Carpool Services for Ride-sharing Platforms: Price and Welfare Implications

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There has been rapid growth in on-demand ride-hailing platforms that serve as an intermediary to match individual service providers (drivers) with consumer demand (riders). Several major players of this market have introduced carpool services that allow passengers heading towards the same direction to share a ride at a discounted fare. In this paper, we develop an analytical model to study the pricing issues of ride-sharing platforms in the presence of carpool services, and their economical and social implications. We show that the carpool service should be provided when its quality and/or the pooling efficiency is high. When the platform finds it optimal to offer the carpool service option, the platform achieves a larger market coverage and the riders are able to enjoy more affordable rides without compromising on service quality. Our analysis reveals that the provision of carpool services benefits the platform and the riders in general, but may hurt the drivers.

Key words: Sharing economy, on-demand two-sided platforms, carpool services, pricing, welfare analysis

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1. Introduction

Recent years have witnessed a phenomenal growth in on-demand ride-hailing platforms (e.g., Uber, Lyft, Grab, Ola and Didi Chuxing) that serve as an intermediary to match individual service providers (drivers) with consumer demand (riders). Instead of planning the provider resources (e.g., drivers, cars, etc) in advance, ride-hailing platforms operate by matching self-scheduling and earning-sensitive drivers with price-sensitive customers in a real time. The rapid growth in popularity and success of these on-demand ride-sharing platforms have been phenomenal. Uber, for instance, is now operating in more than 900 cities globally (Uber 2020) and has achieved 111 million active users by the end of 2019 (DMR 2020a). As another example, Didi Chuxing, the largest on-demand ride-hailing platform in China, processes 10 billion trips per year for more than 550 million users as of 2019 (DMR 2020b).

Several major players of the on-demand ride-hailing market have introduced carpool services, such as UberPool by Uber, Lyft Line by Lyft and Didi Pinche by Didi Chuxing. The carpool service option enables drivers to pick up multiple passengers travelling along similar routes and the riders need to share the ride with each other. The platform typically offers a discounted price for carpool rides. However, this discounted fare often comes at the expense of a lower-quality service because of the lack of privacy and longer trip duration if detours are made to pick up or drop off other passengers. Even before the era of on-demand ride-hailing platforms, carpool has been widely applauded and promoted for their value in reducing the number of vehicles on the road, which helps curtail exhaust pollution and alleviate traffic congestions (see, e.g., Chan and Shaheen 2012). Uber has also promoted UberPool by highlighting the value of their carpool services from the social and environmental perspectives (e.g., UberBlog 2016). However, it is worth noting that the carpool services have also received controversial responses since their introduction. Some riders favor the carpool service because it is less expensive and more affordable than the normal service, and sometimes sharing a ride with others even facilitates new connections (e.g., Jess 2015). On the other hand, one major critique of the carpool service is the potential compromises on privacy, security, and inconvenience involved with riding with a stranger in the closed and confined environment within a car. Moreover, many drivers complain that they work more for UberPool or Lyft Line but get paid less, and receive lower ratings because carpool riders usually receive lower-quality services. Despite the prevalence and controversial perceptions of carpool services in the ride-sharing market, research in the extant literature that rigorously studies their operational, economical and social implications is limited.

In this paper, our primary goal is to model the *carpool* service for the on-demand ride-hailing platforms and investigate the price and welfare implications it bears. More specifically, we consider a monopoly ride-sharing platform who offers both normal (non-pool) and carpool services with vertically differentiated qualities. Riders have heterogeneous values over service quality and would choose the travel option with the highest surplus. Drivers are self-scheduling with heterogeneous reservation wages, and would work for the platform only if the wages distributed by the platform are better than their outside options. To maximize its expected profit, the platform designs its pricing and wage schemes, and matches drivers with riders for both normal and carpool services. We characterize the optimal policy of the platform, study the impact of carpool services on different stakeholders of the market (the platform, riders, and drivers), and draw insights on the conditions under which carpool services are most valuable. Our theoretical analysis is also complemented with computational studies which help strengthen the practical relevance of this work.

1.1. Main Contributions

We next summarize our main results and contributions below.

1.1.1. Price Implications of Carpool Services. We show that the platform should provide the carpool service option when the value deterioration of the carpool service is not too high compared with the normal service, and/or when the pooling efficiency is not too low. We find that the provision of carpool services enables the platform to better utilize the driver capacity and expands its market coverage. Our analysis reveals that, as expected, the optimal price for the carpool services is lower than that of the normal services, which is consistent with the business practice. More interestingly, the presence of carpool services also prompts the platform to charge a lower price for its normal service when both service modes are offered, compared with the benchmark system where only the normal service is available. One may intuit that, by offering an additional service mode, the platform should be in a better position to discriminate between customers with different valuations and can charge a higher price for the (high-quality) normal service. However, our results suggest that the opposite is true. Besides the price discrimination effect, the introduction of the carpool service also has the demand cannibalization effect and the cost reduction effect. On one hand, the carpool service cannibalizes the demand for the normal service, which prompts the platform to charge a lower price for the normal service as well. On the other hand, there also exists a *cost reduction effect* that arises from the drivers' self-scheduling behavior. As we will show later, the supply expansion due to the carpool reduces the labor/capacity cost of the platform. Such a cost reduction effect benefits the normal service as well (because the entire platform uses a shared pool of drivers), and as a result, it also pushes down the price of the normal service. The overall effect of cannibalization and cost reduction together outweighs the price discrimination effect when the platform finds it attractive to introduce the carpool option, resulting in a lower price for the normal service. As a consequence, offering the carpool service enables riders to enjoy more affordable rides without compromising on service quality.

1.1.2. Welfare Implications of Carpool Services. In addition to examining how the provision of carpool services affects the ride-sharing platform's pricing decisions and its bottom line, we are also interested in the welfare implications of the carpool services on different stakeholders. As for the rider surplus, we find that offering carpool services benefits the riders in general. This is intuitive given that riders are provided with an additional service option when the platform offers carpool services. However, we show that the provision of carpool services acts as a double-edged sword for the drivers as it may improve or hurt the driver surplus. On one hand, the presence of carpool services expands market coverage and hence brings in more riders to be matched with drivers and may increase the drivers' earnings. On the other hand, carpool services also enlarge the capacity per driver by pooling multiple passengers into a single ride, which decreases the total need and earning potentials for the drivers. We show that when the drivers' reservation wage is uniformly distributed, the latter outweighs the former and overall the drivers are worse off. As we will detail later, the carpool service should be offered when the pooling efficiency is high enough, and the introduction of the carpool service reduces the capacity/labor cost of the entire platform. This cost reduction is achieved by a lower per-unit-time total wage offered by the platform, which leads to a smaller number of active drivers in equilibrium, as well as a lower driver surplus. Therefore, the platform and the riders may benefit from the carpool service provision at the expense of the drivers. Finally, from the perspective of the entire society, our numerical analysis suggests that the introduction of the carpool services increases the total social welfare in general.

1.2. Related Literature

The emergence and phenomenal growth of the sharing economy in recent years have attracted considerable academic interest from the operations research/operations management community. Our work is closely related to the stream of research that examines how on-demand service platforms can adjust service prices and agent wages to effectively coordinate supply with demand. Banerjee et al. (2015) model the ride-hailing problem as a queueing network where customers arrival and the drivers work hours depend on the real-time dynamic service price. The authors show that dynamic pricing with prices responding instantaneously to demand-supply imbalances does not provide more benefit than the optimal static pricing. In a similar vein, Hu and Zhou (2019b) show that a flat-commission contract can be optimal or near-optimal for the platform compared with the benchmark where the platform is allowed to freely determine the price and wage under various market conditions. Chen and Hu (2019) consider the dynamic pricing decisions of a ride-sharing platform in a strategic environment where the customers and suppliers may wait strategically for better prices. They show that under a thick market with large transaction volume, a waiting-adjusted fixed pricing heuristic is close to optimal. The above papers demonstrate the near-optimality of static pricing, whereas Cachon et al. (2017) and Guda and Subramanian (2019) demonstrate the merit of the surge pricing policy for on-demand service platforms with self-scheduling capacity. Our work in this paper also considers the price and wage optimization problems faced by the on-demand platforms in order to effectively coordinate supply with demand, but with a specific focus on the carpool service, and its operational, economic, and societal implications. We show that the provision of carpool services enables the platform to expand market coverage. Moreover, when the platform finds it optimal to offer the carpool service option, the optimal price of the normal service is reduced compared with the benchmark system where only the normal service is available. In other words, the provision of the carpool services allows customers to pay less without compromising on the service quality.

In addition to examining the price and wage optimization problems, researchers have also explored various operational issues that arise from on-demand service platforms. Bimpikis et al.

(2019a) consider the spatial transition of a ride-sharing network and characterize the value of spatial price discrimination for a ride-sharing platform that serves a network of locations with deterministic demand patterns. The authors show that the pricing policy that uses a fixed commission rate could result in significant profit loss in case of heterogeneous demand patterns across different locations. Bai et al. (2019) use the steady-state equilibrium to characterize the optimal price and wage for a monopolistic on-demand platform where an $M/M/k$ queuing model is used to approximate the waiting time of passengers. They show that the price and wage policy with a fixed payout ratio could capture most of the profit from an optimal policy. Gurvich et al. (2019) use a newsvendor model to study the capacity management problem in sharing marketplaces and find that workers' flexibility to choose their own work schedules reduces worker participation and increases price levels. Taylor (2018) studies how two defining features of an on-demand service platform — congestion-driven delay sensitivity and agent independence — affect the platform's optimal per-service price and wage. Chen and Sheldon (2016) empirically examine the impact of surge pricing (dynamic wages) on the duration for which a driver works on Uber's platform, and find that drivers are more likely to continue working if surge pricing is in effect upon they finish a trip. Hu and Zhou (2019a) consider an intermediary's problem of dynamically matching demand and supply of heterogeneous types in a periodic-review fashion, and they provide sufficient and robustly necessary conditions on matching rewards such that the optimal matching policy follows a priority hierarchy. Chen et al. (2020a) study the impact of bonus strategies on competing two-sided service platforms and their welfare implications. The authors show that bonus competition can lead to opposite impacts on the platforms depending on the market condition.

Our work is also related to the stream of research on carpool services in the operations and transportation literature. Alonso-Mora et al. (2017) present a general model of large-scale ride-sharing systems with carpool services, and develop an algorithm that dynamically generates optimal routes with respect to online demand and vehicle locations. Gopalakrishnan et al. (2017) introduce the new notions of sequential individual rationality (SIR) and sequential fairness (SF) in a cost sharing framework for carpool services. The authors characterize the routes and cost sharing schemes that satisfy SIR and SF. Wen et al. (2017) use reinforcement learning to address the fleet rebalancing needs for carpool services. Under a multi-nomial logit (MNL) model, Cohen and Zhang (2017) show that, with a well-designed profit-sharing contract, it would benefit two competing ride-sharing platforms to partner with each other and jointly offer a new carpool service. A recent paper by Jacob and Roet-Green (2021) develops a queueing-theoretic model and designs an incentive-compatible price-service menu that maximizes the ride sharing platform's revenue at equilibrium. They find that offering both solo and pooled rides is optimal only when the distribution of high- and low-type passengers is not skewed and the congestion (the ratio of passenger-demand to driver-supply) is not very high. Whereas they focus on when the platform should offer the carpool service (or not), our work focuses on understanding the impact of the carpool service on the platform's pricing strategy and the welfare implications on different stakeholders.

Since the carpool service provides customers with an alternative choice with a lower "quality" level than the normal service, our work is also relevant to the extensive marketing literature on the (vertically-differentiated) product line design problem (e.g., Mussa and Rosen 1978, Moorthy 1984, Moorthy and Png 1992, Villas-Boas 1998, Desai 2001, Villas-Boas 2004). In this stream of literature, products are vertically differentiated along the quality attribute, where higher-quality products have higher marginal production costs, and consumers have heterogeneous willingness-to-pay for this quality attribute. The seminal work by Mussa and Rosen (1978) and Moorthy (1984) study a monopolistic firm's vertical product line design decisions (i.e., the quality levels and prices of the offered products) that allow customers to self select which product to purchase in order to maximize their utility, where the consumers' self-selection behavior results in cannibalization among the variants offered. The classic results in this literature show that, compared with the socially efficient solution (i.e, the first-best solution in which the monopolistic firm has precise knowledge of customer valuations and has the ability dictate their choices), the firm may reduce the number of product variants offered and provide lower quality to all customers except those in the highest-valuation segment in order to mitigate the cannibalization effect. Different from most of the papers in this literature, in our model the quality levels of the carpool service and the normal service are exogenous rather than the firm's endogenous decisions. Instead of examining how the consumers' self-selection behavior affects the firm's optimal pricing and quality provision decisions compared with the socially optimal benchmark, we are interested in the impact of offering the carpool service on prices and social welfare relative to the baseline case where only the normal service is available. Our model also differs from this literature in the supply side. A majority of the product line design work assumes that the higher-quality products have higher marginal production costs. In our model, the supply is endogenous and determined by drivers' self-scheduling behavior. The carpool option is not just a "lower quality version" of the normal service, but it also "expands" the supply capacity of the drivers by pooling multiple passengers in a single ride. As we shall show later, the driver capacity efficiency increase due to the introduction of carpools has a labor cost reduction effect that also benefits the normal service, since both service modes share the same pool of drivers on the platform. We remark that a few papers have also incorporated the supply side issues into the classical product line design problem. For example, Netessine and Taylor (2007) shows that more expensive production technology can lead the firm to offer a product line of *higher* average quality at a *lower* average price, thanks to the economies of scale from offering one composite product. Guo and Zhang (2012) shows that a monopolistic firm may reduce the price of the high-end product compared with that under standard second-degree discrimination in order to motivate consumers to deliberate and find out whether the high-end consumption fits their needs.

Finally, in addition to on-demand ride-hailing platforms, other types of online platforms have also been studied in the operations literature, such as e-commerce marketplaces (e.g., Cui et al. 2019b, Zhang et al. 2020, Qi et al. 2020), vacation rental platforms (e.g., Cui et al. 2019a), short-video sharing platforms (e.g., Chen et al. 2020b,c), peer-to-peer product sharing and rental markets (e.g., Benjaafar et al. 2019, Fraiberger and Sundararajan 2017, Jiang and Tian 2018, Li et al. 2017), peer-to-peer service platforms (e.g., Cullen and Farronato 2018), moderating service platforms (e.g., Allon et al. 2012), bike-sharing systems (e.g., Shu et al. 2013, Kabra et al. 2016), and electric car sharing system (e.g., He et al. 2020, 2017). We refer the readers to two excellent recent reviews by Benjaafar and Hu (2020) and Hu (2020) that connect classical operations management theory and models with the new applications of sharing economy and online marketplaces.

The remainder of the paper is organized as follows. We formally introduce our model in Section 2. In Section 3, we examine the operational implications of the carpool services, and in particular its impact on the ride-sharing platform's optimal service provision and pricing decisions. In Section 4, we investigate the welfare implications of the carpool services and how different stakeholders are affected by the introduction of this alternative service option. We summarize and conclude the paper in Section 5 with directions for future research. All proofs are relegated to the Appendix.

2. Model

We consider a ride-sharing platform that offers both *normal services* (i.e., rides with a single destination without carpool) and *carpool services* (i.e., rides shared by multiple passengers with different destinations). The value of a normal service ride to a customer is v_n, whereas that of a carpool ride is v_p. We note that v_n and v_p are on a per-trip basis. Let $\Delta := v_n - v_p > 0$ denote the value difference between these two services per ride, which reflects that the carpool service has a lower "quality" level compared with the normal service, since customers may have to experience a longer waiting time, lower privacy, and less comfortableness if they choose the carpool option. In the same spirit as Bai et al. (2019), we assume that each ride request consists of a certain amount of *service units* to be served by the driver, where a service unit (e.g., travel distance in kilometers/miles, trip time in minutes, or a combination of the two) is the unit of measure that riders get charged and drivers get paid. Let d_n denote the average service units of a normal ride, and let d_p represent the average service units of each rider in a carpool ride. Riders arrive randomly at the platform and request at most one ride service. The platform charges a price rate p_n per service unit for the normal service and a price rate p_p per service unit for the carpool service. For the supply side, the platform pays drivers a wage rate w_n per normal service unit and a wage rate w_p per carpool service unit. We assume that the price rates (p_n, p_p) and wage rates (w_n, w_p) are respectively public information known to the riders and the drivers. Therefore, the "supply" of participating drivers and the "demand" of rider requests are endogenously determined by the platform's pricing decisions — customers either choose a normal service, a carpool service, or leave the platform without requesting any service based on whichever option results in the highest utility, and each driver registered on the platform decides whether or not to work for the platform based on the expected earning compared with their outside option. Hereafter, we call customers and riders interchangeably.

2.1. Customers Ride Request Choice and Effective Arrival Rates

Suppose that for a certain time period (e.g., one hour), the maximum potential demand rate for ride service during this time period is $\bar{\lambda}$. To model the heterogeneity among riders without losing tractability, we assume that there is a continuum of customer types and the *type* of each customer represents her valuation for service quality. Moreover, a rider's *type*, denoted by θ, is independently and uniformly distributed on the interval $[0,1]$, and a type-θ customer's valuation from taking a normal ride is θv_n whereas that from taking a carpool ride is θv_p.

It then follows that the utility of a type-θ customer to request a normal service is $\theta v_n - p_n d_n$, and the utility of a type-θ customer to request a carpool service is $\theta v_p - p_p d_p$. We also normalize a rider's utility from taking the outside option to 0. We assume that customers are rational and make service request choices based on whichever alternative gives them the highest utility. Since $v_n > v_p$, if a type-θ_0 customer chooses to take a normal ride on the platform (i.e., $\theta_0 v_n - d_n p_n \geq \max\{0, \theta_0 v_p - d_p p_p\}$), then any type-$\theta$ customer with $\theta \geq \theta_0$ will take a normal ride as well. Similarly, if a type-θ_1 customer chooses to take a (normal or pooled) ride on the platform (i.e., $\max\{\theta_1 v_n - d_n p_n, \theta_1 v_p - d_p p_p\} \geq 0$), then any type-$\theta$ customer with $\theta \geq \theta_1$ will take a (normal or carpool) ride as well. In other words, there exist two thresholds θ_n and θ_p ($1 \geq \theta_n \geq \theta_p \geq 0$), such that a customer would request a ride service (either normal or pooled) if and only if $\theta \geq \theta_p$ and the customer would request a normal service if and only if $\theta \geq \theta_n$. Therefore, a customer with type θ would choose a normal service if $\theta \in [\theta_n, 1]$, a carpool service if $\theta \in [\theta_p, \theta_n)$, and would leave the platform without requesting any service if $\theta \in [0, \theta_p)$, where θ_n and θ_p are given by the following conditions:

$$\begin{aligned} &\theta_n v_n - p_n d_n = \theta_n v_p - p_p d_p, \\ &\theta_p v_p - p_p d_p = 0. \end{aligned} \tag{1}$$

Let s_n be the fraction of customers who choose a normal ride and s_p be the fraction of customers who choose a carpool ride in equilibrium. It is easy to see that s_n and s_p satisfy the following relationship with θ_n and θ_p:

$$\begin{aligned} &\theta_n = 1 - s_n, \\ &\theta_p = 1 - s_p - s_n. \end{aligned} \tag{2}$$

It is worth noticing that s_n and s_p represent the market share of the normal and carpool services, respectively. Assuming that on average each carpool ride is shared by m passengers, the effective demand arrival rate for normal services λ_n, and that for carpool services λ_p, are given by

$$\begin{aligned} \lambda_n &= \bar{\lambda} s_n, \\ \lambda_p &= \frac{1}{m} \bar{\lambda} s_p. \end{aligned} \tag{3}$$

In view of the one-to-one correspondence between the market shares of the two service modes and their effective demand rates, we shall focus our analysis on (s_n, s_p) instead of (λ_n, λ_p) for mathematical convenience. Moreover, from (1) and (2), the price rates (p_n, p_p) satisfy the following equations:

$$\begin{aligned} p_n &= \frac{(1-s_n)\Delta + (1-s_n-s_p)(v_n-\Delta)}{d_n}, \\ p_p &= \frac{(1-s_p-s_n)(v_n-\Delta)}{d_p}. \end{aligned} \tag{4}$$

2.2. Drivers' Decision and the Number of Active Drivers

Now we consider the self-scheduling drivers' decision on whether or not to work for the platform, depending on wage they can get. Assume that a continuum of drivers with total mass K are registered on the platform. In other words, K represents the maximum number of drivers potentially available to offer a ride service for the platform. Given p_n, p_p, w_n and w_p, let $k \in [0, K]$ be the actual number of drivers who opt to work on the platform, and we assume that the drivers would accept all the ride requests that the platform assigns to them.

The drivers are earnings-sensitive and they would opt to work on the platform if the expected per-unit-time wage is higher than what his outside option would offer. We consider heterogeneous drivers and let $G(\cdot)$ be the cumulative distribution of a driver's reservation earning rate for his outside option. The total per-unit-time wage to all drivers offered by the platform is

$$w_n \lambda_n d_n + w_p \lambda_p d_p' = w_n \bar{\lambda} s_n d_n + \frac{w_p \bar{\lambda} s_p d_p'}{m} = \bar{\lambda} \left(w_n s_n d_n + \frac{w_p s_p d_p'}{m} \right),$$

where d_p' denotes the average service units that a driver provides in a carpool ride[1]. With k active drivers, the expected per-unit-time wage for each driver who participates to work is

$$\frac{\bar{\lambda}}{k} \left(w_n s_n d_n + \frac{w_p s_p d_p'}{m} \right).$$

A driver would participate to offer service if and only if the reservation earning rate of his outside option does not exceed the expected per-unit time wage. Since the drivers are infinitesimal, the total

[1] Here we would like to remark that in general, we have $d_p' \leq m d_p$ since the riders in a carpool ride share a proportion of the ride.

number of active drivers should satisfy $k = KG\left(\frac{\bar{\lambda}}{k}\left(w_n s_n d_n + \frac{w_p s_p d_p'}{m}\right)\right)$. Therefore, in equilibrium, we have

$$G\left(\frac{\bar{\lambda}}{k}\left(w_n s_n d_n + \frac{w_p s_p d_p'}{m}\right)\right) = \frac{k}{K}.$$

Equivalently, (w_n, w_p) satisfy

$$w_n s_n d_n + \frac{w_p s_p d_p'}{m} = \frac{k}{\bar{\lambda}} G^{-1}\left(\frac{k}{K}\right). \tag{5}$$

Equation (5) characterizes the relationship between the wage rates (w_n, w_p) and the number of active drivers k in equilibrium. Let $w := \bar{\lambda}\left(w_n s_n d_n + \frac{w_p s_p d_p'}{m}\right)$. Note that w represents the total wage distributed by the platform per unit time. In view of (5), any choice of the wage rates (w_n, w_p) such that their weighted combination w satisfies $w = kG^{-1}\left(\frac{k}{K}\right)$ will induce k active drivers for the platform in equilibrium.

2.3. Platform's Optimization Problem

We now consider the platform's optimal price and wage decisions. The platform earns an average profit of $(p_n - w_n)d_n$ for each normal ride, and the profit from a carpool ride with an average of m passengers is $mp_p d_p - w_p d_p'$. Therefore, the platform's expected per-unit time profit is equal to

$$\lambda_n(p_n - w_n)d_n + \lambda_p(mp_p d_p - w_p d_p') = \lambda_n p_n d_n + m\lambda_p p_p d_p - (w_n \lambda_n d_n + w_p \lambda_p d_p'). \tag{6}$$

By substituting (4) and (5) into (6), the expected per-unit time profit of the platform as a function of (s_n, s_p, k) is given by

$$\Pi_p(s_n, s_p, k) = \bar{\lambda}\left[((1 - s_n - s_p)(v_n - \Delta) + (1 - s_n)\Delta)s_n + (1 - s_n - s_p)s_p(v_n - \Delta)\right] - kG^{-1}\left(\frac{k}{K}\right). \tag{7}$$

Throughout the paper, we assume that $G(\cdot)$ satisfies the log-concave property[2]. It then can be shown that $C(y) := yG^{-1}(y)$ is convexly increasing in $0 \le y \le 1$.

To better hedge against demand uncertainty and achieve a satisfactory service experience such that customers do not wait too long after submitting a request for ride service, we require that the average driver utilization (i.e., the ratio between the number of drivers in service and the number of drivers that opt to work) cannot exceed a pre-specified threshold $\rho_{\max}$. Let T_n and T_p be the average service time (i.e., average trip length) of a normal ride and a carpool ride, respectively. Since a carpool ride requires additional pick-ups and drop-offs and may necessitate detours, we have $T_n < T_p$. Moreover, we assume that $T_p \le mT_n$ since a carpool ride is usually shared among passengers heading in similar directions and hence its trip duration should not exceed the summation of the

[2] Note that many common probability distributions are log-concave, such as normal distribution, exponential distribution, logistic distribution, chi distribution, and uniform distribution over any convex set.

service times when each one of the m passengers takes a normal ride separately. By Little's law, the average number of drivers in service should be $\lambda_p T_p + \lambda_n T_n = \bar{\lambda}(\frac{1}{m} s_p T_p + s_n T_n)$. Therefore, the service requirement with respect to the average utilization is given by $\frac{\bar{\lambda}(\frac{1}{m} s_p T_p + s_n T_n)}{k} \leq \rho_{\max}$, and hence the platform's optimization problem reads

$$
\begin{aligned}
\Pi_p^* := \max_{s_n, s_p, k} \quad & \Pi_p(s_n, s_p, k) \\
\text{s.t.} \quad & \frac{\bar{\lambda}(\frac{1}{m} s_p T_p + s_n T_n)}{k} \leq \rho_{\max}, \\
& s_n + s_p \leq 1, s_p \geq 0, s_n \geq 0, \\
& 0 \leq k \leq K.
\end{aligned} \tag{8}
$$

The optimal price rates (p_p^*, p_n^*) can be obtained from the optimal solutions (s_n^*, s_p^*, k^*) to the above problem (8) through the identity (4). The optimal per-unit-time total wage offered by the platform w^* can be derived by solving the equilibrium equation $w^* = k^* G^{-1}\left(\frac{k^*}{K}\right)$, and any choice of the wage rates (w_n^*, w_p^*) combination such that $w^* = \bar{\lambda}\left(w_n^* s_n d_n + \frac{w_p^* s_p d_p'}{m}\right)$ is optimal for the platform. For reference, Table 1 below summarizes the relevant notations used in the model.

Table 1 Summary of Notation

$v_n:$	value of a normal service ride
$v_p:$	value of a carpool service ride
$\Delta:$	value difference between the two service modes ($\Delta = v_n - v_p$)
$\theta:$	riders' type, $\theta \sim U[0,1]$
$d_n:$	average service units of a normal ride
$d_p:$	average service units of each rider in a carpool ride
$d_p':$	average service units that a driver provides in a carpool ride
$p_n:$	price rate per service unit for normal service
$p_p:$	price rate per service unit for carpool service
$w_n:$	wage rate per normal service unit for drivers
$w_p:$	wage rate per carpool service unit for drivers
$\bar{\lambda}:$	maximum rider arrival rate
$s_n:$	market share of normal service
$s_p:$	market share of carpool service
$m:$	average number of riders per ride for carpool service
$K:$	number of registered drivers on the platform
$k:$	number of active drivers
$r:$	reservation wage of drivers in the outside option
$G(\cdot):$	CDF of r which is assumed to satisfy the log-concave condition
$\rho_{\max}:$	maximum driver utilization on the platform
$\gamma:$	pooling efficiency of carpool services ($\gamma = m/T_p$)

We conclude this subsection with some discussions on our model assumptions and limitations. First, we have assumed that the average number of passengers in a carpool ride m is exogenous. Admittedly, if the rider request rate for carpool services is high relative to the number of drivers (i.e., the demand-supply ratio is high), it is easier for the platform to match carpool service requests heading to similar directions, which would result in a larger number of riders per shared ride (i.e.,

m) and a longer trip time for carpool services (i.e., T_p). In view of this, our model will be most applicable to the setting where m and T_p are not that sensitive to the demand-supply ratio. In addition, as shown by Equations (7) and (8), and Proposition 3 below, the parameters m and T_p will impact the results only through their ratio $\gamma := m/T_p$. This ratio γ can be considered as the pooling efficiency of the platform, where a higher value of γ suggests that the platform is able to pool more riders together in a single trip (i.e., m is large) without increasing the total trip duration too much (i.e., T_p is not too long). As long as the pooling efficiency is not very sensitive to the demand-supply ratio, all the results and insights in our paper will continue to hold. To facilitate analytical characterizations of the optimal service provision and pricing decisions that lead to clean results, we shall focus on the case where m is exogenously given instead of endogenously determined by the realized demand-supply ratio. Second, our model captures riders' waiting time experience via the average driver utilization constraint, but does not explicitly include waiting time into the riders' utility function. Prior studies in the ride-sharing literature have adopted different modeling approaches regarding the riders' waiting time. Some papers focus more on the operational aspect of ride sharing platforms and directly incorporate waiting time into the utility function of the riders (e.g., Bai et al. 2019, Taylor 2018), while others focus more on the economic and social dimensions of this market and abstract away the operational details such as the (endogenous) waiting time of riders in their utility (e.g., Siddiq and Taylor 2021, Yu et al. 2020). In the ride-sharing literature, the assumption that passengers do not wait in a queue and passenger requests are lost when no cars are available is also common (e.g., Aféche et al. 2018, Banerjee et al. 2015, and Bimpikis et al. 2019b). Our paper characterizes the implications of offering carpool services on the pricing strategy and rider/driver welfare of a ride-sharing platform, which cares more about the economic and social (instead of the operational) aspects of this market. In view of this, and together with model tractability considerations, we choose to not explicitly model the waiting time in riders' utility function, and we shall defer the issue of waiting time to future research.

2.4. Benchmark Model without Carpools

In this subsection, we consider a benchmark model without carpools, i.e., $s_p \equiv 0$. As we shall show later, the comparison between our focal model and this benchmark would lead to interesting insights on the implications of providing carpool services.

In the benchmark, the platform only provides normal services with value v_n, average service units per trip d_n, and average service time per trip T_n. The price rate charged to the riders and wage rate paid to the drivers are respectively $\tilde{p}_n$ and $\tilde{w}_n$. In this case, a type-θ customer would request a normal ride if $\theta \geq \tilde{\theta}_n$, where the threshold $\tilde{\theta}_n$ satisfies $\tilde{\theta}_n v_n - \tilde{p}_n d_n = 0$. Let $\tilde{s}_n$ be the proportion of customers who decide to request a normal service, and the effective arrival rate is $\tilde{\lambda}_n = \bar{\lambda}\tilde{s}_n$. It then follows that $\tilde{\theta}_n = 1 - \tilde{s}_n$ and $\tilde{p}_n = (1 - \tilde{s}_n)v_n/d_n$. With $\tilde{k}_n$ active drivers in equilibrium, the per-unit time wage $\tilde{w}_n$ satisfies the following equation:

$$\tilde{w}_n = \frac{\tilde{k}_n}{\bar{\lambda}\tilde{s}_n d_n} G^{-1}\left(\frac{\tilde{k}_n}{K}\right).$$

The service level constraint requires that the average driver utilization shall not exceed $\rho_{\max}$, i.e., $\bar{\lambda}\tilde{s}_n T_n/\tilde{k}_n \le \rho_{\max}$. The platform's expected per-unit time profit is given by

$$\tilde{\lambda}_n(\tilde{p}_n - \tilde{w}_n)d_n = \bar{\lambda}\tilde{s}_n(1-\tilde{s}_n)v_n - \tilde{\lambda}_n\tilde{w}_n d_n = \bar{\lambda}\tilde{s}_n(1-\tilde{s}_n)v_n - \tilde{k}_n G^{-1}\left(\frac{\tilde{k}_n}{K}\right).$$

Therefore, the platform's optimization problem when only offering normal services is given by

$$\begin{aligned} \tilde{\Pi}_n^* := \max_{\tilde{s}_n, \tilde{k}_n} \quad & \bar{\lambda}\tilde{s}_n(1-\tilde{s}_n)v_n - \tilde{k}_n G^{-1}\left(\frac{\tilde{k}_n}{K}\right) \\ \text{s.t.} \quad & \frac{\bar{\lambda}\tilde{s}_n T_n}{\tilde{k}_n} \le \rho_{\max}, \\ & 0 \le \tilde{s}_n \le 1, \\ & 0 \le \tilde{k}_n \le K. \end{aligned} \tag{9}$$

Let $(\tilde{s}_n^*, \tilde{k}_n^*)$ be the optimal solutions to the above problem. Then the optimal price rate and wage rate can be computed as

$$\tilde{p}_n^* = \frac{(1-\tilde{s}_n^*)v_n}{d_n} \quad \text{and} \quad \tilde{w}_n^* = \frac{\tilde{k}_n^*}{\bar{\lambda}\tilde{s}_n^* d_n} G^{-1}\left(\frac{\tilde{k}_n^*}{K}\right).$$

We also define $\tilde{w}^* := \tilde{w}_n^*\bar{\lambda}\tilde{s}_n^* d_n = \tilde{k}_n^* G^{-1}\left(\frac{\tilde{k}_n^*}{K}\right)$ as the total per-unit-time wage distributed by the platform to drivers without the carpool service option. The following proposition characterizes the impact of model primitives on the market outcome when the platform only offers normal services.

Proposition 1. *(1) If $\bar{\lambda}$ increases, then (a) $\tilde{s}_n^*$ decreases, (b) $\bar{\lambda}\tilde{s}_n^*$ increases, (c) $\tilde{p}_n^*$ increases, (d) $\tilde{k}_n^*$ increases, (e) $\tilde{w}_n^*$ increases; and (f) the optimal profit of the platform $\tilde{\Pi}_n^*$ increases;*
(2) If K increases, then (a) $\tilde{s}_n^*$ increases, (b) $\tilde{p}_n^*$ decreases, (c) $\tilde{k}_n^*$ increases; (d) $\tilde{k}_n^*/K$ decreases; (e) $\tilde{w}_n^*$ decreases; and (f) $\tilde{\Pi}_n^*$ increases.

Proposition 1 shows how the demand (i.e., the rider arrival rate $\bar{\lambda}$) and the supply (i.e., the driver capacity K) would affect the optimal strategy and the optimal profit of the platform when it only offers normal services. In view of Proposition 1, when the rider request arrival rate $\bar{\lambda}$ increases, the platform should correspondingly charge a higher price rate $\tilde{p}_n^*$, and also offer a higher wage rate $\tilde{w}_n^*$ to attract enough drivers to offer rides. The influence of driver capacity is opposite to that of rider arrival rate in the sense that, if there are more registered drivers on the platform, the optimal price rate of the normal service and the wage rate for drivers will both decrease. As for the profit, the platform could earn a higher profit when either the rider arrival rate or the driver capacity is higher.

3. Market Coverage and Pricing

In this section, we analyze the platform's optimization problem (8) to characterize its optimal price and wage decisions. On top of this, we further investigate the operational implications that the carpool services may lead to. Recall that the platform's objective function is given by

$$\Pi_p(s_n,s_p,k)=\bar{\lambda}\left[((1-s_n-s_p)(v_n-\Delta)+(1-s_n)\Delta)s_n+(1-s_n-s_p)s_p(v_n-\Delta)\right]-kG^{-1}\left(\frac{k}{K}\right).$$

Let (s_n^*,s_p^*,k^*) be the optimal solution to problem (8). It is worth noticing that the optimization problem (8) reduces to the benchmark model (9) by letting $s_p\equiv 0$, which immediately implies that the provision of carpool services can help the platform achieve a higher profit.

LEMMA 1. $\Pi_p^*\geq\tilde{\Pi}_n^*$.

Now we focus on the analysis of (8). It's clear that $\Pi_p(s_n,s_p,k)$ is decreasing in k, so at optimality we must have $\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)=\rho_{\max}k^*$. It then follows that the platform's optimization problem reduces to

$$\begin{aligned}(s_n^*,s_p^*)=\arg\max\quad & f_p(s_n,s_p)\\ \text{s.t.}\quad & s_n+s_p\leq 1,\\ & \frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}}\leq K,\\ & s_n,s_p\geq 0,\end{aligned}$$

where

$$f_p(s_n,s_p):=\bar{\lambda}\left[((1-s_n-s_p)(v_n-\Delta)+(1-s_n)\Delta)s_n+(1-s_n-s_p)s_p(v_n-\Delta)\right]-KC\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right).$$

We begin our analysis by establishing the joint concavity of $f_p(\cdot)$ in (s_n,s_p).

LEMMA 2. *$f_p(s_n,s_p)$ is jointly concave in (s_n,s_p).*

Our next result shows how the value difference Δ would affect the structure of the optimal policy for the platform. We show that the proportion of riders who request a normal service s_n^* is increasing in the value difference, whereas the fraction of riders who request a carpool service decreases as the value difference becomes more significant.

PROPOSITION 2. *There exist two thresholds $\underline{\Delta}$ and $\bar{\Delta}$ ($0<\underline{\Delta}<\bar{\Delta}<v_n$), such that $s_n^*\begin{cases}=0, & \text{if } \Delta\in[0,\underline{\Delta}],\\ >0, & \text{if } \Delta\in(\underline{\Delta},v_n];\end{cases}$ and $s_p^*\begin{cases}>0, & \text{if } \Delta\in[0,\bar{\Delta}],\\ =0, & \text{if } \Delta\in(\bar{\Delta},v_n].\end{cases}$ In particular, $\bar{\Delta}=v_n(1-\frac{T_p}{mT_n})$. Moreover, Π_p^* is decreasing in Δ, s_n^* is increasing in Δ, and s_p^* is decreasing in Δ.*

In view of Proposition 2, the optimal service provision strategy of the platform bears an interesting threshold structure. Specifically, if the value difference Δ is small ($\Delta<\underline{\Delta}$), the platform should offer the carpool service alone ($s_p^* > 0$ and $s_n^* = 0$), where the threshold $\underline{\Delta}$ depends on the problem parameters (e.g., $\bar{\lambda}$). If the value difference is moderate ($\Delta \in [\underline{\Delta}, \bar{\Delta}]$), it is optimal for the platform to provide both normal and carpool services ($s_p^* > 0$ and $s_n^* > 0$). If the value difference is large ($\Delta > \bar{\Delta}$), only the normal service should be provided. Furthermore, as the disutility of riders to take a carpool ride increases, the platform should adjust the prices so that the number of riders for normal services increases whereas the number of riders for carpool services decreases. Note that, it is optimal for the platform to offer carpool services if and only if $\frac{\Delta}{v_n} < 1 - \frac{T_p}{mT_n}$, or equivalently, $\frac{v_p}{v_n} > \frac{T_p}{mT_n}$. Recall that the ratio $\gamma := m/T_p$ captures the *pooling efficiency* of the platform, where a higher pooling efficiency means that the platform is able to pool more riders together in a single trip (i.e., m is large) without increasing the total trip duration too much (i.e., T_p is not too long). The condition $\frac{v_p}{v_n} > \frac{1}{\gamma T_n}$ highlights a clear insight that the platform should offer carpool services when the carpool service value v_p (relative to that of the normal service v_n) or the pooling efficiency γ is high. Proposition 2 characterizes the capacity pooling efficiency as a driving force that determines the optimal provision of service modes in the context of ride-sharing platforms.

THEOREM 1. *The total market coverage $s^* := s_n^* + s_p^*$ is decreasing in Δ. Therefore, the provision of carpool services expands market coverage of the platform.*

Theorem 1 proves that the total market coverage of the platform s^* will shrink if the value difference between the two services is larger. According to Proposition 2, we remark that our benchmark model, (9), is a special case of the focal model with $\Delta \geq \bar{\Delta}$. Therefore, Theorem 1 further demonstrates that providing carpool services enables the platform to better leverage its driver capacity and achieve a larger total market coverage, since $s^*(\Delta) \geq s^*(\bar{\Delta}) = s_n^*(\bar{\Delta}) = \tilde{s}_n^*$ for all $\Delta \leq \bar{\Delta}$.

In addition to the market coverage, we next examine the implications of carpool services on the platform's optimal pricing decisions. In particular, we compare the optimal price rates (p_n^*, p_p^*) in the focal model where both service modes are offered, with the optimal price rate $\tilde{p}_n^*$ in the model where the platform does not offer carpool services. Intuitively, it is expected that the optimal price rate for carpool services p_p^* should be lower than the optimal price rate for normal services in the benchmark model $\tilde{p}_n^*$, due to the value difference between the two services ($v_p < v_n$). More interestingly, we show in the following theorem that not only do we have $p_p^* \leq \tilde{p}_n^*$, but the optimal price rate charged for the normal services p_n^* in the presence of carpool services is also lower than its counterpart in the benchmark model $\tilde{p}_n^*$.

THEOREM 2. *(a) For all $\Delta \in [0, \bar{\Delta}]$, $p_p^* \leq \tilde{p}_n^*$. (b) For all $\Delta \in [0, \bar{\Delta}]$, $p_n^* \leq \tilde{p}_n^*$.*

As shown in Theorem 2(a), customers experience a lower quality ride from the carpool service in exchange for a discounted fare p_p^*. Interestingly, if the carpool service is offered, customers who take a normal ride and enjoy the same level of service quality also pay a lower fare p_n^* than the case where only the normal service is offered. This result is somewhat intriguing as one may intuit that, by offering an additional service mode, the platform should be able to discriminate between customers with different preferences over quality and hence can charge a higher price for the high-quality normal service (i.e., the *price discrimination* effect). However, Theorem 2(b) suggests that the opposite is true — the provision of carpool services not only enables the platform to expand market coverage, but also allows riders to enjoy less expensive (normal and pooled) rides. To understand why this is the case, we note that the introduction of the carpool service is not just a "lower quality version" of the normal service, as it also "expands" the capacity of the drivers and "shrinks" the customer demand by combining multiple trips into one. More specifically, the carpool service cannibalizes the demand for the normal service and, which prompts the platform to charge a lower price for the normal service (i.e., the *demand cannibalization* effect). Furthermore, there also exists a *cost reduction* effect that arises from the drivers' self-scheduling behavior. In particular, since the supply expansion due to the carpool outweighs its service value disadvantage (i.e., $\frac{v_p}{v_n} > \frac{T_p}{mT_n}$, the condition under which the carpool service should be offered), the introduction of the carpool service increases the capacity efficiency of the self-scheduling drivers and, as a result, reduces the labor/capacity cost of the platform. Such a cost reduction effect benefits the normal service as well (because the entire platform uses a shared pool of drivers), and therefore, prompts the platform to decrease the price of the normal service. The overall effect of cannibalization and cost reduction together outweighs the price discrimination effect, resulting in a lower price for the normal service when the platform finds it attractive to introduce the carpool option. In other words, the carpool service enables riders to enjoy more affordable rides without compromising on service quality. We also remark that Theorem 2 is consistent with our subsequent analysis in Section 4 that carpool services enhance the rider surplus.

Recall that $\gamma = m/T_p$ can be viewed as the pooling efficiency of the platform. Our next result investigates how the pooling efficiency affects the equilibrium outcome.

PROPOSITION 3. *Assume that $\Delta \in [0, \bar{\Delta}]$. If γ increases, then (a) Π_p^* increases, (b) s_p^* increases, (c) $s^* = s_n^* + s_p^*$ increases, (d) p_n^* decreases, and (e) p_p^* decreases.*

Proposition 3 characterizes the impact of pooling efficiency on the optimal market coverage and pricing policy of the platform. More specifically, as the pooling efficiency increases, the platform would achieve a more expanded market coverage and enjoy a higher profit. Intuitively, a more efficient carpool system that routes drivers to match with riders prompts the platform to increase the usage of carpool services, which in turn also expands the total market coverage. To match demand with supply, the platform decreases the prices p_n^* and p_p^* under a higher pooling efficiency, and therefore customers can enjoy a ride at more affordable prices for both service modes.

Next, we examine the impact of the customer demand rate $\bar{\lambda}$ on the market outcome. We show in the following result that the optimal price rates p_n^* and p_p^* for the normal and carpool services are both monotonically increasing in $\bar{\lambda}$. In other words, the optimal pricing strategy responds to demand spikes with higher prices for both the normal and carpool services.

PROPOSITION 4. *Assume that $\Delta < \bar{\Delta}$ so the carpool service is offered. Then (a) s_n^* is decreasing in $\bar{\lambda}$. (b) There exists some threshold λ_0 such that $s_n^* = 0$ for $\bar{\lambda} \geq \lambda_0$. (c) s_p^* is increasing (resp. decreasing) in $\bar{\lambda}$ for $\bar{\lambda} < \lambda_0$ (resp. $\bar{\lambda} > \lambda_0$). (d) p_n^* and p_p^* are increasing in $\bar{\lambda}$, and p_n^* increases faster than p_p^* does.*

An interesting implication from Proposition 4 is that the proportion of customers who choose the carpool service s_p^* is increasing in the demand arrival rate $\bar{\lambda}$ when both services are provided (i.e., when $s_n^* > 0$), whereas the number of customers who choose the normal service $\lambda_n^* = \bar{\lambda} s_n^*$ decreases to 0 as the rider arrival rate $\bar{\lambda}$ increases. As the rider arrival increases, the platform is gradually incentivizing the riders to shift from normal services to carpool services (s_n^* decreases whereas s_p^* increases in $\bar{\lambda}$ in the range $s_n^* > 0$). Proposition 4 also suggests that it is beneficial to the platform to charge higher prices for both the normal and carpool services when the demand arrival rate increases (p_n^* and p_p^* are both increasing in $\bar{\lambda}$). Furthermore, in the face of a demand spike, the price increase for the carpool service is not as sharp as that for the normal service. Therefore, as shown in Proposition 4(a,c), riders will gradually switch from normal services to carpool services when the demand traffic increases. This is consistent with DidiChuxing's strategy to deal with demand spikes in peak hours. The platform encourages the riders to take the carpool service by putting those requesting carpool services at the front of their rider waiting queue.

3.1. Numerical Experiments

To further illustrate the implications of the carpool services on the platform's pricing decisions and profits, we complement our theoretical analysis with computational studies. We first provide the setup of our numerical experiments. In view of Lemma 1, the provision of carpool service option enables the platform to achieve a higher profit. We next numerically evaluate this profit gain, and investigate when the benefit of providing carpool services will be most significant.

We now describe the setup of our numerical experiments. For simplicity, we use travel distance as a proxy for the service units (d_n, d_p) and assume that the average travel distance is the same at different hours of a day. In our numerical experiments, we fix the value of a normal service ride as $v_n = 2$, the average service time of normal rides and carpool rides as $T_n = 1$ and $T_p = 1.5$, and the average service units (travel distance) of normal rides and carpool rides as $d_n = 1$ and $d_p = 1.2$. The distribution of the drivers' reservation wage rate for outside option is uniformly distributed on $[0, 1]$. The average number of passengers per ride for the carpool services is $m = 2$. Notice that with the above parameters, we have $\bar{\Delta} = v_n(1 - T_p/(mT_n)) = 0.5$. Therefore, we vary the value difference Δ between the normal service and the carpool service in the range $[0, 0.5]$. When $\Delta = 0.5$, our model reduces to the benchmark model that only offers the normal service, i.e., $s_p^* = 0$ and $s_n^* = \tilde{s}_n^*$. For the demand and supply parameters, i.e., the maximum rider arrival rates $\bar{\lambda}$ and the total number of registered drivers K, we calibrate the model parameters based on real Didi's ride data from Bai et al. (2019), which records rides that took place in Hangzhou, China during the time periods between September 7-13 and November 1-30 in 2015.[3] According to Bai et al. (2019), Didi had about 7,800 registered Express/Private drivers in Hangzhou. In our numerical studies, we use the range between 3,000 and 10,000 for the number of registered drivers to cover a large parameter space. For the rider arrival rate, the data from Bai et al. (2019) suggests that the demand rate is relatively stable across the day except during two peak periods in the morning and afternoon rush hours. Similar to Bai et al. (2019), we set the average customer demand rate during peak time periods 7:00-10:00 and 17:00-20:00 as $\bar{\lambda} = 4,000$, and set $\bar{\lambda} = 2,000$ during off-peak periods 10:00-17:00 and 20:00-23:00. The arrival rates from midnight to early morning were omitted Bai et al. (2019) due to incomplete data in the database. In our numerical studies, we set the average arrival rate as $\bar{\lambda} = 500$ between 23:00 and 7:00 for completeness. Table 2a summarizes the above customer arrival pattern.

Time period	λ
7:00-10:00	4,000
10:00-17:00	2,000
17:00-20:00	4,000
20:00-23:00	2,000
23:00-7:00	500

(a) Arrival Pattern I

Time period	λ
7:00-10:00	6,000
10:00-17:00	1,000
17:00-20:00	6,000
20:00-23:00	1,000
23:00-7:00	250

(b) Arrival Pattern II

Table 2 Distribution of Rider Arrival Rate $\bar{\lambda}$

We next examine when carpool services will be the most beneficial to the platform. For various values of the value difference Δ and the total number of registered drivers K, we evaluate the (relative) daily platform profit improvement of adopting carpool services compared with only providing the normal services, i.e., $(\Pi_p^* - \tilde{\Pi}_n^*)/\tilde{\Pi}_n^* \times 100\%$, where Π_p^* is the total daily profit of the platform to offer both normal and carpool services whereas $\tilde{\Pi}_n^*$ is the total daily profit of the platform if only the normal service is provided.

Table 3 summarizes the relative profit improvement of the platform when the carpool service is offered compared with the benchmark model with normal service only where $\rho_{\max}$ is set to 0.8

[3] We refer interested readers to Bai et al. (2019) for the detailed description of the data set.

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta=0$	20.68%	17.02%	14.48%	12.60%	11.16%	10.01%	9.08%	8.31%
$\Delta=0.1$	13.21%	10.12%	7.96%	6.41%	5.27%	4.39%	3.70%	3.13%
$\Delta=0.2$	6.61%	4.50%	3.21%	2.41%	1.89%	1.52%	1.24%	1.04%
$\Delta=0.3$	2.30%	1.50%	1.07%	0.80%	0.62%	0.50%	0.41%	0.34%
$\Delta=0.4$	0.47%	0.31%	0.22%	0.16%	0.13%	0.10%	0.08%	0.07%

Table 3 **Daily Profit Improvement of Adopting Carpool Services (%)** ($v_n=2$, $\rho_{\max}=0.8$, $T_p=1.5$, $T_n=1$, $m=2$, $d_p=1.2$, $d_n=1$, $d'_p=2$, $G(\cdot)\sim U[0,1]$, $\bar{\lambda}$ **is distributed according to Table 2a)**

and $\bar{\lambda}$ follows the distribution in Table 2a throughout the day. We observe from Table 3 that the profit increase becomes more significant and hence the provision of carpool services becomes more valuable when the number of total registered drivers decreases. Intuitively, carpool services can help the platform to enlarge the capacity of the registered drivers, and such benefit is most significant when the driver capacity is limited (i.e., when K is small). Moreover, the numerical results in Table 3 suggest that the profit improvement is more prominent when the value difference Δ between the two service modes becomes smaller. This observation echoes our analytical results in Proposition 2, and suggests that it is to the benefit of the platform to provide high quality carpool services (e.g., by improving route design to reduce detours during multiple pick-ups and drop-offs).

Notice that the mean arrival rate when $\bar{\lambda}$ follows the distribution in Table 2a is 2,000 rides per hour, which is even less than the smallest K that we have tested, and the improvement ranges between 10% and 25% for small to medium values of value difference. When the supply K is far more than the mean demand rate, the platform is still able to achieve a considerable profit gain. These observations suggest that offering the carpool service can provide the platform an operations lever to hedge against the demand variability. Table 4, which summarizes the profit gain when the demand arrival rate $\bar{\lambda}$ has a higher variability throughout the day, further confirms this intuition. More specifically, Table 4 evaluates the relative profit improvement when $\bar{\lambda}$ is distributed according to Table 2b, which has the same mean arrival rate 2,000 rides per hour but with a higher variance than that in Table 2a. Comparing Table 4 with Table 3, we observe that the benefit of offering the carpool service option is more significant when the rider arrival pattern is more volatile.

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta=0$	24.94%	21.62%	19.09%	17.09%	15.46%	14.12%	12.99%	12.02%
$\Delta=0.1$	16.88%	14.21%	12.18%	10.57%	9.25%	8.16%	7.23%	6.44%
$\Delta=0.2$	10.25%	8.16%	6.50%	5.14%	4.12%	3.39%	2.84%	2.41%
$\Delta=0.3$	4.34%	3.02%	2.23%	1.73%	1.38%	1.13%	0.94%	0.80%
$\Delta=0.4$	0.90%	0.62%	0.46%	0.35%	0.28%	0.23%	0.19%	0.16%

Table 4 **Daily Profit Improvement of Adopting Carpool Services (%)** ($v_n=2$, $\rho_{\max}=0.8$, $T_p=1.5$, $T_n=1$, $m=2$, $d_p=1.2$, $d_n=1$, $d'_p=2$, $G(\cdot)\sim U[0,1]$, $\bar{\lambda}$ **is distributed according to Table 2b)**

We have conducted further numerical studies to examine how the maximum utilization $\rho_{\max}$ would affect the profit gain achieved by the provision of carpool services. Table 5 and Table 6 summarize the profit gain of offering the carpool service when $\rho_{\max}$ is equal to 0.7 and 0.9, respectively. As expected, a lower $\rho_{\max}$ decreases the effective supply capacity of each driver and hence the provision of the carpool service becomes more valuable. Under equilibrium, the realized driver utilization is equal to the maximum utilization ρ_{max}, i.e., $\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n) = \rho_{max}k^*$. As is well known in the literature, the expected waiting time of the riders grows exponentially as the server/driver utilization increases to 1 (see, e.g., Bai et al. 2019, Taylor 2018). Therefore, the maximum driver utilization has a material impact on the waiting time of the riders, which is crucial for the rider experience of the platform. In this paper, we assume ρ_{max} is exogenous and it would be an interesting future research direction to endogenize and optimize the maximum driver utilization by directly incorporating the waiting time into riders' utility function.

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	24.41%	20.14%	17.55%	15.41%	13.74%	12.40%	11.30%	10.38%
$\Delta = 0.1$	16.34%	12.98%	10.57%	8.76%	7.34%	6.25%	5.38%	4.67%
$\Delta = 0.2$	8.91%	6.45%	4.80%	3.65%	2.88%	2.34%	1.93%	1.63%
$\Delta = 0.3$	3.35%	2.23%	1.61%	1.22%	0.95%	0.77%	0.64%	0.53%
$\Delta = 0.4$	0.69%	0.46%	0.33%	0.25%	0.19%	0.16%	0.13%	0.11%

Table 5 **Daily Profit Improvement of Adopting Carpool Services (%)** ($v_n = 2$, $\rho_{\max} = 0.7$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d'_p = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ **is distributed according to Table 2a)**

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	17.65%	14.34%	12.09%	10.45%	9.20%	8.22%	7.43%	6.78%
$\Delta = 0.1$	10.66%	7.85%	6.00%	4.72%	3.79%	3.07%	2.53%	2.12%
$\Delta = 0.2$	4.85%	3.15%	2.22%	1.65%	1.28%	1.02%	0.83%	0.69%
$\Delta = 0.3$	1.63%	1.04%	0.73%	0.54%	0.42%	0.33%	0.27%	0.22%
$\Delta = 0.4$	0.33%	0.21%	0.15%	0.11%	0.08%	0.07%	0.05%	0.05%

Table 6 **Daily Profit Improvement of Adopting Carpool Services (%)** ($v_n = 2$, $\rho_{\max} = 0.9$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d'_p = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ **is distributed according to Table 2a)**

4. Rider Surplus, Driver Surplus, and Social Welfare

Besides the on-demand platform's own profit and optimal operational decisions, we are also interested in investigating the social implications of offering the carpool services. In this section, we examine how offering carpool services would affect the welfare of various parties in the system.

In what follows, we first investigate the impact of carpool services on riders' welfare. Given the platform's optimal price rates p_n^* and p_p^*, the total rider surplus is given by

$$\begin{aligned}
RS_p^* =& \bar{\lambda}\left(\mathbb{E}[\theta v_n - p_n^* d_n | \theta \in [1-s_n^*, 1]] + \mathbb{E}[\theta(v_n - \Delta) - p_p^* d_p | \theta \in [1-s_p^*-s_n^*, 1-s_n^*]]\right) \\
=& \bar{\lambda}\left(v_n \mathbb{E}\left[\theta - (1-s_n^*) + \frac{v_n - \Delta}{v_n} s_p^* | \theta \in [1-s_n^*, 1]\right] + (v_n - \Delta)\mathbb{E}[\theta - (1-s_n^*-s_p^*) | \theta \in [1-s_p^*-s_n^*, 1-s_n^*]]\right) \\
=& \bar{\lambda}\left(\frac{1}{2} v_n (s_n^*)^2 + \frac{1}{2}(v_n - \Delta)(s_p^*)^2\right) + \bar{\lambda}(v_n - \Delta) s_n^* s_p^*, \qquad (10)
\end{aligned}$$

where the second equality follows from substituting (4) into the first equation. In view of (10), we use the notation $RS_p^*(\Delta)$ to capture the dependence of the rider surplus on the value difference Δ when both service modes are offered. We use $\tilde{RS}_n^* = RS_p^*(\bar{\Delta})$ to denote the rider surplus in the benchmark model where only the normal service is available. The following result characterizes the comparison between $RS_p^*(\Delta)$ and $\tilde{RS}_n^*$.

PROPOSITION 5. *There exists a threshold $0 \le \underline{\Delta}_r \le \bar{\Delta}$ such that $RS_p^*(\Delta) > \tilde{RS}_n^*$ for $\Delta < \underline{\Delta}_r$. Moreover, if $G(r) = r$, $RS_p^*(\Delta) > \tilde{RS}_n^*$ for all $\Delta < \bar{\Delta}$.*

In view of Proposition 5, the provision of carpool services benefits the riders when the value difference Δ is not too large. If, in addition, the drivers' reservation wage is uniformly distributed (i.e., $G(r) = r$), the total rider surplus is always higher when the platform provides the carpool services. Figure 1 illustrates the relationship between the total rider surplus RS_p^* and the value difference Δ for various values of rider arrival rate $\bar{\lambda}$, where the drivers' reservation rate for the outside option is assumed to be uniformly distributed on $[0,1]$. From Figure 1, we observe that the rider surplus is decreasing in Δ for the entire range $\Delta \in [0, \bar{\Delta}]$. Notice that by Proposition 2, in equilibrium the platform should offer carpool services only when $\Delta < \bar{\Delta}$, and therefore only normal services will be available in the regime where $\Delta \ge \bar{\Delta}$.

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	42.44%	34.84%	29.56%	25.68%	22.70%	20.34%	18.43%	16.85%
$\Delta = 0.1$	31.51%	25.22%	20.76%	16.94%	14.23%	12.20%	10.62%	9.30%
$\Delta = 0.2$	19.53%	14.39%	10.11%	7.53%	5.84%	4.67%	3.82%	3.19%
$\Delta = 0.3$	7.68%	4.89%	3.40%	2.52%	1.94%	1.54%	1.26%	1.05%
$\Delta = 0.4$	1.58%	1.00%	0.70%	0.51%	0.40%	0.31%	0.26%	0.21%

Table 7 Change in Daily Rider Surplus when Adopting Carpool Services (%) ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d_p' = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ is distributed according to Table 2a)

The provision of carpool services has the following two contrasting effects on riders: (a) the (positive) market expansion effect and (b) the (negative) value downgrade effect. First, providing carpool services expands the market coverage of the platform (cf. Theorem 1), which allows more riders to obtain a ride. On the other hand, the carpool service has a lower quality than the normal

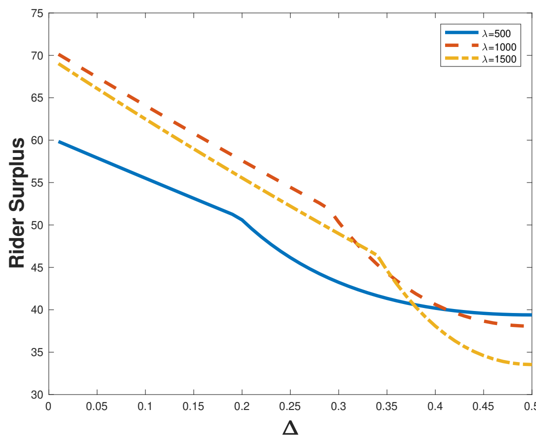

Figure 1 **Rider Surplus** ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d_p' = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\Delta} = v_n(1 - T_p/(mT_n)) = 0.5$, $K = 500$)

service. As a result, the average service quality a rider obtains from the platform is lower in the presence of carpool service. Our results suggest that when the service quality downgrade due to carpool is not too high, offering carpool services benefit the riders. To check the robustness of this insight, we have conducted extensive numerical experiments, and our numerical results in Table 7 suggest that the total rider surplus is monotonically decreasing in Δ and the provision of carpool services is beneficial to the riders.

Since the provision of carpool services expands the market coverage of the platform, one may conjecture that drivers will benefit from the carpool services as well. However, as we elaborate below, the offering of carpool services may actually hurt the drivers' welfare. Note that when both normal and carpool services are offered, the total driver surplus DS_p^* is given by

$$DS_p^* = K\mathbb{E}\left[\frac{w_n\lambda_n d_n + w_p\lambda_p d_p'}{k^*} - r\right]^+ = K\mathbb{E}\left[G^{-1}\left(\frac{k^*}{K}\right) - r\right]^+, \tag{11}$$

where $r \sim G(\cdot)$ is the reservation wage rate of the driver's outside option and $k^* = \frac{\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n)}{\rho_{\max}}$ is the optimal number of active drivers in equilibrium. Analogously, let

$$\tilde{DS}_n^* := K\mathbb{E}\left[G^{-1}\left(\frac{\tilde{k}_n^*}{K}\right) - r\right]^+ \tag{12}$$

denote the total driver surplus for the baseline model without the carpool service option. Figure 2 illustrates how the driver surplus DS_p^* changes with respect to Δ for various values of rider arrival rate $\bar{\lambda}$ when the drivers' reservation wage rate for the outside option is uniformly distributed on $[0,1]$. In view of Figure 2, we observe that the driver surplus is not necessarily monotone in Δ, and the provision of the carpool services may turn out to make the drivers worse off. In what follows, we formalize this observation from Figure 2 when the drivers' reservation wage rate for the outside

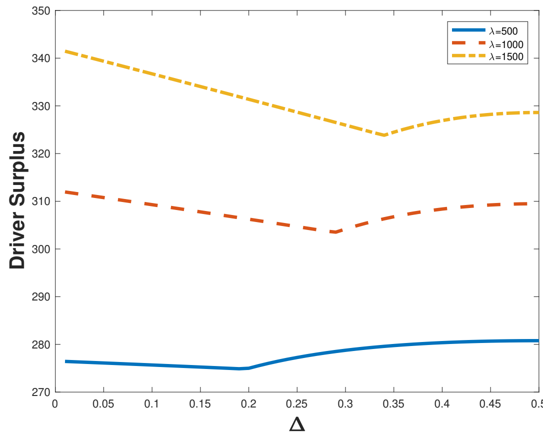

Figure 2 **Driver Surplus** ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d'_p = 2$, $G(\cdot) \sim U[0,1]$, $K = 500$)

option is uniformly distributed. We show in the next proposition that the provision of carpool services leads to lower driver surplus compared with the benchmark where the platform only offers the normal service.

PROPOSITION 6. *Assume that (i) r follows the uniform distribution on $[0,1]$, i.e., $G(r) = r$; and (ii) $\Delta \in (\underline{\Delta}, \bar{\Delta})$, i.e., the platform would offer both normal and carpool services. Then we have (a) $w^* < \tilde{w}^*$; (b) $k^* < \tilde{k}_n^*$; and (c) $DS_p^* < \tilde{DS}_n^*$.*

In light of Proposition 6, when the drivers' reservation wage rate is uniformly distributed and the value difference is in the regime where both normal and carpool services have a positive market share in equilibrium, the drivers would be worse off if the platform offers the carpool services. The provision of the carpool services has the following two opposing effects on drivers: (a) the (positive) market expansion effect and (b) the (negative) demand pooling effect. On one hand, as shown in Theorem 1 and Proposition 3, offering carpool services thickens the market by inducing more riders to hail a ride using the platform ($s_n^* + s_p^* \geq \tilde{s}_n^*$). Therefore, the introduction of the carpool service brings more demand to the drivers and may increase their earnings. On the other hand, carpool services enlarge the capacity of each driver by pooling multiple passengers into a single ride, which decreases the total need and earning potentials for the drivers. Proposition 6 suggests that, when the platform finds it attractive to offer both normal and carpool services, the demand pooling effect outweighs the market expansion effect and the overall impact is harmful to the drivers. To understand why this is the case, note that the carpool service should be offered if and only if the pooling efficiency is sufficiently strong (i.e., $\frac{T_p}{mT_n} < \frac{v_p}{v_n}$). As a result, introducing the carpool service reduces the capacity/labor cost of the entire platform. This cost reduction is achieved by a lower per-unit-time total wage distributed by the platform (i.e., $w^* < \tilde{w}^*$), which leads to a smaller number of active drivers on the platform in equilibrium (i.e., $k^* < \tilde{k}_n^*$). Notice that by (11) and (12), the drivers' surplus is monotonically increasing in the number of active drivers in equilibrium. Therefore, the reduction in the number of active drivers in the presence of carpool services also leads to a lower driver surplus. As shown by Theorem 2, such a labor cost reduction effect of carpool services also prompts the platform to charge a lower price for both normal and carpool services compared with the benchmark case where the carpool service is not available, which helps the riders achieve a larger rider surplus.

To check the robustness of the above insight, we have performed extensive numerical experiments. Table 8 reports how the total driver surplus changes with respect to the service value difference Δ and the driver capacity K, when the drivers' reservation wage rate is uniformly distributed on $[0,1]$ and the rider arrival rate $\bar{\lambda}$ is distributed according to Table 2a. Our numerical results in Table 8 suggest that, the total driver surplus is not necessarily monotone in Δ and the drivers may be worse off in the presence of carpool service. In particular, offering carpool services would make the drivers worse off for all the parameter combinations in our numerical experiments. This result has important practical implications for the operations of the platform. The provision of the carpool services is likely to benefit the platform and the riders, but at the cost of the drivers. Therefore, an important actionable insight from our study is that, the platform may need to carefully redistribute the additional profit from the provision of carpool services to the drivers (e.g., by distributing coupons or bonus rewards) so as to protect their welfare and retain a large enough supply base. We have seen similar ideas adopted in practice that aim to re-balance the interests of ride-sharing platforms and their drivers (see, e.g., Cohen and Zhang 2017).

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	-1.01%	-1.00%	-0.90%	-0.79%	-0.68%	-0.59%	-0.52%	-0.45%
$\Delta = 0.1$	-1.14%	-1.07%	-0.93%	-0.75%	-0.61%	-0.51%	-0.43%	-0.37%
$\Delta = 0.2$	-1.00%	-0.81%	-0.54%	-0.38%	-0.28%	-0.21%	-0.16%	-0.13%
$\Delta = 0.3$	-0.45%	-0.28%	-0.19%	-0.13%	-0.09%	-0.07%	-0.05%	-0.04%
$\Delta = 0.4$	-0.09%	-0.06%	-0.04%	-0.03%	-0.02%	-0.01%	-0.01%	-0.01%

Table 8 Change in Daily Driver Surplus when Adopting Carpool Services (%) ($v_n = 2$**,** $\rho_{\max} = 0.8$**,** $T_p = 1.5$**,** $T_n = 1$**,** $m = 2$**,** $d_p = 1.2$**,** $d_n = 1$**,** $d_p' = 2$**,** $G(\cdot) \sim U[0,1]$**,** $\bar{\lambda}$ **is distributed according to Table 2a)**

Besides the rider and driver surplus, we further examine the impact of the provision of carpool services on the social welfare. The social welfare equals the sum of the rider surplus RS_p^*, the platform's profit Π_p^*, and the driver surplus DS_p^*. It then follows that the social welfare can be computed as follows:

$$\begin{aligned} SW_p^* =& RS_p^* + \Pi_p^* + DS_p^* \\ =& \bar{\lambda}\left(\frac{1}{2}v_n(s_n^*)^2 + \frac{1}{2}(v_n-\Delta)(s_p^*)^2\right) + \bar{\lambda}(v_n-\Delta)s_n^*s_p^* + \bar{\lambda}(p_n^*d_ns_n^* + p_p^*d_ps_p^*) - k^*G^{-1}\left(\frac{k^*}{K}\right) \\ &+ K\mathbb{E}\left[G^{-1}\left(\frac{k^*}{K}\right) - r\right]^+. \end{aligned}$$

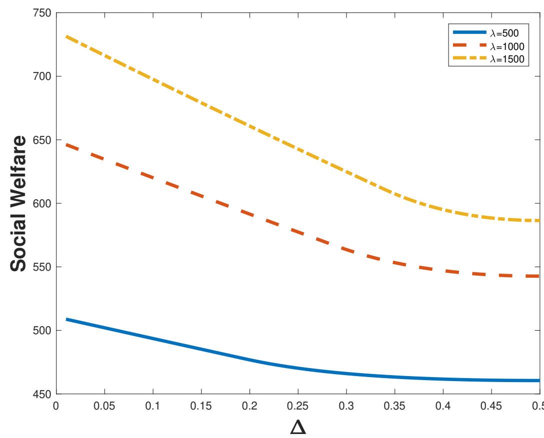

Figure 3 **Social Welfare ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $d_p = 1.2$, $d_n = 1$, $m = 2$, $G(\cdot) \sim U[0,1]$, $K = 500$)**

K	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000
$\Delta = 0$	7.70%	5.61%	4.29%	3.39%	2.76%	2.28%	1.92%	1.64%
$\Delta = 0.1$	4.92%	3.34%	2.36%	1.73%	1.30%	1.00%	0.78%	0.62%
$\Delta = 0.2$	2.46%	1.48%	0.95%	0.65%	0.47%	0.35%	0.26%	0.21%
$\Delta = 0.3$	0.86%	0.50%	0.32%	0.21%	0.15%	0.11%	0.09%	0.07%
$\Delta = 0.4$	0.18%	0.10%	0.06%	0.04%	0.03%	0.02%	0.02%	0.01%

Table 9 **Change in Daily Social Welfare when Adopting Carpool Services (%) ($v_n = 2$, $\rho_{\max} = 0.8$, $T_p = 1.5$, $T_n = 1$, $m = 2$, $d_p = 1.2$, $d_n = 1$, $d_p' = 2$, $G(\cdot) \sim U[0,1]$, $\bar{\lambda}$ is distributed according to Table 2a)**

Figure 3 illustrates how the social welfare SW_p^* changes with respect to Δ for various values of rider arrival rate $\bar{\lambda}$ when the drivers' reservation wage for the outside option is uniformly distributed. As shown in Figure 3, the social welfare is decreasing in Δ and it is to the benefit of the entire society for the platform to offer carpool services. We have further conducted extensive numerical experiments to check the robustness of our results. Table 9 suggests that the total social welfare increases as the value difference Δ becomes smaller, and the provision of carpool services would achieve a higher social welfare. Our analysis and results in this section may prove helpful in providing guidelines for policy makers. The entire society would benefit from the carpool service of ride-sharing platforms from the total social-welfare perspective, so the policy maker may find it beneficial to promote such services. However, not everyone equally benefits from the carpool service, so care must also be taken and appropriate compensation schemes may be needed for platform drivers as they may be worse off with the introduction of the carpool service.

5. Conclusion

Motivated by the increasing popularity of on-demand service platforms with self-scheduling and earning-sensitive service providers and price-sensitive customers, we develop an analytical framework to examine the operational, economical, and social implications of carpool services on ride-hailing platforms. We characterize the optimal price and wage strategy of the platform, and find that it is optimal for the platform to offer the carpool service if its value deterioration relative to the normal service is not too high, and/or when the pooling efficiency is not too low. Providing carpool services enables the platform to achieve a larger market coverage and allows the passengers to pay less for both the normal and carpool services. If the pooling efficiency improves, the platform can further enlarge its market coverage, and decrease the prices. In the presence of carpool services, surge pricing is still optimal, and, as the demand increases, the platform will gradually encourage customers to switch from normal services to carpool services. We show that the provision of carpool services benefits the riders if the value difference between the normal and carpool service modes is not too large. However, drivers may be worse off in the presence of carpool services. In particular, we find that the provision of carpool services will result in a lower driver surplus when the drivers' reservation wage is uniformly distributed.

There are several interesting directions to extend our research. First, we have assumed that the customer's valuation type is uniformly distributed between zero and one. It would be interesting to see whether the results and insights are robust under more general distributions. Second, we have focused on the equilibrium behavior of the system and didn't consider the spatial heterogeneity of riders and drivers for a ride-hailing platform. An interesting future direction is to incorporate the spatial dimension into the joint price and wage optimization problem under demand and supply uncertainty with carpool services. Finally, we have considered a monopolistic platform and ignored competition among platforms. It is not uncommon in practice that there may exist multiple platforms competing for both riders and drivers in the market. Another potential future research direction is to study platform competition in the presence of carpool services, and characterize the optimal price and wage strategies in a competitive setting.

Acknowledgments

The authors would like to thank the department editor, the associate editor, and two anonymous referees for their valuable comments and constructive suggestions that have considerably improved the exposition of this paper. The first author is grateful for the support from the Hong Kong Research Grants Council [General Research Fund 16505418 and 16505819]. The second author is grateful for the support from the National Natural Science Foundation of China [NSFC 71802133], Shanghai Pujiang Talent Program [17PJC074], and Shanghai Eastern Scholar Program [QD2018053].

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Online Appendix to "Carpool Services for Ride-sharing Platforms: Price and Welfare Implications"

The following lemma establishes the convexity of the cost function $C(\cdot)$ and is, therefore, useful throughout the proof of our technical statements.

LEMMA 3. *Assume that $G(\cdot)$ satisfies the log-concave property. Then $C(y) := yG^{-1}(y)$ is convexly increasing in $0 \le y \le 1$.*

Proof. Let $h(x) := \log G(x)$. Since $h(x)$ is concave, we have $h''(x) = \frac{G''(x)\cdot G(x) - (G'(x))^2}{(G(x))^2} \le 0$, which implies

$$G''(x)\cdot G(x) \le (G'(x))^2 \tag{13}$$

To show $C(y)$ is convexly increasing in y, it suffices to show that $C'(y) \ge 0$ and $C''(y) \ge 0$. Since $G(\cdot)$ is non-decreasing and by the inverse function theorem, we have $C'(y) = G^{-1}(y) + y\cdot (G^{-1})'(y) = G^{-1}(y) + \frac{y}{G'(G^{-1}(y))} \ge 0$. It then follows that

$$\begin{aligned}C''(y) &= (G^{-1})'(y) + \frac{G'(G^{-1}(y)) - y\cdot[G''(G^{-1}(y))\cdot(G^{-1})'(y)]}{(G'(G^{-1}(y)))^2}\\ &= \frac{1}{G'(G^{-1}(y))} + \frac{G'(G^{-1}(y)) - y\cdot\frac{G''(G^{-1}(y))}{G'(G^{-1}(y))}}{(G'(G^{-1}(y)))^2} = \frac{2(G'(G^{-1}(y)))^2 - y\cdot G''(G^{-1}(y))}{(G'(G^{-1}(y)))^3} \ge 0\end{aligned}$$

where the last inequality follows from $y = G(x)$ and (13). *Q.E.D.*

Proof of Proposition 1.

We write $\Pi_n(s,k) = \bar\lambda v_n(1-s_n)s_n - KC(k/K)$. Clearly, $\Pi_n(s,k)$ is decreasing in k, so $\bar\lambda T_n\tilde s_n^* = \rho_{\max}\tilde k_n^*$. Plugging this into $\Pi_n(s,k)$, we have that it suffices to solve the optimization problem:

$$\tilde s_n^* = \arg\max_s f(s) := \bar\lambda v_n(1-s)s - KC\left(\frac{\bar\lambda T_n s}{\rho_{\max}K}\right)$$

subject to the constraints $s\in[0,1]$ and $\frac{\bar\lambda T_n s}{\rho_{\max}} \le K$.

When $\bar\lambda$ increases: 1(a) $\tilde s_n^*$ is decreasing in $\bar\lambda$: We have $f'(s) = \bar\lambda v_n(1-2s) - \frac{\bar\lambda T_n}{\rho_{\max}}C'\left(\frac{\bar\lambda T_n s}{\rho_{\max}K}\right)$. Let s^* satisfy $f'(s^*) = 0$, which is unique. We have $\tilde s_n^* = \min\{s^*, (K\rho_{\max})/(\bar\lambda T_n)\}$. It is easy to check that s^* and $(K\rho_{\max})/(\bar\lambda T_n)$ are both decreasing in $\bar\lambda$. Hence, $\tilde s_n^*$ is decreasing in $\bar\lambda$.

1(b) $\bar\lambda\tilde s_n^*$ is increasing in $\bar\lambda$: Let $\bar\lambda s := \lambda$. We have $f(s) = g(\lambda) = v_n\left(1-\frac{\lambda}{\bar\lambda}\right)\lambda - KC\left(\frac{\lambda T_n}{\rho_{\max}K}\right)$. Thus, we have $g'(\lambda) = v_n\left(1-\frac{2\lambda}{\bar\lambda}\right) - \frac{T_n}{\rho_{\max}}C'\left(\frac{T_n\lambda}{\rho_{\max}K}\right)$. Let λ^* satisfies $g'(\lambda^*) = 0$, so $\bar\lambda\tilde s_n^* = \min\{\lambda^*, (K\rho_{\max})/T_n\}$. Since $g'(\lambda^*) = 0$ implies that $\lambda^* \le 0.5\bar\lambda$, λ^* is increasing in $\bar\lambda$. Thus, $\bar\lambda\tilde s_n^*$ is also increasing in $\bar\lambda$.

1(c) $\tilde p_n^*$ is increasing in $\bar\lambda$: It follows immediately that $\tilde p_n^* = (1-\tilde s_n^*)v_n/d_n$ is increasing in $\bar\lambda$.

1(d) $\tilde k_n^*$ is increasing in $\bar\lambda$: Note that $\tilde k_n^* = \bar\lambda\tilde s_n^* T_n/\rho_{\max}$. By (b), $\tilde k_n^*$ is increasing in $\bar\lambda$.

1(e) $\tilde w_n^*$ is increasing in $\bar\lambda$: Note that $\tilde w_n^* = \tilde k_n^*/(\bar\lambda\tilde s_n^* d_n)G^{-1}(\frac{\tilde k_n^*}{K}) = T_n/(\rho_{\max}d_n)G^{-1}(\tilde k_n^*/K)$. Since $\tilde k_n^*$ is increasing in $\bar\lambda$, $\tilde w_n^*$ is also increasing in $\bar\lambda$.

1(f) $\tilde\Pi_n^*$ is increasing in $\bar\lambda$: By the envelope theorem, $\tilde\Pi_n^* = \max\Pi_n(s,k)$ is continuously differentiable in $\bar\lambda$ with $\frac{\partial\tilde\Pi_n^*}{\partial\bar\lambda} = v_n(1-s_b^*)s_b^* > 0$. Thus, $\tilde\Pi_n^*$ is increasing in $\bar\lambda$.

When K increases: 2(a) $\tilde s_n^*$ is increasing in K: As shown in part 1(a), $\tilde s_n^* = \min\{s^*, (K\rho_{\max})/(\bar\lambda T_n)\}$, where s^* satisfies $f'(s^*) = 0$. It is easy to check that s^* and $(K\rho_{\max})/(\bar\lambda T_n)$ are both increasing in K. Hence, $\tilde s_n^*$ is also increasing in K.

2(b) $\tilde{p}_n^*$ is decreasing in K: By part 2(a), it follows immediately that $\tilde{p}_n^* = (1-\tilde{s}_n^*)v_n/d_n$ decreases in K.

2(c) $\tilde{k}_n^*$ is increasing in K: Note that $\tilde{k}_n^* = \bar{\lambda}\tilde{s}_n^* T_n/\rho_{\max}$. By part 2(a), $\tilde{k}_n^*$ is increasing in K.

2(d) $\tilde{k}_n^*/K$ is decreasing in K: Let $z := k/K$. We have $f(s) = h(z) = \bar{\lambda}v_n\left(1 - \frac{\rho_{\max}Kz}{\bar{\lambda}T_n}\right)\frac{\rho_{\max}Kz}{\bar{\lambda}T_n} - KC(z)$. Thus, we have $h'(z) = \bar{\lambda}v_n\left(\frac{\rho_{\max}K}{\bar{\lambda}T_n} - 2\left(\frac{\rho_{\max}K}{\bar{\lambda}T_n}\right)^2 z\right) - KC'(z)$. Let z^* satisfies $h'(z^*) = 0$. By $\frac{\tilde{k}_n^*}{K} = \frac{\bar{\lambda}T_n\tilde{s}_n^*}{\rho_{\max}K}$ and $\tilde{s}_n^* \le 1$, we then have $\tilde{k}_n^*/K = \min\{z^*, 1, \frac{\bar{\lambda}T_n}{\rho_{\max}K}\}$. It is easy to check that if K increases, z^* will decrease. Since $\frac{\bar{\lambda}T_n}{\rho_{\max}K}$ is also decreasing in K, $\tilde{k}_n^*/K$ is decreasing in K.

2(e) $\tilde{w}_n^*$ is decreasing in K. Note that $\tilde{w}_n^* = \tilde{k}_n^*/(\bar{\lambda}\tilde{s}_n^* d_n)G^{-1}(\frac{\tilde{k}_n^*}{K}) = T_n/(\rho_{\max}d_n)G^{-1}(\tilde{k}_n^*/K)$. Since $\tilde{k}_n^*/K$ is decreasing in K, $\tilde{w}_n^*$ is also decreasing in K.

2(f) $\tilde{\Pi}_n^*$ is increasing in K. Since $G^{-1}\left(\frac{k}{K}\right)$ is decreasing K, $\Pi_n(s,k) = \bar{\lambda}v_n(1-s)s - kG^{-1}\left(\frac{k}{K}\right)$ is increasing in K. Furthermore, the constraint $k \le K$ is less tight as K increases. Thus, $\tilde{\Pi}_n^* = \max \Pi_n(s,k)$ is increasing in K as well. *Q.E.D.*

Proof of Lemma 2. We prove joint concavity by showing that the Hessian matrix of $f_p(\cdot)$ is negative semidefinite, or alternatively, its leading principal minors have alternate signs. Taking derivatives and by $v_n = v_p + \Delta$, we have

$$\frac{\partial f_p(s_n,s_p)}{\partial s_n} = \bar{\lambda}[-2v_n s_n - 2s_p v_p + v_n] - \frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p + s_nT_n)}{\rho_{\max}K}\right),$$
$$\frac{\partial f_p(s_n,s_p)}{\partial s_p} = \bar{\lambda}[-2v_p s_n - 2s_p v_p + v_p] - \frac{\bar{\lambda}T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p + s_nT_n)}{\rho_{\max}K}\right).$$

It then follows that $\frac{\partial^2 f_p(s_n,s_p)}{\partial s_n^2} = -2v_n\bar{\lambda} - \frac{\bar{\lambda}^2T_n^2}{\rho_{\max}^2K}C''\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right) \le 0$ because $C(\cdot)$ is convexly increasing. Similarly, we have $\frac{\partial^2 f_p(s_n,s_p)}{\partial s_p^2} = -2v_p\bar{\lambda} - \frac{\bar{\lambda}^2T_p^2}{m^2\rho_{\max}^2K}C''\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right) \le 0$. It remains to show

$$\frac{\partial^2 f_p(s_n,s_p)}{\partial s_n^2}\cdot\frac{\partial^2 f_p(s_n,s_p)}{\partial s_p^2} \ge \left(\frac{\partial^2 f_p(s_n,s_p)}{\partial s_p \partial s_n}\right)^2. \tag{14}$$

It is straightforward to check that (14) holds if and only if

$$2\bar{\lambda}^2v_pv_n + \frac{\bar{\lambda}^3T_p^2\alpha v_n}{m^2\rho_{\max}^2K} + \frac{\bar{\lambda}^3T_n^2\alpha v_p}{\rho_{\max}^2K} \ge 2\bar{\lambda}^2v_p^2 + \frac{2\bar{\lambda}^3T_pT_n\alpha v_p}{m\rho_{\max}^2K}, \tag{15}$$

where $\alpha := C''\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right)$. Since $v_n \ge v_p$ and $\alpha \ge 0$, a sufficient condition for (15) to hold is $T_p^2v_n + m^2T_n^2v_p \ge 2mT_pT_nv_p$, which is clearly true since $v_n \ge v_p$ and $(T_p - mT_n)^2 \ge 0$. *Q.E.D.*

Proof of Proposition 2. We first show that if $\Delta = 0$, $s_n^* = 0$. If $\Delta = 0$,

$$f_p(s_n,s_p) = \bar{\lambda}\left[(1-s_n-s_p)(s_n+s_p)v_n\right] - KC\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p + s_nT_n)}{\rho_{\max}K}\right).$$

Assume to the contrary that $s_n^* > 0$. Let $\epsilon > 0$ be small enough such that $s_n' = s_n^* - \epsilon \ge 0$, $s_p' = s_p^* + \epsilon$. Since $T_n > \frac{T_p}{m}$, we have

$$\frac{\bar{\lambda}(\frac{1}{m}s_p'T_p + s_n'T_n)}{\rho_{\max}K} = \frac{\bar{\lambda}(\frac{1}{m}(s_p^*+\epsilon)T_p + (s_n^*-\epsilon)T_n)}{\rho_{\max}K} = \frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K} + \frac{\bar{\lambda}(\frac{1}{m}T_p - T_n)\epsilon}{\rho_{\max}K} < \frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}.$$

Thus,

$$C\left(\frac{\bar{\lambda}(\frac{1}{m}s_p'T_p + s_n'T_n)}{\rho_{\max}K}\right) < C\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right).$$

In addition, $(1-s_n'-s_p')(s_n'+s_p')v_n=(1-s_n^*-s_p^*)(s_n^*+s_p^*)v_n$. Hence,

$$\begin{aligned}f_p(s_n',s_p') =&\bar\lambda\left[(1-s_n'-s_p')(s_n'+s_p')v_n\right]-KC\left(\frac{\bar\lambda(\frac{1}{m}s_p'T_p+s_n'T_n)}{\rho_{\max}K}\right)\\ >&\bar\lambda\left[(1-s_n^*-s_p^*)(s_n^*+s_p^*)v_n\right]-KC\left(\frac{\bar\lambda(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)=f_p(s_n^*,s_p^*).\end{aligned}$$

Therefore, $s_n^*=0$ if $\Delta=0$.

We now show that if $\Delta=v_n$, $s_p^*=0$. If $\Delta=v_n$, we have $f_p(s_n,s_p):=\bar\lambda\left[(1-s_n)v_ns_n\right]-KC\left(\frac{\bar\lambda(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right)$. Since $C(\cdot)$ is convexly increasing, $f_p(s_n,s_p)$ is decreasing in s_p for all s_n. Therefore, $s_p^*=0$ if $\Delta=v_n$.

Next, we show that s_n^* is increasing in Δ. Assume $\hat\Delta>\Delta$, $\hat f_p(\cdot,\cdot)$ is the profit function associated with $\hat\Delta$, and $(\hat s_n^*,\hat s_p^*)$ is the maximizer of $\hat f_p(\cdot,\cdot)$. Assume to the contrary that $\hat s_n^*<s_n^*$. Then we have $\partial_{s_n}\hat f_p(\hat s_n^*,\hat s_p^*)\le 0\le\partial_{s_n}f_p(s_n^*,s_p^*)$. Therefore,

$$-2\bar\lambda v_ns_n^*-2\bar\lambda(v_n-\Delta)s_p^*+\bar\lambda v_n-\frac{\bar\lambda T_n}{\rho_{\max}}C'\left(\frac{\bar\lambda(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\ge-2\bar\lambda v_n\hat s_n^*-2\bar\lambda(v_n-\hat\Delta)\hat s_p^*+\bar\lambda v_n-\frac{\bar\lambda T_n}{\rho_{\max}}C'\left(\frac{\bar\lambda(\frac{1}{m}\hat s_p^*T_p+\hat s_n^*T_n)}{\rho_{\max}K}\right),$$

which implies that

$$y^*-\hat y^*\le 2v_n(\hat s_n^*-s_n^*)+2(v_n-\hat\Delta)\hat s_p^*-2(v_n-\Delta)s_p^*, \tag{16}$$

where

$$y^*:=\frac{T_n}{\rho_{\max}}C'\left(\frac{\bar\lambda(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\text{ and }\hat y^*:=\frac{T_n}{\rho_{\max}}C'\left(\frac{\bar\lambda(\frac{1}{m}\hat s_p^*T_p+\hat s_n^*T_n)}{\rho_{\max}K}\right).$$

If $\hat s_p^*\le s_p^*$, the convexity of $C(\cdot)$ suggests that $y^*-\hat y^*>0$. Since $\hat s_n^*<s_n^*$, $\hat\Delta>\Delta$, and $\hat s_p^*\le s_p^*$, we have $2v_n(\hat s_n^*-s_n^*)+2(v_n-\hat\Delta)\hat s_p^*-2(v_n-\Delta)s_p^*<0$. This forms a contradiction. Thus, we have $\hat s_p^*>s_p^*$. It then follows that $\partial_{s_p}\hat f_p(\hat s_n^*,\hat s_p^*)\ge 0\ge\partial_{s_p}f_p(s_n^*,s_p^*)$. Therefore, we have $(v_n-\hat\Delta)(1-2\hat s_n^*-2\hat s_p^*)-\frac{T_p}{mT_n}\hat y^*\ge(v_n-\Delta)(1-2s_n^*-2s_p^*)-\frac{T_p}{mT_n}y^*$. It then follows that

$$\begin{aligned}y^*-\hat y^*\ge&\frac{mT_n}{T_p}((v_n-\Delta)(1-2s_n^*-2s_p^*)-(v_n-\hat\Delta)(1-2\hat s_n^*-2\hat s_p^*))\\ \ge&(v_n-\Delta)(1-2s_n^*-2s_p^*)-(v_n-\hat\Delta)(1-2\hat s_n^*-2\hat s_p^*)\\ =&(\hat\Delta-\Delta)+2(v_n-\hat\Delta)\hat s_n^*-2(v_n-\Delta)s_n^*+2(v_n-\hat\Delta)\hat s_p^*-2(v_n-\Delta)s_p^*\\ >&2(v_n-\Delta)(\hat s_n^*-s_n^*)+2(v_n-\hat\Delta)\hat s_p^*-2(v_n-\Delta)s_p^*\\ >&2v_n(\hat s_n^*-s_n^*)+2(v_n-\hat\Delta)\hat s_p^*-2(v_n-\Delta)s_p^*,\end{aligned} \tag{17}$$

where the second inequality follows from $T_n>\frac{1}{m}T_p$ and $s_n^*+s_p^*\le 0.5$ (which will be shown later in (19)), the third inequality follows from $\hat s_n^*<s_n^*$, and the last inequality follows from the assumption that $\hat s_n^*<s_n^*$. Inequality (16) contradicts with inequality (17). Therefore, $\hat s_n^*\ge s_n^*$ if $\hat\Delta>\Delta$.

Next, we show that $\hat s_p^*\le s_p^*$ if $\hat\Delta>\Delta$. Assume to the contrary that $\hat s_p^*>s_p^*$. Then we have $\partial_{s_p}\hat f_p(\hat s_n^*,\hat s_p^*)\ge 0\ge\partial_{s_p}f_p(s_n^*,s_p^*)$, and therefore

$$(v_n-\hat\Delta)(1-2\hat s_n^*-2\hat s_p^*)-\frac{T_p}{mT_n}\hat y^*\ge(v_n-\Delta)(1-2s_n^*-2s_p^*)-\frac{T_p}{mT_n}y^*. \tag{18}$$

We have shown that $\hat s_n^*\ge s_n^*$. Thus, $\hat s_n^*+\hat s_p^*>s_n^*+s_p^*$, $(v_n-\hat\Delta)(1-2\hat s_n^*-2\hat s_p^*)<(v_n-\Delta)(1-2s_n^*-2s_p^*)$, and

$$\hat y^*=\frac{\bar\lambda T_n}{\rho_{\max}}C'\left(\frac{\bar\lambda(\frac{1}{m}\hat s_p^*T_p+\hat s_n^*T_n)}{\rho_{\max}K}\right)>\frac{\bar\lambda T_n}{\rho_{\max}}C'\left(\frac{\bar\lambda(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)=y^*.$$

Therefore,

$$(v_n-\hat{\Delta})(1-2\hat{s}_n^*-2\hat{s}_p^*)-\frac{T_p}{mT_n}\hat{y}^*<(v_n-\Delta)(1-2s_n^*-2s_p^*)-\frac{T_p}{mT_n}y^*.$$

The above inequality contradicts with (18) and hence implies that $\hat{s}_p^*\le s_p^*$ if $\hat{\Delta}>\Delta$.

Next, we show the existence of $\underline{\Delta}$ and $\bar{\Delta}$. Note that if $\Delta=0$ we have $s_p^*>0$, and if $\Delta=v_n$ we have $s_n^*>0$. Since $f_p(s_n,s_p|\Delta)$ is continuously differentiable with respect to (s_n,s_p,Δ), by the maximum theorem, the maximizer $(s_n^*(\Delta),s_p^*(\Delta))$ is continuous in Δ. Therefore, the monotonicity and continuity of s_n^* and s_p^* with respect to Δ yields that there exists $\underline{\Delta}$ and $\bar{\Delta}$ such that

$$s_n^*\begin{cases}=0, & \text{if } \Delta\in[0,\underline{\Delta}],\\ >0, & \text{if } \Delta\in(\underline{\Delta},v_n];\end{cases}\quad\text{and}\quad s_p^*\begin{cases}>0, & \text{if } \Delta\in[0,\bar{\Delta}),\\ =0, & \text{if } \Delta\in[\bar{\Delta},v_n].\end{cases}$$

To show $\bar{\Delta}>\underline{\Delta}$, observe that $s_n^*=s_p^*=0$ is never optimal for any $\Delta\in[0,v_n]$, which immediately implies that $\underline{\Delta}<\bar{\Delta}$. In the remainder of the proof, we show that

$$s_p^*+s_n^*\le 0.5,\ \text{and} \tag{19}$$

$$\bar{\Delta}=v_n\left(1-\frac{T_p}{mT_n}\right). \tag{20}$$

We first show (19). Assume to the contrary that $s_p^*+s_n^*>0.5$. We have

$$\partial_{s_p}f_p(s_n^*,s_p^*)=\bar{\lambda}\left[(v_n-\Delta)(1-2s_n^*-s_p^*)-\frac{T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right]<0,$$

so we must have $s_p^*=0$, and thus $s_n^*>0.5$. Therefore,

$$\partial_{s_n}f_p(s_n^*,s_p^*)=\bar{\lambda}\left[v_n(1-2s_n^*)-\frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right]<0,$$

which implies that $s_n^*=0$, contradicting with $s_n^*>0.5$. We next show (20). It suffices to show that if $\Delta>v_n(1-\frac{T_p}{mT_n})$ (resp. $\Delta<v_n(1-\frac{T_p}{mT_n})$), $s_p^*=0$ (resp. $s_p^*>0$). If $\Delta>v_n(1-\frac{T_p}{mT_n})$ and $s_p^*>0$, the First Order Condition (FOC) with respect to s_p implies that

$$(v_n-\Delta)(1-2s_n^*-2s_p^*)-\frac{T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)=\frac{T_p}{m}\mu_2^*,$$

where μ_2^* is the Lagrangian multiplier with respect to the constraint $\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)\le\rho_{\max}K$. By $\Delta>v_n(1-\frac{T_p}{mT_n})$, we have $\frac{v_n-\Delta}{v_n}<\frac{T_p}{mT_n}$. It then follows that

$$\begin{aligned}\partial_{s_n}f_p(s_n^*,s_p^*)=&\bar{\lambda}\left(-2v_ns_n^*-2(v_n-\Delta)s_p^*+v_n-\frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right)\\ =&\bar{\lambda}\left(-2v_ns_n^*-2(v_n-\Delta)s_p^*+v_n-\frac{m(v_n-\Delta)(1-2s_n^*-2s_p^*)T_n}{T_p}+T_n\mu_2^*\right)\\ >&\bar{\lambda}\left(-2v_ns_n^*-2(v_n-\Delta)s_p^*+v_n-v_n(1-2s_n^*-2s_p^*)+T_n\mu_2^*\right)=2\bar{\lambda}\Delta s_p^*+\bar{\lambda}T_n\mu_2^*>\bar{\lambda}T_n\mu_2^*,\end{aligned}$$

where the first inequality follows from $\frac{v_n-\Delta}{v_n}<\frac{T_p}{mT_n}$. Therefore we have $\partial_{s_n}f_p(s_n^*,s_p^*)-\bar{\lambda}T_n\mu_2^*>0$, which contradicts the FOC that $\partial_{s_n}f_p(s_n^*,s_p^*)-\bar{\lambda}T_n\mu_2^*=0$. If then follows that $s_p^*=0$ if $\Delta>v_n(1-\frac{T_p}{mT_n})$.

If $\Delta<v_n(1-\frac{T_p}{mT_n})$ and $s_p^*=0$, we have that $s_n^*>0$ since both of s_n^* and s_p^* being equal to zero is clearly suboptimal. The FOC with respect to s_n implies that

$$v_n-2v_ns_n^*-\frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}s_n^*T_n}{\rho_{\max}K}\right)=T_n\mu_2^*,$$

and by $\Delta < v_n(1-\frac{T_p}{mT_n})$ we have $\frac{v_n-\Delta}{v_n} > \frac{T_p}{mT_n}$. It then follows that

$$\begin{aligned}
\partial_{s_p} f_p(s_n^*, s_p^*) =& \bar{\lambda}\left((v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right)\\
>& \bar{\lambda}\left(v_n(1-2s_n^*)\frac{T_p}{mT_n} - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right)\\
=& \frac{\bar{\lambda}T_p}{mT_n}\left(v_n(1-2s_n^*) - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{2}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right) = \frac{\bar{\lambda}T_p}{m}\mu_2^*,
\end{aligned}$$

where the inequality follows from $\frac{v_n-\Delta}{v_n} > \frac{T_p}{mT_n}$ and the assumption $s_p^*=0$. Thus, $\partial_{s_p} f_p(s_n^*, s_p^*) - \frac{\bar{\lambda}T_p}{m}\mu_2^* > 0$, which contradicts with $\partial_{s_p} f_p(s_n^*, s_p^*) - \frac{\bar{\lambda}T_p}{m}\mu_2^* = 0$. Therefore, we have $s_p^* > 0$ if $\Delta < v_n(1-\frac{T_p}{mT_n})$. *Q.E.D.*

Proof of Theorem 1. Let $\hat{\Delta} > \Delta$. We need to show that $\hat{s}^* = \hat{s}_n^* + \hat{s}_p^* \le s^* = s_n^* + s_p^*$. Notice that $\hat{s}_n^* \ge s_n^*$ by Proposition 2. If $\hat{s}_n^* = s_n^*$, then we have $\hat{s}^* = \hat{s}_n^* + \hat{s}_p^* \le s^* = s_n^* + s_p^*$ since $\hat{s}_p^* \le s_p^*$ by Proposition 2. Therefore, it remains to consider the case where $\hat{s}_n^* > s_n^*$.

If $\hat{s}_n^* > s_n^*$, we have $\partial_{s_n}\hat{f}_p(\hat{s}_n^*, \hat{s}_p^*) \ge 0 \ge \partial_{s_n} f_p(s_n^*, s_p^*)$, i.e.,

$$-2v_n\hat{s}_n^* - 2(v_n-\hat{\Delta})\hat{s}_p^* - \hat{y}^* \ge -2v_ns_n^* - 2(v_n-\Delta)s_p^* - y^*,$$

where $y^* = \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)$ and $\hat{y}^* = \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}\hat{s}_p^*T_p+\hat{s}_n^*T_n)}{\rho_{\max}K}\right)$. It then follows that

$$2(v_n-\hat{\Delta})(\hat{s}^*-s^*) \le 2\hat{\Delta}(s_n^*-\hat{s}_n^*) + 2(\Delta-\hat{\Delta})s_p^* + y^* - \hat{y}^*.$$

If $y^* \le \hat{y}^*$, then $s^* > \hat{s}^*$ immediately follows from $s_n^* < \hat{s}_n^*$ and $\Delta < \hat{\Delta}$. If $y^* > \hat{y}^*$, the convexity of $C(\cdot)$ implies that $\frac{1}{m}s_p^*T_p + s_n^*T_n > \frac{1}{m}\hat{s}_p^*T_p + \hat{s}_n^*T_n$. Since $(T_p/m) < T_n$, it then follows that $s_p^* - \hat{s}_p^* > \hat{s}_n^* - s_n^*$, or equivalently, $s^* = s_n^* + s_p^* > \hat{s}_n^* + \hat{s}_p^* = \hat{s}^*$. *Q.E.D.*

Proof of Theorem 2. We first show $\underline{p_p^* \le \tilde{p}_n^* \text{ for all } \Delta \in [0, v_n]}$. Note that $p_p^* = (1-s_p^*-s_n^*)(v_n-\Delta)/d_p$ and $\tilde{p}_n^* = (1-\tilde{s}_n^*)v_n/d_n$. By Theorem 1, we have $\tilde{s}_n^* \le s_p^* + s_n^*$ ($\tilde{s}_n^*$ corresponds to $s_n^* + s_p^*$ in the case with $\Delta = v_n$), and $p_p^* \le \tilde{p}_n^*$ follows immediately from $\Delta \ge 0$ and $d_p \ge d_n$. Next, we show that $\underline{p_n^* \le \tilde{p}_n^* \text{ for all } \Delta \in [0, \bar{\Delta})}$. We proceed in two steps. First, we show that $p_n^* \le \tilde{p}_n^*$ when $\Delta \in [\underline{\Delta}, \bar{\Delta})$. Then we show that p_n^* is increasing in Δ on $\Delta \in [0, \underline{\Delta}]$, which would complete the proof.

First, consider the case where $\Delta \in (\underline{\Delta}, \bar{\Delta})$ (i.e., $s_n^* > 0$ and $s_p^* > 0$). Assume, to the contrary, that $p_n^* > \tilde{p}_n^*$, i.e., $(1-s_n^*)v_n - s_p^*(v_n-\Delta) > (1-\tilde{s}_n^*)v_n$. Rearranging terms, we get

$$\tilde{s}_n^* > s_n^* + \frac{v_n-\Delta}{v_n}s_p^* > s_n^* + \frac{T_p}{mT_n}s_p^*,$$

where the second inequality holds because $\Delta < \bar{\Delta}$ implies $\frac{v_n-\Delta}{v_n} > \frac{T_p}{mT_n}$. Note that

$$\begin{aligned}
\partial_{s_n} f_p(s_n^*, s_p^*) =& \bar{\lambda}\left(v_n - 2v_ns_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right)\\
>& \bar{\lambda}\left(v_n - 2v_ns_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}\tilde{s}_n^*T_n}{\rho_{\max}K}\right)\right)\\
>& \bar{\lambda}\left(v_n - 2v_ns_n^* - 2v_n(\tilde{s}_n^*-s_n^*) - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}\tilde{s}_n^*T_n}{\rho_{\max}K}\right)\right)\\
=& \bar{\lambda}\left(v_n - 2v_n\tilde{s}_n^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}\tilde{s}_n^*T_n}{\rho_{\max}K}\right)\right) = f_n'(\tilde{s}_n^*) \ge 0,
\end{aligned} \tag{21}$$

where $f_n(\tilde{s}_n) := \bar{\lambda} v_n(1-\tilde{s}_n)\tilde{s}_n - KC\left(\frac{\bar{\lambda}\tilde{s}_n T_n}{\rho_{\max}K}\right)$ is the profit of the platform which only offers the normal service. In (21), the first inequality follows from $\tilde{s}_n^* > s_n^* + \frac{T_p}{mT_n}s_p^*$, the second inequality follows from $\tilde{s}_n^* > s_n^* + \frac{v_n-\Delta}{v_n}s_p^*$, and the last inequality follows from $\tilde{s}_n^* > 0$. In addition, it is straightforward to check that $\tilde{s}_n^* > s_n^* + \frac{T_p}{mT_n}s_p^*$ implies $\frac{\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n)}{\rho_{\max}K} < 1$. It then follows from (21) that $\partial_{s_n} f_p(s_n^*, s_p^*) > 0$, which contradicts with (s_n^*, s_p^*) being the optimal solution. Therefore, we have $p_n^* \le \tilde{p}_n^*$ when $\Delta \in (\underline{\Delta}, \bar{\Delta})$. Finally, we show $\underline{p_n^* \text{ is increasing in } \Delta \text{ on } \Delta \in [0, \underline{\Delta}]}$. When $\Delta \in [0, \underline{\Delta}]$, we have $s_n^* = 0$ and

$$p_n^* = ((1-s_n^*)\Delta + (1-s_n^*-s_p^*)(v_n-\Delta))/d_n = (\Delta + (1-s_p^*)(v_n-\Delta))/d_n = (v_n - (v_n-\Delta)s_p^*)/d_n.$$

By Proposition 2, s_p^* is decreasing in Δ. Therefore, $(v_n-\Delta)s_p^*$ is decreasing in Δ and it follows that p_n^* is increasing in Δ on $\Delta \in [0, \underline{\Delta}]$.

Proof of Proposition 3. First, it follows immediately from

$$\Pi_p^* = \max\left\{\bar{\lambda}[((1-s_n-s_p)(v_n-\Delta) + (1-s_n)\Delta)s_n + (1-s_n-s_p)(v_n-\Delta)s_p] - KC\left(\frac{\bar{\lambda}(\frac{s_p}{\gamma} + s_n T_n)}{\rho_{\max}K}\right)\right\}$$

that Π_p^* is increasing in γ (as $C(\cdot)$ is decreasing in y). If $\Delta \le \underline{\Delta}$, as shown in Proposition 2, $s_n^* = 0$. It can be easily checked that $\partial_{s_p} f_p(0, s_p) = (v_n-\Delta)(1-2s_p) - \frac{\bar{\lambda}}{\rho_{\max}\gamma}C'\left(\frac{\bar{\lambda}s_p}{\gamma}\right)$ is increasing in s_p, so $f_p(0, s_p)$ is supermodular in (s_p, γ). Hence, s_p^* is increasing in γ. Since $s_n^* = 0$, $s^* = s_n^* + s_p^* = s_p^*$ is increasing in γ, whereas $p_n^* = ((1-s^*)v_p + (1-s_n^*)\Delta)/d_n = ((1-s^*)v_p + \Delta)/d_n$ and $p_p^* = (1-s^*)v_p/d_p$ are decreasing in s^* and thus in γ as well.

We now consider the case $\Delta > \underline{\Delta}$, in which case $s_p^* > 0$ and $s_n^* > 0$. Assume that $\hat{\gamma} > \gamma$, $\hat{f}_p(\cdot,\cdot)$ is the profit function associated with $\hat{\gamma}$, and $(\hat{s}_n^*, \hat{s}_n^*)$ is the optimizer of $\hat{f}_p(\cdot,\cdot)$. We first show $\underline{\hat{s}_p^* \ge s_p^*}$. Assume to the contrary that $\hat{s}_p^* < s_p^*$. Then we have $\partial_{s_p}\hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \le 0 \le \partial_{s_p} f_p(s_n^*, s_p^*)$, or alternatively, $(v_n-\Delta)(1-2\hat{s}_n^* - 2\hat{s}_p^*) - \frac{\hat{y}^*}{\hat{\gamma}T_n} \le (v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{y^*}{\gamma T_n}$, where $\hat{y}^* := \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{\hat{s}_p^*}{\hat{\gamma}} + \hat{s}_n^* T_n)}{\rho_{\max}K}\right)$ and $y^* := \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{s_p^*}{\gamma} + s_n^* T_n)}{\rho_{\max}K}\right)$. Equivalently,

$$\frac{\hat{y}^*}{\hat{\gamma}T_n} - \frac{y^*}{\gamma T_n} \ge 2(v_n-\Delta)(s_n^* - \hat{s}_n^*) + 2(v_n-\Delta)(s_p^* - \hat{s}_p^*). \tag{22}$$

If in addition we have $\hat{s}_n^* \le s_n^*$, the convexity of $C(\cdot)$ suggests that $\hat{y}^* < y^*$. However, (22) implies that $\hat{y}^* > y^*$, which forms a contradiction. Hence, we must have $\hat{s}_n^* > s_n^*$. Thus, $\partial_{s_n}\hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \ge 0 \ge \partial_{s_n} f_p(s_n^*, s_p^*)$, or alternatively, $-2(v_n-\Delta)\hat{s}_p^* + v_n(1-2\hat{s}_n^*) - \hat{y}^* \ge -2(v_n-\Delta)s_p^* + v_n(1-2s_n^*) - y^*$. Equivalently,

$$\hat{y}^* - y^* \le 2(v_n-\Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*). \tag{23}$$

By (22) and $\hat{\gamma}T_n > \gamma T_n > 1$, $\hat{y}^* - y^* > 2(v_n-\Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*)$, which contradicts (23). We have thus shown that $\hat{s}_p^* \ge s_p^*$.

Next, we show that $\underline{\hat{s}_p^* + \hat{s}_n^* \ge s_p^* + s_n^*}$. If $\hat{s}_p^* = s_p^*$, then $\frac{\hat{s}_p^*}{\hat{\gamma}} \le \frac{s_p^*}{\gamma}$. We have

$$\begin{aligned}
\partial_{s_n}\hat{f}_p(s_n^*, \hat{s}_p^*) &= \bar{\lambda}(v_n - 2v_n s_n^* - 2(v_n-\Delta)\hat{s}_p^*) - \frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{\hat{s}_p^*}{\hat{\gamma}} + s_n^* T_n)}{\rho_{\max}K}\right) \\
&\ge \bar{\lambda}(v_n - 2v_n s_n^* - 2(v_n-\Delta)\hat{s}_p^*) - \frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{s_p^*}{\gamma} + s_n^* T_n)}{\rho_{\max}K}\right) = \partial_{s_n} f_p(s_n^*, s_p^*) \ge 0.
\end{aligned}$$

Therefore, we have $\hat{s}_n^* \geq s_n^*$ and hence, $\hat{s}_p^* + \hat{s}_n^* \geq s_p^* + s_n^*$.

Now we consider the case $\hat{s}_p^* > s_p^*$. If $\hat{s}_n^* + \hat{s}_p^* \leq s_n^* + s_p^*$, we must have $\hat{s}_n^* < s_n^*$. Thus, $\partial_{s_n} \hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \leq 0 \leq \partial_{s_n} f_p(s_n^*, s_p^*)$, i.e., $-2(v_n - \Delta)\hat{s}_p^* + v_n(1-2\hat{s}_n^*) - \hat{y}^* \leq -2(v_n-\Delta)s_p^* + v_n(1-2s_n^*) - y^*$. Equivalently,

$$\hat{y}^* - y^* \geq 2(v_n - \Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*) > 0, \tag{24}$$

where the last inequality follows from $\hat{s}_n^* + \hat{s}_p^* \leq s_n^* + s_p^*$. Since $C(\cdot)$ is convex, (24) implies that $\frac{\hat{s}_p^*}{\hat{\gamma}} + T_n \hat{s}_n^* > \frac{s_p^*}{\gamma} + T_n s_n^*$, which is equivalent to that $s_n^* - \hat{s}_n^* < \frac{\hat{s}_p^*}{\hat{\gamma} T_n} - \frac{s_p^*}{\gamma T_n} < \hat{s}_p^* - s_p^*$, where the inequality follows from that $\hat{\gamma} T_n > \gamma T_n > 1$. Thus, $\hat{s}_n^* + \hat{s}_p^* > s_n^* + s_p^*$, contradicting with $\hat{s}_n^* + \hat{s}_p^* \leq s_n^* + s_p^*$. Therefore, we must have $\hat{s}_n^* + \hat{s}_p^* \geq s_n^* + s_p^*$.

Next, we show that $\underline{\hat{p}_p^* \leq p_p^*}$. Note that $\hat{p}_p^* = (v_n - \Delta)(1 - \hat{s}_n^* - \hat{s}_p^*)/d_p$ and $p_p^* = (v_n - \Delta)(1 - \hat{s}_n^* - \hat{s}_p^*)/d_p$, $\hat{p}_p^* \leq p_p^*$ follows immediately from $\hat{s}_n^* + \hat{s}_p^* \geq s_n^* + s_p^*$.

Finally, we show that $\underline{\hat{p}_n^* \leq p_n^*}$. Assume to the contrary that $\hat{p}_n^* > p_n^*$, i.e., $(v_n - \Delta)(1 - \hat{s}_n^* - \hat{s}_p^*) + \Delta(1 - \hat{s}_n^*) > (v_n - \Delta)(1 - s_n^* - s_p^*) + \Delta(1 - s_n^*)$. Hence, $v_n(s_n^* - \hat{s}_n^*) > (v_n - \Delta)(\hat{s}_p^* - s_p^*) > 0$, where the second inequality follows from that $\hat{s}_p^* > s_p^*$. The inequality $\hat{s}_n^* < s_n^*$ implies that $\partial_{s_n} \hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \leq 0 \leq \partial_{s_n} f_p(s_n^*, s_p^*)$, i.e., $-2(v_n - \Delta)\hat{s}_p^* + v_n(1-2\hat{s}_n^*) - \hat{y}^* \leq -2(v_n-\Delta)s_p^* + v_n(1-2s_n^*) - y^*$. Equivalently,

$$\hat{y}^* - y^* \geq 2(v_n - \Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*) > 0, \tag{25}$$

where the last inequality follows from $v_n(s_n^* - \hat{s}_n^*) > (v_n - \Delta)(\hat{s}_p^* - s_p^*) > 0$. Since $C(\cdot)$ is convex, (25) implies that $\frac{\hat{s}_p^*}{\hat{\gamma}} + T_n \hat{s}_n^* > \frac{s_p^*}{\gamma} + T_n s_n^*$, which is equivalent to that $s_n^* - \hat{s}_n^* < \frac{\hat{s}_p^*}{\hat{\gamma} T_n} - \frac{s_p^*}{\gamma T_n} < \frac{\hat{s}_p^* - s_p^*}{\hat{\gamma} T_n}$, where the inequality follows from that $\hat{\gamma} T_n > \gamma T_n$. Since $\Delta < \bar{\Delta}$, $(v_n - \Delta)/v_n > 1/(\hat{\gamma} T_n)$. So we have $\frac{(v_n - \Delta)(\hat{s}_p^* - s_p^*)}{v_n} > \frac{\hat{s}_p^* - s_p^*}{\hat{\gamma} T_n} > s_n^* - \hat{s}_n^*$. This inequality contradicts that $v_n(s_n^* - \hat{s}_n^*) > (v_n - \Delta)(\hat{s}_p^* - s_p^*)$. Therefore, we must have $\hat{p}_n^* \leq p_n^*$. *Q.E.D.*

Proof of Proposition 4. We use $\lambda_p^* := \bar{\lambda} s_p^*$ and $\lambda_n^* := \bar{\lambda} s_n^*$. Notice that when $(v_n - \Delta)/v_n > T_p/(mT_n)$, we have $\Delta < \bar{\Delta}$ and hence $s_p^* > 0$. By the KKT condition (which is both necessary and sufficient for optimality by the joint concavity of $f_p(\cdot)$ and compactness of the feasible region of (s_n, s_p)), we have

$$\bar{\lambda}\left[-2v_p s_n^* - 2\Delta s_n^* - 2s_p^* v_p + v_p + \Delta\right] - \frac{\bar{\lambda} T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right) = \mu_1^* + \bar{\lambda} T_n \mu_2^* - \eta_1^*, \tag{26}$$

$$\bar{\lambda}\left[-2v_p s_n^* - 2s_p^* v_p + v_p\right] - \frac{\bar{\lambda} T_p}{m\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right) = \mu_1^* + \frac{1}{m} \bar{\lambda} T_p \mu_2^* - \eta_2^*, \tag{27}$$

$$\mu_1^*(1 - s_n^* - s_p^*) = 0, \tag{28}$$

$$\mu_2^*\left(\rho_{\max} K - \bar{\lambda}\left(\frac{1}{m} s_p^* T_p + s_n^* T_n\right)\right) = 0, \tag{29}$$

$$\eta_1^* s_n^* = 0, \quad \eta_2^* s_p^* = 0, \tag{30}$$

$$\mu_1^*, \mu_2^*, \eta_1^*, \eta_2^* \geq 0, \tag{31}$$

where $\mu_1^*, \mu_2^*, \eta_1^*$, and η_2^* are the Lagrangian multipliers with respect to the constraints $s_n^* + s_p^* \leq 1$, $\bar{\lambda}(\frac{1}{m} s_n^* T_n + s_p^* T_p) \leq \rho_{\max} K$, $s_n^* \geq 0$ and $s_p^* \geq 0$, respectively. Notice that by (19), $s_n^* + s_p^* < 1$ and hence by complementary slackness condition, we have $\mu_1^* = 0$.

(a) s_n^* is decreasing in $\bar{\lambda}$. Consider $\hat{\bar{\lambda}}$ and $\bar{\lambda}$ with $\hat{\bar{\lambda}} > \bar{\lambda}$. Notice that $(v_n - \Delta)/v_n > T_p/(mT_n)$ and hence $\Delta < \bar{\Delta}$, we have $s_p^* > 0$ and $\hat{s}_p^* > 0$. By (30), $\eta_2^* = \hat{\eta}_2 = 0$. We first consider the case where $\hat{s}_n^*, s_n^* > 0$, and therefore $\eta_1^* = \hat{\eta}_1 = 0$. Then the KKT conditions (26) and (27) imply that:

$$\begin{aligned}
&v_n - 2v_n s_n^* - 2(v_n-\Delta)s_p^* - y^* - \mu_2^* T_n = 0,\\
&(v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{mT_n}y^* - \mu_2^*\frac{T_p}{m} = 0,\\
&v_n - 2v_n \hat{s}_n^* - 2(v_n-\Delta)\hat{s}_p^* - \hat{y}^* - \hat{\mu}_2^* T_n = 0,\\
&(v_n-\Delta)(1-2\hat{s}_n^*-2\hat{s}_p^*) - \frac{T_p}{mT_n}\hat{y}^* - \hat{\mu}_2^*\frac{T_p}{m} = 0,
\end{aligned} \tag{32}$$

where $y^* := \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right)$ and $\hat{y}^* := \frac{T_n}{\rho_{\max}}C'\left(\frac{\hat{\bar{\lambda}}(\frac{1}{m}\hat{s}_p^*T_p + \hat{s}_n^*T_n)}{\rho_{\max}K}\right)$. Observe that both (s_n^*, s_p^*) and $(\hat{s}_n^*, \hat{s}_p^*)$ are located on the line

$$\frac{v_n - 2v_n s_n - 2(v_n-\Delta)s_p}{(v_n-\Delta)(1-2s_n-2s_p)} = \frac{mT_n}{T_p}. \tag{33}$$

If $\hat{s}_n^* - s_n^* = \delta > 0$, then it is easy to check by (33) that $s_p^* > \hat{s}_p^*$ and $s_p^* - \hat{s}_p^* < \delta$. Thus, we have $\hat{s}_n^* + \hat{s}_p^* > s_n^* + s_p^*$ and $\hat{\bar{\lambda}}(\hat{s}_n^* T_n + \frac{1}{m}T_p\hat{s}_p^*) > \bar{\lambda}(s_n^*T_n + \frac{1}{m}s_p^*T_p)$. Hence, $\hat{y}^* > y^*$. Moreover, by the complementary slackness condition (29), $\hat{\mu}_2^* \geq \mu_2^*$. Therefore,

$$(v_n-\Delta)(1-2\hat{s}_n^*-2\hat{s}_p^*) - \frac{T_p}{mT_n}\hat{y}^* - \hat{\mu}_2^*\frac{T_p}{m} < (v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{mT_n}y^* - \mu_2^*\frac{T_p}{m},$$

which contradicts with (32). Hence, in the range of $s_n^* > 0$, s_n^* is decreasing in $\bar{\lambda}$. By the continuity of s_n^*, it is clear that s_n^* is decreasing in $\bar{\lambda}$ for all $\bar{\lambda}$.

(b) There exists a λ_0 such that $s_n^* = 0$ for $\bar{\lambda} \geq \lambda_0$. Note that $\Delta < \bar{\Delta}$ is equivalent to $\frac{v_n - \Delta}{v_n} > \frac{T_p}{mT_n}$. We use $\lambda_p := \bar{\lambda}s_p$ and $\lambda_n := \bar{\lambda}s_n$ as the decision variables. The platform is then to maximize

$$f_p(\lambda_n,\lambda_p) = \left(\left(1-\frac{\lambda_p}{\bar{\lambda}}-\frac{\lambda_n}{\bar{\lambda}}\right)(v_n-\Delta) + \left(1-\frac{\lambda_n}{\bar{\lambda}}\right)\Delta\right)\lambda_n + \left(1-\frac{\lambda_n}{\bar{\lambda}}-\frac{\lambda_p}{\bar{\lambda}}\right)(v_n-\Delta)\lambda_p - KC\left(\frac{\frac{1}{m}\lambda_pT_p + \lambda_nT_n}{\rho_{\max}K}\right), \tag{34}$$

subject to the constraint $0 \leq \lambda_n + \lambda_p \leq \bar{\lambda}$ and $\lambda_nT_n + \lambda_p\frac{T_p}{m} \leq \rho_{\max}K$. We have

$$\begin{aligned}
\partial_{\lambda_n} f_p(\lambda_n^*,\lambda_p^*) &= v_n - 2v_n\frac{\lambda_n^*}{\bar{\lambda}} - 2(v_n-\Delta)\frac{\lambda_p^*}{\bar{\lambda}} - \frac{T_n}{\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right)\\
&= v_n - 2v_ns_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right),
\end{aligned}$$

and

$$\begin{aligned}
\partial_{\lambda_p} f_p(\lambda_n^*,\lambda_p^*) &= (v_n-\Delta)\left(1-\frac{2\lambda_n^*}{\bar{\lambda}}-\frac{2\lambda_p^*}{\bar{\lambda}}\right) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right)\\
&= (v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right).
\end{aligned}$$

Since $\lambda_n^*T_n + \lambda_p^*\frac{T_p}{m} \leq \rho_{\max}K$, it follows that $s_n^* = \frac{\lambda_n^*}{\bar{\lambda}} \leq \frac{\rho_{\max}K}{T_n\bar{\lambda}}$ and $s_p^* = \frac{\lambda_p^*}{\bar{\lambda}} \leq \frac{m\rho_{\max}K}{T_n\bar{\lambda}}$. Therefore, we have $s_n^* \to 0$ and $s_p^* \to 0$ as $\bar{\lambda} \to +\infty$. Because $\Delta < v_n\left(1-\frac{T_p}{mT_n}\right)$, we have

$$v_n - \Delta - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right) > \frac{T_p}{mT_n}\left(v_n - \frac{T_n}{\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right)\right).$$

Therefore, when $\bar{\lambda}$ is sufficiently large (where $s_n^* \to 0$ and $s_p^* \to 0$), we have

$$\begin{aligned}
\partial_{\lambda_p} f_p(\lambda_n^*,\lambda_p^*) - \frac{T_p}{m}\mu_2^* &= (v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right) - \frac{T_p}{m}\mu_2^*\\
&> \frac{T_p}{mT_n}\left(v_n - 2v_ns_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\frac{1}{m}\lambda_p^*T_p + \lambda_n^*T_n}{\rho_{\max}K}\right) - T_n\mu_2^*\right)\\
&= \frac{T_p}{mT_n}(\partial_{\lambda_n} f_p(\lambda_n^*,\lambda_p^*) - T_n\mu_2^*),
\end{aligned} \tag{35}$$

where μ_2^* is the Lagrangian multiplier with respect to the constraint $\lambda_n T_n + \lambda_p \frac{T_p}{m} \le \rho_{\max} K$. Since $s_p^* > 0$ and thus $\lambda_p^* > 0$, the first-order condition $\partial_{\lambda_p} f_p(\lambda_n^*, \lambda_p^*) - \frac{T_p}{m}\mu_2^* = 0$ when $\bar{\lambda}$ is sufficiently large. In this case, (35) implies that $\partial_{\lambda_n} f_n(\lambda_n^*, \lambda_p^*) - T_n \mu_2^* < \frac{mT_n}{T_p}\left(\partial_{\lambda_p} f_p(\lambda_n^*, \lambda_p^*) - \frac{T_p}{m}\right) = 0$. It is straightforward to check that by the KKT condition of optimization problem (34), $\partial_{\lambda_n} f_n(\lambda_n^*, \lambda_p^*) - T_n \mu_2^* < 0$ implies that $\lambda_n^* = 0$. It then follows that $s_n^* = 0$ when $\bar{\lambda}$ is sufficiently large, or, there exists a threshold λ_0, such that $s_n^* = 0$ for $\bar{\lambda} \ge \lambda_0$.

(c) s_p^* is increasing (resp. decreasing) in $\bar{\lambda}$ for $\bar{\lambda} < \lambda_0$ (resp. $\bar{\lambda} > \lambda_0$). Recall that $\lambda_0 := \min\{\bar{\lambda} : s_n^* = 0\}$. If $\bar{\lambda} < \lambda_0$, (s_n^*, s_p^*) satisfies (33). Since s_n^* is decreasing in $\bar{\lambda}$, it is straightforward to check that s_p^* is decreasing in s_n^*, thus increasing in $\bar{\lambda}$ as well. If $\bar{\lambda} > \lambda_0$, then we have $s_n^* = 0$. By Proposition 1, s_p^* is decreasing in $\bar{\lambda}$.

(d) p_n^* and p_p^* are increasing in $\bar{\lambda}$, and $p_n^* d_n - p_p^* d_p$ is increasing in $\bar{\lambda}$. Note that $p_p^* = (v_n - \Delta)(1 - s_n^* - s_p^*)/d_p$. If $\bar{\lambda} < \lambda_0$, $s_n^* > 0$ and (s_n^*, s_p^*) satisfies (33). Since s_n^* is decreasing in $\bar{\lambda}$, it is easy to check, by (33), that $s_n^* + s_p^*$ is decreasing in $\bar{\lambda}$. Thus, $p_p^* = (v_n - \Delta)(1 - s_n^* - s_p^*)/d_p$ is increasing in $\bar{\lambda}$. Furthermore, $p_n^* d_n - p_p^* d_p = (1 - s_n^*)\Delta$ is decreasing in s_n^*, thus increasing $\bar{\lambda}$. Hence, $p_n^* = (p_p^* d_p + (1 - s_n^*)\Delta)/d_n$ is also increasing in $\bar{\lambda}$. *Q.E.D.*

Proof of Proposition 5. It follows from (10) that if $\Delta = 0$, $RS_p^* = \frac{1}{2}\bar{\lambda}(s_p^*)^2$. $\tilde{RS}_n^* = RS_p^*(\bar{\Delta}) = \frac{1}{2}\bar{\lambda}(s_n^*)^2$. We now show that $s_p^*(0) > s_n^*(\bar{\Delta})$. By Theorem 1, $s_p^*(0) + s_n^*(0) > s_p^*(\bar{\Delta}) + s_n^*(\bar{\Delta})$. By Proposition 2, $s_n^*(0) = s_p^*(\bar{\Delta}) = 0$, we have $s_p^*(0) > s_n^*(\bar{\Delta})$, which implies that $RS_p^*(0) > RS_p^*(\bar{\Delta})$. The existence of $\underline{\Delta}_r$ then follows directly from $RS_p^*(\Delta)$ being continuous in Δ.

For the ease of exposition, we normalize $K = 1$, $T_n = 1$, and $v_n = 1$. We also define $\gamma = m/T_p$ and $\eta = v_n - \Delta$. Then, we have the constraints $\gamma > 1$, $\eta < 1$, and $\eta\gamma > 1$. If $G(r) = r$, we first compare $RS_p^*(\Delta)$ with $\tilde{RS}_n^*$ for $\Delta \in (\underline{\Delta}, \bar{\Delta})$. In this case, $s_n^*(\Delta) > 0$. Then, It is straightforward to calculate that

$$\begin{cases} s_n^*(\Delta) = & \frac{1}{2}\left(1 - \frac{\eta\bar{\lambda}(1/\gamma - 1)}{-\eta\bar{\lambda} + (\eta-1)\eta\rho_{\max}^2 + 2\eta\bar{\lambda}/\gamma - \lambda/\gamma^2}\right) \\ s_p^*(\Delta) = & \frac{\bar{\lambda}(1/\gamma - \eta)}{-2\eta\bar{\lambda} + 2(\eta-1)\eta\rho_{\max}^2 + 4\eta\bar{\lambda}/\gamma - 2\bar{\lambda}/\gamma^2} \\ \tilde{s}_n^* = & \frac{\rho_{\max}^2}{2(\bar{\lambda} + \rho_{\max}^2)} \end{cases}$$

Then, we can calculate the difference between the setting with carpool services and that without:

$$RS_p^*(\Delta) - \tilde{RS}_n^* = -\frac{\bar{\lambda}^2(\eta - 1/\gamma)^2(\eta(-\bar{\lambda}^2 + 2(\eta-2)\bar{\lambda}\rho_{\max}^2 + 3(\eta-1)\rho_{\max}^4) + 2\eta\bar{\lambda}(\bar{\lambda} + 2\rho_{\max}^2)/\gamma - \lambda(\lambda + 2\rho_{\max}^2)/\gamma^2)}{4(\lambda + \rho_{\max}^2)^2(\eta\lambda - (\eta-1)\eta\rho_{\max}^2 - 2\eta\bar{\lambda}/\gamma + \bar{\lambda}/\gamma^2)^2}$$

Hence, it suffices to show that

$$\eta(-\bar{\lambda}^2 + 2(\eta-2)\bar{\lambda}\rho_{\max}^2 + 3(\eta-1)\rho_{\max}^4) + 2\eta\bar{\lambda}(\bar{\lambda} + 2\rho_{\max}^2)/\gamma - \lambda(\lambda + 2\rho_{\max}^2)/\gamma^2 < 0.$$

Rearranging the terms, it suffices to show that

$$\bar{\lambda}^2(\eta - 2\eta/\gamma + 1/\gamma^2) > 0, \tag{36}$$

$$2\bar{\lambda}\rho_{\max}^2((2-\eta)\eta - 2\eta/\gamma + 1/\gamma^2) > 0, \tag{37}$$

$$3(1-\eta)\eta\rho_{\max}^4 > 0. \tag{38}$$

To show (36), observe that $\bar{\lambda}^2(\eta - 2\eta/\gamma + 1/\gamma^2) > \bar{\lambda}^2(\eta^2 - 2\eta/\gamma + 1/\gamma^2) = \bar{\lambda}^2(\eta - 1/\gamma)^2 > 0$, where the first inequality follows from $\eta < 1$ and the second from $\eta\gamma > 1$. To show (37), observe that $2\bar{\lambda}\rho_{\max}^2((2-\eta)\eta - 2\eta/\gamma + 1/\gamma^2) > 2\bar{\lambda}\rho_{\max}^2(\eta^2 - 2\eta/\gamma + 1/\gamma^2) = 2\bar{\lambda}\rho_{\max}^2(\eta - 1/\gamma)^2 > 0$, where the first inequality follows from

$(2-\eta)\eta > \eta^2$ for $\eta \in (0,1)$, and the second from $\eta > 1/\gamma$. This proves that if $\Delta \in (\underline{\Delta}, \bar{\Delta})$, $RS_p^*(\Delta) > \tilde{RS}_n^*$. Inequality (38) follows immediately from $0 < \eta < 1$. Putting everything together, we have that $RS_p^*(\Delta) > \tilde{RS}_p^*$ for $\Delta \in [\underline{\Delta}, \bar{\Delta})$.

Finally we show that for the case $\Delta \leq \underline{\Delta}$, $RS_p^*(\Delta) > \tilde{RS}_n^*$. By continuity, if $\Delta = \underline{\Delta}$, $RS_p^*(\Delta) > \tilde{RS}_n^*$. Furthermore, $s_p^*(\Delta)$ is decreasing in Δ (by Proposition 2). Therefore, $RS_p^*(\Delta) = \frac{1}{2}(v_n - \Delta)(s_p^*(\Delta))^2$ is decreasing in Δ. Hence, $RS_p^*(\Delta) > RS_p^*(\underline{\Delta})$ for all $\Delta < \underline{\Delta}$. This concludes the proof of Proposition 5. *Q.E.D.*

Proof of Proposition 6. It is clear from (11) and (12) that the driver surplus is strictly increasing in the number of active drivers k^* in equilibrium, and hence it boils down to analyzing the impact of carpool services on k^* (which is also equivalent to analyzing the impact of carpool services on the per-unit-time wage for the drivers in equilibrium, since $w^* = k^* G^{-1}(k^*/K)$ and G^{-1} is a monotonically increasing function). When $\Delta \in (\underline{\Delta}, \bar{\Delta})$, it follows from Proposition 2 that $s_n^* > 0$ and $s_p^* > 0$. Then by first order conditions $\partial_{s_n} f_p(s_n^*, s_p^*) = 0$ and $\partial_{s_p} f_p(s_n^*, s_p^*) = 0$, it is straightforward to derive that

$$\begin{cases} s_n^* = \dfrac{(\Delta^2 K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)T_p v_n - \Delta m(\bar{\lambda} T_n T_p + K m \rho_{\max}^2 v_n))}{(2\Delta m(\Delta K m \rho_{\max}^2 + \bar{\lambda} T_n(mT_n - 2T_p)) - 2(\Delta K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)^2) v_n)}, \\ s_p^* = \dfrac{\bar{\lambda} m T_n (\Delta m T_n - (mT_n - T_p) v_n)}{2\Delta m(\Delta K m \rho_{\max}^2 + \bar{\lambda} T_n(mT_n - 2T_p)) - 2(\Delta K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)^2) v_n}. \end{cases}$$

Similarly, the first order condition $\partial_{\tilde{s}_n} f_b(\tilde{s}_n^* | \bar{\lambda}) = 0$ implies that

$$\tilde{s}_n^* = \frac{K \rho_{\max}^2 v_n}{2\bar{\lambda} T_n^2 + 2K \rho_{\max}^2 v_n}.$$

Note that

$$\tilde{k}_n^* - k^* = \frac{\bar{\lambda} T_n (\tilde{s}_n^* - s_n^* - (T_p s_p^*)/(mT_n))}{\rho_{\max}}.$$

Therefore, $\tilde{k}_n^* > k^*$ is equivalent to $\tilde{s}_n^* > s_n^* + \frac{T_p s_p^*}{mT_n}$. We next compute $\tilde{s}_n^* - \left(s_n^* + \frac{T_p s_p^*}{mT_n}\right)$ as follows:

$$\begin{aligned} \tilde{s}_n^* - \left(s_n^* + \frac{T_p s_p^*}{mT_n}\right) &= \frac{K \bar{\lambda} \rho_{\max}^2 (\Delta m T_n - (mT_n - T_p) v_n)^2}{2(\bar{\lambda} T_n^2 + K \rho_{\max}^2 v_n)(-\Delta m(\Delta K m \rho_{\max}^2 + \bar{\lambda} T_n(mT_n - 2T_p)) + (\Delta K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)^2) v_n)} \\ &= \frac{K \bar{\lambda} \rho_{\max}^2 (\Delta m T_n - (mT_n - T_p) v_n)^2}{2(\bar{\lambda} T_n^2 + K \rho_{\max}^2 v_n)[\Delta(v_n - \Delta) K m^2 \rho_{\max}^2 + \bar{\lambda}((mT_n - T_p)^2 (v_n - \Delta) + T_p^2 \Delta))]} \\ &> 0, \end{aligned}$$

where the inequality follows from $v_n > \Delta \geq 0$. Therefore, $\tilde{s}_n^* > s_n^* + \frac{T_p s_p^*}{mT_n}$. It then follows that $\tilde{k}_n^* > k^*$, which implies $DS_p^* < \tilde{DS}_n^*$ in view of (11) and (12).

Finally, we show $w^* < \tilde{w}^*$. Note that $w^* = k^* G^{-1}(k^*/K)$ and $\tilde{w}^* = \tilde{k}_n^* G^{-1}(\tilde{k}_n^*/K)$. It then immediately follows from $\tilde{k}_n^* > k^*$ that $\tilde{w}^* > w^*$. *Q.E.D.*