mathbb{E}(\xi_t - \Delta_t(N_t))^+) \\ &\leq -(1 - \eta)(\hat{N}_t - N_t) + \theta\gamma'(N_t)((\hat{N}_t - N_t)) \\ &< [(1 - \theta) - (1 - \theta)]((\hat{N}_t - N_t)) \\ &= 0, \end{aligned} \quad (\text{EC.30})$$

where the first inequality follows from (EC.29) and  $\Delta_t(\hat{N}_t) \leq \Delta_t(N_t)$  (Theorem 4(b)), and the second inequality from the definition of  $\mathcal{N}$ .

Let  $g_t(N_t) := \mathbb{E}[N_{t-1}|N_t] - N_t$ . Clearly,  $g_t(\cdot)$  is continuous in  $N_t$ . On the other hand,  $g_t(0) = \mathbb{E}[N_{t-1}|0] = \theta \mathbb{E}(x_t(0) \wedge D_t(p_t(0), 0)) + (\theta - \sigma) \mathbb{E}(D_t(p_t(0), n) - x_t(0))^+$ . By Theorem 1(a),  $x_t(0) > 0$  and thus  $g_t(0) > 0$ . Since  $\lim_{N_t \rightarrow +\infty} \gamma'(N_t) = 0$ , when  $N_t$  is sufficiently large,  $g_t(N_t) \leq -(1 - \eta)N_t + C$  for some constant  $C$ . Hence,  $\lim_{N_t \rightarrow +\infty} g_t(N_t) = -\infty$ . By the concavity of  $\gamma(\cdot)$  and Lemma EC.4, it is straightforward to check that, if  $g_t(\cdot)$  is decreasing at the point  $N_t$ , it is strictly decreasing at any point  $\hat{N}_t > N_t$ . By (EC.30), there exists a threshold  $\bar{N}_t \leq \mathcal{N}$ , such that  $g_t(N_t) > 0$  on the region  $[0, \bar{N}_t]$  and it is strictly decreasing in  $N_t$  on the region  $[\bar{N}_t, +\infty)$ , with  $\lim_{N_t \rightarrow +\infty} g_t(N_t) = -\infty$ . Therefore, there exists a unique  $\bar{N}_t > \bar{N}_t$  such that  $g_t(N_t) > 0$  for  $N_t < \bar{N}_t$  and  $g_t(N_t) < 0$  for  $N_t > \bar{N}_t$ , i.e.,  $\mathbb{E}[N_{t-1}|N_t] > N_t$  if  $N_t < \bar{N}_t$  and  $\mathbb{E}[N_{t-1}|N_t] < N_t$  if  $N_t > \bar{N}_t$ . Since  $g_t(0) > 0$  and  $\lim_{N_t \rightarrow +\infty} g_t(N_t) = -\infty$ ,  $\bar{N}_t \in (0, +\infty)$ .

**Part (b).** By Theorem 5,  $\Delta_t(N)$  is increasing, whereas  $p_t(N)$  is decreasing, in  $t$  for any  $N \geq 0$ . Therefore,  $\Delta(\cdot) := \lim_{t \rightarrow +\infty} \Delta_t(\cdot)$  and  $p(\cdot) := \lim_{t \rightarrow +\infty} p_t(\cdot)$  is the optimal safety-stock and price in the infinite-horizon discounted reward model ( $T = +\infty$ ). Hence, in this case,

$$N_{t-1} = \eta N_t + \theta(y(N_t) + \xi_t) - \sigma(-\Delta(N_t) + \xi_t)^+,$$

where  $y(N_t) = \mathbb{E}[D_t(p(N_t), N_t)] = \bar{V} - p(N_t) + \gamma(N_t)$ . By Theorem 11.10 in Stoekey et al. (1989), the Markov process  $\{N_t : t \in \mathbb{Z}\}$  is ergodic and, thus, has a stationary distribution  $\nu(\cdot)$ .

Using the same technique as the proof of part (a), we can show that the threshold  $\bar{N}$  exists. The convergence result  $\bar{N} = \lim_{t \rightarrow +\infty} \bar{N}_t$  follows immediately from the monotonicity that  $\Delta_t(\cdot)$  is increasing and  $p_t(\cdot)$  is decreasing in  $t$ .

**Part (c).** It suffices to show that, as  $\eta \uparrow 1$ ,  $\bar{N}_t \uparrow +\infty$  for each  $t$ . Since  $\mathbb{E}[N_{t-1}|N_t] = \eta N_t + \theta y_t(N_t) - \sigma \mathbb{E}(-\Delta_t(N_t) + \xi_t)^+$  is increasing in  $\eta$  for any  $N_t$ . Note that  $\bar{N}_t = \inf\{N_t : \mathbb{E}[N_{t-1}|N_t] < N_t\}$ , so it follows immediately from the monotonicity of  $\mathbb{E}[N_{t-1}|N_t]$  in  $\eta$  that  $\bar{N}_t$  is increasing in  $\eta$ . Hence,  $\bar{N} = \lim_{t \rightarrow +\infty} \bar{N}_t$  is increasing in  $\eta$  as well.

Finally, it remains to show that as  $\eta \uparrow 1$ ,  $\bar{N}_t \uparrow +\infty$ . Assume, to the contrary, that there exists a uniform upper bound  $\Gamma_t \in (0, +\infty)$  such that  $\bar{N}_t < \Gamma_t$  for all  $\eta < 1$ . Hence,

$$g_t(N_t) = -(1-\eta)N_t + y_t(N_t) - \sigma \mathbb{E}(-\Delta_t(N_t) + \xi_t)^+ < 0, \text{ for all } N_t \geq \Gamma_t \text{ and } \eta < 1.$$

Note that  $y_t(N_t) - \sigma \mathbb{E}(-\Delta_t(N_t) + \xi_t)^+$  is the average number of customers who join the network in period  $t$ . Since  $D_t(p_t, N_t) \geq 0$  with probability 1, we have  $y_t(N_t) - \sigma \mathbb{E}(-\Delta_t(N_t) + \xi_t)^+ \geq \underline{y}$  for some  $\underline{y} > 0$ . Therefore, if  $\eta > \max\{1 - \underline{y}/(2\Gamma_t), 0\}$ ,

$$g_t(N_t) = -(1-\eta)N_t + y_t(N_t) - \sigma \mathbb{E}(-\Delta_t(N_t) + \xi_t)^+ > -(1 - (1 - \frac{\underline{y}}{2\Gamma_t}))N_t + \underline{y} = \underline{y}(1 - \frac{N_t}{2\Gamma_t}).$$

Thus, if  $N_t = 1.5\Gamma_t > \Gamma_t$ ,  $g_t(N_t) > 0.25\underline{y} > 0$ , which contradicts that  $g_t(N_t) < 0$  for all  $N_t \geq \Gamma_t$ . Therefore, as  $\eta \uparrow 1$ ,  $\bar{N}_t \uparrow +\infty$  and thus  $\bar{N} = \lim_{t \rightarrow +\infty} \bar{N}_t$  also increases to infinity as  $\eta$  increases to 1. *Q.E.D.*

**Proof of Theorem 8: Part (a).** First  $v_t^w(\cdot, \cdot) \leq v_t(\cdot, \cdot)$  follows immediately from that  $v_t(\cdot, \cdot)$  is the optimal expected profit in periods  $t, t-1, \dots, 1$ .

If  $w \geq t$ , the length of the moving time-window exceeds the total planning horizon length. Therefore, if  $w \geq t$ , the  $w$ -heuristic is the optimal policy and, hence,  $v_t^w(\cdot, \cdot) = v_t(\cdot, \cdot)$ .

It remains to show that if  $w \leq t-2$ ,  $v_t^w(\cdot, \cdot) \leq v_t^{w+1}(\cdot, \cdot)$ . Note that

$$v_t(I_t, N_t) = \mathbb{E}[\sum_{1 \leq \tau \leq t} \alpha^{t-\tau} \{Q_\tau(p_\tau, N_\tau) + \beta \Delta_\tau + \Lambda(\Delta_\tau) + r_n(N_\tau) + cI_\tau\} + \alpha^t I_0 | I_t, N_t],$$

s.t. for each  $1 \leq \tau \leq t$ :

$$p_\tau = p_\tau(N_\tau);$$

$$\Delta_\tau = \Delta_\tau(N_\tau);$$

$$N_{\tau-1} = \eta N_\tau + \theta D_\tau(p_\tau, N_\tau) - \sigma(\xi_\tau - \Delta_\tau(p_\tau, N_\tau))^+ + \epsilon_\tau;$$

$$I_{\tau-1} = \bar{V}_\tau + \gamma(N_\tau) - p_\tau + \Delta_\tau(N_\tau) - D_\tau(p_\tau, N_\tau).$$

Analogously, we have

$$v_t^w(I_t, N_t) = \mathbb{E} \left[ \sum_{1 \leq \tau \leq t} \alpha^{t-\tau} \{Q_\tau(p_\tau, N_\tau) + \beta \Delta_\tau + \Lambda(\Delta_\tau) + r_n(N_\tau) + cI_\tau\} + \alpha^t I_0 | I_t, N_t \right],$$

s.t. for each  $1 \leq \tau \leq w$ :

$$p_\tau = p_\tau(N_\tau);$$

$$\Delta_\tau = \Delta_\tau(N_\tau);$$

for each  $w+1 \leq \tau \leq t$ :

$$p_\tau = p_w(N_\tau);$$

$$\Delta_\tau = \Delta_w(N_\tau);$$

for each  $1 \leq \tau \leq t$ :

$$N_{\tau-1} = \eta N_\tau + \theta D_\tau(p_\tau, N_\tau) - \sigma(\xi_\tau - \Delta_\tau(p_\tau, N_\tau))^+ + \epsilon_\tau;$$

$$I_{\tau-1} = \bar{V}_\tau + \gamma(N_\tau) - p_\tau + \Delta_\tau(N_\tau) - D_\tau(p_\tau, N_\tau);$$

and

$$v_t^{w+1}(I_t, N_t) = \mathbb{E} \left[ \sum_{1 \leq \tau \leq t} \alpha^{t-\tau} \{Q_\tau(p_\tau, N_\tau) + \beta \Delta_\tau + \Lambda(\Delta_\tau) + r_n(N_\tau) + cI_\tau\} + \alpha^t I_0 | I_t, N_t \right],$$

s.t. for each  $1 \leq \tau \leq w+1$ :

$$p_\tau = p_\tau(N_\tau);$$

$$\Delta_\tau = \Delta_\tau(N_\tau);$$

for each  $w+2 \leq \tau \leq t$ :

$$p_\tau = p_w(N_\tau);$$

$$\Delta_\tau = \Delta_w(N_\tau);$$

for each  $1 \leq \tau \leq t$ :

$$N_{\tau-1} = \eta N_\tau + \theta D_\tau(p_\tau, N_\tau) - \sigma(\xi_\tau - \Delta_\tau(p_\tau, N_\tau))^+ + \epsilon_\tau;$$

$$I_{\tau-1} = \bar{V}_\tau + \gamma(N_\tau) - p_\tau + \Delta_\tau(N_\tau) - D_\tau(p_\tau, N_\tau);$$

Since  $I_T \leq x_T(N_t)$ , Theorem 5 implies that  $\Delta_\tau(\cdot) \geq \Delta_{w+1}(\cdot) \geq \Delta_w(\cdot)$  and  $p_\tau(\cdot) \leq p_{w+1}(\cdot) \leq p_w(\cdot)$  for all  $\tau \geq w+1$ . Moreover,  $(\Delta_\tau(\cdot), p_\tau(\cdot))$  is the optimal policy for the dynamic program. Putting everything together, it follows immediately that  $v_t^w(\cdot, \cdot) \leq v_t^{w+1}(\cdot, \cdot) \leq v_t(\cdot, \cdot)$  for all  $t \geq w+2$ . This finishes the proof of part (a).

**Part (b).** The inequality  $v^w(\cdot, \cdot) \leq v^{w+1}(\cdot, \cdot) \leq v(\cdot, \cdot)$  follows from  $v_t^w(\cdot, \cdot) \leq v_t^{w+1}(\cdot, \cdot) \leq v_t(\cdot, \cdot)$  by letting  $T$  approach infinity. Note that  $v_w(\cdot, \cdot) \leq v^w(\cdot, \cdot) \leq v(\cdot, \cdot)$ . Hence,  $\sup |v^w(\cdot, \cdot) - v(\cdot, \cdot)| \leq \sup |v_w(\cdot, \cdot) - v(\cdot, \cdot)|$ . Let  $\mathcal{T}$  be the operator acted on a concave and continuously differentiable function  $f(\cdot, \cdot)$  that satisfies

$$\begin{aligned} \mathcal{T}[f(I, N)] &= \max_{(x, p) \in \hat{\mathcal{F}}(I)} \{R(p, N) + \beta x + \Lambda(x - y(p, N)) + r_n(N) \\ &\quad + \mathbb{E}[\Psi(x_t - y(p, N) - \xi, \eta N + \theta(y(p, N) + \xi) - \sigma(y(p, N) + \xi - x)^+)]\}, \end{aligned}$$

with  $\Psi(x, y) := \alpha \mathbb{E}\{[f(x, y + \epsilon_t) - cx]\}$ ,

$$R(p, N) := (p - \alpha c - b)(\bar{V} + \gamma(N) - p),$$

$$y(p, N) := \bar{V} + \gamma(N) - p.$$

By Theorem 9.6 in Stokey et al. (1989),  $\mathcal{T}$  is a contraction mapping with contraction factor  $\alpha$  under the sup norm. Since  $v(\cdot, \cdot)$  is the fixed point of  $\mathcal{T}$  and  $v_w = \mathcal{T}^w[v_0(\cdot, \cdot)]$ , we have

$$\sup |v_w(\cdot, \cdot) - v(\cdot, \cdot)| \leq \alpha^w \sup |v_0(\cdot, \cdot) - v(\cdot, \cdot)|.$$

Let  $C := \sup |v_0(\cdot, \cdot) - v(\cdot, \cdot)| > 0$  for a given initial state  $(I, N)$ , and  $\delta := -\log(\alpha) > 0$ . Thus, we have

$$\sup |v^w(\cdot, \cdot) - v(\cdot, \cdot)| \leq \sup |v_w(\cdot, \cdot) - v(\cdot, \cdot)| \leq Ce^{-\delta w}. \quad (\text{EC.31})$$

By (EC.31),  $\lim_{w \rightarrow +\infty} v^w(\cdot, \cdot) = v(\cdot, \cdot)$  follows immediately for any initial state  $(I, N)$ . This proves part (b). *Q.E.D.*

**Proof of Lemma 3: Part (a).** Part (a) follows from the same argument as the proof of Lemma EC.2, so we omit its proof for brevity.

**Part (b).** The optimal value function  $v_t^e(I_t, N_t)$  satisfies the following recursive scheme:

$$v_t^e(I_t, N_t) = cI_t + \max_{(x_t, p_t, n_t) \in \hat{\mathcal{F}}_e(I_t)} J_t^e(x_t, p_t, n_t, N_t), \quad (\text{EC.32})$$

where  $\hat{\mathcal{F}}_e(I_t) := [I_t, +\infty) \times [\underline{p}, \bar{p}] \times [0, +\infty)$  denotes the set of feasible decisions and

$$\begin{aligned} J_t^e(x_t, p_t, n_t, N_t) = & R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - y_t(p_t, N_t)) + r_n(N_t) - c_n(n_t) \\ & + \mathbb{E}[\Psi_t^e(x_t - y_t(p_t, N_t) - \xi_t, \eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+ + n_t)], \\ \text{with } \Psi_t^e(x, y) = & \alpha \mathbb{E}\{v_{t-1}^e(x, y + \epsilon_t) - cx\}. \end{aligned} \quad (\text{EC.33})$$

The derivation of (EC.33) is given as follows:

$$\begin{aligned} J_t^e(x_t, p_t, n_t, N_t) &:= -cI_t + \mathbb{E}\{p_t D_t(p_t, N_t) - c(x_t - I_t) - h(x_t - D_t(p_t, N_t))^+ - b(x_t - D_t(p_t, N_t))^- + r_n(N_t) - c_n(n_t) \\ &\quad + \alpha v_{t-1}^e(x_t - D_t(p_t, N_t), \theta D_t(p_t, N_t) + \eta N_t - \sigma(D_t(p_t, N_t) - x_t)^+ + n_t + \epsilon_t) | N_t\}, \\ &= (p_t - \alpha c - b)y_t(p_t, N_t) + (b - (1 - \alpha)c)x_t + r_n(N_t) - c_n(n_t) + \mathbb{E}\{-(h + b)(x_t - y_t(p_t, N_t) - \xi_t)^+ \\ &\quad + \alpha[v_{t-1}^e(x_t - y_t(p_t, N_t) - \xi_t, \eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+ + n_t + \epsilon_t) \\ &\quad - c(x_t - y_t(p_t, N_t) - \xi_t)] | N_t\} \\ &= R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - y_t(p_t, N_t)) + r_n(N_t) - c_n(n_t) \\ &\quad + \mathbb{E}[\Psi_t^e(x_t - y_t(p_t, N_t) - \xi_t, \eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(y_t(p_t, N_t) + \xi_t - x_t)^+ + n_t)]. \end{aligned}$$

We use  $(\hat{x}_t^e(N_t), \hat{p}_t^e(N_t), \hat{n}_t(N_t))$  as the unconstrained optimizer of (EC.33). The same argument as the proof of Lemma EC.2 yields that  $J_t^e(\cdot, \cdot, \cdot, \cdot)$  is jointly concave in  $(x_t, p_t, n_t, N_t)$ . Hence, if  $I_t \leq \hat{x}_t^e(N_t)$ ,  $(x_t^e(I_t, N_t), p_t^e(I_t, N_t), n_t^e(I_t, N_t)) = (\hat{x}_t^e(N_t), \hat{p}_t^e(N_t), \hat{n}_t(N_t))$ ; otherwise  $(I_t > \hat{x}_t^e(N_t))$   $x_t^e(I_t, N_t) = I_t$ .

The same argument as the proof of Lemma 1 implies that  $\mathbb{P}[\hat{x}_t^p(N_t) - D_t(\hat{p}_t^p(N_t), N_t) \leq \hat{x}_{t-1}^p(N_{t-1}) | N_t] = 1$ . Hence, the same argument as the proof of Lemma 2 and its discussions enables us to transform the objective function from  $J_t^e(x_t, p_t, n_t, N_t)$  to  $O_t^e(\Delta_t, p_t, n_t, N_t)$  by letting  $\Delta_t = x_t - \mathbb{E}[D_t(p_t, N_t)] = x_t - y_t(p_t, N_t)$ , and we have that  $(\hat{x}_t^e(N_t), \hat{p}_t^e(N_t), \hat{n}_t(N_t)) = (x_t^e(N_t), p_t^e(N_t), n_t(N_t))$  for all  $N_t \geq 0$ , where  $x_t^e(N_t) = \Delta_t^e(N_t) + y_t(p_t^e(N_t), N_t)$ . Hence, as in Theorem 2, if  $I_T \leq x_T^e(N_T)$ ,  $I_t \leq x_t^e(N_t)$  for all  $t$  with probability 1.

Hence, part (b) follows. *Q.E.D.*

**Proof of Theorem 9: Part (a).** We first show that if (11) holds,  $n_t^*(I_t, N) > 0$  for all  $I_t$ . Observe that, since  $\partial_y \Psi_{t-1}^e(x, y) \geq 0$ ,

$$\partial_{N_{t-1}} v_{t-1}^e(I_{t-1}, N_{t-1}) \geq (\underline{p} - b - \alpha c) \gamma'(N_{t-1}) - \gamma'(N_{t-1}) \Lambda'(\Delta_{t-1}^*) + r'_n(N_{t-1}),$$

where  $\Delta_{t-1}^* := x_{t-1}^{e*}(I_{t-1}, N_{t-1}) - y_{t-1}(I_{t-1}, N_{t-1})$ . The first-order condition with respect to  $x_{t-1}$  yields that  $\Lambda'(\Delta_{t-1}^*) \leq -\beta$  for any realization of  $\xi_t$  and  $\epsilon_t$ . Thus, for any realization of  $\xi_t$  and  $\epsilon_t$ ,

$$\partial_{N_{t-1}} v_{t-1}^e(I_{t-1}, N_{t-1}) \geq (\underline{p} - c) \gamma'(N_{t-1}) + r'_n(N_{t-1}). \quad (\text{EC.34})$$

Therefore, for any  $\Delta_t$  and  $p_t \in [\underline{p}, \bar{p}]$ ,

$$\begin{aligned} \partial_{n_t} O_t^e(\Delta_t, p_t, 0, N) &\geq \alpha \mathbb{E}\{\partial_{N_{t-1}} v_{t-1}^e(x_t - D_t(p_t, N), N_{t-1}) | N_t = N\} - c'_n(0) \\ &\geq \alpha \mathbb{E}\{r'_n(N_{t-1}) + (\underline{p} - c) \gamma'(N_{t-1}) | N_t = N\} - c'_n(0) \\ &\geq \alpha(1 - \iota)[r'_n(\bar{S}(N)) + (\underline{p} - c) \gamma'(\bar{S}(N))] - c'_n(0) \\ &> 0, \end{aligned} \quad (\text{EC.35})$$

where the second inequality follows from (EC.34), and the fourth from the assumption (11). The third inequality of (EC.35) follows from the following inequality:

$$\begin{aligned} \alpha \mathbb{E}[r'_n(N_{t-1}) + (\underline{p} - c) \gamma'(N_{t-1}) | N_t = N] &= \alpha \mathbb{E}_{N_{t-1} \geq \bar{S}(N)}[r'_n(N_{t-1}) + (\underline{p} - c) \gamma'(N_{t-1}) | N_t = N] \\ &\quad + \alpha \mathbb{E}_{N_{t-1} < \bar{S}(N)}[r'_n(N_{t-1}) + (\underline{p} - c) \gamma'(N_{t-1}) | N_t = N] \\ &\geq 0 + \alpha \mathbb{E}_{N_{t-1} < \bar{S}(N)}[r'_n(\bar{S}(N)) + (\underline{p} - c) \gamma'(\bar{S}(N))] \\ &\geq \alpha(1 - \iota)[r'_n(\bar{S}(N)) + (\underline{p} - c) \gamma'(\bar{S}(N))], \end{aligned}$$

where the first inequality follows from the concavity of  $r_n(\cdot)$  and  $\gamma(\cdot)$ , and the second from the definition of  $\bar{S}(N)$ . The inequality (EC.35) yields that  $n_t^*(I_t, N) > 0$  for all  $I_t$ .

Since  $\gamma(\cdot)$  is continuously increasing in  $N_t$ ,  $\bar{S}(N)$  is continuously increasing in  $N$ . The concavity of  $r_n(\cdot)$  and  $\gamma(\cdot)$  implies that  $r'_n(\bar{S}(N))$  and  $\gamma'(\bar{S}(N))$  are continuously decreasing in  $N$ . Therefore, let

$$N^*(\iota) := \max\{N \geq 0 : \alpha(1 - \iota)[r'_n(\bar{S}(N)) + (\underline{p} - c) \gamma'(\bar{S}(N))] > c'_n(0)\}.$$

We have (11) holds for all  $N < N^*(\iota)$ . This completes the proof of part (a).

**Part (b).** Since  $\gamma(\cdot)$  and  $r_n(\cdot)$  are concavely increasing in  $N_t$ ,

$$\partial_{N_{t-1}} v_{t-1}^e(I_{t-1}, N_{t-1}) \leq \partial_{N_{t-1}} v_t^e(I_{t-1}, 0) \leq \left(\sum_{\tau=1}^{t-1} (\alpha\eta)^{\tau-1}\right) (r'_n(0) + (\bar{p} - c) \gamma'(0)).$$

Thus, if  $\alpha(\sum_{\tau=1}^{t-1} (\alpha\eta)^{\tau-1}) (r'_n(0) + (\bar{p} - c) \gamma'(0)) \leq c'_n(0)$ , then for any  $(x_t, p_t, n_t, N_t)$ ,

$$\begin{aligned} \partial_{n_t} J_t^e(x_t, p_t, n_t, N_t) &\leq \alpha \mathbb{E}\{\partial_{N_{t-1}} v_{t-1}^e(x_t - D_t(p_t, N_t), N_{t-1} + n_t) | N_t = N\} - c'_n(0) \\ &\leq \alpha \left(\sum_{\tau=1}^{t-1} (\alpha\eta)^{\tau-1}\right) (r'_n(0) + (\bar{p} - c) \gamma'(0)) - c'_n(0) \\ &\leq 0. \end{aligned}$$

Hence,  $n_t^*(I_t, N_t) = 0$  for all  $(I_t, N_t)$ . This completes the proof of part (b).  $Q.E.D.$

**Proof of Theorem 10: Parts (a)-(c).** We prove parts (a)-(c) together by backward induction. More specifically, we show that if  $\partial_{N_{t-1}}\pi_{t-1}^e(\cdot) \leq \partial_{N_{t-1}}\pi_{t-1}(\cdot)$  for all  $N_{t-1} \geq 0$ , (i)  $p_t^e(N_t) \geq p_t(N_t)$ , (ii)  $\Delta_t^e(N_t) \leq \Delta_t(N_t)$ , (iii)  $x_t^e(N_t) \leq x_t(N_t)$ , and (iv)  $\partial_{N_t}\pi_t^e(\cdot) \leq \partial_{N_t}\pi_t(\cdot)$  for all  $N_t \geq 0$ . Since  $\partial_{N_0}\pi_0^e(\cdot) = \partial_{N_0}\pi_0(\cdot) \equiv 0$ , the initial condition is satisfied. Note that  $\partial_{N_{t-1}}\pi_{t-1}^e(N_{t-1}) \leq \partial_{N_{t-1}}\pi_{t-1}(N_{t-1})$  for all  $N_{t-1} \geq 0$  implies that

$$\partial_y G_t^e(y) = \alpha \mathbb{E}\{\partial_{N_{t-1}}\pi_{t-1}^e(y + \epsilon_t)\} \leq \alpha \mathbb{E}\{\partial_{N_{t-1}}\pi_{t-1}(y + \epsilon_t)\} = \partial_y G_t(y),$$

for all  $y$ .

We first show that  $p_t^e(N_t) \geq p_t(N_t)$  for all  $N_t$ . Assume, to the contrary, that  $p_t^e(N_t) < p_t(N_t)$  for some  $N_t$ . Lemma EC.1 implies that  $\partial_{p_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \leq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \partial_{p_t} Q(p_t^e(N_t), N_t) - \theta \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\ & \leq \partial_{p_t} Q(p_t(N_t), N_t) - \theta \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Since  $Q_t(\cdot, N_t)$  is strictly concave in  $p_t$  and  $p_t^e(N_t) < p_t(N_t)$ ,  $\partial_{p_t} Q(p_t^e(N_t), N_t) > \partial_{p_t} Q(p_t(N_t), N_t)$ . Thus,

$$\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] > \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \quad (\text{EC.36})$$

Note that  $\partial_y G_t^e(\cdot) \leq \partial_y G_t(\cdot)$  for all  $y$ ,  $p_t^e(N_t) < p_t(N_t)$ , and  $n_t(N_t) \geq 0$ . Thus, the concavity of  $G_t^e(\cdot)$  and  $G_t(\cdot)$ , together with the inequality (EC.36), implies that  $\sigma > 0$  and  $\Delta_t^e(N_t) < \Delta_t(N_t)$ . Thus, Lemma EC.1 implies that  $\partial_{\Delta_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \leq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \beta + \Lambda'(\Delta_t^e(N_t)) + \sigma \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\ & \leq \beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\Delta_t^e(N_t)) \geq \Lambda'(\Delta_t(N_t))$  and, thus,

$$\begin{aligned} & \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\ & \leq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.37})$$

Since  $\Delta_t^e(N_t) < \Delta_t(N_t)$  and  $0 \leq \partial_y G_t^e(\cdot) \leq \partial_y G_t(\cdot)$ , it follows immediately that, for any realization of  $\xi_t$ ,

$$\begin{aligned} & \partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t < \Delta_t^e(N_t)\}} \\ & \leq \partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and we have

$$\begin{aligned} & \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t < \Delta_t^e(N_t)\}}] \\ & \leq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}]. \end{aligned} \quad (\text{EC.38})$$

Sum up (EC.37) and (EC.38) and we have:

$$\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \leq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts (EC.36). Therefore,  $p_t^e(N_t) \geq p_t(N_t)$  for all  $N_t$ .

Next, we show that  $\Delta_t^e(N_t) \geq \Delta_t(N_t)$ . If  $\sigma = 0$ , it is straightforward to show that  $\Delta_t^e(N_t) = \Delta_t(N_t) = \Delta_*$ . Hence, we restrict ourselves to the interesting case of  $\sigma > 0$ .

Assume, to the contrary, that  $\Delta_t^e(N_t) > \Delta_t(N_t)$ . Lemma EC.1 implies that  $\partial_{\Delta_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \geq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \beta + \Lambda'(\Delta_t^e(N_t)) + \sigma \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\ & \geq \beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\Delta_t^e(N_t)) \leq \Lambda'(\Delta_t(N_t))$  and, thus,

$$\begin{aligned} & \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\ & \geq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned} \tag{EC.39}$$

The concavity of  $G_t^e(\cdot)$  and  $G_t(\cdot)$  and that  $n_t(N_t) \geq 0$  and  $\partial_y G_t^e(\cdot) \leq \partial_y G_t(\cdot)$  imply that  $y_t(p_t^e(N_t), N_t) < y_t(p_t(N_t), N_t)$  and, thus,  $p_t^e(N_t) > p_t(N_t)$ . Since  $\Delta_t^e(N_t) < \Delta_t(N_t)$ , it follows immediately that, for any realization of  $\xi_t$ ,

$$\begin{aligned} & \partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t < \Delta_t^e(N_t)\}} \\ & \geq \partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and we have

$$\begin{aligned} & \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t < \Delta_t^e(N_t)\}}] \\ & \geq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}]. \end{aligned} \tag{EC.40}$$

Sum up (EC.39) and (EC.40) and we have:

$$\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \geq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \tag{EC.41}$$

By Lemma EC.1,  $p_t^e(N_t) > p_t(N_t)$  yields that  $\partial_{p_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \geq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \partial_{p_t} Q(p_t^e(N_t), N_t) - \theta \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\ & \geq \partial_{p_t} Q(p_t(N_t), N_t) - \theta \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Since  $Q_t(\cdot, N_t)$  is strictly concave in  $p_t$ ,  $\partial_{p_t} Q(p_t^e(N_t), N_t) < \partial_{p_t} Q(p_t(N_t), N_t)$ . Thus,

$$\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] < \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts inequality (EC.41). Therefore,  $\Delta_t^e(N_t) \leq \Delta_t(N_t)$  for any  $N_t$ .

Next, we show  $x_t^e(N_t) \leq x_t(N_t)$ . Note that  $p_t^e(N_t) \geq p_t(N_t)$  implies that  $y_t(p_t^e(N_t), N_t) \leq y_t(p_t(N_t), N_t)$ . Thus,

$$x_t^e(N_t) = y_t(p_t^e(N_t), N_t) + \Delta_t^e(N_t) \leq y_t(p_t(N_t), N_t) + \Delta_t(N_t) = x_t(N_t).$$

Finally, to complete the induction, we show that  $\partial_{N_t} \pi_t^e(N_t) \geq \partial_{N_t} \pi_t(N_t)$  for all  $N_t \geq 0$ . By the envelope theorem,

$$\partial_{N_t} \pi_t^e(N_t) = r'_n(N_t) + (p_t^e(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)],$$

and

$$\partial_{N_t} \pi_t(N_t) = r'_n(N_t) + (p_t(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].$$

If  $p_t^e(N_t) = p_t(N_t)$  and  $\Delta_t^e(N_t) = \Delta_t(N_t)$ ,  $\partial_{N_t} \pi_t^e(N_t) \leq \partial_{N_t} \pi_t(N_t)$  follows immediately from  $\gamma'(N) \geq 0$  and  $\partial_y G_t^e(\cdot) \leq \partial_y G_t(\cdot)$  for all  $y$ .

If  $p_t^e(N_t) = p_t(N_t)$  and  $\Delta_t^e(N_t) < \Delta_t(N_t)$ , Lemma EC.1 yields that  $\partial_{\Delta_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \leq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \beta + \Lambda'(\Delta_t^e(N_t)) + \sigma \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\ & \leq \beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned}$$

The concavity of  $\Lambda(\cdot)$  suggests that  $\Lambda'(\Delta_t^e(N_t)) \geq \Lambda'(\Delta_t(N_t))$  and, thus,

$$\begin{aligned} & \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\ & \leq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}]. \end{aligned} \tag{EC.42}$$

Since  $\Delta_t^e(N_t) < \Delta_t(N_t)$ , it follows immediately that, for any realization of  $\xi_t$ ,

$$\begin{aligned} & \partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t < \Delta_t^e(N_t)\}} \\ & \leq \partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}. \end{aligned}$$

Integrate over  $\xi_t$  and we have

$$\begin{aligned} & \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t < \Delta_t^e(N_t)\}}] \\ & \leq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}]. \end{aligned} \tag{EC.43}$$

Sum up (EC.42) and (EC.43) and we have:

$$\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \leq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \tag{EC.44}$$

Plugging (EC.44) into the formulas of  $\partial_{N_t} \pi_t^e(\cdot)$  and  $\partial_{N_t} \pi_t(\cdot)$ , we have the inequality  $\partial_{N_t} \pi_t^e(N_t) \leq \partial_{N_t} \pi_t(N_t)$  follows immediately from  $p_t^e(N) = p_t(N)$ .

If  $p_t^e(N_t) > p_t(N_t)$ , Lemma EC.1 yields that  $\partial_{p_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \geq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$ , i.e.,

$$\begin{aligned} & \partial_{p_t} Q_t(p_t^e(N_t), N_t) - \theta \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\ & \geq \partial_{p_t} Q_t(p_t(N_t), N_t) - \theta \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)], \end{aligned}$$

i.e.,

$$\begin{aligned} & \bar{V}_t + c - 2p_t^e(N_t) + \gamma(N_t) - \theta \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\ & \leq \bar{V}_t + c - 2p_t(N_t) + \gamma(N_t) - \theta \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \end{aligned}$$

Thus,

$$\begin{aligned} & (p_t^e(N_t) - p_t(N_t)) + \theta(\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\ & \quad - \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)] \\ & \geq p_t^e(N_t) - p_t(N_t) \\ & > 0. \end{aligned}$$

Moreover,

$$\begin{aligned} & \mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\ & \quad - \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)] \\ & \geq \frac{2}{\theta}(p_t^e(N_t) - p_t(N_t)) \\ & > 0. \end{aligned}$$



Therefore,

$$\begin{aligned}
\partial_{N_t} \pi_t^e(N_t) - \partial_{N_t} \pi_t(N_t) &= ((p_t^e(N_t) - p_t(N_t)) + \theta(\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+]) \\
&\quad - \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)] \gamma'(N_t) \\
&\quad + \eta(\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+]) \\
&\quad - \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)] \\
&\geq 0.
\end{aligned}$$

Hence,  $\partial_{N_t} \pi_t^e(N_t) \geq \partial_{N_t} \pi_t(N_t)$  for all  $N_t$ . This completes the induction and, thus, the proof of parts (a)-(c).

**Part (d).** Note that  $\pi_t(\cdot)$  is the normalized optimal profit with the Bellman equation (9) and feasible decision set  $\{(x_t, p_t, n_t) : \Delta_t \in \mathbb{R}, p_t \in [\underline{p}, \bar{p}], n_t = 0\} \subset \mathcal{F}_e$ , which is the feasible decision set associated with the profit  $\pi_t^e(\cdot)$ . Thus,  $\pi_t^e(N_t) \geq \pi_t(N_t)$  for all  $t$  and any  $N_t \geq 0$ . If  $n_t(N_t) > 0$ , we must have  $\pi_t^e(N_t) > \pi_t(N_t)$ . Otherwise there are two lexicographically different policies (one with  $n_t(N_t) = 0$  and the other with  $n_t(N_t) > 0$ ) that generate the same optimal normalized profit  $\pi_t(N_t)$ . This contradicts the assumption that the lexicographically smallest policy is selected. Thus,  $\pi_t^e(N_t) > \pi_t(N_t)$ , which establishes part (d). *Q.E.D.*

**Proof of Theorem 11: Part (a).** We show part (a) by backward induction. More specifically, we show that if  $\sigma = \eta = 0$  and  $v_{t-1}(\cdot, \cdot)$  is supermodular in  $(I_{t-1}, N_{t-1})$ ,  $v_t(\cdot, \cdot)$  is supermodular in  $(I_t, N_t)$ . Since  $v_0(I_0, N_0) = cI_0$ , the initial condition is satisfied.

Since supermodularity is preserved under expectation,  $\bar{\Psi}_t(x, y) := \alpha \mathbb{E}\{[v_{t-1}(x - \xi_t, y + \theta\xi_t + \epsilon_t) - cx]\}$  is supermodular in  $(x, y)$ . Let  $y_t = \bar{V}_t - p_t + \gamma(N_t)$ . Observe that

$$\begin{aligned}
J_t(x_t, p_t, N_t) &= R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t)) + \bar{\Psi}_t(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t))) \\
&= (\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta x_t + \Lambda(x_t - y_t) + \bar{\Psi}_t(x_t - y_t, \theta y_t).
\end{aligned}$$

Hence,

$$v_t(I_t, N_t) = cI_t + \max_{(x_t, y_t) \in \mathcal{F}'_t(I_t, N_t)} \{(\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta x_t + \Lambda(x_t - y_t) + \bar{\Psi}_t(x_t - y_t, \theta y_t)\},$$

where  $\mathcal{F}'_t(I_t, N_t) := \{(x_t, y_t) : x_t \geq I_t, y_t \in [\bar{V}_t + \gamma(N_t) - \bar{p}, \bar{V}_t + \gamma(N_t) - \underline{p}]\}$ . Because  $\gamma(\cdot)$  is increasing in  $N_t$ ,  $\Lambda(\cdot)$  is concave, and  $\bar{\Psi}_t(\cdot, \cdot)$  is concave and supermodular,  $(\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta x_t + \Lambda(x_t - y_t) + \bar{\Psi}_t(x_t - y_t, \theta y_t)$  is supermodular in  $(x_t, y_t, N_t)$ . Moreover, it's straightforward to verify that the feasible set  $\{(x_t, y_t, I_t, N_t) : N_t \geq 0, (x_t, y_t) \in \mathcal{F}'_t(I_t)\}$  is a lattice in  $\mathbb{R}^4$ . Therefore,  $v_t(I_t, N_t)$  is supermodular in  $(I_t, N_t)$ . This completes the induction and, thus, the proof of part (a).

**Part (b).** The continuity results in parts (b)-(e) all follow from the joint concavity and continuous differentiability of  $J_t(\cdot, \cdot, \cdot)$  in  $(x_t, p_t, N_t)$ . Since  $x_t^*(I_t, N_t) = \max\{I_t, x_t(N_t)\}$ ,  $x_t^*(I_t, N_t)$  is increasing in  $I_t$ . Moreover, because the objective function  $(\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta x_t + \Lambda(x_t - y_t) + \bar{\Psi}_t(x_t - y_t, \theta y_t)$  is supermodular in  $(x_t, y_t, N_t)$ ,  $x_t^*(I_t, N_t)$  is increasing in  $N_t$  as well. This proves part (b).

**Part (c).** If  $I_t \leq x_t(N_t)$ ,  $p_t^*(I_t, N_t) = p_t(N_t)$ , which is independent of  $I_t$ . If  $I_t > x_t(N_t)$ ,  $x_t^*(I_t, N_t) = I_t$  and, thus,

$$J_t(x_t^*(I_t, N_t), p_t, N_t) = R_t(p_t, N_t) + \beta I_t + \Lambda(I_t - \bar{V}_t + p_t - \gamma(N_t)) + \bar{\Psi}_t(I_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t))). \quad (\text{EC.45})$$

Since  $\Lambda(\cdot)$  is concave and  $\bar{\Psi}_t(\cdot, \cdot)$  is concave and supermodular,  $J_t(x_t^*(I_t, N_t), p_t, N_t)$  is submodular in  $(I_t, p_t)$ . Hence,  $p_t^*(I_t, N_t)$  is decreasing in  $I_t$  for all  $(I_t, N_t)$ . By Theorem 4(d), if  $I_t \leq x_t(N_t)$ ,  $p_t^*(I_t, N_t) = p_t(N_t)$  is increasing in  $N_t$ . If  $I_t > x_t(N_t)$ , we observe from (EC.45) that  $J_t(x_t^*(I_t, N_t), p_t, N_t)$  is supermodular in  $(p_t, N_t)$ . Hence,  $p_t^*(I_t, N_t)$  is increasing in  $N_t$  for all  $(I_t, N_t)$ . This proves part (c).

**Part (d).** If  $I_t \leq x_t(N_t)$ ,  $y_t^*(I_t, N_t) = y_t(N_t)$ , which is independent of  $I_t$ . If  $I_t > x_t(N_t)$ ,  $x_t^*(I_t, N_t) = I_t$  and, thus,

$$J_t(x_t^*(I_t, N_t), p_t, N_t) = (\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta I_t + \Lambda(I_t - y_t) + \bar{\Psi}_t(I_t - y_t, \theta y_t).$$

Since  $\Lambda(\cdot)$  is concave and  $\Psi_t(\cdot, \cdot)$  is concave and supermodular,  $J_t(x_t^*(I_t, N_t), p_t, N_t)$  is supermodular in  $(I_t, y_t)$  and its domain is a sublattice of  $\mathbb{R}^2$ . Hence,  $y_t^*(I_t, N_t)$  is increasing in  $I_t$  for all  $(I_t, N_t)$ . By Theorem 4(d), if  $I_t \leq x_t(N_t)$ ,  $y_t^*(I_t, N_t) = y_t(N_t)$  is increasing in  $N_t$ . If  $I_t > x_t(N_t)$ ,  $x_t^*(I_t, N_t) = I_t$  and, thus,  $J_t(x_t^*(I_t, N_t), p_t, N_t) = (\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta I_t + \Lambda(I_t - y_t) + \bar{\Psi}_t(I_t - y_t, \theta y_t)$ . The supermodularity of  $J_t(x_t^*(I_t, N_t), p_t, N_t)$  in  $(y_t, N_t)$  follows directly from that  $\gamma(\cdot)$  is increasing in  $N_t$ . Moreover, the feasible set  $\{(y_t, N_t) : y_t \in [\bar{V}_t + \gamma(N_t) - \bar{p}, \bar{V}_t + \gamma(N_t) - \underline{p}]\}$  is clearly a sublattice of  $\mathbb{R}^2$ . Therefore,  $y_t^*(I_t, N_t)$  is increasing in  $N_t$  for all  $(I_t, N_t)$ . This proves part (d).

**Part (e).** If  $I_t \leq x_t(N_t)$ , optimizing (9) yields that  $\Delta_t^*(I_t, N_t) = \Delta_*$  is independent of  $I_t$  and  $N_t$ . If  $I_t > x_t(N_t)$ , since  $I_t - \Delta_t = y_t$ ,

$$J_t(x_t^*(I_t, N_t), p_t, N_t) = (\bar{V}_t + \gamma(N_t) + \Delta_t - I_t - \alpha c - b)(I_t - \Delta_t) + \beta I_t + \Lambda(\Delta_t) + \bar{\Psi}_t(\Delta_t, \theta(I_t - \Delta_t)).$$

Since  $\bar{\Psi}_t(\cdot, \cdot)$  is concave and supermodular,  $J_t(x_t^*(I_t, N_t), p_t, N_t)$  is supermodular in  $(I_t, \Delta_t)$ . Moreover, the feasible set  $\{(I_t, \Delta_t) : \Delta_t \in [I_t - \bar{V}_t - \gamma(N_t) + \underline{p}, I_t - \bar{V}_t - \gamma(N_t) + \bar{p}]\}$  is clearly a sublattice of  $\mathbb{R}^2$ . Hence,  $\Delta_t^*(I_t, N_t)$  is increasing in  $I_t$  for all  $(I_t, N_t)$ . Moreover, since  $\Delta_t^*(I_t, N_t) = I_t - y_t^*(I_t, N_t)$ , by part (d),  $\Delta_t^*(I_t, N_t)$  is decreasing in  $N_t$ . This proves part (e). *Q.E.D.*

## EC.2. Additional Discussions on Assumption 1

In this section, we present some additional discussions on the key technical assumption of our model, Assumption 1, which assumes that  $R_t(\cdot, \cdot)$  is jointly concave on its domain. Specifically, we present the necessary and sufficient conditions for  $R_t(\cdot, \cdot)$  to be jointly concave in Section EC.2.1. Then, in Section EC.2.2, we give concrete examples of concave  $R_t(\cdot, \cdot)$  functions for specific forms of the network externalities function  $\gamma(\cdot)$ . Finally, Section EC.2.3 numerically examines the robustness of our analytically results for non-concave  $R_t(\cdot, \cdot)$  functions.

### EC.2.1. Necessary and Sufficient Conditions for Assumption 1

First, we give the necessary and sufficient condition for the joint concavity of  $R_t(\cdot, \cdot)$ .

LEMMA EC.3. *Assumption 1 holds for period  $t$ , if and only if, for all  $N_t \geq 0$ ,*

$$-2(\underline{p} - \alpha c - b)\gamma''(N_t) \geq (\gamma'(N_t))^2. \quad (\text{EC.46})$$

**Proof:** Since  $\gamma(\cdot)$  is twice continuously differentiable,  $R_t(\cdot, \cdot)$  is twice continuously differentiable, and jointly concave in  $(p_t, N_t)$  if and only if the Hessian of  $R_t(\cdot, \cdot)$  is negative semi-definite, i.e.,  $\partial_{p_t}^2 R_t(p_t, N_t) \leq 0$ ,

and  $\partial_{p_t}^2 R_t(p_t, N_t) \partial_{N_t}^2 R_t(p_t, N_t) \geq (\partial_{p_t} \partial_{N_t} R_t(p_t, N_t))^2$ , where  $\partial_{p_t}^2 R_t(p_t, N_t) = -2$ ,  $\partial_{N_t}^2 R_t(p_t, N_t) = (p_t - b - \alpha c) \gamma''(N_t)$ , and  $\partial_{p_t} \partial_{N_t} R_t(p_t, N_t) = \gamma'(N_t)$ . Hence,  $R_t(\cdot, \cdot)$  is jointly concave on  $[\underline{p}, \bar{p}] \times [0, +\infty)$  if and only if  $-2(p_t - b - \alpha c) \gamma''(N_t) \geq (\gamma'(N_t))^2$  for all  $(p_t, N_t)$ . Since  $-2(p_t - b - \alpha c) \gamma''(N_t) \geq -2(\underline{p} - b - \alpha c) \gamma''(N_t)$ ,  $-2(p_t - b - \alpha c) \gamma''(N_t) \geq (\gamma'(N_t))^2$  for all  $(p_t, N_t)$  if and only if  $-2(\underline{p} - b - \alpha c) \gamma''(N_t) \geq (\gamma'(N_t))^2$  for all  $N_t \geq 0$ . *Q.E.D.*

The necessary and sufficient condition for Assumption 1 characterized by inequality (EC.46) offers little insight on what Assumption 1 means in practice. Thus, we give a simpler condition for Assumption 1.

**LEMMA EC.4.** *Let  $M := \sup\{-(\gamma'(N_t))^2/\gamma''(N_t) : N_t \geq 0\}$ .  $R_t(\cdot, \cdot)$  is jointly concave if and only if  $\underline{p} \geq \alpha c + b + \frac{M}{2}$ .*

**Proof:** By Lemma EC.3, if Assumption 1 holds  $-2(\underline{p} - \alpha c - b) \gamma''(N_t) \geq (\gamma'(N_t))^2$ . Since  $\gamma''(N_t) \leq 0$ , (EC.46) implies that  $\underline{p} - \alpha c - b \geq -\frac{1}{2}((\gamma'(N_t))^2/\gamma''(N_t))$  for any  $N_t \geq 0$ . Taking supreme, we have that  $\underline{p} \geq \alpha c + b + \frac{M}{2}$ .

If  $\underline{p} \geq \alpha c + b + \frac{M}{2}$ , we have  $-2(\underline{p} - \alpha c - b) \gamma''(N_t) \geq -M \gamma''(N_t)$  for any  $N_t$ . Since  $M = \sup\{-(\gamma'(N_t))^2/\gamma''(N_t) : N_t \geq 0\}$ ,  $M \geq -(\gamma'(N_t))^2/\gamma''(N_t)$  for any  $N_t \geq 0$ . Therefore,  $-M \gamma''(N_t) \geq (\gamma'(N_t))^2$  for any  $N_t$ . Putting everything together, we have  $-2(\underline{p} - \alpha c - b) \gamma''(N_t) \geq -M \gamma''(N_t) \geq (\gamma'(N_t))^2$ . By Lemma EC.3,  $R_t(\cdot, \cdot)$  is jointly concave. *Q.E.D.*

### EC.2.2. Examples of Concave $R_t(\cdot, \cdot)$ Functions

We continue our discussion by giving some concrete examples of jointly concave  $R_t(\cdot, \cdot)$  functions (see Assumption 1). We characterize the necessary and sufficient conditions under which  $R_t(\cdot, \cdot)$  is jointly concave for some specific forms of the network externalities function  $\gamma(\cdot)$ . We discuss three families of  $\gamma(\cdot)$ : (a) exponential functions; (b) power functions; and (c) logarithm functions. We demonstrate that the necessary and sufficient conditions characterized in Lemmas EC.4 and EC.3 can be satisfied by these simple  $\gamma(\cdot)$  functions under certain conditions, which are presented in model primitives and easy to verify.

First, we specify the functional form of  $\gamma(\cdot)$  as  $\gamma(N_t) = \gamma_0 - \gamma_0 \exp(-kN_t)$  for  $N_t \geq 0$  ( $\gamma_0, k > 0$ ). First, we compute the first and second order derivatives of  $\gamma(\cdot)$ :

$$\begin{cases} \gamma'(N_t) = k\gamma_0 \exp(-kN_t), \\ \gamma''(N_t) = -k^2\gamma_0 \exp(-kN_t). \end{cases} \quad (\text{EC.47})$$

Note that  $-\frac{(\gamma'(N_t))^2}{\gamma''(N_t)} = \gamma_0 \exp(-kN_t) \leq \gamma_0$ . Hence, the necessary condition characterized in Lemma EC.4 for  $R_t(\cdot, \cdot)$  to be jointly concave is satisfied for this family of  $\gamma(\cdot)$ 's. Next we characterize the necessary and sufficient condition for  $R_t(\cdot, \cdot)$  to be jointly concave for exponential network externalities functions.

**LEMMA EC.5.** *If  $\gamma(N_t) = \gamma_0 - \gamma_0 \exp(-kN_t)$  ( $\gamma_0, k > 0$ ), we have  $R_t(\cdot, \cdot)$  is jointly concave in  $(p_t, N_t)$  if and only if*

$$2(\underline{p} - \alpha c - b) \geq \gamma_0. \quad (\text{EC.48})$$

**Proof:** Plug (EC.47) into (EC.46), and we have that  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if

$$2(\underline{p} - \alpha c - b) k^2 \gamma_0 \exp(-kN_t) \geq k^2 \gamma_0^2 \exp(-2kN_t), \text{ for any } N_t \geq 0. \quad (\text{EC.49})$$

Direct algebraic manipulation yields that (EC.49) is equivalent to that  $2(\underline{p} - \alpha c - b) \exp(kN_t) \geq \gamma_0$  for any  $N_t \geq 0$ . Therefore,  $R_t(\cdot, \cdot)$  is jointly concave if and only if (EC.48) holds. *Q.E.D.*

Lemma EC.5 specifies the necessary and sufficient conditions characterized in Lemmas EC.4 and EC.3 in the case with exponential network externalities functions. In short,  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if  $\underline{p}$  is sufficiently large relative to  $\gamma_0$ , which is equivalent to that the price elasticity of demand is sufficiently high compared with the network size elasticity of demand.

Next, we specify the functional form of  $\gamma(\cdot)$  as  $\gamma(N_t) = \gamma_0 - \gamma_0(N_t + 1)^{-k}$  for  $N_t \geq 0$  ( $\gamma_0, k > 0$ ). First, we compute the first and second order derivatives of  $\gamma(\cdot)$ :

$$\begin{cases} \gamma'(N_t) = k\gamma_0(N_t + 1)^{-k-1}, \\ \gamma''(N_t) = -k(k+1)\gamma_0(N_t + 1)^{-k-2}. \end{cases} \quad (\text{EC.50})$$

Note that for  $N_t \geq 0$ ,  $-\frac{(\gamma'(N_t))^2}{\gamma''(N_t)} = \frac{k\gamma_0}{k+1}(N_t + 1)^{-k} \leq \frac{k\gamma_0}{k+1}$ . Hence, the necessary condition characterized in Lemma EC.4 for  $R_t(\cdot, \cdot)$  to be jointly concave is satisfied. Next we characterize the necessary and sufficient condition for  $R_t(\cdot, \cdot)$  to be jointly concave for power network externalities functions.

LEMMA EC.6. *If  $\gamma(N_t) = \gamma_0 - \gamma_0(N_t + 1)^{-k}$  for  $N_t \geq 0$  ( $\gamma_0, k > 0$ ), we have  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if*

$$2(\underline{p} - \alpha c - b)(k+1) \geq \gamma_0 k. \quad (\text{EC.51})$$

**Proof:** Plug (EC.50) into (EC.46), and we have that  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if

$$2(\underline{p} - \alpha c - b)k(k+1)\gamma_0(N_t + 1)^{-k-2} \geq k^2\gamma_0^2(N_t + 1)^{-2k-2}, \text{ for any } N_t \geq 0. \quad (\text{EC.52})$$

Direct algebraic manipulation yields that (EC.52) is equivalent to  $2(\underline{p} - \alpha c - b)(k+1)(N_t + 1)^k \geq k\gamma_0$  for all  $N_t \geq 0$ . Therefore,  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if (EC.51) holds. *Q.E.D.*

Lemma EC.6 specifies the necessary and sufficient conditions characterized in Lemmas EC.4 and EC.3 in the case with power network externalities functions. As in the case with exponential network externalities functions,  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if  $\underline{p}$  is sufficiently large relative to  $\gamma_0$ , which is equivalent to that the price elasticity of demand is sufficiently high compared with the network size elasticity of demand.

Finally, we specify the functional form of  $\gamma(\cdot)$  as  $\gamma(N_t) = \gamma_0 \log(N_t + 1)$  ( $\gamma_0 > 0$ ). First, we compute the first and second order derivatives of  $\gamma(\cdot)$ :

$$\begin{cases} \gamma'(N_t) = \frac{\gamma_0}{N_t + 1}, \\ \gamma''(N_t) = -\frac{\gamma_0}{(N_t + 1)^2}. \end{cases} \quad (\text{EC.53})$$

Note that  $-\frac{(\gamma'(N_t))^2}{\gamma''(N_t)} = \gamma_0$  for all  $N_t$ . Hence, the necessary condition characterized in Lemma EC.4 for  $R_t(\cdot, \cdot)$  to be jointly concave is satisfied for this family of  $\gamma(\cdot)$ 's. Next we characterize the necessary and sufficient condition for  $R_t(\cdot, \cdot)$  to be jointly concave for logarithm network externalities functions.

LEMMA EC.7. *If  $\gamma(N_t) = \gamma_0 \log(N_t + 1)$  ( $\gamma_0 > 0$ ), we have  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if:*

$$2(\underline{p} - \alpha c - b) \geq \gamma_0. \quad (\text{EC.54})$$

**Proof:** Plug (EC.53) into (EC.46), and we have that  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if

$$2(\underline{p} - \alpha c - b) \frac{\gamma_0}{(N_t + 1)^2} \geq \frac{\gamma_0^2}{(N_t + 1)^2}. \quad (\text{EC.55})$$

Direct algebraic manipulation yields that (EC.55) is equivalent to  $2(\underline{p} - \alpha c - b) \geq \gamma_0$  for all  $N_t \geq 0$ . Therefore,  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if (EC.54) holds. *Q.E.D.*

Lemma EC.7 specifies the necessary and sufficient conditions characterized in Lemmas EC.4 and EC.3 in the case power network externalities functions. As in the cases with exponential and power network externalities functions,  $R_t(\cdot, \cdot)$  is jointly concave on its domain if and only if  $\underline{p}$  is sufficiently large relative to  $\gamma_0$ , which is equivalent to that the price elasticity of demand is sufficiently high compared with the network size elasticity of demand.

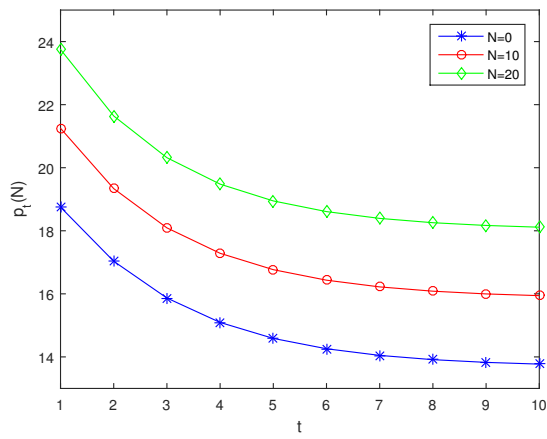
Lemmas EC.5-EC.7 confirm our previous insight delivered by Lemma EC.4 that when the price elasticity of demand (i.e.,  $|\frac{d\mathbb{E}[D_t(p_t, N_t)]/\mathbb{E}[D_t(p_t, N_t)]}{dp_t/p_t}|$ ) is sufficiently high relative to the *network size elasticity of demand* (i.e.,  $|\frac{d\mathbb{E}[D_t(p_t, N_t)]/\mathbb{E}[D_t(p_t, N_t)]}{dN_t/N_t}|$ ),  $R_t(\cdot, \cdot)$  is jointly concave in  $(p_t, N_t)$  on its domain. Therefore, Assumption 1 can be satisfied for a wide variety of network externalities function  $\gamma(\cdot)$ 's. To conclude this section, we remark that the above method can be easily adapted to characterize the conditions under which  $R_t(\cdot, \cdot)$  is jointly concave with other families of network externalities functions.

### EC.2.3. Robustness Check with Non-Concave $R_t(\cdot, \cdot)$ Functions

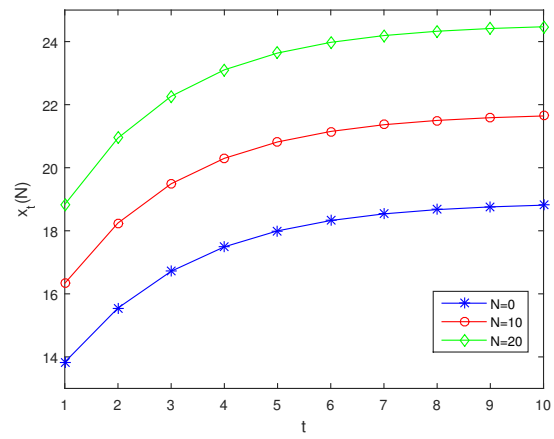
Note that with the linear network externalities function  $\gamma(\cdot)$  (see Sections 5.3), Assumption 1 does not hold. We adopt the same numerical setup as Section 5.3 to check the robustness of our analytical results when Assumption 1 does not hold. In particular, we have numerically verified that the optimal policy in each period is still a base-stock/list-price policy. Moreover, throughout our simulations, the key sample-path property (7) continues to hold and, thus, the state-space of the dynamic program can be decoupled. In all our numerical experiments, Theorems 3, 4, 5, 6 and 7 are still valid. Therefore, the numerical experiments demonstrate the robustness of the insights derived from our analytical model when Assumption 1 is no longer satisfied. For example, Figures EC.1-EC.2 plot the optimal list-price and base-stock level with parameters  $k = \theta = \eta = 0.5$ . Figure EC.1 illustrates that the optimal list price  $p_t(\cdot)$  is increasing in the network size  $N_t$  and decreasing in the time index  $t$ . Thus, the theoretic predictions of Theorem 4(a) and Theorem 5(c) are robust when  $R_t(\cdot, \cdot)$  is non-concave. Similarly, Figure EC.2 depicts the optimal base-stock level  $x_t(\cdot)$  and validates the robustness of Theorem 5(b) when Assumption 1 is violated.

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**Figure EC.1** Optimal List-Price  $p_t(N)$



**Figure EC.2** Optimal Base-Stock Level  $x_t(N)$