Interest Rates

Interest Rates Measure How Cash Flows Are Translated Across Time

- Financial transactions involve cash payment at one point of time in exchange for cash receipt at another point of time.
- The excess of the amount received over the amount paid originally is called the **interest**.
- The interest divided by the original amount is the **interest rate** for that investment for that period.
- It is customary to specify the time period with the rate as investments for different time periods generally offer different interest rates.
- If an investment of $50 returns $60 after 2 years, then the interest rate is ($60 - $50)/$50 = 20% over two years.
- If the interest rate is given, we can find out the final amount to which an initial investment grows:

$$Future\ value\ after\ one\ period = Initial\ amount\ \times (1 + rate\ per\ period)$$

- Suppose an amount is invested for multiple periods and the same interest rate applies each period. The amount earned after the first period is reinvested for the second period, the amount earned after the second period is reinvested for the third period, and so on.
- The original amount along with the cumulative interest is received after the final period.
- The interest earned in one period itself earns interest in the next period. This is called **compounding**.
- Suppose the period length is t, the amount invested in A, the interest rate per period is r, and there are n periods. Then, the amount grows to
  - A × (1+r) after one period
  - A × $(1+r)^2$ after two periods
  - A × $(1+r)^3$ after three periods, and
  - A × $(1+r)^n$ after n periods. This represents an interest rate of $(1+r)^n$ - 1 over n periods.
- Thus, interest rate of r per period (repeatedly) is equivalent to interest rate of $(1+r)^n$ - 1 over n periods.

$$Future\ value\ after\ n\ periods = Initial\ amount\ \times (1 + rate\ per\ period)^n$$

Note: The above formula is valid only if the interest rate is same each period. It applies even if n is not an integer.

Example 1: How much does \$100 grow to over 5 years if the interest rate is 5% every six months? How much does \$273 grow to over six months if the interest rate is 1% every month? How much does \$1000 grow to over one day if the interest rate is 0.2% per week?

Interest Rate Conversion

Interest rates over different periods are **equivalent** if they result in the same future value of a specified amount over a specified time period.

Example 2: Which of the following pairs of interest rates are equivalent?

1. 5% every six months, 10.25% every year
2. 1% every month, 13.2% every year
3. 2% every quarter, 4.04% every six months

- An interest rate of r over time period t is equivalent to interest rate of $(1 + r)^{T/t} - 1$ over time period T. Here T can be less than or greater than t.
- Thus, an interest rate can be specified in multiple equivalent ways: per year rate, per quarter rate, per day rate, etc.

$$\begin{aligned} &Interest\ rate\ conversion: Rate\ of\ r\ over\ time\ interval\ t \\ &\equiv\ Rate\ of\ (1+r)^{\frac{T}{t}} - 1\ over\ time\ interval\ T \end{aligned}$$

Example 3: Convert 3.5% per six months to equivalent annual rate. Convert 3.5% per six months to equivalent monthly rate.

Terminology

- The time period over which interest is earned is called the **compounding period**.
- The number of compounding periods per year is called the **compounding frequency**.
- Semiannual compounding refers to a compounding period of six months while quarterly compounding refers to a compounding frequency of four per year.
- When compounding period is not a year, it is common to mention the interest rate as a per annum rate (called **stated rate** or **Annual Percentage Rate, APR**) along with the compounding frequency.
- The stated rate is calculated as the interest rate per compounding period multiplied by the compounding frequency.

$$\begin{aligned} &Stated\ rate\ per\ annum \\ &\qquad = Interest\ rate\ per\ compounding\ period\ \times \#compounding\ periods\ per\ year \end{aligned}$$

- It is important to realize that the stated rate is not the interest rate one will earn over a year.
- Rather the stated rate is just a way of expressing the interest rate per compounding period.
- The interest rate per compounding period equals the stated rate multiplied by the number of years in the compounding period.
- The formulas provided above can then be used to determine equivalent interest rate over any other period.

$$
\begin{aligned}
&\mathit{Interest\ rate\ per\ compounding\ period} \\
&\qquad = \mathit{Stated\ rate\ per\ annum} \times \#\mathit{years\ in\ compounding\ period} \\
&\qquad = \mathit{Stated\ rate\ per\ annum}/\#\mathit{compounding\ periods\ per\ year}
\end{aligned}
$$

Example 4: Convert 12% per annum compounded semiannually to an equivalent annual rate. Convert 8% per annum compounded quarterly to an equivalent monthly rate.

Continuous Compounding

- An interest rate is expressed as a **continuously compounded interest rate** if the compounding frequency approaches the limit of infinity.
- If the interest rate is R per annum with compounding frequency of n, the equivalent annual rate is $(1 + R/n)^n - 1$.
- As n approaches infinity, this number approaches $e^R - 1$, where e is Euler's constant, approximately equal to 2.71828.

$$Future\ value\ after\ n\ years = Initial\ amount\ \times e^{n \times continuously\ compounded\ rate\ per\ annum}$$

$$Interest\ rate\ over\ n\ years = \ e^{n \times continuously\ compounded\ rate\ per\ annum} - 1$$

Example 5: If the interest rate is 8% per year continuously compounded, what is the interest rate per quarter? If the interest rate is 10% per annum with semiannual compounding, what is the continuously compounded interest rate?

Multiple Interest Rates

- Interest rate for an investment or loan depends on
  - the horizon of the investment or loan – a three-month rate can be different from a six-month rate.
  - the date of the investment or loan – the three-month rate from January to April can be different from the three-month rate from February to May.
  - the point of time when the rate is fixed – for a three-month investment or loan from February to May, the interest rate fixed a month earlier in January may be different from the rate that would be fixed in February.
  - the risk of the loan – loans secured with collateral may have lower interest rates than unsecured loans.
- In a **floating-rate loan**, the payments through the life of the loan change as interest rate changes.
- In a **fixed-rate loan**, the interest rate for all future payments is predetermined.

Common Interest Rates

- Treasury rates
    - Treasury rates are interest rates investors earn on government issued securities such as Treasury bills and Treasury bonds.
    - These securities are considered risk-free, so these rates are good candidates for risk-free rates.
    - However, interest rates on these securities are affected by the special tax and regulatory provisions associated so treasury rates are no longer used as measures of risk-free rate.
- LIBOR
    - London Interbank Offered Rate (LIBOR) is an unsecured short-term borrowing rate between banks.
    - There are many LIBOR rates corresponding to different currencies and different investing periods. LIBOR rates are used as benchmark interest rates for many derivative products.
    - They were widely used as the risk-free rate in derivatives pricing.
    - However, after the financial crisis of 2008, vulnerabilities identified in the process by which LIBOR is determined have revealed that the rate is susceptible to manipulation.

- Overnight rates
    - Overnight rates are the rates at which banks borrow from and lend to each other on an overnight basis.
    - In the United States, banks can earn overnight rates on funds deposited with Federal Reserve or pay overnight rates on funds borrowed from the Federal Reserve.
    - The weighted average rate of these transactions is called the effective **federal funds rate**.
    - The corresponding rate in the United Kingdom is sterling overnight index average (**SONIA**) and in the eurozone is euro overnight index average (**EONIA**).

- Repo rates
    - In a repurchase agreement (repo), a financial institution sells securities to another financial institution and agrees to buy back the securities at a later date.
    - This lets the first financial institution effectively borrow funds.
    - The exchange of cash for securities reduces the credit risk – the risk that one party will renege on its promise.
    - The most common repo rate is for overnight repo.
    - The secured nature of transaction makes repo rates low-risk rates and these rates tend to be smaller than unsecured rates such as LIBOR.

Swaps

- A swap is an agreement in which two parties agree to exchange two series of cash flows.
- In an **interest rate swap**, one party takes a loan with a fixed interest rate by getting cash today and promising to pay **fixed interest rate** and principal back in the future.
- The other party takes a different loan of the same size with variable interest rate by getting cash today and promising to pay **variable interest rate** and principal back in the future.
- The cash flows are **netted** so that there is no net transfer of cash today and principal amounts are not transferred at maturity. Just the difference between the fixed interest rate and the variable interest rate is used to determine the cash flows between the parties.
- The variable interest rate is tied to a benchmark rate such as the LIBOR rate.
- Instead of considering the interest payments to be for the same long-term loan, each interest payment is considered for a short-term loan and the loans are considered to be rolled over.
- This assumption makes no difference to the cash flows but reduces the credit risk. If one party defaults on its interest payment (not principal because principals are never exchanged), the loans are not renewed.

- The fixed interest rate agreed upon represents the expectations about the value of the variable rate over a long period but with much lower credit risk.
- This rate, called the **swap rate** can be used as the risk-free rate for derivative contracts.
- As an example, a five-year swap rate can be the fixed rate agreed upon for a five-year swap in which interest payments are exchanged every three months and the variable interest payment is based on three-month LIBOR.

Overnight Indexed Swaps

- In an overnight indexed swap, the variable interest rate is based on overnight interest rates and thus represents little credit risk.
- However, payments are exchanged at longer intervals such as one month or three months.
- The variable interest for this period is calculated using the geometric average of the overnight interest rates for the period.
- The fixed rate agreed upon reflects expectations of future overnight rates and is called the overnight indexed swap rate or OIS rate.

Risk-free Rate Developments

- Different versions of LIBOR and related rates (collectively IBORs) are being replaced with alternative measures of risk-free rates.
- International Swaps and Derivatives Association (ISDA) coordinates these transitions.
- Different countries or regional economic systems have determined their own measures of risk-free rates.
- In some cases, these rates are published by central banks (for example, New York Federal Reserve Bank or Bank of Japan) and in others by a large financial institution (such as Refinitiv in Canada).
- These new measures will be used in all new derivative contracts.
- In addition, ISDA also specifies how older contracts still in existence should be modified to replace LIBOR with new rates (fallback rates).

- Most new overnight risk-free rates are based on overnight lending transactions (such as repo transactions) between large financial institutions.
- Longer-term risk-free rates are swap rates associated with long-term swaps on these overnight rates.

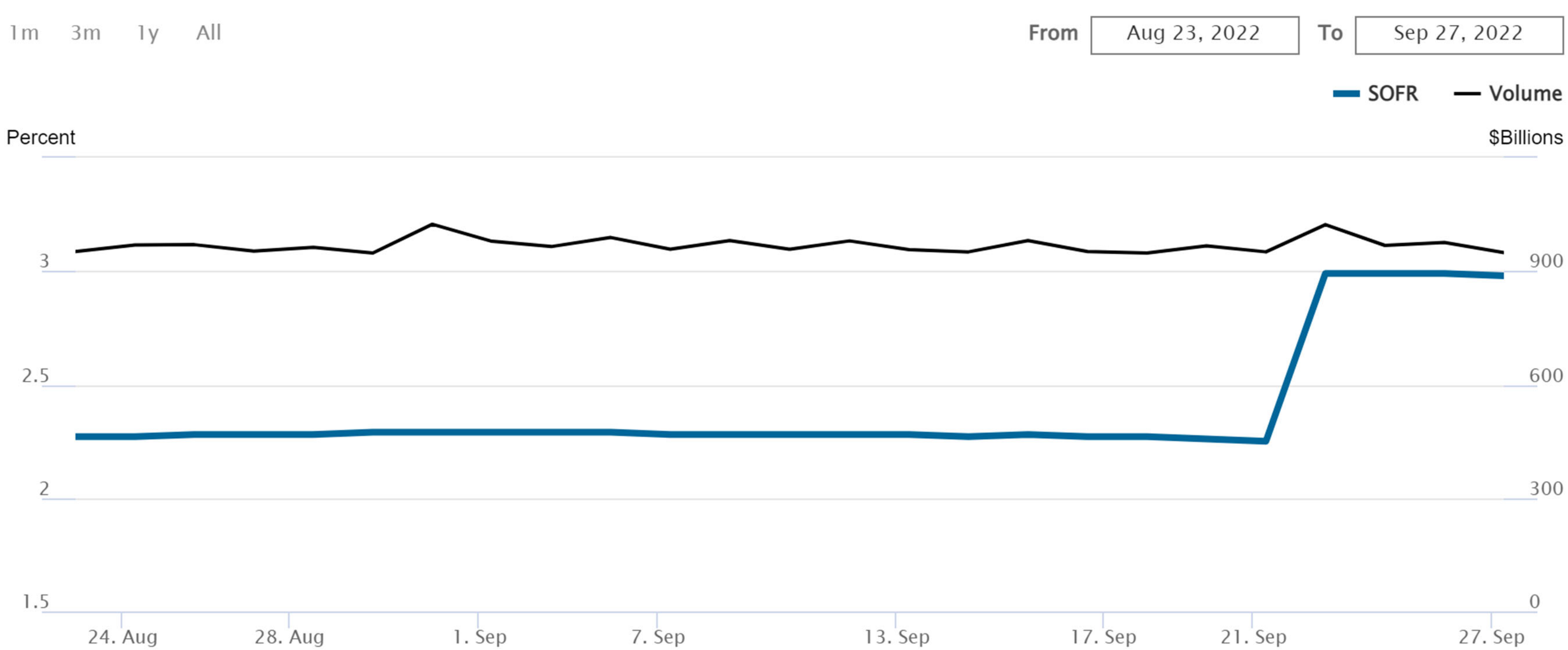
Currency	IBOR	Fallback Rate
AUD	BBSW	AONIA
CAD	CDOR	CORRA
CHF	CHF LIBOR	SARON
EUR	EUR LIBOR/EURIBOR	€STR
GBP	GBP LIBOR	SONIA
HKD	HIBOR	HONIA
JPY	JPY LIBOR/TIBOR/Euroyen TIBOR	TONA
USD	USD LIBOR	SOFR

New York Fed Reference Rates

- The New York Fed produces a number of reference rates for money markets.
- Unsecured Rates
  - Effective Federal Funds Rate (EFFR) is calculated using data on overnight federal funds transactions of domestic banks and U.S. branches and agencies of foreign banks.
  - Overnight Bank Funding Rate (OBFR) is calculated using the same federal funds transaction data that are included in the EFFR, as well as certain overnight Eurodollar transactions and certain overnight deposit transactions that are placed at domestic bank branches, including unsecured borrowings of U.S. dollars booked at international banking facilities and at offshore branches that are managed or controlled by a U.S. banking office.
- Secured Rates
  - Tri-Party General Collateral Rate (TGCR) provides a measure of the rate on overnight, tri-party GC repo transactions secured by Treasury securities that are executed between counterparties that know each other's identity at the time of the trade, and is calculated based on data collected from the Bank of New York Mellon, excluding GCF Repo.
  - Broad General Collateral Rate (BGCR) provides a measure of the rate on overnight Treasury general collateral (GC) repo transactions, and is calculated based on the same tri-party repo transactions used for the TGCR, plus General Collateral Finance (GCF) repo transactions cleared through the FICC's GCF Repo service.

- Secured Overnight Financing Rate (SOFR) provides a broad measure of the general cost of financing Treasury securities overnight, and is calculated based on the data used for the BGCR, plus transactions cleared through the Fixed Income Clearing Corporation's (FICC) Delivery-versus-Payment repo service.

SECURED OVERNIGHT FINANCING RATE CHART



Term Structure

- The date at which cash is borrowed or invested and the date at which cash is returned uniquely determine the interest rate (for a given risk, loans of different risks can carry different rates even with the same dates).
- However, if the dates are changed, interest rates may change.
- For example, the interest rate for a three-month investment today may be different from the interest rate for a three-month investment next year.
- And the interest rate for a three-month investment today is different from the interest rate for a six-month investment today.
- Term structure refers to a series of interest rates corresponding to different investment or borrowing horizons, all starting at the same time. See the following table for an example of term structure of treasury yields.

Date	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
09/20/2022	2.57	3.05	3.35	3.86	4.03	3.96	3.94	3.75	3.69	3.57	3.83	3.59
09/21/2022	2.59	3.06	3.31	3.86	4.08	4.02	3.98	3.74	3.65	3.51	3.73	3.50
09/22/2022	2.73	3.09	3.29	3.87	4.08	4.11	4.12	3.91	3.84	3.70	3.90	3.65
09/23/2022	2.67	3.07	3.24	3.85	4.15	4.20	4.21	3.96	3.85	3.69	3.87	3.61
09/26/2022	2.73	3.14	3.39	3.95	4.17	4.27	4.37	4.15	4.06	3.88	4.01	3.72
09/27/2022	2.71	3.14	3.35	3.91	4.16	4.30	4.39	4.21	4.14	3.97	4.15	3.87

Zero rates and Bond Pricing

- The zero-rate for time period T at time t is the interest rate paid or earned on an investment with an initial investment at time t and a single final repayment at time t+T.
- **Zero coupon bonds** (ZCBs) are bonds that pay no coupons but a final principal amount at maturity.
- The price of such a bond is less than the principal amount.
- The difference represents interest and can be used to measure zero rates.
- Alternatively, zero rates can be used to price bonds.
- For bonds with coupons, bond price is the sum of prices of several zero-coupon bonds, one for each cash flow of the original bond.

- Consider a bond being priced at date 0.
- The bond pays coupon $C_1, C_2, \ldots, C_N$ at dates $t_1, t_2, \ldots, t_N$, respectively, and principal P at date t_N.
- Let $r_1, r_2, \ldots, r_N$ be the zero rates at time 0 corresponding to dates $t_1, t_2, \ldots, t_N$, respectively.
- Then bond price at date 0 is the sum of present values of the coupons and the principal, where each present value is calculated using the corresponding zero rate.
- If dates are measured in years and interest rates are annually compounded annual rates, then

$$P_0 = \frac{C_1}{(1+r_1)^{t_1}} + \frac{C_2}{(1+r_2)^{t_2}} + \cdots \frac{C_N}{(1+r_N)^{t_N}} + \frac{P}{(1+r_N)^{t_N}}.$$

- If dates are measured in years and interest rates are continuously compounded annual rates, then

$$P_0 = C_1 e^{-r_1 t_1} + C_2 e^{-r_2 t_2} + \cdots C_N e^{-r_N t_N} + P e^{-r_N t_N}.$$

- The reverse calculation can be used to determine zero rates from bond prices.
- In general, to determine n rates, n prices are necessary.
- Prices of shorter-maturity bonds are used to determine shorter maturity zero rates.
- These are then used as inputs in determining longer-maturity zero rates using prices of longer-maturity bonds. This process is called **bootstrapping**.

Example 6: Price the following bonds using the given zero rates. Assume coupon payments are semiannual.

Bond	Principal ($)	Maturity (years)	Coupon Rate (% per year)	Price ($)
A	1000	0.5	0	
B	1000	1	0	
C	1000	1.5	4	
D	1000	2	6	

Maturity (years)	Zero rate (continuously compounded, % per annum)
0.5	5.2
1	5.5
1.5	5.6
2	5.6

Example 7: Determine zero rates given the prices of the following bonds. Assume coupon payments are semiannual.

Bond	Principal ($)	Maturity (years)	Coupon Rate (% per year)	Price ($)
A	1000	0.5	0	962.71
B	1000	1	0	925.43
C	1000	1.5	4	943.78
D	1000	2	6	961.88

Maturity (years)	Zero rate (continuously compounded, % per annum)
0.5	
1	
1.5	
2	

Yield to Maturity

- One bond price can be used to determine only one interest rate.
- If a bond has multiple cash flows, then the bond price is inadequate to determine all zero rates corresponding to the different cash flow dates.
- However, if we assume that all rates are same, then we can calculate this common rate.
- This rate is the bond's yield to maturity.
- The interpretation is that if all interest rates were fixed, then the common value would have to be the yield-to-maturity for the bond price to be correct.
- Of course, this is just a hypothetical. The yield-to-maturity can be considered a complicated average of the different zero rates used to price the bond.

Example 8: Determine the yield to maturity of the bonds in Example 7.

Forward Rates

- A forward rate is the interest rate one must pay for borrowing or earn from investing in a future period.
- This interest rate reflects demand and supply today, is locked in today, and is reflected in current prices.
- Notice this is different from waiting until that period arrives and borrowing or investing at the then current interest rate. The interest rate that will prevail in the future is uncertain.
- However, a forward rate is determined today with no uncertainty.
- The prevailing interest rate in the future may be lower or higher than the forward rate.

- Forward rates can be determined from zero rates based on absence of arbitrage arguments.
- Forward rates for an investment starting at time t_1 from today and ending at time t_2 from today are determined by comparing the zero rates r_1 and r_2 for times t_1 and t_2, respectively.
- A $1 loan for time duration t_2 should require the same repayment as a $1 loan for time duration t_1 at the zero rate r_1 rolled over to a loan from time t_1 to t_2 at the forward rate f.
- We assume all rates are mentioned as continuously compounded rates. Then,

$$e^{r_2 t_2} = e^{r_1 t_1} e^{f(t_2 - t_1)}$$

- This equation can be solved to get

$$f = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1}$$

Note: The above formula applies only when interest rates are continuously compounded. If the interest rates are not continuously compounded, forward rates can be calculated by equating (1) the future value of a longer-term loan calculated using the zero rate to (2) the future value of a shorter-term loan using zero rate multiplied by the future value for the remaining term using the forward rate.

Example 9: Calculate forward rates using the zero rates given in Example 6.

Example 10: Calculate zero rates using the following forward rates.

Start of investment (years)	Six-month forward rate (continuously compounded, % per annum)
0	4.5
0.5	4.7
1	4.9
1.5	5.0

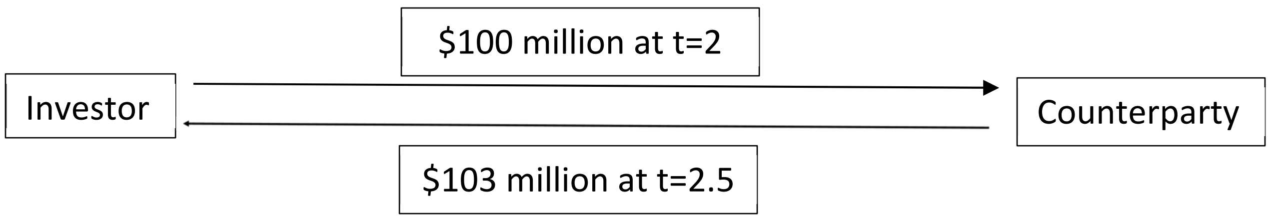
Forward Rate Agreements

- A forward rate agreement (FRA) is an over-the-counter contract used to lock in an interest rate for borrowing or lending in a future period.
- We have seen that forward rates are implied by current zero rates. For example, current interest rates for one-year loan and two-year loans can be used to infer the forward rate for one-year borrowing one year from today.
- One can lock-in a forward rate by entering into two transactions – borrowing for one duration and lending for a different duration, possibly with two counterparties.
- An FRA achieves the same result in form of a contract with a single counterparty.
- The parties agree to borrow and lend a fixed amount for a fixed duration on a fixed future date at a fixed interest rate.

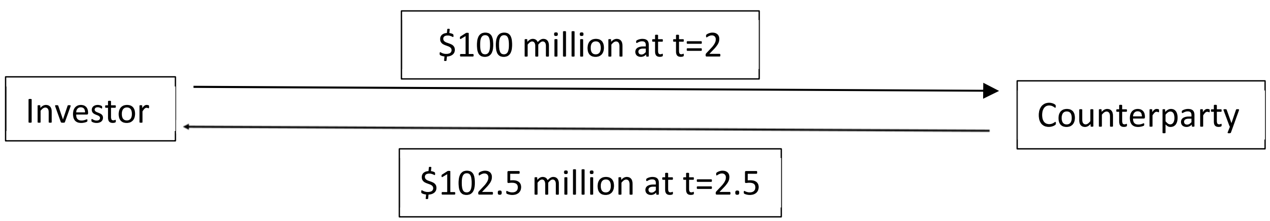
- When an FRA is initiated, the fixed interest rate is chosen to be the appropriate forward rate and the FRA has zero value, both to the borrower and to the lender, because the alternative of borrowing and lending also results in locking in the forward rate.
- However, when the period specified in the FRA starts. the actual interest rate may be different from the rate specified in the FRA.
- If the actual rate is higher, the borrower benefits from FRA and if the actual rate is lower, the lender benefits from the FRA.
- Instead of actually borrowing or lending, at the end of the period, the party that benefits from the FRA can receive an amount equal to the difference in the interest amounts based on the actual rate and the FRA rate. Or the present value of that difference can be transferred at the beginning of the period. This is how FRAs are settled.
- FRAs must also specify which interest rate will be used for settlement as there are multiple interest rates for different credit risks. Most FRAs are based on LIBOR.

Example 11: An investor enters into an FRA that allows the investor to lend \$100 million for a six-month period starting two years from today at 5% per annum compounded semiannually. What is the investor's payoff from FRA two years later, if at that time the interest rate for a six-month loan is 6% per annum compounded semiannually?

Let us measure time t in years with t=0 the time at which the investor enters into the FRA. The prevailing interest at t=2 is 6% per annum compounded semiannually so the investor can achieve the following cash flows without FRA:



The FRA enables the investor to achieve the following cash flows:



Comparing the two sets of cash flows, the net impact of the FRA on the investor is:

$$\text{Investor} \xrightarrow[\text{\$0.5 million at t=2.5}]{} \text{Counterparty}$$

Calculating the present value using the current rate of 6% per annum compounded semiannually, this is equivalent to:

$$\text{Investor} \xrightarrow[\text{\$485,437 at t=2}]{} \text{Counterparty}$$

That is, the investor must pay the counterparty \$485,437 two years after entering into FRA. The following calculation leads to the same conclusion:

$$\text{Investor's payoff} = \frac{\$100{,}000{,}000 \times \frac{0.05}{2} - \$100{,}000{,}000 \times \frac{0.06}{2}}{\left(1+\frac{0.06}{2}\right)^{1}} = -\$485{,}437$$

Valuation of FRA

- The value of a contract is the net gain or loss (in present value terms) from the contract.
- That is, it is the benefit from having a position in the contract relative to not having that position in the contract.
- When a party takes a position in a contract, it pays a price that equals the value of the contract to the party.
- Contracts such as forward contracts and FRAs do not require any initial investment because the terms are chosen so that the initial value of the contract is zero. Subsequently, the value may become positive or negative.
- To value a contract, we compare the cash flows with a position in the contract to comparable cash flows in market transactions without a position in contract, and value the difference.

- Consider an FRA
    - at time 0
    - that **pays** (to lender) an interest rate of R_K
    - on a loan for an amount L
    - that starts at time T_1
    - and matures at time T_2.
- In the absence of the FRA, an investor can lock in the forward rate R_F for the loan.
- The FRA earns the additional interest of $L(R_K - R_F)(T_2 - T_1)$ after time T_2.
- The value of the FRA is the present value of this additional interest.
- If R_2 is the continuously compounded (zero) interest rate for time T_2, then the value of the FRA is

$$Value\ of\ FRA\ to\ Lender: L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}$$

- Now consider an FRA that
  - **charges** (to borrower)
  - an interest rate of R_K
  - on a loan for an amount L
  - that starts after time T_1
  - and matures after time T_2.
- In the absence of the FRA, an investor can lock in the forward rate R_F for the loan.
- The FRA's incremental cash flow is $-L(R_K - R_F)(T_2 - T_1)$ after time T_2.
- The value of the FRA is the present value of this incremental cash flow. If R_2 is the continuously compounded (zero) interest rate for time T_2, then the value of the FRA is

$$Value\ of\ FRA\ to\ Borrower: L(R_F - R_K)(T_2 - T_1)e^{-R_2 T_2}$$

Example 12: Suppose interest rates are as given in Example 6. Value an FRA that allows an investor to get an interest rate of 6% per annum with semiannual compounding on a six-month loan for $100 million starting one year from today.

The FRA promises a rate of 6% per annum with semiannual compounding. Let us calculate the corresponding forward rate:

$$f = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1} = \frac{0.056 \times 1.5 - 0.055 \times 1}{1.5 - 1} = 5.8\%\ per\ annum\ c.c.$$

To value the FRA, we first convert it to a semiannually-compounded rate: $2 \times (e^{0.058 \times 0.5} - 1) = 5.884919\%$ per annum compounded semiannually. The value of the FRA then is

$$\$100{,}000{,}000 \times (0.06 - 0.05884919)(1.5 - 1)e^{-0.056 \times 1.5} = \$52{,}905$$

Theories of the Term Structure of Interest Rates

- The term structure tends to be upward sloping most of the times.
- That is, longer-duration loans carry a higher interest rate than shorter-duration loans.
- However, the yield curve can sometimes be downward-sloping or inverted.
- Some investors consider this a sign that the economy is expected to worsen in the future.
- The following theories of term structure attempt to explain interest rate differences over different horizons.

Expectations theory

- This theory states that long-term rates reflect future short-term rates.
- For example, the interest rate for a two-year loan reflects the interest rate for a one-year loan and the expected interest rate for one-year loan prevailing one year later.
- Based on this theory, forward rates equal expected future zero rates.

Market segmentation theory

- The interest rates for different horizons depend on the demand and supply for loans of that duration.
- These interest rates differ because different kinds of large investors prefer to invest or borrow at different horizons.

Liquidity preference theory

- Investors prefer liquidity – availability of cash.
- As a result, lenders prefer to lend for short horizon and borrowers prefer to borrow for long horizons.
- The mismatch in preferences of lenders and borrowers results in higher long-term interest rates.
- The theory predicts that forward rates are higher than the expected value of the corresponding zero rates in the future.