Problem Set IV Solutions

QF 430: Introduction to Derivatives
Due Friday, November 11

Please submit neatly handwritten or typed answers. You can turn in paper submissions in class or submit a single pdf file electronically through Canvas. Show your steps or reasoning.

1 Securitization

Problem 1.1. This problem examines how correlation impacts cash flows of securitization tranches. Borrowers Yash and Zara each are supposed to pay off their loans of $100,000 each next year. If a borrower does not default, then he or she repays the loan in full. If a borrower defaults, 30% of the loan amount is recovered. The bank which issued the loans has bundled the two loans and sold the cash flow to investors Alice and Bob. Of the total cash flow from the two borrowers, Alice will first receive all cash flow up to $140,000, and Bob will receive any remaining cash flow.

(a) What is the cash flow to Alice in each of the following four outcomes? Ignore discounting for this problem.

Alice's Cash Flow	Zara defaults	Zara does not default
Yash defaults		
Yash does not default		

(b) What is the cash flow to Bob in each of the following four outcomes?

Bob's Cash Flow	Zara defaults	Zara does not default
Yash defaults		
Yash does not default		

(c) Yash and Zara are both likely to default with probability 10%. Suppose their defaults are independent so the probabilities of the four outcomes are given in the following table. Calculate the expected cash flow of Alice and the expected cash flow of Bob.

	Zara defaults	Zara does not default
Yash defaults	1%	9%
Yash does not default	9%	81%

(d) Again, Yash and Zara default with probability 10% each, but defaults are imperfectly positively correlated such that the probabilities of the four outcomes are given in the following table. Calculate the expected cash flow of Alice and the expected cash flow of Bob.

	Zara defaults	Zara does not default
Yash defaults	5%	5%
Yash does not default	5%	85%

(e) Again, Yash and Zara default with probability 10% each, but defaults are perfectly positively correlated so the probabilities of the four outcomes are given in the following table. Calculate the expected cash flow of Alice and the expected cash flow of Bob.

	Zara defaults	Zara does not default
Yash defaults	10%	0%
Yash does not default	0%	90%

Solution.

(a)

Alice's Cash Flow	Zara defaults	Zara does not default
Yash defaults	\$60,000	\$130,000
Yash does not default	\$130,000	\$140,000

(b)

Bob's Cash Flow	Zara defaults	Zara does not default
Yash defaults	\$0	\$0
Yash does not default	\$0	\$60,000

(c) Alice's expected payoff $= 0.81 \times \$140,000 + 0.18 \times \$130,000 + 0.01 \times \$60,000 =$ $\boxed{\textbf{\$137,400}}$.

Bob's expected payoff $= 0.81 \times \$60,000 + 0.18 \times \$0 + 0.01 \times \$0 =$ $\boxed{\textbf{\$48,600}}$.

(d) Alice's expected payoff $= 0.85 \times \$140,000 + 0.1 \times \$130,000 + 0.05 \times \$60,000 =$ $\boxed{\textbf{\$135,000}}$.

Bob's expected payoff $= 0.85 \times \$60,000 + 0.1 \times \$0 + 0.05 \times \$0 =$ $\boxed{\textbf{\$51,000}}$.

(e) Alice's expected payoff $= 0.9 \times \$140,000 + 0.1 \times \$60,000 =$ $\boxed{\textbf{\$132,000}}$.

Bob's expected payoff $= 0.9 \times \$60,000 + 0.1 \times \$0 =$ $\boxed{\textbf{\$54,000}}$.

Note that in each case the total expected cash flow is \$186,000. However, more of this expected cash flow is received by the senior trancheholder Alice when correlation across defaults is low. When correlation across defaults increases, Alice's expected cash flow decreases. During the Financial crisis of 2007-2008, holders of senior tranches discovered that default correlations were higher than they anticipated. ■

Problem 1.2. Download the prospectus for Ford Credit's Asset Backed Security deal closed in September 2022 from https://www.ford.com/finance/content/dam/abs-reports-pdf/ford/us/public-retail-securitization/ford-credit-auto-owner-trusts/prospectuses/Ford_Retail_2022-C_As-Printed_Final_Prospectus.pdf. Next, download the latest investor report for the same deal from https://www.ford.com/finance/content/dam/abs-reports-pdf/ford/us/public-retail-securitization/ford-credit-auto-owner-trusts/investor-reports/2022/september/22-C_2022-10-17.pdf. You only need to focus on pages 1, 6-8, and 10-12 of the prospectus.

(a) How many loans (receivables) are backing the deal? What is the average principal balance (not the original amount financed)? What percentage of the assets backing the deal are receivables from car loans?

(b) What is the principal amount and the interest rate for the most senior (Class A-1) notes? What are the corresponding figures for the most junior notes?

(c) How much money do the underwriters keep (underwriter discount)?

(d) Based on the waterfall structure (Transactions Payments Diagram on page 8 of prospectus), what is the next use of cash flows after second priority principal payment has been made?

(e) Verify the interest amount calculation for Class A-1 notes on page 2 of the investor report. The principal balance is the beginning of period balance (page 3). The payment date is October 17, 2022 (page 1). There is no previous payment date but the period starts from the closing date of September 23, 2022. You found the interest rate in part b above and the interest rate day convention is on page 11 of the prospectus.

(f) The total available funds of \$64,134,887.70 on October 17, 2022 were used for servicing fees, interest payments, and principal payments (see page 2 of investor report). What amount was paid for each?

Solution.

(a) There are **52,430** loans. The average principal balance is **\$33,288.25**. **4.53%** of the loans are car loans. (Source: Prospectus, page 12).

(b) The principal amount and the interest rate for Class A-1 notes are **\$304.74 million** and **3.633%**. The principal amount and the interest rate for Class C notes are **\$31.57 million** and **5.22%** (Source: Prospectus, page 1).

(c) The underwriters' discount is **\$2,847,303** (Source: Prospectus, page 1).

(d) After second priority principal payment, interest due on the Class C notes is paid.

(e) Class A-1 notes accrue interest on an "actual/360" basis. There are 24 actual days from September 23, 2022 to October 17, 2022. The principal balance at the beginning of the period was \$304,740,000. Interest on Class A-1 notes = $\$304,740,000 \times 3.633\% \times 24/360$ = **\$738,080.28**.

(f) The payments for servicing fees, interest payments, and principal payments were **\$1,454,419.22**, **\$4,101,122.83**, and **\$58,579,345.65** (Source: page 2 of investor report).

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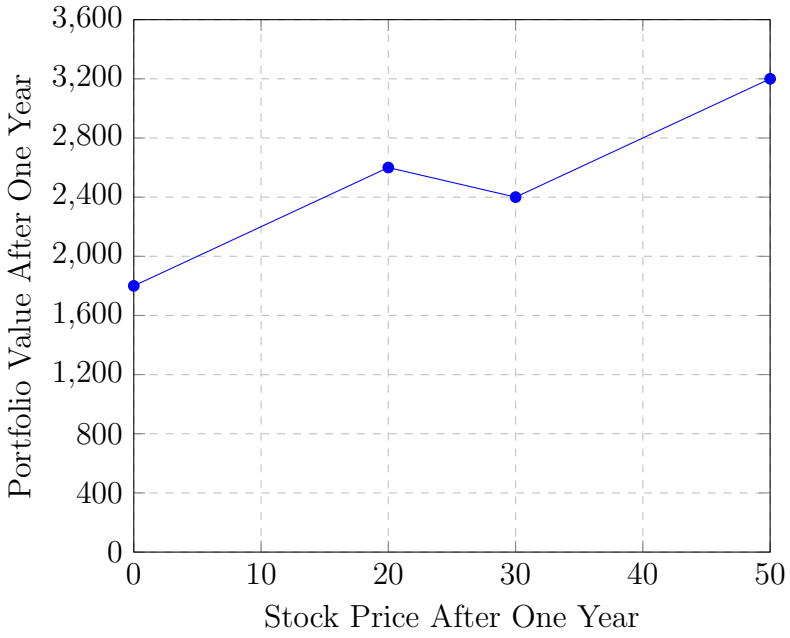
2 Options Markets

Problem 2.1. An investor buys 100 shares of a stock, sells 60 call options on the stock with strike price of \$20 and buys 60 put options on the stock with strike price of \$30. All options are one-year European options. Draw a diagram illustrating the value of the investor's portfolio as a function of the stock price after one year.

Solution.



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Problem 2.2. Options on Tesla Stock (ticker TSLA) are traded on Chicago Board Options Exchange (CBOE). These are American style options. The price quotes at November 2, 2022 for six options on Tesla have been mixed up. Match each price quote with the corresponding option. Bid and ask quotes are better indicators of option price than last sale which may be a stale price. You can assume that prices follow the no-arbitrage relations discussed in

the class. Tesla stock price was $224.90 at a time close to the time the following quotes are provided.

Option	Type	Expiration	Strike
1	Call	11/4/2022	225
2	Put	11/4/2022	225
3	Call	11/4/2022	227.5
4	Put	11/4/2022	227.5
5	Call	11/11/2022	225
6	Put	11/11/2022	227.5

Quote	Bid/Ask Mean	Bid	Ask	Volume	Open Interest	Delta
A	6.30	6.25	6.35	8130	4330	-0.5650
B	9.80	9.75	9.85	1643	1151	-0.5230
C	4.00	3.95	4.05	12724	6818	0.4373
D	4.975	4.95	5.00	27139	10722	-0.4850
E	5.125	5.10	5.15	12610	7637	0.5150
F	8.85	8.80	8.90	1496	1592	0.5247

Fill in the following table and explain your reasoning.

Option	Quote
1	
2	
3	
4	
5	
6	

Solution. The matching of options and quotes follows:

Option	Quote
1	**E**
2	**D**
3	**C**
4	**A**
5	**F**
6	**B**

Price quotes C, E, and F with positive Deltas are for calls. Price quotes A, B, and D with negative Deltas are for puts. The most expensive call quote F must be for Nov 11 call with strike price of $225 (as call price increases with maturity and decreases with strike). Among C and E, the more expensive E must be for the call with lower strike of $225 and the cheaper one C must be for the higher strike of $227.5.
The most expensive put quote B must be for the Nov 11 put with strike 227.5 (as put price increases with maturity and increases with strike). Among A and D, the more expensive A

must be for the put with the higher strike of \$227.5 and the cheaper one D must be for the lower strike of \$225. ■

3 Option Price Properties

Problem 3.1. A European call option on a stock with a strike price of \$50 and expiring in six months is trading at \$14. A European put option on the stock with the same strike price and expiration as the call option is trading at \$2. The current stock price is \$60 and a \$1 dividend is expected in three months. Zero coupon risk-free bonds with face value of \$100 and maturing after 3 months and 6 months are trading at \$99 and \$98, respectively. Identify the arbitrage opportunity open to a trader.

Solution. The put-call parity with known dividends requires that

$$\text{Call Price} + \text{PV of Dividends} + \text{PV of Strike} = \text{Put Price} + \text{Stock Price}.$$

But on this case, we find that

$$\begin{aligned}
&\text{Call Price} + \text{PV of Dividends} + \text{PV of Strike} \\
&= 14 + 1 \times 99/100 + 50 \times 98/100 = 63.99 \\
&> 2 + 60 \\
&= \text{Put Price} + \text{Stock Price}.
\end{aligned}$$

The trader should **sell call** for \$14, **borrow** \$49.99, **buy put** for \$2, and buy **stock** for \$60. This results in a net cash flow of \$1.99. The loan of \$49.99 can be considered to be a loan of \$0.99 that will be paid through the dividend of \$1 on the stock after three months and a loan of \$49.00 that will be paid with \$50 after six months. These \$50 come from selling the stock after six months. If the stock is worth more than \$50 at that time, we will sell the stock for \$50 to the call owner. If the stock is worth less than \$50 at that time, we will exercise the put to sell the stock for \$50. ■

Problem 3.2. Suppose that c_1, c_2, and c_3 are the prices of European call options with strike prices K_1, K_2, and K_3, respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that

$$c_2 \leq 0.5(c_1 + c_3)$$

That is option prices are convex in strike price. (Hint: Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2. Show that its future payoffs are non-negative so the portfolio's current price must be non-negative.)

Solution. Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2. At maturity, this portfolio's value will be non-negative.

Stock Price	Value from K_1 Call	Value from K_2 Calls	Value from K_3 Call	Total
$S_T < K_1$	0	0	0	0
$K_1 < S_T < K_2$	$S_T - K_1$	0	0	$S_T - K_1 > 0$
$K_2 < S_T < K_3$	$S_T - K_1$	$-2(S_T - K_2)$	0	$(2K_2 - K_1 - K_3) + (K_3 - S_T) > 0$
$S_T > K_3$	$S_T - K_1$	$-2(S_T - K_2)$	$S_T - K_3$	$2K_2 - K_1 - K_3 = 0$

The cost of setting up this portfolio is $c_1 + c_3 - 2c_2$. If $c_2 > 0.5(c_1 + c_3)$, setting up the portfolio will result in **positive cash flow** and the preceding table shows that there will be **no losses the future**. This is an arbitrage opportunity. Hence, we must have $c_2 \leq 0.5(c_1 + c_3)$. ■

Problem 3.3. A stock that does *not* pay dividend is trading at \$54. A European call option with strike price of \$60 and maturing in one year is trading at \$22. An American call option with strike price of \$60 and maturing in one year is trading at \$24. You can borrow or lend money at any time at risk-free rate of 5% per annum with continuous compounding. Devise an arbitrage strategy.

Solution. American and European calls options on non-dividend-paying stocks must trade at the same price. Since American call is priced higher, **sell an American call** and **buy a European call**. This results in a cash flow of \$24 − \$22 = \$2. If American call is not exercised, there is no risk of loss. If American call is exercised at maturity, exercise European call at the same time for an offsetting transaction with zero total cash flow. If American call is exercised before maturity, borrow a stock to sell it to American call owner for \$60. Invest \$60. Get back \$60 plus interest at maturity, use \$60 to exercise European call and return the borrowed stock and keep any interest as extra profit. ■

Problem 3.4. Consider a stock that does *not* pay dividend. A one-year European put option with strike \$40 is trading at \$12 and a one-year European put option with strike \$60 is trading at \$31.50. The risk-free interest rate is 5% per annum with continuous compounding. Construct an arbitrage strategy.

Solution. The maximum difference between the cash flows from the two options at maturity can be \$20 when both options are exercised. Therefore, the difference in the prices of the two options should be no more than present value of \$20, that is $20e^{-0.05} = \$19.02$. The difference between \$12 and \$31.50, \$19.50 is too high. **Sell the option trading at \$31.50** and **buy the option trading at \$12** and invest \$19.02. The net cash flow is \$0.48. The amount invested will grow to \$20 after a year. If the stock price is below \$40, both options are exercised and \$20 will be sufficient to cover the need for \$20 difference in exercise prices. If the stock price is between \$40 and \$60, the option sold is exercised. To pay \$60 to the option owner, sell the stock that you get from option exercise. The proceeds plus \$20 are enough to pay \$60. If the stock price is above \$60, neither option is exercised and you keep that \$20. ■