Solution to Problem Set III

QF 430: Introduction to Derivatives
Due Tuesday, April 14

- Please submit neatly handwritten or typed answers. You can submit electronically a **single pdf** file through Canvas.
- While you may have done your work in one or more spreadsheets, do not turn in spreadsheets. The final product is a text document, possibly with equations, formulas, tables, graphs etc.
- Show your steps or reasoning.
- When you show an arbitrage strategy, in addition to describing the initial trades, you must also show that the strategy cannot result in any losses.
- A spreadsheet is provided for Black-Scholes pricing of European options.
- START WORK EARLY. The problems take time.

1 Securitization

Problem 1.1. *Borrowers Yash and Zara each are supposed to pay off their loans of \$100,000 each next year. The bank which issued the loans has bundled the two loans and sold the cash flow to investors Alice and Bob. Alice will receive first \$120,000 and Bob will receive any remaining cash flow.*
If a borrower does not default, then he or she repays the loan in full. If a borrower defaults, 40% of the loan amount is recovered. Yash and Zara are both likely to default with probability 10%. Calculate the expected cash flows of Alice and Bob (no discounting needed) under each of the following probability distributions. Hint: Do not calculate total expected cash flow and then divide it. First allocate cash flows for each set of loan outcomes and then determine expected values.

(a) *Defaults are independent.*

	Zara defaults	Zara does not default
Yash defaults	1%	9%
Yash does not default	9%	81%

(b) *Defaults are imperfectly positively correlated.*

	Zara defaults	Zara does not default
Yash defaults	5%	5%
Yash does not default	5%	85%

(c) *Defaults are perfectly positively correlated.*

	Zara defaults	Zara does not default
Yash defaults	10%	0%
Yash does not default	0%	90%

Solution.

(a) When no one defaults, Alice gets \$120,000, Bob gets \$80,000. When one borrower defaults, Alice gets \$120,000, Bob gets \$20,000. When both borrowers default, Alice gets \$80,000, Bob gets \$0.

Alice's expected payoff $= 0.81 \times \$120,000 + 0.18 \times \$120,000 + 0.01 \times \$80,000 = \boxed{\mathbf{\$119{,}600}}$.

Bob's expected payoff $= 0.81 \times \$80,000 + 0.18 \times \$20,000 + 0.01 \times \$0 = \boxed{\mathbf{\$68{,}400}}$.

(b) Alice's expected payoff $= 0.85 \times \$120,000 + 0.1 \times \$120,000 + 0.05 \times \$80,000 = \boxed{\mathbf{\$118{,}000}}$.

Bob's expected payoff $= 0.85 \times \$80,000 + 0.1 \times \$20,000 + 0.05 \times \$0 = \boxed{\mathbf{\$70{,}000}}$.

(c) Alice's expected payoff $= 0.9 \times \$120,000 + 0.1 \times \$80,000 = \boxed{\mathbf{\$116{,}000}}$.

Bob's expected payoff $= 0.9 \times \$80,000 + 0.1 \times \$0 = \boxed{\mathbf{\$72{,}000}}$.

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Problem 1.2. *Provide two economic benefits of securitization. For each, explain why a person or an organization is better off with securitization and why the same outcome can not be achieved without securitization. Mention one adverse consequence of securitization.*

Solution. Benefits:

(a) Securitization spreads risks across a large set of investors, resulting in better risk sharing. Investors improve risk-return tradeoffs.

(b) Securitization improves social welfare by allowing investments (such as launching a new business, construction of houses, roads, or bridges, research and development) that are good (positive NPV) investments but cannot get financed in absence of securitization.

(c) Securitization improves liquidity and allows investor in a project or cause to exit the investment by selling the security. In contrast, a lender finds it difficult to sell loans owned in absence of securitization.

(d) Price of a securitized product aggregates information of multiple investors and can be used as a metric of the performance of the underlying business or project. This allows others to make better investment decisions.

Adverse consequences:

(a) Risk sharing dilutes the incentive of any investor to perform due diligence and monitoring of borrower and can result in poor investments.

(b) Large separation between borrowers and investors makes investors reduces investors' awareness of the risks of securitized products.

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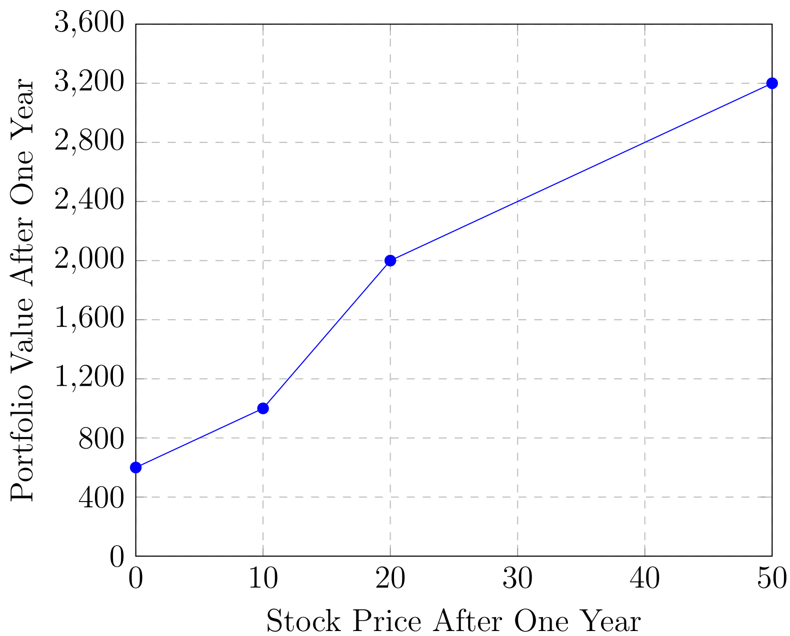
2 Options Markets

Problem 2.1. *An investor buys 100 shares of a stock, shorts 60 call options on the stock with strike price of $20 and buys 60 put options on the stock with strike price of $10. All options are one-year European options. Draw a diagram illustrating the value of the investor's portfolio as a function of the stock price after one year.*

Solution.



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Problem 2.2. *Options on S&P 500 index are traded on Chicago Board Options Exchange (CBOE). These are European style options. On March 30, 2020, S&P 500 index value was 2626.65. The price quotes for six options on S&P 500 have been mixed up. Match each price quote with the corresponding option. You can assume that longer maturity options are more* *valuable. Bid and ask quotes are better indicators of option price than last sale which may be a stale price.*

Option	Ticker	Type	Expiration	Strike
1	SPX200417C02000000	Call	4/17/2020	2000
2	SPX200417P02000000	Put	4/17/2020	2000
3	SPX200619C02000000	Call	6/19/2020	2000
4	SPX200619P02000000	Put	6/19/2020	2000
5	SPX200619C02500000	Call	6/19/2020	2500
6	SPX200619P02500000	Put	6/19/2020	2500

Quote	Last Sale	Net	Bid	Ask	Volume	Implied Volatility	Delta	Gamma	Open Interest
A	602.7	0	665	671.9	0	0.579	0.8679	0.0003	36122
B	57.3	-14.35	56.3	57.7	7155	0.5933	-0.1364	0.0003	71122
C	628.7	39.4	627.7	635.1	4351	0.7552	0.9558	0.0002	32025
D	162.5	-33.05	164.3	166.6	6322	0.455	-0.3775	0.0007	71207
E	11.8	-12.7	11	11.7	8098	0.8133	-0.0554	0.0002	71701
F	275.47	23.57	273.8	280.5	2167	0.4487	0.623	0.0007	51442

Fill in the following table and explain your reasoning.

Option	Quote
1	
2	
3	
4	
5	
6	

Solution. The matching of options and quotes follows:

Option	Quote
1	**C**
2	**E**
3	**A**
4	**B**
5	**F**
6	**D**

Price quotes A, C, and F with positive Deltas are for calls. Price quotes B, D, and E with negative Deltas are for puts. The most valuable call is in price quote A. It must be for the June call with strike of 2000. Among the other two call quotes C and F, the lower quote F must be for June call with strike of 2500. The quote is too low for April call with strike of 2000 (call value cannot be less than spot minus present value of strike, i.e., 2626.65 minus present value of 2000). Among puts, April put with strike of 2000 is less valuable than June put with strike of 2500, which is less valuable than June put with strike of 2000. ■

3 Option Price Properties

Problem 3.1. *A European call option on a stock with a strike price of \$50 and expiring in six months is trading at \$14. A European put option on the stock with the same strike price and expiration as the call option is trading at \$2. The current stock price is \$60 and a \$1 dividend is expected in three months. Zero coupon risk-free bonds with face value of \$100 and maturing after 3 months and 6 months are trading at \$99 and \$98, respectively. Identify the arbitrage opportunity open to a trader.*

Solution. The put-call parity with known dividends requires that

$$\text{Call Price} + \text{PV of Dividends} + \text{PV of Strike} = \text{Put Price} + \text{Stock Price}.$$

But on this case, we find that

$$\begin{aligned}&\text{Call Price} + \text{PV of Dividends} + \text{PV of Strike}\\ &= 14 + 1 \times 99/100 + 50 \times 98/100 = 63.99\\ &> 2 + 60\\ &= \text{Put Price} + \text{Stock Price}.\end{aligned}$$

The trader should $\boxed{\textbf{sell call}}$ for \$14, $\boxed{\textbf{borrow}}$ \$49.99, $\boxed{\textbf{buy put}}$ for \$2, and $\boxed{\textbf{stock}}$ for \$60. This results in a net cash flow of \$1.99. Out of the loan amount of \$49.99, \$0.99 will be repaid using the dividend of \$1 in three months. The remaining \$ should be repaid with \$50 after six months. If the stock price after six months is more than \$50, call will be exercised, otherwise put will be exercised. Either way the trader will sell the stock for \$50 and use that to repay the loan. ■

Problem 3.2. *Suppose that p_1, p_2, and p_3 are the prices of European put options with strike prices K_1, K_2, and K_3, respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that*

$$p_2 \leq 0.5(p_1 + p_3)$$

(Hint: Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2.)

Solution. Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2. At maturity, this portfolio's value will be non-negative.

Stock Price	Value from K_1 Put	Value from K_2 Puts	Value from K_3 Put	Total
$S_T < K_1$	$K_1 - S_T$	$-2(K_1 - S_T)$	$K_3 - S_T$	$K_1 + K_3 - 2K_2 > 0$
$K_1 < S_T < K_2$	0	$-2(K_1 - S_T)$	$K_3 - S_T$	$(K_1 + K_3 - 2K_2) + (S_T - K_1) > 0$
$K_2 < S_T < K_3$	0	0	$K_3 - S_T$	$K_3 - S_T > 0$
$S_T > K_3$	0	0	0	0

If $p_2 > 0.5(p_1 + p_3)$, setting up the portfolio will result in $\boxed{\textbf{positive cash flow}}$ and the preceding table shows that there will be $\boxed{\textbf{no losses the future}}$. This is an arbitrage opportunity. Hence, we must have $p_2 \leq 0.5(p_1 + p_3)$. ■

Problem 3.3. *A stock that does not pay dividend is trading at \$20. A European call option with strike price of \$15 and maturing in one year is trading at \$6. An American call option with strike price of \$15 and maturing in one year is trading at \$8. You can borrow or lend money at any time at risk-free rate of 5% per annum with continuous compounding. Devise an arbitrage strategy.*

Solution. American and European calls options on non-dividend-paying stocks must trade at the same price. Since American call is priced higher, $\boxed{\textbf{sell an American call}}$ and $\boxed{\textbf{buy a European call}}$. This results in a cash flow of $\$8 - \$6 = \$2$. If American call is not exercised, there is no risk of loss. If American call is exercised at maturity, exercise European call at the same time for an offsetting transaction with zero total cash flow. If American call is exercised before maturity, borrow a stock to sell it to American call owner for \$15. Invest \$15. Get back \$15 plus interest at maturity, use \$15 to exercise European call and return the borrowed stock and keep any interest as extra profit. ■

Problem 3.4. *Consider a stock that does not pay dividend. A one-year European put option with strike \$40 is trading at \$2.40 and a one-year European put option with strike \$50 is trading at \$12.30. The risk-free interest rate is 5% per annum with continuous compounding. Construct an arbitrage strategy.*

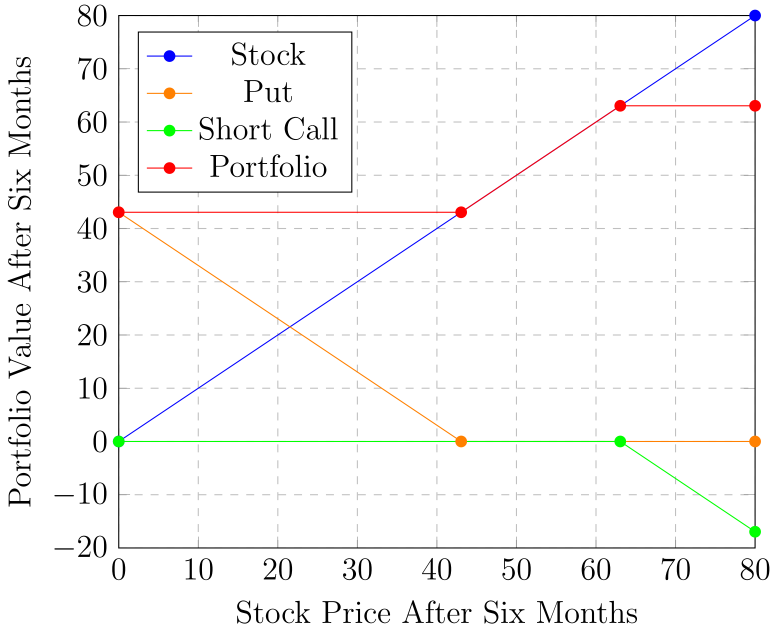
Solution. The maximum difference between the cash flows from the two options at maturity can be \$10 when both options are exercised. Therefore, the difference in the prices of the two options should be no more than present value of \$10. that is $10e^{-0.05} = \$9.51$. The difference between \$12.30 and \$2.30, \$9.90 is too high. $\boxed{\textbf{Sell the option trading at \$12.30}}$ and $\boxed{\textbf{buy the option trading at \$2.40}}$ and invest \$9.51. The net cash flow is \$0.39. The amount invested will grow to \$10 after a year. If the stock price is below \$40, both options are exercised and \$10 will be sufficient to cover the need for \$10 difference in exercise prices. If the stock price is between \$40 and \$50, the option sold is exercised. Get a stock, sell it. The proceeds plus \$10 are enough to pay \$50. If the stock price is above \$50, neither option is exercised. ■

4 Financial Engineering with Options

Problem 4.1. *You own a stock trading at \$50. The stock does not pay any dividend. You are worried about the risk of stock price movement but cannot sell or short the stock. You want to reduce your exposure to stock price movement but do not want to incur any cash expense for doing so. Use a collar so that the value of your hedged stock after six months will lie within a range. That is, lock a price range between A and B such that your position will be worth A if stock price after six months is less than A, B if stock price after six months exceeds B, and equal to the stock price if stock price after six months is between A and B. What collar strategy will provide you a \$20 range (that is, B - A = 20) at zero cost? Ignore commissions or transaction costs. The risk-free rate is 5% per annum with continuous compounding and the volatility of the stock is 40%. Structure the collar using European options. Assume option prices are given by Black-Scholes formulas.*

Solution. Buy a put for strike A and sell a call for strike $A+20$. This will result in the desired portfolio value. Choose K so that the put and the call have the same price. Using Black-

Scholes pricing formula and trial and error, $A = 43.05$. That is, **buy a put with strike of \$43.05** and **sell a call with strike of \$63.05**. Both are priced at \$2.112.



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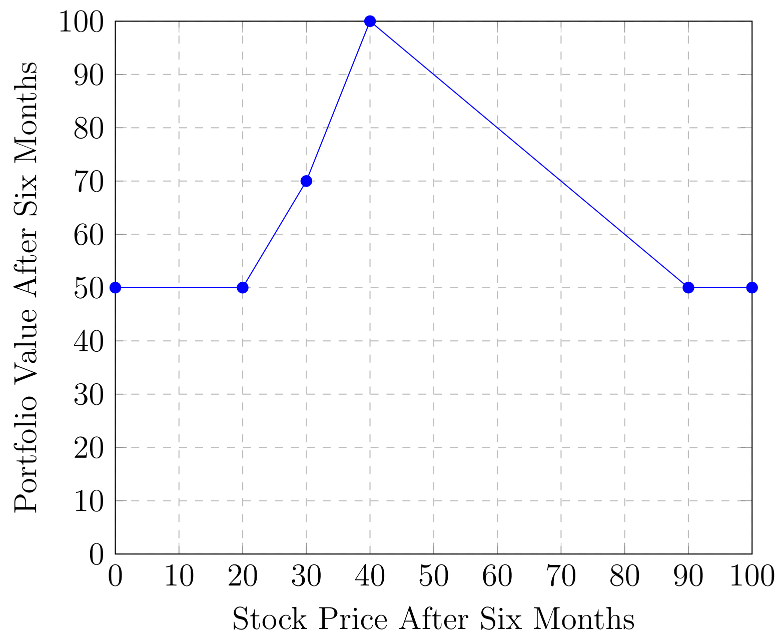
Problem 4.2. *A stock is trading at \$50. An investor wants you to design her a portfolio whose value after one year will depend on the price of the stock at that time as shown in the figure below. The portfolio value varies linearly between circular markers. How can you structure such a portfolio for her? How much will it cost her? Assume option prices are given by Black-Scholes formulas. Assume that the stock does not pay dividend and its volatility is 40%. The risk-free rate is 5% per annum with continuous compounding. Ignore transaction costs and any other fees. To ensure that your answer is correct, check the value of the portfolio at maturity for a few values of stock price. You don't need to report these checks.*



Solution. $\boxed{\textbf{Buy a bond}}$ that matures after six months with face value of $\boxed{\textbf{\$50}}$. $\boxed{\textbf{Buy two calls}}$ with strike price of $\boxed{\textbf{\$20}}$. $\boxed{\textbf{Buy another call}}$ with strike price of $\boxed{\textbf{\$30}}$. $\boxed{\textbf{Sell four calls}}$ with strike price of $\boxed{\textbf{\$40}}$. $\boxed{\textbf{Buy a call}}$ with strike price of $\boxed{\textbf{\$90}}$. All options are European and mature in six months. The bond price is \$48.7655. Call prices are \$30.4948, \$20.8611, \$12.2873, and \$0.1622 for strikes of \$20, \$30, \$40, and \$90, respectively. The net cost of the portfolio is $\boxed{\textbf{\$81.6294}}$.

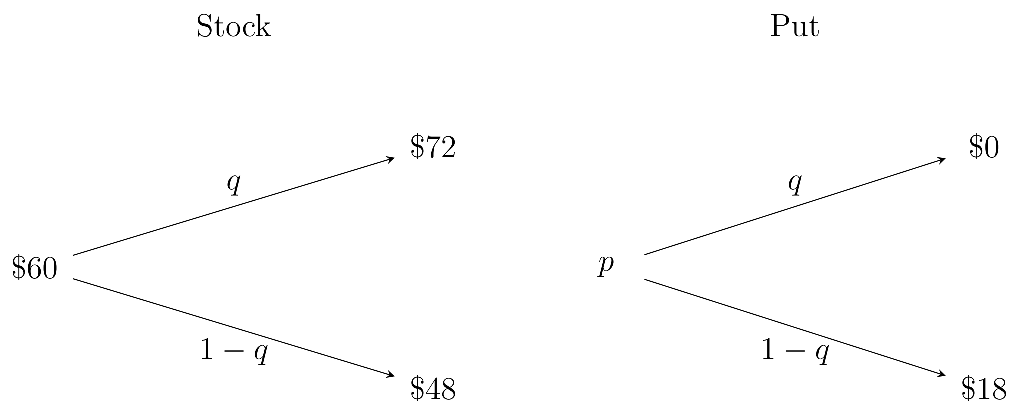
There can be other portfolios with the same payoffs. Here is one. $\boxed{\textbf{Buy a bond}}$ that matures after six months with face value of $\boxed{\textbf{\$50}}$. $\boxed{\textbf{Buy one put}}$ with strike price of $\boxed{\textbf{\$90}}$. $\boxed{\textbf{Sell four puts}}$ with strike price of $\boxed{\textbf{\$40}}$. $\boxed{\textbf{Buy one put}}$ with strike price of $\boxed{\textbf{\$30}}$. $\boxed{\textbf{Buy two puts}}$ with strike price of $\boxed{\textbf{\$20}}$. All options are European and mature in six months. The bond price is \$48.7655. Put prices are \$0.0010, \$0.1204, \$1.2997, and \$37.9401 for strikes of \$20, \$30, \$40, and \$90, respectively. The net cost of the portfolio is $\boxed{\textbf{\$81.6294}}$. Or based on Put-Call Parity, the bond can be replaced with a stock, a short call and a long put, both with strikes of \$50.

There was a typo in the problem earlier. If the investment horizon is taken to be a year instead of six months, the bond price is \$47.5615, call prices are \$31.0069, \$22.0069, \$14.4882, and \$1.0682 for strikes of \$20, \$30, \$40, and \$90, respectively, put prices are \$0.0315, \$0.5438, \$2.5374, and \$36.6788 for strikes of \$20, \$30, \$40, and \$90, respectively, and portfolio cost is \$74.6975. ■

5 Pricing Options with Binomial Trees

Problem 5.1. *A stock that not pay any dividend is currently trading at \$60. After one period, the price will either increase by 20% or decrease by 20%. The risk-free rate is 8% per period (not continuously compounded). Consider a European put option on the stock with* *strike price of \$66. Set up a replicating portfolio and price the option.*

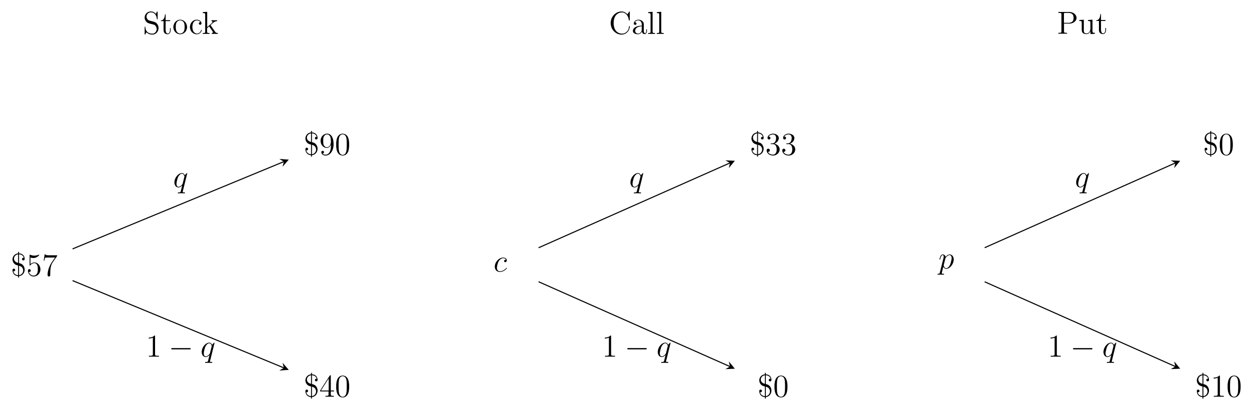
Solution. The stock and the option binomial trees follow. The put will be worth $\max(66 - 72, 0) = \boxed{\mathbf{\$0}}$ if the stock price rises to \$72 and $\max(66 - 48, 0) = \boxed{\mathbf{\$18}}$ if the stock price falls to \$48.



The put delta is $\frac{0-18}{72-48} = -\frac{3}{4}$. Consider a portfolio with $\boxed{\textbf{one put and } \frac{3}{4} \textbf{ stocks}}$. Its value after one period will be \$54 if the stock price rises to \$72 ($\frac{3}{4} \times 72 + 0$) or falls to \$48 ($\frac{3}{4} \times 48 + 18$). Its current price equals the present value of \$54, that is, $54/1.08 = \$50$. The price of $\frac{3}{4}$ stocks is $60 \times \frac{3}{4} = \$45$ so the value of the put is $50 - 45 = \boxed{\mathbf{\$5}}$. ■

Problem 5.2. *A stock that not pay any dividend is currently priced at \$57. After one period, the stock price will increase to \$90 or decrease to \$40. A bond that will pay \$100 after one period is trading at \$95. Find the risk-neutral probabilities of stock price up and down movements. Value an at-the-money call option. Value a European put option with strike \$50.*

Solution. The stock and the option binomial trees follow. The call will be worth $\max(90 - 57, 0) = \boxed{\mathbf{\$33}}$ if the stock price rises to \$90 and $\max(40 - 57, 0) = \boxed{\mathbf{\$0}}$ if the stock price falls to \$48. The put will be worth $\max(50 - 90, 0) = \boxed{\mathbf{\$0}}$ if the stock price rises to \$90 and $\max(50 - 40, 0) = \boxed{\mathbf{\$10}}$ if the stock price falls to \$40.



The risk-neutral probability that the stock price rises to $90 is

$$q = \frac{57 \times \frac{100}{95} - 40}{90 - 40} = \boxed{\mathbf{0.4}}.$$

Call value is

$$(0.4 \times 33 + 0.6 \times 0) \times \frac{95}{100} = \boxed{\mathbf{\$12.54}}$$

and the put value is

$$(0.4 \times 0 + 0.6 \times 10) \times \frac{95}{100} = \boxed{\mathbf{\$5.70}}.$$

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Problem 5.3. *A stock that does not pay dividend is trading at $100. The stock price will increase by 10% or decrease by 10% in one year. After that, the stock price will increase or decrease by 20% in the second year. The risk-free interest rate is 5% per annum with continuous compounding. Value a two-year American put option with strike price of $102. Note that risk-neutral probabilities or replicating portfolios may differ across two periods. You must also check for early exercise.*

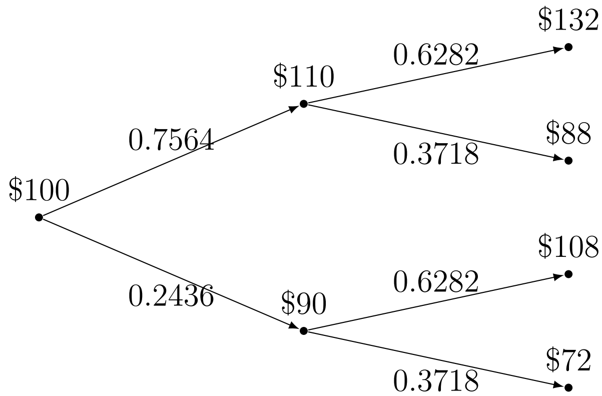
Solution. The risk-neutral probability of stock price increase in the first year is

$$\frac{e^{0.05\times 1} - 0.9}{1.1 - 0.9} = \boxed{\mathbf{0.7564}}.$$

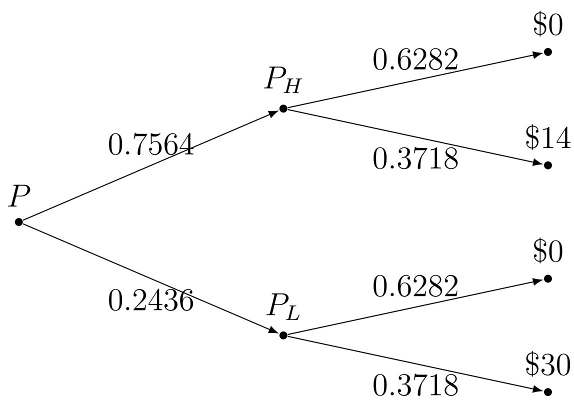
The risk-neutral probability of stock price increase in the second year is

$$\frac{e^{0.05\times 1} - 0.8}{1.2 - 0.8} = \boxed{\mathbf{0.6282}}.$$

The stock tree follows:



The American put option tree follows:



If the stock price is \$110 after one year, the value of holding on to the put option is

$$(0 \times 0.6282 + 14 \times 0.3718)e^{-0.05\times 1} = \$4.9516.$$

The option is out-of-money so early exercise is not optimal. Option value is $P_H = \boxed{\mathbf{\$4.9516}}$.
If the stock price is \$90 after one year, the value of holding on to the put option is

$$(0 \times 0.6282 + 30 \times 0.3718)e^{-0.05\times 1} = \$10.6106.$$

The value from exercising the option is higher, $\max(102 - 90, 0) = \$12$, so early exercise is optimal and the option value is $P_L = \boxed{\mathbf{\$12}}$. The value of holding on to the put option today is

$$(4.9516 \times 0.7564 + 12 \times 0.2436)e^{-0.05\times 1} = \$6.3437.$$

The value from exercising the option is lower, $\max(102 - 100, 0) = \$2$ so early exercise is not optimal. Option value is $P = \boxed{\mathbf{\$6.3437}}$. ■