Option Valuation Using Binomial Trees

Conceptual Approach of Option Valuation

- A common method of valuation is **discounted cash flow valuation**.
- This method requires two inputs: **expected cash flows** and the **discount rate**, which depends on the risk of the cash flows.
- Both inputs are **subjective** and people who disagree on these inputs will arrive at different estimates of value.

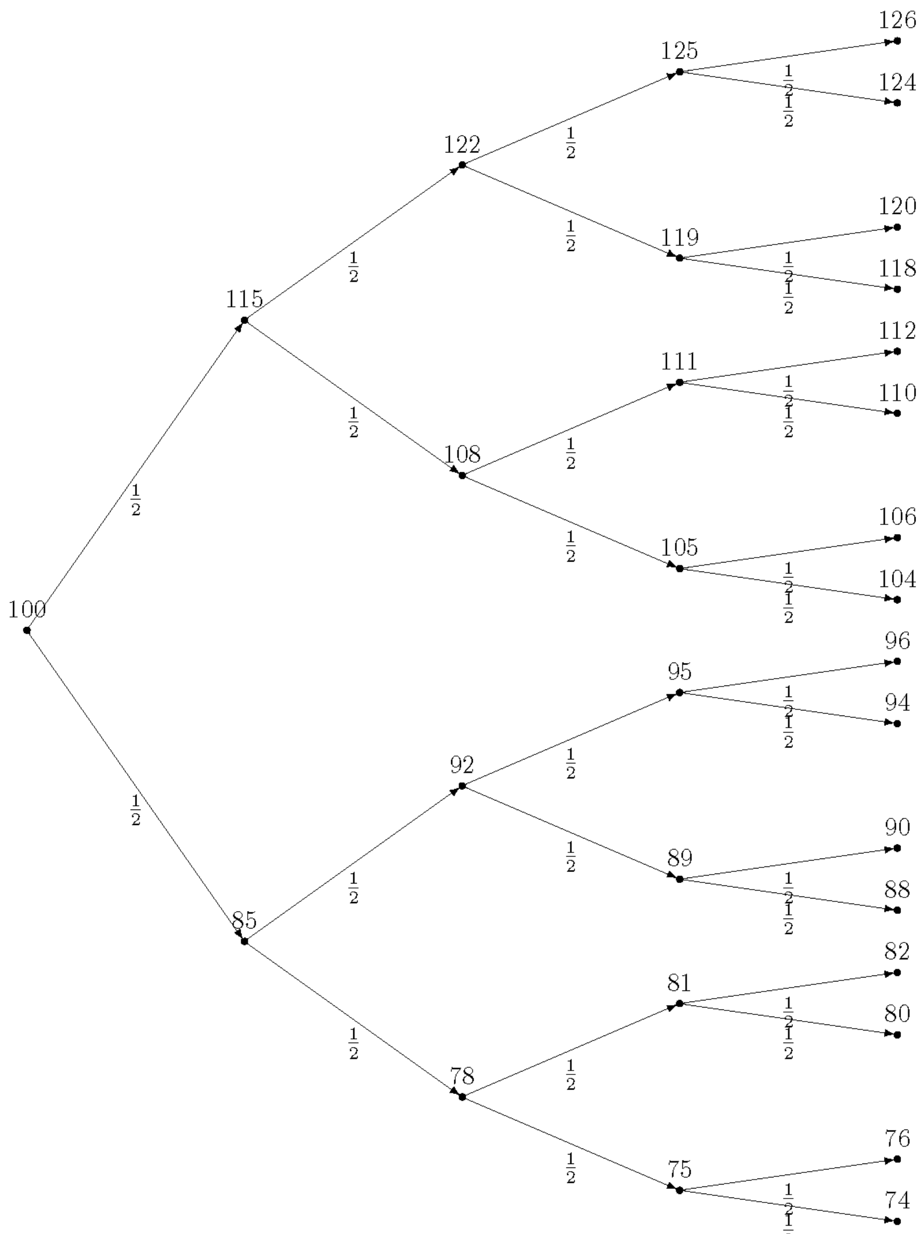
Relative Pricing

- For example, estimate a price-earnings multiple from other comparable stocks and multiply this multiple with the earnings of a stock to come up with price of the stock.
- Such a method assumes that investors' beliefs about future cash flows and risk are already built into observed prices.
- Rather than estimating these beliefs, one directly uses market prices of some assets to price other assets.
- Derivatives are natural candidates for relative pricing methods because they are closely related to the underlying asset.
- The price of the underlying asset incorporates investors' beliefs about how the underlying asset will perform in the future.
- To price a derivative, we create two portfolios that will have similar cash flows in future and then argue that the two should have the same price otherwise there will be arbitrage opportunities.

Binomial Tree

- A binomial tree is used to represent time evolution of the price of the underlying asset.
- Time is divided into discrete periods and prices are considered only at discrete steps of time, such as, 1 year, 1 quarter, 12 days, 3 hours, etc.
- The binomial tree method assumes that given the price in a period, there are only two possible values for the price in the next period.
- The tree consists of nodes, representing possible prices, connected by arrows.
- The nodes are arranged in levels – leftmost node is level 0, representing the initial time, the nodes immediately to the right of level 0 constitute level 1 and represent prices after one time step, and similarly, prices at successive time steps are represented by successive levels consisting of nodes immediately to the right of the previous level.
- Each price (except in the last level) is connected by arrow to the two possible prices it can lead to in the next level.

Example 1: The initial price of a stock is \$100. Draw a four-step binomial tree to represent the evolution of stock price over a year. Assume that the stock price increases or decreases by \$15 in the first quarter, by \$7 in the second quarter, by \$3 in the third quarter, and by \$1 in the fourth quarter. Assume the probability of price increase or decrease is 0.5.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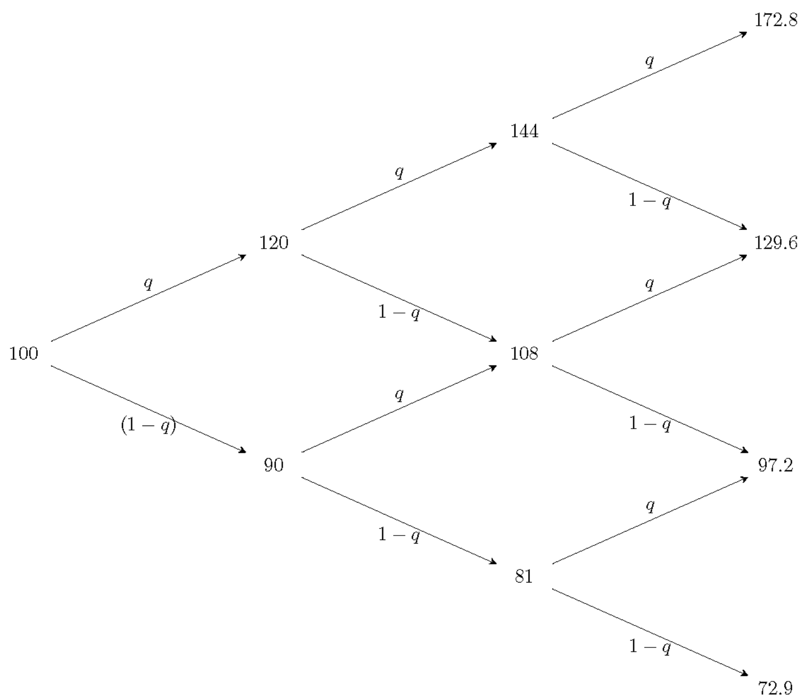
Recombining Binomial Tree

- A recombining binomial tree is a binomial tree in which a larger price change followed by a smaller price change leads to the same price as a smaller price change followed by a larger price change.
- This means that there can be multiple ways of starting from an initial price and reaching a final price.
- The benefit of these trees is that each new level increases the number of price values by only 1 so the number of price values in level n is $n + 1$.

Example 2: The initial price of a stock is \$100. Draw a four-step binomial tree to represent the evolution of stock price over a year. Assume that the stock price increases by 20% or decreases by 10% in each quarter. Assume the probability of price increase is q.

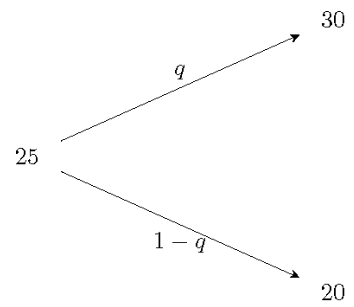


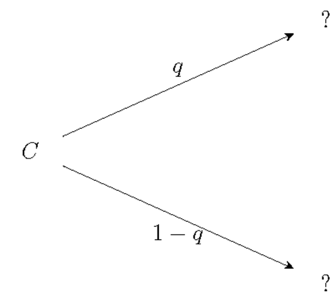
Riskless Portfolio Approach

- In this valuation method, we combine the underlying asset and the option to create a riskless portfolio.
- This means that the cash flow of the portfolio in the future is known with certainty.
- Any riskless security earns risk-free rate so this portfolio’s price can be calculated by discounting the portfolio’s future cash flow at risk-free rate.

Example 3: Suppose a stock is worth \$25 today and will either be worth 20% more or 20% less in one year. The risk-free rate is 5% per annum continuously compounded. What is the value of an at-the-money European call option that matures after one year?

The binomial trees for the prices of the stock and the call option are shown below:





We solve these valuation problems backward by first determining the value of the option at maturity. If stock price is \$30, the call option is worth $max(30-25,0) = 5$. If stock price is \$20, the call option is worth $max(20-25,0) = 0$.

Create a riskless portfolio using stocks and call options. Suppose we choose x stocks and y options. Then the cash flows in future will be $30x + 5y$ and $20x$. This portfolio will be riskless if the two cash flows are identical:

$$30x + 5y = 20x$$

This means $\frac{x}{y}$. That is, for each long option, we must short ___ shares of stock or for each short option, we must long _____ shares of stock. The number ___ is the option's **delta** after one year. This value can be calculated as:

$$Option\ delta\ = \frac{Difference\ in\ Option\ Cash\ Flows}{Difference\ in\ Stock\ Cash\ Flows}$$

We can verify that $(5 - 0)/(30 - 20) = 0.5$. There are many values of x and y which will work. Let is assume we have one call option and we short 0.5 shares of stock That is, $y = 1, x = -0.5$. The final payoff of this portfolio is sure to be -10. So the current value of this portfolio can be calculated by discounting at risk-free rate of 5%.

Current value of portfolio = ______________________________.

But since the portfolio consists of 1 call and -0.5 shares of stock, its value equals $C - 0.5 \times 25 = C - 12.5$. So we can solve

$$C - 12.5 = -9.5123$$

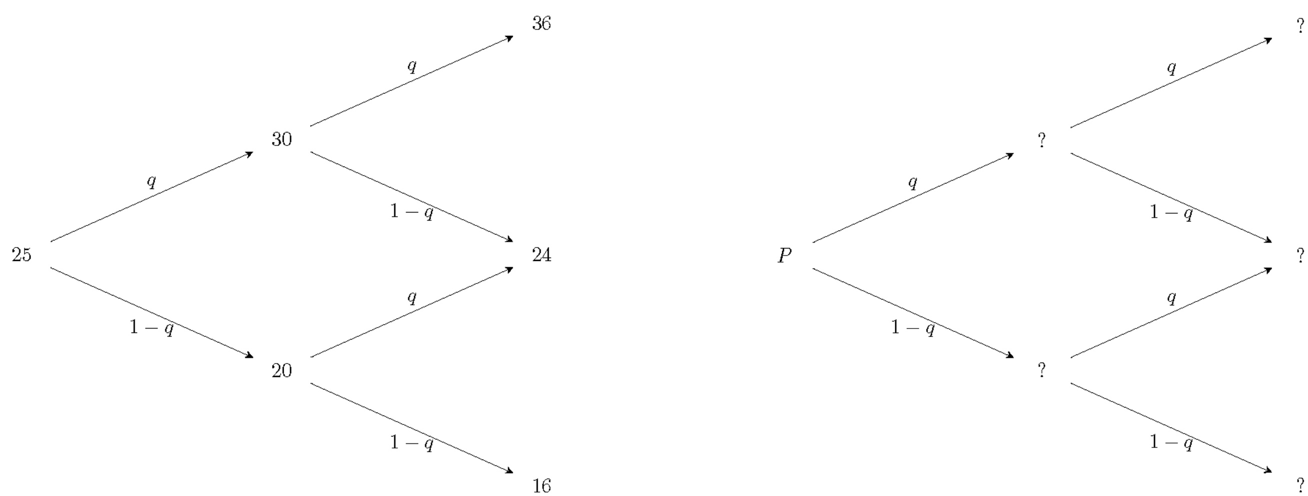
to get $C\ =$ ______________.

Multiple Periods

- The above approach can be used for binomial trees with multiple levels.
- Proceed backwards from maturity and moving left, repeatedly determine the option price one level earlier.
- By increasing the number of periods, we can
    1. value options with longer maturity, or
    2. value a fixed maturity option with greater accuracy.

Example 4: Suppose a stock is worth \$25 today and each year its price increases by 20% or decreases by 20%. The stock does not pay any dividends. The risk-free rate is 5% per annum continuously compounded. What is the value of a European put option with exercise price of \$26 that matures after two years?

The binomial trees for the prices of the stock and the put option are shown below:



Start by first determining the value of the option at maturity. If stock price is \$36, the put option is worth $max(26-36,0)=$ ___. If stock price is \$24, the put option is worth $max(26-24,0)=$ ____. If stock price is \$16, the put option is worth $max(26-16,0)=$ ____.

Consider the node after one year at which the stock price is \$30. How many shares of stock must be combined with an option to create a riskless portfolio?

We first calculate the option delta for the following year as $(0 - 2)/(36 - 24) =$ ____. So a portfolio of 1 put option and 1/6 share will be a riskless portfolio.

You can verify that the value of this portfolio at maturity is fixed at \$6. So the value of the portfolio when it is created at the time stock price is \$30 should be ________________.

Since this value must equal the value of a put option and the price of 1/6 share of stock, we have $P_1 + \frac{30}{6} = 5.7074$ which gives $P_1 =$ ____________.

Next, consider the node after one year at which the stock price is \$20. Now the option delta for the following year is $(2 - 10)/(24 - 16) =$ ______.

So a portfolio of 1 put option and 1 share will be a riskless portfolio. You can verify that the value of this portfolio at maturity is fixed at \$26.

The value of the portfolio when it is created at the time stock price is \$20 should be $26e^{-0.05\times 1} = \$$___________.

Since this value must equal the value of a put option and the price of 1 share of stock, we have $P_2 + 20 = 24.7320$ which gives $P_2 =$ ___________.

Finally, consider the initial node at which the stock price is \$25. The option delta for the following year is $(0.7074 - 4.7320)/(30 - 20) =$ ___________.

So a portfolio of 1 put option and 0.4025 share will be a riskless portfolio. You can verify that the value of this portfolio after one year is fixed at \$__________.

So the initial value of the portfolio should be $12.7811e^{-0.05\times 1} =$ \$___________. Since this value must equal the value of a put option and the price of 0.4025 share of stock, we have $P + 0.4025 \times 25 = 12.1578$ which gives $P =$ __________.

Risk Neutral Probability Approach

- There are two difficulties in discounted cash flow valuation of options:
    1. The probability distribution of future value of the underlying asset is not known
    2. The appropriate discount rate is not known
- **Risk neutral valuation** allows us to overcome both problems.
- Insight behind risk neutral valuation: We can create a combination of the derivative and the underlying asset in which the price movements exactly offset each other so that the portfolio's cash flow is riskless.
- We considered examples of this earlier in the riskless portfolio approach.
- No discount rate other than risk-free rate is used so investor beliefs or risk aversion aren't necessary if the price of the underlying asset is known.
- That means that even if investors in the world were risk neutral, they should arrive at the same price of the derivative as the real investors, which may be risk-averse.
- We pretend that investors are risk-neutral to solve the above two problems.
- Since risk-neutral investors demand risk-free rate from all assets, the future value of the underlying asset must grow at the risk-free rate.
- Moreover, we can use risk-free rate to discount option cash flows.

- Suppose the underlying asset's price changes from S_0 to S_h or S_l in time Δt.
- In a risk-neutral world, investors assign probabilities to these two future prices so that the expected return on the underlying asset equals the risk-free rate r.
- Let q be the **risk-neutral probability** assigned to price S_h. Then,

$$S_0 e^{r \times \Delta t} = q S_h + (1 - q) S_l,$$

or,

$$q = \frac{S_0 e^{r \times \Delta t} - S_l}{S_h - S_l}.$$

- The risk-neutral probabilities depend only on the prices chosen in the binomial tree of the underlying asset and on the risk-free rate.
- They do not depend on the derivative. This means that risk-neutral probabilities need to be calculated once and can be used to value multiple derivatives.
- It is important to understand that even though risk-neutral probabilities can be used to value derivatives, these are not real-world probabilities because in the real world few assets grow at risk-free rate.

Example 5: Suppose a stock is worth \$25 today and will either be worth 20% more or 20% less in one year. The risk-free rate is 5% per annum continuously compounded. What is the value of an at-the-money European call option that matures after one year?

The binomial trees for the prices of the stock and the call option are shown below:

We first determine the risk-neutral probability that the stock price will increase to \$30. It is given by

$$q = \frac{S_0 e^{r\times\Delta t} - S_l}{S_h - S_l} = \frac{25e^{0.05\times1} - 20}{30 - 20} = \underline{\hspace{3cm}}.$$

At maturity, if stock price is \$30, the call option is worth $max(30 - 25,0) = 5$ and if stock price is \$20, the call option is worth $max(20 - 25,0) = 0$. We determine the current value of call option by calculating expected value at maturity using risk-neutral probabilities and then discounting at risk-free rate:

$$C = \{0.6282 \times 5 + (1 - 0.6282) \times 0\}e^{-0.05\times 1} = \$\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Multiple Periods

- The risk-neutral probability approach can be used for binomial trees with multiple levels by proceeding backwards from maturity and moving left, repeatedly determining the option price one level earlier.
- In general, risk-neutral probabilities may differ at different nodes and need to be calculated separately for each node.
- However, if we create the binomial tree in a particular way, the same risk-neutral probabilities can be used at all nodes. This requires that
    1. The underlying asset does not pay any dividend.
    2. In each step, the stock price changes from S to Su or Sd where $d = 1/u$.
- For such a tree, the risk-neutral probability that the stock price will change from S to Su is given by

$$q = \frac{e^{r \times \Delta t} - d}{u - d}.$$

Example 6: Suppose a stock is worth \$25 today and each year its price increases by 20% or decreases by 20%. The stock does not pay any dividends. The risk-free rate is 5% per annum continuously compounded. What is the value of a European put option with exercise price of \$26 that matures after two years?

The risk-neutral probability that the stock price increases by 20% in a year is given by

$$q = \frac{e^{r\times\Delta t} - d}{u - d} = \frac{e^{0.05\times1} - 0.8}{1.2 - 0.8} = \underline{\hspace{3cm}}.$$

At maturity, if stock price is \$36, the put option is worth $max(26-36,0)=0$, if stock price is \$24, the put option is worth $max(26-24,0)=2$, and if stock price is \$16, the put option is worth $max(26-16,0)=10$. The put option value after one year, when the stock price is \$30, is calculated as

$$P_1 = \{0.6282\times0 + (1-0.6282)\times2\}e^{-0.05\times1} = \$\underline{\hspace{3cm}}.$$

The put option value after one year, when the stock price is \$20, is calculated as

$$P_1 = \{0.6282\times2 + (1-0.6282)\times10\}e^{-0.05\times1} = \$\underline{\hspace{3cm}}.$$

The initial put option value is calculated as

$$P = \{0.6282 \times 0.7074 + (1 - 0.6282) \times 4.7320\} e^{-0.05 \times 1} = \$\_\_\_\_\_\_\_\_\_\_.$$

Valuation of American Options

- American options can be easily valued using binomial tree method by checking at each node whether it is optimal to exercise the option and get intrinsic value or to hold the option to realize higher expected value from possibility of future exercise.
- The difference from European options is that each time we move back in time, the earlier option value calculated using riskless portfolio approach or the risk-neutral probability approach must be compared with intrinsic value of the option and the higher of the two values is the option value.

Example 7: Suppose a stock is worth \$25 today and each year its price increases by 20% or decreases by 20%. The stock does not pay any dividends. The risk-free rate is 5% per annum continuously compounded. What is the value of an American put option with exercise price of \$26 that matures after two years?

The difference from Example 6 is that now we must also consider the possibility of early exercise at nodes before maturity. From example 6, if the stock price is \$30 after one year, and the put option is not exercised, then its value is

$$\{0.6282 \times 0 + (1 - 0.6282) \times 2\}e^{-0.05\times 1} = \$0.7074.$$

Exercising the option is unprofitable because the option is out-of-money so it is optimal to hold on to the option and its value is $P_1 = \$0.7074$. If the stock price is \$20 after one year and the put option is not exercised, its value is

$$\{0.6282 \times 2 + (1 - 0.6282) \times 10\}e^{-0.05\times 1} = \$4.7320.$$

However, exercising the option results in $max(26 - 20,0) = \$6$ which is more than $\$4.7320$ so it is optimal to exercise the option early and its value is $P_2 = \$6$. The initial put option value, if it is not exercised, is

$$P = \{0.6282 \times 0.7074 + (1 - 0.6282) \times 6\}e^{-0.05\times 1} = \$2.5448.$$

Exercising the option results in $max(26 - 25,0) = \$1$ which is less than $\$2.5448$ so it is optimal to hold the option and option value is $\$2.5448$. The difference of $\$1.5448$ from the intrinsic value of $1 is sometimes called the **time value of the option**.

Creating Binomial Tree

- Binomial tree method can be used to value real-life options.
- The methodology is simple but the real difficulty is in choosing the right parameters.
- To create a simple binomial tree, you need to determine the number of steps in the tree (n) and the factor by which price increases (u) or decreases (d) in each time step.
- The condition $d = 1/u$ means that only one of u or d needs to be estimated.
- However, if we choose too high u (too low d), stock price will move a lot in each time step and this higher volatility will cause us to overvalue options.
- Similarly, if we choose too low u (too high d), stock price will move very little in each time step and the low volatility will cause us to undervalue options.
- Thus, it is important to choose u and d to match stock price volatility. Finally, to determine the risk-neutral probabilities of u and d, we need the risk-free rate.

- We need the following information to estimate option value using binomial trees:
    1. The payoff of the option: For call and put options, it is enough to know the exercise price K and whether the option is call or put ($max(S-K,0)$ for call and $max(K-S,0)$ for put). The payoff may be more complicated for a fancy derivative
    2. Whether the option is American or European
    3. The spot price of the underlying asset (S_0)
    4. The time to maturity (T)
    5. The volatility of the return on the underlying asset (σ)
    6. The risk-free rate (r)
    7. Any dividends or any other income from the underlying asset during the life of the option or equivalent dividend yield (y)
    8. The number of time steps in the binomial tree (n)
- Note that there are alternative ways of creating binomial trees.
- These alternative trees may result in different option values if only a few time steps are considered.

- However, as the number of time steps becomes large, all kinds of binomial trees will converge to the same value as long as the trees are created to match the risk-free rate and the volatility.
- The reason people use different kinds of trees is that some trees are easy to create and understand while some other trees, even though not so easy to understand, achieve convergence in fewer time steps.

Cox Ross Rubinstein Specification

The length of each time step is

$$\Delta t = \frac{T}{n}.$$

The multiples by which stock price moves up and down are given by:

$$u = e^{\sigma\sqrt{\Delta t}},$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}},$$

and risk-neutral probability of u is

$$q = \frac{e^{(r-y)\Delta t} - d}{u - d}.$$

Jarrow Rudd Risk Neutral Specification

The length of each time step is

$$\Delta t = \frac{T}{n}.$$

The multiples by which stock price moves up and down are given by:

$$u = e^{(r-y-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}},$$

$$d = e^{(r-y-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}},$$

and risk-neutral probability of u is

$$q = \frac{e^{(r-y)\Delta t} - d}{u - d}.$$

Implied Volatility

- The binomial tree method assumes a certain pattern by which stock price grows. Is it realistic?
- The ultimate test of a valuation method is how closely it matches the market prices. If the binomial tree method does not match the market price, then we can conclude:
    1. The binomial tree method is not perfect or
    2. The parameters used were incorrect.
- Both of these possibilities may be responsible for pricing error.
- However, since binomial tree method is attractive because of the ease of implementing it, practitioners prefer to use this method.
- In face of any discrepancy with traded prices, they adjust the parameters of the model so that the calculated price matches the market price.
- Why use the binomial tree method in the first place if you already know the market price and are adjusting parameters to match market price?
- The answer is that we may have market prices for some options and not others on an underlying asset. We can choose parameters to best fit available market prices and then use these parameters to price other options.

- The basic idea is that even if the binomial tree method is not perfect, the required adjustment in parameters to match price is same for all options.
- Practitioners often adjust the volatility of the underlying asset to match market prices. This volatility is called **implied volatility**.
- That is, a binomial tree is implemented and adjusted for different values of volatility until the price calculated matches the market price.
- The volatility at which this match is achieved is called the implied volatility. Now this tree with implied volatility can be used to price other derivatives.

Volatility Smile

- If the binomial tree method is correct then with appropriate value of volatility, one should be able to price all derivatives on a stock.
- In other words, the implied volatility of a stock calculated using market prices of different derivatives should be same.
- In practice, the implied volatility tends to be lower for options which are close to at-the-money and tends to be higher for deep in-the-money or deep out-of money options.
- A plot of implied volatility against the strike price is called volatility smile because the shape of the graph is like a smile.
- Volatility smile indicates that binomial tree method is not perfect in matching prices of all traded options.

Sample Visual Basic Code

```
'This function returns option price using CRR Binomial tree

'parameters are:

'CallPutFlag - use "c" for call and "p" for put option

'S - spot price

'K - option strike

'T - option maturity

'r - risk free rate

'v - volatility

'ExerciseType - use "a" for American and "e" for european

'N - no of time steps for the binomial tree
```

```
Public Function CRR_Price(CallPutFlag, S, K, T, r, v, ExerciseType, N) As Double
    S0 = S
    If CallPutFlag = "c" Then
        CallPutFlag = 1
    Else
        CallPutFlag = -1
    End If
    dt = T / N
    u = Exp(v * dt ^ 0.5) 'size of up jump
    d = Exp(-v * dt ^ 0.5) 'size of down jump
    p1 = (u - Exp(r * dt)) / (u - d) 'probability of up jump
    p2 = 1 - p1 'probability of down jump
    ReDim Smat(1 To N + 1, 1 To N + 1) 'hlods stock prices
```

```
Smat(1, 1) = S0
For i = 1 To UBound(Smat, 1) - 1
    Smat(1, i + 1) = Smat(1, i) * Exp(v * dt ^ 0.5)
    For j = 2 To i + 1
        Smat(j, i + 1) = Smat(j - 1, i) * Exp(-v * dt ^ 0.5)
    Next j
Next i
```

```
ReDim Cmat(1 To N + 1, 1 To N + 1)


For i = 1 To N + 1
Cmat(i, N + 1) = Application.Max(CallPutFlag * (Smat(i, N + 1) - K), 0)
Next i
For i = UBound(Smat, 2) - 1 To 1 Step -1
    For j = 1 To i
        present_value = Exp(-r * dt) * (p2 * Cmat(j, i + 1) + p1 * Cmat(j + 1, i + 1))
        immediate_val = CallPutFlag * (Smat(j, i) - K)
        If ExerciseType = "a" Then
            Cmat(j, i) = Application.Max(present_value, immediate_val)
        Else
            Cmat(j, i) = Application.Max(present_value, 0)
```

```
        End If
      Next j
    Next i
    CRR_Price = Cmat(1, 1)
End Function
```