Mathematics of Option Pricing

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September 2022

Organization

Organization

Casual Description of Terminology

- Ω: the set of all possibilities
- ω: one possibility
- Event: a collection of possibilities, a subset of Ω. Each even has one or more ωs
- Sigma-algebra: a collection of events of interest
- Probability of a single *omega* is the chance that that particular ω is picked from Ω
- Probability of an event is the probability that an ω in that event is picked
- Random variable: a variable that takes a numberic value based on ω

Sigma-Algebra

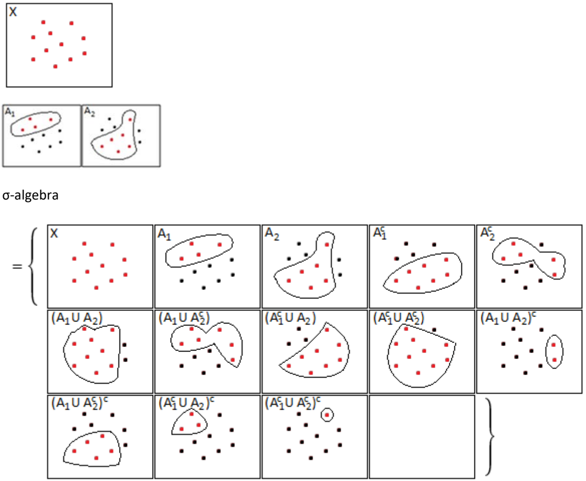
Definition 1

Let Ω be a nonempty set, and let $\mathcal{F}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a σ-algebra (or σ-field) if:

1. the empty set $\emptyset$ belongs to $\mathcal{F}$,
2. whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$, and
3. whenever a sequence of sets $A_1, A_2, \ldots$ belongs to $\mathcal{F}$, then their union $\cup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$.

- Note that the union of some sets, indicated by $\cup$ is a bigger set including everything in those individual sets.
- On the other hand, the intersection of some sets, indicated by $\cap$, is a smaller set including contents common to all individual sets.

Example of a σ-algebra



Source: https://www.studieportalen.dk/forums/thread.aspx?id=1841432

Figure: The smallest σ-algebra of X that includes A_1 and A_2

Sigma-Algebra

- Typically, Ω represents the set of all possibilities and each element ω in Ω is a distinct possibility.
- An outcome is picking of a single ω in Ω. The outcome is random if we are not sure which ω is picked.
- A σ-algebra is used to define how much information we have about a random outcome.
- Suppose we are interested in identifying an unknown element ω of the set Ω. If our information is represented by a σ-algebra, then for each set A in the σ-algebra, we know whether ω lies in that set or not.

Sigma-Algebra

- The information is like what one learns in a game of twenty questions where one guesses an object by asking a series of questions with yes or no answers. In this case, the questions determine whether ω lies in a set or not.
- The finer the division of Ω in the σ-algebra, the more we can narrow down the possibilities for ω. However, we may not be able to pin down ω.
- For example, suppose we narrow down the identity of ω to be either ω_1 or ω_2. We will not be able to figure out which one if some sets in σ-algebra have both ω_1 and ω_2 while the rest have none of them.

Probability Space

Definition 2

Let Ω be a nonempty set, and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A probability measure $\mathbb{P}$ is a function that, to every $A \in \mathcal{F}$, assigns a number in $[0, 1]$, called the probability of A and written $\mathbb{P}(A)$. We require:

1. $\mathbb{P}(\Omega) = 1$, and
2. whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n). \tag{1}$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Events

- Each subset of Ω is called an event. However, a probability measure $(\Omega, \mathcal{F}, \mathbb{P})$ does not define probability for each event, only for events that are included in the σ-algebra $\mathcal{F}$.
- Notice that the definition of σ-algebra is designed to include all events whose probability can be inferred from the probability of other events in the σ-algebra.
- If we know the probability of one event, then we also know the probability of the complement of that event. That explains the second condition in Definition 1.
- If we know probabilities of some disjoint events, then the probability of their union can be calculated as the sum of their individual probabilities. That explains the third condition in Definition 1.

Example 1

- *Suppose a coin is tossed infinitely many times with outcomes H (head) or T (tail) in each toss. Let Ω be the set of possible outcomes. Suppose p is the probability of a head, $q = 1 - p$, and tosses are independent.*
- *Suppose A_H is the set of all sequences beginning with H, A_T is the set of all sequences beginning with T, A_{HH} is the set of all sequences beginning with HH, A_{HT} is the set of all sequences beginning with HT, and so on.*
- *Then, one possible σ-algebra is*

$$\mathcal{F}_0 = \{\emptyset, \Omega\}$$

We have $\mathbb{P}(\emptyset) = \quad , \quad \mathbb{P}(\Omega) =$

Example II

Example 1

- *A finer σ-algebra is $\mathcal{F}_1 = \{\emptyset, \Omega, A_H, A_T\}$ with probabilities*

$$\mathbb{P}(\emptyset) = \qquad , \quad \mathbb{P}(\Omega) = \qquad , \quad \mathbb{P}(A_H) = \qquad , \quad \mathbb{P}(A_L) =$$

- *What if we add $A_{HH}, A_{HT}, A_{TH}, A_{TT}$ to the σ-algebra? Verify that we get*

$$\begin{aligned}\mathcal{F}_2 = &\{\emptyset, \Omega, A_H, A_T, A_{HH}, A_{HT}, A_{TH}, A_{TT}, A_{HH}^c, A_{HT}^c,\\ &A_{TH}^c, A_{TT}^c, A_{HH} \cup A_{TH}, A_{HH} \cup A_{TT}, A_{HT} \cup A_{TH},\\ &A_{HT} \cup A_{TT}\}.\end{aligned}$$

- *Can you determine the probabilities?*

Example III

Example 1

- *Note that there can be multiple probability measures associated with the same* σ*-algebra.*
- *If* $\mathcal{F}_1 = \{\emptyset, \Omega, A_H, A_T\}$ *and the coin is fair, the probability measure is* $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1, \mathbb{P}(A_H) = \mathbb{P}(A_T) = 0.5$.
- *However, there can be another probability measure based on the assumption that the coin is biased and head turns up with 51% probability.*
- *Under this probability measure,* $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1, \mathbb{P}(A_H) = 0.51, \mathbb{P}(A_T) = 0.49$.

Organization

Random Variables

- Random variables are random numbers that depend on some random event. For example, if you toss a coin ten times, the number of heads in the ten coins is a random variable and it depends on the random outcomes of the ten tosses.

Definition 3

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable is a real-valued function defined on Ω with the property that for every Borel-subset B of $\mathcal{R}$, the subset of Ω given by

$$\{\omega \in \Omega; X(\omega) \in B\}$$

is in the σ-algebra $\mathcal{F}$.

Borel Sets

- Borel-subset means every reasonable set you can think of.
- The simplest Borel sets are open sets. For example, the open set (2.67,783) represents numbers greater than 2.67 and less than 7.83.
- Borel sets also include unions of open sets such as (2.67,783) or (-5.76,4.29) or intersections of open sets, such as set of numbers common to both (2.67,783) and (-5.76,4.29).
- The unions and intersections can be carried out an infinite (but countable) number of times. Thus, examples of Borel subsets include numbers between -25 and -19 or between -2 and 73.8, the number 5 by itself, or all negative numbers or 6.36 and all numbers greater than 10.

Random Variables

- Intuitively, this definition says that a random variable's value must depend on random events that we can identify.
- For example, if our random events are defined by ten tosses of a coin, then the number of heads is a random variable because we can determine questions like whether the number of heads is three based on the outcomes of tosses.
- However, the year that the coin was printed or the amount of rainfall tomorrow is not a random variable because it cannot be determined based on the random events (outcomes of ten tosses).

Example

Example 2

Consider the infinite coin-toss space from the previous example.
Define X to be the number of heads in first two tosses.
Then, $X = 0$ for the event A_{TT}, $X = 1$ for the event $A_{HT} \cup A_{TH}$, and $X = 2$ for the event A_{HH}.
X can be a random variable only if all three events are included in the σ-algebra.
Thus, X is a random variable with σ-algebra $\mathcal{F}_2$ but not with σ-algebras $\mathcal{F}_0$ or $\mathcal{F}_1$.

Expected Value

Definition 4

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The expected value of X is defined to be

$$\mathbb{E}X = \int_{\Omega} X(\omega)\mathrm{d}\mathbb{P}(\omega). \tag{2}$$

- The above definition means that for a random variable with finitely many possible values, multiply each possible value X_i of X with the probability $\mathbb{P}(\omega)$ of the event ω that $X = X_i$ and then sum these products.
- This gives the expected value of X. If X can take infinitely many values, then calculating expected value requires integration instead of summation.

Example

Example 3

Consider the random variable X in the previous example. It takes value 0 for the event A_{TT} whose probability is $\frac{1}{4}$. It takes value 1 for the event $A_{HT} \cup A_{TH}$ whose probability is $\frac{1}{2}$. It takes value 2 for the event A_{HH} whose probability is $\frac{1}{4}$. The expected value of X is then,

$$0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1.$$

Properties of Expected Value

If X and Y are two random variables and c is a constant, then

$$\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y, \text{ and } \mathbb{E}(cX) = c\mathbb{E}X, \tag{3}$$

but $\mathbb{E}(XY)$ does not equal $\mathbb{E}X\mathbb{E}Y$ in general. To see this, in the previous example, $\mathbb{E}(XX) = \mathbb{E}(X^2) = \frac{3}{2}$ but $\mathbb{E}X\mathbb{E}X = 1$.

Example I

Example 4

A box has two red (r) marbles and one green (g) marble. One marble is taken out and its color is noted. Then another marble is taken out and its color is noted.
Three outcomes are possible. The sample space is $\{(r,r),(r,g),(g,r)\}$.
There are eight possible events (sets of outcomes):

$$\{\}, \{(r,r)\}, \{(r,g)\}, \{(g,r)\}, \{(r,r),(r,g)\}, \{(r,r),(g,r)\},$$
$$\{(r,g),(g,r)\}, \{(r,r),(r,g),(g,r)\}$$

Example II

Example 4

Consider random variable X, the number of red marbles taken out. Its value can be 1 or 2.
Sigma-algebra for X must consist of an event with all outcomes for which X is 1: $\{(r,g),(g,r)\}$.
Another event includes all outcomes for which X is 2: $\{(r,r)\}$.
Each event considered is the complement of the other.
The union of both is $\{(r,r),(r,g),(g,r)\}$. The complement of this last event is $\{\}$. So the sigma-algebra must include

$$\{\},\{(r,r)\},\{(r,g),(g,r)\},\{(r,r),(r,g),(g,r)\}$$

Further unions and complements do not add any other events.
Note that if the included events have either both or none of the outcomes (r,g) and (g,r) because the value of X is the same for both.

Equivalent Probability Measures

Definition 5

Let Ω be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of Ω. Two probability measures $\mathbb{P}$ and $\overline{\mathbb{P}}$ are said to be **equivalent** if they agree which sets in $\mathcal{F}$ have probability zero.

- There can be multiple probability measures for an event space. An example would be views of different people about how the stock market would behave.
- The above definition states that two probability measures are equivalent if they agree on zero-probability events.
- In the context of people's beliefs about stock price movement, two beliefs will be equivalent if they disagree on probabilities of most events but if one thinks an event is impossible (or zero probability), then the other also thinks so.

Equivalent Probability Measures

- All investors whose beliefs form equivalent measures must agree on whether a strategy is an arbitrage strategy or not.
- In an arbitrage strategy, there are some positive probability events with positive profits and all events with negative profits have zero probability.
- Since investors whose beliefs form equivalent probability measures agree on which events have positive probability and which events have zero probability, they must agree on whether a trading strategy is an arbitrage strategy or not.

Radon-Nikodym Theorem

Theorem 1

(Radon-Nikodym) Let $\mathbb{P}$ *and* $\overline{\mathbb{P}}$ *be equivalent probability measures defined on* $(\Omega, \mathcal{F})$*. Then there exists an almost surely positive random variable* Z *such that* $\mathbb{E}Z = 1$ *and*

$$\overline{\mathbb{P}}(A) = \int_A Z(\omega)\mathrm{d}\mathbb{P}(\omega) \text{ for every } A \in \mathcal{F}. \tag{4}$$

Radon-Nikodym Theorem

- This result states that the probabilities assigned by the measure $\overline{\mathbb{P}}$ can be obtained by multiplying the probabilities assigned by the measure $\mathbb{P}$ with a positive random variable Z.
- Since expected value of Z is 1, sometimes Z will exceed 1 and sometimes Z will be less than 1.
- This means that some probabilities will be higher under one measure and some probabilities will be higher under another measure.
- This result suggests that multiplication with a positive random variable with mean one is a recipe for moving from one set of beliefs (probability measure) to another such that both beliefs agree on no-arbitrage prices.
- This result is important for risk-neutral method of derivative valuation. The idea is that absence of arbitrage implies a risk neutral measure exists and vice versa.

Organization

Random Walk Example I

Example 5

- *Consider a gambler who makes a living by flipping a fair coin and making fair bets of \$1 each time. Then the gambler's winnings can be represented with a sequence X_i of independent random variables with the probability distribution:*

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

- *Let S_n be the cumulative gain or loss of the gambler after n bets:*

$$S_n = X_0 + X_1 + \ldots + X_n.$$

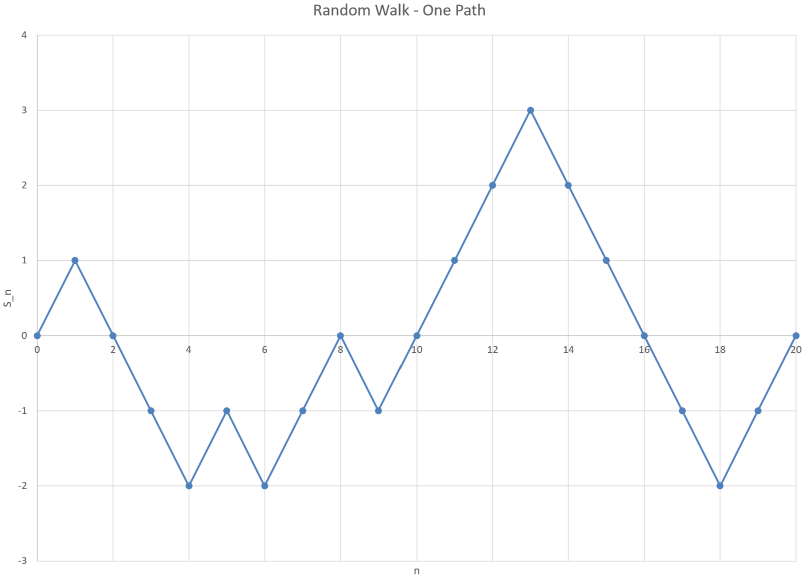
S_n is an example of a random walk.

Random Walk Example II

Example 5

- *If the cumulative winnings of the gambler after n bets are* S_n*, the winning after* $n+1$ *bets are equally likely to be* $S_n + 1$ *or* $S_n - 1$*.*
- *Therefore, after n bets, the expected value of* S_{n+1} *is then same as* S_n*.*
- *Repeating this argument, after* $n+1$ *bets, the expected value of* S_{n+2} *is* S_{n+1}*.*
- *Combining these two, after n bets, the expected value of* S_{n+2} *is* S_n*.*
- *This argument can be extended to show that after n bets, the expected value of* S_{n+k} *equals* S_n *for any positive integer* k*.*
- *Random walk has a memoryless property. That is, future changes to random walk are unrelated to past outcomes.*

A Sample Random Walk Path



Continuous Time Stochastic Process

1. The idea of a random walk can be extended to Brownian motions in continuous time.
2. The value of a continuous time stochastic process changes continuously with time.
3. A standard Brownian motion, or any continuous time stochastic process, is
    - a continuous time process and
    - stochastic
4. If we suppress the stochastic part, a continuous time process is just something that can vary continuously with time, for example the week of the day or the number of minutes on the clock, etc.
5. If we suppress time evolution of a continuous time stochastic process and focus only on its value at a fixed time point t_1, then a continuous time stochastic process is simply a random variable.

Definition 6

A continuous-time stochastic process $\{B_t : 0 \leq T\}$ is called a **Standard Brownian Motion** on $[0, T]$ if it has the following four properties:

1. $B_0 = 0$
2. The increments of B_t are independent; that is, for any finite set of times $0 \leq t_1 < t_2 < \ldots < t_n < T$, the random variables

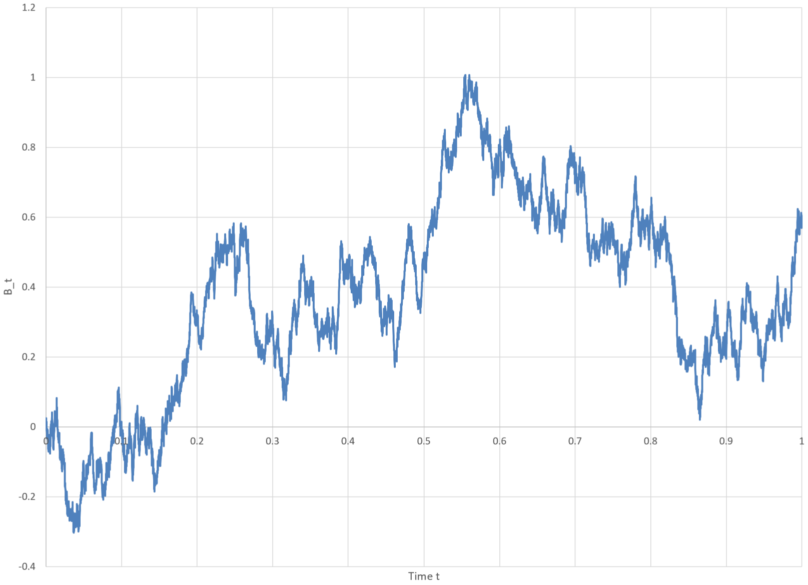
$$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}$$

   are independent.
3. For any $0 \leq s \leq t < T$, the increment $B_t - B_s$ has the Gaussian (normal) distribution with mean 0 and variance $t - s$.
4. For all ω in a set of probability one, $B_t(\omega)$ is a continuous function of t.

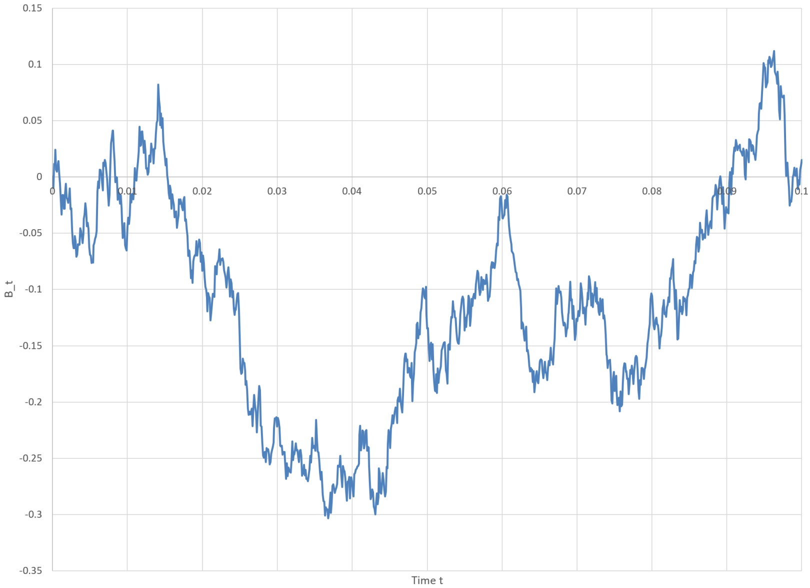
Brownian Motion

1. We can think of a standard Brownian motion as an infinite collection of paths.
2. Each path is associated with a different random event ω in the sample space Ω. That is, each time a standard Brownian motion is realized, a particular ω is picked from Ω and the path corresponding to that ω is realized.
3. The change in value of Brownian motion over time t is a normally distributed random variable with zero mean and variance t.

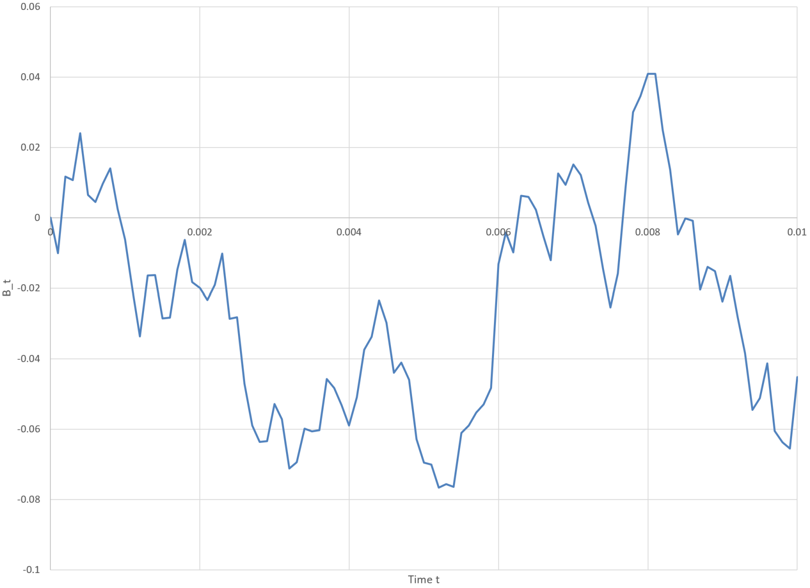
Sample Brownian motion path - Time 0 to 1



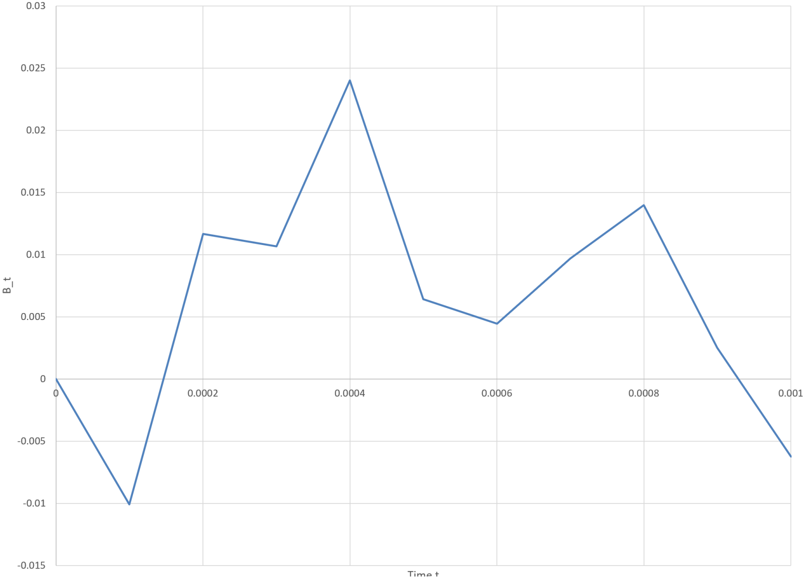
Brownian motion path - Time 0 to 0.1



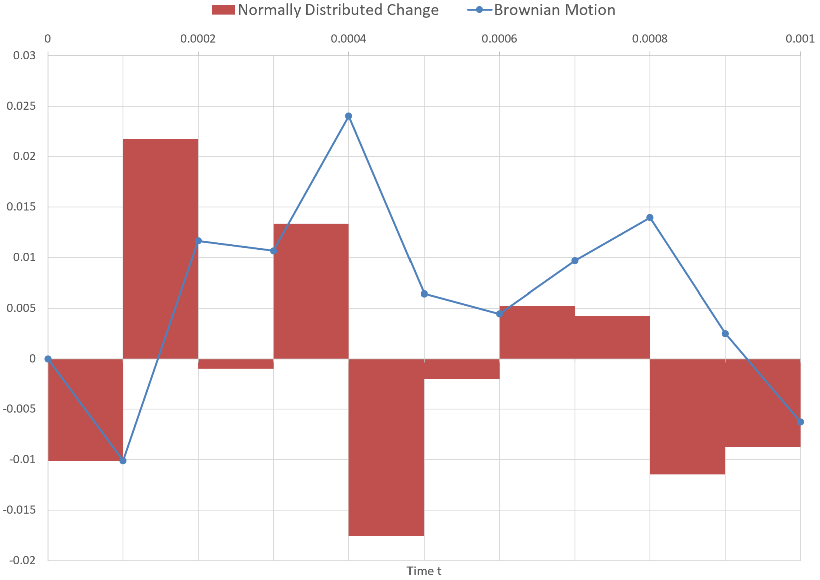
Brownian motion path - Time 0 to 0.01



Brownian motion path - Time 0 to 0.001



Brownian motion path and changes - Time 0 to 0.001



Martingales

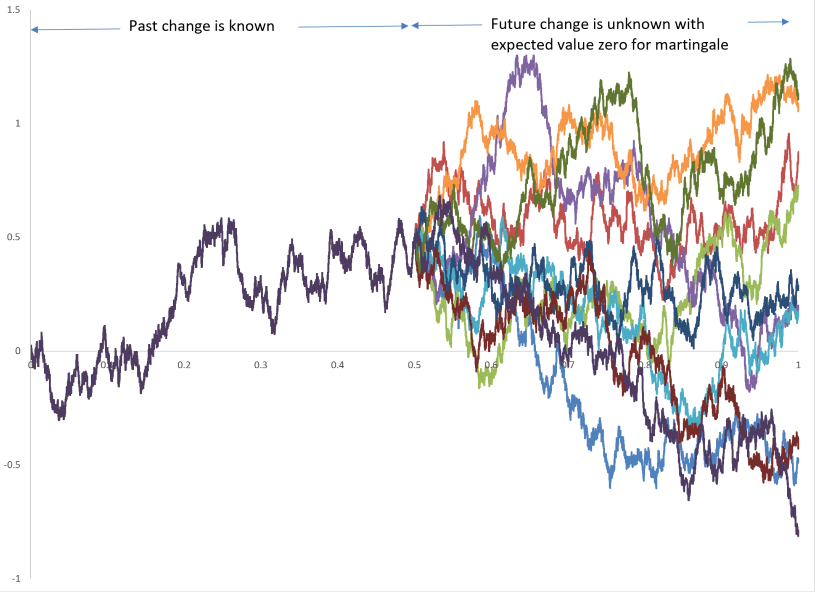
1. Even after observing the path up to some time t, one cannot distinguish between that path from another path that happens to be identical up to time t but diverges after that.
2. Definition 6 requires that future changes in a standard Brownian motion are unrelated to and cannot be predicted based on past changes. Furthermore, all changes in standard Brownian motion are normally distributed.

Definition 7

A stochastic process is a **martingale** if expected value of B_τ at time $t < \tau$ is B_t.

3. A standard Brownian motion is a martingale because future changes are on average zero.

Brownian motion is Martingale



Scaled and Inverted Brownian Motions

Theorem 2

If $\{B_t : t \geq 0\}$ is a standard Brownian motion, then for any $a > 0$, the scaled process defined by

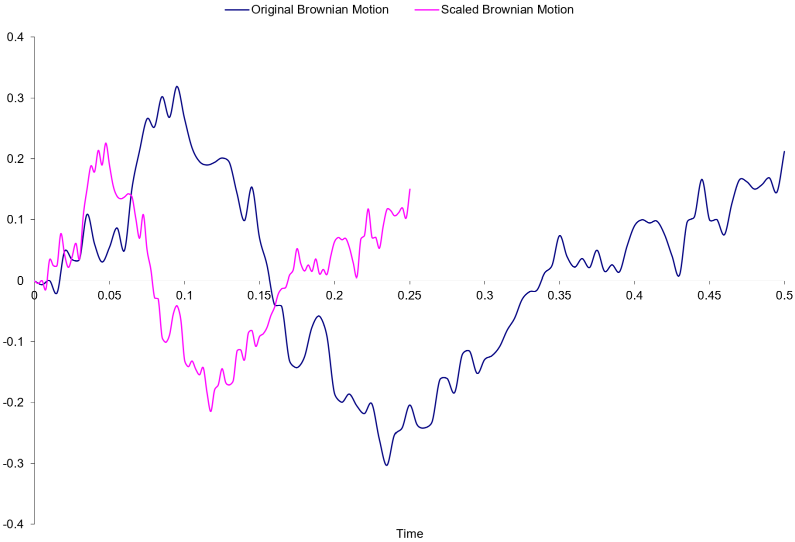
$$X_t = \frac{1}{\sqrt{a}} B_{at} \text{ for } t \geq 0 \tag{5}$$

and the inverted process defined by

$$Y_0 = 0 \text{ and } Y_t = tB_{1/t} \text{ for } t > 0 \tag{6}$$

are both standard Brownian motions.

Scaled Brownian Motion



Reflected Brownian Motion

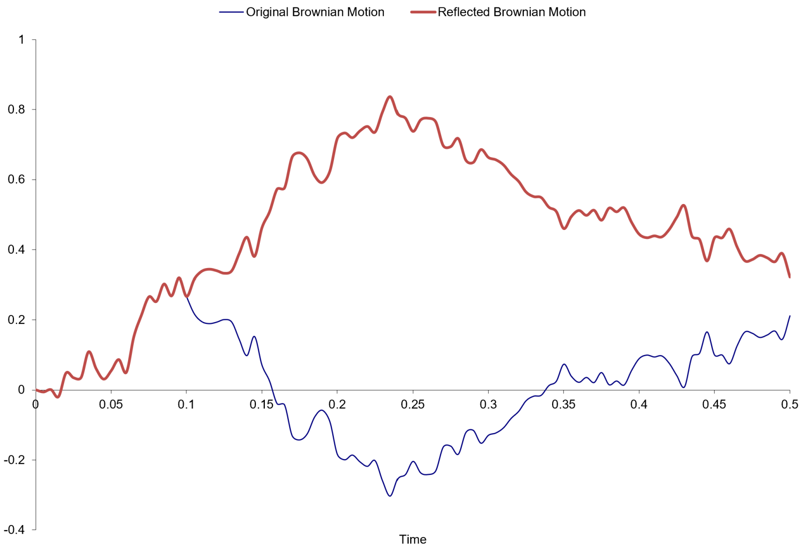
Theorem 3

(Reflection Principle) If $\{B_t : t \geq 0\}$ is a standard Brownian motion and $\bar{t}$ is a fixed time, then the process $\{\overline{B}_t : t \geq 0\}$ defined by

$$\overline{B}_t = \begin{cases} B_t & \text{if } t < \bar{t} \\ B_{\bar{t}} - (B_t - B_{\bar{t}}) & \text{if } t \geq \bar{t} \end{cases} \tag{7}$$

is again a standard Brownian motion.

Reflected Brownian Motion



Organization

Stochastic Differential Equations

- If the price of the underlying stock follows a stochastic process, so does the price of a derivative on the stock.
- To price a derivative, we create portfolios of underlying stock and derivative and analyze the returns.
- Financial principles such as law of one price or absence of arbitrage impose conditions on the return on the underlying stock and the return on the derivative.
- These restrictions take the form of an equation relating small changes in stock price, small changes in derivative price, and small changes in time.
- Equations relating small changes in quantities are called differential equations.

- When the changes are in stochastic quantities (stock price or derivative price in our case), these equations are called **stochastic differential equations**.
- Stochastic differential equations can also be expressed as **stochastic integrals** which express the change in a stochastic quantity by integrating small changes in that quantity.
- An **Itô integral** is a stochastic integral that integrates with respect to a Brownian motion.

Itô integral

Definition 8

An Itô integral is an integral of the form

$$I(f)(\omega) = \int_0^T f(\omega, t)\mathrm{d}B_t. \tag{8}$$

Properties of Stochastic Differentials

- The following properties will be very useful in our analysis of stochastic differential equations and stochastic integrals:

$$
\begin{aligned}
\mathrm{d}B_t^2 &= \mathrm{d}t \\
\mathrm{d}B_t \mathrm{d}t &= 0 \\
\mathrm{d}t^2 &= 0
\end{aligned} \tag{9}
$$

- Here $\mathrm{d}t$ represents an infinitesimal time interval and $\mathrm{d}B_t$ the change in the Brownian motion during that time interval.
- The approximations in the above equation are at the scale of $\mathrm{d}t$. That is, as $\mathrm{d}t$ is made smaller and approaches zero, quantities that approach zero even faster than $\mathrm{d}t$ are assumed to be zero.

Itô's Formula

- The following important formula states that an Itô integral can be evaluated using only a few partial derivatives, ignoring the partial derivatives that are zero based on the previous equation.

Theorem 4

(Itô's Formula) For a function $f(t,x)$ differentiable in t and twice differentiable in x,

$$\begin{aligned} f(t,B_t) =& f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s,B_s)\mathrm{d}B_s + \int_0^t \frac{\partial f}{\partial t}(s,B_s)\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)\mathrm{d}s. \end{aligned} \tag{10}$$

Itô's Formula

- The notation $\frac{\partial f}{\partial x}$ represents the partial derivative of f with respect to x, the ratio of the change in f to the change in x for a small change in x, keeping all other inputs to f fixed.
- The formula states that for a function of time and Brownian motion, if its value is available at one point, its value can be obtained at any other point by starting from the original value and integrating the effect of changes in the Brownian motion, integrating the effect of changes in time, and integrating the effect of squared changes in the Brownian motion (which equal change in time).
- The theorem differs from the corresponding results in non-stochastic calculus in having the third term on the right, a term not required when the integration is with respect to a deterministic process.

Itô's Formula

- Itô's formula can also be expressed in differential form as

$$\mathrm{d}f(t, B_t) = \frac{\partial f}{\partial x}(t, B_t)\mathrm{d}B_t + \frac{\partial f}{\partial t}(t, B_t)\mathrm{d}t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)\mathrm{d}t. \tag{11}$$

- Notice that in the last term, $\mathrm{d}B_t \cdot \mathrm{d}B_t$ has been substituted with $\mathrm{d}t$ following the properties of stochastic differentials mentioned earlier.
- If f depends on time and another stochastic quantity (that depends on Brownian motion, instead of being Brownian motion as above), the corresponding formula is

$$\mathrm{d}f(t, S_t) = \frac{\partial f}{\partial x}(t, S_t)\mathrm{d}S_t + \frac{\partial f}{\partial t}(t, S_t)\mathrm{d}t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t)\mathrm{d}S_t \cdot \mathrm{d}S_t. \tag{12}$$

Organization

Arithmetic Brownian Motion

- Standard Brownian motions can be used to create other continuous stochastic processes.
- An **arithmetic Brownian motion** (ABM) is a process X_t which follows

$$\mathrm{d}X_t = \alpha \mathrm{d}t + \sigma \mathrm{d}B_t \tag{13}$$

- An ABM is a generalization of the standard Brownian motion.
- The first term captures time trend and α is called the drift.
- The second term, the stochastic component with no time trend, is called diffusion and σ is the volatility.
- A standard Brownian motion is a special arithmetic Brownian motion with drift $\alpha = 0$ and volatility $\sigma = 1$.
- Like standard Brownian motion, changes in the value of an ABM over different time intervals are independent and normally distributed. Unlike standard Brownian motion, the mean and the variance do not necessary equal zero and the time interval, respectively.

Arithmetic Brownian Motion Example

Example 6

- *Consider an arithmetic Brownian motion:*

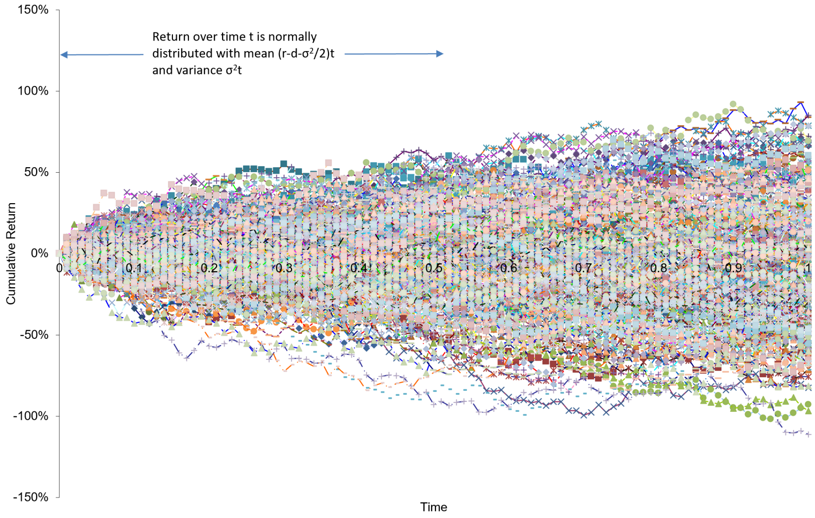
$$\mathrm{d}X_t = 0.05\mathrm{d}t + 0.4\mathrm{d}B_t \tag{14}$$

- *What is the drift of X_t?*
- *What is the volatility of X_t?*
- *Suppose $X_0 = 10$. What is the expected value of X_2?*
- *What is the variance of X_2?*
- *What is the standard deviation of X_2?*

Arithmetic Brownian Motion

- Logarithmic stock return is random and can be approximated as normally distributed but not necessarily zero on average.
- Stock returns are therefore, not modeled as standard Brownian motion which has zero drift.
- It is common to model a stock's return as an arithmetic Brownian motion with α the expected return on the stock and σ the volatility of the stock's return.
- Since an arithmetic return is not bounded and can reach arbitrarily high or arbitrarily low value, it is not a good choice for modeling stock prices which cannot be negative.
- However, the log return of the stock price can be used to recover the stock price.
- We model a stock's price as the process whose log is an ABM. Such a process is called a geometric Brownian motion.

Return as Arithmetic Brownian Motion



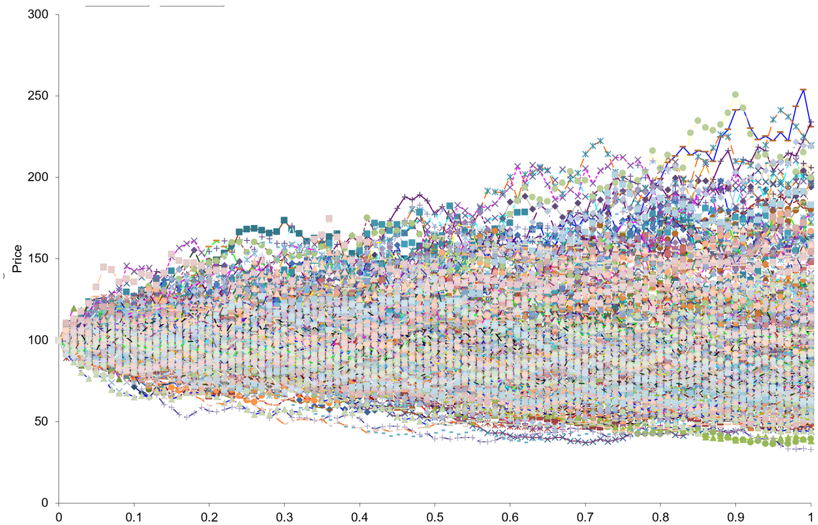
Geometric Brownian Motion

- A **geometric Brownian motion** (GBM) is a process X_t which follows

$$\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t \tag{15}$$

- Compare the stochastic differential equation of a GBM to that of an ABM. The right side is multiplied by stock price in the equation for GBM.
- Thus, changes in an ABM do not depend on current level while changes in a GBM are proportional to the current level.
- The log of a GBM is an ABM.
- Properties of a GBM make it a popular choice for modeling a stock's price.
- A GBM that starts as a positive value remains positive.
- Moreover, the return on a GBM is an ABM so if stock price is modeled as a GBM, stock returns over different time intervals are independent (market efficiency) and are normally distributed.

Price as Geometric Brownian Motion



Mean Reverting Process

- A **mean reverting process** is of the form

$$\mathrm{d}X_t = \kappa(\mu - X_t)\mathrm{d}t + \sigma X_t^{\gamma}\mathrm{d}B_t \tag{16}$$

- While stock prices do not exhibit mean reversion, a mean-reverting process can be a good modeling choice for quantities which exhibit a trend to revert to mean.

GBM and Probability Distribution

- We now consider the distribution of stock price changes if the stock price is modeled as a geometric Brownian motion.
- Suppose a stock's price at time t, S_t, is S_0 at time 0 and follows a geometric Brownian motion with drift μ and volatility σ.
- Then, the log of stock price $\ln(S_t)$ starts at $\ln(S_0)$ and follows an arithmetic Brownian motion with drift $\mu - \sigma^2/2$ and volatility σ.
- The log of the stock price $\ln(S_T)$ is normally distributed.
- This is the same as saying that the stock price S_T is lognormally distributed.
- The exact distribution for the change in log of stock price is:

$$\ln(S_T) - \ln(S_0) \sim \phi\left[(\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right]. \tag{17}$$

GBM and Probability Distribution of Log Price

- The change in log stock price, $\ln(S_T) - \ln(S_0)$ is normally distributed with mean $(\mu - \frac{\sigma^2}{2})T$ and variance $\sigma^2 T$ or standard deviation $\sigma\sqrt{T}$.
- The above equation can be rewritten as

$$\ln\left(\frac{S_T}{S_0}\right) \sim \phi\left[(\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right]. \tag{18}$$

- The log of future stock price is then normally distributed as follows:

$$\ln(S_T) \sim \phi\left[\ln(S_0) + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right]. \tag{19}$$

- The normal distribution is symmetric so the mean and median are the same. Thus, mean or median value of $\ln(S_T)$ is $\ln(S_0) + (\mu - \frac{\sigma^2}{2})T$.

GBM and Probability Distribution of Price

- Percentiles of stock price can be obtained readily by taking exponential of the percentiles of the log of stock price.
- The *p*-th percentile value of stock price is the exponential of the *p*-th percentile value of log of stock price.
- In particular, the median value of stock price is the exponential of the median value of the log of stock price.
- However, the mean of stock price is not the exponential of the mean of log of stock price.
- The reason is that log is not a linear function. The stock price increase from an increase λ in the log of stock price is higher than the stock price decrease from a decrease of λ in the log of stock price.
- The mean and variance of the stock price are given by

$$\mathbb{E}(S_T) = S_0 e^{\mu T} \tag{20}$$

and

$$\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1). \tag{21}$$

GBM and Probability Distribution Example I

Example 7

- *Consider a stock with initial price of \$50, (continuously compounded) expected return of 6.5% per annum and volatility of 30%.*
- *The log of stock price after one year will be normally distributed with mean and median of* $\ln(50) + (0.065 - 0.3^2/2) * 1$ *and volatility* 0.3.
- *The expected stock price after one year is* $50e^{0.065*1}$.

GBM and Probability Distribution Example II

Example 7

- *The following figure shows the probability distributions of log of stock price and of stock price.*
- *The horizontal axis is the probability percentile, the left vertical axis is for log of stock price plotted as dashed line and the right vertical axis is for stock price, plotted as solid line.*
- *The vertical line in the middle is at 50th percentile and intersects the two plots at their median values.*
- *The mean for the log of stock price is the same as median because its distribution is symmetric.*
- *However, the mean of the stock price exceeds the median because the stock price rise to the right is much steeper than the stock price fall to the left.*

Probability Distribution Example Plots

Organization

- We will now model a continuous-time arbitrage argument to derive Black-Scholes formula for pricing European call options.
- The most important assumptions behind this model:
  - stock price follows geometric Brownian motion
  - all assets can be traded in any amount with no transaction costs
  - investors agree on stock and bond prices.
- Let S_t denote the price of a non-dividend-paying stock at time t and β_t denote the price of a riskless bond at time t. The time dynamics of these processes are modeled as:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \tag{22}$$

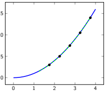
and

$$\mathrm{d}\beta_t = r\beta_t \mathrm{d}t. \tag{23}$$

- Note that we assume that the stock price follows a geometric Brownian motion and the bond price follows a deterministic process with exponential growth at risk-free rate.
- The arguments in the following derivation apply to any general European derivative, not just a call option.
- Suppose the payoff of the derivative at expiration time T is given by $h(S_T)$.
- For a European call option with strike price K, $h(S_T) = (S_t - K)^+ \equiv \max(S_T - K, 0)$. To price this derivative, we consider a self-financing replicating portfolio.

Self-Financing Replicating Portfolio

- By the absence of arbitrage argument, derivative price must equal the cost of setting up a replicating portfolio whose payoff equals the derivative payoff when the derivative expires.
- A static portfolio of stocks and bonds cannot replicate derivative payoff, which is non-linear in stock and bond prices.



- Since a nonlinear relation can be treated as linear for small changes, a portfolio of stocks and bonds can replicate change in derivative value over a small time interval.
- The replicating portfolio is rebalanced continuously by buying and selling securities so that it continues to replicate the derivative payoff over the next short time interval.
- Since a European derivative has no cash flow before maturity, the trades for the dynamic rebalancing of the replicating portfolio should not yield net positive or negative cash flow.
- Any cash required for buying securities to the replicating portfolio must be obtained by selling some other securities. Such a portfolio is called a self-financing portfolio.

Change in Value of Replicating Portfolio

- Suppose the replicating portfolio consists of a_t units of stock and b_t units of bond at time t. The portfolio value at time t is
$$V_t = a_t S_t + b_t \beta_t. \tag{24}$$
- The boundary condition at expiration is that the replicating portfolio replicate the option payoff
$$V_T = h(S_T). \tag{25}$$
- The change in the value (24) of the portfolio during a small time interval $\mathrm{d}t$ is $S_t \mathrm{d}a_t + a_t \mathrm{d}S_t + \beta_t \mathrm{d}b_t + b_t \mathrm{d}\beta_t$.
- Note that we assume that the stock does not pay dividends.
- The self-financing condition that no cash be invested or withdrawn from the portfolio for trades implies $S_t \mathrm{d}a_t + \beta_t \mathrm{d}b_t = 0$.
- Then, the change in value of the portfolio arises only from the changes in stock price and bond price:
$$\mathrm{d}V_t = a_t \mathrm{d}S_t + b_t \mathrm{d}\beta_t. \tag{26}$$

Stochastic Process for Replicating Portfolio - I

- We will arrive at two forms of the stochastic differential equation that governs changes in the value of the replicating portfolio. We will then use Itô calculus and match coefficients of differential equations.
- We use our expression for portfolio value as a function of stock price and time: $V_t = f(t, S_t)$. Substituting (22) and (23) in (26), we get

$$\begin{aligned} \mathrm{d}V_t &= a_t(\mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t) + b_t(r\beta_t \mathrm{d}t) \\ &= (a_t \mu S_t + r b_t \beta_t)\mathrm{d}t + a_t \sigma S_t \mathrm{d}B_t. \end{aligned} \tag{27}$$

Stochastic Process for Replicating Portfolio - II

- Applying Itô's formula (12) to the stochastic process $V_t = f(t, S_t)$ and then substituting (22), we get another expression for $\mathrm{d}V_t$:

$$\begin{aligned} \mathrm{d}V_t &= f_t(t, S_t)\mathrm{d}t + \frac{1}{2} f_{xx}(t, S_t)\mathrm{d}S_t \cdot \mathrm{d}S_t + f_x(t, S_t)\mathrm{d}S_t \\ &= \left\{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t)\sigma^2 S_t^2 + f_x(t, S_t)\mu S_t \right\} \mathrm{d}t \\ &\quad + f_x(t, S_t)\sigma S_t \mathrm{d}B_t. \end{aligned} \tag{28}$$

- In the above equation, the symbol x in the partial derivatives f_x and f_{xx} stands for stock price and $f_x \equiv \frac{\partial f}{\partial x}$, $f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}$, and $f_t \equiv \frac{\partial f}{\partial t}$.

Number of Stocks in Replicating Portfolio

- Matching the coefficients of $\mathrm{d}B_t$ in (27) and (28), we get

$$a_t = f_x(t, S_t). \tag{29}$$

- To interpret this equation, note that the left side is the number of stocks in the replicating portfolio.
- The right side is the derivative of the option price with respect to stock price or the sensitivity of the derivative price to the stock price.
- Recall that we call this quantity **delta** of the derivative.
- The above equation states that the number of stocks held in a replicating portfolio at any time equals the delta of the derivative at that time.

Number of Bonds in Replicating Portfolio

- Next, we match the coefficients of $\mathrm{d}t$ in (27) and (28) to get

$$f_x(t, S_t)\mu S_t + rb_t\beta_t = f_t(t, S_t) + \frac{1}{2}f_{xx}(t, S_t)\sigma^2 S_t^2 + f_x(t, S_t)\mu S_t. \tag{30}$$

- After canceling $f_x(t, S_t)\mu S_t$ from both sides, we can solve for b_t:

$$b_t = \frac{1}{r\beta_t}\left\{f_t(t, S_t) + \frac{1}{2}f_{xx}(t, S_t)\sigma^2 S_t^2\right\}. \tag{31}$$

- The above equation gives the number of bonds in the replicating portfolio at any point of time. Since we didn't specify the face value of bonds, a more meaningful quantity is the investment $b_t\beta_t$ in bonds:

$$\frac{1}{r}\left\{f_t(t, S_t) + \frac{1}{2}f_{xx}(t, S_t)\sigma^2 S_t^2\right\}. \tag{32}$$

Black-Scholes Partial Differential Equation

- We can now write the value of the portfolio as the sum of the value of the stocks and the bonds in the portfolio:

$$
\begin{aligned}
f(t, S_t) = V_t &= a_t S_t + b_t \beta_t \\
&= f_x(t, S_t) S_t + \frac{1}{r} \left\{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S_t^2 \right\}.
\end{aligned}
\tag{33}
$$

- Using the symbol x for stock price S_t, the above equation can be written as Black-Scholes Partial Differential Equation (PDE):

$$
f_t(t, x) = -\frac{1}{2}\sigma^2 x^2 f_{xx}(t, x) - rx f_x(t, x) + rf(t, x) \tag{34}
$$

with the terminal boundary condition

$$
f(T, x) = h(x) \text{ for all } x. \tag{35}
$$

Black-Scholes Partial Differential Equation

- There are four terms in the Black-Scholes PDE (34), representing the derivative value and its partial derivatives.
- The term $f(t,x)$ is the value of the derivative. The term $f_t(t,x)$ is the rate of change of value of the derivative with the passage of time and is called the **theta** (Θ) of the derivative.
- The term $f_x(t,x)$ is the sensitivity of the derivative price to the price of the underlying stock and is called the **delta** (Δ) of the derivative.
- Finally, the term $f_{xx}(t,x)$ is the second derivative of the derivative price with respect to the stock price or the rate of change of the derivative's delta with respect to the stock price and is called the derivative's **gamma** (Γ).
- Using Π to indicate the derivative value and S to indicate stock price, the Black-Scholes PDE can be written as

$$\Theta = -\frac{1}{2}\sigma^2 S^2 \Gamma - rS\Delta + r\Pi. \tag{36}$$

Organization

Heat Equation

- The Black-Scholes PDE (34) is similar to a very well-studied equation with applications in science, engineering, and finance.
- This equation, called heat equation or diffusion equation, is

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}. \tag{37}$$

- The equation considers a quantity u that depends on time and on an underlying quantity x such that the time-sensitivity of quantity u is proportional to the second derivative of u with respect to the underlying quantity x.
- Like all differential equations, the exact solution of the above equation depends on a boundary condition.

Solution of Heat Equation

- Suppose the boundary condition specifies the initial value of u at time 0 as a known function of the underlying quantity x:

$$u(0, x) = h(x). \tag{38}$$

- Then, the solution of the heat equation is

$$u(t, x) = \int_{-\infty}^{\infty} h(y) \frac{1}{2\sqrt{\pi \lambda t}} e^{-(x-y)^2/4\lambda t} \mathrm{d}y. \tag{39}$$

Connecting Heat Equation and Black-Scholes PDE

- With a few mathematical tricks, the Black-Scholes PDE (34) can be transformed into the heat equation.
- One of these tricks is changing the time variable from current time t to remaining time to maturity $\tau = T - t$ because the boundary condition for Black-Scholes PDE is a terminal condition (35) whereas the solution (39) of the heat equation requires an initial condition (38).
- With the change of variable, the function h in (35) plays the same role as the the function h in (38).
- For a call option, we replace h with the intrinsic value of the call in (35) to get

$$h(x) = f(T, x) = (x - K)^{+} \text{ for all } x. \tag{40}$$

Black-Scholes Formula for Call

- The solution of the Black-Scholes PDE (34) with the boundary condition (40) can then be obtained by performing the integration in (39) with appropriate adjustments.
- The solution decomposes into two integrals, each of which can be converted to an integral of the normal probability density (of the form e^{-x^2}).
- The result is the Black-Scholes formula for the price of a European call option:

$$\begin{aligned} f(t, S_t) = & S_t \Phi\left(\frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\ & - Ke^{-r\tau}\Phi\left(\frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \end{aligned} \tag{41}$$

- Here, Φ is the cumulative normal distribution function. Sometimes the symbol N is used instead of the symbol Φ.

Black-Scholes Formula for Put

- The put-call parity or an alternative boundary condition can be used to derive the formula for the price of a European put option:

$$
\begin{aligned}
& K e^{-r\tau} \Phi\left(-\frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
& - S_t \Phi\left(-\frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \qquad (42)
\end{aligned}
$$

Dividends

- If the underlying stock has a known dividend yield q, these formulas can be adjusted to get European call price

$$\begin{aligned} S_t e^{-q\tau} \Phi \left(\frac{\log(S_t/K) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right) \\ - K e^{-r\tau} \Phi \left(\frac{\log(S_t/K) + (r - q - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right). \end{aligned} \tag{43}$$

- The European put price is

$$\begin{aligned} K e^{-r\tau} \Phi \left(-\frac{\log(S_t/K) + (r - q - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right) \\ - S_t e^{-q\tau} \Phi \left(-\frac{\log(S_t/K) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right). \end{aligned} \tag{44}$$

Organization

Options on Stock Indices

- Options on stock indices, called index options, are often traded on exchanges.
- These options can be used for portfolio insurance to limit downside risk.
- One practical issue in such risk management is determining the number of index options to use.
- If the stock index and the portfolio have a one-to-one return relationship on average (one percent increase in one is usually associated with one percent increase in the other), then the notional amount of options must equal the value of the portfolio being hedged.
- However, if the relationship is not one-to-one, the notional amount of options used for hedging equals the value of the portfolio multiplied by the beta of the portfolio.

Options on Stock Indices

- Index options can be priced just like options on individual stocks by using the dividend yield on the index in place of stock dividend yield.
- As with all derivative pricing, the dividend yield must be based on expected dividends whose ex-dividend dates lie between pricing date and the expiration of the option.
- Note that index volatility is usually less than the average volatility of individual stocks in the index.
- Option on an index is therefore, cheaper than options on individual stocks.
- Often, the process of calculating index option price from inputs is reversed.
- That is, the prices of index options are used to infer the market's expectations of the dividend yield on the index.

Currency Options

- Currency options are mostly traded in over-the-counter market. A call or a put allows the option owner to buy or sell a specified amount of a foreign currency at a pre-specified exchange rate.
- These options can also be priced like options on a stock with the modification that the interest rate on the foreign currency replaces the dividend yield on the underlying stock.
- Again, the task of pricing can be reversed to infer implied volatility about the exchange rate from the option price.

Organization

Futures Options

- A futures option is an option with a futures contract as the underlying asset.
- That is, a call (put) gives the option holder the right to buy (sell) a futures contract at a fixed price. These options are naturally related to the options on the asset underlying the futures contract.
- For example, a call on a futures contract on stock XYZ is related to a call on stock XYZ.
- Traders may sometimes prefer to use futures options rather than options directly on underlying assets if the futures market is more liquid and allows transactions with lower transaction costs.

Futures Options

- Black's model (from the same Fischer Black that was a co-discoverer of Black-Scholes formula) is a popular method for pricing Futures options.
- It assumes that the futures price follow lognormal distribution like stock prices in the Black-Scholes model.
- The pricing formulas are similar to Black-Scholes formulas with the difference that the stock price is replaced by the Futures price F_0, the dividend yield with the risk-free interest rate and stock volatility with futures price volatility.

Futures Options

- The price c of a European call option on futures and the price p of a European put option on futures are given by

$$c = e^{-rT}\left[F_0 N(d_1) - KN(d_2)\right], \tag{45}$$

$$p = e^{-rT}\left[KN(-d_2) - F_0 N(-d_1),\right] \tag{46}$$

where

$$d_1 = \frac{\ln F_0/K + \sigma^2 T/2}{\sigma\sqrt{T}}, d_2 = \frac{\ln F_0/K - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

- σ is the volatility of the futures price, T is the time to maturity of the option, and N (same as Φ) is the cumulative normal distribution function.

Organization

Risk-Neutral Pricing

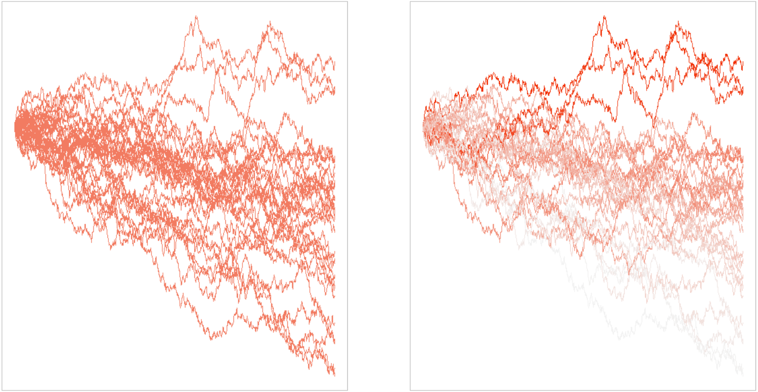
- A powerful technique for valuing derivatives is risk-neutral pricing. The technique can be used when the payoffs of the assets being priced can be replicated by dynamically trading other assets with known prices.
- The technique can be used when there are enough assets with related payoffs to pin down payoff of the asset being priced.
- For example, a stock and a bond can be traded to pin down price of a call or put option on the stock. In this case, risk neutral pricing can be used.
- If other assets cannot completely capture the risk of the asset being priced, risk-neutral pricing cannot be used.
- The risk-neutral technique assumes that investors do not assign any risk premium to the risk of the underlying stock.
- That means that the stock, any derivatives on the stock, and riskless bonds (or even risky bonds), all earn expected return of risk-free rate. This assumption simplifies math.

Risk-Neutral Pricing

- We assume that asset prices are Brownian motions (not necessarily standard Brownian motion).
- A particular history of the price of a stock is an example of a sample path of a Brownian motion.
- The Brownian motion in itself is not a single path, but a collection of infinitely many sample paths or alternative stock price histories, only one of which is realized at random.
- **Drift** and **volatility** are two properties of a Brownian motion. It turns out that the volatility of a Brownian motion can be inferred precisely from just one sample path.
- For a Brownian motion with volatility σ, if the sample path of length T is divided into n subperiods and the change in each subperiod is squared and all these squares added up, the result will converge to $\sigma^2 T$ as n becomes large.
- This means that a Brownian motion's volatility is unique and there can be no disagreements about it. The same is not true for drift.

Risk-Neutral Pricing

- The drift of a Brownian motion captures the rate at which Brownian motion increases on average.
- This cannot be determined from one sample path. It depends on the aggregation of all sample paths and the probabilities of these paths.
- If one assigns higher probability to sample paths trending higher and lower probability to sample paths trending lower, drift will be higher than if one did the opposite.
- Thus, drift depends on the probabilities assigned to different sample paths.
- Two investors that disagree on the probabilities of different sample paths may arrive at different values of drift.



30 paths of a Brownian motion with negative drift

The same paths reweighted according to the Girsanov formula

Figure: Visualisation of the Girsanov theorem — The left side shows a Wiener process with negative drift under a canonical measure P; on the right side each path of the process is colored according to its likelihood under the martingale measure Q. The density transformation from P to Q is given by the Girsanov theorem. Source: Wikipedia

Risk-Neutral Pricing

- A change in the probabilities of different sample paths is a change in the probability measure.
- A change in probability measure can change the drift of a Brownian motion without changing the volatility of the Brownian motion.
- For example, a change in probability measure can transform a stock price process with expected return of 5% and volatility of 30% to a process with expected return of 10% and volatility of 30%.
- But what values of drift can we obtain with a change of probability measure?
- A remarkable mathematical result called **Girsanov's theorem** states that any value of drift can be obtained (without changing volatility) with an appropriate change of probability measure.

Girsanov's Theorem

- Girsanov's theorem essentially allows one to transform the drift of a Brownian motion by using an appropriate probability measure.
- Suppose there is a Brownian motion with a given drift μ_1 under a probability measure P and we want to express it as a Brownian motion with drift μ_2.
- Girsanov's theorm states that there exists a probability measure Q under which the Brownian motion has drift μ_2.
- Recall that Radon-Nikodym theorem (Theorem 1) suggests that one can move from one probability measure to an equivalent probability measure by multiplying probabilities with a positive random variable with zero mean.
- That is, we scale up some probabilities and scale down some other probabilities (total probability has to remain unchanged at one).
- In continuous time, the multiplication should be with a martingale stochastic process.

Girsanov's Theorem

- Given any Brownian motion with a nonzero drift under a probability measure, Girsanov's theorem specifies an equivalent probability measure under which the Brownian motion is a **martingale** (has no drift).
- We can use also this method to find an equivalent probability measure under which the original Brownian motion has any desired drift, such as risk-free rate.
- How does this help in pricing derivatives? Note that the price of a European derivative security is the solution to equations (34) and (35).
- A solution that satisfies these equations is correct, regardless of how one arrives at the solution.
- A "trick" to solving these equations is based on the observation that the equations depend on σ, the stock price volatility but the drift of the stock price is absent from these equations.

Risk-Neutral Pricing

- Two investors who disagree on their beliefs about the drift of stock price movement will arrive at the same derivative price as long as they agree on volatility.
- This means we can delegate derivative pricing to an investor who believes (or pretends) that stock price drift equals risk-free rate.
- The catch is that the volatility shouldn't change. Girsanov's theorem guarantees such a probability measure.
- This provides an intuitive way to solve (34) and (35) from the point of view of a risk-neutral investor who requires an expected return equal to the risk-free return from the riskless bond and from the stock underlying the derivative.
- That is, the investor expects a drift equal to the risk-free rate from both assets.
- Since a derivative on stock can be expressed as a combination of stock and bond, the drift of the derivative's price must also be risk-free rate.

Risk-Neutral Pricing

- A risk-neutral investor can use risk neutral pricing with two features:
    1. the stock value grows at the risk-free rate on average and
    2. all cash flows are discounted at the risk-free rate.
- The first feature allows the investor to predict stock price probability distribution at option expiration.
- This distribution is used to calculate the derivative cash flow distribution at expiration and therefore, the expected value of the derivative cash flow at expiration.
- The second feature allows the investor to discount the derivative's expected cash flow to determine derivative price today.

Risk-Neutral Pricing

- One application of risk neutral pricing is the construction of multiple step binomial trees for numerical pricing of derivatives.
- The probabilities for up and down movement of stock price are calculated using risk-neutral pricing assumption.
- We now apply this principle derivative pricing under Black-Scholes assumption about stock price movement.

Risk-Neutral Pricing

- Consider the price of a derivative security at time t with stock price x.
- Let the derivative expire at time $T = t + \tau$ and let h be the derivative's payoff at expiration.
- Using the lognormal distribution (19) for log of stock price at expiration, $y = \ln(S_T)$, and replacing drift μ with risk-free rate r, the derivative price is

$$f(t,x) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} h(e^y) e^{-\frac{1}{2}\left(\frac{y-\ln(x)-(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)^2} \mathrm{d}y. \quad (47)$$

- Substituting $z = \frac{y-\ln(x)-(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}$, we get

$$f(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[e^{-r\tau} h(xe^{z\sigma\sqrt{\tau}+(r-\sigma^2/2)\tau})\right] e^{-\frac{z^2}{2}} \mathrm{d}z. \quad (48)$$

- For a European call option with strike K, substituting $h(x) = \max(x - K, 0)$, the price is given by

$$c(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\max(xe^{z\sigma\sqrt{\tau} - \sigma^2\tau/2} - Ke^{-r\tau}, 0) \right] e^{-\frac{z^2}{2}} \mathrm{d}z. \tag{49}$$

- This expression provides Black-Scholes formula with some algebra

Risk-Neutral Pricing

- Risk-neutral pricing is powerful as it can be used to numerically calculate the expectation in (48) for any European derivative, by simulating a normal random variable or by any other method.
- The derivative price obtained this way must agree with the derivative price that the investor will obtain as solution to (34) and (35) and therefore, all investors will agree with the price obtained this way.
- Note that this method would not have worked if equations (34) and (35) involved stock price drift.
- Investors don't have to be actually risk neutral to use this method.
- If a no-arbitrage condition states a certain relation between the prices of securities A, B, and C, then all investors must agree on that condition even though they may disagree on future prices of A, B, and C.

Organization

Feynman-Kac

- We have seen two methods of pricing derivatives.
- One is deriving Black-Scholes Partial Differential Equation (PDE) and then seeking a mathematical solution to the PDE (34) with the terminal condition (35).
- The other is risk neutral method, which motivates binomial tree pricing, and yields (48).
- Since both give the same solution, what is the link between the two?
- *Feynman-Kac Formula* provides such a link between PDEs with boundary conditions and many general stochastic processes.
- We will not present the result here. However, the result has important implications for pricing derivatives.
- Suppose no-arbitrage arguments are used to derive a partial differential equation for a derivative.

Feynman-Kac

- Then, one pricing method is to solve that PDE numerically with some boundary conditions.
- An alternative method is to apply Feynman-Kac formula to transform the price into an expectation and calculate the expectation.
- Numerical methods exist for solving PDEs and simulations can be used to numerically calculate expectation.
- For any derivative pricing problem, we can use the method that is more convenient for that problem.