Problem Set VI Solutions

QF 430: Introduction to Derivatives
Due Wednesday, December 14

Please submit neatly handwritten or typed answers. You can submit electronically a single pdf file through Canvas. Show your steps or reasoning. Do not round too much in intermediate calculation steps.

1 Black-Scholes Model and Risk-Neutral Valuation

Problem 1.1.

(a) Let $f(t,x)$ be the price at time t of a European derivative security on a stock with price x. Write the Black-Scholes partial differential equation (PDE) that f must satisfy. Assume risk-free rate is r and the volatility of the stock is σ. What does each term in the equation mean?

(b) The Black-Scholes PDE does not specify whether the derivative is a call option, a put option, or some other derivative. How do you incorporate the derivative payoff when using Black-Scholes PDE to price a derivative?

(c) (Not graded, optional for those interested) There is an exotic European-style derivative whose price equals $Ze^{kt}x^2$ where t is time, x is stock price, and Z and k are two constants. The stock price follows a geometric Brownian motion with volatility σ equal to 20% per annum. The risk-free rate is 3% per annum continuously compounded. Determine k. The following calculus facts will be useful. If a does not depend on x and b and k do not depend on t, then,

$$\frac{\partial ax^2}{\partial x} = 2ax, \frac{\partial^2 ax^2}{\partial x^2} = 2a, \text{ and } \frac{\partial be^{kt}}{\partial t} = bke^{kt}.$$

Solution.

(a) The Black-Scholes partial differential equation is

$$f_t(t,x) = -\frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) - rxf_x(t,x) + rf(t,x)$$

or

$$rf(t,x) = f_t(t,x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) + rxf_x(t,x)$$

where $f(t,x)$ is the value of the derivative when the underlying asset value is x and time is t, $f_t(t,x)$ is the rate of change of value of the derivative with the passage of time, $f_x(t,x)$ is the sensitivity of the derivative price to the price of the underlying asset, $f_{xx}(t,x)$ is the second derivative of the derivative price with respect to the price of the underlying asset, σ is volatility of the return on the underlying asset, and r is the risk-free rate.

(b) The derivative payoff, which distinguishes different derivatives such as calls and puts, is specified as a terminal boundary condition. The derivative price depends on the solution of the Black-Scholes PDE along with the terminal boundary condition.

(c) Substituting the expression for price and its partial derivatives in the Black-Scholes PDE, we get

$$0.03 \times Ze^{kt}x^2 = Zke^{kt}x^2 + \frac{1}{2} \times 0.2^2 \times 2Ze^{kt}x^2 + 0.03 \times 2Ze^{kt}x^2$$

which simplifies to

$$0.03 \times Ze^{kt}x^2 = (k + 0.1) \times Ze^{kt}x^2.$$

$k = -0.07$ and the derivative price is $Ze^{-0.07t}x^2$.

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Problem 1.2. A non-dividend-paying stock is trading at \$75 and has volatility of 40% per annum. Consider an option on the stock with strike price \$70 and maturity six months. The risk-free rate is 5% per annum (continuously compounded).

(a) What is the price of the option if it is a European call? Show your calculations.

(b) What is the price of the option if it is a European put? Show your calculations.

(c) What is the price of the option if it is an American call?

(d) Suppose you have some private information. That information will soon become public and stock price will increase by 20% to \$90. You can invest in stock to make a 20% profit. If you instead invest in call options priced in part (a), what percentage profit can you earn?

Solution.

(a) The price is given by Black-Scholes formula:

$$\begin{aligned}
&S\Phi\left(\frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}\Phi\left(\frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
&= 75\Phi\left(\frac{\log(\frac{75}{70}) + (0.05 + \frac{0.4^2}{2})0.5}{0.4\sqrt{0.5}}\right) - 70e^{-0.05\times 0.5}\Phi\left(\frac{\log(\frac{75}{70}) + (0.05 - \frac{0.4^2}{2})0.5}{0.4\sqrt{0.5}}\right) \\
&= 75\Phi(0.47374) - 70e^{-0.025}\Phi(0.19089) \\
&= 75 \times 0.68216 - 70e^{-0.025} \times 0.57570 = \$11.858.
\end{aligned}$$

(b) The price can be obtained using the Black-Scholes formula for put:

$$
\begin{aligned}
&Ke^{-r\tau}\Phi\left(-\frac{\log(S/K)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)-S\Phi\left(-\frac{\log(S/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)\\
&=70e^{-0.05\times0.5}\Phi\left(-\frac{\log(\frac{75}{70})+(0.05-\frac{0.4^2}{2})0.5}{0.4\sqrt{0.5}}\right)-75\Phi\left(-\frac{\log(\frac{75}{70})+(0.05+\frac{0.4^2}{2})0.5}{0.4\sqrt{0.5}}\right)\\
&=70e^{-0.025}\Phi(-0.19089)-75\Phi(-0.47374)\\
&=70e^{-0.025}\times0.42430-75\times0.31784=\$5.130.
\end{aligned}
$$

Alternatively, using put-call parity, put price equals call price plus bond price minus stock price:

$$11.858+70e^{-0.05\times0.5}-75=\$5.130.$$

(c) The price of the American call equals the price of the European call on a non-dividend-paying stock so the price is \$11.858.

(d) The price of call option will change to:

$$
\begin{aligned}
&90\Phi\left(\frac{\log(\frac{90}{70})+(0.05+\frac{0.4^2}{2})0.5}{0.4\sqrt{0.5}}\right)-70e^{-0.05\times0.5}\Phi\left(\frac{\log(\frac{90}{70})+(0.05-\frac{0.4^2}{2})0.5}{0.4\sqrt{0.5}}\right)\\
&=90\Phi(1.11834)-70e^{-0.025}\Phi(0.83550)\\
&=90\times0.86829-70e^{-0.025}\times0.79828=\$23.646.
\end{aligned}
$$

Your profit will be (\$23.646 − \$11.858)/\$11.858 × 100% = 99.41%

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Problem 1.3. Consider Black-Scholes formulae for prices of European call and put options with strike K each, maturity T each on a non-dividend-paying stock with price S and volatility σ, with risk-free rate r. The formulas are written in terms of quantities d_1 and d_2 used to calculate the probabilities of the normal distribution. If the volatility of the stock becomes large and approaches infinity,

(a) what values do d_1 and d_2 approach?

(b) what value does the call price approach?

(c) what value does the put price approach?

Solution.

(a) $d_1=\frac{\log(S/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}=\frac{\log(S/K)+r\tau}{\sigma\sqrt{\tau}}+\frac{1}{2}\sigma\sqrt{\tau}$. The first term approaches zero and the second term approaches ∞ as σ approaches ∞, so d_1 approaches ∞.

$d_2=\frac{\log(S/K)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}=\frac{\log(S/K)+r\tau}{\sigma\sqrt{\tau}}-\frac{1}{2}\sigma\sqrt{\tau}$. The first term approaches zero and the second term approaches $-\infty$ as σ approaches ∞, so d_2 approaches $-\infty$.

(b) As d_1 approaches ∞, $\Phi(d_1)$ approaches 1. As d_2 approaches $-\infty$, $\Phi(d_2)$ approaches 0. Call price given by $S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$ approaches stock price S. The intuition is that future stock price will either be very high with a low probability or almost zero with a high probability. If it is very high, you will get the stock and what you pay as the exercise price will be negligible in comparison. If stock price is very low in future, you won't miss anything by not getting the stock. So the option is as valuable as having the stock.

(c) As d_1 approaches ∞, $\Phi(-d_1)$ approaches 0. As d_2 approaches $-\infty$, $\Phi(-d_2)$ approaches 1. Put price given by $Ke^{-r\tau}\Phi(-d_2) - S\Phi(-d_1)$ approaches present value of exercise price $Ke^{-r\tau}$. The intuition is that future stock price will either be very high with a low probability or almost zero with a high probability. If it is almost zero, your payoff equals almost the full strike price. If the stock price is very high, you get nothing but the probability of that is very low. So the option is as valuable as having the present value of the strike price.

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Problem 1.4. Derivatives can be valued using risk neutral valuation. The spreadsheet "Risk Neutral Valuation.xlsx" simulates 100,000 values of stock price (assuming log normal distribution) at expiration of derivative. Each stock price results in a derivative payoff. These derivative payoffs are discounted and averaged to get derivative value. The cells in column C currently calculate payoff of a European call with strike price \$50. Value a different derivative under the following assumptions:
Change the current stock price in cell G1 to 20 + the number of the first letter of your last name + the number of the second letter of your last name. Assume A = 1, B = 2, and so on until Z = 26. For example, change current stock price to 39 if your name is Jane Doe.
Change the formula in cells in column C to value a derivative whose payoff after six months is zero if the stock price after six months is less than the current stock price and is the square of the difference in prices if the stock price after six months is more than today's stock price. For example, if the stock price today is \$50 and the stock price at expiration is \$47.2, you will get \$0. If the stock price at expiration is \$53.6, you will get $\$3.6^2 = \$12.96$.
The risk-free rate is 5% per annum compounded continuously and the stock volatility is 20% per annum. Provide the stock price and the calculated derivative value. Also provide an image of the first few (about 5) lines of the spreadsheet or provide values in a table as follows:

Current Stock Price	
Derivative value	

Simulation Number	Price at Maturity	Derivative Payoff
1		
2		
3		
4		
5		

Solution. The following table gives derivative price D (in \$) for each possible current price D (in \$). The value provided by risk-neutral pricing simulation may differ from these prices by at most 3 percent.

S	D	S	D	S	D	S	D	S	D
22	7.14	32	15.08	42	25.85	52	39.75	62	56.34
23	7.78	33	15.98	43	27.12	53	41.37	63	58.07
24	8.43	34	16.92	44	28.39	54	42.72	64	60.23
25	9.18	35	17.98	45	29.8	55	44.42	65	62.32
26	9.94	36	19.07	46	31.09	56	46.02	66	64.09
27	10.68	37	20.12	47	32.51	57	47.68	67	66.04
28	11.48	38	21.21	48	33.84	58	49.39	68	67.76
29	12.33	39	22.29	49	35.32	59	51.19	69	69.85
30	13.23	40	23.54	50	36.69	60	52.83	70	72.02
31	14.14	41	24.73	51	38.2	61	54.59	71	73.99

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Problem 1.5. Volatility smile or volatility skew refers to the relation between implied volatility and strike price (Hull chapter 19). Quotes for selected call options on Meta stock on December 6, 2022 are provided below. Determine implied volatility for each option and plot as a function of strike price. Meta does not pay dividends. Assume Meta stock price is \$114.12. Assume the interest rate is 3.81% per annum with continuous compounding. The expiration date of all options is December 30, 2022. To determine implied volatility, input all parameters in Black-Scholes formula and adjust volatility until the price given by the formula matches the price quote. If you use Excel, Goal Seek function can be helpful. The Excel formula =DAYS(DATE(2022,12,30),DATE(2022,12,6)) returns the number of days to expiration. You may use the spreadsheet "Black-Scholes Price.xlsx".

Strike	Price
105	12.56
108	9.31
111	7.66
114	5.75
117	4.60
120	3.10
123	2.42

Solution. The implied volatility values are in the following table:

Strike price (\$)	105	108	111	114	117	120	123
Implied volatility (per annum)	61.5%	48.9%	50.5%	47.6%	49.2%	45.5%	47.2%

A plot follows.

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2 Option Risk Measures

Problem 2.1. A stock is trading at \$40 and its volatility is 50%. You own a put option with a strike price of \$52 and one year to maturity. The risk-free rate is 2% per annum compounded continuously.

(a) How many shares of stock should you buy or short so that a small change in stock price has no effect on your portfolio (option and the position in stocks)?

(b) Given the portfolio you chose in (a), if stock price increases by a large amount (such as \$40 to \$50), would the value of the portfolio stay same, increase or decrease?

(c) Given the portfolio you chose in (a), if stock price decreases by a large amount (such as \$40 to \$30), would the value of the portfolio stay same, increase or decrease?

Solution.

(a) Put delta is

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\begin{aligned}
& -\Phi\left(-\frac{\log(S/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
&= -\Phi\left(-\frac{\log(40/52)+(0.02+\frac{1}{2}\times 0.5^2)\times 1}{0.5\sqrt{1}}\right) \\
&= -\Phi(0.23473) = -0.59279.
\end{aligned}
$$

You should buy 0.59279 shares of stock.

(b) Delta hedging assumes a linear relation between option price and stock price such that if stock price changes, option price changes by an amount equal to the product of the stock price change and the put delta. However option price is convex in stock price so if stock price increases by a large amount, the final option price is more than what delta hedging assumes. Thus, the portfolio value increases. If stock price increases from \$40 to \$50, the delta hedged portfolio's value increases from \$39.18 (put price of 15.47 plus 0.59279 times stock price of 40) to \$40.10 (put price of 10.46 plus 0.59279 times stock price of 50).

(c) Delta hedging assumes a linear relation between option price and stock price such that if stock price changes, option price changes by an amount equal to the product of the stock price change and the put delta. However option price is convex in stock price so if stock price decreases by a large amount, the final option price is more than what delta hedging assumes. Thus, the portfolio value increases. If stock price decreases from \$40 to \$30, the delta hedged portfolio's value increases from \$39.18 to \$40.18 (put price of 22.39 plus 0.59279 times stock price of 30).

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Problem 2.2. A trader holds a derivative whose value depends on the price of a stock. To manage this risk, the trader takes positions (long or short) in the underlying stock and in two options on the stock. The Delta, Gamma, and Vega of the derivative and the two options are given below. What position in the two options and the stock (in addition to the derivative already held) makes the trader's combined position Delta neutral, Gamma neutral, and Vega neutral? Assume you can buy and short any number, including fractional units. Recall that the Greek parameters are additive: Greek for 5 options is 5 times Greek of an option, Greek for a shorted security is negative of the Greek for the security, and Greek for two securities in a portfolio is the sum of the Greeks for the two securities. Also recall that a stock's Gamma and Vega are zero.

	Delta	Gamma	Vega
Derivative	-0.5	600	-240
Option 1	0.05	20	4
Option 2	-0.04	30	8

Solution. The Delta, Gamma, and Vega of stock are 1, 0, and 0. If n_0 stocks, n_1 option 1, and n_2 option 2 are added to make the portfolio Delta neutral, Gamma neutral, and Vega neutral, then, the equations for Delta, Gamma, and Vega of the combined position are:

$$\begin{array}{rcccccccc} -0.5 & + & n_0 & + & 0.05n_1 & - & 0.04n_2 & = & 0, \\ 600 & & & + & 20n_1 & + & 30n_2 & = & 0, \\ -240 & & & + & 4n_1 & + & 8n_2 & = & 0. \end{array}$$

Dividing the middle equation by 5, we get

$$4n_1 + 6n_2 = -120.$$

The third equation can be rewritten as

$$4n_1 + 8n_2 = 240.$$

Subtracting from the last equation the previous one, we get $n_2 = 180$. Substituting in the last equation, we get $n_1 = -300$. Substituting values of n_1 and n_2 in the first equation, we get $n_0 = 22.7$. The trader should buy 22.7 shares of stock, short 300 option 1, and buy 180 option 2. ■

3 Credit Derivatives

Problem 3.1. A credit default swap provides protection on bonds with principal value of \$100 million. The CDS spread is 60 basis points per year paid semiannually, that is through payments of 30 basis points every six months. The CDS is settled in cash. A default occurs after 1 year and five months. The price of the cheapest to deliver bond is 40% of its face value shortly after the default. What are the cash flows of the CDS from the perspective of the CDS buyer? List each amount with its timing.

Solution. The cash flows are:

- At 6 months: CDS premium = \$100 million $\times 0.30\%$ = \$300,000 is paid by the protection buyer to the protection seller.
- At 1 year: CDS premium of \$300,000 is paid by the protection buyer to the protection seller.
- At 1 year, 5 months: Insurance payment = \$100 million $\times(1 - 40\%)$ = \$60 million is paid by the protection seller to the protection buyer. CDS premium of $\$300,000 \times 5/6 = \$250,000$ is paid by the protection buyer to the protection seller. Thus, the net cash flow is a payment of \$59.75 million by the protection seller to the protection buyer.

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Problem 3.2. Calculate the credit default swap spread for a 3-year CDS where the spread is paid at the end of each year and the accrued spread is paid at the time of default. Assume that defaults can occur in the middle of every year, that is, after 0.5, 1.5, or 2.5 years. Assume that the beliefs about default are as follows:

- There will be a default after 0.5 years with probability 2%
- There will be a default after 1.5 years with probability 3%
- There will be a default after 2.5 years with probability 4%
- There will be no default within three years with probability 91%

Assume that the term structure of the risk-free rate is flat at 1% per annum continuously compounded rate and that the recovery rate is 40%.

Solution. The principal amount does not matter for this problem so for convenience we assume that the principal amount is \$1. Suppose the CDS spread is s. The present value of expected CDS premium payments calculation follows:

Time (years)	Probability	Amount	Present Value
0.5	0.02	0.5s	$0.01se^{-0.01\times0.5} = 0.00995s$
1	0.98	s	$0.98se^{-0.01\times1} = 0.970429s$
1.5	0.03	0.5s	$0.015se^{-0.01\times1.5} = 0.014777s$
2	0.95	s	$0.95se^{-0.01\times2} = 0.931189s$
2.5	0.04	0.5s	$0.02se^{-0.01\times2.5} = 0.019506s$
3	0.91	s	$0.91se^{-0.01\times3} = 0.883105s$
Total			2.8288s

The present value of CDS insurance payments calculation follows:

Time (years)	Probability	Amount	Present Value
0.5	0.02	0.6	$0.012e^{-0.01\times0.5} = 0.01194$
1.5	0.03	0.6	$0.018e^{-0.01\times1.5} = 0.017732$
2.5	0.04	0.6	$0.024e^{-0.01\times2.5} = 0.023407$
Total			0.05308

Equating the present value of CDS premium payments to CDS insurance payments, we get $2.8288s = 0.05308$ or $s = 0.01876$ or 1.876%. For a rough estimate, the average default probability per year is 3% and recovery rate is 40% so CDS premium should be around $3\% \times (1 - 40\%) = 1.8\%$. ∎