Risk Measures

- The risks of an option or a portfolio in general are the factors on which the value of the option or the portfolio depends.
- For call or put options on stocks, option value depends on the price of the underlying stock, the time to maturity, the risk-free rate, and the volatility of the underlying stock.
- Investors may want to
  - identify risks
  - decide how much risk they are willing to take
  - reduce the risks
- A risk measure of an option (or a portfolio) with respect to a parameter x is the ratio of the change in the value of the option (or portfolio) to a small change in x. The risk measures for options are traditionally represented by Greek letters.

Applications of Risk Measures

- If an investor has a position in an option and wants to neutralize its risk, the investor can take another position in some other assets with opposite risk measure.
- An investor can create an option position synthetically without actually buying the option by creating a portfolio using other assets with the same risk measures as the desired option.
- We saw in our discussion of put-call parity that a synthetic stock can be created using calls, puts, and bonds.

Covered and Naked Positions

- Consider an investor who has written a call option on a stock.
- The investor's position is covered if the investor owns the stock that can be delivered if the call is exercised.
- The investor's position is naked if the investor does not hold the stock.
- A covered position hedges the investor's exposure if the call is exercised.
- A naked position covers the investor if the call is not exercised.
- However, the investor cannot be sure if the call will be exercised or not so both naked and covered positions are extreme positions and provide imperfect hedge.

Example: A stock is trading at $25. After one period, the stock price will increase to $30 or decrease to $20. Consider a short position in a call option with strike price of $25. What will be the future values of your portfolio if your portfolio consists of

a) Naked call
b) Covered call

If the call is naked, the portfolio value equals the value of the short call, and it will be worth -$5 if the stock price rises to $30 and worth $0 if the stock price falls to $20.

If the call is covered, the portfolio value equals the value of the short call and the stock. It will be worth -$5 + $30 = $25 if the stock price rises to $30 and worth $0 + $20 = $20 if the stock price falls to $20.

Thus, a naked short call position results in a loss if the stock price rises, while a covered short call position results in a loss if the stock price falls. None is riskless.

Stop-Loss Strategy

- Both naked call and covered call are imperfect hedges whose effectiveness depends on the moneyness of the option.
- A dynamic strategy changes portfolio as the moneyness of the option changes.
- Simplest dynamic strategy:
  - covered position when the option is in the money
  - naked position when the option is out-of-money
  - to hedge a short position in a call, buy the underlying stock when its price rises above the exercise price and sell when its price falls below the exercise price
  - this is called stop-loss strategy.
- The stop-loss strategy is not very effective in practice because it is difficult to predict if the stock price will increase or decrease relative to exercise price.
- This means that stock will be purchased when the price is already more than exercise price and stock will be sold at prices below exercise price. This will lead to loss on average. This is not a problem as gain/loss in trading offset gain/loss on short call position.
- The problem is that offsetting is not perfect and total gain/loss is uncertain.
- This uncertainty is undesirable as a hedging strategy should reduce risk.

Example: See Excel spreadsheet for simulation of stop-loss strategy.

Delta

- The Delta (Δ) of an option measures the sensitivity of the option with respect to the stock price change.
- It measures the ratio of the change in the value of the option to the change in stock price when the change in stock price is infinitesimal and other parameters are kept unchanged.
- Recall that the value of a European call option is given by the Black-Scholes formula:

$$d_1 = \frac{ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

$$c = SN(d_1) - Xe^{-rT}N(d_2)$$

$$p = Xe^{-rT}N(-d_2) - SN(-d_1)$$

- Taking derivative of call price with respect to stock price S, we get

$$\Delta = N(d_1)$$

- For a European put option,

$$\Delta = -N(-d_1) = N(d_1) - 1$$

Note: The expressions provided for option risk measures assume Black-Scholes pricing formula. If an option does not follow Black-Scholes assumptions or if the option is not American, explicit formulas for option risk measures may not exist. In such cases, numerical methods such as binomial tree pricing can be used to estimate option risk measures.

Dynamic Delta Hedging

- To hedge an option exposure against stock price movements, a position in stocks is held and adjusted dynamically as the delta of the option changes.
- The idea in hedging is to make the delta of the portfolio zero.
- The underlying stock has a delta of +1.
- If there is a long position in a call, hedging it requires shorting delta number of stocks.
- To synthetically create a call position, one must buy delta number of stocks.
- As option delta changes with time, the number of stocks held or shorted must also change.
- With frequent trading, most of the risk associated with stock price can be mitigated.
- However, trading may involve transaction costs.
- In practice, the frequency of trading is based on a compromise between the risk of position and the transaction costs.

Example: A European call option on a non-dividend paying stock has a delta of 0.6. If you have shorted 50 contracts (each contract is for 100 shares), what position in stocks would you take to Delta hedge your portfolio? Alternatively, what position in corresponding puts can you take? If the stock price increases, will you buy stock or sell stock?

Delta of the short position in 50 contracts = $-0.6 \times 50 \times 100 = -3000$

To make portfolio Delta-neutral, we need a position with Delta = 3000. Since each share of stock has a Delta of +1, we need to buy 3000 shares of stock.

Alternatively, we could hedge using put options. Delta of a put option = Delta of a call option – 1 = 0.6 – 1 = -0.4. Number of put options needed to make the portfolio Delta-neutral is 3000/(-0.4) = -7500. That is, 7,500 put options should be shorted. If each put contract is for 100 shares, we need to short 75 put contracts.

Theta

- The Theta (Θ) of an option measures the sensitivity of the option with respect to the passage of time.
- Theta of options is usually negative.
- For a European call option on a non-dividend paying stock:

$$\Theta = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

- and for the corresponding put,

$$\Theta = -\frac{S_0 N'^{(d_1)}\sigma}{2\sqrt{T}} + rKe^{-rT}N(d_2)$$

Gamma

- Delta hedging is perfect only if it is carried out continuously.
- In practice, if hedging position is updated at an interval, the delta of the position changes in that interval making hedge imperfect.
- Gamma (Γ) of an option measures the sensitivity of the delta of the option with respect to the stock price.
- It is a measure of the convexity of the option price as a function of stock price.
- For a European call or put option on a non-dividend paying stock:

$$\Gamma = -\frac{N'(d_1)}{S_0\sqrt{T}}$$

- The positive value of gamma shows that delta of an option increases as the underlying stock price increases.
- That is why a Delta hedged portfolio must be dynamically adjusted.

- Gamma and Theta of an option are related so that the change in the value of a delta-neutral portfolio is approximately:

$$\Delta\Pi = \Theta\Delta t + \frac{1}{2}\Gamma\Delta S^2$$

Vega

- The Vega of a portfolio is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset.
- The Vega of the underlying asset itself is zero.
- For a European call or put option, Vega is given by

$$\mathrm{V} = S_0\sqrt{T}N'(d_1)$$

- The volatility is forward-looking volatility.
- A popular measure of market's estimate of future volatility is the volatility index **VIX** which estimates 30-day volatility based on prices of options on S&P 500 index.
- The VIX index is often referred to as the "Fear Index."

Rho

- The rho of a portfolio is the rate of change of the value of the portfolio with respect to the interest rate.
- For a European call option on a non-dividend paying stock:

$$\rho = KTe^{-rT}\sqrt{T}N(d_2)$$

- For a European put option on a non-dividend paying stock:

$$\rho = -KTe^{-rT}\sqrt{T}N(-d_2)$$

You have written a put option on a stock. What can you do to hedge your exposure?

A. Long stock
B. Short stock

Short stock (A put option has negative Delta but a short position in put has a positive Delta. So we need a position with negative Delta. A short position in stock has negative Delta.)

You have written a call option on a stock. What can you do to hedge your exposure?

A. Long stock
B. Short stock

Long stock (A call option has positive Delta but a short position in call has negative Delta. So we need a position with a positive Delta. A long position in stock has positive Delta.)