Computing roadmaps in unbounded smooth real algebraic sets II: algorithm and complexity

Rémi Prébet
Department of Mathematics, KU Leuven, Leuven, Belgium
remi.prebet@kuleuven.be

Mohab Safey El Din
Sorbonne Université, LIP6 CNRS UMR 7606, Paris, France
Mohab.Safey@lip6.fr

Éric Schost
University of Waterloo, David Cheriton School of Computer Science, Waterloo ON, Canada
eschost@uwaterloo.ca

February 5, 2024

Abstract

A roadmap for an algebraic set V defined by polynomials with coefficients in some real field, say $\mathbb{R}$, is an algebraic curve contained in V whose intersection with all connected components of $V \cap \mathbb{R}^n$ is connected. These objects, introduced by Canny, can be used to answer connectivity queries over $V \cap \mathbb{R}^n$ provided that they are required to contain the finite set of query points $\mathcal{P} \subset V$; in this case,we say that the roadmap is associated to $(V, \mathcal{P})$.

In this paper, we make effective a connectivity result we previously proved, to design a Monte Carlo algorithm which, on input *(i)* a finite sequence of polynomials defining V (and satisfying some regularity assumptions) and *(ii)* an algebraic representation of finitely many query points $\mathcal{P}$ in V, computes a roadmap for $(V, \mathcal{P})$. This algorithm generalizes the nearly optimal one introduced by the last two authors by dropping a boundedness assumption on the real trace of V.

The output size and running times of our algorithm are both polynomial in $(nD)^{n \log d}$, where D is the maximal degree of the input equations and d is the dimension of V. As far as we know, the best previously known algorithm dealing with such sets has an output size and running time polynomial in $(nD)^{n \log^2 n}$.

1 Introduction

Let $\mathbf{Q}$ be a real field and let $\mathbf{R}$ (resp. $\mathbf{C}$) be a real (resp. algebraic) closure of $\mathbf{Q}$. One can think about $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ instead, for the sake of understanding. Further, $n \geq 0$ is an integer which stands for the dimension of the ambient space in which we compute roadmaps. In this document we deal with sets in $\mathbf{R}^n$ and $\mathbf{C}^n$ defined by polynomial equations with coefficients in $\mathbf{Q}$, that are referred to as respectively algebraic sets and real algebraic sets defined over $\mathbf{Q}$. We refer to [33, 14] and [8] for precise definition and properties of these sets. Considering sets in $\mathbf{R}^n$ defined by polynomial equations and inequalities defines the class of semi-algebraic sets; we refer to [8, 3] for a comprehensive study of these sets and their properties.

In particular, semi-algebraic and real algebraic sets can be decomposed into finitely many semi-algebraically connected components by [8, Theorem 2.4.4.]. Counting these components

[21, 35] or answering connectivity queries over these sets [32] finds many applications in e.g. robotics [9, 11, 12].

Following [9, 10], such computational issues are tackled by computing objects called *roadmaps* and introduced by Canny in [9]. In this paper, we focus on the case of real algebraic sets.

Given an algebraic set $V \subset \mathbf{C}^n$ and a finite set of query points $\mathcal{P} \subset V$, both defined over $\mathbf{Q}$, a *roadmap* $\mathcal{R}$ associated to $(V, \mathcal{P})$ is an algebraic curve which is contained in V, which contains $\mathcal{P}$, and whose intersection with each semi-algebraically connected component of $V \cap \mathbf{R}^n$ is non-empty and semi-algebraically connected.

Given a polynomial system defining V, the effective construction of roadmaps relies on connectivity statements which allow one to construct real algebraic subsets of $V \cap \mathbf{R}^n$, of smaller dimension, having a connected intersection with the connected components of $V \cap \mathbf{R}^n$. Such statements in [29, 4, 5] make the assumption that V has finitely many singular points and that $V \cap \mathbf{R}^n$ is bounded. In [27], a generalization was obtained by dropping the boundedness assumption. In this paper, we design a Monte Carlo algorithm for computing roadmaps based on this latter result, assuming regularity assumptions on the system defining V. Under those assumptions, this improves the state of the art complexity.

Prior works Canny provided the first algorithms for computing roadmaps; we call such algorithms *roadmap algorithms*. Suppose that $V \subset \mathbf{C}^n$ is defined by s polynomials of degree at most D. Canny obtained in [9, 10] a Monte Carlo roadmap algorithm using $(sD)^{O(n^2)}$ arithmetic operations in $\mathbf{Q}$. A deterministic version is also given, with a runtime $(sD)^{O(n^4)}$. This striking and important result was then reconsidered and improved in [35, 20, 22] (among others) to obtain in [2] a deterministic algorithm using $(sD)^{O(n^2)}$ field operations; this was the state-of-the-art for decades. All these algorithms are based on the same following geometric solving pattern. First a curve, defined as the critical locus of a projection on a plane, is computed; it meets all semi-algebraically connected components of the set under study. Next, connectivity failures are repaired by slicing our set with appropriate hyperplanes, performing recursive calls over these slices.

The algorithm designed in [29] is the first one to be based on a different geometric solving pattern, thanks to an innovative geometric connectivity theorem (under assumptions on the input variety, in particular boundedness). This theorem gives much more freedom in the way to construct roadmaps; in particular, it allows one to choose critical loci of higher dimension, which makes it possible to slice the input with sections of smaller dimension. This yields a better balance between the dimensions of these geometric objects, reducing the depth of the recursion. The algorithm in [29] is a first prototype of this new family of roadmap algorithms; it is Monte Carlo and runs in time $(nD)^{O(n\sqrt{n})}$; the algorithm in [5] has similar runtime, but drops all assumptions from [29]. In [4], the authors provide a deterministic algorithm running in time $(nD)^{O(n \log^2 n)}$, for an output roadmap with degree $(nD)^{O(n \log n)}$. By re-introducing some regularity and boundedness assumption, the first Monte Carlo roadmap algorithm running in time $(nD)^{O(n \log d)}$ is given in [30], where d is the dimension of the input algebraic set. Also, explicit constants in the big-O exponent are given, showing that the algorithm runs in time sub-quadratic in the degree bound of the output.

This leaves open the problem of dropping the regularity and boundedness assumptions in this last algorithm. Doing so requires a number of ingredients, starting with connectivity results which do not make use of such assumptions. In [27], we showed how to drop the boundedness assumption. In this article, we put this in practice, and design a roadmap algorithm for smooth real algebraic sets using a number of arithmetic operations which is similar to the one of [30], without the boundedness assumption.

1.1 Data representations

Before entering into a detailed description of this complexity result, we start by recalling the data representations we use to encode the input and output of the algorithm we design further.

Straight-line programs Input polynomials will be represented as *straight-line programs*, which is a flexible way of representing multivariate polynomials as a division and loop-free sequences of operations. Formally, a straight-line program Γ of length E, computing polynomials in $\mathbf{Q}[\mathbf{X}]$, with $\mathbf{X} = x_1, \dots, x_n$, is a sequence $\Gamma = (\gamma_1, \dots, \gamma_E)$ such that for all $1 \leq i \leq E$, one of the two following holds:

- $\gamma_i = \lambda_i$ with $\lambda_i \in \mathbf{Q}$;
- $\gamma_i = (\mathsf{op}_i, a_i, b_i)$ with $\mathsf{op}_i \in \{+, -, \times\}$ and $-n+1 \leq a_i, b_i < i$.

To Γ we associate polynomials $G_{-n+1}, \dots, G_E$ such that $G_i = x_{i+n}$ for $-n+1 \leq i \leq 0$, and for $i \geq 1$:

- if $\gamma_i = \lambda_i$ then $G_i = \lambda_i$;
- if $\gamma_i = (\mathsf{op}_i, a_i, b_i)$ then $G_i = G_{a_i} \, \mathsf{op}_i \, G_{b_i}$.

Then we say that Γ computes some polynomials $f_1, \dots, f_c \in \mathbf{Q}[\mathbf{X}]$ if it holds that $\{f_1, \dots, f_c\} \subset \{G_{-n+1}, \dots, G_E\}$.

Example 1. We give an illustrating example presented in [23, Section 1.1]. For $m \in \mathbb{N}^*$, a straight-line program computing $x_1^{2^m}$ in $\mathbf{Q}[x_1, x_2]$ is given by taking

$$\begin{cases} \gamma_1 & = & (\times, -1, -1) \\ \gamma_2 & = & (\times, 1, 1) \\ & \vdots & \\ \gamma_m & = & (\times, m-1, m-1) \end{cases}$$

where we associate $G_1 = x_1^2$ to γ_1, $G_2 = G_1^2 = x_1^4$ to γ_2 and so on with $G_m = G_{m-1}^2 = x_1^{2^m}$ which is associated to γ_m. Such a program has length m, while the dense and sparse representations of $x_1^{2^m}$ have respective length $2^m + 1$ and 2. But remark that a straight-line program computing $(x_1 + x_2)^{2^m}$ can be obtained by setting $\gamma_1 = (+, -1, 0)$, which computes $x_1 + x_2$, and adding $\gamma_{m+1} = (\times, m, m)$ at the end. The latter modification increments the length by one, while the sparse representation has now length $\binom{2^m}{2}$.

Because of the good behaviour of such a representation with respect to linear changes of variables, it is used as input in many algorithms for solving polynomial systems [23, 16, 18, 17, 19, 25]. It is not restrictive since any polynomial of degree D in n variables can be computed with a straight-line program of length $O(D^n)$ by simply evaluating and summing all its monomials. By convention, we note $\Gamma^0 = (0)$ the straight-line program of length 1 that computes the zero polynomial.

Zero-dimensional parametrizations A finite set of points defined over $\mathbf{Q}$ is represented using zero-dimensional parametrizations. A *zero-dimensional parametrization* $\mathscr{P}$ with coefficients in $\mathbf{Q}$ consists of:

- polynomials $(\omega, \rho_1, \dots, \rho_n)$ in $\mathbf{Q}[u]$ where u is a new variable, ω is a monic square-free polynomial and it holds that $\deg(\rho_i) < \deg(w)$,

- a linear form $\mathfrak{l}$ in variables $x_1, \ldots, x_n$,

such that

$$\mathfrak{l}(\rho_1, \ldots, \rho_n) = u \mod \omega.$$

Such a data-structure encodes the finite set of points, denoted by $\mathsf{Z}(\mathscr{P})$ defined as follows

$$\mathsf{Z}(\mathscr{P}) = \{(\rho_1(\vartheta), \ldots, \rho_n(\vartheta)) \in \mathbf{C}^n \mid \omega(\vartheta) = 0\}.$$

According to this definition, the roots of ω are exactly the values taken by $\mathfrak{l}$ on $\mathsf{Z}(\mathscr{P})$. We define the *degree* of such a parametrization $\mathscr{P}$ as the degree of the polynomial ω, which is exactly the cardinality of $\mathsf{Z}(\mathscr{P})$. By convention we note $\mathscr{P}_\emptyset = (1)$ the zero-dimensional parametrization that encodes the empty set.

Such parametrizations will be used to encode input query points and internally in our roadmap algorithm to manipulate finite sets of points.

One-dimensional parametrizations Algebraic curves defined over $\mathbf{Q}$ will be represented using one-dimensional rational parametrizations. A *one-dimensional rational parametrization* $\mathscr{R}$ with coefficients in $\mathbf{Q}$ is a couple as follows:

- polynomials $(\omega, \rho_1, \ldots, \rho_n)$ in $\mathbf{Q}[u, v]$ where u and v are new variables, ω is a monic square-free polynomial and with $\deg(\rho_i) < \deg(w)$,

- linear forms $(\mathfrak{l}, \mathfrak{l}')$ in the variables $x_1, \ldots, x_n$,

such that

$$\mathfrak{l}(\rho_1, \ldots, \rho_n) = u\frac{\partial \omega}{\partial u} \mod \omega$$

and

$$\mathfrak{l}'(\rho_1, \ldots, \rho_n) = v\frac{\partial \omega}{\partial u} \mod \omega.$$

Such a data-structure encodes the algebraic curve, denoted by $\mathsf{Z}(\mathscr{R})$, defined as the Zariski closure of the following constructible set

$$\left\{\left(\frac{\rho_1(\vartheta,\eta)}{\partial\omega/\partial u(\vartheta,\eta)}, \ldots, \frac{\rho_n(\vartheta,\eta)}{\partial\omega/\partial u(\vartheta,\eta)}\right) \in \mathbf{C}^n \mid \omega(\vartheta,\eta) = 0, \frac{\partial \omega}{\partial u}(\vartheta,\eta) \neq 0\right\}.$$

We define the *degree* $\deg(\mathscr{R})$ of such a parametrization $\mathscr{R}$ as the degree of $\mathsf{Z}(\mathscr{R})$ (that is, the maximum of the cardinalities of the finite sets obtained by intersecting $\mathsf{Z}(\mathscr{R})$ with a hyperplane). Any algebraic curve C can be described as $C = \mathsf{Z}(\mathscr{R})$, for some one-dimensional rational parametrization that satisfies $\deg(\omega) = \deg(\mathscr{R})$ (that is, the degree of C); guaranteeing this is the reason why we use rational functions with the specific denominator $\partial\omega/\partial u$. This will always be our choice in the sequel; in this case, storing a one-dimensional parametrization $\mathscr{R}$ of degree δ involves $O(n\delta^2)$ coefficients.

Our algorithm computes a roadmap R of an algebraic set V, that has, by definition, dimension one. The output is given by means of a one-dimensional parametrization of R.

1.2 Contributions

Recall that an algebraic set V can be uniquely decomposed into finitely many *irreducible components*. When all these components have the same dimension d, we say that V is *d-equidimensional*. For a set of polynomials $\boldsymbol{f} \subset \mathbf{C}[\mathbf{X}]$, with $\mathbf{X} = x_1, \ldots, x_n$, we denote by $\boldsymbol{V}(\boldsymbol{f}) \subset \mathbf{C}^n$ the vanishing locus of $\boldsymbol{f}$, and by $\mathcal{O}(\boldsymbol{f})$ its complement.

Given an algebraic set $V \subset \mathbf{C}^n$, we denote by $\boldsymbol{I}(V)$ the *ideal of* V, that is the ideal of $\mathbf{C}[\mathbf{X}]$ of polynomials vanishing on V. For $\boldsymbol{y} \in \mathbf{C}^n$, we denote by $\mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{f})$ the Jacobian matrix of the polynomials $\boldsymbol{f}$ evaluated at $\boldsymbol{y}$. For V d-equidimensional in $\mathbf{C}^n$, those points $\boldsymbol{y} \in V$ at which the Jacobian matrix of a finite set of generators of $\boldsymbol{I}(V)$ has rank $n-d$ are called *regular* points and the set of those points is denoted by $\mathrm{reg}(V)$. The others are called *singular* points; the set of singular points of V (its singular locus) is denoted by $\mathrm{sing}(V)$ and is an algebraic subset of V.

We say that $(f_1, \ldots, f_p) \subset \mathbf{Q}[\mathbf{X}]$ is a *reduced regular sequence* if the following holds for every $i \in \{1, \ldots, p\}$:

- the algebraic set $\boldsymbol{V}(f_1, \ldots, f_i) \subset \mathbf{C}^n$ is either equidimensional of dimension $n-i$ or empty,
- the ideal $\langle f_1, \ldots, f_i \rangle$ is radical.

In the following main result, and in all this work, we design, and also use known algorithms, whose success relies on the successive choice of several parameters $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots$ in affine spaces $\mathbf{Q}^{a_1}, \mathbf{Q}^{a_2}, \ldots$ These algorithms are probabilistic in the sense that, in any such case, there exist non-zero polynomials $\Delta_1, \Delta_2, \ldots$, such that the algorithm is successful if $\Delta_i(\boldsymbol{\lambda}_i) \neq 0$ for all i. These algorithms are *Monte Carlo*, as we cannot always guarantee correctness of the output with reasonable complexity. Nevertheless, in cases when we can detect failure, our procedures will output fail (though not returning fail does not guarantee correctness).

Our main result is the following one. Below, the soft-O notation $O\tilde{~}(g)$ denotes the class $O(g \log_2(g)^a)$ for some constant $a > 0$, where $\log_2$ is the binary logarithm function.

Theorem 1.1. *Let $\boldsymbol{f} = (f_1, \ldots, f_p) \subset \mathbf{Q}[x_1, \ldots, x_n]$ be a reduced regular sequence, let D be bounding the degrees of the f_i's and suppose that Γ is a straight-line program of length E evaluating $\boldsymbol{f}$. Assume that $\boldsymbol{V}(\boldsymbol{f}) \subset \mathbf{C}^n$ has finitely many singular points.*

Let $\mathscr{P}$ be a zero-dimensional parametrization of degree μ with $\mathsf{Z}(\mathscr{P}) \subset V(\boldsymbol{f})$. There exists a Monte Carlo algorithm which, on input Γ and $\mathscr{P}$ computes a one-dimensional parametrization $\mathscr{R}$ of a roadmap of $(\boldsymbol{V}(\boldsymbol{f}), \mathsf{Z}(\mathscr{P}))$ of degree

$$\mathscr{B} = \mu\, n^{4d\log_2(d)+O(d)}\ D^{3n\log_2(d)+O(n)} = \mu(nD)^{O(n\log_2(d))},$$

where $d = n - p$, using $\underline{E\mathscr{B}^3 \text{ arithmetic operations in } \mathbf{Q}}$.

Hence, we dropped the boundedness assumption on $\boldsymbol{V}(\boldsymbol{f}) \cap \mathbf{R}^n$ made in [30, Theorem 1.1], still keeping a complexity similar to the algorithm presented in [30]. Note that the arithmetic complexity statement above is cubic in the *degree* bound $\mathscr{B}$ on the output; the output *size* itself is $O\left(n\mathscr{B}^2\right)$ elements in $\mathbf{Q}$. Hence, as in [30], our runtime is sub-quadratic in the bound on the output size. Note that a complexity bound with a comprehensive exponent, that is without using big Oh notation, is given below in Theorem 3.8, of which the above main result is a direct consequence.

Our algorithm works as follows:

- it starts by considering the critical locus W associated to some special polynomial map $\boldsymbol{\varphi}$ with image in $\mathbf{R}^2$;
- next, it uses a slight variant of the algorithm in [30], to deal with some slices $V(\boldsymbol{f}) \cap \boldsymbol{\varphi}^{-1}(v_i)$ for some $v_1, \ldots, v_\ell$ in $\mathbf{R}$.

We will see that the map $\boldsymbol{\varphi}$ depends on some parameters in $\mathbf{C}^N$, for some $N \geq 0$, and that there exists a non-zero polynomial G such that when choosing these parameters in the open set $\mathcal{O}(G)$, one can apply the connectivity result in [27], which does not need any boundedness assumption. This is where first elements of randomization are needed. A second element comes from the use of a variant of [30], which is also a Monte Carlo algorithm.

2 Preliminaries

2.1 Minors, rank and submatrices

We present here some technical results on the minors and the rank of a certain class of matrices that will occur in this paper, when dealing with particular cases and incidence varieties in Section 4.

Lemma 2.1. *Let $q \geq 1$ and $1 \leq c \leq p$ be integers. Let A, B, C be respectively $c \times p$, $c \times q$ and $q \times p$ matrices with coefficients in a commutative ring R such that M_1 and M_2 are the following $(c+q) \times (q+p)$ matrices:*

$$M_1 = \begin{bmatrix} B & A \\ I_q & \mathbf{0} \end{bmatrix} \qquad \text{and} \qquad M_2 = \begin{bmatrix} \mathbf{0} & A \\ I_q & C \end{bmatrix},$$

where I_q is the identity matrix of size q. Let $m \in R$ and e be in $\{0, \ldots, c\}$; then, for $k = 1, 2$ the following conditions are equivalent:

1. *m is the determinant of a $(q+e)$-submatrix of M_k that contains I_q;*
2. *$(-1)^{qe} m$ is an e-minor of A.*

In this case, if $1 \leq i_1 \leq \cdots \leq i_e \leq c$ and $1 \leq j_1, \ldots, j_e \leq p$ are the indices of respectively the rows and the columns of A selected in item 2, then the corresponding rows and columns in M_k are of respective indices

$$1 \leq i_1 \leq \cdots \leq i_e \leq c+1 \leq \cdots \leq c+q \quad \text{and} \quad 1 \leq \cdots \leq q \leq q+j_1 \leq \cdots \leq q+j_e.$$

Proof. The determinant of any submatrix of M_i containing I_q can be reduced, up to the sign $(-1)^{qe}$, to a minor of A by using the cofactor expansion with respect to the last q rows of M_1 (resp. the first q columns of M_2). Conversely, any e-minor of A is a $(q+e)$-minor of M_k, by extending the associated submatrix of A to a submatrix of M_k containing I_q. The correspondence between indices stated above is then straightforward. □

Lemma 2.2. *With the notation of Lemma 2.1, if R is a field, then* $\operatorname{rank}(M_k) = \operatorname{rank}(A) + q \geq q$ *for $k = 1, 2$.*

Proof. For $k = 1$, performing row operations allows us to replace B by the zero matrix, after which the claim becomes evident. For $k = 2$, use column operations. □

2.2 Polynomial maps, generalized polar varieties and fibers

Let $Z \subset \mathbf{C}^n$ be an equidimensional algebraic set and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_m)$ be a finite set of polynomials of $\mathbf{C}[\mathbf{X}]$; we still denote by $\boldsymbol{\varphi}\colon Z \to \mathbf{C}^m$ the restriction of the polynomial map induced by $\boldsymbol{\varphi}$ to Z. For $\boldsymbol{y} \in \operatorname{reg}(Z)$, we say that $\boldsymbol{y} \in Z$ is a *critical point* of $\boldsymbol{\varphi}$ if

$$d_{\boldsymbol{y}}\boldsymbol{\varphi}(T_{\boldsymbol{y}}Z) \neq \mathbf{C}^m,$$

where $d_{\boldsymbol{y}}\boldsymbol{\varphi}$ is the differential of $\boldsymbol{\varphi}$ at $\boldsymbol{y}$. We will denote by $W^\circ(\boldsymbol{\varphi}, Z)$ the set of the critical points of $\boldsymbol{\varphi}$ on Z, and by $W(\boldsymbol{\varphi}, Z)$ its Zariski closure. A *critical value* is the image of a critical point by $\boldsymbol{\varphi}$.

Besides, we let $K(\boldsymbol{\varphi}, Z) = W^\circ(\boldsymbol{\varphi}, Z) \bigcup \operatorname{sing}(Z)$ be the set of *singular points* of $\boldsymbol{\varphi}$ on Z. When Z is defined by a reduced regular sequence $\boldsymbol{f} = (f_1, \ldots, f_c)$, $K(\boldsymbol{\varphi}, Z)$ is then defined as the intersection of Z with the set of points of $\mathbf{C}^n$ where the Jacobian matrix of $(\boldsymbol{f}, \varphi)$ has rank at most $c+m-1$ (see [30, Lemma A.2]).

For $1 \leq i \leq m$, we set

$$\begin{array}{rccc} \boldsymbol{\varphi}_i\colon & \mathbf{C}^n & \to & \mathbf{C}^i \\ & \boldsymbol{y} & \mapsto & (\varphi_1(\boldsymbol{y}), \dots, \varphi_i(\boldsymbol{y})). \end{array}$$

Given the maps $(\boldsymbol{\varphi}_i)_{1\leq i\leq m}$, we denote $W^\circ(\boldsymbol{\varphi}_i, Z)$, $W(\boldsymbol{\varphi}_i, Z)$ and $K(\boldsymbol{\varphi}_i, Z)$ by respectively $W^\circ_{\boldsymbol{\varphi}}(i, Z)$, $W_{\boldsymbol{\varphi}}(i, Z)$ and $K_{\boldsymbol{\varphi}}(i, Z)$. For $i = 0$, we let $\mathbf{C}^0$ be a singleton of the form $\mathbf{C}^0 = \{\bullet\}$, and $\boldsymbol{\varphi}_0\colon \boldsymbol{y} \in \mathbf{C}^n \to \bullet \in \mathbf{C}^0$ be the unique possible map. Then for all $\boldsymbol{y} \in \mathbf{C}^0$, $\boldsymbol{\varphi}_0^{-1}(\boldsymbol{y}) = \mathbf{C}^n$; we set $W^\circ_{\boldsymbol{\varphi}}(0, Z) = W_{\boldsymbol{\varphi}}(0, Z) = \emptyset$. The sets $W_{\boldsymbol{\varphi}}(i, Z)$, for $0 \leq i \leq m$ are called the *generalized polar varieties associated to φ on Z*.

The main result we state in this subsection is the following (the somewhat lengthy proof is in Section 5). It establishes some genericity properties of generalized polar varieties associated to a class of polynomial maps. It is a generalization of [28, Theorem 1], which only deals with projections.

Proposition 2.3. *Let $V \subset \mathbf{C}^n$ be a d-equidimensional algebraic set with finitely many singular points and θ be in $\mathbf{C}[\mathbf{X}]$. Let $2 \leq \mathfrak{r} \leq d+1$. For $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{\mathfrak{r}})$ in $\mathbf{C}^{\mathfrak{r}n}$, we define $\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1), \dots, \varphi_{\mathfrak{r}}(\mathbf{X}, \boldsymbol{\alpha}_{\mathfrak{r}}))$, where for $2 \leq j \leq \mathfrak{r}$*

$$\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1) = \theta(\mathbf{X}) + \sum_{k=1}^{n} \alpha_{1,k} x_k \quad \textit{and} \quad \varphi_j(\mathbf{X}, \boldsymbol{\alpha}_j) = \sum_{k=1}^{n} \alpha_{j,k} x_k.$$

Then, there exists a non-empty Zariski open subset $\Omega_{\mathsf{l}}(V, \theta, \mathfrak{r}) \subset \mathbf{C}^{\mathfrak{r}n}$ such that for every $\boldsymbol{\alpha} \in \Omega_{\mathsf{l}}(V, \theta, \mathfrak{r})$ and $i \in \{1, \dots, \mathfrak{r}\}$, the following holds:

1. *either $W_{\boldsymbol{\varphi}}(i, V)$ is empty or $(i-1)$-equidimensional;*
2. *the restriction of $\boldsymbol{\varphi}_{i-1}$ to $W_{\boldsymbol{\varphi}}(i, V)$ is a Zariski-closed map;*
3. *for any $\boldsymbol{z} \in \mathbf{C}^{i-1}$, the fiber $K_{\boldsymbol{\varphi}}(i, V) \cap \boldsymbol{\varphi}_{i-1}^{-1}(\boldsymbol{z})$ is finite.*

The connectivity result [27, Theorem 1.1] makes use of generalized polar varieties satisfying these properties, but also of fibers of polynomial maps.

Remark 1. Let $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_{\mathfrak{r}})$ be polynomials in $\mathbf{C}[\mathbf{X}]$ and an integer $1 \leq e \leq \mathfrak{r}$. Given an algebraic set $V \subset \mathbf{C}^n$ and a set $Q \subset \mathbf{C}^e$, the *fiber of V over Q with respect to $\boldsymbol{\varphi}$* is the set $V_{|\boldsymbol{\varphi}_e \in Q} = V \cap \boldsymbol{\varphi}_e^{-1}(Q)$. We say that V *lies over Q with respect to $\boldsymbol{\varphi}$* if $\boldsymbol{\varphi}_e(V) \subset Q$. Finally, for $\boldsymbol{z} \in \mathbf{C}^e$, the set $V_{|\boldsymbol{\varphi}_e \in \{\boldsymbol{z}\}}$ will be denoted by $V_{|\boldsymbol{\varphi}_e = \boldsymbol{z}}$.

2.3 Charts and atlases of algebraic sets

We say that an algebraic set is *complete intersection* if it can be defined by a number of equations equal to its codimension. Not all algebraic sets are complete intersections; for instance determinantal varieties and, consequently, a whole class of generalized polar varieties, are a prototype of non complete intersections. This creates complications to control the complexity of algorithms manipulating generalized polar varieties recursively.

However, we may use local representations which describe Zariski open subsets of an algebraic set with a number of equations equal to its codimension.

Such local representations are obtained by considering *locally closed sets*. We say that a subset V° of $\mathbf{C}^n$ is locally closed if there exist an open $\mathcal{O}$ and a closed Zariski subset Z of $\mathbf{C}^n$ such that $V^\circ = Z \cap \mathcal{O}$. In that case, the dimension of V° is the dimension of its Zariski closure V, and V° is said to be equidimensional if V is. In this situation, we define $\mathrm{reg}(V^\circ) = \mathrm{reg}(V) \cap V^\circ$ and $\mathrm{sing}(V^\circ) = \mathrm{sing}(V) \cap V^\circ$, and V° is said to be non-singular if $\mathrm{reg}(V^\circ) = V^\circ$. For $\boldsymbol{f} = (f_1, \ldots, f_p) \subset \mathbf{C}[\mathbf{X}]$ with $p \leq n$, we define the locally closed set $\boldsymbol{V}^\circ_{\mathrm{reg}}(\boldsymbol{f})$ as the set of all $\boldsymbol{y}$ where the Jacobian matrix $\mathrm{Jac}(\boldsymbol{f})$ of $\boldsymbol{f}$ has full rank p. We will denote by $\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{f})$ the Zariski closure of $\boldsymbol{V}^\circ_{\mathrm{reg}}(\boldsymbol{f})$.

A *chart* associated to an algebraic set $V \subset \mathbf{C}^n$ can be seen as a local representation of V by another locally closed subset of V that is smooth and in complete intersection. We recall hereafter the definitions introduced in [30, Section 2.5], which we slightly generalize. Below, for a polynomial m in $\mathbf{C}[\mathbf{X}]$, recall that we write $\mathcal{O}(m) = \mathbf{C}^n - \boldsymbol{V}(m)$.

Definition 2.4 (Charts of algebraic sets). *Let $1 \leq e \leq \mathfrak{r} \leq n+1$ be integers and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_\mathfrak{r}) \subset \mathbf{C}[\mathbf{X}]$. Let $Q \subset \mathbf{C}^e$ be a* finite *set and $V, S \subset \mathbf{C}^n$ be algebraic sets lying over Q with respect to $\boldsymbol{\varphi}$. We say that a pair of the form $\chi = (m, \boldsymbol{h})$ with m and $\boldsymbol{h} = (h_1 \ldots, h_c)$ in $\mathbf{C}[\mathbf{X}]$ is a* chart *of $(V, Q, S, \boldsymbol{\varphi})$ if the following holds:*

(C_1) *$\mathcal{O}(m) \cap V - S$ is non-empty;*

(C_2) *$\mathcal{O}(m) \cap V - S = \mathcal{O}(m) \cap \boldsymbol{V}(\boldsymbol{h})_{|\boldsymbol{\varphi}_e \in Q} - S$;*

(C_3) *$e + c \leq n$;*

(C_4) *for all $\boldsymbol{y} \in \mathcal{O}(m) \cap V - S$, $\mathrm{Jac}_{\boldsymbol{y}}([\boldsymbol{h}, \boldsymbol{\varphi}_e])$ has full rank $c + e$.*

When $\boldsymbol{\varphi} = (x_1, \ldots, x_\mathfrak{r})$ defines the canonical projection, one will simply refer to χ as a chart of (V, Q, S), and if $e = 0$ as a chart of (V, S) (no matter what $\boldsymbol{\varphi}$ is).

The first condition C_1 ensures that χ is not trivial, and the following ones ensure that χ is a smooth representation of $V - S$ in complete intersection (for V equidimensional, S contains the singular points of V). This is a generalization of [30, Definition 2.2] in the sense that, if $\boldsymbol{\varphi} = (x_1, \ldots, x_n)$, one recovers the same definition.

Lemma 2.5. *Let $1 \leq \mathfrak{r} \leq n+1$ be integers and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_\mathfrak{r}) \subset \mathbf{C}[\mathbf{X}]$. Let $V, S \subset \mathbf{C}^n$ be two algebraic sets with V d-equidimensional and let $\chi = (m, \boldsymbol{h})$, with $\boldsymbol{h} = (h_1, \ldots, h_c)$, be a chart of (V, S). Then, for $1 \leq i \leq \mathfrak{r}$ and $\boldsymbol{y} \in \mathcal{O}(m) \cap V - S$, $\boldsymbol{y}$ lies in $W_{\boldsymbol{\varphi}}(i, V)$ if and only if $\mathrm{Jac}_{\boldsymbol{y}}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ does not have full rank $c + i$.*

Proof. Let $\boldsymbol{y} \in \mathcal{O}(m) \cap V - S$. By [30, Lemma A.8], $\boldsymbol{y} \in \mathrm{reg}(V)$, so that $\boldsymbol{y}$ lies in $W_{\boldsymbol{\varphi}}(i, V)$ if and only if it lies in $W^\circ_{\boldsymbol{\varphi}}(i, V)$. Besides, by [30, Lemma A.7], $T_{\boldsymbol{y}}V$ coincide with $\ker \mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{h})$. Hence, by definition $\boldsymbol{y}$ lies in $W_{\boldsymbol{\varphi}}(i, V)$ if and only if $d_{\boldsymbol{y}}\boldsymbol{\varphi}_i(\ker \mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{h})) \neq \mathbf{C}^i$. But the latter, is equivalent to saying that the matrix $\mathrm{Jac}_{\boldsymbol{y}}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ does not have full rank $c + i$. □

A straightforward rewriting of Lemma 2.5 is the following which provides a local description of a polar variety by means of a critical locus on a variety defined by a complete intersection.

Lemma 2.6. *Reusing the notation of Lemma 2.5, it holds that the sets $W_{\boldsymbol{\varphi}}(i, V)$ and $W^\circ_{\boldsymbol{\varphi}}(i, \boldsymbol{V}_{reg}(\boldsymbol{h}))$ coincide in $\mathcal{O}(m) - S$.*

Together with the notion of charts, we define atlases as a collection of charts that cover the whole algebraic set we consider.

Definition 2.7 (Atlases of algebraic sets). *Let $1 \leq e \leq \mathfrak{r} \leq n+1$ be integers and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_\mathfrak{r}) \subset \mathbf{C}[\mathbf{X}]$. Let $Q \subset \mathbf{C}^e$ be a* finite *set and $V, S \subset \mathbf{C}^n$ be algebraic sets lying over Q with respect to $\boldsymbol{\varphi}$. Let $\boldsymbol{\chi} = (\chi_j)_{1 \leq j \leq s}$ with $\chi_j = (m_j, \boldsymbol{h}_j)$ for all j. We say that $\boldsymbol{\chi}$ is an* atlas *of $(V, Q, S, \boldsymbol{\varphi})$ if the following holds:*

(A_1) *$s \geq 1$;*

(A_2) *for each $1 \leq j \leq s$, χ_i is a chart of $(V, Q, S, \boldsymbol{\varphi})$;*

(A_3) *$V - S \subset \bigcup_{1 \leq j \leq s} \mathcal{O}(m_j)$.*

When $\boldsymbol{\varphi} = (x_1, \ldots, x_{\mathfrak{r}})$ is defines the canonical projection, one simply refers to $\boldsymbol{\chi}$ as an atlas of (V, Q, S), and if $e = 0$ as an atlas of (V, S).

Here the definition is the same as [30, Definition 2.3]. Note that, according to [30, Lemma A.13], there exists an atlas of $(V, \operatorname{sing}(V))$ for any equidimensional algebraic set V.

2.4 Charts and atlases for generalized polar varieties

We deal now with the geometry of generalized polar varieties (under genericity assumptions) and show how to define charts and atlases for them. In this whole subsection, we let $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{n+1}) \subset \mathbf{C}[\mathbf{X}]$; for $1 \leq i \leq n$, we denote by $\boldsymbol{\varphi}_i$ the sequence $(\varphi_1, \ldots, \varphi_i)$ and, by a slight abuse of notation, the polynomial map it defines.

Definition 2.8. *Let $\boldsymbol{h} = (h_1, \ldots, h_c) \subset \mathbf{C}[\mathbf{X}]$ with $1 \leq c \leq n$ and let $i \in \{1, \ldots, n - c + 1\}$. Let m'' be a $(c + i - 1)$-minor of $\operatorname{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ containing the rows of $\operatorname{Jac}(\boldsymbol{\varphi}_i)$. We denote by $\mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')$ the sequence of $(c + i)$-minors of $\operatorname{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ obtained by successively adding the missing row and a missing column of $\operatorname{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ to m''. This sequence has length $n - c - i + 1$.*

Then, given a chart $\chi = (m, \boldsymbol{h})$ of some algebraic set V, we can define a candidate for being a chart of generalized polar varieties associated to $\boldsymbol{\varphi}_i$ and $V(\boldsymbol{h}) \cap \mathcal{O}(m)$.

Definition 2.9. *Let $V, S \subset \mathbf{C}^n$ be two algebraic sets, $\chi = (m, \boldsymbol{h})$ be a chart of (V, S), with $\boldsymbol{h}$ of length c and $i \in \{1, \ldots, n - c + 1\}$. For every c-minor m' of $\operatorname{Jac}(\boldsymbol{h})$ and every $(c + i - 1)$-minor m'' of $\operatorname{Jac}(\boldsymbol{h}, \boldsymbol{\varphi}_i)$ containing the rows of $\operatorname{Jac}(\boldsymbol{\varphi}_i)$, we define $W_{\mathrm{chart}}(\chi, m', m'', \boldsymbol{\varphi}_i)$ as the couple:*

$$W_{\mathrm{chart}}(\chi, m', m'', \boldsymbol{\varphi}_i) = \Big(mm'm'',\ \big(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')\big) \Big)$$

Then, the definition of the associated atlas comes naturally. Let $V, S \subset \mathbf{C}^n$ be two algebraic sets with V d-equidimensional, $\boldsymbol{\chi} = (\chi_j)_{1 \leq j \leq s}$ be an atlas of (V, S) (with $\chi_j = (m_j, \boldsymbol{h}_j)$) and $i \in \{1, \ldots, d + 1\}$. Since V is d-equidimensional, by [30, Lemma A.12], all the sequences of polynomials $\boldsymbol{h}_j$ have same cardinality $c = n - d$.

Definition 2.10. *We define $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, i)$ as the sequence of all charts $W_{\mathrm{chart}}(\chi_j, m', m'', \boldsymbol{\varphi}_i)$ for every $j \in \{1, \ldots, s\}$, every c-minor m' of $\operatorname{Jac}(\boldsymbol{h}_j)$ and every $(c{+}i{-}1)$-minor m'' of $\operatorname{Jac}(\boldsymbol{h}_j, \boldsymbol{\varphi}_i)$ containing the rows of $\operatorname{Jac}(\boldsymbol{\varphi}_i)$, for which $\mathcal{O}(m_j m' m'') \cap W_{\boldsymbol{\varphi}}(i, V) - S$ is not empty.*

These constructions generalize the ones introduced in [30, Section 3.1] in the following sense: for $\boldsymbol{\varphi} = (x_1, \ldots, x_{n+1})$, except for some trivial cases, the objects we just defined match the ones in [30, Definition 3.1 to 3.3], possibly up to signs (which have no consequence). The next lemma makes this more precise; in this lemma, we write $\boldsymbol{\pi} = (x_1, \ldots, x_{n+1})$ and $\boldsymbol{\pi}_i = (x_1, \ldots, x_i)$.

Lemma 2.11. *Let V, S be algebraic sets and $\boldsymbol{h} = (h_1, \ldots, h_c) \subset \mathbf{C}[\mathbf{X}]$. Let $1 \leq i \leq n - c + 1$ and m'' be a $(c + i - 1)$-minor of $\operatorname{Jac}(\boldsymbol{h}, \boldsymbol{\pi}_i)$, containing the rows of $\operatorname{Jac}(\boldsymbol{\pi}_i)$. Then either $m'' = 0$ or*

1. *$\mu'' = (-1)^{i(c-1)} m''$ is a $(c - 1)$-minor of $\operatorname{Jac}(\boldsymbol{h}, i)$;*

2. $\mathcal{H}_{\boldsymbol{\pi}}(\boldsymbol{h}, i, m'') = (-1)^{ic} H$, *where* H *is the* $(n-c-i+1)$*-sequence of* c*-minors of* $\mathrm{Jac}(\boldsymbol{h}, i)$ *obtained by successively adding the missing row and the missing columns of* $\mathrm{Jac}(\boldsymbol{h}, i)$ *to* μ''*;*

3. *if* $\chi = (\boldsymbol{h}, m)$ *is a chart of* (V, S)*, then for every* c*-minor* m' *of* $\mathrm{Jac}(\boldsymbol{h})$*,*

$$W_{\mathrm{chart}}(\chi, m', m'') = W_{\mathrm{chart}}(\chi, m', (-1)^{i(c-1)}\mu'')$$

which is $\big(mm'm'', (\boldsymbol{h}, (-1)^{ic}H)\big)$ *with* H *as above.*

Assume, in addition, that V *is* d*-equidimensional, with* $d = n - c$*. Let* $\boldsymbol{\chi} = (\chi_j)_{1 \le j \le s}$ *be an atlas of* (V, S)*, with* $\chi_j = (\boldsymbol{h}_j, m_j)$*, and let* c *be the common cardinality of the* $\boldsymbol{h}_j$*'s. Then*

4. $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\pi}, i)$ *is the sequence of all* $W_{\mathrm{chart}}(\chi_j, m', (-1)^{i(c-1)}\mu'')$*, for* $j \in \{1, \ldots, s\}$ *and for* m', μ'' *respectively a* c*-minor of* $\mathrm{Jac}(\boldsymbol{h}_j)$ *and a* $(c-1)$*-minor of* $\mathrm{Jac}(\boldsymbol{h}_j, i)$ *for which* $\mathcal{O}(m_j m' \mu'') \cap W(\boldsymbol{\pi}_i, V) - S$ *is not empty.*

Proof. According to Lemma 2.1, up to the sign $(-1)^{i(c-1)}$, the $(c-1)$-minors of $\mathrm{Jac}(\boldsymbol{h}, i)$ are exactly the $(i+c-1)$-minors of $\mathrm{Jac}(\boldsymbol{h}, \boldsymbol{\pi}_i)$ containing the identity matrix $I_i = \mathrm{Jac}(\boldsymbol{\pi}_i)$, since

$$\mathrm{Jac}_{x_1,\ldots,x_n}(\boldsymbol{h}, \boldsymbol{\pi}_i) = \begin{bmatrix} \mathrm{Jac}_{x_1,\ldots,x_i}(\boldsymbol{h}) & \mathrm{Jac}_{x_{i+1},\ldots,x_n}(\boldsymbol{h}) \\ I_i & \mathbf{0} \end{bmatrix}.$$

Since m'' contains the rows of $\mathrm{Jac}(\boldsymbol{\pi}_i) = [I_i \;\; \mathbf{0}]$, either it actually contains I_i or it is zero, as a zero row appears. We assume the first case; then, by the discussion above, $\mu'' = (-1)^{i(c-1)} m''$ is the determinant of a $(c-1)$-submatrix M of $\mathrm{Jac}(\boldsymbol{h}, i) = \mathrm{Jac}_{x_{i+1},\ldots,x_n}(\boldsymbol{h})$.

The row and columns of $\mathrm{Jac}(\boldsymbol{h}, i)$ that are not in M have respective indices $1 \le k \le c$ and $1 \le \ell_1 \le \ldots \le \ell_{n-i-c+1} \le n$. Since m'' contains I_i, the rows and columns of $\mathrm{Jac}(\boldsymbol{h}, \boldsymbol{\pi}_i)$ that are not in m'' have respective indices $1 \le k' \le c$ and $i+1 \le \ell'_1 \le \ldots \le \ell'_{n-c-i+1} \le n$. Then, according to Lemma 2.1, for all $1 \le j \le n-c-i+1$,

$$k = k' \quad \text{and} \quad \ell_j = \ell'_j - i.$$

Hence, by Lemma 2.1, the $(c+i)$-minors obtained by adding the missing row and the missing columns of $\mathrm{Jac}(\boldsymbol{h}, \boldsymbol{\pi}_i)$ to the submatrix used to define m'' are exactly the c-minors of $\mathrm{Jac}(\boldsymbol{h}, i)$ obtained by adding the missing row and the missing columns of $\mathrm{Jac}(\boldsymbol{h}, i)$ to μ'', up to a factor $(-1)^{ic}$. This gives the second statement. The third statement is then nothing but the definition of $W_{\mathrm{chart}}(\chi, m, m'')$.

Finally, consider an atlas $\boldsymbol{\chi}$ of (V, S). By Lemma 2.1, for $j \in \{1, \ldots, s\}$, all $(c-1)$-minors μ'' of $\mathrm{Jac}(\boldsymbol{h}_j, i)$ are, up to sign, $(c+i-1)$-minors of $\mathrm{Jac}(\boldsymbol{h}_j, \boldsymbol{\pi}_i)$ built with the rows of $\mathrm{Jac}(\boldsymbol{\pi}_i)$. Conversely, let $j \in \{1, \ldots, s\}$, m' be a c-minor of $\mathrm{Jac}(\boldsymbol{h}_j)$ and let m'' be a $(c+i-1)$-minor of $\mathrm{Jac}(\boldsymbol{h}_j, \boldsymbol{\pi}_i)$ containing the rows of $\mathrm{Jac}(\boldsymbol{\pi}_i)$. Then either $m'' = 0$, so that $\mathcal{O}(m'')$ and then $\mathcal{O}(m_j m' m'') \cap W(\boldsymbol{\pi}_i, V) - S$ is empty, or $\mu'' = (-1)^{i(c-1)} m''$ is a $(c-1)$-minor of $\mathrm{Jac}(\boldsymbol{h}_j, i)$. Hence, according to the third item, for $j \in \{1, \ldots, s\}$ and any c-minor m' of $\mathrm{Jac}(\boldsymbol{h}_j)$, the sequences of

- all $W_{\mathrm{chart}}(\chi_j, m', m'')$ for every $(c+i-1)$-minor m'' of $\mathrm{Jac}(\boldsymbol{h}, \boldsymbol{\pi}_i)$ containing the rows of $\mathrm{Jac}(\boldsymbol{\pi}_i)$, for which $\mathcal{O}(m_j m' m'') \cap W(\boldsymbol{\pi}_i, V) - S$ is not empty, and

- all $W_{\mathrm{chart}}(\chi_j, m', (-1)^{i(c-1)} m'')$ for every $(c-1)$-minor μ' of $\mathrm{Jac}(\boldsymbol{h}_j, i)$ for which $\mathcal{O}(m_j m' m'') \cap W(\boldsymbol{\pi}_i, V) - S$ is not empty,

are equal to $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\pi}, i)$. $\square$

We can now state the main result of this subsection, which we prove in Section 6. This is, a generalization of [30, Proposition 3.4] which only deals with the case of projections.

Proposition 2.12. *Let $V, S \subset \mathbf{C}^n$ be two algebraic sets with V d-equidimensional and S finite, and let $\boldsymbol{\chi}$ be an atlas of (V, S). For $2 \le \mathfrak{r} \le d+1$, let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{\mathfrak{r}})$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{\mathfrak{r}})$, and for $1 \le j \le \mathfrak{r}$, let $\boldsymbol{\alpha}_j = (\alpha_{j,1}, \ldots, \alpha_{j,n}) \in \mathbf{C}^n$ and*

$$\varphi_j(\mathbf{X}, \boldsymbol{\alpha}_j) = \theta_j(\mathbf{X}) + \sum_{k=1}^{n} \alpha_{j,k} x_k + \xi_j(\boldsymbol{\alpha}_j) \in \mathbf{C}[\mathbf{X}].$$

where $\theta_j \in \mathbf{C}[\mathbf{X}]$ and $\xi_j : \mathbf{C}^n \to \mathbf{C}$ is a polynomial map, with coefficients in $\mathbf{C}$.

There exists a non-empty Zariski open subset $\Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, S, \boldsymbol{\theta}, \boldsymbol{\xi}, \mathfrak{r}) \subset \mathbf{C}^{\mathfrak{r}n}$ such that for every $\boldsymbol{\alpha} \in \Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, S, \boldsymbol{\theta}, \boldsymbol{\xi}, \mathfrak{r})$, writing $\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}), \ldots, \varphi_{\mathfrak{r}}(\mathbf{X}, \boldsymbol{\alpha}))$, the following holds. For i in $\{1, \ldots, \mathfrak{r}\}$, either $W_{\boldsymbol{\varphi}}(i, V)$ is empty or

1. *$W_{\boldsymbol{\varphi}}(i, V)$ is an equidimensional algebraic set of dimension $i-1$;*
2. *if $2 \le i \le (d+3)/2$, then $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, i)$ is an atlas of $(W_{\boldsymbol{\varphi}}(i, V), S)$ and $\mathrm{sing}(W_{\boldsymbol{\varphi}}(i, V)) \subset S$.*

We end this subsection with a statement we use further for the proof of our main algorithm; it addresses the special case $i = 2$.

Proposition 2.13. *Let $V \subset \mathbf{C}^n$ be a d-equidimensional algebraic set with $d \ge 1$ and $\mathrm{sing}(V)$ finite. Let $\theta \in \mathbf{C}[\mathbf{X}]$, and for $i \in \{1, 2\}$, let $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,n})$ in $\mathbf{C}^n$ and*

$$\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1) = \theta(\mathbf{X}) + \sum_{k=1}^{n} \alpha_{1,k} x_k \quad \text{and} \quad \varphi_2(\mathbf{X}, \boldsymbol{\alpha}_2) = \sum_{k=1}^{n} \alpha_{2,k} x_k.$$

Then, there exists a non-empty Zariski open subset $\Omega_{\mathsf{K}}(V, \theta) \subset \mathbf{C}^{2n}$ such that for every $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in \Omega_{\mathsf{K}}(V, \theta)$, and $\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1), \varphi_2(\mathbf{X}, \boldsymbol{\alpha}_2))$, the following holds. Either $W_{\boldsymbol{\varphi}}(2, V)$ is empty or

1. *$W_{\boldsymbol{\varphi}}(2, V)$ is 1-equidimensional;*
2. *the sets $W^{\circ}_{\boldsymbol{\varphi}}(1, W_{\boldsymbol{\varphi}}(2, V))$, $W_{\boldsymbol{\varphi}}(1, W_{\boldsymbol{\varphi}}(2, V))$ and $K_{\boldsymbol{\varphi}}(1, W_{\boldsymbol{\varphi}}(2, V))$ are finite.*

Proof. Let $\boldsymbol{\chi}$ be an atlas of $(V, \mathrm{sing}(V))$, as obtained by applying [30, Lemma A.13]. Let $\Omega_{\mathsf{K}}(V, \boldsymbol{\theta})$ be the intersection of the non-empty Zariski open subsets $\Omega_{\mathsf{I}}(V, \theta, 2)$ and $\Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, \mathrm{sing}(V), (\theta, 0), (0), 2)$ of $\mathbf{C}^{2n}$, obtained by applying Propositions 2.3 and 2.12 respectively (recall that we assume $d \ge 1$). From now on, choose $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in \Omega_{\mathsf{K}}(V, \theta)$ and let $\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1), \varphi_2(\mathbf{X}, \boldsymbol{\alpha}_2))$. In the following, we denote $W_{\boldsymbol{\varphi}}(2, V)$ by W_2. Suppose W_2 is non-empty, otherwise the result trivially holds.

Since $\boldsymbol{\alpha} \in \Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, \mathrm{sing}(V), (\boldsymbol{\theta}, 0), (0), 2)$ and $2 \le (d+3)/2$ for $d \ge 1$, then, by Proposition 2.12, W_2 is equidimensional of dimension 1 and $\mathrm{sing}(W_2) \subset \mathrm{sing}(V)$ is finite. Hence, $K_W = W_{\boldsymbol{\varphi}}(1, W_2)$ is well defined and the following inclusion holds

$$K_W \subset \bigcup_{z \in \boldsymbol{\varphi}_1(K_W)} W_2 \cap \boldsymbol{\varphi}_1^{-1}(z)$$

By an algebraic version of Sard's lemma from [30, Proposition B.2], we deduce that $\boldsymbol{\varphi}_1(W_{\boldsymbol{\varphi}}(1, W_2))$ is finite. Besides, since $\boldsymbol{\alpha} \in \Omega_{\mathsf{I}}(V, \theta, 2)$ then, by Proposition 2.3, $\boldsymbol{\varphi}_1^{-1}(z) \cap W_2$ is finite for any $z \in \mathbf{C}$.

Hence, as a set contained in a finite union of finite sets, K_W is finite, and so are $W^{\circ}_{\boldsymbol{\varphi}}(1, W_2)$ and $K_{\boldsymbol{\varphi}}(1, W_2) = K_W \cup \mathrm{sing}(W_2)$. $\square$

2.5 Charts and atlases for fibers of polynomial maps

We now study the regularity and dimensions of fibers of some generic polynomial maps over algebraic sets. The construction we introduce below is quite similar to the one in [30], but a bit more general.

Definition 2.14. *Let $V, S \subset \mathbf{C}^n$ be two algebraic sets with V d-equidimensional, and let $\boldsymbol{\chi} = (\chi_j)_{1 \le j \le s}$ be an atlas of (V, S). Let $1 \le e \le \mathfrak{r} \le n+1$ be integers and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_{\mathfrak{r}}) \subset \mathbf{C}[\mathbf{X}]$. For $Q \subset \mathbf{C}^e$ we define $F_{\text{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\varphi})$ as the sequence of all $\chi_j = (m_j, \boldsymbol{h}_j)$ such that $\mathcal{O}(m_j) \cap F_Q - S_Q$ is not empty, where*

$$F_Q = V_{|\boldsymbol{\varphi}_e \in Q} \quad \textit{and} \quad S_Q = \big(S \cup W_{\boldsymbol{\varphi}}(e, V)\big)_{|\boldsymbol{\varphi}_e \in Q}.$$

The above definition is a direct generalization of [30, Definition 3.6], where $\boldsymbol{\varphi} = (x_1, \dots, x_n)$. The main result of this subsection is the following proposition, which we prove in Section 7.

Proposition 2.15. *Let $V, S \subset \mathbf{C}^n$ be two algebraic sets with V d-equidimensional and S finite. Let $\boldsymbol{\chi}$ be an atlas of (V, S). Let $2 \le \mathfrak{r} \le d+1$ and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_{\mathfrak{r}}) \subset \mathbf{C}[\mathbf{X}]$. For $2 \le j \le d$, let $\boldsymbol{\alpha}_j = (\alpha_{j,1}, \dots, \alpha_{j,n}) \in \mathbf{C}^n$ and*

$$\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1) = \theta(\mathbf{X}) + \sum_{k=1}^{n} \alpha_{1,k} x_k \quad \textit{and} \quad \varphi_j(\mathbf{X}, \boldsymbol{\alpha}_j) = \sum_{k=1}^{n} \alpha_{j,k} x_k$$

where $\theta \in \mathbf{C}[\mathbf{X}]$.

There exists a non-empty Zariski open subset $\Omega_{\mathsf{F}}(\boldsymbol{\chi}, V, S, \theta, \mathfrak{r}) \subset \mathbf{C}^{\mathfrak{r}n}$ such that for every $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{\mathfrak{r}}) \in \Omega_{\mathsf{F}}(\boldsymbol{\chi}, V, S, \theta, \mathfrak{r})$ and writing

$$\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1), \dots, \varphi_{\mathfrak{r}}(\mathbf{X}, \boldsymbol{\alpha}_{\mathfrak{r}})),$$

the following holds. Let $0 \le e \le d$, $Q \in \mathbf{C}^e$ a finite subset and F_Q and S_Q be as in Definition 2.14. Then either F_Q is empty or

1. *S_Q is finite;*

2. *V_Q is an equidimensional algebraic set of dimension $d - e$;*

3. *$F_{\text{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\varphi})$ is an atlas of (F_Q, S_Q) and $\operatorname{sing}(F_Q) \subset S_Q$.*

3 The algorithm

3.1 Overall description

Recall that $\mathbf{X}$ denotes a sequence of n indeterminates $x_1, \dots, x_n$. In this document, we also consider a family $\mathbf{A} = (a_{i,j})_{1 \le i,j \le n}$ of n^2 new indeterminates, which stand for generic parameters. For $1 \le i, j \le n$, we note $a_i = (a_{i,1}, \dots, a_{i,n})$, so that $\mathbf{A}_{\le i}$ represents the subfamily $(a_1, \dots, a_i)$. An element $\boldsymbol{\alpha} \in \mathbf{C}^{in}$ will often be represented as a vector of length i of the form $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_i)$, with all $\boldsymbol{\alpha}_j = (\alpha_{j,1}, \dots, \alpha_{j,n}) \in \mathbf{C}^n$.

Then, as suggested by Propositions 2.3, 2.12, 2.13 and 2.15, we will consider polynomials of the form:

$$\phi_i(\mathbf{X}, a_i) = \theta_i(\mathbf{X}) + \sum_{j=1}^{n} a_{i,j} x_j + \xi_i(a_i) \in \mathbf{R}[\mathbf{X}, \mathbf{A}]. \tag{1}$$

where $1 \leq i \leq n$, $\theta_i \in \mathbf{R}[\mathbf{X}]$ and $\xi_i \in \mathbf{R}[\mathbf{A}]$. We will choose θ_i so that the polynomial map ϕ_i inherits some useful properties. For instance, taking $\theta_i = x_1^2 + \cdots + x_n^2$, for any $\boldsymbol{\alpha}_i$ in $\mathbf{R}^n$, the polynomial map associated to $\phi_i(\mathbf{X}, \boldsymbol{\alpha}_i)$ is proper and bounded from below on $\mathbf{R}^n$.

Hereafter, we describe, on an example, the core idea of the the strategy that guided the design of our algorithm and the choice of data structures.

Example 2. Consider the algebraic set $V = \boldsymbol{V}(f) \subset \mathbb{C}^4$ defined as the vanishing locus of the polynomial

$$f = \sum_{i=1}^{4} (x_i^3 - x_i) - 1 \in \mathbb{Q}[x_1, x_2, x_3, x_4].$$

We want to compute a roadmap of $(V, \emptyset)$ (or simply V). Following the strategy we designed in our previous work (see [27, Section 5]), V must satisfy some regularity properties, that is

(H_1) V is d-equidimensional, $d \geq 2$, and $\mathrm{sing}(V)$ is finite.

The first part of the assumption can be satisfied by computing an equidimensional decomposition of V, which can be done within the complexity bounds considered in this work (see e.g. [26] for the best-known complexity bound for a probabilistic algorithm). However, it is worth noting that this increases the degrees of the generators.

The condition $d \geq 2$ is not restrictive, as the case $d = 1$ is trivial for roadmap computations. The smoothness assumption is more restrictive. Indeed, it can be satisfied using deformation techniques, such as done in [4, 5], but these steps would not fit, as such, in our complexity bounds.

Let us check that, in our example, V satisfies H_1. We will describe further a subroutine $\mathsf{SingularPoints}$, to compute $\mathrm{sing}(V)$ as long as this holds.

Checking (H_1). As an hypersurface, V is irreducible, and then equidimensional, of dimension 3. The partial derivatives of f, $\frac{\partial f}{\partial x_i} = 3x_i^2 - 1$, for $1 \leq i \leq 4$, do not simultaneously vanish on V. Hence, $\mathrm{sing}(V) = \emptyset$, and V satisfies assumption (H_1).

Following a particular case of [27, Section 5], we want to choose a sequence of polynomial $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n)$ in $\mathbf{Q}[\mathbf{X}]$ such that the following holds:

(H_2) the restriction of $\boldsymbol{\varphi}_1$ to $V \cap \mathbb{R}^n$ is proper and bounded below;

(H_3) $W_2 = W_{\boldsymbol{\varphi}}(2, V)$ is 1 equidimensional, and smooth outside $\mathrm{sing}(V)$;

(H_4) for any $z \in \mathbf{C}$, $V_{|\boldsymbol{\varphi}_1 = z}$ is $(d-1)$-equidimensional;

(H_5) $K_{\boldsymbol{\varphi}}(1, W_2)$ is finite.

In addition, let $K = K_{\boldsymbol{\varphi}}(1, W_2) \cup \mathrm{sing}(V)$ and $F = V \cap \boldsymbol{\varphi}_1^{-1}(\boldsymbol{\varphi}_1(K))$. We require that

(H_6) $\mathcal{P}_W = F \cap W_2$ is finite.

Then, under the above assumptions if $\mathcal{R}_F$ is a roadmap of $(F, \mathcal{P}_W)$, then $W_2 \cup \mathcal{R}_F$ is a roadmap of V. This statement is a consequence of both [29, Proposition 2] and [27, Theorem 1.1], and will be properly stated and proved in Proposition 3.9. This splits the problem of computing a roadmap of V into the computation of representations of W_2, F and $\mathcal{P}_W$, and a roadmap of $(F, \mathcal{P}_W)$. Since $F \cap \mathbb{R}^n$ is bounded, by assumption (H_2), the latter computation can be done using the algorithm of [30].

We describe this process more precisely with our example. Each step consisting in checking the assumptions, and computing the associated objects.

Checking ($\mathsf{H}_{2/3}$). Set first $\boldsymbol{\varphi} = \left(\sum_{i=1}^4 x_i^2,\ x_2,\ x_3,\ x_4\right)$. The restriction of $\boldsymbol{\varphi}_1$ to $V \cap \mathbf{R}^4$ is proper and non-negative. We could then compute a representation of $W_2 = W_{\boldsymbol{\varphi}}(2, V)$, before computing one for its singular locus $\operatorname{sing}(W_2)$. However, the latter singular set is not empty, while $\operatorname{sing}(V)$ is. This contradicts the assumptions needed in [27, Theorem 1.1] and the strategy for computing a roadmap of V designed in [27, Section 5] might fail.

Following Propositions 2.3, 2.12, 2.13 and 2.15 from the preliminaries, we propose the following. To prevent these regularity failures, and to satisfy all assumptions of [27, Theorem 1.1], while keeping the properties of $\boldsymbol{\varphi}$, we add to $\boldsymbol{\varphi}_1$ a linear form; here we take $x_1 - x_4$, but in general it should be taken with random coefficients.

Hence, consider now the sequence $\boldsymbol{\varphi}$ of polynomials maps

$$\boldsymbol{\varphi} = \left(\sum_{i=1}^{4} x_i^2 + x_1 - x_4,\ x_2,\ x_3,\ x_4\right),$$

whose restriction to $\mathbf{R}^4$ is still proper and bounded below, by construction. If the linear form we added has been sufficiently randomly chosen, Proposition 2.12 claims that W_2 satisfies assumption (H_3).

Using Gröbner basis computations on a determinantal ideal defining W_2, we compute a representation of W_2, and next $\operatorname{sing}(W_2)$, that turns out to be empty, as requested. More generally, computing the two previous sets efficiently is the purpose of the algorithm SolvePolar, presented further in Lemma 3.4.

Checking (H_4). By Proposition 2.15, this assumption holds if we have added to $\boldsymbol{\varphi}$ a linear form that is generic enough. Using the Jacobian criterion, we can check that in our case, for any $z \in \mathbb{C}$, the fiber $F_z = V \cap \boldsymbol{\varphi}_1^{-1}(z)$ is an equidimensional algebraic set of dimension 2 (if it is not empty). Moreover the singular locus of F_z is contained in the finite set $W_{\boldsymbol{\varphi}}(1, V)$. Computing the latter set is tackled by the subroutine Crit, presented in Lemma 3.3.

Checking (H_5). We also need to check the finiteness and compute the set $K_{\boldsymbol{\varphi}}(1, W_2)$. If $\boldsymbol{\varphi}$ is generic enough, the finiteness is ensured by Proposition 2.13; computing this set is the purpose of the algorithm CritPolar, presented in Lemma 3.6. In our case, there are finitely many (more precisely 129) such points, and 23 of them are real.

Checking (H_6). We need to compute the set $K = K_{\boldsymbol{\varphi}}(1, W_2) \cup \operatorname{sing}(V)$. As the two members of the unions have been computed by the algorithms CritPolar and SingularPoints, respectively, one can compute this union using the procedure Union from [30, Lemma J.3] (also presented in the next subsection).

Then, for $\boldsymbol{\varphi}$ generic enough, Proposition 2.3 ensures that the last assumption holds. The computation of $\mathcal{P}_W$ boils down to computing finitely many fibers on the restriction of $\boldsymbol{\varphi}_1$ to W_2. This is the purpose of the algorithm FiberPolar, presented in Lemma 3.6.

At this point, we have computed representations of W_2 and $\mathcal{P}_W$, and ensured that all assumptions of [27, Theorem 1.1] are satisfied. Hence, one only need to compute a roadmap of $(F, \mathcal{P}_W)$. This is the purpose of algorithm RoadmapBounded, presented in Proposition 3.7.

3.2 Subroutines

Our main algorithm (Algorithm 1) makes use of several subroutines which allow us to manipulate zero-dimensional and one-dimensional parametrizations, polar varieties and fibers of polynomial maps in order to make [27, Theorem 1.1] effective.

As a reminder, in this document, we manipulate subroutines that involve selecting suitable parameters in $\mathbf{Q}^i$, for various $i \geq 1$. These algorithms are probabilistic, which means that for any choice of (say) i parameters we have to do, there exists a non-zero polynomial Δ, such that for $\boldsymbol{\lambda} \in \mathbf{Q}^i$, success is achieved if $\Delta(\boldsymbol{\lambda}) \neq 0$. It is also important to note that these algorithms are considered Monte Carlo, as their output's correctness cannot be guaranteed within a reasonable complexity. In certain cases, where we can identify errors, we require our procedures to output fail. However, not returning fail does not guarantee correctness.

Let $1 \leq c \leq n$, and $\boldsymbol{f} = (f_1, \ldots, f_c)$ be a sequence of polynomials in $\mathbf{R}[\mathbf{X}]$. We say that $\boldsymbol{f}$ satisfies assumption (A) if

(A) : $\boldsymbol{f}$ is a reduced regular sequence, with $d = n - c \geq 2$, and $\operatorname{sing}(\boldsymbol{V}(\boldsymbol{f}))$ is finite.

In particular, the zero-set of $\boldsymbol{f}$ in $\mathbf{C}^n$ is then either empty or d-equidimensional.

3.2.a Basic subroutines

The first two subroutines we use are described in [30] and are used to compute $\operatorname{sing}(V(\boldsymbol{f}))$ (on input a straight-line program evaluating $\boldsymbol{f}$) and to compute a rational parametrization encoding the union of zero-dimensional sets or the union of algebraic curves. They are both Monte Carlo algorithms, in the sense described above, and can output fail in case errors have been detected during the execution. However, in case of success, the following holds.

- $\mathsf{SingularPoints}$, described in [30, Section J.5.4], takes as input a straight-line program Γ that evaluates polynomials $\boldsymbol{f} \in \mathbf{C}[\mathbf{X}]$ satisfying assumption (A) and outputs a zero-dimensional parametrization of $\operatorname{sing}(\boldsymbol{V}(\boldsymbol{f}))$.
- Union, described in [30, Lemma J.3] (resp. [30, Lemma J.8]), takes as input two zero-dimensional (resp. one-dimensional) parametrizations $\mathscr{P}_1$ and $\mathscr{P}_2$ and outputs a zero-dimensional (resp. one-dimensional) parametrization encoding $\mathsf{Z}(\mathscr{P}_1) \cup \mathsf{Z}(\mathscr{P}_2)$.

We now describe basic subroutines performing elementary operations on straight-line program and zero-dimensional parametrizations. The first one allows us to generate a generic polynomial with a prescribed structure.

Lemma 3.1. *Let $1 \leq i \leq n$ and $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_i) \in \mathbf{C}^{in}$. Then there exists an algorithm PhiGen which takes as input $\boldsymbol{\alpha}$ and returns in time $O(n)$ a straight-line program $\Gamma^{\boldsymbol{\varphi}}$ of length $O(n)$ computing in $\mathbf{Q}[\mathbf{X}]$:*

$$\varphi_1 = \sum_{k=1}^{n} x_k^2 + \alpha_{1,k} x_k \quad \text{and} \quad \varphi_j = \sum_{k=1}^{n} \alpha_{j,k} x_k \quad \text{for } 2 \leq j \leq i.$$

Proof. Given the constants $\boldsymbol{\alpha}$, it suffices to generate a straight-line program that computes the linear forms $\sum_{k=1}^{n} \boldsymbol{\alpha}_{j,k} x_k$, for $j = 1, \ldots, i$, and adds the quadratic form $\sum_{k=1}^{n} x_k^2$ to the first one. This takes linear time, and the result straight-line program has linear length. $\square$

Next, we present a procedure computing the image of a zero-dimensional parametrization by a polynomial map, given as a straight-line program, generalizing the subroutine $\mathsf{Projection}$ from [30, Lemma J.5]. The proof of the next lemma is given in Subsection 4.1.

Lemma 3.2. *Let $\mathscr{P}$ be a zero-dimensional parametrization of degree κ such that $\mathsf{Z}(\mathscr{P}) \subset \mathbf{C}^n$ and let $\Gamma^{\boldsymbol{\varphi}}$ be a straight-line program of length E' computing polynomials $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_i)$. There exists a Monte Carlo algorithm Image which, on input $\Gamma^{\boldsymbol{\varphi}}$, $\mathscr{P}$ and $j \in \{1, \ldots, i\}$, outputs either fail or a zero-dimensional parametrization $\mathscr{Q}$, of degree at most κ, using*

$$O^{\sim}\left((n^2\kappa + E')\kappa\right)$$

operations in $\mathbf{Q}$. In case of success, $\mathsf{Z}(\mathscr{Q}) = \boldsymbol{\varphi}_j(\mathsf{Z}(\mathscr{P}))$.

3.2.b Subroutines for polar varieties

The next subroutines are used to compute generalized polar varieties and quantities related to them. The proof of all statements below can be found in Subsection 4.4. In this subsection, we fix $1 \leq c \leq n-2$ and we refer to the following objects:

- sequences of polynomials $\boldsymbol{g} = (g_1, \ldots, g_c)$ and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ all of them in $\mathbf{Q}[x_1, \ldots, x_n]$, of degrees bounded by D, such that $\boldsymbol{g}$ satisfies assumption (A); we note $d = n - c$;

- straight-line programs Γ and $\Gamma^{\boldsymbol{\varphi}}$, of respective lengths E and E', computing respectively $\boldsymbol{g}$ and $\boldsymbol{\varphi}$;

- zero-dimensional parametrizations $\mathscr{S}$ and $\mathscr{Q}''$, of respective degrees σ and κ'', describing finite sets $S \subset \mathbf{C}^n$ and $Q'' \subset \mathbf{C}$, such that $\operatorname{sing}(\boldsymbol{V}(\boldsymbol{g})) \subset S$ (the $''$ superscripts we use here match those used in the algorithm);

- an atlas $\boldsymbol{\chi}$ of $(\boldsymbol{V}(\boldsymbol{g}), S)$, given by [30, Lemma A.13], as S is finite and contains $\operatorname{sing}(\boldsymbol{V}(\boldsymbol{g}))$.

We start with the subroutine Crit, which is used for computing critical and singular points of some polynomial map, again under some regularity assumption. These critical points are nothing but zero-dimensional polar varieties.

Lemma 3.3. *Assume that $K_{\boldsymbol{\varphi}}(1, \boldsymbol{V}(\boldsymbol{g}))$ is finite. There exists a Monte Carlo algorithm Crit which takes as input Γ, $\Gamma^{\boldsymbol{\varphi}}$ and $\mathscr{S}$, and outputs either fail or a zero-dimensional parametrization $\mathscr{S}_F$, with coefficients in $\mathbf{Q}$, of degree at most*

$$\binom{n+1}{d} D^{c+2}(D-1)^d + \sigma$$

such that, in case of success, $\mathsf{Z}(\mathscr{S}_F) = K_{\boldsymbol{\varphi}}(1, \boldsymbol{V}(\boldsymbol{g})) \cup S$. It uses

$$O\tilde{~}\left(E''(n+2)^{4d+8} D^{2n+3}(D-1)^{2d} + n\sigma^2\right)$$

operations in $\mathbf{Q}$, where $E'' = E + E'$.

We now tackle higher dimensional cases, with the subroutine $\mathsf{SolvePolar}$ which, under some assumptions, computes one-dimensional parametrization encoding one-dimensional generalized polar varieties.

Lemma 3.4. *Let $W = W_{\boldsymbol{\varphi}}(2, \boldsymbol{V}(\boldsymbol{g}))$ and assume that one of the following holds*

- *W is empty, or*

- *W is 1-equidimensional, with $\operatorname{sing}(W) \subset S$, and $W_{\mathrm{atlas}}(\boldsymbol{\chi}, \boldsymbol{V}(\boldsymbol{g}), S, \boldsymbol{\varphi}, 2)$ is an atlas of (W, S).*

Then, there exists a Monte Carlo probabilistic algorithm $\mathsf{SolvePolar}$ which takes as input Γ, $\Gamma^{\boldsymbol{\varphi}}$ and $\mathscr{S}$ and which outputs either fail or a one-dimensional parametrization $\mathscr{W}_2$, with coefficients in $\mathbf{Q}$, of degree at most

$$\delta = (n+c+4)D^{c+2}(D-1)^d(c+2)^d,$$

such that, in case of success, $\mathsf{Z}(\mathscr{W}_2) = W$. It uses at most

$$O\tilde{~}\left((n+c)^3(E'' + (n+c)^3)D\delta^3 + (n+c)\delta\sigma^2\right)$$

operations in $\mathbf{Q}$, where $E'' = E + E'$.

The subroutine $\mathsf{CritPolar}$ is devoted to compute critical points of the restriction of some polynomial map to a generalized polar variety of dimension at most one. It generalizes the subroutine W_1 from [30, Proposition 6.4].

Lemma 3.5. *Let $W = W_{\boldsymbol{\varphi}}(2, \boldsymbol{V}(\boldsymbol{g}))$ and assume that either W is empty, or*

- *W is 1-equidimensional, with $\operatorname{sing}(W) \subset S$, and $W_{\text{atlas}}(\boldsymbol{\chi}, \boldsymbol{V}(\boldsymbol{g}), S, \boldsymbol{\varphi}, 2)$ is an atlas of (W, S),*
- *and $W_{\boldsymbol{\varphi}}(1, W)$ is finite.*

There exists a Monte Carlo algorithm $\mathsf{CritPolar}$ which takes as input Γ, $\Gamma^{\boldsymbol{\varphi}}$ and $\mathscr{S}$ and which outputs either fail or a zero-dimensional parametrization $\mathscr{K}$, with coefficients in $\mathbf{Q}$, such that $\mathsf{Z}(\mathscr{K}) = W_{\boldsymbol{\varphi}}(1, W) \cup S$ using at most

$$O^{\sim}\left((n+c)^{12} E'' D^3 \delta^2 + (n+c)\sigma^2\right)$$

operations in $\mathbf{Q}$, where $E'' = E + E'$, and $\delta = (n+c+4)D^{c+2}(D-1)^d(c+2)^d$. Moreover $\mathscr{K}$ has degree at most $\delta(n+c+4)D + \sigma$.

Finally, we consider the subroutine $\mathsf{FiberPolar}$ which, given polynomials defining a generalized polar variety of dimension at most one, the polynomial map $\boldsymbol{\varphi}$ and a description of Q'', computes the fibers of the polynomial map $\boldsymbol{\varphi}$ over Q'' on the polar variety.

Lemma 3.6. *Let $W = W_{\boldsymbol{\varphi}}(2, \boldsymbol{V}(\boldsymbol{g}))$ and assume that either W is empty, or*

- *W is 1-equidimensional, with $\operatorname{sing}(W) \subset S$, and $W_{\text{atlas}}(\boldsymbol{\chi}, \boldsymbol{V}(\boldsymbol{g}), S, \boldsymbol{\varphi}, 2)$ is an atlas of (W, S);*
- *$W \cap \boldsymbol{\varphi}_1^{-1}(Q'')$ is finite.*

There exists a Monte Carlo algorithm $\mathsf{FiberPolar}$ which takes as input Γ, $\Gamma^{\boldsymbol{\varphi}}$, $\mathscr{S}$ and $\mathscr{Q}''$ and which outputs either fail or a zero-dimensional parametrization $\mathscr{Q}$, with coefficients in $\mathbf{Q}$, such that $\mathsf{Z}(\mathscr{Q}) = \left(W \cap \boldsymbol{\varphi}_1^{-1}(Q'')\right) \cup S$, using at most

$$O^{\sim}\left((n+c)^4\left(E'' + (n+c)^2\right) D \kappa''^2 \delta^2 + (n+c)\sigma^2\right)$$

operations in $\mathbf{Q}$, where $E'' = E + E'$, and $\delta = (n+c+4)D^{c+2}(D-1)^d(c+2)^d$. Moreover, $\mathscr{Q}$ has degree at most $\kappa''\delta + \sigma$.

3.2.c Subroutines for computing roadmaps in the bounded case

As seen above, in Example 2, we are ultimately led to compute a roadmap for a bounded real algebraic set (this set is given as fibers over finitely many algebraic points of the restriction of a polynomial map to our input). To do so, we call the algorithm $\mathsf{RoadmapRecLagrange}$ from [30], which internally uses similar techniques but with projections (where $\boldsymbol{\varphi} = \boldsymbol{\pi}$). The description and the complexity analysis of this procedure are given in Subsection 4.5. The subtlety comes from the fact that, in [30], the correction and complexity estimate of $\mathsf{RoadmapRecLagrange}$ are given for an input consisting of polynomials $\boldsymbol{f}$ defining an algebraic set $V = \boldsymbol{V}(\boldsymbol{f})$; here, we need an algorithm that works for an input given as fibers of a polynomial map. More precisely, we prove the following result in Subsection 4.5.

Proposition 3.7. *Let Γ and $\Gamma^{\boldsymbol{\varphi}}$ be straight-line programs, of respective length E and E', computing respectively sequences of polynomials $\boldsymbol{g} = (g_1, \ldots, g_p)$ and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ in $\mathbf{Q}[x_1, \ldots, x_n]$, of degrees bounded by D. Assume that $\boldsymbol{g}$ satisfies (A). Let $\mathscr{Q}$ and $\mathscr{S}_Q$ be zero-dimensional parametrizations of respective degrees κ and σ that encode finite sets $Q \subset \mathbf{C}^e$ (for some $0 < e \leq n$) and $S_Q \subset \mathbf{C}^n$, respectively. Let $V = \boldsymbol{V}(\boldsymbol{g})$ and $F_Q = V_{|\boldsymbol{\varphi}_e \in Q}$, and assume that*

- *F_Q is equidimensional of dimension $d-e$, where $d = n-p$;*
- *$F_{\text{atlas}}(\boldsymbol{\chi}, V, Q, \boldsymbol{\varphi})$ is an atlas of (F_Q, S_Q), and $\operatorname{sing}(F_Q) \subset S_Q$;*
- *the real algebraic set $F_Q \cap \mathbf{R}^n$ is bounded.*

Consider additionally a zero-dimensional parametrization $\mathscr{P}$ of degree μ encoding a finite subset $\mathcal{P}$ of F_Q, which contains S_Q. Assume that $\sigma \leq ((n+e)D)^{n+e}$.

There exists a probabilistic algorithm RoadmapBounded which takes as input $((\Gamma, \Gamma^{\boldsymbol{\varphi}}, \mathscr{Q}, \mathscr{S}), \mathscr{P})$ and which, in case of success, outputs a roadmap of $(F_Q, \mathcal{P})$, of degree

$$O^{\sim}\Big((\mu+\kappa)16^{3d_F}(n_F \log_2(n_F))^{2(2d_F+12\log_2(d_F))(\log_2(d_F)+5)} D^{(2n_F+1)(\log_2(d_F)+3)}\Big)$$

where $n_F = n+e$ and $d_F = d-e$, and using

$$O^{\sim}\Big(\mu'^3 16^{9d_F} E''(n_F \log_2(n_F))^{(12d_F+24\log_2(d_F))(\log_2(d_F)+6)} D^{(6n_F+3)(\log_2(d_F)+4)}\Big)$$

arithmetic operations in $\mathbf{Q}$ (where $\mu' = \mu+\kappa$ and $E'' = E+E'+e$).

3.3 Description of the main algorithm

Now, we describe the main algorithm to compute roadmaps of smooth unbounded real algebraic sets. In addition to the subroutines mentioned above, we define Random as a procedure that takes as input a set X and returns a random element in X. Together with PhiGen, it allows us to generate "generic enough" polynomial maps so that the results of the previous section do apply (Propositions 2.3, 2.12, 2.13 and 2.15).

Algorithm 1 Roadmap algorithm for smooth unbounded real algebraic sets.

Input: ▷ a straight-line program Γ that evaluates polynomials $\boldsymbol{f} = (f_1, \ldots, f_c) \subset \mathbf{Q}[\mathbf{X}]$, satisfying assumption (A); we note $V = \boldsymbol{V}(\boldsymbol{f})$;

▷ a zero-dimensional parametrization $\mathscr{P}_0$ encoding a finite set $\mathcal{P}_0 \subset V$.

Output: a one-dimensional parametrization $\mathscr{R}$ encoding a roadmap of $(V, \mathcal{P}_0)$.

```
1:  𝒮   ← SingularPoints(Γ);                 // Z(𝒮) = sing(V);
2:  𝒫   ← Union(𝒫₀, 𝒮);                      // P := Z(𝒫) = P₀ ∪ sing(V)
3:  α   ← Random(Q^{2n});
4:  Γ^φ ← PhiGen(α);                          // Γ^φ computes φ = (||X||² + ⟨α₁, X⟩, ⟨α₂, X⟩)
5:  𝒲₂  ← SolvePolar(Γ, Γ^φ, 𝒮);             // W₂ := Z(𝒲₂) = W_φ(2, V);
6:  𝒦   ← CritPolar(Γ, Γ^φ, 𝒫);              // K := Z(𝒦) = W_φ(1, W₂) ∪ P₀ ∪ sing(V);
7:  𝒬   ← Image(Γ^φ, 1, 𝒦);                  // Q := Z(𝒬) = φ₁(K);
8:  𝒫_F ← FiberPolar(Γ, Γ^φ, 𝒬, 𝒫);          // Z(𝒫_F) = [W₂ ∪ P₀ ∪ sing(V)] ∩ φ₁⁻¹(Q);
9:  𝒮_F ← Crit(Γ, Γ^φ, 𝒮)                    // Z(𝒮_F) = K_φ(1, V);
10: ℛ_F ← RoadmapBounded((Γ, Γ^φ, 𝒬, 𝒮_F), 𝒫_F)
                                              // Z(ℛ_F) is a roadmap of (V ∩ φ₁⁻¹(Q), Z(𝒫_F));
11: return Union(𝒲₂, ℛ_F)                     // W₂ ∪ Z(ℛ_F) is a roadmap of (V, P₀).
```

3.4 Correctness and complexity estimate

This subsection is devoted to the proof of the following theorem, which directly implies Theorem 1.1.

Theorem 3.8. *Let Γ be a straight-line program of length E evaluating polynomials $\boldsymbol{f} = (f_1, \ldots, f_c)$ of degrees bounded by D, satisfying* (A)*. Let $\mathscr{P}_0$ be a zero-dimensional parametrization of degree μ encoding a finite subset of $\boldsymbol{V}(\boldsymbol{f}) \subset \mathbf{C}^n$. Then there exists a non-empty Zariski open $\Omega \subset \mathbf{C}^{2n}$ such that the following holds.*

Let $\boldsymbol{\alpha} \in \mathbf{Q}^{2n}$ the vector randomly chosen in the execution of Algorithm 1, then if $\boldsymbol{\alpha} \in \Omega$, and if the calls to the subroutines

SingularPoints, Union, SolvePolar, CritPolar, Image, FiberPolar, Crit and RoadmapBounded

are successful then, on inputs Γ and $\mathscr{P}_0$, Algorithm 1 either returns a one-dimensional parametrization of degree

$$O^{\sim}\left(\mu 16^{3d}(n\log_2(n))^{2(2d-2+12\log_2(d-1))(\log_2(d-1)+6)} D^{(2n+4)(\log_2(d-1)+4)}\right)$$

using

$$O^{\sim}\left(\mu^3 16^{9d} E(n\log_2(n))^{6(2d+12\log_2(d-1))(\log_2(d-1)+7)} D^{3(2n+4)(\log_2(d-1)+5)}\right)$$

arithmetic operations in $\mathbf{Q}$, with $d = n - c$.

In case of success, its output describes a roadmap of $(\boldsymbol{V}(\boldsymbol{f}), \mathsf{Z}(\mathscr{P}_0))$.

The correctness of Algorithm 1 relies mainly on the conjunction of [27, Theorem 1.1] and [29, Proposition 2], that form the following statement, with slightly stronger assumptions, which hold in our context.

Proposition 3.9. *Let $V \subset \mathbf{C}^n$ be a $\mathbf{Q}$-algebraic set of dimension $d \geq 2$, and let $\mathcal{P}_0$ be a finite subset of V. Let $\varphi = (\varphi_1, \varphi_2) \subset \mathbf{R}[\mathbf{X}]$ and $W = W_{\boldsymbol{\varphi}}(2, V)$. Suppose that the following holds:*

(H_1) *V is equidimensional and* $\mathrm{sing}(V)$ *is finite;*

(H_2) *the restriction of $\boldsymbol{\varphi}_1$ to $\boldsymbol{V} \cap \mathbf{R}^n$ is a proper map bounded from below;*

(H_3) *W is either empty or 1-equidimensional and smooth outside* $\mathrm{sing}(V)$*;*

(H_4) *for any $\boldsymbol{y} \in \mathbf{C}^2$, the set $V \cap \boldsymbol{\varphi}_1^{-1}(\boldsymbol{y})$ is either empty or $(d-1)$-equidimensional;*

(H_5) *$K_{\boldsymbol{\varphi}}(1, W)$ is finite.*

Let further $K = K_{\boldsymbol{\varphi}}(1, W) \cup \mathcal{P}_0 \cup \mathrm{sing}(V)$ and $F = V \cap \boldsymbol{\varphi}_1^{-1}(\boldsymbol{\varphi}_1(K))$.
Assume in addition that

(H_6) *$\mathcal{P}_W = F \cap W$ is finite.*

If $\mathcal{R}_F$ is a roadmap of $(F, \mathcal{P}_0 \cup \mathcal{P}_W)$, then $W \cup \mathcal{R}_F$ is a roadmap of $(V, \mathcal{P}_0)$.

Proof. Remark first that the so-called assumptions A, P and B from the connectivity result from [27, Theorem 1.1] are direct consequences of assumptions H_1 to H_4. Besides, $W_{\boldsymbol{\varphi}}(1, V) \subset K_{\boldsymbol{\varphi}}(1, W)$ and $\mathrm{sing}(W) \subset \mathrm{sing}(V)$, by [30, Lemma A.5] together with assumption H_3. Hence, one can write

$$K = W_{\boldsymbol{\varphi}}(1, V) \cup S \cup \mathrm{sing}(V).$$

where $S = W_{\boldsymbol{\varphi}}(1, W) \cup \mathcal{P}_0$. By H_5, S is a finite subset of V, that intersects every semi-algebraically connected component of $W_{\boldsymbol{\varphi}}(1, W) \cap \mathbf{R}^n$ by definition. Hence, S satisfies assumption C of [27, Theorem 1.1]. By application of this latter result, $W \cup F$ has then a non-empty and semi-algebraically connected intersection with every semi-algebraically connected component of $V \cap \mathbf{R}^n$ and it contains $\mathcal{P}_0$ by construction.

Moreover, by H_6, $F \cap W$ is finite, so that by [29, Proposition 2], the following holds. If $\mathcal{R}_W$ and $\mathcal{R}_F$ are roadmaps of respectively $(W, \mathcal{P}_0 \cup \mathcal{P}_W)$ and $(F, \mathcal{P}_0 \cup \mathcal{P}_W)$, then $\mathcal{R}_W \cup \mathcal{R}_F$ is a roadmap of $(V, \mathcal{P}_0)$. But remark that W is a roadmap of $(W, \mathcal{P}_W)$ since W has dimension one. Besides, [29, Proposition 2] can be slightly generalized as only one of $\mathcal{R}_W$ or $\mathcal{R}_F$ must contain $\mathcal{P}_0$. Hence, taking $\mathcal{R}_W = W$ allows us to conclude. □

Proof of Theorem 3.8. Let Γ and $\mathscr{P}_0$ be the inputs of Algorithm 1 and assume that Γ evaluates polynomials $\boldsymbol{f} = (f_1, \ldots, f_c)$ satisfying assumption A. Let $V = \boldsymbol{V}(\boldsymbol{f})$ and $\mathcal{P}_0 = \mathsf{Z}(\mathscr{P}_0)$.

Recall that we assume all calls to the subroutines $\mathsf{SingularPoints}$, Union, $\mathsf{SolvePolar}$, $\mathsf{CritPolar}$, Image, $\mathsf{FiberPolar}$, Crit and $\mathsf{RoadmapBounded}$ do succeed.

Steps 1-2 By [30, Proposition J.35], the procedure $\mathsf{SingularPoints}$ outputs a zero-dimensional parametrization $\mathscr{S}$ describing $\mathrm{sing}(V)$ using $O\tilde{~}(ED^{4n+1})$ operations in $\mathbf{Q}$. By [30, Proposition I.1] (or [31, Proposition 3]) $\mathscr{S}$ has degree at most

$$\sigma_{\mathscr{S}} = \binom{n-1}{c-1} D^c (D-1)^d = \binom{n-1}{d} D^c (D-1)^d \in O(n^d D^n)$$

Then, according to [30, Lemma J.3] and our assumptions, the procedure Union outputs a zero-dimensional parametrization $\mathscr{P}$ of degree at most

$$\delta_{\mathscr{P}} = \mu + \sigma_{\mathscr{S}} \; = \; O(\mu + n^d D^n), \quad \text{using} \quad O\tilde{~}(n(\mu^2 + n^{2d} D^{2n})) \text{ operations in } \mathbf{Q}$$

which describes $\mathcal{P} := \mathcal{P}_0 \cup \mathrm{sing}(V)$.

Besides, since V is equidimensional, there exists, by [30, Lemma A.13], an atlas $\boldsymbol{\chi}$ of $(V, \mathrm{sing}(V))$. According to Definition 2.7, $\boldsymbol{\chi}$ is an atlas of $(V, \mathcal{P})$ as well.

Steps 3-4 By definition of the procedure Random, $\boldsymbol{\alpha}$ is an arbitrary element of $\mathbf{Q}^{2n}$, and according to Lemma 3.1, Γ^{φ} is a straight-line program of length $E' = 6n - 2 \; = \; O(n)$, which evaluates $\varphi = (\theta(\mathbf{X}) + \langle \boldsymbol{\alpha}_1, \mathbf{X} \rangle, \langle \boldsymbol{\alpha}_2, \mathbf{X} \rangle)$, where $\theta = x_1^2 + \cdots + x_n^2$. In particular, $E'' := E + E' \; = \; O(E + n)$.

Let Ω be the intersection of the following four non-empty Zariski open subsets of $\mathbf{C}^{2n}$:

$$\Omega_{\mathsf{I}}(V, \theta, 2),\ \Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, \mathrm{sing}(V), (\theta, 0), (0), 2),\ \Omega_{\mathsf{K}}(V, \theta) \text{ and } \Omega_{\mathsf{F}}(\boldsymbol{\chi}, V, \mathrm{sing}(V), \theta, 2),$$

defined respectively by Propositions 2.3, 2.12, 2.13 and 2.15 applied to V, $\boldsymbol{\varphi}$ and possibly $\boldsymbol{\chi}$. The set Ω is a non-empty Zariski open subset of $\mathbf{C}^{2n}$ as well, and for now on, we suppose that $\boldsymbol{\alpha} \in \Omega$.

Step 5 Let $W = W_{\boldsymbol{\varphi}}(2, V)$. Since $\boldsymbol{\alpha} \in \Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, \mathrm{sing}(V), (\theta, 0), (0), 2)$, by Proposition 2.12, either W is empty or it is equidimensional of dimension 1, with $\mathrm{sing}(W) \subset \mathrm{sing}(V)$. Moreover, in the latter case, since $(d + 3)/2 \geq 2$ by assumption, $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, \mathrm{sing}(V), \varphi, 2)$ is an atlas of $(W, \mathrm{sing}(V))$.

Hence, by Lemma 3.4 and our assumptions, $\mathsf{SolvePolar}$ returns a one-dimensional parametrization $\mathscr{W}_2$, of degree at most

$$\delta = (n+c+4)D^{c+2}(D-1)^d(c+2)^d \; = \; O(n^{d+1}D^{n+2}),$$

such that $\mathsf{Z}(\mathscr{W}_2) = W$, using at most

$$O\tilde{}\left((n+c)^3(E+(n+c)^3)D\delta^3 + (n+c)\delta\sigma_{\mathscr{S}}^2\right) \; = \; O\tilde{}\left(n^{3d+4}(E+n^3)D^{3n+7}\right)$$

operations in $\mathbf{Q}$.

Steps 6-7 Since we assume $\boldsymbol{\alpha} \in \Omega_{\mathsf{K}}(V,\theta)$, Proposition 2.13 states that either W is empty or it is equidimensional of dimension 1, and $W_{\boldsymbol{\varphi}}(1,W)$ is finite. Moreover, since $\boldsymbol{\alpha} \in \Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, \mathrm{sing}(V), (\theta,0), (0), 2)$, we deduce by Proposition 2.12 that $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, \mathcal{P}, \varphi, 2)$ is an atlas of $(W,\mathcal{P})$, as W is 1-equidimensional or empty and $\mathcal{P}_0$ is finite.

Let $K = W_{\boldsymbol{\varphi}}(1,W) \cup \mathcal{P}$. By Lemma 3.5, $\mathsf{CritPolar}$ returns either fail or a zero-dimensional parametrization $\mathscr{K}$, of degree at most

$$\delta_{\mathscr{K}} = \delta(n+c+4)D + \delta_{\mathscr{P}} \; = \; O\left(n^{d+2}D^{n+3} + \mu\right),$$

using at most

$$O\tilde{}\left((n+c)^{12}(E+n)D^3\delta^2 + (n+c)\delta_{\mathscr{P}}^2\right) \; = \; O\tilde{}\left(n^{2d+14}(E+n)D^{2n+7} + n\mu^2\right)$$

operations in $\mathbf{Q}$. Moreover, by assumption, $\mathscr{K}$ describes K. Finally, let $Q = \boldsymbol{\varphi}_1(K)$ then, by Lemma 3.2 and our assumptions, on input $\Gamma^{\boldsymbol{\varphi}}$, $\mathscr{K}$ and $j=1$, procedure Image outputs a zero-dimensional parametrization $\mathscr{Q}$, of degree less than $\delta_{\mathscr{K}}$, such that, in case of success, $\mathsf{Z}(\mathscr{Q}) = Q$. Moreover, since by Lemma 3.1, $\Gamma^{\boldsymbol{\varphi}}$ has length in $O(n)$, then the execution of Image uses at most

$$O\tilde{}\left((n^2\delta_{\mathscr{K}} + n)\delta_{\mathscr{K}}\right) \; = \; O\tilde{}\left(n^{2d+6}D^{2n+6}\right)$$

operations in $\mathbf{Q}$.

Step 8 Since $\boldsymbol{\alpha} \in \Omega_{\mathsf{I}}(V,\theta,2)$, by Proposition 2.3, $W \cap \boldsymbol{\varphi}_1^{-1}(z)$ is finite for any $z \in \mathbf{C}$. In particular, $W \cap \boldsymbol{\varphi}_1^{-1}(Q)$ is finite, since $Q = \mathsf{Z}(\mathscr{Q})$ is. Besides, as seen above, $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, \mathcal{P}, \varphi, 2)$ is an atlas of $(W,\mathcal{P})$ since $\boldsymbol{\alpha}$ lies in $\Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, \mathrm{sing}(V), (\theta,0), (0), 2)$.

Let $\mathcal{P}_F = [W \cap \boldsymbol{\varphi}_1^{-1}(Q)] \cup \mathcal{P}$. By Lemma 3.6 and our assumptions, $\mathsf{FiberPolar}$ outputs a zero-dimensional parametrization $\mathscr{P}_F$, of degree bounded by

$$\mu_{\mathscr{P}_F} = \delta_{\mathscr{K}}\delta + \delta_{\mathscr{P}} \; = \; O(n^{2d+3}D^{2n+5} + \mu),$$

using at most $O\tilde{}\left((n+c)^4(E+(n+c)^2)D\delta_{\mathscr{K}}^2\delta^2 + (n+c)\delta_{\mathscr{P}}^2\right)$ operations in $\mathbf{Q}$ which is in

$$O\tilde{}\left(n^{4d+10}(E+n^2)D^{4n+10} + n\mu^2\right)$$

and such that $\mathscr{P}_F$ describes $\mathcal{P}_F$. Besides, remark that by definition $\varphi(\mathcal{P}) \subset \varphi(Q)$ so that $\mathcal{P}_F = [W \cup \mathcal{P}] \cap \boldsymbol{\varphi}_1^{-1}(Q)$.

Step 9 Since $\boldsymbol{\alpha} \in \Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, \operatorname{sing}(V), (\theta, 0), (0), 2)$, by Proposition 2.12, $W_{\boldsymbol{\varphi}}(1, V)$ is finite. Besides, under assumption (A), V is equidimensional with finitely many singular points. Let $\mathcal{S}_F = K_{\boldsymbol{\varphi}}(1, V)$. By Lemma 3.3 and our assumptions, Crit outputs a zero-dimensional parametrization $\mathscr{S}_F$, which describes $\mathcal{S}_F$, of degree bounded by

$$\sigma_{\mathscr{S}_F} = \binom{n+1}{d} D^{c+2}(D-1)^d \;=\; O(n^d D^{n+2})$$

using at most

$$O\tilde{}\left((n+2)^{4d+8}(E+n)D^{2n+3}(D-1)^{2d} + n\sigma_{\mathscr{S}}^2\right) \;=\; O\tilde{}\left(n^{4d+8}(E+n)D^{4n+3}\right)$$

operations in $\mathbf{Q}$.

Step 10 Since $\boldsymbol{f}$ satisfies assumption (A), the ideal $\langle \boldsymbol{f} \rangle$ generated by the polynomials in $\boldsymbol{f}$ is radical. Besides, the restriction of $\boldsymbol{\varphi}_1$ to $\boldsymbol{V}(\boldsymbol{f}) \cap \mathbf{R}^n$ is naturally proper and bounded from below by $\sum_{i=1}^n \boldsymbol{\alpha}_i^2/4$. Hence, as $Q = \mathsf{Z}(\mathscr{Q})$ is finite, $Q \cap \mathbf{R}$ is bounded and so is

$$V \cap \mathbf{R}^n \cap \boldsymbol{\varphi}_1^{-1}(Q \cap \mathbf{R}^2) = V \cap \boldsymbol{\varphi}_1^{-1}(Q) \cap \mathbf{R}^n,$$

as $\boldsymbol{\varphi} \subset \mathbf{Q}[\mathbf{X}]$, since $\boldsymbol{\alpha} \in \mathbf{Q}^{2n}$ by above.

Let $F_Q = V \cap \boldsymbol{\varphi}_1^{-1}(Q)$. Since $\boldsymbol{\alpha} \in \Omega_{\mathsf{F}}(\boldsymbol{\chi}, V, \operatorname{sing}(V), \theta, 2)$, by Proposition 2.15 F_Q is either empty or equidimensional of dimension $d-1$, with $\operatorname{sing}(F_Q) \subset S_Q$, where

$$S_Q := \operatorname{sing}(V) \cup [W_{\boldsymbol{\varphi}}(1, V) \cap \boldsymbol{\varphi}_1^{-1}(Q)] = K_{\boldsymbol{\varphi}}(1, V),$$

since $\boldsymbol{\varphi}_1(K_{\boldsymbol{\varphi}}(1, V)) \subset \boldsymbol{\varphi}_1(Q)$.
Moreover, in the latter case, the sequence $F_{\text{atlas}}(\boldsymbol{\chi}, V, Q, \operatorname{sing}(V), \boldsymbol{\varphi})$ is an atlas of (F_Q, S_Q). The zero-dimensional parametrizations $\mathscr{P}_F$ and $\mathscr{S}_F$ describe respectively finite sets $\mathcal{P}_F$ and $\mathcal{S}_F$ such that

$$S_Q = \mathcal{S}_F \subset \mathcal{P}_F \subset F_Q,$$

and $\mathscr{S}_F$ has degree $\sigma_{\mathscr{S}_F} \leq (nD)^{n+2}$. Finally, recall that $\mathscr{Q}$ and $\mathscr{P}_F$ both have degree bounded by $O\tilde{}\left(\mu + n^{2d+3}D^{2n+5}\right)$. Hence, according to Proposition 3.7, and after a few straightforward simplifications, we deduce that RoadmapBounded either outputs fail or a one-dimensional parametrization $\mathscr{R}_F$ of degree at most

$$\mathscr{B}_{\mathscr{R}_F} = O\tilde{}\left(\mu 16^{3d}(n\log_2(n))^{2(2d-2+12\log_2(d-1))(\log_2(d-1)+6)} D^{(2n+4)(\log_2(d-1)+4)}\right),$$

using

$$O\tilde{}\left(\mu^3 16^{9d} E(n\log_2(n))^{6(2d+12\log_2(d-1))(\log_2(d-1)+7)} D^{3(2n+4)(\log_2(d-1)+5)}\right)$$

operations in $\mathbf{Q}$. Moreover, in case of success, $\mathscr{R}_F$ describes a roadmap of $(F_Q, \mathcal{P}_F)$.

Step 11 Remark that $\mathscr{W}_2$ and $\mathscr{R}_F$ both have degree at most $\mathscr{B}_{\mathscr{R}_F}$. hence, by [30, Lemma J.8], on input $\mathscr{W}_2$ and $\mathscr{R}_F$, Union either outputs fail or a one-dimensional parametrization of degree at most $O\tilde{}(\mathscr{B}_{\mathscr{R}_F})$ using $O\tilde{}(n\mathscr{B}_{\mathscr{R}_F}^3)$ operations in $\mathbf{Q}$. Therefore, the complexity of this step is bounded by the one of previous step. Moreover, in case of success, the output describes $W \cup F_Q$.

It follows that under assumption (A), all assumptions from Proposition 3.9 are satisfied. Hence, since $\mathsf{Z}(\mathscr{R}_F)$ is a roadmap of $(F_Q, \mathcal{P}_F)$ and $\mathcal{P}_F = \mathcal{P} \cup (F_Q \cap W)$, by Proposition 3.9, Algorithm 1 returns a roadmap of $(V, \mathcal{P})$. Since $\mathcal{P}$ contains $\mathcal{P}_0$, the output is a roadmap of $(V, \mathcal{P}_0)$ as well.

In conclusion, if $\boldsymbol{\alpha} \in \Omega$ and all calls to the subroutines are successful then, on input Γ and $\mathscr{P}_0$ such that assumption (A) is satisfied, Algorithm 1 outputs a one-dimensional parametrization encoding a roadmap of $(V, \mathcal{P}_0)$. Moreover this parametrization has degree bounded by $\mathscr{B}_{\mathscr{R}_F}$ and all steps have complexity bounded by the one of Step 10. Since these bounds match the ones given in the statement of Theorem 3.8, we are done. $\square$

Our main result, namely Theorem 1.1, is a direct consequence of Theorem 3.8 since, if $n-c<2$ then $\boldsymbol{V}(\boldsymbol{f})$ is a roadmap of $(\boldsymbol{V}(\boldsymbol{f}), \mathsf{Z}(\mathscr{P}))$.

Remark 2. Remark that, as long as the restriction of $\boldsymbol{\varphi}_1$ to $\boldsymbol{V}(\boldsymbol{f}) \cap \mathbf{R}^n$ is proper and bounded below, the above proof still holds. This could allow one, a more *ad-hoc* choice for $\boldsymbol{\varphi}$.

4 Subroutines

4.1 Proof of Lemma 3.2

Lemma 4.1. *Let Γ and $\Gamma^{\boldsymbol{\varphi}}$ be straight-line programs of respective lengths E and E' computing sequences of polynomials respectively $\boldsymbol{f}$ and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_i)$ in $\mathbf{Q}[x_1, \dots, x_n]$. Then there exists an algorithm* IncSLP *which takes as input Γ, $\Gamma^{\boldsymbol{\varphi}}$ and returns a straight-line program $\widetilde{\Gamma}$ of length*

$$E + E' + i,$$

that evaluates $\boldsymbol{f}^{\boldsymbol{\varphi}} = (\boldsymbol{f}, \varphi_1 - e_1, \dots, \varphi_i - e_i)$ in $\mathbf{Q}[\mathbf{E}, \mathbf{X}]$, where $\mathbf{E} = (e_1, \dots, e_i)$ are new variables.

Proof. Up to reordering, we can suppose that the polynomials $\varphi_1, \dots, \varphi_i$ correspond to the respective indices $E' - i + 1, \dots, E'$ in $\Gamma^{\boldsymbol{\varphi}}$. Let $1 \leq j \leq N$, then the straight-line program

$$\Gamma^{\boldsymbol{\varphi}-\mathbf{E}} = \Big(\Gamma^{\boldsymbol{\varphi}},\ (+,\ E'-i+1,\ -n-i+1), \dots, (+,\ E',\ -n)\Big)$$

has length $E' + i$ and computes $(\varphi_1 - e_1, \dots, \varphi_i - e_i)$ in $\mathbf{Q}[e_1, \dots, e_i, x_1, \dots, x_n]$. Finally let

$$\Gamma' = (\Gamma, \Gamma^{\boldsymbol{\varphi}-\mathbf{E}}),$$

then Γ' is a straight-line program of length $E+E'+i$, that computes $\boldsymbol{f}^{\boldsymbol{\varphi}} = (\boldsymbol{f}, \varphi_1 - e_1, \dots, \varphi_i - e_i)$ in $\mathbf{Q}[e_1, \dots, e_i, x_1, \dots, x_n]$. $\square$

Let $1 \leq i \leq n$ be integers and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_i) \subset \mathbf{C}[\mathbf{X}]$, and set

$$\begin{array}{rccc} \Psi_{\boldsymbol{\varphi}}\colon & \mathbf{C}^n & \to & \mathbf{C}^{i+n} \\ & \boldsymbol{y} & \mapsto & (\boldsymbol{\varphi}(\boldsymbol{y}), \boldsymbol{y}) \end{array}.$$

Then $\Psi_{\boldsymbol{\varphi}}$ is an isomorphic embedding of algebraic sets, with inverse the projection on the last n coordinates. We call $\Psi_{\boldsymbol{\varphi}}$ the *incidence isomorphism associated to $\boldsymbol{\varphi}$*.

Let $V \subset \mathbf{C}^n$ be a d-equidimensional algebraic set with $1 \leq d \leq n$. Then $V^{\boldsymbol{\varphi}} = \Psi_{\boldsymbol{\varphi}}(V) \subset \mathbf{C}^{i+n}$ is called the *incidence variety associated to V with respect to $\boldsymbol{\varphi}$*, or in short, the incidence variety of $(V, \boldsymbol{\varphi})$.

Finally, we note $\boldsymbol{\pi} = (e_1, \dots, e_i)$ so that for $0 \leq j \leq i$, $\boldsymbol{\pi}_j$ is the canonical projection on the first j coordinates in $\mathbf{C}^{i+n}$. The following lemma is immediate, and illustrates the main feature that motivates the introduction of incidence varieties.

Lemma 4.2. *For any $0 \leq j \leq i$, the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "\Psi_{\boldsymbol{\varphi}}"] \arrow[dr, "\boldsymbol{\varphi}_j"'] & V^{\boldsymbol{\varphi}} \arrow[d, "\boldsymbol{\pi}_j"] \\ & \mathbf{C}^j \end{tikzcd}.$$

Lemma 4.3. *Let $\mathscr{Q}$ be a zero-dimensional parametrization of degree κ such that $\mathsf{Z}(\mathscr{Q}) \subset \mathbf{C}^n$ and let $\Gamma^{\boldsymbol{\varphi}}$ be a straight-line program of length E' which evaluates polynomials $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_i)$. There exists an algorithm* $\mathsf{IncParam}$ *which takes as input $\mathscr{Q}$, $\Gamma^{\boldsymbol{\varphi}}$ and returns a zero-dimensional parametrization $\widetilde{\mathscr{Q}}$ of degree κ and encoding $\Psi_{\boldsymbol{\varphi}}(\mathsf{Z}(\mathscr{Q})) \subset \mathbf{C}^{i+n}$, where $\Psi_{\boldsymbol{\varphi}}$ is the incidence isomorphism associated to $\boldsymbol{\varphi}$, using*

$$O^{\sim}\left(E'\kappa\right)$$

operations in $\mathbf{Q}$.

Proof. Write $\mathscr{Q} = ((q, v_1, \dots, v_n), \mathfrak{l})$ following the definition of zero-dimensional parametrizations given in the introduction. Since

$$\mathsf{Z}(\mathscr{Q}) = \{(v_1(\boldsymbol{t}), \dots, v_n(\boldsymbol{t})) \mid q(\boldsymbol{t}) = 0\}$$

then $\Psi_{\boldsymbol{\varphi}}(\mathsf{Z}(\mathscr{Q}))$ is

$$\Big\{\Big(\varphi_1\big(v_1(\boldsymbol{t}), \dots, v_n(\boldsymbol{t})\big), \dots, \varphi_i\big(v_1(\boldsymbol{t}), \dots, v_n(\boldsymbol{t})\big), v_1(\boldsymbol{t}), \dots, v_n(\boldsymbol{t})\Big) \mid q(\boldsymbol{t}) = 0\Big\}.$$

Let $e_1, \dots, e_i$ be new indeterminates and $\mathfrak{l}'(e_1, \dots, e_i, x_1, \dots, x_n) = \mathfrak{l}(x_1, \dots, x_n)$ and for all $1 \leq j \leq i$, let $w_j = \varphi_j(v_1, \dots, v_n) \mod q \in \mathbf{Q}[t]$. Then we claim that $\widetilde{\mathscr{Q}} = ((q, w_1, \dots, w_i, v_1, \dots, v_n), \mathfrak{l})$ is a zero-dimensional parametrization of $\Psi_{\boldsymbol{\varphi}}(\mathsf{Z}(\mathscr{Q}))$. Indeed for all $1 \leq j \leq i$, $\deg(w_j) < \deg(q)$ and

$$\mathfrak{l}'\big(w_1, \dots, w_i, v_1, \dots, v_n\big) = \mathfrak{l}\big(v_1, \dots, v_n\big) = t.$$

Besides, computing $\widetilde{\mathscr{Q}}$ is done by evaluating $\Gamma^{\boldsymbol{\varphi}}$ at $v_1, \dots, v_n$ doing all operations modulo q; this can be done using $O^{\sim}(E'\kappa)$ operations in $\mathbf{Q}$. □

We can now prove Lemma 3.2.

Proof of Lemma 3.2 Let $\Psi_{\boldsymbol{\varphi}}$ be the incidence isomorphism associated to $\boldsymbol{\varphi}$. By Lemma 4.2, the image of $\mathsf{Z}(\mathscr{P})$ by $\boldsymbol{\varphi}_j$, can be obtained by projecting the incidence variety $\Psi_{\boldsymbol{\varphi}}(\mathsf{Z}(\mathscr{P}))$ on the first j coordinates.

Hence the algorithm Image can be performed as follows. First, according to Lemma 4.3, there exists an algorithm $\mathsf{IncParam}$ which, on input $\mathscr{P}$ and $\Gamma^{\boldsymbol{\varphi}}$, computes a zero-dimensional parametrization $\widetilde{\mathscr{P}}$ of degree κ, encoding $\Psi_{\boldsymbol{\varphi}}(\mathsf{Z}(\mathscr{P})) \subset \mathbf{C}^{j+n}$, and using $O^{\sim}(E'\kappa)$ operations in $\mathbf{Q}$. Secondly, according to [30, Lemma J.5.], there exists an algorithm $\mathsf{Projection}$ which, on input $\widetilde{\mathscr{P}}$ and $j \in \{1, \dots, i\}$, computes a zero-dimensional parametrization $\mathscr{Q}$ encoding

$$\boldsymbol{\pi}_j(\widetilde{\mathscr{P}}) = \boldsymbol{\pi}_j\left(\Psi_{\boldsymbol{\varphi}}(\mathsf{Z}(\mathscr{P}))\right) = \boldsymbol{\varphi}_j(\mathsf{Z}(\mathscr{P})),$$

using $O^{\sim}\left(n^2\kappa^2\right)$ operations in $\mathbf{Q}$. □

4.2 Auxiliary results for generalized polar varieties

We reuse the notation introduced in the previous subsection. Let $\mathbf{E} = (e_1 \dots, e_i)$ new indeterminates. Recall that $V \subset \mathbf{C}^n$ is a d-equidimensional algebraic set.

Lemma 4.4. *Let $\boldsymbol{h} \subset \mathbf{C}[\mathbf{X}]$ be a set of generators of $\boldsymbol{I}(V)$. Then*

$$\boldsymbol{h}^{\boldsymbol{\varphi}} = (\boldsymbol{h}, \varphi_1 - e_1, \dots, \varphi_i - e_1) \subset \mathbf{C}[\mathbf{E}, \mathbf{X}]$$

is a set of generator of $\boldsymbol{I}(V^{\boldsymbol{\varphi}}) \subset \mathbf{C}[\mathbf{E}, \mathbf{X}]$, which is equidimensional of dimension d.

Proof. Remark that by Lemma 2.2, for any $(\boldsymbol{t}, \boldsymbol{y}) \in V^{\boldsymbol{\varphi}}$,

$$\operatorname{rank} \operatorname{Jac}_{\boldsymbol{t},\boldsymbol{y}}(\boldsymbol{h}^{\boldsymbol{\varphi}}) = \operatorname{rank} \begin{bmatrix} \mathbf{0} & \operatorname{Jac}_{\boldsymbol{y}}(\boldsymbol{h}) \\ -I_i & \operatorname{Jac}_{\boldsymbol{y}}(\boldsymbol{\varphi}) \end{bmatrix} = \operatorname{rank} \operatorname{Jac}_{\boldsymbol{y}}(\boldsymbol{h}) + i,$$

so that for all $\boldsymbol{y} \in \operatorname{reg}(V)$, since $\operatorname{Jac}(\boldsymbol{h})$ has rank $n-d$ at $\boldsymbol{y}$, then $\operatorname{Jac}(\boldsymbol{h}^{\boldsymbol{\varphi}})$ has rank $n-d+i$ at $\Psi_{\boldsymbol{\varphi}}(\boldsymbol{y})$. Hence, since $\operatorname{reg}(V)$ is Zariski dense in V, by [29, Lemma 15] $\langle \boldsymbol{h}^{\boldsymbol{\varphi}} \rangle$ is an equidimensional radical ideal of dimension d.

Besides, let $(\boldsymbol{t}, \boldsymbol{y}) \in \mathbf{C}^n$, then $\boldsymbol{h}^{\boldsymbol{\varphi}}(\boldsymbol{t}, \boldsymbol{y}) = 0$ if and only if $\boldsymbol{h}(\boldsymbol{y}) = 0$ and $\boldsymbol{\varphi}(\boldsymbol{y}) = \boldsymbol{t}$ that is $(\boldsymbol{t}, \boldsymbol{y}) \in V^{\boldsymbol{\varphi}}$ since $\boldsymbol{h}$ generates $\boldsymbol{I}(V)$. Hence $\boldsymbol{V}(\langle \boldsymbol{h}^{\boldsymbol{\varphi}} \rangle) = V^{\boldsymbol{\varphi}}$ so that by Hilbert's Nullstellensatz [14, Theorem 1.6],

$$\boldsymbol{I}(V^{\boldsymbol{\varphi}}) = \sqrt{\langle \boldsymbol{h}^{\boldsymbol{\varphi}} \rangle} = \langle \boldsymbol{h}^{\boldsymbol{\varphi}} \rangle .$$

□

The following lemma shows an important consequence of Lemma 4.2 for polar varieties.

Lemma 4.5. *For $0 \leq j \leq i$, the restriction of $\Psi_{\boldsymbol{\varphi}}$ induces an isomorphism between $W_{\boldsymbol{\varphi}}(j, V)$ (resp. $K_{\boldsymbol{\varphi}}(j, V)$) and $W(\boldsymbol{\pi}_j, V^{\boldsymbol{\varphi}})$ (resp. $K(\boldsymbol{\pi}_j, V^{\boldsymbol{\varphi}})$).*

Proof. Let $\boldsymbol{h}$ be generators of $\boldsymbol{I}(V)$. By Lemma 4.4, $\boldsymbol{h}^{\boldsymbol{\varphi}}$ are generators of $\boldsymbol{I}(V)$. Let $\boldsymbol{y} \in V$, $\boldsymbol{y}^{\boldsymbol{\varphi}} = \Psi_{\boldsymbol{\varphi}}(\boldsymbol{y}) \in V^{\boldsymbol{\varphi}}$ and $0 \leq j \leq i$. Then by Lemma 2.2,

$$\operatorname{rank} \operatorname{Jac}_{\boldsymbol{y}^{\boldsymbol{\varphi}}}([\boldsymbol{h}^{\boldsymbol{\varphi}}, \boldsymbol{\pi}_j]) = \operatorname{rank} \begin{bmatrix} & \mathbf{0} & \operatorname{Jac}_{\boldsymbol{y}}(\boldsymbol{h}) \\ & -I_i & \operatorname{Jac}_{\boldsymbol{y}}(\boldsymbol{\varphi}) \\ I_j & \mathbf{0} & \mathbf{0} \end{bmatrix} = \operatorname{rank} \operatorname{Jac}_{\boldsymbol{y}}([\boldsymbol{h}, \boldsymbol{\varphi}_j]) + i, \qquad (2)$$

where I_ℓ denotes the $\ell \times \ell$ identity matrix. Since both V and $V^{\boldsymbol{\varphi}}$ are d-equidimensional, then by [30, Lemma A.2], $K_{\boldsymbol{\varphi}}(j, V)$ and $K(\boldsymbol{\pi}_j, V^{\boldsymbol{\varphi}})$ are the sets of points $\boldsymbol{y} \in V$ and $\boldsymbol{y}^{\boldsymbol{\varphi}} \in V^{\boldsymbol{\varphi}}$ where respectively

$$\operatorname{Jac}_{\boldsymbol{y}}([\boldsymbol{h}, \boldsymbol{\varphi}_j]) < n - d + j \quad \text{and} \quad \operatorname{Jac}_{\boldsymbol{y}^{\boldsymbol{\varphi}}}([\boldsymbol{h}^{\boldsymbol{\varphi}}, \boldsymbol{\pi}_j]) < n + i - d + j.$$

Hence by (2), the two conditions are equivalent and then it holds that

$$\Psi_{\boldsymbol{\varphi}}(K_{\boldsymbol{\varphi}}(j, V)) = K(\boldsymbol{\pi}_j, V^{\boldsymbol{\varphi}}) \qquad \text{for all } 0 \leq j \leq i.$$

In particular, for $j = 0$, $\Psi_{\boldsymbol{\varphi}}(\operatorname{sing}(V)) = \operatorname{sing}(V^{\boldsymbol{\varphi}})$, so that for all $0 \leq j \leq i$,

$$\Psi_{\boldsymbol{\varphi}}(W^\circ_{\boldsymbol{\varphi}}(j, V)) = W^\circ(\boldsymbol{\pi}_j, V^{\boldsymbol{\varphi}}).$$

Since $\Psi_{\boldsymbol{\varphi}}$ is an isomorphism of algebraic sets, it is a homeomorphism for the Zariski topology, so that it maps the Zariski closure of sets to the Zariski closure of their image. Hence, we can conclude that $\Psi_{\boldsymbol{\varphi}}(W_{\boldsymbol{\varphi}}(j, V)) = W(\boldsymbol{\pi}_j, V^{\boldsymbol{\varphi}})$ for all $0 \leq j \leq i$. □

Lemma 4.6 (Chart and atlases)**.** *Let $1 \leq e \leq n$, $Q \subset \mathbf{C}^e$ be a finite set and S be an algebraic set such that V and S lie over Q with respect to $\boldsymbol{\varphi}$. By a slight abuse of notation, we denote equally $m \in \mathbf{C}[\mathbf{X}]$ when seen in $\mathbf{C}[\mathbf{E}, \mathbf{X}]$. Then, the following holds.*

1. *Let $\chi = (m, \boldsymbol{h}) \subset \mathbf{C}[\mathbf{X}]$ be a chart of $(V, Q, S, \boldsymbol{\varphi})$, then $\chi^{\boldsymbol{\varphi}} = (m, \boldsymbol{h}^{\boldsymbol{\varphi}}) \subset \mathbf{C}[\mathbf{E}, \mathbf{X}]$ is a chart of $(V^{\boldsymbol{\varphi}}, Q, S^{\boldsymbol{\varphi}}, \boldsymbol{\pi})$, where $S^{\boldsymbol{\varphi}} = \Psi_{\boldsymbol{\varphi}}(S)$.*

2. *Let $\boldsymbol{\chi} = (\chi_j)_{1 \leq j \leq s}$ be an atlas of $(V, Q, S, \boldsymbol{\varphi})$, then if $\boldsymbol{\chi}^{\boldsymbol{\varphi}} = (\chi_j^{\boldsymbol{\varphi}})_{1 \leq j \leq s}$ as defined in the previous item, $\boldsymbol{\chi}^{\boldsymbol{\varphi}}$ is an atlas of $(V^{\boldsymbol{\varphi}}, Q, S^{\boldsymbol{\varphi}}, \boldsymbol{\pi})$.*

Proof. We start with the first statement. Let Q, S and $\chi = (m, \boldsymbol{h})$ be as in the statement. Then, it holds that:

C_1 : Let $\boldsymbol{y} \in \mathcal{O}(m) \cap V - S$, which is non-empty by property C_1 of χ. Then by definition $\Psi_{\boldsymbol{\varphi}}(\boldsymbol{y}) \in V^{\boldsymbol{\varphi}}$, and since $\Psi_{\boldsymbol{\varphi}}$ is an isomorphism on $V^{\boldsymbol{\varphi}}$, $\Psi_{\boldsymbol{\varphi}}(\boldsymbol{y}) \notin S^{\boldsymbol{\varphi}}$. Finally since $m \in \mathbf{C}[\mathbf{X}]$, then $m(\Psi_{\boldsymbol{\varphi}}(\boldsymbol{y})) = m(\boldsymbol{y}) \neq 0$ so that $\mathcal{O}(m) \cap V^{\boldsymbol{\varphi}} - S^{\boldsymbol{\varphi}}$ is not empty.

C_2 : Note that since $m \in \mathbf{C}[\mathbf{X}]$, $\Psi_{\boldsymbol{\varphi}}(\mathcal{O}(m))$ is defined by $m \neq 0$. By a slight abuse of notation, we still denote this Zariski open set $\mathcal{O}(m)$. Hence, it follows from the definition of $\Psi_{\boldsymbol{\varphi}}$ that $\Psi_{\boldsymbol{\varphi}}(\mathcal{O}(m) \cap V - S) = \mathcal{O}(m) \cap V^{\boldsymbol{\varphi}} - S^{\boldsymbol{\varphi}}$. Besides, by Lemma 4.2, $\boldsymbol{\pi}_e \circ \Psi_{\boldsymbol{\varphi}}$ and $\boldsymbol{\varphi}_e$ coincide on V. Then
$$Z_{|\boldsymbol{\varphi}_e \in Q} = \Psi_{\boldsymbol{\varphi}}(Z)_{|\boldsymbol{\pi}_e \in Q} \text{ for any } Z \subset V.$$
Finally, as seen in the proof of Lemma 4.4, $\Psi_{\boldsymbol{\varphi}}(\boldsymbol{V}(\boldsymbol{h})) = \boldsymbol{V}(\boldsymbol{h}^{\boldsymbol{\varphi}})$. Hence by property C_2 of χ,
$$\mathcal{O}(m) \cap V^{\boldsymbol{\varphi}} - S^{\boldsymbol{\varphi}} = \Psi_{\boldsymbol{\varphi}}(\mathcal{O}(m) \cap \boldsymbol{V}(\boldsymbol{h})_{|\boldsymbol{\varphi}_e \in Q} - S) = \mathcal{O}(m) \cap \boldsymbol{V}(\boldsymbol{h}^{\boldsymbol{\varphi}})_{|\boldsymbol{\pi}_e \in Q} - S^{\boldsymbol{\varphi}},$$
since $\mathcal{O}(m) \cap \boldsymbol{V}(\boldsymbol{h})_{|\boldsymbol{\varphi}_e \in Q} - S$ is a subset of V.

C_3 : Let c be the cardinality of $\boldsymbol{h}$, then $\boldsymbol{h}^{\boldsymbol{\varphi}}$ has cardinality $c + i$. Hence by property C_3 of χ, $e + c + i \leq i + n$ as required.

C_4 : Finally let $\boldsymbol{y}^{\boldsymbol{\varphi}} = (\boldsymbol{t}, \boldsymbol{y}) \in \mathcal{O}(m) \cap V^{\boldsymbol{\varphi}} - S^{\boldsymbol{\varphi}}$, we know from above that $\boldsymbol{y} \in \mathcal{O}(m) \cap V - S$, so that by property C_4 of χ, $\mathrm{Jac}_{\boldsymbol{y}}[\boldsymbol{h}, \boldsymbol{\varphi}_e]$ has full rank $c + e$. But by equality (2) in the proof of Lemma 4.5, this means that $\mathrm{Jac}_{\boldsymbol{y}^{\boldsymbol{\varphi}}}([\boldsymbol{h}^{\boldsymbol{\varphi}}, \boldsymbol{\pi}_e])$ has full rank $c + i + e$ as required.

We have shown that charts can be transferred to incidence varieties, let us now prove that this naturally gives rise to atlases. Consider an atlas $\boldsymbol{\chi} = (\chi_j)_{1 \leq j \leq s}$ of $(V, Q, S, \boldsymbol{\varphi})$, and let $\boldsymbol{\chi}^{\boldsymbol{\varphi}} = (\chi_j^{\boldsymbol{\varphi}})_{1 \leq j \leq s}$, where for all $1 \leq j \leq s$, $\chi_j^{\boldsymbol{\varphi}}$ is defined from χ_j as above. We proved that $\boldsymbol{\chi}^{\boldsymbol{\varphi}}$ is an atlas of $(V^{\boldsymbol{\varphi}}, Q, S^{\boldsymbol{\varphi}}, \boldsymbol{\pi})$.

Property A_1 is straightforward, and A_2 is given by the first statement of this lemma which we just proved. Finally, since $\Psi_{\boldsymbol{\varphi}}(V - S) = V^{\boldsymbol{\varphi}} - S^{\boldsymbol{\varphi}}$, then for any $\boldsymbol{y}^{\boldsymbol{\varphi}} = (\boldsymbol{t}, \boldsymbol{y}) \in V^{\boldsymbol{\varphi}} - S^{\boldsymbol{\varphi}}$, by property A_3 of $\boldsymbol{\chi}$, there exists $1 \leq j \leq s$ such that $m_j(\boldsymbol{y}^{\boldsymbol{\varphi}}) = m_j(\boldsymbol{y}) \neq 0$. Then $\boldsymbol{\chi}^{\boldsymbol{\varphi}}$ satisfies property A_3 of atlases. $\square$

We deduce the following results for two important particular cases.

Lemma 4.7. *Let $S \subset \mathbf{C}^n$ be an algebraic set, $\chi = (m, \boldsymbol{h})$ and $\boldsymbol{\chi} = (\chi_j)_{1 \leq j \leq s}$ be respectively a chart and an atlas of (V, S), and let $\chi^{\boldsymbol{\varphi}}$ and $\boldsymbol{\chi}^{\boldsymbol{\varphi}}$ the chart and atlas constructed from respectively χ and $\boldsymbol{\chi}^{\boldsymbol{\varphi}}$ as in Lemma 4.6. The following holds.*

1. *If $\boldsymbol{h}$ has cardinality c, then for any c-minor m' of $\mathrm{Jac}(\boldsymbol{h})$ and any $(c+i-1)$-minor m'' of $\mathrm{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$, containing the rows of $\mathrm{Jac}(\boldsymbol{\varphi}_i)$, the following holds. If $W_{\mathrm{chart}}(\chi, m', m'')$ is a chart of $\mathscr{W} = (W_{\boldsymbol{\varphi}}(i, V), S)$, then $W_{\mathrm{chart}}(\chi^{\boldsymbol{\varphi}}, m', m'')$ is a chart of $\mathscr{W}^{\boldsymbol{\varphi}} = (W(\boldsymbol{\pi}_i, V^{\boldsymbol{\varphi}}), S^{\boldsymbol{\varphi}})$.*

2. *If $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, i)$ is an atlas of $\mathscr{W}$ then, $W_{\mathrm{atlas}}(\boldsymbol{\chi}^{\boldsymbol{\varphi}}, V^{\boldsymbol{\varphi}}, S^{\boldsymbol{\varphi}}, \boldsymbol{\pi}, i)$ is an atlas of $\mathscr{W}^{\boldsymbol{\varphi}}$.*

Proof. Let m' and m'' be respectively a c-minor of $\mathrm{Jac}(\boldsymbol{h})$ and a $(c+i-1)$-minor of $\mathrm{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$, containing the rows of $\mathrm{Jac}(\boldsymbol{\varphi}_i)$. Assume that
$$W_{\mathrm{chart}}(\chi, m', m'', \boldsymbol{\varphi}) = \Big(mm'm'',\ \big(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')\big) \Big)$$

is a chart of $\mathscr{W}$. By C_1, $\mathcal{O}(mm'm'') \cap W_{\boldsymbol{\varphi}}(i, V) - S$ is not empty, so that m' and m'' are not identically zero. Since

$$\mathrm{Jac}(\boldsymbol{h}^{\boldsymbol{\varphi}}) = \begin{pmatrix} \mathbf{0} & \mathrm{Jac}(\boldsymbol{h}) \\ -I_i & \mathrm{Jac}(\boldsymbol{\varphi}_i) \end{pmatrix},$$

Lemma 2.1 shows that m' is a $(c+i)$-minor of $\mathrm{Jac}(\boldsymbol{h}^{\boldsymbol{\varphi}})$ and m'' is a $(c+i+i-1)$-minor of $\mathrm{Jac}(\boldsymbol{h}^{\boldsymbol{\varphi}}, \boldsymbol{\pi}_i)$ containing $I_i = \mathrm{Jac}(\boldsymbol{\pi}_i)$. Hence, according to Definition 2.9,

$$W_{\mathrm{chart}}(\chi^{\boldsymbol{\varphi}}, m', m'') = \left(mm'm'', (\boldsymbol{h}^{\boldsymbol{\varphi}}, \mathcal{H}_{\boldsymbol{\pi}}(\boldsymbol{h}^{\boldsymbol{\varphi}}, i, m''))\right),$$

where, by definition, $\mathcal{H}_{\boldsymbol{\pi}}(\boldsymbol{h}^{\boldsymbol{\varphi}}, i, m'')$ is the sequence of $(c+i+i)$-minors of $\mathrm{Jac}([\boldsymbol{h}^{\boldsymbol{\varphi}}, \boldsymbol{\pi}_i])$ obtained by successively adding the missing row and the missing columns of $\mathrm{Jac}([\boldsymbol{h}^{\boldsymbol{\varphi}}, \boldsymbol{\pi}_i])$ to m''.

But, since $m'' \neq 0$, Lemma 2.11 implies that $\mathcal{H}_{\boldsymbol{\pi}}(\boldsymbol{h}^{\boldsymbol{\varphi}}, i, m'')$ is, as well, the sequence of $(c+i)$-minors obtained by successively adding the missing row and the missing columns of $\mathrm{Jac}(\boldsymbol{h}^{\boldsymbol{\varphi}}, i) = \mathrm{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i)$ to m''. We deduce that

$$\mathcal{H}_{\boldsymbol{\pi}}(\boldsymbol{h}^{\boldsymbol{\varphi}}, i, m'') = \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''),$$

so that if $\boldsymbol{g} = (\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''))$, then $\boldsymbol{g}^{\boldsymbol{\varphi}} = (\boldsymbol{h}^{\boldsymbol{\varphi}}, \mathcal{H}_{\boldsymbol{\pi}}(\boldsymbol{h}^{\boldsymbol{\varphi}}, i, m''))$.
Hence $W_{\mathrm{chart}}(\chi^{\boldsymbol{\varphi}}, m', m'')$ is the chart constructed from $W_{\mathrm{chart}}(\chi, m', m'', \boldsymbol{\varphi})$ in Lemma 4.6, and since, by Lemma 4.5, $\Psi_{\boldsymbol{\varphi}}(W_{\boldsymbol{\varphi}}(i, V)) = W(\boldsymbol{\pi}_i, V^{\boldsymbol{\varphi}})$, the first statement of Lemma 4.6 implies that $W_{\mathrm{chart}}(\chi^{\boldsymbol{\varphi}}, m', m'')$ is a chart of $\mathscr{W}^{\boldsymbol{\varphi}}$.

To prove the second assertion, remark that by Lemma 2.11 (third assertion), $W_{\mathrm{atlas}}(\boldsymbol{\chi}^{\boldsymbol{\varphi}}, V^{\boldsymbol{\varphi}}, S^{\boldsymbol{\varphi}}, \boldsymbol{\pi}, i)$ is the sequence of all those $W_{\mathrm{chart}}(\chi_j^{\boldsymbol{\varphi}}, m', m'')$, for $j \in \{1, \ldots, s\}$ and for m', m'' respectively a $c+i$-minor of $\mathrm{Jac}(\boldsymbol{h}_j^{\boldsymbol{\varphi}})$ and a $(c+i-1)$-minor of $\mathrm{Jac}(\boldsymbol{h}_j^{\boldsymbol{\varphi}}, i)$ for which $\mathcal{O}(m_j m'm'') \cap W(\boldsymbol{\pi}_i, V^{\boldsymbol{\varphi}}) - S$ is not empty.

As seen above, the polynomials m' and m'' are actually c-minors of $\mathrm{Jac}(\boldsymbol{h}_j)$ and $(c+i-1)$-minors of $\mathrm{Jac}([\boldsymbol{h}_j^{\boldsymbol{\varphi}}, \boldsymbol{\varphi}_i])$, and in the first point, we prove that $W_{\mathrm{chart}}(\chi_j^{\boldsymbol{\varphi}}, m', m'')$ is the chart constructed in the first point of Lemma 4.6 from $W_{\mathrm{chart}}(\chi_j, m', m'')$. Hence $W_{\mathrm{atlas}}(\boldsymbol{\chi}^{\boldsymbol{\varphi}}, V^{\boldsymbol{\varphi}}, S^{\boldsymbol{\varphi}}, \boldsymbol{\pi}, i)$ is exactly the atlas constructed from $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, i)$ in the second item of Lemma 4.6.
In conclusion, by Lemma 4.6, if $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, i)$ is an atlas of $\mathscr{W}$, then $W_{\mathrm{atlas}}(\boldsymbol{\chi}^{\boldsymbol{\varphi}}, V^{\boldsymbol{\varphi}}, S^{\boldsymbol{\varphi}}, \boldsymbol{\pi}, i)$ is an atlas of $\mathscr{W}^{\boldsymbol{\varphi}}$. □

Lemma 4.8. *Let $1 \leq e \leq n$, $Q \subset \mathbf{C}^e$ be a finite set and S be an algebraic set such that V and S lie over Q with respect to $\boldsymbol{\varphi}$. Let further*

$$\mathscr{F} = \left(V_{|\boldsymbol{\varphi}_e \in Q},\ (S \cup W_{\boldsymbol{\varphi}}(e, V))_{|\boldsymbol{\varphi}_e \in Q}\right)$$

and

$$\mathscr{F}^{\boldsymbol{\varphi}} = \left(V^{\boldsymbol{\varphi}}_{|\boldsymbol{\pi}_e \in Q},\ (S^{\boldsymbol{\varphi}} \cup W(\boldsymbol{\pi}_e, V^{\boldsymbol{\varphi}}))_{|\boldsymbol{\pi}_e \in Q}\right).$$

Let $\chi = (m, \boldsymbol{h})$ and $\boldsymbol{\chi} = (\chi_j)_{1 \leq j \leq s}$ be respectively a chart and an atlas of $(V, Q, S, \boldsymbol{\varphi})$ and let $\chi^{\boldsymbol{\varphi}}$ and $\boldsymbol{\chi}^{\boldsymbol{\varphi}}$ the chart and atlas constructed from respectively χ and $\boldsymbol{\chi}^{\boldsymbol{\varphi}}$ as in Lemma 4.6.

If $F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\varphi})$ is an atlas of $\mathscr{F}$ then $F_{\mathrm{atlas}}(\boldsymbol{\chi}^{\boldsymbol{\varphi}}, V^{\boldsymbol{\varphi}}, Q, S^{\boldsymbol{\varphi}}, \boldsymbol{\pi})$ is an atlas of $\mathscr{F}^{\boldsymbol{\varphi}}$.

Proof. Without loss of generality one can assume that $S \subset V$. Since by Lemma 4.2, $\boldsymbol{\pi}_e \circ \Psi_{\boldsymbol{\varphi}}$ and $\boldsymbol{\varphi}_e$ coincide on V, then

$$\Psi_{\boldsymbol{\varphi}}\left((S \cup W_{\boldsymbol{\varphi}}(e, V))_{|\boldsymbol{\varphi}_e \in Q}\right) = (S^{\boldsymbol{\varphi}} \cup W(\boldsymbol{\pi}_e, V^{\boldsymbol{\varphi}}))_{|\boldsymbol{\pi}_e \in Q}.$$

Hence, for any $1 \leq j \leq s$, $\mathcal{O}(m_j) \cap V^{\boldsymbol{\varphi}}_{|\boldsymbol{\pi}_e \in Q} - (S^{\boldsymbol{\varphi}} \cup W(\boldsymbol{\pi}_e, V^{\boldsymbol{\varphi}}))_{|\boldsymbol{\pi}_e \in Q}$ coincides with

$$\Psi_{\boldsymbol{\varphi}}\left(\mathcal{O}(m_j) \cap V_{|\boldsymbol{\varphi}_e \in Q} - (S \cup W_{\boldsymbol{\varphi}}(e, V))_{|\boldsymbol{\varphi}_e \in Q}\right),$$

so that these sets are not-empty for the same j's in $\{1, \ldots, s\}$.
Hence $F_{\mathrm{atlas}}(\boldsymbol{\chi}^{\boldsymbol{\varphi}}, V^{\boldsymbol{\varphi}}, Q, S^{\boldsymbol{\varphi}}, \boldsymbol{\pi})$ is the atlas constructed from $F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\pi})$ in Lemma 4.6.

In conclusion, by the second assertion of Lemma 4.6, if $F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\pi})$ is an atlas of $\mathscr{F}$ then $F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\pi})$ is an atlas of $\mathscr{F}^{\boldsymbol{\varphi}}$. □

4.3 Lagrange systems

We present here a simplified version of generalized Lagrange systems defined in [30, Section 5.2] to encode polar varieties and provide equivalent results adapted to our case. As we only use a simplified version (involving a single block of Lagrange multipliers), we simply call them Lagrange systems.

4.3.a Definitions

The following is nothing but a simplified version of [30, Definition 5.3].

Definition 4.9. *A* Lagrange system *is a triple* $L = (\Gamma, \mathscr{Q}, \mathscr{S})$ *where*

- Γ *is a straight-line program evaluating a sequence of polynomials* $\boldsymbol{F} = (\boldsymbol{f}, \boldsymbol{g}) \subset \mathbf{Q}[\mathbf{X}, \boldsymbol{L}]$, *where*
  - $\mathbf{X} = (X_1, \dots, X_n)$ *and* $\boldsymbol{L} = (L_1, \dots, L_m)$*;*
  - $\boldsymbol{f} = (f_1, \dots, f_p) \subset \mathbf{Q}[\mathbf{X}]$ *and* $\boldsymbol{g} = (g_1, \dots, g_q) \subset \mathbf{Q}[\mathbf{X}, \boldsymbol{L}]$ *with* $\deg_{\boldsymbol{L}} \boldsymbol{g} \leq 1$*;*
- $\mathscr{Q}$ *is a zero-dimensional parametrization with coefficients in* $\mathbf{Q}$, *with* $Q = \mathsf{Z}(\mathscr{Q}) \subset \mathbf{C}^e$*;*
- $\mathscr{S}$ *is a zero-dimensional parametrization with coefficients in* $\mathbf{Q}$, *with* $S = \mathsf{Z}(\mathscr{S}) \subset \mathbf{C}^n$ *lying over* Q*;*
- $(n+m) - (p+q) \geq e$.

We also define N *and* P *as respectively the number of variables and equations, so that*

$$N = n + m, \quad P = p + q \quad \text{and} \quad d = N - e - P \geq 0.$$

One checks that such a Lagrange system is also a generalized Lagrange system in the sense of [30, Definition 5.3]. We can then define the same objects associated to such systems as follows. We denote by $\pi_{\mathbf{X}} \colon \mathbf{C}^N \to \mathbf{C}^n$ the projection on the variables associated to $\mathbf{X}$ in any set of $\mathbf{C}^N$ defined by equations in $\mathbf{C}[\mathbf{X}, \boldsymbol{L}]$.

Definition 4.10. *Let* $L = (\Gamma, \mathscr{Q}, \mathscr{S})$ *be a Lagrange system and all associated data defined in Definition 4.9. We define the following objects:*

- *the* type of L *is the triple* $T = (\boldsymbol{n}, \boldsymbol{p}, e)$ *where* $\boldsymbol{n} = (n, m)$ *and* $\boldsymbol{p} = (p, q)$*;*
- $\mathscr{U}(L) = \pi_{\mathbf{X}}\Big(\boldsymbol{V}(\boldsymbol{F})_{|\boldsymbol{\pi}_e \in Q} - \pi_{\mathbf{X}}^{-1}(S)\Big) \subset \mathbf{C}^n$
- $\overline{\mathscr{U}(L)}^{Z} \subset \mathbf{C}^n$ *the Zariski closure of* $\mathscr{U}(L)$.

Then we say that L *defines* $\overline{\mathscr{U}(L)}^{Z}$.

(We see here that Lagrange systems are nothing but generalized Lagrange systems of type $(1, \boldsymbol{n}, \boldsymbol{p}, e)$, in the sense of [30]). We now define local and global normal forms, that can be seen as equivalent to charts and atlases for Lagrange systems, replacing the notion of complete intersection by the one of normal form presented below.

For any non-zero polynomial M of a polynomial ring $\mathbf{C}[\mathbf{Y}]$ we denote by $\mathbf{C}[\mathbf{Y}]_M$ the localization of $\mathbf{C}[\mathbf{Y}]$ at M, that is the of all g/M^j where $\boldsymbol{g} \in \mathbf{C}[\mathbf{Y}]$ and $j \in \mathbb{N}$.

Definition 4.11. *For a non-zero $M \in \mathbf{Q}[\mathbf{X}]$ and polynomials $\boldsymbol{H} \subset \mathbf{Q}[\mathbf{X}, \boldsymbol{L}]_M$, we say that $\boldsymbol{H}$* is in normal form in $\mathbf{Q}[\mathbf{X}, \boldsymbol{L}]$ *if these polynomials have the form*

$$\boldsymbol{H} = (h_1, \ldots, h_c, L_1 - \rho_1, \ldots, L_m - \rho_m),$$

where the h_j's are in $\mathbf{Q}[\mathbf{X}]$ and the ρ_j's are in $\mathbf{Q}[\mathbf{X}]_M$. We call $\boldsymbol{h} = (h_1, \ldots, h_c)$ and $\boldsymbol{\rho} = (L_j - \rho_j)_{1 \le j \le m}$ respectively the $\mathbf{X}$- and $\boldsymbol{L}$-components of $\boldsymbol{H}$.

Definition 4.12. *A* local normal form *of a Lagrange system $L = (\Gamma, \mathscr{Q}, \mathscr{S})$ is the data of $\psi = (\mathfrak{m}, \mathfrak{d}, \boldsymbol{h}, \boldsymbol{H})$ that satisfies the following conditions:*

- L_1 $\mathfrak{m}, \mathfrak{d} \in \mathbf{Q}[\mathbf{X}] - \{0\}$ *and $\boldsymbol{H}$ is in normal form in $\mathbf{Q}[\mathbf{X}, \boldsymbol{L}]_{\mathfrak{m}\mathfrak{d}}$ with $\mathbf{X}$-component $\boldsymbol{h} = (h_1, \ldots, h_c)$;*
- L_2 *$\boldsymbol{H}$ and $\boldsymbol{F}$ have the same cardinality $n - c = N - P$;*
- L_3 *$\langle \boldsymbol{F}, \boldsymbol{I}(Q) \rangle = \langle \boldsymbol{H}, \boldsymbol{I}(Q) \rangle$ in $\mathbf{Q}[\mathbf{X}, \boldsymbol{L}]_{\mathfrak{m},\mathfrak{d}}$;*
- L_4 *$(\mathfrak{m}, \boldsymbol{h})$ is a chart of (V, Q, S);*
- L_5 *$\mathfrak{d}$ does not vanish on $\mathcal{O}(\mathfrak{m}) \cap \mathscr{U}(L)$.*

Given such a local normal form ψ we will note $\chi = (\mathfrak{m}, \boldsymbol{h})$ the associated chart.

As for atlases and charts, we define now global normal forms using local normal forms. The definition takes into consideration a family $\mathscr{Y} = (Y_1, \ldots, Y_r)$ of algebraic sets; this is specifically needed to help us prove correctness of the main algorithm.

Definition 4.13. *A* global normal form *of a Lagrange system $L = (\Gamma, \mathscr{Q}, \mathscr{S})$ is the data of $\boldsymbol{\psi} = (\psi_j)_{1 \le j \le s}$ such that:*

- G_1 *each $\psi_j = (\mathfrak{m}_j, \mathfrak{d}_j, \boldsymbol{h}_j, \boldsymbol{H}_j)$ is a local normal form;*
- G_2 *$\boldsymbol{\chi} = ((\mathfrak{m}_j, \boldsymbol{h}_j))_{1 \le j \le s}$ is an atlas of (V, Q, S).*

Let further $\mathscr{Y} = (Y_1, \ldots, Y_r)$ be algebraic subsets of $\mathbf{C}^n$. A global normal form of $(L; \mathscr{Y})$ is the data of a global normal form $\boldsymbol{\psi} = (\psi_j)_{1 \le j \le s}$ of L such that for all $1 \le j \le s$ and $1 \le k \le r$:

- G_3 *for any irreducible component Y of Y_k contained in V and such that $\mathcal{O}(\mathfrak{m}_j) \cap Y - S$ is not empty, $\mathcal{O}(\mathfrak{m}_j \mathfrak{d}_j) \cap Y - S$ is not empty.*

We say that L (resp. $(L; \mathscr{Y})$) has the global normal form property *if there exists a global normal form $\boldsymbol{\psi}$ of L (resp. $(L; \mathscr{Y})$) and we will note $\boldsymbol{\chi}$ the associated atlas.*

4.3.b Lagrange system for polar varieties

We give here a slightly different version of results presented in [30, Section 5.5]. We first recall the construction of [30, Definition 5.11] adapted to our more elementary case.

Definition 4.14. *Let $L = (\Gamma, (1), \mathscr{S})$ be a Lagrange system whose type is $((n, 0), (p, 0), 0)$, let $\boldsymbol{f} \subset \mathbf{C}[\mathbf{X}]$ be the polynomials which are evaluated by Γ and let $i \in \{1, \ldots, n-p\}$.*

Let $\boldsymbol{L} = (L_1, \ldots, L_p)$ be new indeterminates, for $\boldsymbol{u} = (u_1, \ldots, u_p) \in \mathbf{Q}^p$, define

$$\boldsymbol{F}_{\boldsymbol{u}} = \Big(\boldsymbol{f},\ \mathsf{Lagrange}(\boldsymbol{f}, i, \boldsymbol{L}),\ u_1 L_1 + \cdots + u_p L_p - 1\Big),$$

where $\mathsf{Lagrange}(\boldsymbol{f}, i, \boldsymbol{L})$ denotes the entries of

$$\begin{bmatrix} L_1 & \cdots & L_p \end{bmatrix} \cdot \mathrm{Jac}(\boldsymbol{f}, i).$$

We define $W_{\mathrm{lag}}(L, \boldsymbol{u}, i)$ as the triplet $(\Gamma_{\boldsymbol{u}}, \mathscr{Q}, \mathscr{S})$, where $\Gamma_{\boldsymbol{u}}$ is a straight-line program that evaluates $\boldsymbol{F}_{\boldsymbol{u}}$, it is a Lagrange system of type $((n, p), (p, n - i + 1), 0)$.

We can now prove an analog of [30, Proposition 5.13].

Proposition 4.15. *Let $V, S \subset \mathbf{C}^n$ be two algebraic sets with V d-equidimensional and S finite. Let $\boldsymbol{\chi}$ be an atlas of (V, S) and let $i \in \{2, \ldots, (d+3)/2\}$. Write $W = W(\boldsymbol{\pi}_i, V)$ and assume that the following holds. Either W is empty or it is equidimensional of dimension $i-1$, with $\operatorname{sing}(W) \subset S$, and $W_{\text{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\pi}, i)$ is an atlas of (W, S).*

Let $L = (\Gamma, (1), \mathscr{S})$ be a Lagrange system such that $V = \overline{\mathscr{U}(L)}$ and $S = \mathsf{Z}(\mathscr{S})$. Let $\mathscr{Y} = (Y_1, \ldots, Y_r)$ be algebraic sets in $\mathbf{C}^n$ and let finally $\boldsymbol{\psi}$ be a global normal form for $(L; (W, \mathscr{Y}))$ such that $\boldsymbol{\chi}$ is the associated atlas of (V, S). There exists a non-empty Zariski open subset $\mathscr{I}(L, \boldsymbol{\psi}, \mathscr{Y})$ of $\mathbf{C}^p$ such that for all $\boldsymbol{u} \in \mathscr{I}(L, \boldsymbol{\psi}, \mathscr{Y}) \cap \mathbf{Q}^p$, the following holds:

- *$W_{\text{lag}}(L, \boldsymbol{u}, i)$ is a Lagrange system that defines W;*
- *if $W \neq \emptyset$, then $(W_{\text{lag}}(L, \boldsymbol{u}, i); \mathscr{Y})$ has a global normal form whose atlas is $W_{\text{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\pi}, i)$.*

Proof. The statement of this proposition is identical to [30, Proposition 5.13] except that, in [30, Proposition 5.13], our assumptions on W are replaced by a generic linear change of variables on V. [30, Proposition 5.13] claims the same statements on $V^{\mathbf{A}}$ where $\mathbf{A}$ is assumed to lie in a non-empty Zariski open set $\mathscr{G}_1(\boldsymbol{\chi}, V, \emptyset, S, i)$ defined in [30, Proposition 3.4].

In the proof of [30, Proposition 5.13], the fact that $\mathbf{A}$ lies in $\mathscr{G}_1(\boldsymbol{\chi}, V, \emptyset, S, i)$ allows one to assume that the statements of [30, Proposition 3.4] but also [30, Lemma B.12] hold. In our proposition stated above, according to Lemma 2.11, the assumptions on W are exactly the conclusion of [30, Proposition 3.4], while [30, Lemma B.12] is nothing but a consequence of these facts. Therefore, under these assumptions, the proof of [30, Proposition 5.13] can be replicated, *mutatis mutandis*, for V instead of $V^{\mathbf{A}}$, and constitutes a valid proof for the above statement. $\square$

4.3.c Lagrange system for fibers

Definition 4.16. *Let $L = (\Gamma, (1), \mathscr{S})$ be a Lagrange system whose type is $((n,0),(p,0),0)$ and let $e \in \{1, \ldots, n-p\}$. Let $\mathscr{Q}''$ be a zero-dimensional parametrization that encodes a finite set $Q'' \subset \mathbf{C}^e$ and let $\mathscr{S}''$ be a zero-dimensional parametrization that encodes a finite set $S'' \subset \mathbf{C}^n$ lying over Q''. We define $F_{\text{lag}}(L, \mathscr{Q}'', \mathscr{S}'')$ as the triplet $(\Gamma, \mathscr{Q}'', \mathscr{S}'')$, it is a Lagrange system of type $((n,0),(p,0),e)$.*

As in the previous paragraph, we state an analogue of [30, Proposition 5.16] where we replaced the assumption of a generic linear change of variables by the assumptions that such a change of variables allows us to satisfy. In addition, we handle here the more general situation where, using the notation below, $W = W(\boldsymbol{\pi}_e, V^{\boldsymbol{\varphi}})$, as the case $W = W(\boldsymbol{\pi}_{e+1}, V^{\boldsymbol{\varphi}})$ considered in [30] can be deduced from the former.

Proposition 4.17. *Let $V, S \subset \mathbf{C}^n$ be two algebraic sets with V d-equidimensional and S finite. Let $\boldsymbol{\chi}$ be an atlas of (V, S) and let $e \in \{2, \ldots, (d+3)/2\}$. Define $W = W(\boldsymbol{\pi}_e, V^{\boldsymbol{\varphi}})$ and let $\mathscr{Q}''$ and $\mathscr{S}''$ be zero-dimensional parametrizations with coefficients in $\mathbf{Q}$ that respectively encode a finite set $Q'' \subset \mathbf{C}^e$ and $S'' = S \cup W_{|\boldsymbol{\pi}_e \in Q''}$ and let $V'' = V_{|\boldsymbol{\pi}_e \in Q''}$. Assume that S'' is finite and, either V'' is empty or its is equidimensional of dimension $d-e$, with $\operatorname{sing}(V'')$ contained in S'', and $F_{\text{atlas}}(\boldsymbol{\chi}, V, S, \mathscr{Q}'', \boldsymbol{\pi})$ is an atlas of (V'', Q'', S'').*

Let $L = (\Gamma, (1), \mathscr{S})$ be a Lagrange system such that $V = \overline{\mathscr{U}(L)}$ and $S = \mathsf{Z}(\mathscr{S})$. Let $\mathscr{Y} = (Y_1, \ldots, Y_r)$ be algebraic sets in $\mathbf{C}^n$ and let finally $\boldsymbol{\psi}$ be a global normal form for $(L; (V'', \mathscr{Y}))$ such that $\boldsymbol{\chi}$ is the associated atlas of (V, S). Then the following holds:

- *$F_{\text{lag}}(L, \mathscr{Q}'', \mathscr{S}'')$ is a Lagrange system that defines V'';*

- *if $V'' \neq \emptyset$, then $(F_{\text{lag}}(L, \mathscr{Q}'', \mathscr{S}''); \mathscr{Y})$ has a global normal form whose atlas is $F_{\text{atlas}}(\boldsymbol{\chi}, V, Q'', S, \boldsymbol{\pi})$.*

Proof. As above, the statement of this proposition is identical to the one in [30, Proposition 5.16], except that the assumptions on S'' and V'' are replaced by a generic change of variables on V. Indeed, [30, Proposition 5.16] claims the same statements as we do on $V^{\mathbf{A}}$, where $\mathbf{A}$ is assumed to lie in a non-empty Zariski open set $\mathscr{G}_3(\boldsymbol{\chi}, V, \emptyset, S, e)$ defined in [30, Proposition 3.7].

In the proof of [30, Proposition 5.17], the fact that $\mathbf{A} \in \mathscr{G}_3(\boldsymbol{\chi}, V, \emptyset, S, e)$ allows us to assume that the statements of [30, Proposition 3.7] but also [30, Lemma C.1] hold. In the case of the proposition stated above, the assumptions on S'' and V'' are exactly the statement of [30, Proposition 3.7], while [30, Lemma C.1] is nothing but a consequence of these facts. Again, under these assumptions, the proof of [30, Proposition 5.17] can be replicated, *mutatis mutandis*, for V instead of $V^{\mathbf{A}}$, and constitutes a valid proof for the above statement. ☐

4.4 Proofs of Lemmas 3.3, 3.4, 3.5 and 3.6

As done in Subsection 3.2.b, we fix $1 \leq c \leq n-2$ and we refer to the following objects:

- sequences of polynomials $\boldsymbol{g} = (g_1, \dots, g_c)$ and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ in $\mathbf{Q}[\mathbf{X}]$, of maximal degrees D, such that $\boldsymbol{g}$ satisfies assumption A that is: $\boldsymbol{g}$ is a reduced regular sequence and $\text{sing}(\boldsymbol{V}(\boldsymbol{g}))$ is finite;
- straight-line programs Γ and $\Gamma^{\boldsymbol{\varphi}}$, of respective lengths E and E', computing respectively $\boldsymbol{g}$ and $\boldsymbol{\varphi}$;
- the equidimensional algebraic set $V = \boldsymbol{V}(\boldsymbol{g})$, of dimension $d = n - c$, defined by $\boldsymbol{g}$;
- zero-dimensional parametrizations $\mathscr{S}$ and $\mathscr{Q}''$, of respective degrees σ and κ'', describing finite sets $S \subset \mathbf{C}^n$ and $Q'' \subset \mathbf{C}$, such that $\text{sing}(V) \subset S$;
- an atlas $\boldsymbol{\chi}$ of (V, S), given by [30, Lemma A.13], as S is finite and contains $\text{sing}(V)$.

Let $\Psi_{\boldsymbol{\varphi}}$ be the incidence isomorphism associated to $\boldsymbol{\varphi}$ and let $\boldsymbol{g}^{\boldsymbol{\varphi}}$ as defined in Lemma 4.4, so that $\widetilde{V} := \boldsymbol{V}(\boldsymbol{g}^{\boldsymbol{\varphi}}) = \Psi_{\boldsymbol{\varphi}}(V)$. According to Lemmas 4.4 and 4.5, $\widetilde{V} \subset \mathbf{C}^{2+n}$ is equidimensional with finitely many singular points.

Lemma 4.18. *Let $\mathscr{Y} = (Y_1, \dots, Y_r)$ be algebraic sets in $\mathbf{C}^n$. There exists an algorithm such that, on input $\Gamma, \mathscr{S}$ and $\Gamma^{\boldsymbol{\varphi}}$, runs using at most $O\tilde{~}(E'\sigma)$ operations in $\mathbf{Q}$, and outputs*

- *$\widetilde{\Gamma}$, a straight-line program of length $E + E' + 2$, computing $\boldsymbol{g}^{\boldsymbol{\varphi}}$,*
- *$\widetilde{\mathscr{S}}$, a zero-dimensional parametrization of degree σ, encoding $\widetilde{S} = \Psi_{\boldsymbol{\varphi}}(S)$,*

such that the Lagrange system $\widetilde{L} = (\widetilde{\Gamma}, (1), \widetilde{\mathscr{S}})$ of type $((2+n, 0), (2+c, 0), 0)$ defines $\widetilde{V}$, and $(\widetilde{L}, \mathscr{Y})$ has a global normal form.

Proof. By Lemmas 4.1 and 4.3, there exist algorithms IncSLP and $\mathsf{IncParam}$ respectively, which, on input $\Gamma, \mathscr{S}$ and $\Gamma^{\boldsymbol{\varphi}}$, output $\widetilde{\Gamma}$ and $\widetilde{\mathscr{S}}$ as described in the statement, using at most $O\tilde{~}(E'\sigma)$ operations in $\mathbf{Q}$. Let $\widetilde{L} = (\widetilde{\Gamma}, (1), \widetilde{\mathscr{S}})$. By Lemma 4.4, $\boldsymbol{g}^{\boldsymbol{\varphi}}$ is a reduced regular sequence as $\boldsymbol{g}$ is. Then, according to [30, Proposition 5.10], $\widetilde{L}$ defines a Lagrange system that defines $\widetilde{V}$ and $\boldsymbol{\psi} = ((1, 1, \boldsymbol{g}^{\boldsymbol{\varphi}}, \boldsymbol{g}^{\boldsymbol{\varphi}}))$ is a global normal form of $(\widetilde{L}, \mathscr{Y})$. ☐

We deduce an algorithm for computing critical points on V.

Proof of Lemma 3.3 By Lemmas 4.4, 4.2 and 4.5, $W_{\boldsymbol{\varphi}}(1, V)$ can be obtained by projecting the incidence polar variety $W(\boldsymbol{\pi}_1, \widetilde{V})$ on the last n coordinates. Computing a parametrization of the latter set can then be done using the algorithm W_1 of [30, Proposition 6.3] on the Lagrange system given by [30, Proposition 5.10].

According to Lemma 4.18, we can compute a Lagrange system $\widetilde{L}$ of type $((2+n,0),(2+c,0),0)$, with the global normal form property, that defines $\widetilde{V}$. Hence, by [30, Proposition 6.4], there exists a Monte Carlo algorithm W_1 which, on input $\widetilde{L}$, either fails or returns a zero-dimensional parametrization $\widetilde{\mathscr{W}_1}$ which describes it using at most

$$O\tilde{\ }\left((E+E')(n+2)^{4d+8}D^{2n+3}(D-1)^{2d}+n\sigma^2\right)$$

operations in $\mathbf{Q}$. Moreover, in case of success, $\widetilde{\mathscr{W}_1}$ describes $W(\boldsymbol{\pi}_1, \widetilde{V}) - \widetilde{S}$, with the notation of Lemma 4.18. Besides, by [30, Proposition I.1] (or [31, Proposition 3]) the degree of $K(\boldsymbol{\pi}_1, \widetilde{V})$ is upper bounded by

$$\tbinom{n+1}{c+1}D^{c+2}(D-1)^d = \tbinom{n+1}{d}D^{c+2}(D-1)^d.$$

Finally, by Lemma 4.5, $W_{\boldsymbol{\varphi}}(1, V)$ can be obtained by projecting $W(\boldsymbol{\pi}_1, \widetilde{V})$ on the last n coordinates and taking the union with S. This is done by performing the subroutines $\mathsf{Projection}$ and Union [30, Lemma J.3 and J.5], which uses at most

$$O\tilde{\ }\left(n^2\tbinom{n+1}{c+1}^2D^{2c+4}(D-1)^{2d}+n\sigma^2\right)$$

operations in $\mathbf{Q}$. □

In the following, we consider the polar varieties $W = W_{\boldsymbol{\varphi}}(2, V)$ and $\widetilde{W} = W(\boldsymbol{\pi}_2, \widetilde{V})$ so that, by Lemma 4.5, $\widetilde{W} = \Psi_{\boldsymbol{\varphi}}(W)$.

Lemma 4.19. *Let $\mathscr{Y} = (Y_1, \ldots, Y_r)$ be algebraic sets in $\mathbf{C}^n$. There exists a Monte Carlo algorithm which, on input $\Gamma, \mathscr{S}$ and $\Gamma^{\boldsymbol{\varphi}}$, runs using at most $O\tilde{\ }(E'\sigma + n(E+E'))$ operations in $\mathbf{Q}$, and outputs a Lagrange system $\widetilde{L_W}$ of type*

$$\big((2+n,\ 2+c),\ (2+c,\ n+1),\ 0\big).$$

Either W is empty or assume that W is 1-equidimensional, with $\operatorname{sing}(W) \subset S$, and $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, 2)$ is an atlas of (W, S). Then, in case of success, $\widetilde{L_W}$ defines $W(\boldsymbol{\pi}_2, \widetilde{V})$ and $(\widetilde{L_W}, \mathscr{Y})$ has a global normal form.

Proof. According to Lemma 4.18, one can compute, using $O\tilde{\ }(E'\sigma)$ operations in $\mathbf{Q}$, a Lagrange system $\widetilde{L}$ of type $((2+n,0),(2+c,0),0)$, defining $\widetilde{V}$, and such that $(\widetilde{L},(\widetilde{W},\mathscr{Y}))$ has a global normal form $\boldsymbol{\psi}$.

Let $\boldsymbol{u}$ be an arbitrary element of $\mathbf{Q}^{c+2}$ (such an element can be provided by the procedure Random we mentioned in Subsection 3.3) and let $\widetilde{L_W} = W_{\mathrm{Lagrange}}(\widetilde{L}, \boldsymbol{u}, 2)$. According to Definition 4.14, $\widetilde{L_W}$ is a Lagrange system of type

$$\big((2+n,\ 2+c),\ (2+c,\ n+1),\ 0\big).$$

Computing $\widetilde{L_W}$ boils down to apply Baur-Strassen's algorithm [6] to obtain a straight-line program evaluating the Jacobian matrix associated to $\boldsymbol{g}, \boldsymbol{\varphi}$ as in the proof of [30, Lemma O.1].

By assumption, either W is empty, and so is $\widetilde{W}$, or W is equidimensional of dimension 1, with $\operatorname{sing}(W) \subset S$. Then, by Lemma 4.5, $\widetilde{W}$ is equidimensional of dimension 1, with $\operatorname{sing}(\widetilde{W}) \subset \Psi_{\boldsymbol{\varphi}}(S) = \widetilde{S}$. Moreover, as $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, 2)$ is an atlas of (W, S) then, by Lemma 4.7, $W_{\mathrm{atlas}}(\boldsymbol{\chi}^{\boldsymbol{\varphi}}, \widetilde{V}, \widetilde{S}, \boldsymbol{\pi}, 2)$ is an atlas of $(\widetilde{W}, \widetilde{S})$.

Therefore, by Proposition 4.15, there exists a non-empty Zariski open subset $\mathscr{I}(\widetilde{L}, \boldsymbol{\psi}, \mathscr{Y})$ of $\mathbf{C}^p$ such that, if $\boldsymbol{u} \in \mathscr{I}(\widetilde{L}, \boldsymbol{\psi}, \mathscr{Y})$ then, either $\widetilde{W} \neq \emptyset$ or $(\widetilde{L_W}, \mathscr{Y})$ admits a global normal form. In both cases, $\widetilde{L_W}$ is a Lagrange system that defines $\widetilde{W}$. □

Proof of Lemma 3.4 According to Lemmas 4.4, 4.2 and 4.5, $W_{\boldsymbol{\varphi}}(2, V)$ can be obtained by projecting the incidence polar variety $W(\boldsymbol{\pi}_2, \widetilde{V})$ on the last n coordinates. Computing a parametrization of the latter set can then be done using the algorithm SolveLagrange of [30, Proposition 6.3] on the Lagrange system given by Proposition 4.15.

By Lemma 4.19, we can compute a Lagrange system $\widetilde{L_W}$ defining $W(\boldsymbol{\pi}_2, \widetilde{V})$, that admits a global normal form. Then, by [30, Proposition 6.3], there exists a Monte Carlo algorithm SolveLagrange which, on input $\widetilde{L_W}$, either fails or returns a one-dimensional parametrization $\widetilde{\mathscr{W}}$ of degree at most

$$\delta = (n + c + 4)D^{c+2}(D - 1)^d(c + 2)^d,$$

describing $\overline{\mathscr{U}(\widetilde{L_W})}$, which is exactly $\widetilde{W}$ by Proposition 4.15. Moreover, by [30, Proposition 6.3], the execution of SolveLagrange uses at most

$$O\tilde{~}\big((n + c)^3(E + E' + (n + c)^3)D\delta^3 + (n + c)\delta\sigma^2\big)$$

operations in $\mathbf{Q}$. Finally, by Lemma 4.5, W can be obtained by projecting $\widetilde{W}$ on the last n coordinates. Hence, running Projection, with input $\widetilde{\mathscr{W}}$ and n, we get a one-dimensional parametrization $\mathscr{W}$, of degree at most δ, encoding W. According to [30, Lemma J.9], the latter operation costs at most $O\tilde{~}\big(n^2\delta^3\big)$ operations in $\mathbf{Q}$. □

Proof of Lemma 3.5 By Lemma 4.19, we can compute a Lagrange system $\widetilde{L_W}$ defining $W(\boldsymbol{\pi}_2, \widetilde{V})$, such that $(\widetilde{L_W}; W(\boldsymbol{\pi}_1, \widetilde{W}))$ has the global normal form property. Hence, by [30, Proposition 6.4], there exists a Monte Carlo algorithm $\mathsf{W_1}$ which, on input $\widetilde{L_W}$, either fails or returns a zero-dimensional parametrization $\widetilde{\mathscr{K}}$ of degree at most $\delta(n + c)D$, where

$$\delta = (n + c + 4)D^{c+2}(D - 1)^d(c + 2)^d,$$

describing $W(\boldsymbol{\pi}_1, \overline{\mathscr{U}(\widetilde{L_W})}) - \widetilde{S}$, which is exactly $W(\boldsymbol{\pi}_1, \widetilde{W}) - \widetilde{S}$ by Proposition 4.15. Moreover, by [30, Proposition 6.3], the execution of $\mathsf{W_1}$ uses at most

$$O\tilde{~}\big((n + c)^{12}(E + E')D^3\delta^2 + (n + c)\sigma^2\big)$$

operations in $\mathbf{Q}$. Finally, by Lemma 4.5, $W_{\boldsymbol{\varphi}}(1, W)$ can be obtained by projecting $W(\boldsymbol{\pi}_1, \widetilde{W})$ on the last n coordinates and taking the union with S. This is done using the subroutines Projection and Union which, according to [30, Lemma J.3 and J.5], use at most $O\tilde{~}\big((n + c)^4D^2\delta^2 + n\sigma^2\big)$ operations in $\mathbf{Q}$. □

Proof of Lemma 3.6 By Lemma 4.19, we can compute a Lagrange system $\widetilde{L_W}$ defining $W(\boldsymbol{\pi}_2, \widetilde{V})$, such that $(\widetilde{L_W}; \widetilde{W} \cap \boldsymbol{\pi}_1^{-1}(\widetilde{Q''}))$ has the global normal form property. Hence, by [30, Proposition 6.5], there exists a Monte Carlo algorithm Fiber which, on input $\widetilde{L_W}$, either fails or returns a zero-dimensional parametrization $\widetilde{\mathscr{F}}$ of degree at most $\kappa''\delta$ where

$$\delta = (n + c + 4)D^{c+2}(D - 1)^d(c + 2)^d,$$

describing $[\overline{\mathscr{U}(\widetilde{L_W})} \cap \boldsymbol{\pi}_1^{-1}(\widetilde{Q''})] - \widetilde{S}$, which is exactly $[\widetilde{W} \cap \boldsymbol{\pi}_1^{-1}(\widetilde{Q''})] - \widetilde{S}$ by Proposition 4.15. Moreover, by [30, Proposition 6.3], the execution of FiberPolar uses at most

$$O\tilde{~}\big((n + c)^4\big[E + E' + (n + c)^2\big]D(\kappa'')^2\delta^2 + (n + c)\sigma^2\big)$$

operations in $\mathbf{Q}$,according to [30, Definition 6.1]. Finally, by Lemma 4.5, $W \cap \boldsymbol{\varphi}_1^{-1}(Q'')$ can be obtained by projecting $\widetilde{W} \cap \boldsymbol{\pi}_1^{-1}(\widetilde{Q''})$ on the last n coordinates and taking the union with S.. This is done, using the subroutines Projection and Union which, according to [30, Lemma J.3 and J.5], use at most $O\tilde{~}\big((n + c)^2(\kappa'')^2\delta^2 + n\sigma^2\big)$ operations. □

4.5 Proof of Proposition 3.7

This paragraph is devoted to prove Proposition 3.7. We recall its statement below.

Proposition (3.7)**.** *Let Γ and $\Gamma^{\boldsymbol{\varphi}}$ be straight-line programs, of respective length E and E', computing respectively polynomials $\boldsymbol{g} = (g_1, \ldots, g_p)$ and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ in $\mathbf{Q}[x_1, \ldots, x_n]$, of degrees bounded by D. Assume that $\boldsymbol{g}$ satisfies* (A)*. Let $\mathscr{Q}$ and $\mathscr{S}_Q$ be zero-dimensional parametrizations of respective degrees κ and σ that encode finite sets $Q \subset \mathbf{C}^e$ (for some $0 < e \leq n$) and $S_Q \subset \mathbf{C}^n$, respectively. Let $V = \boldsymbol{V}(\boldsymbol{g})$ and $F_Q = V_{|\boldsymbol{\varphi}_e \in Q}$, and assume that*

- *F_Q is equidimensional of dimension $d - e$, where $d = n - p$;*
- *$F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, \boldsymbol{\varphi})$ is an atlas of (F_Q, S_Q), and $\mathrm{sing}(F_Q) \subset S_Q$;*
- *the real algebraic set $F_Q \cap \mathbf{R}^n$ is bounded.*

Consider additionally a zero-dimensional parametrization $\mathscr{P}$ of degree μ encoding a finite subset $\mathcal{P}$ of F_Q, which contains S_Q. Assume that $\sigma \leq ((n+e)D)^{n+e}$.

There exists a probabilistic algorithm RoadmapBounded *which takes as input $((\Gamma, \Gamma^{\boldsymbol{\varphi}}, \mathscr{Q}, \mathscr{S}), \mathscr{P})$ and which, in case of success, outputs a roadmap of $(F_Q, \mathcal{P})$, of degree*

$$O^{\sim}\left((\mu+\kappa)16^{3d_F}(n_F\log_2(n_F))^{2(2d_F+12\log_2(d_F))(\log_2(d_F)+5)}D^{(2n_F+1)(\log_2(d_F)+3)}\right),$$

where $n_F = n + e$ and $d_F = d - e$, and using

$$O^{\sim}\left(\mu'^3 16^{9d_F}E''(n_F\log_2(n_F))^{(12d_F+24\log_2(d_F))(\log_2(d_F)+6)}D^{(6n_F+3)(\log_2(d_F)+4)}\right)$$

arithmetic operations in $\mathbf{Q}$ where $\mu' = (\mu + \kappa)$ and $E'' = (E + E' + e)$.

We start by proving a variant of this result for when $\boldsymbol{\varphi}$ encodes projections. Then, using incidence varieties and the associated subroutines, we will generalize it to arbitrary polynomial maps.

4.5.a The particular case of projections

We study here algorithm RoadmapRecLagrange from [30, Section 7.1]. It takes as input a Lagrange system $L_\rho = (\Gamma_\rho, \mathscr{Q}_\rho, \mathscr{S}_\rho)$ having the global normal form property, and a zero-dimensional parametrization $\mathscr{P}_\rho$, where $\mathsf{Z}(\mathscr{Q}_\rho)$ lies in $\mathbf{C}^{e_\rho}$, for some $e_\rho > 0$; the output is a roadmap for the algebraic set defined by L_ρ, and $\mathsf{Z}(\mathscr{P}_\rho)$. The following proposition ensures correction and describes the related complexity. The discussion is entirely similar to that of [30, Proposition O.7], but the analysis done there assumed that $\mathscr{Q}_\rho$ was empty and had $e_\rho = 0$ (the notation we use, with objects subscripted by ρ, is directly taken from there, in order to facilitate the comparison). In what follows, let $\mathfrak{x}_1, \ldots, \mathfrak{x}_m$, where $m \geq 0$, be new indeterminates.

Proposition 4.20. *Let $\boldsymbol{f} = (f_1, \ldots, f_{p_\rho}) \subset \mathbf{Q}[\mathfrak{x}_1, \ldots, \mathfrak{x}_m]$ be given by a straight-line program Γ_ρ of length E_ρ with $\deg(f_i) \leq D$ for $1 \leq i \leq p_\rho$, let $\mathscr{Q}_\rho$ and $\mathscr{S}_\rho$ be zero-dimensional parametrizations which have respective degrees κ_ρ and σ_ρ and encode finitely many points in respectively $\mathbf{C}^{e_\rho}$ (for some $e_\rho > 0$) and in $\mathbf{C}^m$. Assume that the Lagrange system $L_\rho = (\Gamma_\rho, \mathscr{Q}_\rho, \mathscr{S}_\rho)$ has the global normal form property. Let $d_\rho = m - p_\rho - e_\rho$, hence the dimension of $V(\Gamma_\rho)_{|\boldsymbol{\pi}_{e_\rho} \in \mathsf{Z}(\mathscr{Q}_\rho)}$.*

Consider a zero-dimensional parametrization $\mathscr{P}_\rho$ of degree μ_ρ such that $\mathsf{Z}(\mathscr{P}_\rho)$ is a finite subset of $V(\Gamma_\rho)_{|\boldsymbol{\pi}_{e_\rho} \in \mathsf{Z}(\mathscr{Q}_\rho)}$ which contains $\mathsf{Z}(\mathscr{S}_\rho)$. Assume that $\sigma_\rho \leq (mD)^m$.

There exists a Monte Carlo algorithm RoadmapRecLagrange *which takes as input* $((\Gamma_\rho, \mathscr{Q}_\rho, \mathscr{S}_\rho), \mathscr{P}_\rho)$ *and which, in case of success, outputs a roadmap for* $(V(\Gamma_\rho)_{|\boldsymbol{\pi}_{e_\rho} \in Z(\mathscr{Q}_\rho)}, \mathscr{P}_\rho)$ *of degree*

$$O\tilde{}\left((\mu_\rho + \kappa_\rho) 16^{3d_\rho} (m \log_2(m))^{2(2d+12\log_2(d_\rho))(\log_2(d_\rho)+5)} D^{(2m+1)(\log_2(d_\rho)+3)}\right)$$

using

$$O\tilde{}\left((\mu_\rho + \kappa_\rho)^3 16^{9d_\rho} E_\rho (m \log_2(m))^{(12d+24\log_2(d_\rho))(\log_2(d_\rho)+6)} D^{(6m+3)(\log_2(d_\rho)+4)}\right)$$

arithmetic operations in $\mathbf{Q}$.

Proof. Since, by assumption, L_ρ has the global normal form property, one can call the algorithm RoadmapRecLagrange from [30, Section 7.1] on input $L_\rho = (\Gamma_\rho, \mathscr{Q}_\rho, \mathscr{S}_\rho)$ and $\mathscr{P}_\rho$. This algorithm computes data-structures, which are called generalized Lagrange systems, that encode:

- a polar variety in $V(\Gamma_\rho)_{|\boldsymbol{\pi}_{e_\rho} \in \mathsf{Z}(\mathscr{Q}_\rho)}$ of dimension $\widetilde{d} - 1 \simeq d_\rho/2$ for $\widetilde{d} = \lfloor \frac{d_\rho+3}{2} \rfloor$;
- appropriate fibers in $V(\Gamma_\rho)_{|\boldsymbol{\pi}_{e_\rho} \in \mathsf{Z}(\mathscr{Q}_\rho)}$ of dimension $d_\rho - (\widetilde{d} - 1) \simeq d_\rho/2$.

A generalized Lagrange system (see [30, Definition 5.3]) is encoded by a triplet $L = (\Gamma, \mathscr{Q}, \mathscr{S})$ such that Γ is a straight-line program that evaluates some polynomials, say $\boldsymbol{F} = (\boldsymbol{f}, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s)$ where

- $\boldsymbol{f}$ lies in $\mathbf{Q}[\mathfrak{X}]$, with $\mathfrak{X} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_m)$;
- $\boldsymbol{f}_i$ lies in $\mathbf{Q}[\mathfrak{X}, \boldsymbol{L}_1, \ldots, \boldsymbol{L}_i]$ and has length p_i, where the $\boldsymbol{L}_j$'s are sequences of extra variables of length m_j (these are called blocks of Lagrange multipliers);
- for any $f_{i,j}$ in $\boldsymbol{f}_i$, the degree of $f_{i,j}$ in $\boldsymbol{L}_j$ is at most 1 for $1 \leq i \leq p_i$ and $1 \leq j \leq i$.

Also, $\mathscr{Q}$ (resp. $\mathscr{S}$) is a zero-dimensional parametrization encoding points in $\mathbf{C}^e$ (resp. $\mathbf{C}^m$).

The algebraic set of $\mathbf{C}^m$ defined by $L = (\Gamma, \mathscr{Q}, \mathscr{S})$ is the Zariski closure of the projection on the $\mathfrak{X}$-space of $V(\boldsymbol{F})_{|\boldsymbol{\pi}_{\mathfrak{X},e} \in \mathsf{Z}(\mathscr{Q})} \setminus \pi_{\mathfrak{X}}^{-1}(\mathsf{Z}(\mathscr{S}))$.

Short description of RoadmapRecLagrange. From a generalized Lagrange system L satisfying the global normal form property and encoding some algebraic set X, one can build a generalized Lagrange system encoding a polar variety W over X using [30, Definition 5.11 and Proposition 5.13], which satisfies the global normal form property, up to some generic enough linear change of coordinates and some restriction on the dimension of W. Additionally, given finitely many base points $Q' \subset \mathbf{C}^{e'}$ encoded by a zero-dimensional parametrization $\mathscr{Q}'$, [30, Definition 5.14 and Proposition 5.16] show how to deduce from L and $\mathscr{Q}'$ a generalized Lagrange system for $X_{|\boldsymbol{\pi}_{e'} \in Q'}$ satisfying the global normal form property, again assuming the coordinate system is generic enough.

Maintaining the global normal form property allows us to call recursively RoadmapRecLagrange. All in all, these computations are organised in a binary tree $\mathcal{T}$, whose root is denoted by ρ. Each child node τ encodes computations performed by a recursive call with input some generalized Lagrange system $L_\tau = (\Gamma_\tau, \mathscr{Q}_\tau, \mathscr{S}_\tau)$ and some zero-dimensional parametrization $\mathscr{P}_\tau$ encoding some control points. Both L_τ and $\mathscr{P}_\tau$ have been computed by the parent node. Correctness is proved in [30, Section N.3]. Further, we denote by κ_τ, σ_τ and μ_τ the respective degrees of $\mathscr{Q}_\tau$, $\mathscr{S}_\tau$ and $\mathscr{P}_\tau$.

The dimension of $\mathcal{V}(L_\tau)$ is denoted by d_τ. Calling RoadmapRecLagrange with input L_τ sets $\widetilde{d}_\tau = \lfloor \frac{d_\tau+3}{2} \rfloor$ and computes

(a) a generalized Lagrange system L'_τ which encodes the polar variety $W = W(e_\tau, d'_\tau, \mathcal{V}(L_\tau)^{\mathbf{A}})$, where $\mathbf{A}$ is randomly chosen;

(b) a zero-dimensional parametrization $\mathscr{B}_\tau$ which encodes the union of $\mathsf{Z}(\mathscr{P})^{\mathbf{A}}$ with $W(e_\tau, 1, W)$; we denote its degree by β_τ; note that by construction (see [30], $\mathsf{Z}(\mathscr{B}_\tau)$ contains $\mathsf{Z}(\mathscr{S}_\tau)$);

(c) a zero-dimensional parametrization $\mathscr{Q}''_\tau$ which encodes the projection of $\mathscr{B}_\tau$ on the e''_τ first coordinates (with $e''_\tau = e_\tau + \widetilde{d}_\tau - 1$); we denote its degree by κ''_τ;

(d) a zero-dimensional parametrization $\mathscr{P}'_\tau$ encoding $\mathsf{Z}(\mathscr{P}_\tau)^{\mathbf{A}} \cup Y_\tau$ with $Y_\tau = V(\mathcal{V}(L'_\tau))_{|\boldsymbol{\pi}_{e''_\tau} \in \mathsf{Z}(\mathscr{Q}''_\tau)}$ and a zero-dimensional parametrization $\mathscr{P}''_\tau$ which encodes those points of $\mathsf{Z}(\mathscr{P}'_\tau)$ which project on $\mathsf{Z}(\mathscr{Q}''_\tau)$; further we denote their degrees by μ'_τ and μ''_τ, the degree of Y_τ will be denoted by γ_τ;

(e) zero-dimensional parametrizations $\mathscr{S}'_\tau$ and $\mathscr{S}''_\tau$ of respective degrees σ'_τ and σ''_τ which do encode $\mathsf{Z}(\mathscr{S}_\tau)^{\mathbf{A}} \cup Y_\tau$ and those points of $\mathsf{Z}(\mathscr{S}''_\tau)$ which project on $\mathsf{Z}(\mathscr{Q}''_\tau)$; note that by construction, $\mathsf{Z}(\mathscr{S}'_\tau)$ and $\mathsf{Z}(\mathscr{S}''_\tau)$ are contained in $\mathsf{Z}(\mathscr{P}'_\tau)$ and $\mathsf{Z}(\mathscr{P}''_\tau)$ respectively;

(f) and a generalized Lagrange system L''_τ which encodes $\mathcal{V}(L_\tau)_{|\boldsymbol{\pi}_{e''_\tau} \in \mathsf{Z}(\mathscr{Q}''_\tau)}$.

The recursive calls of $\mathsf{RoadmapRecLagrange}$ are then performed on $(L'_\tau, \mathscr{P}'_\tau)$ and $(L''_\tau, \mathscr{P}''_\tau)$.

For a given generalized Lagrange system L_τ corresponding to some node τ, the number of blocks of Lagrange multipliers is denoted by k_τ. The total number of variables (resp. polynomials) lying in $\mathbf{Q}[\mathfrak{X}, \boldsymbol{L}_1, \ldots, \boldsymbol{L}_i]$ for $i \le k_\tau$ is denoted by $M_{i,\tau}$ (resp. $P_{i,\tau}$). By construction, for $i = 0$, we have $P_{0,\tau} = p_\rho$. For $i = k_\tau$, we denote $M_{k_\tau,\tau}$ (resp. $P_{k_\tau,\tau}$) by M_τ (resp. P_τ).

As in [30, Section 6.1], we attach to each such generalized Lagrange system the quantity

$$\delta_\tau = (P_\tau + 1)^{k_\tau} D^p (D-1)^{m-e_\tau-p_\rho} \prod_{i=0}^{k_\tau-1} M_{i+1,\tau}^{M_{i,\tau}-e_\tau-P_{i,\tau}}.$$

We establish below that the degree of $\mathcal{V}(L_\tau)$ is bounded by $\kappa_\tau \delta_\tau$.

Complexity analysis The complexity of $\mathsf{RoadmapRecLagrange}$ is analysed in [30, Section O], assuming that $e_\rho = 0$ (see [30, Proposition O.7]). This is done by proceeding in two steps:

- *Step (i)* proves some elementary bounds on the number of variables and polynomials (the m_i's and the p_i's) involved in the data-structures encoding these polar varieties and fibers in the recursive calls (see [30, Section O.1]);
- *Step (ii)* proves uniform degree bounds for the parametrizations $\mathscr{P}'_\tau$, $\mathscr{P}''_\tau$, $\mathscr{B}_\tau$, $\mathscr{Q}'_\tau$, $\mathscr{Q}''_\tau$, as well as $\mathscr{S}'_\tau, \mathscr{S}''_\tau$ where τ ranges over all nodes of the binary tree $\mathcal{T}$. Uniform degree bounds are also given for all $\mathcal{V}(L_\tau)$.

  These degree bounds are used in combination with the complexity estimates of [30, Section 6.2] for solving generalized Lagrange systems and [30, Sections J.1 and J.2] which do depend polynomially on these bounds and the ones established in *(i)*.

Since the total number of nodes is $O(m)$, it suffices to take m times the sum of all costs established by *(ii)*. Hereafter, we slightly extend this analysis when $e_\rho > 0$, following the same reasoning, which we recall step by step by highlighting the main (and tiny) differences.

Step (i). We start with step *(i)*. Both [30, Lemma O.1] and [30, Lemma O.2] control the lengths of the straight-line programs, the numbers of blocks of Lagrange multipliers and their lengths, as well as the numbers of polynomials and total number of variables remain valid, assuming $e_\rho = 0$. Their proofs are based on how these quantity evolve when building generalized Lagrange systems encoding polar varieties and fibers (see [30, Lemmas 5.12 and 5.15]). This is not changed in our context where the initial call to RoadmapRecLagrange is done with some base points $\mathsf{Z}(\mathscr{Q}_\rho)$ with $e_\rho > 0$ because for each note τ, we take $\widetilde{d}_\tau = \lfloor \frac{d_\tau+3}{2} \rfloor$ as in [30]. This implies that the conclusions of [30, Lemma O.1] and [30, Lemma O.2] still hold when taking $d_\rho = m - p_\rho - e_\rho$.

All in all, we deduce that:

- the maximum number of blocks of Lagrange multipliers and the depth of $\mathcal{T}$ are bounded by $\lceil \log_2(d_\rho) \rceil$
- All straight-line programs have length bounded by $4m^{4+2\log_2(d_\rho)}(E_\rho + m^4)$
- the total number of variables for the generalized Lagrange system L_τ is bounded by $(m^2)^{\frac{d_\rho}{h_\tau}+1}$ where h_τ is the height of the node τ.

Step (ii). We can now investigate Step *(ii)*. The two main quantities to consider are

$$\boldsymbol{\delta} = 16^{d_\rho+2} m^{2d_\rho+12\log_2(d_\rho)} D^m$$

and

$$\boldsymbol{\zeta} = (\mu_\rho + \kappa_\rho) 16^{2(d_\rho+3)} (m \log_2(m))^{2(2d_\rho+12\log_2(d_\rho))} D^{(2m+1)(\log_2(d_\rho+2))}.$$

The first step is to prove that for any node τ, the degree of $\mathcal{V}(L_\tau)$ is dominated by $\kappa_\tau \boldsymbol{\delta}$. Using the global normal form property, [30, Propositions 5.13 and 6.2] prove that the degree of $\mathcal{V}(L_\tau)$ is upper bounded by $\kappa_\tau \delta_\tau$. Recall that, by definition,

$$\delta_\tau = (P_\tau + 1)^{k_\tau} D^p (D-1)^{m-e_\tau-p_\rho} \prod_{i=0}^{k_\tau-1} M_{i+1,\tau}^{M_{i,\tau}-e_\tau-P_{i,\tau}}.$$

[30, Lemma O.4] shows that the above left-hand side quantity is dominated by $\boldsymbol{\delta}$, using the results of Step *(i)* which we proved to still hold. We then deduce that the degree of $\mathcal{V}(L_\tau)$ is upper bounded by $\kappa_\tau \boldsymbol{\delta}$.

[30, Lemma O.5] establishes recurrence formulas for the quantities β_τ, γ_τ, $\mu_\tau + \kappa_\tau$ and σ_τ when τ ranges in the set of nodes of the binary tree $\mathcal{T}$. It states that, letting τ' and τ'' be the two children of τ, β_τ, γ_τ, $\mu_{\tau'} + \kappa_{\tau'}$, $\mu_{\tau''} + \kappa_{\tau''}$, $\sigma_{\tau'}$ and $\sigma_{\tau''}$ are bounded above by $2\boldsymbol{\delta}^2 \zeta_\tau (\mu_\tau + \kappa_\tau)$ where $\zeta_\tau = \left(m^2 \log_2(m) D\right)^{\frac{d_\rho}{2^{h_\tau}}+1}$ (here h_τ is the height of τ) in the context of [30] with $e_\rho = 0$ and assuming that $\mathsf{Z}(\mathscr{S}_\tau)$ is contained in $\mathsf{Z}(\mathscr{P}_\tau)$ for any node τ of $\mathcal{T}$ (this is used to prove the statements on $\sigma_\tau, \sigma_{\tau'}$ and $\sigma_{\tau''}$). In the context of [30], we have $\mathsf{Z}(\mathscr{S}_\rho) = \emptyset$. In our context, we still take $\widetilde{d}_\tau = \lfloor \frac{d_\tau+3}{2} \rfloor$ as in [30], hence the structure of our binary tree $\mathcal{T}$ is the same as the one in [30]. Also we assume that $\mathsf{Z}(\mathscr{S}_\rho)$ is contained in $\mathsf{Z}(\mathscr{P}_\tau)$ and that its degree is bounded by $(mD)^m$. This is enough to transpose the recursion performed in the proof of [30, Lemma O.5 and Proposition O.3] and deduce that μ_τ, κ_τ and σ_τ are bounded by $\boldsymbol{\zeta}$ when τ ranges over the set of nodes of $\mathcal{T}$.

The runtime estimates in [30, Section O.3] to compute the parametrizations and generalized Lagrange systems in steps (a) to (f) above are then the same (they depend on $\boldsymbol{\delta}$, $\boldsymbol{\zeta}$ and the above bounds on deduced at Step *(i)*). The statements of [30, Lemmas O.8, O.9, O.10 and O.11] can then be applied here *mutatis mutandis* which, as in [30, Section O.3], allow us to deduce the same statement as [30, Proposition O.7], i.e. that the total runtime lies in

$$O^{\sim}\left((\mu_\rho + \kappa_\rho)^3 16^{9d_\rho} E_\rho (m \log_2(m))^{6(2d+12\log_2(d_\rho))(\log_2(d_\rho)+6)} D^{3(2m+1)(\log_2(d_\rho)+4)}\right)$$

and outputs a roadmap of degree in

$$O^{\sim}\left((\mu_\rho + \kappa_\rho)16^{3d_\rho}(m\log_2(m))^{2(2d+12\log_2(d_\rho))(\log_2(d_\rho)+5)}D^{(2m+1)(\log_2(d_\rho)+3)}\right).$$

$\square$

4.5.b Proof of Proposition 3.7

To prove Proposition 3.7, we now show how to return to the case of projections from the general one, before calling the procedure RoadmapRecLagrange, whose complexity is analysed in Proposition 4.20.

Consider the notations introduced in the statement of the proposition. In the following let $\Psi_{\boldsymbol{\varphi}_e}$ be the incidence isomorphism associated to $\boldsymbol{\varphi}_e$ and let $\boldsymbol{g}^{\boldsymbol{\varphi}_e}$ as defined in Lemma 4.4, so that $\widetilde{V} := \boldsymbol{V}(\boldsymbol{g}^{\boldsymbol{\varphi}_e}) = \Psi_{\boldsymbol{\varphi}_e}(V)$. According to Lemma 4.4 and 4.5, $\widetilde{V} \subset \mathbf{C}^{e+n}$ is equidimensional with finitely many singular points. Additionally, let $\widetilde{F_Q} = \Psi_{\boldsymbol{\varphi}_e}(F_Q)$ and $\widetilde{S_Q} = \Psi_{\boldsymbol{\varphi}_e}(S_Q)$, so that $\widetilde{F_Q} = \widetilde{V}_{|\boldsymbol{\pi}_e \in Q}$, according to Lemma 4.2.

Lemma 4.21. *There exists an algorithm such that, on input Γ, $\Gamma^{\boldsymbol{\varphi}}$, $\mathscr{Q}$ and $\mathscr{S}$ as above, runs using at most $O^{\sim}(E'\sigma)$ operations in $\mathbf{Q}$, and outputs a Lagrange system $\widetilde{L_F}$ of type*

$$\big((e+n,0),\ (e+c,0),\ e\big).$$

Under the assumptions of Proposition 3.7, $\widetilde{L_F}$ has a global normal form, and defines $\widetilde{F_Q}$.

Proof. According to Lemma 4.18, we can compute a Lagrange system $\widetilde{L}$ of type $((e+n,0),(e+c,0),0)$, with the global normal form property, that defines $\widetilde{V}$. Let $\widetilde{L_F} = F_{\mathrm{lag}}(\widetilde{L},\mathscr{Q},\mathscr{S})$, as defined in Definition 4.16, it is a Lagrange system of type $(((e+n,0),\ (e+c,0),\ e)$.

By assumptions of Proposition 3.7, either F_Q is empty, and so is $\widetilde{F_Q}$, or F_Q is equidimensional of dimension $d-e$, with $\mathrm{sing}(F_Q) \subset S_Q$. Then, by Lemma 4.5, $\widetilde{F_Q}$ is equidimensional of dimension $d-e$, with $\mathrm{sing}(\widetilde{F_Q}) \subset \Psi_{\boldsymbol{\varphi}}(S_Q) = \widetilde{S_Q}$. Moreover, as $F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S_Q, \boldsymbol{\varphi})$ is an atlas of (F_Q, S_Q) then, by Lemma 4.8, $F_{\mathrm{atlas}}(\boldsymbol{\chi}^{\boldsymbol{\varphi}}, \widetilde{V}, \widetilde{S_Q}, \boldsymbol{\pi})$ is an atlas of $(\widetilde{F_Q}, \widetilde{S_Q})$.

Hence, by Proposition 4.17, either $\widetilde{F_Q} = \emptyset$ or $\widetilde{L_F}$ admits a global normal form. $\square$

Suppose now that the Lagrange system $\widetilde{L_F}$, given by Lemma 4.21 has been computed. According to Lemma 4.3, one can compute a zero-dimensional parametrization $\widetilde{\mathscr{P}}$, encoding $\widetilde{\mathcal{P}} = \Psi_{\boldsymbol{\varphi}_e}(\mathcal{P})$, within the same complexity bound. One checks, by assumption, that $\widetilde{S_Q} \subset \widetilde{\mathcal{P}} \subset \widetilde{F_Q}$ and that $\widetilde{S_Q}$ has degree bounded by $((n+e)D)^{n+e}$

Therefore, according to Proposition 4.20, with $m = n+e$, there exists a Monte Carlo algorithm RoadmapRecLagrange which, on input $\widetilde{L_F}$ and $\widetilde{\mathscr{P}}$, outputs, in case of success, a roadmap $\widetilde{\mathscr{R}_{F_Q}}$ of $(\widetilde{F_Q}, \widetilde{\mathcal{P}})$ of degree

$$O^{\sim}\left((\mu+\kappa)16^{3d_F}(n_F\log_2(n_F))^{2(2d_F+12\log_2(d_F))(\log_2(d_F)+5)}D^{(2n_F+1)(\log_2(d_F)+3)}\right),$$

where $n_F = n+e$ and $d_F = d-e$, and using

$$O^{\sim}\left(\mu'^3 16^{9d_F}E''(n_F\log_2(n_F))^{6(2d_F+12\log_2(d_F))(\log_2(d_F)+6)}D^{3(2n_F+1)(\log_2(d_F)+4)}\right)$$

arithmetic operations in $\mathbf{Q}$ with $\mu' = (\mu+\kappa)$ and $E'' = (E+E'+e)$.

Finally, let $\mathscr{B}_{\mathrm{RM}}$ be the degree bound, given above, on the roadmap $\widetilde{\mathscr{R}_{F_Q}}$ of $(\widetilde{F_Q}, \widetilde{Q})$ output by RoadmapRecLagrange. Then, by [30, Lemma J.9], one can compute the projection $\mathscr{R}_{F_Q}$, of

$\widetilde{\mathscr{R}_{F_Q}}$, on the last n variables. The complexity of such step is bounded by $O\tilde{\ }\left(n_F^2\mathscr{B}_{\mathrm{RM}}^3\right)$, that is bounded by

$$O\tilde{\ }\left((\mu+\kappa)^3 16^{9d_F}(n_F\log_2(n_F))^{(12d_F+24\log_2(d_F))(\log_2(d_F)+6)}D^{(6n_F+3)(\log_2(d_F)+3)}\right)$$

operations in $\mathbf{Q}$. Finally, since Ψ_{φ_e} is an isomorphism of algebraic sets, it induces a one-to-one homeomorphic correspondence between the semi-algebraically connected components of $\widetilde{F_Q}\cap\mathbf{R}^{n_F}$ and $F_Q\cap\mathbf{R}^n$ by [27, Lemma 2.1]. Therefore, $\mathscr{R}_{F_Q}$ is a roadmap of $(F_Q,\mathcal{P})$.

5 Proof of Proposition 2.3: finiteness of fibers

We recall the statement of the proposition we address to prove.

Proposition (2.3). *Let $V\subset\mathbf{C}^n$ be a d-equidimensional algebraic set with finitely many singular points and θ be in $\mathbf{C}[\mathbf{X}]$. Let $2\leq\mathfrak{r}\leq d+1$. For $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_{\mathfrak{r}})$ in $\mathbf{C}^{\mathfrak{r}n}$, we define $\boldsymbol{\varphi}=(\varphi_1(\mathbf{X},\boldsymbol{\alpha}_1),\ldots,\varphi_{\mathfrak{r}}(\mathbf{X},\boldsymbol{\alpha}_{\mathfrak{r}}))$, where for $2\leq j\leq\mathfrak{r}$*

$$\varphi_1(\mathbf{X},\boldsymbol{\alpha}_1)=\theta(\mathbf{X})+\sum_{k=1}^n\alpha_{1,k}x_k \quad \textit{and} \quad \varphi_j(\mathbf{X},\boldsymbol{\alpha}_j)=\sum_{k=1}^n\alpha_{j,k}x_k.$$

Then, there exists a non-empty Zariski open subset $\Omega_{\mathsf{I}}(V,\theta,\mathfrak{r})\subset\mathbf{C}^{\mathfrak{r}n}$ such that for every $\boldsymbol{\alpha}\in\Omega_{\mathsf{I}}(V,\theta,\mathfrak{r})$ and $i\in\{1,\ldots,\mathfrak{r}\}$, the following holds:

1. *either $W_{\boldsymbol{\varphi}}(i,V)$ is empty or $(i-1)$-equidimensional;*
2. *the restriction of $\boldsymbol{\varphi}_{i-1}$ to $W_{\boldsymbol{\varphi}}(i,V)$ is a Zariski-closed map;*
3. *for any $\boldsymbol{z}\in\mathbf{C}^{i-1}$, the fiber $K_{\boldsymbol{\varphi}}(i,V)\cap\boldsymbol{\varphi}_{i-1}^{-1}(\boldsymbol{z})$ is finite.*

The rest of this section is devoted to the proof of this result. We first establish a general lower bound on the dimension of the non-empty generalized polar varieties. This is a direct generalization of [30, Lemma B.5 & B.13].

Lemma 5.1. *Let $\mathfrak{K}$ be an algebraically closed field, and let $V\subset\mathfrak{K}^n$ be a d-equidimensional algebraic set. Then, for any $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_{d+1})\subset\mathfrak{K}[\mathbf{X}]$, and any $1\leq i\leq d+1$, all irreducible components of $W_{\boldsymbol{\phi}}(i,V)$ have dimension at least $i-1$.*

Proof. Since V is d-equidimensional, the case $i=d+1$ is immediate; assume now that $i\leq d$. According to [30, Lemma A.13], there exists an atlas $\boldsymbol{\chi}=(\chi_j)_{1\leq j\leq s}$ of $(V,\operatorname{sing}(V))$. For $1\leq j\leq s$, let $\chi_j=(m_j,\boldsymbol{h}_j)$. By [30, Lemma A.12], $\boldsymbol{h}_j$ has cardinality $c=n-d$. According to Lemma 2.5, fix $j\in\{1,\ldots,s\}$, the following holds in $\mathcal{O}(m_j)-\operatorname{sing}(V)$,

$$W_{\boldsymbol{\phi}}(i,V) \ = \ \{\boldsymbol{y}\in\boldsymbol{V}_{\mathrm{reg}}^{\circ}(\boldsymbol{h}_j)\mid\operatorname{rank}(\operatorname{Jac}_{\boldsymbol{y}}(\boldsymbol{h}_j,\boldsymbol{\phi}_i)<c+i\}=W_{\boldsymbol{\phi}}^{\circ}(i,\boldsymbol{V}_{\mathrm{reg}}^{\circ}(\boldsymbol{h}_j)). \qquad (3)$$

Let $\boldsymbol{y}\in W_{\boldsymbol{\phi}}^{\circ}(i,V)=W_{\boldsymbol{\phi}}(i,V)-\operatorname{sing}(V)$. Since $\boldsymbol{y}\in V$, there exists $j\in\{1,\ldots,s\}$ such that $\boldsymbol{y}\in\mathcal{O}(m_j)$. Hence, by (3), in $\mathcal{O}(m_j)-\operatorname{sing}(V)$, the irreducible component of $W_{\boldsymbol{\phi}}(i,V)$ containing $\boldsymbol{y}$ is the same as the irreducible component of the Zariski closure of $W_{\boldsymbol{\phi}}^{\circ}(i,\boldsymbol{V}_{\mathrm{reg}}^{\circ}(\boldsymbol{h}_j))$ containing $\boldsymbol{y}$. Since these irreducible components are equal over a non-empty Zariski open set, they have same dimension by [33, Theorem 1.19]. Hence, proving that this common dimension is at least $i-1$ allows us to conclude.

Let $\mathfrak{m}\subset\mathfrak{K}[\mathbf{X}]$ be the ideal generated by the $(c+i)$-minors of $\operatorname{Jac}[\boldsymbol{h}_j,\boldsymbol{\phi}_i]$. Then,

$$W_{\boldsymbol{\phi}}^{\circ}(i,\boldsymbol{V}_{\mathrm{reg}}^{\circ}(\boldsymbol{h}_j))=\boldsymbol{V}_{\mathrm{reg}}^{\circ}(\boldsymbol{h}_j)\cap\boldsymbol{V}(\mathfrak{m})$$

which is contained in the algebraic set $\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j)\cap\boldsymbol{V}(\mathfrak{m})$. We assume that $\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j)\cap\boldsymbol{V}(\mathfrak{m})$ is not empty otherwise the statement of the proposition trivially holds.

Note that any irreducible component Z of $\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j)\cap\boldsymbol{V}(\mathfrak{m})$, has an ideal of definition $\mathfrak{p}$ in $\mathfrak{K}[\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j)]$ that is an isolated prime component of the determinantal ideal $\mathfrak{m}\cdot\mathfrak{K}[\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j)]$. Then by [13, Theorem 3], $\mathfrak{p}$ has height at most $n-c-(i-1)$ so that the codimension of Z in $\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j)$ is at most $n-c-(i-1)$. Since $\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j)$ has dimension $n-c$, the dimension of Z is then at most $i-1$.

One concludes by observing that, any irreducible component of the Zariski closure of $W^{\circ}_{\boldsymbol{\phi}}(i,\boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h}_j))$ is the union of irreducible components of $\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j)\cap\boldsymbol{V}(\mathfrak{m})$. $\square$

5.1 An adapted Noether normalization lemma

Consider an algebraically closed field $\mathfrak{K}$, let $\boldsymbol{f}=(f_1,\dots,f_m)\colon\mathfrak{K}^n\to\mathfrak{K}^m$ be a polynomial map and $V\subset\mathfrak{K}^n$ and let $Y\subset\mathfrak{K}^m$ be algebraic sets such that $\boldsymbol{f}(V)\subset Y$. Finally, consider the restriction $\widetilde{\boldsymbol{f}}:V\to Y$ of $\boldsymbol{f}$, and recall that the pullback $\widetilde{\boldsymbol{f}}^*$ of $\widetilde{\boldsymbol{f}}$ is defined by

$$\begin{array}{rccc}\widetilde{\boldsymbol{f}}^*\colon & \mathfrak{K}[Y]=\mathfrak{K}[y_1,\dots,y_m]/\boldsymbol{I}(Y) & \longrightarrow & \mathfrak{K}[V]=\mathfrak{K}[x_1,\dots,x_n]/\boldsymbol{I}(V)\\ & g & \longmapsto & g\circ\boldsymbol{f}\end{array}.$$

Definition 5.2 ([33, Section 5.3])**.** *We say that the restriction $\widetilde{\boldsymbol{f}}$ of $\boldsymbol{f}$ is a* finite map *if*

1. *$\boldsymbol{f}(V)$ is dense in Y, which is equivalent to $\widetilde{\boldsymbol{f}}^*$ being injective;*
2. *the extension $\mathfrak{K}[Y]\hookrightarrow\mathfrak{K}[V]$ induced by $\widetilde{\boldsymbol{f}}^*$ is integral.*

The following lemma shows that to verify such conditions, we may not have to work over an algebraically closed field: if V and Y are defined over a subfield $\mathbf{K}$ of $\mathfrak{K}$, finiteness of $\widetilde{\boldsymbol{f}}$ is equivalent to the pullback $\mathbf{K}[\mathbf{Y}]/\boldsymbol{I}(Y)\to\mathbf{K}[\mathbf{X}]/\boldsymbol{I}(V)$ being injective and integral.

Lemma 5.3. *Let $\mathbf{K}\subset\boldsymbol{L}$ be two fields, let I,J be ideals in respectively $\mathbf{K}[\mathbf{Y}]=\mathbf{K}[y_1,\dots,y_m]$ and $\mathbf{K}[\mathbf{X}]=\mathbf{K}[x_1,\dots,x_n]$ and let I',J' be their extensions in respectively $\boldsymbol{L}[\mathbf{Y}]$ and $\boldsymbol{L}[\mathbf{X}]$. Let finally $\boldsymbol{f}=(f_1,\dots,f_m)$ be in $\mathbf{K}[\mathbf{X}]$, such that for g in I, $g\circ\boldsymbol{f}$ is in J.*

Consider the ring homomorphisms

$$\zeta_{\mathbf{K}}:\mathbf{K}[\mathbf{Y}]/I\to\mathbf{K}[\mathbf{X}]/J \text{ and } \zeta_{\boldsymbol{L}}:\boldsymbol{L}[\mathbf{Y}]/I'\to\boldsymbol{L}[\mathbf{X}]/J',$$

that both map y_j to f_j, for all j. Then, $\zeta_{\mathbf{K}}$ is injective, resp. integral, if and only if $\zeta_{\boldsymbol{L}}$ is.

Proof. Injectivity of $\zeta_{\mathbf{K}}$ is equivalent to equality between ideals $I=(J\mathbf{K}[\mathbf{Y},\mathbf{X}]+\langle y_1-f_1,\dots,y_m-f_m\rangle)\cap\mathbf{K}[\mathbf{Y}]$; similarly, injectivity of $\zeta_{\boldsymbol{L}}$ is equivalent to $I'=(J'\boldsymbol{L}[\mathbf{Y},\mathbf{X}]+\langle y_1-f_1,\dots,y_m-f_m\rangle)\cap\boldsymbol{L}[\mathbf{Y}]$. These properties can be determined by Gröbner basis calculations; since the generators of I,J are the same as those of I',J', they are thus equivalent.

Next, integrality of $\zeta_{\mathbf{K}}$ directly implies that of $\zeta_{\boldsymbol{L}}$. Conversely, integrality of $\zeta_{\boldsymbol{L}}$ is equivalent to the existence of polynomials $G_1,\dots,G_n$ in $\boldsymbol{L}[y_1,\dots,y_m,s]$, all monic in s, such that $G_j(f_1,\dots,f_m,x_j)$ is in J' for all j. If we assume that such polynomials exist, we can then linearize these membership equalities, reducing such properties to the existence of a solution to certain linear systems with entries in $\mathbf{K}$. Since we know that solutions exist with entries in $\boldsymbol{L}$, some must also exist with entries in $\mathbf{K}$. This then yields integrality of $\zeta_{\mathbf{K}}$. $\square$

The Noether normalization lemma says that for V r-dimensional and $Y=\mathfrak{K}^r$, the restriction of a generic linear mapping $\mathfrak{K}^n\to\mathfrak{K}^m$ to V is finite. We give here a proof of this lemma adapted to our setting, where the shape of the projections we perform is made explicit. We start with a statement for ideals rather than algebraic sets.

Proposition 5.4 (Noether normalization). *Let $\mathbf{K}$ be a field, let J be an ideal in $\mathbf{K}[\mathbf{X}]$ and let r be the dimension of its zero-set over an algebraic closure of $\mathbf{K}$. Let further $\boldsymbol{a}$ be $r(n-r)$ new indeterminates. Then the $\mathbf{K}(\boldsymbol{a})$-algebra homomorphism*

$$\begin{array}{rccc} \zeta_{\boldsymbol{a}}\colon & \mathbf{K}(\boldsymbol{a})[z_1,\dots,z_r] & \longrightarrow & \mathbf{K}(\boldsymbol{a})[\mathbf{X}]/J\mathbf{K}(\boldsymbol{a})[\mathbf{X}] \\ & z_j & \longmapsto & x_j + \sum_{k=1}^{n-r} a_{j,k}\, x_{r+k} \mod J \end{array}$$

is injective and makes $\mathbf{K}(\boldsymbol{a})[\mathbf{X}]/J\mathbf{K}(\boldsymbol{a})[\mathbf{X}]$ integral over $\mathbf{K}(\boldsymbol{a})[z_1,\dots,z_r]$.

Proof. We proceed by induction on the number n of variables. The case $n=0$ is straightforward. Assume now that $n>0$ and that the statement holds for $k<n$ variables. Remark that if $J=\{0\}$, we have $r=n$, $\zeta_{\boldsymbol{a}}$ is the isomorphism $\mathbf{K}[z_1\dots,z_n]\to\mathbf{K}[\mathbf{X}]$ mapping z_i to x_i for all i (which is then integral); in this case, we are done.

Assume now that $J\neq\{0\}$, and let f be non-zero in J. Let δ be the total degree of f and let $\boldsymbol{\ell}=(\ell_1,\dots,\ell_{n-1})$ be new indeterminates. Writing $f=\sum_{i_1,\dots,i_n} c_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n}$, the leading coefficient of $f(x_1+\ell_1 x_n,\dots,x_{n-1}+\ell_{n-1}x_n,x_n)$ in x_n is

$$\sum_{i_1+\cdots+i_n=\delta} c_{i_1,\dots,i_n}\ell_1^{i_1}\cdots\ell_{n-1}^{i_{n-1}} = f_\delta(\ell_1,\dots,\ell_{n-1},1)$$

where f_δ is the homogeneous degree-δ component of f. Therefore, taking $F=f_\delta(\ell_1,\dots,\ell_{n-1},1)$, F is not the zero polynomial and the polynomial

$$\frac{1}{F} f(x_1+\ell_1x_n,\dots,x_{n-1}+\ell_{n-1}x_n,x_n)\in\mathbf{K}(\boldsymbol{\ell})[\mathbf{X}]$$

is monic in x_n. Let further J' be the extension of J to $\mathbf{K}(\boldsymbol{\ell})[\mathbf{X}]$, let $\mathbf{Y}=(y_1,\dots,y_{n-1})$ be new indeterminates and consider the $\mathbf{K}(\boldsymbol{\ell})$-algebra homomorphism

$$\begin{array}{rccc} \tau\colon & \mathbf{K}(\boldsymbol{\ell})[\mathbf{Y}] & \longrightarrow & \mathbf{K}(\boldsymbol{\ell})[\mathbf{X}] \\ & y_j & \longmapsto & x_j-\ell_j x_n \end{array};$$

the contraction $J'^c=\tau^{-1}(J')$ is an ideal in $\mathbf{K}(\boldsymbol{\ell})[\mathbf{Y}]$. For $1\le j\le n-1$, let $[y_j]=y_j \mod J'^c$ and for $1\le k\le n$ let $[x_j]=x_j \mod J'$. Then let

$$\begin{array}{rccc} [\tau]\colon & \mathbf{K}(\boldsymbol{\ell})[\mathbf{Y}]/J'^c & \longrightarrow & \mathbf{K}(\boldsymbol{\ell})[\mathbf{X}]/J' \\ & [y_j] & \longmapsto & [x_j]-\ell_j[x_n] \end{array}$$

and

$$g(s)=\frac{1}{F}f\big([y_1]+\ell_1 s,\dots,[y_{n-1}]+\ell_{n-1}s,s\big)\in\Big(\mathbf{K}(\boldsymbol{\ell})[\mathbf{Y}]/J'^c\Big)[s];$$

this is a monic polynomial in s. If we extend $[\tau]$ to a $\mathbf{K}(\boldsymbol{\ell})$-algebra homomorphism $\mathbf{K}(\boldsymbol{\ell})[\mathbf{Y}]/J'^c\,[s]\to\mathbf{K}(\boldsymbol{\ell})[\mathbf{X}]/J'\,[s]$, g satisfies

$$[\tau](g)\big([x_n]\big)=\frac{1}{F}f([x_1],\dots,[x_n])=0,$$

since $f\in J$ by assumption. Since $[\tau]$ is by construction injective, it makes $\mathbf{K}(\boldsymbol{\ell})[\mathbf{X}]/J'$ an integral extension of $\mathbf{K}(\boldsymbol{\ell})[\mathbf{Y}]/J'^c$ (the integral dependence relation for $[x_j]$, for $j<n$, is obtained by replacing s by $(s-[y_j])/\ell_j$ in g and clearing denominators).

In particular, these two rings have the same Krull dimension [24, Corollary 2.13]. This latter dimension is the same as that of $\mathbf{K}[\mathbf{X}]/J$ (because it can be read off a Gröbner basis of J, and such Gröbner bases are also Gröbner bases of J'), that is, r. In other words, the zero-set of J'^c over an algebraic closure of $\mathbf{K}(\boldsymbol{\ell})$ has dimension r.

Then we can apply the induction hypothesis to $J'^c \subset \mathbf{K}(\boldsymbol{\ell})[\mathbf{Y}]$. If we let $\boldsymbol{b} = (b_{i,j})_{1 \le i \le r, 1 \le j \le n-1-r}$ be $r(n-1-r)$ new indeterminates, and introduce $\mathbf{Z} = (z_1, \ldots, z_r)$, the $\mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})$-algebra homomorphism

$$\begin{array}{rccc} \eta_{\boldsymbol{b}}\colon & \mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{Z}] & \longrightarrow & \mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{Y}]/J'^c\mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{Y}] \\ & z_j & \longmapsto & [y_j] + \displaystyle\sum_{k=1}^{n-1-r} b_{j,k}\, [y_{r+k}] \end{array}$$

is thus injective and realizes an integral extension of the polynomial ring $\mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{Z}]$. On the other hand, by Lemma 5.3, the extended map

$$[\tau]^e : \mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{Y}]/J'^c\mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{Y}] \longrightarrow \mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{X}]/J'\mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{X}]$$

remains injective and integral. By transitivity, it follows that the $\mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})$-algebra homomorphism

$$\begin{array}{rccc} [\tau]^e \circ \eta_{\boldsymbol{b}}\colon & \mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{Z}] & \longrightarrow & \mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{X}]/J'\mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})[\mathbf{X}] \\ & z_j & \longmapsto & [x_j] + \displaystyle\sum_{k=1}^{n-r} m_{j,k}\, [x_{r+k}] \end{array},$$

where for all $1 \le j \le r$,

$$\boldsymbol{m}_j = \left(b_{j,1}, \ldots, b_{j,n-1-r},\ -\ell_j - \sum_{k=1}^{n-1-r} b_{j,k}\, \ell_{r+k} \right),$$

is injective and integral as well. In particular, the restriction of $[\tau]^e \circ \eta_{\boldsymbol{b}}$ to a mapping $\mathbf{K}(\boldsymbol{m})[\mathbf{Z}] \to \mathbf{K}(\boldsymbol{m})[\mathbf{X}]/J\mathbf{K}(\boldsymbol{m})[\mathbf{X}]$ is still injective and integral, by Lemma 5.3 (here, we write $\mathbf{K}(\boldsymbol{m}) = \mathbf{K}(m_{1,1}, \ldots, m_{r(n-r)})$).

Letting $\boldsymbol{a}$ be $r(n-r)$ new indeterminates, we observe that $\iota : a_{i,j} \mapsto m_{i,j}$ defines a $\mathbf{K}$-isomorphism $\mathbf{K}(\boldsymbol{a}) \to \mathbf{K}(\boldsymbol{m}) \subset \mathbf{K}(\boldsymbol{\ell}, \boldsymbol{b})$, since the entries of $\boldsymbol{m}$ are $\mathbf{K}$-algebraically independent. The conclusion follows. ☐

Corollary 5.5. *Let $V \subset \mathbf{C}^n$ be an r-dimensional algebraic set. Let $\boldsymbol{a} = (a_{i,j})_{1 \le i \le r, 1 \le j \le n-r}$ be $r(n-r)$ new indeterminates and let $V_{\mathfrak{K}} \subset \mathfrak{K}^n$ be the extension of V to the algebraic closure $\mathfrak{K}$ of $\mathbf{K} = \mathbf{C}(\boldsymbol{a})$. Then the restriction $\widetilde{\boldsymbol{f}} : V_{\mathfrak{K}} \to \mathfrak{K}^r$ of the polynomial map $\boldsymbol{f} = (f_1, \ldots, f_r)$ given by*

$$f_j = x_j + \sum_{k=1}^{n-r} a_{j,k}\, x_{r+k}, \quad 1 \le j \le r$$

is finite.

Proof. Let J be the defining ideal of V in $\mathbf{C}[\mathbf{X}]$. Letting $\mathbf{Z} = z_1, \ldots, z_r$ be r new indeterminates, the previous proposition shows that $\widetilde{\boldsymbol{f}}^* : \mathbf{C}(\boldsymbol{a})[\mathbf{Z}] \to \mathbf{C}(\boldsymbol{a})[\mathbf{X}]/J\mathbf{C}(\boldsymbol{a})[\mathbf{X}]$ is injective and integral. By Lemma 5.3, we further deduce that it is also the case for the extension of $\boldsymbol{f}^* : \mathfrak{K}[\mathbf{Z}] \to \mathfrak{K}[\mathbf{X}]/J\mathfrak{K}[\mathbf{X}]$.

Because $\mathbf{C}$ is algebraically closed, J remains radical in $\mathfrak{K}[\mathbf{X}]$, so that $J\mathfrak{K}[\mathbf{X}]$ is the defining ideal of $V_{\mathfrak{K}}$, and we are done. ☐

5.2 Finiteness on polar varieties

In this section, we prove the core of the Proposition 2.3, by proving finiteness properties on the restriction of the considered morphisms to their associated polar varieties.

Proposition 5.6. *Let $V \subset \mathbf{C}^n$ be a d-equidimensional algebraic set with finitely many singular points and let $\theta \in \mathbf{C}[\mathbf{X}]$. For $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{d+1})$ in $\mathbf{C}^{(d+1)n}$, and for $2 \leq j \leq d+1$, let*

$$\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1) = \theta(\mathbf{X}) + \sum_{k=1}^{n} \alpha_{1,k} x_k \quad \textit{and} \quad \varphi_j(\mathbf{X}, \boldsymbol{\alpha}_j) = \sum_{k=1}^{n} \alpha_{j,k} x_k.$$

Then for any $1 \leq i \leq d+1$, there exists a non-empty Zariski open set $\Omega_i \subset \mathbf{C}^{(d+1)n}$ such that if $\boldsymbol{\alpha} \in \Omega_i$ and $\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1), \ldots, \varphi_{d+1}(\mathbf{X}, \boldsymbol{\alpha}_{d+1}))$, then the restriction of $\boldsymbol{\varphi}_{i-1}$ to $W_{\boldsymbol{\varphi}}(i, V)$ is a finite map.

Proof. Let $\boldsymbol{a} = (\boldsymbol{a}_i)_{1 \leq j \leq d+1}$, with $\boldsymbol{a}_j = (a_{j,1}, \ldots, a_{j,n})$ for all j, be $(d+1)n$ new indeterminates, and let $\mathbf{C}(\boldsymbol{a})$ be the field of rational fractions in the entries of $\boldsymbol{a}$. We let $\mathfrak{K}$ be the algebraic closure of $\mathbf{C}(\boldsymbol{a})$, and we denote by $V_{\mathfrak{K}} \subset \mathfrak{K}^n$ the extension of V to $\mathfrak{K}$. Let further

$$\phi_1(\mathbf{X}, a_1) = \theta(\mathbf{X}) + \sum_{k=1}^{n} a_{1,k} x_k \quad \text{and} \quad \phi_j(\mathbf{X}, a_j) = \sum_{k=1}^{n} a_{j,k} x_k, \quad 2 \leq j \leq d+1$$

and define $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_{d+1})$ in $\mathbf{C}(\boldsymbol{a})[\mathbf{X}]$; as before, for $1 \leq i \leq d+1$, we write $\boldsymbol{\phi}_i = (\phi_1, \ldots, \phi_i)$. We will prove the following property, which we call $\mathcal{P}(i)$, by decreasing mathematical induction, for $i = d+1, \ldots, 1$:

$\mathcal{P}(i)$: the restriction of $\boldsymbol{\phi}_{i-1}$ to $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})$ is a finite map.

Let us first see how to deduce the proposition from this claim; hence, we start by fixing i in $1, \ldots, d+1$ and assume that $\mathcal{P}(i)$ holds.

Since $\boldsymbol{\phi}_i$ and $V_{\mathfrak{K}}$ are defined by polynomials with coefficients in $\mathbf{C}(\boldsymbol{a})$, it is also the case for $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})$ by [30, Lemma A.2]. Then $\mathcal{P}(i)$ shows (via the discussion preceding Lemma 5.3) that the pullback $\widetilde{\boldsymbol{\phi}}^*_{i-1} : \mathbf{C}(\boldsymbol{a})[z_1, \ldots, z_{i-1}] \to \mathbf{C}(\boldsymbol{a})[\mathbf{X}]/\boldsymbol{I}(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))$ is injective and integral.

- Injectivity means that the ideal generated by $\boldsymbol{I}(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))$ and the polynomials $z_1 - \phi_1, \ldots, z_{i-1} - \phi_{i-1}$ in $\mathbf{C}(\boldsymbol{a})[z_1, \ldots, z_{i-1}, \mathbf{X}]$ has a trivial intersection with $\mathbf{C}(\boldsymbol{a})[z_1, \ldots, z_{i-1}]$. Then, this remains true for the restriction of $\boldsymbol{\varphi}_{i-1}$ to $W_{\boldsymbol{\varphi}}(i, V)$ for $\boldsymbol{\alpha}$ in a non-empty Zariski-open set in $\mathbf{C}^{(d+1)n}$. For instance, it is enough to ensure that the numerators and denominators of the coefficients of all polynomials appearing in a lexicographic Gröbner basis computation for the ideal above, in $\mathbf{C}(\boldsymbol{a})[z_1, \ldots, z_{i-1}, \mathbf{X}]$, do not vanish at $\boldsymbol{\alpha}$.

- Integrality means that there exist n monic polynomials $P_1, \ldots, P_n$ in $\mathbf{C}(\boldsymbol{a})[z_1, \ldots, z_{i-1}][s]$ such that all polynomials

$$P_j\left(\varphi_1, \ldots, \varphi_{i-1}, x_j\right), \quad 1 \leq j \leq n$$

  belong to $\boldsymbol{I}(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})$ in $\mathbf{C}(\boldsymbol{a})[\mathbf{X}]$. Taking $G \in \mathbf{C}[\boldsymbol{a}]$ as the least common multiple of the denominators of all coefficients that appear in these membership relations, we see that for $\boldsymbol{\alpha}$ in $\mathbf{C}^{(d+1)n}$, if $G(\boldsymbol{\alpha}) \neq 0$, $\boldsymbol{\varphi}_{i-1}$ makes $\mathbf{C}[\mathbf{X}]/\boldsymbol{I}(W_{\boldsymbol{\varphi}}(i, V))$ integral over $\mathbf{C}[z_1, \ldots, z_{i-1}]$.

Base case: $\boldsymbol{i = d+1}$ We prove $\mathcal{P}(d+1)$. As $T_{\boldsymbol{y}} V_{\mathfrak{K}}$ has dimension d, for every $\boldsymbol{y} \in \mathrm{reg}(V_{\mathfrak{K}})$, the polar variety $W_{\boldsymbol{\phi}}(d+1, V_{\mathfrak{K}})$ is nothing but $V_{\mathfrak{K}}$ (since the latter only admits finitely many singular points); hence, we have to prove that the restriction of $\boldsymbol{\phi}_d$ to $V_{\mathfrak{K}}$ is finite.

Let $y_1, \ldots, y_d$ be new variables and consider the algebraic set $V' \subset \mathbf{C}^{d+n}$ defined by $y_1 - \theta, y_2, \ldots, y_d$ and all polynomials f, for f in $\boldsymbol{I}(V)$; as above, we denote its extension to $\mathfrak{K}^{d+n}$ by $V'_{\mathfrak{K}}$. Apply Corollary 5.5 to V' (which is still of dimension d): we deduce that the restriction of $\boldsymbol{\phi}_d$ to $V'_{\mathfrak{K}}$ is finite. Since $V'_{\mathfrak{K}}$ and $V_{\mathfrak{K}}$ are isomorphic (since $V'_{\mathfrak{K}}$ is a graph above $V_{\mathfrak{K}}$), we are done with this case.

Induction step: $\boldsymbol{1 \leq i \leq d}$ Assume now that $\mathcal{P}(i+1)$ holds. Thus, the restriction of $\boldsymbol{\phi}_i$ to a mapping $W_{\boldsymbol{\phi}}(i+1, V_{\mathfrak{K}}) \to \mathfrak{K}^i$ is finite. By [33, Theorem 1.12] this restriction is a Zariski-closed map so that, since $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}) \subset W_{\boldsymbol{\phi}}(i+1, V_{\mathfrak{K}})$, $\boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})) \subset \mathfrak{K}^i$ is an algebraic set and the restriction of $\boldsymbol{\phi}_i$ to a mapping $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}) \to \boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))$ is finite as well.

Let $\mathbf{Y} = (y_1, \ldots, y_i)$ be new indeterminates. Because these sets are defined over $\mathbf{C}(\boldsymbol{a})$, we deduce that the pullback $\mathbf{C}(\boldsymbol{a})[\mathbf{Y}]/\boldsymbol{I}(\boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))) \to \mathbf{C}(\boldsymbol{a})[\mathbf{X}]/\boldsymbol{I}(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))$ that maps y_j to φ_j (for all $j \leq i$) is injective and integral (Lemma 5.3).

On another hand, by the theorem on the dimension of the fibers [33, Theorem 1.25], for any irreducible component C of $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})$ and for a generic $\boldsymbol{y} \in \boldsymbol{\phi}_i(C)$,

$$\dim C - \dim \boldsymbol{\phi}_i(C) = \dim \boldsymbol{\phi}_i^{-1}(\boldsymbol{y}) \cap C = 0$$

since, as a finite map, the restriction of $\boldsymbol{\phi}_i$ to $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})$ has finite fibers. By an algebraic version of Sard's theorem [30, Proposition B.2]

$$\dim\ \boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})) \leq i-1,$$

so that $\dim W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}) \leq i-1$ as well. Together with Lemma 5.1, this proves that both $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})$ and its image $\boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))$ are either empty or equidimensional of dimension $i-1$. If they are empty, there is nothing to do, so suppose it is not the case.

Let $\mathbf{Z} = (z_1, \ldots, z_{i-1})$ and $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_{i-1})$ be new indeterminates. Since $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})$, and thus its image $\boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))$, are defined over $\mathbf{C}(\boldsymbol{a})$, we can apply Noether normalization to $\boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))$ (Proposition 5.4) with coefficients in $\mathbf{C}(\boldsymbol{a})$, and deduce that the $\mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})$-algebra homomorphism

$$\begin{array}{rccl} \zeta\colon & \mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})[\mathbf{Z}] & \longrightarrow & \mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})[\mathbf{Y}]/\boldsymbol{I}(\boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))) \\ & z_j & \longmapsto & y_j + \ell_j y_i \mod \boldsymbol{I}(\boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))) \end{array}$$

is injective and integral. Besides, we deduce from Lemma 5.3 that after scalar extension, the ring homomorphism

$$\mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})[\mathbf{Y}]/\boldsymbol{I}(\boldsymbol{\phi}_i(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))) \to \mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})[\mathbf{X}]/\boldsymbol{I}(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}))$$

that maps y_j to φ_j (for all $j \leq i$) is still injective and integral. If we set

$$\psi_j = \phi_j + \ell_j \phi_i \quad \text{for } 1 \leq j \leq i-1 \quad \text{and} \quad \psi_j = \phi_j \quad \text{for } i \leq j \leq d+1, \tag{4}$$

and finally $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_{d+1}) \subset \mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})[\mathbf{X}]$, then, by transitivity,

$$\begin{array}{rccl} \boldsymbol{\psi}_{i-1}\colon & \mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})[\mathbf{Z}] & \longrightarrow & \mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})[\mathbf{X}]/\boldsymbol{I}(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})) \\ & z_j & \longmapsto & \psi_j(\mathbf{X}) \mod \boldsymbol{I}(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})) \end{array}$$

is injective and integral as well.

Since the first i entries of $\boldsymbol{\psi}$ are elementary row operations of the first i entries of $\boldsymbol{\phi}$, we deduce that $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}}) = W_{\boldsymbol{\psi}}(i, V_{\mathfrak{K}})$. Besides, injecting the definition of the ϕ_j's in (4), one gets that $\boldsymbol{\psi}(\mathbf{X}) = \boldsymbol{\phi}(\mathbf{X}, \boldsymbol{m})$, where

$$\boldsymbol{m} = (\boldsymbol{a}_1 + \ell_1 \boldsymbol{a}_i, \dots, \boldsymbol{a}_{i-1} + \ell_{i-1}\boldsymbol{a}_i, \boldsymbol{a}_i, \dots, \boldsymbol{a}_{d+1})$$

is a vector of $(d+1)n$ $\mathbf{C}$-algebraically independent elements of $\mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})$. Through the isomorphism $\mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell}) \to \mathbf{C}(\boldsymbol{m})$, we see that

$$\begin{array}{rccc} \boldsymbol{\phi}_{i-1}\colon & \mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})[\mathbf{Z}] & \longrightarrow & \mathbf{C}(\boldsymbol{a}, \boldsymbol{\ell})[\mathbf{X}]/\boldsymbol{I}(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})) \\ & z_j & \longmapsto & \phi_j(\mathbf{X}) \mod \boldsymbol{I}(W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})) \end{array}$$

is injective and integral. From Lemma 5.3, we see that this precisely gives that the restriction of $\boldsymbol{\phi}_{i-1}$ to $W_{\boldsymbol{\phi}}(i, V_{\mathfrak{K}})$ is finite. This ends the proof of the induction step, and, by mathematical induction, of the proposition. □

5.3 Proof of the main proposition

We conclude by proving Proposition 2.3, which is a direct consequence of the previous results. Let V, θ and $2 \leq \mathfrak{r} \leq d+1$ as given in the statement of the proposition.

Let Ω be the non-empty Zariski open subset of $\mathbf{C}^{(d+1)n}$ obtained as the intersection, for all $1 \leq i \leq d+1$, of the Ω_i's given by application of Proposition 5.6. Let $\boldsymbol{a} = (\boldsymbol{a}_i)_{1 \leq i \leq d+1}$, where $\boldsymbol{a}_i = (a_{i,1}, \dots, a_{i,n})$ be $(d+1)n$ new indeterminates. By definition, there exists $\boldsymbol{f} = (f_1, \dots, f_p) \subset \mathbf{C}[\boldsymbol{a}]$, such that $\Omega = \mathbf{C}^{(d+1)n} - \boldsymbol{V}(\boldsymbol{f})$. Then, let $\Omega_{\mathsf{I}}(V, \theta, \mathfrak{r})$ be the projection on the first $\mathfrak{r}n$ coordinates of Ω, it is the union, for all $\boldsymbol{\alpha}'' \in \mathbf{C}^{(d+1-\mathfrak{r})n}$, of the non-empty Zariski open sets

$$\mathbf{C}^{\mathfrak{r}n} - \boldsymbol{V}\left(\boldsymbol{f}(\boldsymbol{a}', \boldsymbol{\alpha}'')\right),$$

where $\boldsymbol{a}' = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_{\mathfrak{r}})$, hence a non-empty Zariski open subset of $\mathbf{C}^{\mathfrak{r}n}$.

Let $\boldsymbol{\alpha}' \in \Omega_{\mathsf{I}}(V, \theta, \mathfrak{r})$ and $\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}'_1), \dots, \varphi_{\mathfrak{r}}(\mathbf{X}, \boldsymbol{\alpha}'_{\mathfrak{r}}))$. Let $i \in \{1, \dots, \mathfrak{r}\}$ then, there exists $\boldsymbol{\alpha}'' \in \mathbf{C}^{(d+1-\mathfrak{r})n}$ such that $(\boldsymbol{\alpha}', \boldsymbol{\alpha}'') \in \Omega_i$. Therefore by Proposition 5.6, the restriction of $\boldsymbol{\varphi}_{i-1}$ to $W_{\boldsymbol{\varphi}}(i, V)$ is finite.

In particular, by [33, Section 5.3], the restriction of $\boldsymbol{\varphi}_{i-1}$ to $W_{\boldsymbol{\varphi}}(i, V)$ is a Zariski-closed map that has finite fibers. Moreover, since $\operatorname{sing}(V)$ is finite, we deduce that $K_{\boldsymbol{\varphi}}(i, V) \cap \boldsymbol{\varphi}_{i-1}^{-1}(\boldsymbol{z})$ is finite for any $\boldsymbol{z} \in \mathbf{C}^{i-1}$. Finally, as a consequence, and by [33, Theorem 1.12 and 1.25], $W_{\boldsymbol{\varphi}}(i, V)$ is equidimensional of dimension $i-1$. It is worth noting that the latter can also be seen as a consequence of [33, Theorem 1.25] and Lemma 5.1.

6 Proof of Proposition 2.12: atlases for polar varieties

This section is devoted to prove Proposition 2.12, that we recall below.

Proposition (2.12)**.** *Let $V, S \subset \mathbf{C}^n$ be two algebraic sets with V d-equidimensional and S finite and $\boldsymbol{\chi}$ be an atlas of (V, S). For $2 \leq \mathfrak{r} \leq d+1$, let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{\mathfrak{r}})$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{\mathfrak{r}})$, and for $1 \leq j \leq \mathfrak{r}$, let $\boldsymbol{\alpha}_j = (\alpha_{j,1}, \dots, \alpha_{j,n}) \in \mathbf{C}^n$ and*

$$\varphi_j(\mathbf{X}, \boldsymbol{\alpha}_j) = \theta_j(\mathbf{X}) + \sum_{k=1}^{n} \alpha_{j,k} x_k + \xi_j(\boldsymbol{\alpha}_j) \in \mathbf{C}[\mathbf{X}].$$

where $\theta_j \in \mathbf{C}[\mathbf{X}]$ and $\xi_j\colon \mathbf{C}^n \to \mathbf{C}$ is a polynomial map, with coefficients in $\mathbf{C}$.

There exists a non-empty Zariski open subset $\Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, S, \boldsymbol{\theta}, \boldsymbol{\xi}, \mathfrak{r}) \subset \mathbf{C}^{\mathfrak{r}n}$ such that for every $\boldsymbol{\alpha} \in \Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, S, \boldsymbol{\theta}, \boldsymbol{\xi}, \mathfrak{r})$, writing $\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}), \dots, \varphi_{\mathfrak{r}}(\mathbf{X}, \boldsymbol{\alpha}))$, the following holds. For i in $\{1, \dots, \mathfrak{r}\}$, either $W_{\boldsymbol{\varphi}}(i, V)$ is empty or

1. $W_{\boldsymbol{\varphi}}(i, V)$ is an equidimensional algebraic set of dimension $i-1$;

2. if $2 \leq i \leq (d+3)/2$, then $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, i)$ is an atlas of $(W_{\boldsymbol{\varphi}}(i, V), S)$ and $\mathrm{sing}(W_{\boldsymbol{\varphi}}(i, V)) \subset S$.

6.1 Regularity properties

In this subsection, we fix the three integers $(d, \mathfrak{r}, i)$ such that $2 \leq \mathfrak{r} \leq d+1 \leq n+1$ and $1 \leq i \leq \mathfrak{r}$.

For $1 \leq j \leq i$, let $a_j = (a_{j,1}, \ldots, a_{j,n})$ be new indeterminates, and let $\mathbf{A} = (a_j)_{1 \leq j \leq i}$. For $1 \leq j \leq i$, we will also denote by $\mathbf{A}_{\leq j}$, the subfamily $(a_1, \ldots, a_j)$. Finally, we consider sequences $\boldsymbol{h} = (h_1, \ldots, h_c) \subset \mathbf{C}[\mathbf{X}]$, where $c = n-d$, and $\phi = (\phi_1, \ldots, \phi_i)$ such that

$$\phi_j(\mathbf{X}, a_j) = \theta_j(\mathbf{X}) + \sum_{k=1}^{n} a_{j,k} x_k + \xi_j(a_j) \in \mathbf{C}[\mathbf{X}, \mathbf{A}_{\leq j}],$$

for $1 \leq j \leq i$. We start by investigating the regular situation. The first step towards the proof of Proposition 2.12 is to establish the following statement.

Proposition 6.1. *There exists a non-empty Zariski open set $\Omega_i^{\boldsymbol{h}} \subset \mathbf{C}^{in}$, such that for all $\boldsymbol{\alpha} \in \Omega_i^{\boldsymbol{h}}$, and $\boldsymbol{\varphi} = \big(\phi_1(\mathbf{X}, \boldsymbol{\alpha}_1), \ldots, \phi_i(\mathbf{X}, \boldsymbol{\alpha}_i)\big) \subset \mathbf{C}[\mathbf{X}]$, the following holds:*

1. for all $\boldsymbol{y} \in \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h})$, there exists a c-minor m' of $\mathrm{Jac}(\boldsymbol{h})$ such that $m'(\boldsymbol{y}) \neq 0$;

2. the irreducible components of $W_{\boldsymbol{\varphi}}(i, \boldsymbol{V}_{reg}(\boldsymbol{h}))$ have dimension less than $i-1$;

Assume now that $i \leq (d+3)/2$, and let m' be any c-minor of $\mathrm{Jac}(\boldsymbol{h})$ and let m'' be any $(c+i-1)$-minors of $\mathrm{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ containing the rows of $\mathrm{Jac}(\boldsymbol{\varphi}_i)$. Then, the following holds:

3. for all $\boldsymbol{y} \in \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h})$ there exists m'' as above, such that $m''(\boldsymbol{y}) \neq 0$;

4. $W^{\circ}_{\boldsymbol{\varphi}}(i, \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h}))$ is defined on $\mathcal{O}(m'm'')$ by the vanishing of the polynomials in $(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''))$;

5. $\mathrm{Jac}(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''))$ has full rank $n-(i-1)$ on $\mathcal{O}(m'm'') \cap W^{\circ}_{\boldsymbol{\varphi}}(i, \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h}))$.

6.1.a Rank estimates

We start by proving some genericity results on the ranks of some Jacobian matrix. Two direct consequences (namely Corollaries 6.3 and 6.4) of Proposition 6.2 below will establish the third statement of Proposition 6.1.

Let $1 \leq p \leq n-1$ and $M(\mathbf{X}, \mathbf{A}_{\leq 1})$ be a $p \times n$ matrix with coefficients in $\mathbf{C}[\mathbf{X}, \mathbf{A}_{\leq 1}]$. For $1 \leq j \leq i$, let

$$J_j(\mathbf{X}, \mathbf{A}_{\leq j}) = \begin{bmatrix} & M(\mathbf{X}, \mathbf{A}_{\leq 1}) & \\ \partial_{x_1}\phi_1(\mathbf{X}, a_{1,1}) & \cdots & \partial_{x_n}\phi_1(\mathbf{X}, a_{1,n}) \\ \vdots & & \vdots \\ \partial_{x_1}\phi_j(\mathbf{X}, a_{j,1}) & \cdots & \partial_{x_n}\phi_j(\mathbf{X}, a_{j,n}) \end{bmatrix},$$

where for all $1 \leq k \leq i$ and $1 \leq \ell \leq n$, $\partial_{x_\ell}\phi_k = \frac{\partial \theta_k(\mathbf{X})}{\partial x_\ell} + a_{k,\ell} \in \mathbf{C}[\mathbf{X}, a_{k,\ell}]$. Proposition 6.2 below generalizes [30, Proposition B.6]. Our proof follows the same pattern as the one of [30, Proposition B.6].

Proposition 6.2. *Assume that there exists a non-empty Zariski open subset $\mathscr{E}_0 \subset \mathbf{C}^n$ such that for all $(\boldsymbol{y}, \boldsymbol{\alpha}) \in \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h}) \times \mathscr{E}_0$, the matrix $M(\boldsymbol{y}, \boldsymbol{\alpha})$ has full rank p. Then, for every*

$$1 \leq j \leq \min\{\, i,\, c - p + (d+3)/2 \,\},$$

there exists a non-empty Zariski open subset $\mathscr{E}_i \subset \mathbf{C}^{in}$ such that for all $(\boldsymbol{y}, \boldsymbol{\alpha}) \in \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h}) \times \mathscr{E}_i$,

$$\operatorname{rank} M(\boldsymbol{y}, \boldsymbol{\alpha}) = p \quad \textit{and} \quad \operatorname{rank} J_i(\boldsymbol{y}, \boldsymbol{\alpha}) \geq p + j - 1.$$

Before proving the above proposition, we first give two direct consequences of it, whose conjunction proves the third item of Proposition 6.1. Taking $M = \operatorname{Jac}(\boldsymbol{h})$, the next lemma is a direct consequence of the definition of $\boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h})$.

Corollary 6.3. *If $1 \leq i \leq (d+3)/2$ then, there exists a non-empty Zariski open subset $\mathscr{E}'_i \subset \mathbf{C}^{in}$ such that for all $(\boldsymbol{y}, \boldsymbol{\alpha}) \in \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h}) \times \mathscr{E}'_i$, the matrix $\operatorname{Jac}_{(\boldsymbol{y},\boldsymbol{\alpha})}([\boldsymbol{h}, \phi])$ has rank at least $c + i - 1$.*

Besides we deduce the following more subtle consequence.

Corollary 6.4. *If $1 \leq i \leq (n + c + 1)/2$ then, there exists a non-empty Zariski open subset $\mathscr{E}''_i \subset \mathbf{C}^{in}$ such that for all $(\boldsymbol{y}, \boldsymbol{\alpha}) \in \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h}) \times \mathscr{E}''_i$, the matrix $\operatorname{Jac}_{(\boldsymbol{y},\boldsymbol{\alpha})}(\phi)$ has full rank i.*

Proof. Take $M = \operatorname{Jac}(\phi_1)$. The matrix $\operatorname{Jac}(\phi_1)$ has not full rank if, and only if, all the derivatives of ϕ_1 vanish at this point. Following the proof strategy of Lemma 6.6, let

$$Z^{\circ} = Z \cap \boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h}) \subset \mathbf{C}^{n+n} \quad \text{where} \quad Z = \boldsymbol{V}\left(\boldsymbol{h}, \frac{\partial \phi_1}{\partial x_1}, \ldots, \frac{\partial \phi_1}{\partial x_n}\right).$$

whose following Jacobian matrix, has full rank $c + n$ at any $(\boldsymbol{y}, \boldsymbol{\alpha}) \in Z^{\circ}$

$$\operatorname{Jac}_{(\mathbf{X}, a_{1,1}, \ldots a_{1,n})}\left(\boldsymbol{h}, \frac{\partial \phi_1}{\partial x_1}, \ldots, \frac{\partial \phi_1}{\partial x_n}\right) = \left[\begin{array}{c|ccc} \operatorname{Jac}(\boldsymbol{h}) & & \mathbf{0} & \\ \hline * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{array}\right].$$

Hence, by the Jacobian criterion [14, Theorem 16.19], Z° is either empty or a d-equidimensional locally closed set. Since $d < n$ by assumption, then the projection of Z° on the variables $\mathbf{A}_{\leq 1}$ is a proper subset of $\mathbf{C}^n$ and taking $\mathscr{E}_0$ as its complement allows us to conclude.

Indeed, for any $1 \leq i \leq (n+2)/2$, by Proposition 6.2, there exists a non-empty Zariski open subset $\mathscr{E}_i$ of $\mathbf{C}^{in}$ such that for all $(\boldsymbol{y}, \boldsymbol{\alpha}) \in \boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h}) \times \mathscr{E}_i$,

$$\operatorname{rank} \operatorname{Jac}_{(\boldsymbol{y},\boldsymbol{\alpha})}(\phi_1, \ldots, \phi_i) = \operatorname{rank} \operatorname{Jac}_{(\boldsymbol{y},\boldsymbol{\alpha})}(\phi_1, \phi_1, \ldots, \phi_i) = 1 + i - 1 = i.$$

□

The rest of this paragraph is devoted to the proof of Proposition 6.2. Following the construction of the proof of [30, Proposition B.6], we proceed by induction on j. For all $1 \leq j \leq \min\{i, \lfloor c - p + (d+3)/2\}$, we denote by R_j the statement of Proposition 6.2.

Base case: $j = 1$ By assumption, there exists a non-empty Zariski open subset $\mathscr{E}_0 \subset \mathbf{C}^n$ such that for all $(\boldsymbol{y}, \boldsymbol{\alpha}) \in \boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h}) \times \mathscr{E}_0$, the matrix $M(\boldsymbol{y}, \boldsymbol{\alpha}_1)$ has full rank p. Therefore, the matrix J_1, containing M, has rank at least p. This proves that R_1 holds.

Induction step: $2 \leq j \leq \min\{\, i,\, c - p + (d+3)/2 \,\}$ Assume that R_{j-1} holds, and let us prove that so does R_j. Let $\mathfrak{M}$ be the set of ordered pairs $\mathfrak{m} = (\mathfrak{m}_r, \mathfrak{m}_c)$ where

- $\{1, \ldots, p\} \subset \mathfrak{m}_r \subset \{1, \ldots, p+j-1\}$
- $\mathfrak{m}_c \subset \{1, \ldots, n\}$
- $|\mathfrak{m}_r| = |\mathfrak{m}_c| = p + j - 2$

Then, for each such $\mathfrak{m}$, let $J_{\mathfrak{m}}$ be the square submatrix of J_j obtained by selecting the rows and the columns in respectively $\mathfrak{m}_r$ and $\mathfrak{m}_c$. Such a submatrix can also be obtained by removing from J_i, $n-p-j+2$ columns and and two rows, which includes the last row. Besides, let $\delta_{\mathfrak{m}} \in \mathbf{C}[\mathbf{X}, \mathbf{A}_{\leq j-1}]$ be the determinant of $J_{\mathfrak{m}}$, that is the $(p+j-2)$-minor of J_j associated to $\mathfrak{m}$. Finally, let Sub_j be the subset of $\mathfrak{m} \in \mathfrak{M}$ such that there exists $(\boldsymbol{y}, \boldsymbol{\alpha}) \in \boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h}) \times \mathbf{C}^{jn}$ such that $\delta_{\mathfrak{m}}(\boldsymbol{y}, \boldsymbol{\alpha}) \neq 0$.

Lemma 6.5. *The set* Sub_j*, thus defined, is not empty.*

Proof. By induction assumption R_{j-1}, there exists a non-empty Zariski open subset $\mathscr{E}_{j-1} \subset \mathbf{C}^{(j-1)n}$ such that for all $(\boldsymbol{y}, \boldsymbol{\alpha}') \in \boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h}) \times \mathscr{E}_{j-1}$, the matrix $J_{j-1}(\boldsymbol{y}, \boldsymbol{\alpha}')$ has rank at least $p+j-2$ and $M(\boldsymbol{y}, \boldsymbol{\alpha}')$ has full rank p. We deduce that there exists a non-zero $(p+j-2)$-minor of $J_{j-1}(\boldsymbol{y}, \boldsymbol{\alpha}')$ containing the rows of $M(\boldsymbol{y}, \boldsymbol{\alpha}')$. Then, by definition of $\mathfrak{M}$,

$$\forall (\boldsymbol{y}, \boldsymbol{\alpha}') \in \boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h}) \times \mathscr{E}_{j-1},\ \exists \mathfrak{m} \in \mathfrak{M}, \quad \delta_{\mathfrak{m}}(\boldsymbol{y}, \boldsymbol{\alpha}') \neq 0, \tag{5}$$

where $\delta_{\mathfrak{m}} \in \mathbf{C}[\mathbf{X}, \mathbf{A}_{\leq j-1}]$. This proves, in particular, that Sub_j is not empty, as neither $\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h})$ nor $\mathscr{E}_{j-1}$ is empty. $\square$

We now prove the following lemma, which is the key step in the proof of R_j.

Lemma 6.6. *For all* $\mathfrak{m} \in \mathsf{Sub}_j$*, there exists a non-empty Zariski open subset* $\mathfrak{E}_{\mathfrak{m}} \subset \mathbf{C}^{jn}$ *such that, for all* $(\boldsymbol{y}, \boldsymbol{\alpha}) \in \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h}) \times \mathfrak{E}_{\mathfrak{m}}$*, if* $\delta_{\mathfrak{m}}(\boldsymbol{y}, \boldsymbol{\alpha}) \neq 0$*, then* $J_j(\boldsymbol{y}, \boldsymbol{\alpha})$ *has rank at least* $p+j-1$*.*

Proof. Take $\mathfrak{m}$ in Sub_j. We proceed to show that the subset of the $\boldsymbol{\alpha} \in \mathbf{C}^{jn}$ such that, for all $\boldsymbol{y} \in \boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h})$, $\delta_{\mathfrak{m}}(\boldsymbol{y}, \boldsymbol{\alpha}) \neq 0$ and $J_j(\boldsymbol{y}, \boldsymbol{\alpha})$ has rank at most $p+j-2$ is a *proper algebraic subset* of $\mathbf{C}^{jn}$. Then, taking the complement will give us $\mathfrak{E}_{\mathfrak{m}}$.

Up to reordering, assume that the rows and columns of J_j that are not in $J_{\mathfrak{m}}$ are the ones of respective indices $p+j-1, p+j$ (the last two rows) and $p-j+3, \ldots, n$ (the last $n-p+j-2$ columns). In other words, $(p+k, \ell) \notin \mathfrak{m}_r \times \mathfrak{m}_c$ for all $k \in \{j-1, j\}$ and $\ell \in \{p-j+3, \ldots, n\}$. For such k, ℓ, we denote by $\delta_{k,\ell}$ the minor of J_j obtained by adding to $J_{\mathfrak{m}}$ the row and column indexed by respectively $p+k$ and ℓ. Let $\mathbf{A}''$ be the subset of elements of $\mathbf{A}_{\leq}$ formed by the $2(n-p-j+2)$ indeterminates

$$a_{j-1,p-j+3}, \ldots, a_{j-1,n} \quad \text{and} \quad a_{j,p-i+3}, \ldots, a_{j,n},$$

and let $\mathbf{A}' = \mathbf{A}_{\leq j} - \mathbf{A}''$. Remark then that for any such $k \in \{j-1, j\}$ and $\ell \in \{p-j+3, \ldots, n\}$, by cofactor expansion there exists a polynomial $g_{k,\ell} \in \mathbf{C}[\mathbf{X}, \mathbf{A}']$ such that

$$\delta_{u,v} = \delta_{\mathfrak{m}} \cdot \frac{\partial \phi_k}{\partial x_\ell}(\mathbf{X}, a_{k,\ell}) +\ g_{k,\ell}(\mathbf{X}, \mathbf{A}') \tag{6}$$

Let $\boldsymbol{\delta}$ be the sequence of the $2(n-p-j+2)$ minors $\delta_{k,\ell}$. We proceed to prove that, the set of specialization values $\boldsymbol{\alpha} \in \mathbf{C}^{jn}$ of the genericity parameters (the entries of $\mathbf{A}_{\leq j}$), such that all these minors in $\boldsymbol{\delta}(\mathbf{X},\boldsymbol{\alpha})$ are identically zero but not $\delta_{\mathfrak{m}}(\mathbf{X},\boldsymbol{\alpha})$, is a proper algebraic subset of $\mathbf{C}^{jn}$. Hence, let t a new indeterminate and consider the locally closed set

$$Z^\circ = Z \cap \boldsymbol{V}^\circ_{\mathrm{reg}}(\boldsymbol{h}) \subset \mathbf{C}^{n+jn+1} \quad \text{where} \quad Z = \boldsymbol{V}\left(\boldsymbol{h}, \boldsymbol{\delta}, 1 - t\delta_{\mathfrak{m}}\right).$$

One observes that if $(\boldsymbol{y},\boldsymbol{\alpha},t) \in Z^\circ$ then $\boldsymbol{y} \in \boldsymbol{V}^\circ_{\mathrm{reg}}(\boldsymbol{h})$, $\delta_{\mathfrak{m}}(\boldsymbol{y},\boldsymbol{\alpha}) \neq 0$ and all the $\delta_{k,\ell}$'s vanish.

We claim first that Z° is not empty. Indeed, since $\mathfrak{m} \in \mathsf{Sub}_j$, there exists $(\boldsymbol{y},\boldsymbol{\alpha}) \in \boldsymbol{V}^\circ_{\mathrm{reg}}(\boldsymbol{h}) \times \mathbf{C}^{jn}$ such that $\delta_{\mathfrak{m}}(\boldsymbol{y},\boldsymbol{\alpha}) \neq 0$. Since $\delta_{\mathfrak{m}} \in \mathbf{C}[\mathbf{X},\mathbf{A}']$, it is independent of the entries of $\mathbf{A}''$. Besides, for any $k \in \{j-1, j\}$ and $\ell \in \{p-j+3,\ldots,n\}$,

$$\frac{\partial \phi_k}{\partial x_\ell}(\boldsymbol{y}, a_{k,\ell}) = \frac{\partial \theta_k}{\partial x_\ell}(\mathbf{X}) + a_{k,\ell} \in \mathbf{C}[\mathbf{X}][\mathbf{A}''] \tag{7}$$

is a non-constant polynomial in the entries of $\mathbf{A}''$. Then, according to (6), for every such k,ℓ, one can choose $\alpha_{k,\ell} \in \mathbf{C}$ such that $\delta_{k,\ell}(\boldsymbol{y},\boldsymbol{\alpha}',\alpha_{k,\ell}) = 0$. Let $\widetilde{\boldsymbol{\alpha}}$ be the element of $\mathbf{C}^{jn}$ obtained by this choice, then

$$\left(\, \boldsymbol{y},\ \widetilde{\boldsymbol{\alpha}},\ 1/\delta_{\mathfrak{m}}(\boldsymbol{y},\widetilde{\boldsymbol{\alpha}})\,\right) \in Z^\circ.$$

We deduce that Z° is non-empty. We now estimate the dimension of Z° . According to (6) and (7) the following Jacobian matrix has full rank $c + 2(n-p-j+2)+1$ at every point of Z°:

$$\mathrm{Jac}_{(\mathbf{X},\mathbf{A}',\mathbf{A}'',t)}(\boldsymbol{h},\boldsymbol{\delta},1-t\delta_{\mathfrak{m}}) = \left[\begin{array}{c|ccc|ccc|c} \mathrm{Jac}(\boldsymbol{h}) & & \mathbf{0} & & & \mathbf{0} & & 0 \\ \hline * & * & & * & \delta_{\mathfrak{m}} & 0 & \vdots & 0 \\ & & & & 0 & \ddots & 0 & \vdots \\ * & & * & & & 0 & \delta_{\mathfrak{m}} & 0 \\ \hline * & * & & * & * & * & * & \delta_{\mathfrak{m}} \end{array}\right].$$

Therefore, by the Jacobian criterion [14, Theorem 16.19], Z° is an equidimensional locally closed set of dimension $jn-(n-p)+2(j-2)$. Let $Z' \subset \mathbf{C}^{jn}$ be the Zariski closure of the projection of Z° on the coordinates associated to the variables $\mathbf{A}$, then

$$\dim Z' \leq \dim Z^\circ = jn + d - 2(n-p) + 2(j-2) < jn \qquad \text{since } j \leq c - p + (d+3)/2.$$

Hence Z' is a proper algebraic set of $\mathbf{C}^{jn}$, so that its complement $\mathscr{E}_{\mathfrak{m}}$ a non-empty Zariski open subset of $\mathbf{C}^{jn}$. Further, for any $(\boldsymbol{y},\boldsymbol{\alpha}) \in \boldsymbol{V}^\circ_{\mathrm{reg}}(\boldsymbol{h}) \times \mathscr{E}_{\mathfrak{m}}$ such that $\delta_{\mathfrak{m}}$ does not vanish at $(\boldsymbol{y},\boldsymbol{\alpha})$, the point $\big(\boldsymbol{y},\boldsymbol{\alpha},1/\delta_{\mathfrak{m}}(\boldsymbol{y},\boldsymbol{\alpha})\big)$ is not in Z°, otherwise $\boldsymbol{\alpha}$ would be in Z'. Hence, there exists (k,ℓ) as above such that $\delta_{k,\ell}(\boldsymbol{y},\boldsymbol{\alpha}) \neq 0$, so that $J_j(\boldsymbol{y},\boldsymbol{\alpha})$ has a non-zero $(c+j-1)$-minor, and then, has rank at least $c+j-1$. This proves the lemma. □

We can now conclude on the induction step as follows. Since, by Lemma 6.5, Sub_j is not empty, let

$$\mathscr{E}_j = (\mathscr{E}_{j-1} \times \mathbf{C}^n) \cap \bigcap_{\mathfrak{m} \in \mathsf{Sub}_j} \mathfrak{E}_{\mathfrak{m}},$$

where the $\mathfrak{E}_{\mathfrak{m}}$ are the non-empty Zariski open sets given by Lemma 6.6. Remark first that $\mathscr{E}_j$ is a non-empty Zariski open subset of $\mathbf{C}^{jn}$ since it is a finite intersection of non-empty Zariski open sets. Let $(\boldsymbol{y},\boldsymbol{\alpha}',\boldsymbol{\alpha}_j) \in \boldsymbol{V}^\circ_{\mathrm{reg}}(\boldsymbol{h}) \times \mathscr{E}_j$, as seen in (5), there exists $\mathfrak{m}_0 \in \mathsf{Sub}_j$ such that $\delta_{\mathfrak{m}_0}(\boldsymbol{y},\boldsymbol{\alpha}') \neq 0$. By construction, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}',\boldsymbol{\alpha}_j)$ belongs to $\mathfrak{E}_{\mathfrak{m}_0}$ so that, by Lemma 6.6, $J_j(\boldsymbol{y},\boldsymbol{\alpha})$ has rank at least $p+j-1$. Besides, since $\boldsymbol{\alpha}' \in \mathscr{E}_{j-1}$, $M(\boldsymbol{y},\boldsymbol{\alpha}')$ has full rank p.

In conclusion, we proved that R_j, which the induction step, and, by mathematical induction, this proves Proposition 6.2.

6.1.b Dimension estimates

In this paragraph, we aim to prove the second point of Proposition 6.1, using transversality results. Let

$$\begin{array}{rccccccccl}\Phi\colon & \mathbf{C}^n & \times & \mathbf{C}^{in} & \times & \mathbf{C}^c & \times & \mathbf{C}^i & \longrightarrow & \mathbf{C}^c \times \mathbf{C}^n \\ & (\ \boldsymbol{y} & , & \boldsymbol{\alpha} & , & \boldsymbol{\lambda} & , & \boldsymbol{\vartheta}\ ) & \longmapsto & \Big(\boldsymbol{h}(\boldsymbol{y}), {}^t[\boldsymbol{\lambda}, \boldsymbol{\vartheta}] \cdot \mathrm{Jac}_{(\boldsymbol{y},\boldsymbol{\alpha})}(\boldsymbol{h}, \phi)\Big)\end{array}$$

and for any $\boldsymbol{\alpha} \in \mathbf{C}^{in}$, let $\Phi_{\boldsymbol{\alpha}} = (\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}) \mapsto \Phi(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}, \boldsymbol{\alpha})$. The interest of such a map is illustrated by the following lemma. Let $\mathscr{A} \subset \mathbf{C}^{n+in+c+i}$ be the Zariski open subset of the elements $(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta})$ where $\boldsymbol{\lambda} \neq \mathbf{0}$ and $\mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{h})$ has full rank.

Lemma 6.7. *Let $\boldsymbol{\alpha} \in \mathbf{C}^{in}$ and*

$$W^\circ_{\boldsymbol{\alpha}} = \Big\{\boldsymbol{y} \in \mathbf{C}^n \mid \boldsymbol{y} \in \boldsymbol{V}^\circ_{reg}(\boldsymbol{h}) \quad \textit{and} \quad \operatorname{rank} \mathrm{Jac}_{(\boldsymbol{y},\boldsymbol{\alpha})}(\boldsymbol{h}, \phi) \leq c + i - 1\Big\}.$$

Then $W^\circ_{\boldsymbol{\alpha}} = \pi_{\mathbf{X}}\big(\mathscr{A} \cap \Phi^{-1}_{\boldsymbol{\alpha}}(\mathbf{0})\big)$.

Proof. Let $\boldsymbol{\alpha} \in \mathbf{C}^{in}$ and $\boldsymbol{y} \in \boldsymbol{V}^\circ_{\mathrm{reg}}(\boldsymbol{h})$. Then $\boldsymbol{y} \in W^\circ_{\boldsymbol{\alpha}}$ if and only if $\mathrm{Jac}_{(\boldsymbol{y},\boldsymbol{\alpha})}(\boldsymbol{h}, \phi)$ has not full rank, which is equivalent to having a non-zero vector in its co-kernel by duality. Besides, since $\boldsymbol{y} \in \boldsymbol{V}^\circ_{\mathrm{reg}}(\boldsymbol{h})$, the matrix $\mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{h})$ has full rank. Hence $\boldsymbol{y}$ belongs to $W^\circ_{\boldsymbol{\alpha}}$ if and only if there exists a non-zero vector $(\boldsymbol{\lambda}, \boldsymbol{\vartheta}) \in \mathbf{C}^{c+i}$ such that $\Phi(\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}) = 0$ and $\mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{h})$ has full rank. Finally $\boldsymbol{\vartheta}$ cannot be zero otherwise $\mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{h})$ would have a non-trivial left-kernel (containing $\boldsymbol{\lambda}$), and then would not be full rank. □

Lemma 6.8. *Let $\mathscr{A} \subset \mathbf{C}^{n+c+i}$ be the Zariski open subset of the elements $(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta})$ where $\boldsymbol{\vartheta} \neq \mathbf{0}$ and $\mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{h})$ has full rank. There exists a non-empty Zariski open subset $\mathscr{D}_i \subset \mathbf{C}^{in}$ such that for all $\boldsymbol{\alpha} \in \mathscr{D}_i$, $\mathrm{Jac}_{(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\vartheta})} \Phi_{\boldsymbol{\alpha}}$ has full rank $c + n$ at any $(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}) \in \mathscr{A} \cap \Phi^{-1}_{\boldsymbol{\alpha}}(\mathbf{0})$.*

Proof. We have

$$\mathrm{Jac}_{(\mathbf{X},\lambda,\vartheta,a_1,\ldots,a_i)} \Phi = \left[\begin{array}{c|c|ccc} \mathrm{Jac}(\boldsymbol{h}) & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline * & * & \vartheta_1 I_n & \cdots & \vartheta_i I_n \end{array}\right]$$

where I_n is the identity matrix of size n. Let $\boldsymbol{\alpha} \in \mathbf{C}^{in}$, and $(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}) \in \mathscr{A}$ such that the above Jacobian matrix has full rank $c+n$ at $(\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\vartheta})$. Hence $\mathbf{0}$ is a regular value of Φ on $\mathscr{A} \times \mathbf{C}^{in}$. Therefore, by the Thom's weak transversality theorem [30, Proposition B.3], there exists a non-empty Zariski open subset $\mathscr{D}_i \subset \mathbf{C}^{in}$ such that for all $\boldsymbol{\alpha} \in \mathscr{D}_i$, $\mathbf{0}$ is a regular value of $\Phi_{\boldsymbol{\alpha}}$ on $\mathscr{A}$. In other words, for all $\boldsymbol{\alpha} \in \mathscr{D}_i$, the matrix $\mathrm{Jac}\, \Phi_{\boldsymbol{\alpha}}$ has full rank $c + n$ over $\mathscr{A} \cap \Phi^{-1}_{\boldsymbol{\alpha}}(\mathbf{0})$. □

Lemma 6.9. *Let $\mathscr{D}_i \subset \mathbf{C}^{in}$ be the non-empty Zariski subset defined in Lemma 6.8. Then, for all $\boldsymbol{\alpha} \in \mathscr{D}_i$, $W^\circ_{\boldsymbol{\alpha}}$ has dimension at most $i - 1$.*

Proof. Let $\boldsymbol{\alpha} \in \mathscr{D}_i$ and suppose that $W^\circ_{\boldsymbol{\alpha}}$ is not empty. Then, according to Lemma 6.7, $\mathscr{A} \cap \Phi^{-1}_{\boldsymbol{\alpha}}(\mathbf{0})$ is non-empty as well. By Lemma 6.8 and [30, Lemma A.1], $\mathscr{A} \cap \Phi^{-1}_{\boldsymbol{\alpha}}(\mathbf{0})$ is a non-singular equidimensional locally closed set and

$$\dim\left(\mathscr{A} \cap \Phi^{-1}_{\boldsymbol{\alpha}}(\mathbf{0})\right) = n + c + i - (c + n) = i.$$

Let C be the Zariski closure of $\mathscr{A} \cap \Phi^{-1}_{\boldsymbol{\alpha}}(\mathbf{0})$ and $(C_j)_{1\leq j\leq \ell}$ be its irreducible components. For all $1 \leq j \leq \ell$, let T_j be the Zariski closure of $\pi_{\mathbf{X}}(C_j)$. Since $W^\circ_{\boldsymbol{\alpha}} \subset \bigcup_{1\leq j\leq \ell} T_j$, it is enough to prove that $\dim T_j \leq i - 1$ for all $1 \leq j \leq \ell$.

Fix $1 \leq j \leq \ell$. The restriction $\pi_{\mathbf{X}}: C_j \to T_j$ is a dominant regular map between two irreducible algebraic sets. Then one can apply the theorem on the dimension of fibers from [33, Theorem 1.25] and claim that there exists a non-empty Zariski open subset Ω_1 of T_j such that

$$\forall \boldsymbol{z} \in \Omega_1,\ \dim\left(\pi_{\mathbf{X}}^{-1}(\boldsymbol{z}) \cap C_j\right) = \dim C_j - \dim T_j = i - \dim T_j. \tag{8}$$

Then it is enough to prove that $\dim(\pi_{\mathbf{X}}^{-1}(\boldsymbol{z}) \cap C_j) \geq 1$. Let $J' = \{1 \leq k \leq \ell \mid T_k = T_j\}$. Then it holds that

$$\Omega_2 = T_j - \bigcup_{k \notin J'} T_k$$

is a non-empty Zariski open subset of T_j. Besides, for all $\boldsymbol{z} \in \Omega_2$, $\pi_{\mathbf{X}}^{-1}(\boldsymbol{z}) \cap C_j = \pi_{\mathbf{X}}^{-1}(\boldsymbol{z}) \cap C$ which is the Zariski closure of $\pi_{\mathbf{X}}^{-1}(\boldsymbol{z}) \cap \mathscr{A} \cap \Phi_{\boldsymbol{\alpha}}^{-1}(\mathbf{0})$ if and only if $\boldsymbol{z} \in W_{\boldsymbol{\alpha}}^{\circ}$ (otherwise it is empty).

However, by definition, $C_j' = \mathscr{A} \cap \Phi_{\boldsymbol{\alpha}}^{-1}(\mathbf{0}) \cap C_j$ is a non-empty Zariski open subset of C_j, and then $\pi_{\mathbf{X}}(C_j')$ is a non-empty Zariski subset of T_j. Since it contains $\pi_{\mathbf{X}}(C_j')$, the set $\Omega_3 = W_{\boldsymbol{\alpha}}^{\circ} \cap T_j$ is a non-empty Zariski open subset of T_j as well.

Now, let $\Omega = \Omega_1 \cap \Omega_2 \cap \Omega_3$, it is a non-empty (Zariski open) subset of T_j, and let $\boldsymbol{z} \in \Omega$. Since $\boldsymbol{z}$ is in Ω_3, it is in $W_{\boldsymbol{\alpha}}^{\circ}$ by definition. Besides, $\boldsymbol{z} \in \Omega_2$, so that

$$\dim(\pi_{\mathbf{X}}^{-1}(\boldsymbol{z}) \cap C_j) = \dim\left(\pi_{\mathbf{X}}^{-1}(\boldsymbol{z}) \cap \mathscr{A} \cap \Phi_{\boldsymbol{\alpha}}^{-1}(\mathbf{0})\right).$$

Since $\boldsymbol{z} \in \Omega_1$, together with (8), one gets that

$$\forall \boldsymbol{z} \in \Omega, \quad \boldsymbol{z} \in W_{\boldsymbol{\alpha}}^{\circ} \quad \text{and} \quad \dim T_j = i - \dim\left(\pi_{\mathbf{X}}^{-1}(\boldsymbol{z}) \cap \mathscr{A} \cap \Phi_{\boldsymbol{\alpha}}^{-1}(\mathbf{0})\right), \tag{9}$$

Let $\boldsymbol{z} \in \Omega$, remark that

$$\pi_{\mathbf{X}}^{-1}(\boldsymbol{z}) \cap \mathscr{A} \cap \Phi_{\boldsymbol{\alpha}}^{-1}(\mathbf{0}) = \{\boldsymbol{z}\} \times \left(E_{\boldsymbol{z}} \cap \mathcal{O}(\vartheta_{e+1}, \dots, \vartheta_i)\right)$$

where $E_{\boldsymbol{z}}$ is a linear subspace of $\mathbf{C}^{c+i}$. Indeed, $E_{\boldsymbol{z}}$ is defined by homogeneous linear equations in the entries of $(\boldsymbol{\lambda}, \boldsymbol{\vartheta})$. Since $\boldsymbol{z} \in W_{\boldsymbol{\alpha}}^{\circ} \subset \pi_{\mathbf{X}}(\mathscr{A} \cap \Phi_{\boldsymbol{\alpha}}^{-1}(\mathbf{0}))$, there exists a non-zero $(\boldsymbol{\lambda}, \boldsymbol{\vartheta}) \in \mathbf{C}^{c+i}$ such that $(\boldsymbol{z}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}) \in \mathscr{A} \cap \Phi_{\boldsymbol{\alpha}}^{-1}(\mathbf{0})$. Then $E_{\boldsymbol{z}}$ contains a non-zero vector, so that $\dim E_{\boldsymbol{z}} \geq 1$. Finally, injecting this inequality in (9) leads to $\dim T_j \leq i - 1$ as required. $\square$

6.1.c Proof of Proposition 6.1.

We can now tackle the proof of the main proposition of this subsection. Recall that we have fixed three integers $(d, \mathfrak{r}, i)$ such that $2 \leq \mathfrak{r} \leq d+1 \leq n+1$ and $1 \leq i \leq \mathfrak{r}$. Moreover, we consider polynomials $\boldsymbol{h} = (h_1, \dots, h_c)$ in $\mathbf{C}[\mathbf{X}]$, where $c = n - d$. Finally, let $\phi = (\phi_1, \dots, \phi_i)$, such that

$$\phi_j(\mathbf{X}, a_j) = \theta_j(\mathbf{X}) + \sum_{k=1}^{n} a_{j,k} x_k + \xi_j(a_j) \in \mathbf{C}[\mathbf{X}, \mathbf{A}_{\leq j}],$$

for all $1 \leq j \leq i$. Let $\Omega_i^{\boldsymbol{h}}$ be the non-empty Zariski open subset of $\mathbf{C}^{in}$ defined by

$$\Omega_i^{\boldsymbol{h}} = \begin{cases} \mathscr{D}_i \cap \mathscr{E}_i' \cap \mathscr{E}_i'' & \text{if } i \leq (d+3)/2; \\ \mathscr{D}_i & \text{else,} \end{cases}$$

where $\mathscr{D}_i$, $\mathscr{E}_i'$ and $\mathscr{E}_i''$ are the non-empty Zariski open sets given respectively by Lemma 6.8, Corollaries 6.3 and 6.4. Note that the assumptions of Corollary 6.4 since $d \leq n - 1$.

Now let $\boldsymbol{\alpha} \in \Omega_i^{\boldsymbol{h}}$ and $\boldsymbol{\varphi} = (\phi_1(\mathbf{X}, \boldsymbol{\alpha}), \dots, \phi_i(\mathbf{X}, \boldsymbol{\alpha}))$. The first item of the proposition is a direct consequence definition of $\boldsymbol{V}_{\mathrm{reg}}^{\circ}(\boldsymbol{h})$. Besides, according to [30, Lemma A.2], the set $W_{\boldsymbol{\alpha}}^{\circ}$ defined in Lemma 6.7 is nothing but $W_{\boldsymbol{\varphi}}^{\circ}(i, \boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}))$, whose Zariski closure is $W_{\boldsymbol{\varphi}}^{\circ}(i, \boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}))$, by definition. Hence, since $\boldsymbol{\alpha} \in \mathscr{D}_i$, the second item is exactly the statement of Lemma 6.9.

Suppose now that $i \leq (d+3)/2$, so that $\boldsymbol{\alpha} \in \mathscr{E}_i' \cap \mathscr{E}_i''$. Hence, by Corollaries 6.3 and 6.4, for all $\boldsymbol{y} \in \boldsymbol{V}_{\mathrm{reg}}^{\circ}(\boldsymbol{h})$,

$$\operatorname{rank} \operatorname{Jac}_{\boldsymbol{y}}(\boldsymbol{\varphi}_i) = i \quad \text{and} \quad \operatorname{rank} \operatorname{Jac}_{\boldsymbol{y}}(\boldsymbol{h}, \boldsymbol{\varphi}_i) \geq c + i - 1.$$

Hence, there exists a $(c+i-1)$-minor m'' of $\operatorname{Jac}_{\boldsymbol{y}}(\boldsymbol{h}, \boldsymbol{\varphi}_i)$, containing the rows of $\operatorname{Jac}(\boldsymbol{\varphi}_i)$, that does not vanish at $\boldsymbol{y}$. This proves the third item.

In the remaining we proceed to prove the last two items. Let m' be a c-minor of $\operatorname{Jac}(\boldsymbol{h})$ and m'' be a $(c+i-1)$-minor of $\operatorname{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ containing the rows of $\operatorname{Jac}(\boldsymbol{\varphi}_i)$. Assume, without loss of generality, that m'' is not the zero polynomial. The next lemma establishes the second to last item of Proposition 6.1.

Lemma 6.10. *Let m' and m'' as above. The set $W_{\boldsymbol{\varphi}}^{\circ}(i, \boldsymbol{V}_{reg}^{\circ}(\boldsymbol{h}))$ is defined on $\mathcal{O}(m'm'')$ by the vanishing set of the polynomials $(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''))$. Equivalently,*

$$\mathcal{O}(m'm'') \ \cap \ W_{\boldsymbol{\varphi}}^{\circ}(i, \boldsymbol{V}_{reg}^{\circ}(\boldsymbol{h})) = \mathcal{O}(m'm'') \ \cap \ \boldsymbol{V}\big(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')\big).$$

Proof. Inside the Zariski open set $\mathcal{O}(m')$, the matrix $\operatorname{Jac}(\boldsymbol{h})$ has full rank, which implies by [30, Lemma A.2] that

$$\mathcal{O}(m') \cap W_{\boldsymbol{\varphi}}^{\circ}(i, \boldsymbol{V}_{\mathrm{reg}}^{\circ}(\boldsymbol{h})) = \mathcal{O}(m') \cap \Big\{\boldsymbol{y} \in \boldsymbol{V}(\boldsymbol{h}) \mid \operatorname{rank}(\operatorname{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i]) < c + i\Big\}.$$

Besides, by the exchange lemma of [1, Lemma 1], if m is a $(c+i)$-minor of $\operatorname{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$, then one can write

$$m''m = \sum_{j=1} \varepsilon_j m_j m_j'' \quad \text{where } \varepsilon_j = \pm 1 \quad \text{and} \quad N \in \{1, \dots, d-i+1\}$$

and where m_j'' (resp. m_j) is obtained by successively adding to m'' (resp. removing to m) the missing row and a missing column of $\operatorname{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ that are in m. Remark that, for such a m, all the m_j'''s are in $\mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')$, by definition.

Hence, for all $\boldsymbol{y} \in \boldsymbol{V}(\boldsymbol{h})$, if $m''(\boldsymbol{y}) \neq 0$, then all the $(c+i)$-minors of $\operatorname{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ vanish at $\boldsymbol{y}$ if and only if all the polynomials of $\mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')$ vanish at $\boldsymbol{y}$. In other words:

$$\mathcal{O}(m'm'') \cap W_{\boldsymbol{\varphi}}^{\circ}(i, \boldsymbol{V}_{\mathrm{reg}}^{\circ}(\boldsymbol{h})) = \mathcal{O}(m'm'') \cap \boldsymbol{V}\big(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')\big).$$

□

In order to prove the the last item of Proposition 6.1, we need introduce Lagrange systems for general polynomial applications. This generalizes, in some sense, the construction of [30, Subsection 5.1], also presented in Subsection 4.3.

Let $L_1, \dots, L_c$ and $T_1, \dots, T_i$ be new indeterminates, since $m'' \neq 0$, consider the ring of rational fractions $\mathbf{C}[\mathbf{X}, L_1, \dots, L_c, T_1, \dots, T_i]_{m''}$ of the form $f/(m'')^r$, for $f \in \mathbf{C}[\mathbf{X}, L_1, \dots, L_c, T_1, \dots, T_i]$ and $r \in \mathbb{N}$. This the localization ring at the multiplicative set $\{(m'')^r \mid r \in \mathbb{N}\}$.

Let $\mathcal{I}_W$ the ideal of $\mathbf{C}[\mathbf{X}, L_1, \dots, L_c, T_1, \dots, T_i]_{m''}$ generated by the entries of

$$\boldsymbol{h}, \quad [L_1, \dots, L_c, T_1, \dots, T_i] \cdot \begin{bmatrix} \operatorname{Jac}(\boldsymbol{h}) \\ \operatorname{Jac}(\boldsymbol{\varphi}_i) \end{bmatrix}.$$

The following lemma is an immediate generalization of [30, Proposition 5.2.].

Lemma 6.11. *Let $1 \leq \iota \leq c$ such that the index of the row of* $\mathrm{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$ *not in m'' has index ι. Then there exist $(\lambda_j)_{1\leq j\neq\iota\leq c}$ and $(\tau_j)_{1\leq j\leq i}$ in $\mathbf{C}[\mathbf{X}]_{m''}$ such that $\mathcal{I}_W$ is generated by the entries of*

$$\boldsymbol{h}, \quad L_\iota \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''), \quad (L_j - \lambda_j L_\iota)_{1\leq j\neq\iota\leq c}, \quad (T_j - \tau_j L_\iota)_{1\leq j\leq i}. \tag{10}$$

Proof. For the sake of simplicity, suppose that m'' is the lower-left minor of $\mathrm{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$, so that $\iota = 1$. Then $\mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')$ is the sequence of minors obtained by adding the first row and columns in the ones of index $c+i, \ldots, n$. We denote these minors by $M_1, \ldots, M_{n-c-i+1}$. Then, we write

$$\mathrm{Jac}(\boldsymbol{h}, \boldsymbol{\varphi}_i) = \begin{pmatrix} \boldsymbol{u}_{1,c+i-1} & \boldsymbol{w}_{1,n-c-i+1} \\ \boldsymbol{m}_{c+i-1,c+i-1} & \boldsymbol{v}_{c+i-1,n-c-i+1} \end{pmatrix}$$

such that $m'' = \det(\boldsymbol{m})$ and the indices are the dimensions of the submatrices. As m'' is not zero, it is a unit of $\mathbf{C}[\mathbf{X}, L_1, \ldots, L_c, T_1, \ldots, T_i]_{m''}$, so that $\boldsymbol{m}$ has an inverse with coefficients in the same ring, given by m''^{-1} and the cofactor matrix of $\boldsymbol{m}$. Hence $\mathcal{I}_W$ is generated by the entries of $\boldsymbol{h}$ and

$$\begin{aligned}
&[L_1, \ldots, L_c, T_1, \ldots, T_i] \cdot \mathrm{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i]) \cdot \begin{bmatrix} \boldsymbol{m}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} I_{c+i-1} & -\boldsymbol{v} \\ \boldsymbol{0} & 1 \end{bmatrix} \\
= \quad &[L_1, \ldots, L_c, T_1, \ldots, T_i] \cdot \begin{bmatrix} \boldsymbol{u}\boldsymbol{m}^{-1} & \boldsymbol{w} - \boldsymbol{u}\boldsymbol{m}^{-1}\boldsymbol{v} \\ I_{c+i-1} & \boldsymbol{0} \end{bmatrix},
\end{aligned}$$

where I_{c+i-1} is the identity matrix of size $c+i-1$. The first $c-1$ entries are the $L_j - [\boldsymbol{u}\boldsymbol{m}^{-1}]_j L_1$ for $1 < j \leq c$ and the i followings are the $T_j - [\boldsymbol{u}\boldsymbol{m}^{-1}]_j L_1$ for $1 \leq j \leq i$. Hence taking $(\boldsymbol{\lambda}, \boldsymbol{\tau}) = \boldsymbol{u}\boldsymbol{m}^{-1}$ gives the last terms in (10).

Finally, since $\boldsymbol{m}$ is invertible, we can compute the minors $M_1, \ldots, M_{n-c-i+1}$ of $\mathrm{Jac}(\boldsymbol{h}, \boldsymbol{\varphi}_i)$, using the block structure we described above (see e.g. [7, Proposition 2.8.3] and [34, Theorem 1]) to obtain that for all $1 \leq j \leq n-c-i+1$,

$$M_j = (-1)^{c+i-1} m'' [\boldsymbol{w} - \boldsymbol{u}\boldsymbol{m}^{-1}\boldsymbol{v}]_j.$$

Hence, the last $n-c-i+1$ entries are $L_1 M_1/m'', \ldots, L_1 M_{n-c-i+1}/m''$ up to sign, we are done. $\square$

The next lemma ends the proof of the last item Proposition 6.1, and then conclude the proof of the whole proposition.

Lemma 6.12. *The Jacobian matrix of the polynomials in $(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''))$ has full rank $n - (i-1)$ at every point of the set $\mathcal{O}(m'm'') \cap W^{\circ}_{\boldsymbol{\varphi}}(i, \boldsymbol{V}^{\circ}_{reg}(\boldsymbol{h}))$.*

Proof. Recall that $\boldsymbol{\varphi} = (\phi_1(\mathbf{X}, \boldsymbol{\alpha}), \ldots, \phi_i(\mathbf{X}, \alpha))$, where $\boldsymbol{\alpha} \in \Omega_i^{\boldsymbol{h}}$. Then, remark that

$$\left(\boldsymbol{h}(\mathbf{X}), \quad [L_1, \ldots, L_c, T_1, \ldots, T_i] \cdot \begin{bmatrix} \mathrm{Jac}_{\mathbf{X}}(\boldsymbol{h}) \\ \mathrm{Jac}_{\mathbf{X}}(\boldsymbol{\varphi}_i) \end{bmatrix}\right) = \Phi_{\boldsymbol{\alpha}}\Big(\mathbf{X}, L_1, \ldots, L_c, T_1, \ldots, T_i\Big),$$

where $\Phi_{\boldsymbol{\alpha}}$ is the polynomial map considered in Lemma 6.8. Let $(\lambda_j)_{1\leq j\neq\iota\leq c}$ and $(\tau_j)_{1\leq j\leq i}$ in $\mathbf{C}[\mathbf{X}]_{m''}$ given by Lemma 6.11.

Now fix $\boldsymbol{y} \in \mathcal{O}(m'm'') \cap W_{\boldsymbol{\varphi}}(i, \boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h}))$, and let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_j)_{1\leq j\leq c}$ and $\boldsymbol{\vartheta} = (\boldsymbol{\vartheta}_j)_{1\leq j\leq i}$ where

$$\begin{aligned}
\boldsymbol{\lambda}_\iota = 1 \quad \text{and} \quad &\boldsymbol{\lambda}_j = \lambda_j(\boldsymbol{y}) \text{ for all } 1 \leq j \neq \iota \leq c, \\
&\boldsymbol{\vartheta}_j = \tau_j(\boldsymbol{y}) \text{ for all } 1 \leq j \leq i.
\end{aligned}$$

These are well defined since $m''(\boldsymbol{y}) \neq 0$. Since $\boldsymbol{h}$ and $\mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')$ vanish at $\boldsymbol{y}$, by Lemma 6.10, all the polynomials in (10) vanish at $(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta})$. Moreover, according to Lemma 6.11 and the above remark, the polynomials in (10) and the entries of $\Phi_{\boldsymbol{\alpha}}(\mathbf{X}, L_1, \ldots, L_c, T_1, \ldots, T_i)$ generates the same ideal $\mathcal{I}_W$ in $\mathbf{C}[\mathbf{X}]_{m''}$. Hence, since $m''(\boldsymbol{y}) \neq 0$, the entries of $\Phi_{\boldsymbol{\alpha}}$ vanish at $(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta})$ as well, that is $\Phi_{\boldsymbol{\alpha}}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}) = \mathbf{0}$

Besides, since $\boldsymbol{y} \in \mathcal{O}(m')$, $\mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{h})$ has full rank. Then, $\boldsymbol{\vartheta}$ cannot be zero, since $\boldsymbol{\lambda} \neq \mathbf{0}$ $\mathrm{Jac}(\boldsymbol{h})$ has a trivial left-kernel. Hence, according to the notation of Lemma 6.8, $(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}) \in \mathscr{A} \cap \Phi_{\boldsymbol{\alpha}}^{-1}(\mathbf{0})$. Therefore, by Lemma 6.8, $\mathrm{Jac}\,\Phi_{\boldsymbol{\alpha}}$ has full rank $n + c$ at $(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\vartheta})$, as $\boldsymbol{\alpha} \in \mathscr{D}_i \subset \Omega_i^{\boldsymbol{h}}$.

Finally, remark that the sequence of polynomials in (10) has length $n + c$. Hence, since the latters generate the same ideal than the entries of the entries of $\Phi_{\boldsymbol{\alpha}}(\mathbf{X}, L_1, \ldots, L_c, T_1, \ldots, T_i)$, their Jacobian matrix has full rank $n + c$ at this point as well. Computing this Jacobian matrix the latter rank statement amounts to the Jacobian matrix of

$$\Big(\boldsymbol{h},\ \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')\Big)$$

having full rank $n - (i - 1)$ at $\boldsymbol{y}$. $\square$

6.2 Proof of Proposition 2.12

Let $V, S \subset \mathbf{C}^n$ be be two algebraic sets with V d-equidimensional and S finite, and let $\boldsymbol{\chi} = (\chi_j)_{1 \leq j \leq s}$ be an atlas of (V, S) with $\chi_j = (m_j, \boldsymbol{h}_j)$ for $1 \leq j \leq s$. According to [30, Lemma A.12], all the $\boldsymbol{h}_j$'s have same cardinality $c = n - d$.

Besides, let $2 \leq \mathfrak{r} \leq d + 1$ and the sequences $\boldsymbol{\theta} = (\theta_1, \ldots, \xi_{\mathfrak{r}})$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{\mathfrak{r}})$ in $\mathbf{C}[\mathbf{X}]$. For $1 \leq j \leq \mathfrak{r}$, let $\boldsymbol{\alpha}_j = (\alpha_{j,1}, \ldots, \alpha_{j,n}) \in \mathbf{C}^n$ and

$$\varphi_j(\mathbf{X}, \boldsymbol{\alpha}_j) = \theta_j(\mathbf{X}) + \sum_{k=1}^{n} \alpha_{j,k} x_k + \xi_j(\boldsymbol{\alpha}_j) \in \mathbf{C}[\mathbf{X}].$$

Then, for $1 \leq i \leq \mathfrak{r}$, we can apply Proposition 6.1 to the sequences $\boldsymbol{h}_j$, $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$, there exist a non-empty Zariski open subset $\Omega(\boldsymbol{h}_j, i)$ of $\mathbf{C}^{in}$ such that for all $\boldsymbol{\alpha} \in \Omega(\boldsymbol{h}_j, i)$, the sequence $\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}), \ldots, \varphi_i(\mathbf{X}, \boldsymbol{\alpha}))$ satisfies the statements of Proposition 6.1. Then we define the following non-empty Zariski open subset of $\mathbf{C}^{\mathfrak{r}n}$,

$$\Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, S, \boldsymbol{\theta}, \boldsymbol{\xi}, \mathfrak{r}) = \bigcap_{1 \leq i \leq \mathfrak{r}} \ \bigcap_{1 \leq j \leq s} \Omega(\boldsymbol{h}_j, i) \times \mathbf{C}^{(\mathfrak{r}-i)n}.$$

Fix now $\boldsymbol{\alpha} \in \Omega_{\mathsf{W}}(\boldsymbol{\chi}, V, S, \boldsymbol{\theta}, \boldsymbol{\xi}, \mathfrak{r})$ and $\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}), \ldots, \varphi_{\mathfrak{r}}(\mathbf{X}, \boldsymbol{\alpha}))$. From now on, fix also $1 \leq i \leq \mathfrak{r}$ and suppose that $W_{\boldsymbol{\varphi}}(i, V)$ is not empty. In the following, and for conciseness, we might identify $\Omega(\boldsymbol{h}_j, i)$ to $\Omega(\boldsymbol{h}_j, i) \times \mathbf{C}^{(\mathfrak{r}-i)n}$. in a straightforward way. We start with the first item statement of Proposition 2.12. Again, it is proved through the properties of atlases, but when i is restricted to some values.

Lemma 6.13. *The algebraic set $W_{\boldsymbol{\varphi}}(i, V)$ is equidimensional of dimension $i - 1$.*

Proof. By Lemma 2.6, for all $1 \leq j \leq s$, as χ_j is a chart of (V, S) then,

$$\mathcal{O}(m_j) \cap W_{\boldsymbol{\varphi}}(i, V) - S \ = \ \mathcal{O}(m_j) \cap W^{\circ}_{\boldsymbol{\varphi}}(i, \boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j)) - S.$$

Let $\boldsymbol{y} \in W_{\boldsymbol{\varphi}}(i, V) - S$. Since $\boldsymbol{y} \in V$, by property A_3 of the atlas $\boldsymbol{\chi}$, there exists $j \in \{1, \ldots, s\}$ such that $\boldsymbol{y} \in \mathcal{O}(m_j)$. Hence, by the above equality, in $\mathcal{O}(m_j) - S$, the irreducible component of $W_{\boldsymbol{\varphi}}(i, V)$ containing $\boldsymbol{y}$ coincides with the one of $W_{\boldsymbol{\varphi}}(i, \boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}_j))$ containing $\boldsymbol{y}$. Since these irreducible components are equal over a non-empty Zariski open set, they have the same dimension (see e.g. [15, Proposition 10.(1)]). By the second item of Proposition 6.1, since $\boldsymbol{\alpha} \in \Omega(\boldsymbol{h}_j, i)$, this dimension is less than $i-1$.

We just showed that the Zariski closure of $W_{\boldsymbol{\varphi}}(i, V) - S$ has dimension less than $i-1$. If $i = 1$, since S is finite this means that $W_{\boldsymbol{\varphi}}(i, V)$ is finite as well and we are done. If $i \geq 2$, then by Lemma 5.1, the irreducible components of $W_{\boldsymbol{\varphi}}(i, V)$ have dimension at least $i-1 \geq 1$ so that the Zariski closure of $W_{\boldsymbol{\varphi}}(i, V) - S$ is $W_{\boldsymbol{\varphi}}(i, V)$. Hence the irreducible components of $W_{\boldsymbol{\varphi}}(i, V)$ have dimension exactly $i-1$ □

We now prove a strict generalization of [30, Lemma B.12.] which gives the key arguments for the proof of the second item statement of Proposition 2.12.

Lemma 6.14. *Let $\chi = (m, \boldsymbol{h})$ be a chart of (V, S). Then for any c-minor m' of $\mathrm{Jac}(\boldsymbol{h})$ and any $(c+i-1)$-minor m'' of $\mathrm{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$, containing the rows of $\mathrm{Jac}(\boldsymbol{\varphi}_i)$, the following holds.*

1. *The sets $\mathcal{O}(mm'm'') \cap W_{\boldsymbol{\varphi}}(i, V) - S$ and $\mathcal{O}(mm'm'') \cap \boldsymbol{V}(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m'')) - S$ coincides;*
2. *if they are not empty, then $W_{\mathrm{chart}}(\chi, m', m'')$ is a chart of $(W_{\boldsymbol{\varphi}}(i, V), S)$.*

Moreover, if $i \leq (d+3)/2$ then the following holds.

3. *The sets $\mathcal{O}(mm'm'') - S$, for all m', m'' as above, cover $\mathcal{O}(m) \cap V - S$;*
4. *the sets $\mathcal{O}(mm'm'') - S$, for all m', m'' as above, $\mathcal{O}(m) \cap W_{\boldsymbol{\varphi}}(i, V) - S$.*

Proof. By Lemma 2.6, since χ is a chart of (V, S),

$$\mathcal{O}(m) \cap W_{\boldsymbol{\varphi}}(i, V) - S = \mathcal{O}(m) \cap W^{\circ}_{\boldsymbol{\varphi}}(i, \boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h})) - S.$$

Besides, by the second to last item of Proposition 6.1, $W^{\circ}_{\boldsymbol{\varphi}}(i, \boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}))$ is defined in $\mathcal{O}(m'm'')$ by the vanishing of the polynomials $(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''))$, so that

$$\mathcal{O}(mm'm'') \cap W_{\boldsymbol{\varphi}}(i, V) - S = \mathcal{O}(mm'm'') \cap \boldsymbol{V}(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}_j, i, m'')) - S. \tag{11}$$

The first item is proved. Suppose now that the former sets are not-empty, we proceed to prove that $W_{\mathrm{chart}}(\chi, m', m'')$ is a chart of $(W_{\boldsymbol{\varphi}}(i, V), S)$. Property C_1 holds by assumption, while property C_2 of $W_{\mathrm{chart}}(\chi, m', m'')$ is exactly equation (11). Besides, since $(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}_j, i, m''))$ has length $n - i - 1 \leq n$, then C_3 holds as well. Finally, by the last item of Proposition 6.1, $\mathrm{Jac}(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''))$ has full rank on

$$\mathcal{O}(m'm'') \cap W^{\circ}_{\boldsymbol{\varphi}}(\boldsymbol{V}_{\mathrm{reg}}(\boldsymbol{h}), V).$$

Then, by (11), $\mathrm{Jac}(\boldsymbol{h}, \mathcal{H}_{\boldsymbol{\varphi}}(\boldsymbol{h}, i, m''))$ has full rank on $\mathcal{O}(mm'm'') \cap W_{\boldsymbol{\varphi}}(i, V) - S$. This proves that $W_{\mathrm{chart}}(\chi, m', m'')$ satisfies the last property C_4 of charts and the second statement of the lemma is proved.

Suppose now that $i \leq (d+3)/2$ and let $\boldsymbol{y} \in \mathcal{O}(m) \cap V - S$. Then, by property C_4 of χ, $\mathrm{Jac}(\boldsymbol{h})$ has full rank in $\boldsymbol{y}$, so that $\boldsymbol{y} \in \boldsymbol{V}^{\circ}_{\mathrm{reg}}(\boldsymbol{h})$. Therefore, by the first and third item of Proposition 6.1, there exists a c-minor m' of $\mathrm{Jac}(\boldsymbol{h})$ and a $(c+i-1)$-minor m'' of $\mathrm{Jac}([\boldsymbol{h}, \boldsymbol{\varphi}_i])$, containing the rows of $\mathrm{Jac}(\boldsymbol{\varphi}_i)$, such that $(m'm'')(\boldsymbol{y}) \neq 0$. Hence $\boldsymbol{y} \in \mathcal{O}(mm'm'') - S$ and the third item of the lemma is proved.

Finally, if $\boldsymbol{y} \in \mathcal{O}(m) \cap W_{\boldsymbol{\varphi}}(i, V) - S$, then one still has $\boldsymbol{y} \in \mathcal{O}(mm'm'') - S$, as $W_{\boldsymbol{\varphi}}(i, V) \subset V$. This proves the last item. □

We can now prove the second statement of Proposition 2.12. With the above lemmas, it is mainly a matter of verification. Suppose that $2 \leq i \leq (d+3)/2$. We prove that $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, i)$ is an atlas of $(W_{\boldsymbol{\varphi}}(i, V), S)$. In the following, for $1 \leq j \leq s$, we refer to m'_j and m''_j as respectively a c-minor of $\mathrm{Jac}(\boldsymbol{h}_j)$ and a $(c+i-1)$-minor of $\mathrm{Jac}([\boldsymbol{h}_j, \boldsymbol{\varphi}_i])$, containing the rows of $\mathrm{Jac}(\boldsymbol{\varphi}_i)$.

A_1 : Since, by Lemma 6.13, $W_{\boldsymbol{\varphi}}(i, V)$ has dimension at least 1, it is not contained in S. In particular, there exists $1 \leq j \leq s$ such that $\mathcal{O}(m_j) \cap W_{\boldsymbol{\varphi}}(i, V) - S$ is not empty. Hence, by the third item of Lemma 6.14, there exist minors m'_j and m''_j such that $\mathcal{O}(m_j m'_j m''_j) \cap W_{\boldsymbol{\varphi}}(i, V) - S$ is not empty.

A_2 : For m_j,m'_j and m''_j as in the previous item, since $\mathcal{O}(m_j m'_j m''_j) \cap W_{\boldsymbol{\varphi}}(i, V) - S$ is not empty, then the second item of Lemma 6.14 shows that the sequence $W_{\mathrm{chart}}(\chi_j, m'_j, m''_j)$ is a chart of $(W_{\boldsymbol{\varphi}}(i, V), S)$.

A_3 : Let $\boldsymbol{y} \in W_{\boldsymbol{\varphi}}(i, V) - S$, by property A_3 of $\boldsymbol{\chi}$ there exists $1 \leq j \leq s$ such that $\boldsymbol{y} \in \mathcal{O}(m_j)$. Then, by the third item of Lemma 6.14, there exist m'_j and m''_j as in the previous points such that $\boldsymbol{y} \in \mathcal{O}(m_j m'_j m''_j)$. In particular $\mathcal{O}(m_j m'_j m''_j) \cap W_{\boldsymbol{\varphi}}(i, V) - S$ is not empty.

Hence $W_{\mathrm{atlas}}(\boldsymbol{\chi}, V, S, \boldsymbol{\varphi}, i)$ is an atlas of $(W_{\boldsymbol{\varphi}}(i, V), S)$, and since we proved that $W_{\boldsymbol{\varphi}}(i, V)$ is equidimensional, then by [30, Lemma A.12] $\mathrm{sing}(W_{\boldsymbol{\varphi}}(i, V)) \subset S$.

7 Proof of Proposition 2.15: atlases for fibers

This section is devoted to the proof of Proposition 2.15. We recall its statement below.

Proposition (2.15)**.** *Let $V, S \subset \mathbf{C}^n$ be two algebraic sets with V d-equidimensional and S finite. Let $\boldsymbol{\chi}$ be an atlas of (V, S). Let $2 \leq \mathfrak{r} \leq d+1$ and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{\mathfrak{r}}) \subset \mathbf{C}[\mathbf{X}]$. For $2 \leq j \leq d$, let $\boldsymbol{\alpha}_j = (\alpha_{j,1}, \ldots, \alpha_{j,n}) \in \mathbf{C}^n$ and*

$$\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1) = \theta(\mathbf{X}) + \sum_{k=1}^{n} \alpha_{1,k} x_k \quad \text{and} \quad \varphi_j(\mathbf{X}, \boldsymbol{\alpha}_j) = \sum_{k=1}^{n} \alpha_{j,k} x_k$$

where $\theta \in \mathbf{C}[\mathbf{X}]$.

There exists a non-empty Zariski open subset $\Omega_{\mathsf{F}}(\boldsymbol{\chi}, V, S, \theta, \mathfrak{r}) \subset \mathbf{C}^{\mathfrak{r}n}$ such that for every $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{\mathfrak{r}}) \in \Omega_{\mathsf{F}}(\boldsymbol{\chi}, V, S, \theta, \mathfrak{r})$ and writing

$$\boldsymbol{\varphi} = (\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1), \ldots, \varphi_{\mathfrak{r}}(\mathbf{X}, \boldsymbol{\alpha}_{\mathfrak{r}})),$$

the following holds. Let $0 \leq e \leq d$, $Q \in \mathbf{C}^e$ a finite subset and F_Q and S_Q be as in Definition 2.14. Then either F_Q is empty or

1. *S_Q is finite;*
2. *V_Q is an equidimensional algebraic set of dimension $d - e$;*
3. *$F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\varphi})$ is an atlas of (F_Q, S_Q) and $\mathrm{sing}(F_Q) \subset S_Q$.*

Let V, S and $\boldsymbol{\chi} = (\chi_j)_{1 \leq j \leq s}$ be as above, with $\chi_j = (m_j, \boldsymbol{h}_j)$ for $1 \leq j \leq s$. Consider and integer $2 \leq \mathfrak{r} \leq d+1$, we show in the following that it suffices to take $\Omega_{\mathsf{F}}(\boldsymbol{\chi}, V, Q, S, \theta, \mathfrak{r})$ as the non-empty Zariski open subset $\Omega_{\mathsf{I}}(V, \theta, \mathfrak{r})$ of $\mathbf{C}^{\mathfrak{r}n}$ obtained by the application of Proposition 2.3 to V, θ and $\mathfrak{r}$.

Let $\boldsymbol{\alpha} \in \Omega_{\mathsf{F}}(\boldsymbol{\chi}, V, S, \theta, \mathfrak{r})$ and $\varphi = (\boldsymbol{\varphi}_1(\mathbf{X}, \boldsymbol{\alpha}), \ldots, \boldsymbol{\varphi}_{\mathfrak{r}}(\mathbf{X}, \boldsymbol{\alpha}))$ where for $2 \leq j \leq \mathfrak{r}$,

$$\varphi_1(\mathbf{X}, \boldsymbol{\alpha}_1) = \theta(\mathbf{X}) + \sum_{k=1}^{n} \alpha_{1,k} x_k \quad \text{and} \quad \varphi_j(\mathbf{X}, \boldsymbol{\alpha}_j) = \sum_{k=1}^{n} \alpha_{j,k} x_k$$

For $1 \leq e \leq \mathfrak{r} - 1$, let $Q \subset \mathbf{C}^e$ be a finite set and F_Q, S_Q as in Definition 2.14. Suppose also that F_Q is not empty. We start with the following lemma, proving local statements on the fibers. It is a direct generalization of [30, Lemma C.1].

Lemma 7.1. *Let $1 \leq j \leq s$ and $m = m_j$, $\boldsymbol{h} = \boldsymbol{h}_j$ and $\chi = (m, \boldsymbol{h})$. Then either $\mathcal{O}(m) \cap F_Q$ is empty or χ is a chart of $(F_Q, Q, S_Q, \boldsymbol{\varphi})$, and S_Q is finite.*

Proof. Remark first that since $\boldsymbol{\alpha} \in \Omega(V, \theta)$, then by Proposition 2.3, the set

$$S_Q = \big(S \cup W_{\boldsymbol{\varphi}}(e, V)\big) \cap \boldsymbol{\varphi}_e^{-1}(Q)$$

is finite, since S and Q are. Assume now that $\mathcal{O}(m) \cap F_Q$ is not empty, and let us prove that χ is a chart of $(F_Q, Q, S_Q, \boldsymbol{\varphi})$.

- C_1 : This holds by assumption.
- C_2 : By property C_2 of χ, the sets F_Q and $\boldsymbol{V}(\boldsymbol{h})_{|\boldsymbol{\varphi}_e \in Q}$ coincide in $\mathcal{O}(m) - S$. But since $S \subset S_Q$ in $\boldsymbol{\varphi}_e^{-1}(Q)$ then these sets coincide in $\mathcal{O}(m) - S_Q$ as well.
- C_3 : Since V is d-equidimensional, then by [30, Lemma A.12], $c = n - d$. Hence, since $e \leq \mathfrak{r} - 1 \leq d$, the inequality $e + c \leq n$ holds.
- C_4 : Finally, let $\boldsymbol{y} \in \mathcal{O}(m) \cap F_Q - S_Q$. Since $\boldsymbol{y} \notin S_Q$ then $\boldsymbol{y} \notin W_{\boldsymbol{\varphi}}(e, V) \cap \boldsymbol{\varphi}_e^{-1}(Q)$, but since $\boldsymbol{y} \in \boldsymbol{\varphi}_e^{-1}(Q)$ then actually $\boldsymbol{y} \notin W_{\boldsymbol{\varphi}}(e, V)$. Hence since $\boldsymbol{y} \in \mathcal{O}(m)$, then by Lemma 2.5, $\mathrm{Jac}_{\boldsymbol{y}}(\boldsymbol{h}, \boldsymbol{\varphi}_e)$ has full rank $c + e$.

All the properties of charts being satisfied, we are done. □

We now proceed to prove Proposition 2.15. The first statement is given by Lemma 7.1. If $e = d$, then the second statement is satisfied by the last item Proposition 2.3, since $K_{\boldsymbol{\varphi}}(d+1, V) = V$. Assume now that $e < d$. By Krull's principal ideal Theorem [14, Theorem B] or equivalently the theorem on the dimension of fibers [33, Theorem 1.25], all irreducible components of F_Q have dimension at least $d - e > 0$.

We now prove the last statement that is that $F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\varphi})$ is an atlas of $(F_Q, Q, S_Q, \boldsymbol{\varphi})$:

- A_1 : Since F_Q has positive dimension and S_Q is finite, then $F_Q - S_Q$ is not empty. Since $F_Q \subset V$, then by property A_3 of $\boldsymbol{\chi}$, there exists $1 \leq j \leq s$ such that $\mathcal{O}(m_j) \cap F_Q - S_Q$ is not empty.
- A_2 : Let $1 \leq j \leq s$ such that $\mathcal{O}(m_j) \cap F_Q - S_Q$ is not empty, then by Lemma 7.1, χ_j is a chart of $(F_Q, Q, S_Q, \boldsymbol{\varphi})$. Since the elements of $F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\varphi})$ are exactly such χ_j, we are done.
- A_3 : Finally let $\boldsymbol{y} \in F_Q - S_Q$, since $\boldsymbol{y} \in \boldsymbol{\varphi}_e^{-1}(Q)$ then $\boldsymbol{y} \notin S$. Since $F_Q \subset V$, then by property A_3 of $\boldsymbol{\chi}$, there exists $1 \leq j \leq s$ such that $\boldsymbol{y} \in \mathcal{O}(m_j)$. In particular, $\mathcal{O}(m_j) \cap F_Q - S_Q$ is non-empty, so that $\chi_j \in F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\varphi})$.

Hence $F_{\mathrm{atlas}}(\boldsymbol{\chi}, V, Q, S, \boldsymbol{\varphi})$ is an atlas of $(F_Q, Q, S_Q, \boldsymbol{\varphi})$. In particular, since V is d-equidimensional, all the $\boldsymbol{h}_j$'s have same cardinality $c = n - d$ by [30, Lemma A.12]. Hence by [30, Lemma A.11], $F_Q - S_Q$ is a non-singular $(d - e)$-equidimensional locally closed set. Since F_Q has positive dimension and S_Q is finite, we deduce that F_Q is the Zariski closure of $F_Q - S_Q$ and then, is a $(d - e)$-equidimensional algebraic set, smooth outside S_Q. This concludes the proof of Proposition 2.15.

Acknowledgments

The first author is supported by the FWO grant G0F5921N. The second author is supported by ANR grants ANR-18-CE33-0011 Sesame, and ANR-19-CE40-0018 De Rerum Natura, the joint ANR-FWF ANR-19-CE48-0015 ECARP project, the joint French–Austrian project Eagles (ANR-22-CE91-0007 & FWF I6130-N) and the Grant FA8665-20-1-7029 of the EOARD-AFOSR. The third author was supported by an NSERC Discovery Grant.

References

[1] B. Bank, M. Giusti, J. Heintz, and G. M. Mbakop. Polar varieties and efficient real elimination. *Mathematische Zeitschrift*, 238(1):115–144, 2001.

[2] S. Basu, R. Pollack, and M.-F. Roy. Computing roadmaps of semi-algebraic sets on a variety. *Journal of the American Mathematical Society*, 13(1):55–82, 2000.

[3] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in Real Algebraic Geometry*. 10. Springer, Berlin, Heidelberg, 2 edition, 2006.

[4] S. Basu and M.-F. Roy. Divide and conquer roadmap for algebraic sets. *Discrete & Computational Geometry*, 52(2):278–343, 2014.

[5] S. Basu, M.-F. Roy, M. Safey El Din, and É. Schost. A baby step–giant step roadmap algorithm for general algebraic sets. *Foundations of Computational Mathematics*, 14(6):1117–1172, 2014.

[6] W. Baur and V. Strassen. The complexity of partial derivatives. *Theoretical computer science*, 22(3):317–330, 1983.

[7] D. S. Bernstein. *Matrix Mathematics*. Princeton University Press, Princeton, 2009.

[8] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*, volume 36. Springer Science & Business Media, 2013.

[9] J. Canny. *The complexity of robot motion planning*. MIT press, 1988.

[10] J. Canny. Computing roadmaps of general semi-algebraic sets. *The Computer Journal*, 36(5):504–514, 1993.

[11] J. Capco, M. Safey El Din, and J. Schicho. Robots, computer algebra and eight connected components. In *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*, pages 62–69, 2020.

[12] J. Capco, M. Safey El Din, and J. Schicho. Positive dimensional parametric polynomial systems, connectivity queries and applications in robotics. *Journal of Symbolic Computation*, 115:320–345, 2023.

[13] J. A. Eagon and D. G. Northcott. Ideals defined by matrices and a certain complex associated with them. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 269(1337):188–204, 1962.

[14] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150. Springer-Verlag, New York, 1995.

[15] W. Fulton. *Algebraic curves, An Introduction to Algebraic Geometry*, volume 54. 2008.

[16] M. Giusti, J. Heintz, J. E. Morais, J. Morgenstem, and L. M. Pardo. Straight-line programs in geometric elimination theory. *Journal of pure and applied algebra*, 124(1-3):101–146, 1998.

[17] M. Giusti, J. Heintz, J. E. Morais, and L. M. Pardo. When polynomial equation systems can be "solved" fast? In *International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes*, pages 205–231. Springer, 1995.

[18] M. Giusti, J. Heintz, J. E. Morais, and L. M. Pardo. Le rôle des structures de données dans les problèmes d'élimination. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 325(11):1223–1228, 1997.

[19] M. Giusti, G. Lecerf, and B. Salvy. A gröbner free alternative for polynomial system solving. *Journal of complexity*, 17(1):154–211, 2001.

[20] L. Gournay and J.-J. Risler. Construction of roadmaps in semi-algebraic sets. *Applicable Algebra in Engineering, Communication and Computing*, 4(4):239–252, 1993.

[21] D. Grigoryev and N. Vorobjov. Counting connected components of a semialgebraic set in subexponential time. *Computational Complexity*, 2:133–186, 06 1992.

[22] J. Heintz, M.-F. Roy, and P. Solernó. Single exponential path finding in semi-algebraic sets, part ii: The general case. In *Algebraic geometry and its applications*, pages 449–465. Springer, 1994.

[23] T. Krick. Straight-line programs in polynomial equation solving. *Foundations of computational mathematics: Minneapolis*, 312:96–136, 2002.

[24] E. Kunz. *Introduction to Commutative Algebra and Algebraic Geometry.* Friedr. Vieweg & Sohn, 1985.

[25] G. Lecerf. Computing an equidimensional decomposition of an algebraic variety by means of geometric resolutions. In *Proceedings of the 2000 international symposium on Symbolic and algebraic computation*, pages 209–216, 2000.

[26] G. Lecerf. Computing the equidimensional decomposition of an algebraic closed set by means of lifting fibers. *Journal of Complexity*, 19(4):564–596, 2003.

[27] R. Prébet, M. Safey El Din, and Éric Schost. Computing roadmaps in unbounded smooth real algebraic sets i: Connectivity results. *Journal of Symbolic Computation*, 120:102234, 2024.

[28] M. Safey El Din and É. Schost. Polar varieties and computation of one point in each connected component of a smooth real algebraic set. In *Proceedings of the 2003 international symposium on Symbolic and algebraic computation*, pages 224–231, 2003.

[29] M. Safey El Din and É. Schost. A baby steps/giant steps probabilistic algorithm for computing roadmaps in smooth bounded real hypersurface. *Discrete & Computational Geometry*, 45(1):181–220, 2011.

[30] M. Safey El Din and É. Schost. A nearly optimal algorithm for deciding connectivity queries in smooth and bounded real algebraic sets. *Journal of the ACM (JACM)*, 63(6):1–37, 2017.

[31] M. Safey El Din and Éric Schost. Bit complexity for multi-homogeneous polynomial system solving—application to polynomial minimization. *Journal of Symbolic Computation*, 87:176–206, 2018.

[32] J. T. Schwartz and M. Sharir. On the "piano movers" problem. ii. general techniques for computing topological properties of real algebraic manifolds. *Advances in applied Mathematics*, 4(3):298–351, 1983.

[33] I. R. Shafarevich. *Basic algebraic geometry*, volume 1. Springer, 2013.

[34] J. R. Silvester. Determinants of block matrices. *The Mathematical Gazette*, 84(501):460–467, 2000.

[35] N. Vorobjov and D. Y. Grigoriev. Determination of the number of connected components of a semi-algebraic set in subexponential time. *Dokl. Akad. Nauk SSSR*, 314(5), 1990.