Solution to Problem Set IV

QF 430: Introduction to Derivatives
Due Thursday, April 30

- Please submit neatly handwritten or typed answers. Submit electronically a **single pdf** file through Canvas.
- While you may have done your work in one or more spreadsheets, do not turn in spreadsheets. The final product is a text document, possibly with equations, formulas, tables, graphs etc.
- Show your steps or reasoning.
- Two spreadsheets are provided: one for Black-Scholes pricing and another for risk-neutral valuation.

1 Random Variables

Problem 1.1. *The price of a stock, presently \$40, can increase each day by \$1 or decrease each day by \$1. We observe the process for a total of 3 days. Use* $\{$ *and* $\}$ *to enclose sets. Use parentheses to enclose a sequence of prices. For example, (50,60,70) refers to an outcome where prices are 50, 60, and 70 over three days.*

(a) What is the sample space Ω *of all possible outcomes over three days?*

(b) Define X *to be the stock price at the end of three days. If* X *is a random variable on measure space* $(\Omega, \mathcal{F})$*, what events must* σ*-algebra* $\mathcal{F}$ *include? Recall each event is a set of zero, one, or multiple outcomes. For example,* $\{$*(41,42,43), (41,42,41), (41,40,41), (39,40,41)*$\}$ *is the event corresponding to* X *being 43 or 41.*

(c) Define another random variable Y *to be the value of the stock after 2 days. If* Y *is a random variables on measure space* $(\Omega, \mathcal{F})$*, what events must* σ*-algebra* $\mathcal{F}$ *include?*

(d) Suppose X *and* Y *are defined on probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *where probability of \$1 increase or \$1 decrease in stock price on any day is 0.5 each. Under probability measure* $\mathbb{P}$*, what is the probability that* $X = 41$*?*

(e) Under probability measure $\mathbb{P}$*, are* X *and* Y *independent? Two random variables are independent if the probability that* $X = x$ *and* $Y = y$ *equals the probability that* $X = x$ *multiplied by the probability that* $Y = y$ *for all possible values of* x *and* y*.*

Solution.

(a) $\{$(41,42,43), (41,42,41), (41,40,41), (41,40,39), (39,40,41), (39,40,39), (39,38,39), (39,38,37)$\}$

(b) $\mathcal{F}$ must include

- $\{\}$
- $\{$(41,42,43)$\}$
- $\{$((41,42,41), (41,40,41), (39,40,41)$\}$
- $\{$(41,40,39), (39,40,39), (39,38,39)$\}$
- $\{$(39,38,37)$\}$
- $\{$(41,42,43), (41,42,41), (41,40,41), (39,40,41)$\}$
- $\{$(41,42,43), (41,40,39), (39,40,39), (39,38,39)$\}$
- $\{$(41,42,43), (39,38,37)$\}$
- $\{$(41,42,41), (41,40,41), (41,40,39), (39,40,41), (39,40,39), (39,38,39)$\}$
- $\{$(41,42,41), (41,40,41), (39,40,41), (39,38,37)$\}$
- $\{$(41,40,39), (39,40,39), (39,38,39), (39,38,37)$\}$
- $\{$(41,42,43), (41,42,41), (41,40,41), (41,40,39), (39,40,41), (39,40,39), (39,38,39)$\}$
- $\{$(41,42,43), (41,42,41), (41,40,41), (39,40,41), (39,38,37)$\}$
- $\{$(41,42,43), (41,40,39), (39,40,39), (39,38,39), (39,38,37)$\}$
- $\{$(41,42,41), (41,40,41), (41,40,39), (39,40,41), (39,40,39), (39,38,39), (39,38,37)$\}$
- $\{$(41,42,43), (41,42,41), (41,40,41), (41,40,39), (39,40,41), (39,40,39), (39,38,39), (39,38,37)$\}$

(c) $\mathcal{F}$ must include

- $\{\}$
- $\{$(41,42,43), (41,42,41)$\}$
- $\{$(41,40,41), (41,40,39), (39,40,41), (39,40,39)$\}$
- $\{$(39,38,39), (39,38,37)$\}$
- $\{$(41,42,43), (41,42,41), (41,40,41), (41,40,39), (39,40,41), (39,40,39)$\}$
- $\{$(41,42,43), (41,42,41), (39,38,39), (39,38,37)$\}$
- $\{$(41,40,41), (41,40,39), (39,40,41), (39,40,39), (39,38,39), (39,38,37)$\}$
- $\{$(41,42,43), (41,42,41), (41,40,41), (41,40,39), (39,40,41), (39,40,39), (39,38,39), (39,38,37)$\}$

(d) The probability of $\{$((41,42,41), (41,40,41), (39,40,41)$\}$ is $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$.

(e) X and Y are not independent as X is more likely to be high if Y is high. Formally, the probability that $X = 43$ is $\frac{1}{8}$, the probability that $Y = 38$ is $\frac{1}{4}$ but the probability that $X = 43$ and $Y = 38$ is 0, not $\frac{1}{8}\times\frac{1}{4}$.

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2 Stochastic Processes and LogNormal Distribution

Problem 2.1. *A stock is trading at \$100. Assume that the stock price follows lognormal distribution. The continuously compounded expected return on the stock is 5% per annum and the volatility of the stock return is 40% per annum. Let S be the stock price after six months.*

(a) *What is the mean and standard deviation of* $\ln(S)$ *(natural log of S)?*

(b) *What is the median (50th percentile value) value of* $\ln(S)$*?*

(c) *What is the median value of S?*

Solution.

(a) If the stock price follows geometric Brownian motion with drift μ and volatility σ then log of stock price follows arithmetic Brownian motion with drift $\mu - \sigma^2/2$ and volatility σ. The change in log of stock price over a time interval T has mean $(\mu - \sigma^2/2)T$, variance $\sigma^2 T$ and standard deviation $\sigma\sqrt{T}$. The mean of $\ln(S)$ is $\ln(100) + (0.05 - 0.4^2/2) \times 0.5 = 4.5902$ and the standard deviation is $0.4\sqrt{0.5} = 0.2828$.

(b) For the normal distribution, the median is same as the mean, $\ln(S)$ is $\ln(100) + (0.05 - 0.4^2/2) \times 0.5 = 4.5902$.

(c) Median of $\ln(S)$ is $\ln(100) + (0.05 - 0.4^2/2) \times 0.5$ so median of $S = e^{\ln(S)}$ is $e^{\ln(100)+(0.05-0.4^2/2)\times 0.5} = 100e^{(0.05-0.4^2/2)\times 0.5} = \$98.5112$.

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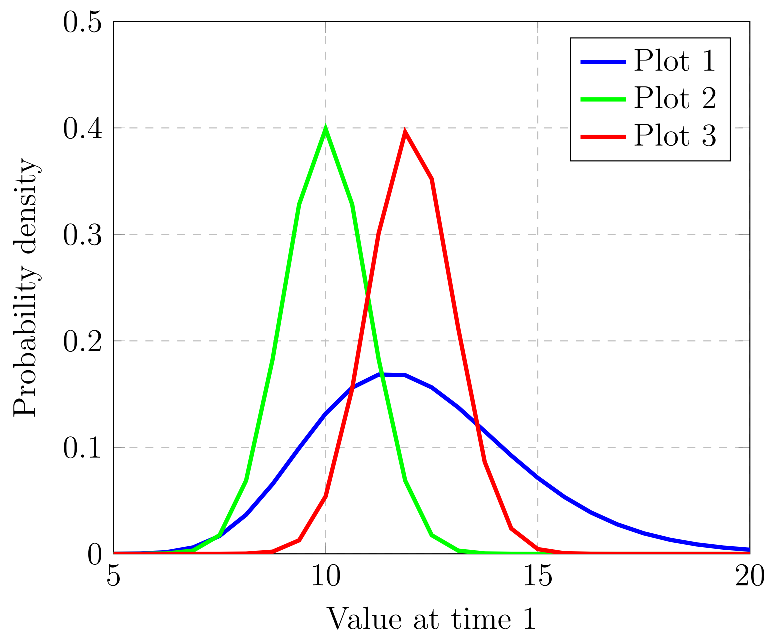
Problem 2.2. *The value of a stochastic process at time 0 is 10. Identify which of the following plots represents the probability distribution of the stochastic process at time 1 under each of the following cases:*

(a) *The stochastic process is a standard Brownian motion.*

(b) *The stochastic process is an arithmetic Brownian motion.*

(c) *The stochastic process is a geometric Brownian motion.*

Explain your reasoning.



Solution.

(a) A standard Brownian motion has zero drift so the mean at time 1 must also equal 10. The probability density of the value at a given time is normally distributed, which is symmetric around the mean. Thus, Plot 2 represents a standard Brownian motion.

(b) In an arithmetic Brownian motion, the probability density of the value at a given time is normally distributed and symmetric around the mean but the drift can be nonzero so the mean at time 1 need not equal 10. Thus, Plot 3 represents an arithmetic Brownian motion.

(c) In a geometric Brownian motion, the probability density of the value at a given time is lognormally distributed, which is not symmetric. Thus, Plot 1 represents a geometric Brownian motion.

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3 Black-Scholes Model and Risk-Neutral Valuation

Problem 3.1.

(a) What is the Black-Scholes partial differential equation (PDE) for the price $f(t, S_t)$ at time t of a European derivative security on a stock with price S_t? Specify the meaning of the terms or symbols in the equation.

(b) Note that the Black-Scholes PDE does not specify whether the derivative is a call option, a put option, or some other derivative. How do you incorporate the derivative payoff when using Black-Scholes PDE to price a derivative?

(c) *(Optional, no points) There is an exotic European-style derivative whose price at time t equals $Ze^{kt}S_t^2$ if the stock price at time t is S_t, with Z and k two constants. The stock price follows a geometric Brownian motion with volatility σ equal to 40% per annum. The risk-free rate is 5% per annum continuously compounded. Determine k. The following calculus facts will be useful:*

- *If a does not depend on S_t, then, derivative of aS_t^2 with respect to S_t is $2aS_t$:*

$$\frac{\partial a S_t^2}{\partial S_t} = 2aS_t.$$

- *If a does not depend on S_t, then second derivative of aS_t^2 with respect to S_t is $2a$:*

$$\frac{\partial^2 a S_t^2}{\partial S_t^2} = 2a.$$

- *If b does not depend on t, then, derivative of be^{kt} with respect to t is bke^{kt}:*

$$\frac{\partial b e^{kt}}{\partial t} = bke^{kt}.$$

Solution.

(a) The Black-Scholes partial differential equation is

$$f_t(t,x) = -\frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) - rxf_x(t,x) + rf(t,x)$$

or

$$rf(t,x) = f_t(t,x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) + rxf_x(t,x)$$

where $f(t,x)$ is the value of the derivative when the underlying asset value is x and time is t, $f_t(t,x)$ is the rate of change of value of the derivative with the passage of time, $f_x(t,x)$ is the sensitivity of the derivative price to the price of the underlying asset, $f_{xx}(t,x)$ is the second derivative of the derivative price with respect to the price of the underlying asset, σ is volatility of the return on the underlying asset, and r is the risk-free rate.

(b) The derivative payoff, which distinguishes different derivatives such as calls and puts, is specified as a terminal boundary condition. The derivative price depends on the solution of the Black-Scholes PDE along with the terminal boundary condition.

(c) Substituting the expression for price and its partial derivatives in the Black-Scholes PDE (stock price symbol S_t can be replaced with x), we get

$$0.05 \times Ze^{kt}S_t^2 = Zke^{kt}S_t^2 + \frac{1}{2} \times 0.4^2 \times 2Ze^{kt}S_t^2 + 0.05 \times 2Ze^{kt}S_t^2$$

which simplifies to

$$0.05 \times Ze^{kt}S_t^2 = (k + 0.26) \times Ze^{kt}S_t^2.$$

$k = -0.21$ and the derivative price is $Ze^{-0.21t}S_t^2$

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Problem 3.2. *A non-dividend-paying stock is trading at 72 and has volatility of 30% per annum. Consider an option on the stock with strike price \$75 and maturity six months. The risk-free rate is 2% per annum (continuously compounded).*

(a) *What is the price of the option if it is a European call?*

(b) *What is the price of the option if it is a European put?*

(c) *What is the price of the option if it is an American call?*

Solution.

(a) The price is given by Black-Scholes formula:

$$\begin{aligned}
&S\Phi\left(\frac{\log(S/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)-Ke^{-r\tau}\Phi\left(\frac{\log(S/K)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)\\
&=72\Phi\left(\frac{\log(\frac{72}{75})+(0.02+\frac{0.3^2}{2})0.5}{0.3\sqrt{0.5}}\right)-75e^{-0.02\times0.5}\Phi\left(\frac{\log(\frac{72}{75})+(0.02-\frac{0.3^2}{2})0.5}{0.3\sqrt{0.5}}\right)\\
&=72\Phi(-0.03923)-75e^{-0.01}\Phi(-0.25136)\\
&=72\times0.48435-75e^{-0.01}\times0.40077=\$5.115.
\end{aligned}$$

(b) The price can be obtained using the Black-Scholes formula for put:

$$\begin{aligned}
&Ke^{-r\tau}\Phi\left(-\frac{\log(S/K)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)-S\Phi\left(-\frac{\log(S/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)\\
&=75e^{-0.02\times0.5}\Phi\left(-\frac{\log(\frac{72}{75})+(0.02-\frac{0.3^2}{2})0.5}{0.3\sqrt{0.5}}\right)-72\Phi\left(-\frac{\log(\frac{72}{75})+(0.02+\frac{0.3^2}{2})0.5}{0.3\sqrt{0.5}}\right)\\
&=75e^{-0.01}\Phi(0.25136)-72\Phi(0.03923)\\
&=75e^{-0.01}\times0.59923-72\times0.51565=\$7.369.
\end{aligned}$$

Alternatively, using put-call parity, put price equals call price plus bond price minus stock price:

$$5.115+75e^{-0.02\times0.5}-72=\$7.369.$$

(c) The price of the American call equals the price of the European call on a non-dividend-paying stock so the price is \$5.115.

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Problem 3.3. *Consider Black-Scholes formulae for prices of European call and put options with strike K each, maturity T each on a non-dividend-paying stock with price S and volatility σ, with risk-free rate r. The formulas are written in terms of quantities d_1 and d_2 used to calculate the probabilities of the normal distribution. If the volatility of the stock becomes large and approaches infinity,*

(a) *what values do d_1 and d_2 approach?*

(b) *what value does the call price approach?*

(c) *what value does the put price approach?*

Solution.

(a) $d_1 = \frac{\log(S/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = \frac{\log(S/K)+r\tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}$. The first term approaches zero and the second term approaches ∞ as σ approaches ∞, so d_1 approaches ∞.

$d_2 = \frac{\log(S/K)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = \frac{\log(S/K)+r\tau}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau}$. The first term approaches zero and the second term approaches $-\infty$ as σ approaches ∞, so d_2 approaches $-\infty$.

(b) As d_1 approaches ∞, $\Phi(d_1)$ approaches 1. As d_2 approaches $-\infty$, $\Phi(d_2)$ approaches 0. Call price given by $S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$ approaches stock price S. The intuition is that future stock price will either be very high with a low probability or almost zero with a high probability. If it is very high, you will get the stock and what you pay as the exercise price will be negligible in comparison. If stock price is very low in future, you won't miss anything by not getting the stock. So the option is as valuable as having the stock.

(c) As d_1 approaches ∞, $\Phi(-d_1)$ approaches 0. As d_2 approaches $-\infty$, $\Phi(-d_2)$ approaches 1. Put price given by $Ke^{-r\tau}\Phi(-d_2) - S\Phi(-d_1)$ approaches present value of exercise price $Ke^{-r\tau}$. The intuition is that future stock price will either be very high with a low probability or almost zero with a high probability. If it is almost zero, your payoff equals almost the full strike price. If the stock price is very high, you get nothing but the probability of that is very low. So the option is as valuable as having the present value of the strike price.

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Problem 3.4. *Derivatives can be valued using risk neutral valuation. The spreadsheet "Risk Neutral Valuation.xlsx" simulates 10,000 values of stock price (assuming log normal distribution) at expiration of derivative. Each stock price results in a derivative payoff. These derivative payoffs are discounted and averaged to get derivative value. The formula in cells in column C is set to price a European call with strike price of \$50 but can be changed to price any other derivative. Value a derivative under the following assumptions:*
The current stock price (to be specified in cell G1 of the spreadsheet) is 20 + the number of the first letter of your last name + the number of the second letter of your last name. Assume A = 1, B = 2, and so on until Z = 26. For example, current stock price is 39 if your name is Jane Doe.
Change the formula in cells in column C to value a derivative which will expire after six months. Its payoff will equal the absolute difference between the stock price at that time and today's stock price, up to a maximum of \$5. For example, if the stock price today is \$50 and the stock price at expiration is \$47.2, you will get \$2.8. If the stock price at expiration is \$53.6, you will get \$3.6. If the stock price at expiration is \$30 or \$90, you will get \$5.
The risk-free rate is 5% per annum compounded continuously and the stock volatility is 20% per annum. What is the derivative value? Provide an image of the first few (about 5) lines of the spreadsheet or fill in the following tables:

Simulation Number	Price at Maturity	Derivative Payoff
1		
2		
3		
4		
5		

Current Stock Price	
Time to maturity (years)	
Volatility	
Risk-free rate (c.c.)	
Derivative value	

Solution. The following table gives derivative price D (in \$) for each possible current price D (in \$). The value provided by risk-neutral pricing simulation should be within 2 cents of these prices.

S	D	S	D	S	D	S	D	S	D
22	2.301	32	2.933	42	3.338	52	3.61	62	3.802
23	2.378	33	2.982	43	3.37	53	3.632	63	3.818
24	2.451	34	3.029	44	3.401	54	3.653	64	3.834
25	2.522	35	3.073	45	3.43	55	3.674	65	3.849
26	2.589	36	3.116	46	3.459	56	3.694	66	3.864
27	2.653	37	3.157	47	3.486	57	3.714	67	3.879
28	2.714	38	3.196	48	3.513	58	3.732	68	3.893
29	2.773	39	3.234	49	3.538	59	3.751	69	3.906
30	2.829	40	3.27	50	3.563	60	3.768	70	3.92
31	2.882	41	3.304	51	3.587	61	3.786	71	3.933

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Problem 3.5. *Volatility smile or volatility skew refers to the relation between implied volatility and strike price (Hull chapter 19). Quotes for selected options on Facebook stock on April 17, 2020 are provided below. Determine implied volatility for call options and plot as a function of strike price. Use midpoint of bid and ask quotes for option price. Facebook does not pay dividends. Assume the interest rate is 0.5% per annum with annual compounding (not continuous compounding). The expiration date of all options is June 19, 2020. To determine implied volatility, input all parameters in Black-Scholes formula and adjust volatility until the price given by the formula matches the price quote. If you use Excel, Goal Seek function can be helpful. The Excel formula =DAYS(DATE(2020,6,19),DATE(2020,4,17)) returns the number of days to expiration. You may use the spreadsheet "Black-Scholes Price.xlsx".*

FB Facebook, Inc. Class A Common Stock

DATA AS OF APR 17, 2020

$179.24 2.99 1.7%

FB > FB OPTION CHAIN

Facebook, Inc. Class A Common Stock (FB) Option Chain

MONTH JUN 2020 DATE 06/19/2020

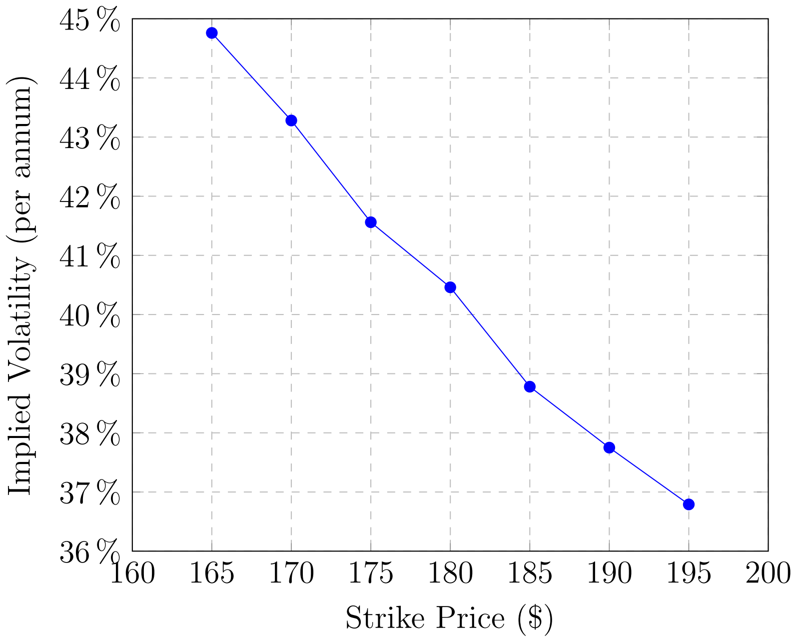
CALLS							PUTS					
LAST	CHANGE	BID	ASK	VOLUME	OPEN INT.	STRIKE	LAST	CHANGE	BID	ASK	VOLUME	OPEN INT.
21.00	+2.88	20.35	22.05	49	2700	165.00	7.10	-1.22	6.20	7.05	86	4763
18.04	+1.83	17.25	18.25	71	5183	170.00	8.37	-1.43	8.10	8.65	188	12909
14.55	+1.20	14.20	14.80	212	4915	175.00	10.45	-1.40	10.00	10.55	154	9635
11.65	+1.16	11.45	12.00	799	9107	180.00	12.75	-3.25	12.10	12.70	279	7582
9.20	+0.90	8.60	9.60	380	5527	185.00	15.30	-1.37	14.25	15.30	79	6090
6.91	+0.56	6.80	7.20	404	8996	190.00	17.72	-3.08	17.35	18.20	56	8284
5.22	+0.57	4.65	5.85	584	5365	195.00	21.95	-3.51	20.15	22.35	23	6134

EXPIRES: 06/19/2020 LAST TRADE: $179.24 (AS OF APR 17, 2020)

Solution. The implied volatility values are in the following table:

Strike price ($)	165	170	175	180	185	190	195
Implied volatility (per annum)	44.8%	43.3%	41.6%	40.5%	38.8%	37.8%	36.8%

A plot follows.



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4 Option Risk Measures

Problem 4.1.

(a) Define the hedge performance measure specified by the textbook (Hull, chapter 17) to assess the performance of a hedging strategy.

(b) Why can this performance measure not be reduced to zero in practice for hedging a position in an option?

Solution.

(a) The hedge performance measure is the standard deviation of the cost of hedging a position to the price of the hedged position.

(b) Reducing standard deviation of cost to zero requires a perfect hedge. Such a hedge for an option requires continuous trading. In practice, continuous trading results in transaction costs that increase cost of hedging and the uncertainty in the size of the transaction costs prevents standard deviation from declining to zero.

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Problem 4.2. *A stock is trading at \$42 and its volatility is 35%. You own a put option with a strike price of \$45 and one year to maturity. The risk-free rate is 2% per annum compounded continuously.*

(a) How many shares of stock should you buy or short so that a small change in stock price has no effect on your portfolio (put and the position in stocks)?

(b) *Given the portfolio you chose in (a), if stock price increases by a large amount (such as \$42 to \$50), would the value of the portfolio stay same, increase or decrease?*

(c) *Given the portfolio you chose in (a), if stock price decreases by a large amount (such as \$42 to \$35), would the value of the portfolio stay same, increase or decrease?*

Solution.

(a) Put delta is

$$\begin{aligned}
&\Phi\left(\frac{\log(S/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)-1\\
&=\Phi\left(\frac{\log(42/45)+(0.02+\frac{1}{2}\times 0.35^2)\times 1}{0.35\sqrt{1}}\right)-1\\
&=\Phi(0.03502)-1=-0.46803.
\end{aligned}$$

You should buy 0.46803 shares of stock.

(b) Delta hedging assumes a linear relation between option price and stock price such that if stock price increases, put option price decreases and the change equals the product of the stock price change and the put delta. However option price is convex in stock price so if stock price increases by a large amount, the decrease in put price is less than what delta hedging assumes. Thus, the portfolio value increases. If stock price increases from \$42 to \$50, the delta hedged portfolio's value increases from \$27.5068 to \$28.3029.

(c) Delta hedging assumes a linear relation between option price and stock price such that if stock price decreases, put option price increases and the change equals the product of the stock price change and the put delta. However option price is convex in stock price so if stock price decreases by a large amount, the increase in put price is more than what delta hedging assumes. Thus, the portfolio value increases. If stock price decreases from \$42 to \$30, the delta hedged portfolio's value increases from \$27.5068 to \$29.5526.

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Problem 4.3. *A trader holds a complicated portfolio whose value depends on the price of a stock. The trader wants to manage this risk. To do so, the trader can take positions (long or short) in the underlying stock and in two options on the stock. The Delta, Gamma, and Vega of the portfolio and the two options are given below. What position in the two options and the stock (in addition to the portfolio already held) can make the trader's combined position Delta neutral, Gamma neutral, and Vega neutral? Assume you can buy and short any number, including fractional units.*

	Delta	Gamma	Vega
Portfolio	0.6	-800	-300
Option 1	-0.02	20	4
Option 2	0.1	40	7

Solution. The Delta, Gamma, and Vega of stock are 1, 0, and 0. If n_0 stocks, n_1 option 1, and n_2 option 2 are added to make the portfolio Delta neutral, Gamma neutral, and Vega neutral, then, the equations for Delta, Gamma, and Vega of the combined position are:

$$\begin{array}{rcrcrcrcr} 0.6 & + & n_0 & - & 0.02n_1 & + & 0.1n_2 & = & 0, \\ -800 & & & + & 20n_1 & + & 40n_2 & = & 0, \\ -300 & & & + & 4n_1 & + & 7n_2 & = & 0. \end{array}$$

Dividing the middle equation by 5, we get

$$4n_1 + 8n_2 = 160$$

and the bottom equation can be rewritten as

$$4n_1 + 7n_2 = 300.$$

Subtracting the last equation from the previous one, we get $n_2 = -140$. Substituting in the last equation, we get $n_1 = 320$. Substituting values of n_1 and n_2 in the first equation, we get $n_0 = 19.8$. The trader should buy 19.8 shares of stock, buy 320 option 1, and short 140 option 2. ■