CeSNAC: Certified Symbolic-Numeric Algorithms for Connectivity on semi-algebraic sets

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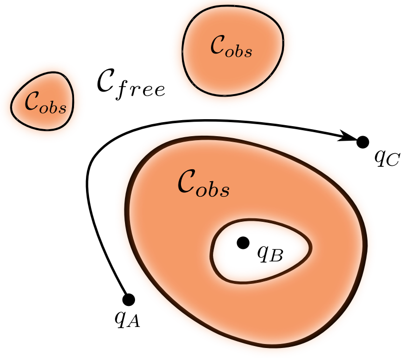
- **when?** spring 2026 (4 – 6 months)



Context and Motivation

Semi-algebraic sets are subsets of $\mathbb{R}^n$ defined by finitely many polynomial equalities and inequalities. They form a rich language for problems including robotics [Can88, CSS23], optimization [FRPM06], program verification [GHMM23] and dynamical systems [WR13].

A fundamental algorithmic question across these domains concerns *connectivity queries*: given two points $p, q \in S$, are they in the same connected component of a semi-algebraic set S? This is a *decision problem*, whose answer must be exact. Any numerical uncertainty or rounding error may lead to an incorrect conclusion [LaV06]. On the other hand, purely symbolic algorithms, though exact, often suffer from prohibitive computational costs and memory requirements [BPR00].

This motivates the development of **certified symbolic-numeric algorithms**: hybrid methods combining the efficiency of numerical computation with the mathematical guarantees of symbolic approaches. Here, *certification* does not refer to formal verification, but to *computer-aided proofs* that establish mathematically valid bounds on numerical errors. Such bounds ensure that the computed result corresponds to an actual mathematical solution within a rigorously proven neighborhood. By doing so, one retains both computational efficiency and mathematical reliability, providing guarantees comparable to purely symbolic methods.

State of the art. The approach originates from [Hon10], which introduced a geometric method for deciding connectivity in sets of the form $\{f \neq 0\}$, where f is a polynomial, using gradient flows of suitably chosen smooth "routing" functions r. The correctness of this principle was later formalized in [HRSS20], which proved that, under appropriate regularity assumptions, the connectivity of such sets can be decided by following trajectories of ∇r between critical points. However, that work assumes the existence of a subroutine capable of computing and *certifying* these trajectories. This key component is still left to be developed.

The most recent contribution [CHHS25] extends the framework to smooth, locally closed algebraic sets of the form $\{g = 0, f \neq 0\}$ and proposes a prototype implementation based on numerical algebraic geometry. While this implementation demonstrates the feasibility of the method, it remains non-certified: numerical path tracking is used heuristically, without guarantees that the computed trajectories correspond to true gradient flows. The remaining step, and the focus of this internship, is therefore to design a **certified numerical subroutine** for gradient-path tracking, completing the theoretical and practical foundation of the algorithm.

Scientific Objective

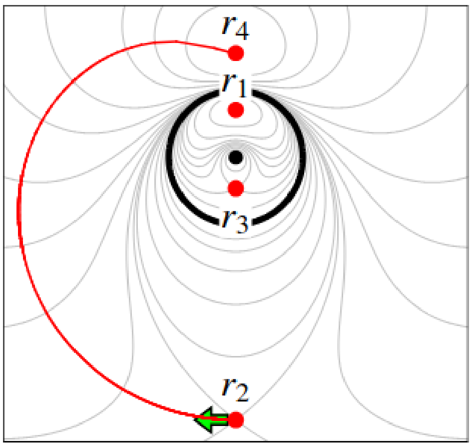
This internship focuses on *certifying a symbolic-numeric algorithm for connectivity queries on open semi-algebraic sets*, following the framework proposed in [Hon10, HRSS20, CHHS25]. More precisely:

- construct a *routing function* g whose gradient flow ($\dot{x} = \nabla r(x)$) connects points within the same connected component of $\{f \neq 0\}$;
- develop a numerical subroutine that tracks these trajectories to determine a candidate destination reached by the flow;
- use a posteriori validation methods to *certify* these trajectories, i.e., to prove that the numerically computed trajectory corresponds to an actual solution of the gradient flow within a guaranteed bound.

Certification in this context thus provides a mathematically rigorous control on numerical errors, bridging symbolic and numerical computation. A long-term perspective would be to formalize these computer-aided proofs within a theorem prover, turning them into full formal verifications. But this is a priori not the objective of the internship.

The key subproblem, denoted Destination, consists in reliably determining which critical point is reached by integrating a gradient flow starting from a given point and direction. This is numerically challenging because trajectories can lie on manifolds of measure 0, requiring tight control on approximation errors.

Example. Consider the routing function: $r(x, y) = \dfrac{(-2x^2 + x^4 - 2y^2 + 2x^2y^2 + y^2)^2}{(x^2 + (y-1)^2 + 1)^5}$.



The black curves represent the boundary of the region $\{r \neq 0\}$. The gray lines show the level sets of r, along which the gradient field ∇r is depicted. The red dots mark the equilibrium points (critical points, where $\nabla r = 0$), whose relative connectivity is to be determined. Highlighted in red is a trajectory of the gradient flow starting at the critical point r_2 in the initial direction indicated by the green arrow; its Destination is the critical point r_4.

Methodology and Expected Work

The internship will aim to:

1. **Analyze** the existing symbolic algorithm (based on [HRSS20, CHHS25]) and identify the components requiring certification;

2. **Design and implement** a certified numerical subroutine for the gradient flow system

$$\phi'(t) = \nabla r(\phi(t)), \quad \phi(0) = p,$$

ensuring correctness in trajectory tracking, using methods from validated numerics [Tuc11];

3. **Integrate** symbolic and numerical components into a unified prototype algorithm Connectivity, deciding whether two points lie in the same connected component of a smooth semi-algebraic set;

4. **Evaluate** the approach experimentally on benchmark examples, comparing symbolic, numerical, and certified symbolic-numeric strategies.

Depending on the progress and results, this work may naturally extend into a **PhD project**, for instance by developing formal verification of the certified numerical routines or exploring more intricate connectivity problems.

Prerequisites

The internship is aimed at a Master's student in theoretical computer science or applied mathematics. The following skills and knowledge will be useful:

- **Algorithmic foundations:** strong background in algorithms, discrete mathematics, and formal reasoning;
- **Symbolic computation:** familiarity with polynomial manipulation and basic algebraic structures (fields, ideals, varieties);
- **Numerical methods:** some exposure to numerical solvers or scientific computation (e.g. solving ODEs, root-finding);
- **Programming skills:** experience with a scientific programming language, preferably Julia;
- **Theoretical interest:** curiosity for topics such as real algebraic geometry, dynamical systems, and a posteriori certification.

The project is well suited for a student interested in bridging symbolic computation, numerical analysis, and mathematical verification.

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