Solution to Problem Set II

QF 430: Introduction to Derivatives
Due Tuesday, March 10

- Please submit neatly handwritten or typed answers. You can submit electronically a **single pdf** file through Canvas.
- While you may have done your work in one or more spreadsheets, do not turn in spreadsheets. The final product is a text document, possibly with equations, formulas, tables, graphs etc.
- Show your steps or reasoning.
- Do not round too much in intermediate calculation steps. Aim for accuracy of at least four decimal places in interest rates (0.0337 or 3.37% should not be approximated as 3.4%). For currency amounts, relative accuracy matters. So \$98,473.52 can be approximated as \$98,474 but \$3.52 cannot be approximated as \$4. In problems where arbitrage strategy profit/loss depends on a comparison of two prices, maintain extra precision in intermediate steps. For example, when comparing against a given price of \$98,473.80, approximating \$98,473.52 with \$98,474 can lead to opposite inference.
- START WORK EARLY. The problems take time.

1 Forward and Futures Prices

Problem 1.1. *The risk-free rate of interest is 3% per annum with continuous compounding, and the dividend yield on a stock index is 1.8% per annum. The current value of the index is 500. What is the six-month futures price?*

Solution. The six month futures price is

$$500e^{(0.03-0.018)\times 0.5} = 503.009.$$

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Problem 1.2. *The risk-free rate of interest is 5% per annum with continuous compounding. A stock is trading at \$60. The stock pays dividend of \$0.50 every quarter. The next dividend will be paid after 1 month. What is the six-month futures price of the stock?*

Solution. The six month futures price is

$$(60 - 0.5e^{-0.05\times 1/12} - 0.5e^{-0.05\times 4/12})e^{0.05\times 0.5} = \boxed{\mathbf{60.5042}}.$$

■

Problem 1.3. *The risk-free rate of interest is 4% per annum with continuous compounding, and the fixed dividend yield on a stock index is 2% per annum. The three-month stock index futures price is $80 and the six-month stock index futures price is $81. Describe an arbitrage strategy.*

Solution. The ratio of the two futures prices is $F_0^{6M}/F_0^{3M} = 81/80 = 1.0125$. Based on interest rate and dividend yield, this ratio should be

$$\frac{F_0^{6M}}{F_0^{3M}} = \frac{S_0 e^{(0.04-0.02)\times 0.5}}{S_0 e^{(0.04-0.02)\times 0.25}} = e^{(0.04-0.02)\times 0.25} = 1.005.$$

We conclude that the six-month futures price is too high relative to the three-month futures price (this may be because six-month futures price is too high while three-month price is correct or six-month futures price is correct but three-month futures price is too low or both prices may be incorrect). The maxim of buy low, sell high suggests the following arbitrage strategy. For some positive integer N, take a $\boxed{\textbf{long position in } N \textbf{ three-month futures}}$, take a $\boxed{\textbf{short position in } N \textbf{ six-month futures}}$, and lock in a $\boxed{\textbf{loan of } \$80N}$ to be borrowed after three months and repaid after six months. The borrowed amount will let you buy (the stocks in) the stock index after three months using long futures contracts. Your interest liability on loan increases at the rate of 4% per annum continuously compounded but part of that is offset with the dividend income earned at the rate of 2% per annum continuously compounded, resulting in a net repayment amount growth of 2% per annum continuously compounded. Six months from today, sell (the stocks in) the stock index at $\$81N$ using short futures contracts and repay the loan amount of $\$80Ne^{0.02\times 0.25} = \$80.401N$ for a $\boxed{\textbf{net profit of } \$0.599N \textbf{ after six months}}$, equivalent to a $\boxed{\textbf{net profit of } \$0.599Ne^{-0.04\times 0.5} = \$0.587N \textbf{ today}}$. ■

Problem 1.4. *The three-month interest rates in the United Kingdom and the United States are 0.7% and 1.6% per annum, respectively, with continuous compounding. The spot price of the British pound sterling is $1.3000. The futures price for a contract deliverable in three months is $1.3050. What arbitrage opportunities does this create?*

Solution. The theoretical futures price is

$$\$1.3000e^{(0.016-0.007)\times 3/12} = \$1.3029.$$

The actual futures price is too high. This suggests that an arbitrageur should buy British pound sterlings and short British pound sterling futures. Specifically, $\boxed{\textbf{borrow \$1.3}}$, $\boxed{\textbf{buy £1 for \$1.3}}$, $\boxed{\textbf{lend £1}}$, and $\boxed{\textbf{short three-month futures contract on } £1e^{0.007\times 0.25}}$. After three months, get back $£1e^{0.007\times 0.25}$ from pounds lent, sell using short futures contract to get $\$1e^{0.007\times 0.25} \times 1.305 = \$1.3073$ and repay the \$ loan amount of $\$1.3e^{0.016\times 0.25} = \$1.3052$, for a $\boxed{\textbf{net profit of \$0.0021}}$. This strategy can be scaled up. ■

Problem 1.5. *Gold price is $1602.4 per ounce. The storage costs are $2.16 per ounce per year payable quarterly in advance. Assuming that interest rates are 2% per annum with continuous compounding for all maturities, calculate the futures price of gold for delivery in six months.*

Solution. The present value of the storage costs for six months are

$$0.54 + 0.54e^{-0.02\times 0.25} = 1.0773 \text{ per ounce.}$$

The six-month futures price is

$$F_0 = (1602.4 + 1.0773)e^{0.02\times 0.5} = \boxed{\textbf{\$1619.593 \ \ per ounce}}.$$

■

Problem 1.6. *When a known future cash outflow in a foreign currency is hedged by a company using a forward contract, there is no foreign exchange risk. When it is hedged using futures contracts, the daily settlement process does leave the company exposed to some risk. Explain the nature of this risk. In particular, consider whether the company is better off using a futures contract or a forward contract when*

(a) *The value of the foreign currency falls rapidly during the life of the contract*

(b) *The value of the foreign currency rises rapidly during the life of the contract*

(c) *The value of the foreign currency first rises and then falls back to its initial value*

(d) *The value of the foreign currency first falls and then rises back to its initial value*

Assume that the forward price equals the futures price.

Solution. In total the gain or loss under a futures contract is equal to the gain or loss under the corresponding forward contract. However the timing of the cash flows is different. When the time value of money is taken into account, a futures contract may prove to be more valuable or less valuable than a forward contract. Of course the company does not know in advance which will work out better. The long forward contract provides a perfect hedge. The long futures contract provides a slightly imperfect hedge.

(a) In this case, the $\boxed{\textbf{forward contract would lead to a slightly better outcome}}$. The company will make a loss on its hedge. If the hedge is with a forward contract, the whole of the loss will be realized at the end. If it is with a futures contract, the loss will be realized day by day throughout the contract. On a present value basis, the former is preferable.

(b) In this case, the $\boxed{\textbf{futures contract would lead to a slightly better outcome}}$. The company will make a gain on the hedge. If the hedge is with a forward contract, the gain will be realized at the end. If it is with a futures contract, the gain will be realized day by day throughout the life of the contract. On a present value basis, the latter is preferable.

(c) In this case, the $\boxed{\textbf{futures contract would lead to a slightly better outcome}}$. This is because it would involve positive cash flows early and negative cash flows later.

(d) In this case, the $\boxed{\textbf{forward contract would lead to a slightly better outcome}}$. This is because, in the case of the futures contract, the early cash flows would be negative and the later cash flow would be positive.

$\blacksquare$

Problem 1.7. *The current USD/Swiss franc exchange rate is 1.0200 dollars per Swiss franc. The one-year forward exchange rate is 1.0440 dollars per Swiss franc. The one-year USD interest rate is 1.6% per annum continuously compounded. Estimate the one-year Swiss franc interest rate.*

Solution. If the one-year Swiss franc interest rate is r_f, then

$$1.0440 = 1.02000e^{(0.016-r_f)\times 1}$$

so that

$$0.016 - r_f = \ln(\frac{1.044}{1.02})$$

or

$$r_f = 0.016 - \ln(\frac{1.044}{1.02}) = -0.00726 = \boxed{\textbf{-0.726\% pacc}}.$$

$\blacksquare$

2 Interest Rate Derivatives

Problem 2.1. *It is March 3, 2020. The quoted price of a U.S. government bond with a 5% per annum coupon (paid semiannually) is 108-15. The bond matures on May 16, 2028.*

(a) What is the cash price?

(b) How does your answer change if it is a corporate bond?

Solution. (a) The actual number of days between the last coupon date of November 16, 2019 and March 3, 2020 is 108. The number of days between the last coupon date of November 16, 2019 and the next coupon date of May 16, 2020 is 182. The accrued interest for the government bond is $2.5 \times 108/182 = 1.4835$. The cash price of the bond is $108\frac{15}{32} + 1.4835 = \boxed{\mathbf{109.9523}}$.

(b) Using a 30/360 day count, the number of days between the last coupon date of November 16, 2019 and March 3, 2020 is 107 and the number of days between the last coupon date of November 16, 2019 and the next coupon date of May 16, 2020 is 180. The accrued interest for the corporate bond is $2.5 \times 107/180 = 1.4861$. The cash price of the bond is $108\frac{15}{32} + 1.4861 = \boxed{\mathbf{109.9549}}$.

$\blacksquare$

Problem 2.2. *The price of a 90-day Treasury bill is quoted as 10.80. What continuously compounded return (on an actual/365 basis) does an investor earn on the Treasury bill for the 90-day period?*

Solution. The cash price of the Treasury bill (assuming face value of \$100) is

$$\$100 - \frac{90}{360} \times \$10.8 = \$97.30.$$

The annualized continuously compounded return is

$$\frac{365}{90}\ln(\frac{100}{97.3}) = 0.11105 = \boxed{\textbf{11.105\% pacc}}.$$

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Problem 2.3. *Suppose that the Treasury bond futures price is 97-19. Which of the following four bonds is cheapest to deliver?*

Bond	Quoted Price	Conversion Factor
1	93-6	0.9547
2	98-21	1.0088
3	107-12	1.1019
4	118-29	1.2215

Solution. The cheapest-to-deliver bond is the one for which

$$\text{Quoted Price} - \text{Futures Price} \times \text{Conversion Factor}$$

is least. Calculating this cost for each of the 4 bonds, we get

$$\begin{aligned}
&\text{Bond 1: } 93\frac{6}{32} - 97\frac{19}{32} \times 0.9547 &&= 93.1875 - 97.59375 \times 0.9547 &&= \boxed{\mathbf{0.0147}} \\
&\text{Bond 2: } 98\frac{21}{32} - 97\frac{19}{32} \times 1.0088 &&= 98.65625 - 97.59375 \times 1.0088 &&= \boxed{\mathbf{0.2037}} \\
&\text{Bond 3: } 107\frac{12}{32} - 97\frac{19}{32} \times 1.1019 &&= 107.375 - 97.59375 \times 1.1019 &&= \boxed{\mathbf{-0.1636}} \\
&\text{Bond 4: } 118\frac{29}{32} - 97\frac{19}{32} \times 1.2215 &&= 118.90625 - 97.59375 \times 1.2215 &&= \boxed{\mathbf{-0.3045}}.
\end{aligned}$$

$\boxed{\textbf{Bond 4}}$ is the cheapest to deliver. ■

Problem 2.4. *Download the treasury conversion factors spreadsheet from CME group website at* [*https://www.cmegroup.com/trading/interest-rates/treasury-conversion-factors.html*](https://www.cmegroup.com/trading/interest-rates/treasury-conversion-factors.html)*. Open the tab with conversion factors and scroll down to the table for U.S. Treasury Bond Futures Contracts. Each row provides conversion factors for a specific bond. Use the first row from the table. Find the conversion factors for the futures contract with the two earliest delivery months. Fill in the bond information, the delivery months, and the conversion factors below. Next, perform and show your conversion factor calculations to verify both conversion factor values.*

Coupon	
Maturity Date	
CUSIP	
Delivery Month 1	
Conversion Factor 1	
Delivery Month 2	
Conversion Factor 2	

Solution. The details of the specified conversion factors from the spreadsheet follow.

Coupon	4.5
Maturity Date	2/15/36
CUSIP	912810FT0
Delivery Month 1	March 2020
Conversion Factor 1	0.8484
Delivery Month 2	June 2020
Conversion Factor 2	0.85

For the March 2020 delivery month, the maturity of the bond is rounded down from February 15, 2035 to the nearest multiple of three-months from March 1, 2020. The standardized bond matures on December 1, 2035, has face value of \$100 and pays coupons of \$2.25 on June 1 and December 1 every year. The next coupon is paid on June 1, 2020 and there are 31 coupon payments after that. Assuming yield to be 6% pasc, the bond price immediately after the coupon payment on June 1, 2020 is

$$\frac{2.25}{0.03}\left(1-\frac{1}{1.03^{31}}\right)+\frac{100}{1.03^{31}}=\$84.9997.$$

Adding the coupon payable on June 1, 2020 and discounting back, the cash price on March 1, 2020 is

$$\frac{84.9997+2.25}{1.03^{0.5}}=\$85.9697.$$

Subtracting the accrued coupon, the quoted price is

$$85.9697-\frac{2.25}{2}=\$84.8447.$$

The conversion factor is $84.8447/100 = \boxed{\mathbf{0.8484}}$.
For the June 2020 delivery month, the maturity of the bond is rounded down from February 15, 2035 to the nearest multiple of three-months from June 1, 2020. The standardized bond matures on December 1, 2035, has face value of \$100 and pays coupons of \$2.25 on June 1 and December 1 every year. There are 31 coupons payments with the next coupon payable six months after June 1, 2020. Assuming yield to be 6% pasc, the cash price (and quoted price) of the bond on June 1, 2020 is

$$\frac{2.25}{0.03}\left(1-\frac{1}{1.03^{31}}\right)+\frac{100}{1.03^{31}}=\$84.9997.$$

The conversion factor is $84.9997/100 = \boxed{\mathbf{0.8500}}$. ■

Problem 2.5. *It is March 6, 2020. The cheapest-to-deliver bond in a September 2020 Treasury bond futures contract is a 5% coupon bond, and delivery is expected to be made on September 30, 2020. Coupon payments on the bond are made on January 1 and July 1 each year. The rate of interest with continuous compounding is 3% per annum for all maturities. The conversion factor for the bond is 0.8851. The current quoted bond price is \$113.82. Calculate the quoted futures price for the contract.*

Solution. There have been 65 days since the previous coupon date of January 1, 2020. There are 182 days between the previous coupon date of January 1, 2020 and the next coupon date of July 1, 2020. Therefore, the cash bond price is $S_0 = 113.82 + 65/182 \times 2.50 = \$114.7129$. The only coupon on the bond before delivery of the futures contract is a coupon of \$2.50 to be received after 117 days. The present value of the coupon is $2.5e^{-0.03\times 117/365} = 2.4761$. The delivery on the futures contract will happen after 179 days. The cash futures price if it were written on the 5% bond would therefore, be

$$(114.7129 - 2.4761)e^{0.03\times\frac{179}{365}} = \$113.9003.$$

At delivery there are 121 days of accrued interest and the number of days from the previous coupon date of July 1, 2040 to January 1, 2041 is 184. The quoted futures if the contract were written on the 5% bond would therefore, be

$$113.9003 - 2.50 \times \frac{121}{184} = 112.2426.$$

The quoted price is

$$112.2426/0.8851 = \boxed{\mathbf{126.8135}}.$$

■

Problem 2.6. *Portfolio A consists of a two-year zero-coupon bond with a face value of \$3,000 and a 10-year zero-coupon bond with a face value of \$7,000. Portfolio B consists of a 6.88-year zero-coupon bond with a face value of \$10,000. The current yield on all bonds is 5% per annum.*

(a) Show that both portfolios have the same duration.

(b) Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same (difference is less than 0.01%).

(c) What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?

Solution. (a) The duration of Portfolio A is

$$\frac{2 \times 2,000e^{-0.05\times 2} + 10 \times 7,000e^{-0.05\times 10}}{2,000e^{-0.05\times 2} + 7,000e^{-0.05\times 10}} = \boxed{\mathbf{6.88}}$$

Since this is also the duration of Portfolio B, the two portfolios have the same duration.

(b) The value of Portfolio A is

$$2,000e^{-0.05\times 2} + 7,000e^{-0.05\times 10} = \$6,960.227.$$

When yields increase by 10 basis points its value becomes

$$2,000e^{-0.051\times 2} + 7,000e^{-0.051\times 10} = \$6,912.558.$$

The percentage change in value is

$$\frac{6,912.558 - 6,960.227}{6960.227} = \boxed{\mathbf{-0.685\%}}.$$

The value of Portfolio B is

$$10,000e^{-0.05\times 6.88} = \$7,089.289.$$

When yields increase by 10 basis points its value becomes

$$10,000e^{-0.051\times 6.88} = \$7,040.682.$$

The percentage change in value is

$$\frac{7,040.682 - 7,089.289}{7,089.289} = \boxed{\textbf{-0.686\%}}.$$

The percentage changes in the values of the two portfolios for a 10 basis point increase in yields are the same (the difference is less than 0.01%).

(c) When yields increase by 5%, the value of Portfolio A becomes

$$2,000e^{-0.10\times 2} + 7,000e^{-0.10\times 10} = \$5,031.348.$$

The percentage change in value is

$$\frac{5,031.348 - 6,960.227}{6960.227} = \boxed{\textbf{-27.713\%}}.$$

When yields increase by 5%, the value of Portfolio B becomes

$$10,000e^{-0.10\times 6.88} = \$5,025.802.$$

The percentage change in value is

$$\frac{5,025.802 - 7,089.289}{7,089.289} = \boxed{\textbf{-29.107\%}}.$$

Since the magnitude of the percentage change in value of Portfolio A is less than that of Portfolio B, Portfolio A has a greater convexity and less interest rate risk.

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Problem 2.7. *A portfolio manager plans to use a Treasury bond futures contract to hedge a bond portfolio over the next three months. The portfolio is worth $54 million and will have a duration of 6 years in three months. The futures price is 120, and each futures contract is on $100,000 of bonds. The bond that is expected to be cheapest to deliver will have a duration of 11 years at the maturity of the futures contract.*

(a) What position in futures contracts is required?

(b) Suppose that all rates increase over the three months, but long-term rates increase less than short-term and medium-term rates. What is the effect of this on the performance of the hedge?

Solution. (a) The number of short futures contracts required is

$$\frac{54,000,000 \times 6}{120,000 \times 11} = 245.45$$

Rounding to the nearest whole number, $\boxed{\textbf{245 contracts should be shorted}}$.

(b) In this case $\boxed{\textbf{the gain on the short futures position is likely to be less than the loss on the bond portfolio}}$. This is because the gain on the short futures position depends on the size of the movement in long-term rates and the loss on the bond portfolio depends on the size of the movement in medium-term rates. Duration-based hedging assumes that the movements in the two rates are the same.

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3 Swaps

Problem 3.1. *A financial institution has agreed to pay 4% per annum and to receive three-month LIBOR in return in an interest rate swap. The notional principal is $80 million and payments are exchanged every three months. The swap has a remaining life of 11 months. Three-month forward LIBOR for all maturities is currently 4.8% per annum. The three-month LIBOR rate one month ago was 4.4% per annum. OIS rates for all maturities are currently 4.5% with continuous compounding. All other rates are compounded quarterly. What is the value of the swap?*

Solution. We can value the swap as a series of forward rate agreements (FRAs). In each FRA, the fixed rate is 4% paqc. The floating rate is 4.4% paqc for the first FRA and the forward rate of 4.8% for subsequent FRAs. The cash flows are discounted using OIS rates. The value of the swap is

$$80,000,000 \times (4.4\% - 4\%) \times 0.25e^{-0.045\times 2/12} + 80,000,000 \times (4.8\% - 4)\% \times 0.25e^{-0.045\times 5/12} +$$
$$80,000,000 \times (4.8\% - 4\%) \times 0.25e^{-0.045\times 8/12} + 80,000,000 \times (4.8\% - 4\%) \times 0.25e^{-0.045\times 11/12}$$
$$= \boxed{\textbf{\$545,236}}$$

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Problem 3.2. *A U.S. company wants to borrow sterling at a fixed rate of interest. A British company wants to borrow U.S. dollars at a fixed rate of interest. They have been quoted the following interest rates (per annum):*

	US Dollars	Sterling
US company	9.4%	8.8%
British company	8.8%	7.6%

Design a swap that will net a bank, acting as intermediary, 10 basis points per annum and that will produce a gain of 25 basis points per annum for each of the two companies.

Solution. The spread between the interest rates offered to the US company and the British company is 1.2% (or 120 basis points) on sterling loans and 0.6% (or 60 basis points) on U.S. dollar loans. The total benefit to all parties from the swap is therefore, 120-60=60 basis points.

It is therefore, possible to design a swap which will earn 10 basis points for the bank while making each company 25 basis points better off than they would be by going directly to financial markets. One possible swap is shown in Figure 1.

Figure 1: One possible swap

The US company borrows at an effective rate of 8.55% per annum in sterling. The British company borrows at an effective rate of 8.55% per annum in US dollars. Each pays 0.25% lower rate than what it would have paid by directly borrowing in that currency. The bank earns 95 basis points in sterling and pays 85 basis points in dollars for a net 10 basis-point spread. The way in which currency swaps such as this operate is as follows. Principal amounts in dollars and sterling that are roughly equivalent are chosen. These principal amounts flow in the opposite direction to the arrows at the time the swap is initiated. Interest payments then flow in the same direction as the arrows during the life of the swap and the principal amounts flow in the same direction as the arrows at the end of the life of the swap. The swap can be structured in different ways by increasing or decreasing all the rates that the financial institution pays or receives by the same percentage points.
Note that the bank is exposed to some exchange rate risk in the swap. It earns 95 basis points in sterling and pays 85 basis points in U.S. dollars. This exchange rate risk could be hedged using forward contracts. ■

Problem 3.3. *Suppose that the term structure of risk-free interest rates is flat in the United States and Australia. The USD interest rate is 5% per annum and the AUD rate is 6% per annum. The current value of the AUD is 0.65 USD. Under the terms of a swap agreement, a financial institution pays 5% per annum in AUD and receives 4% per annum in USD. The principals in the two currencies are $13 million USD and 20 million AUD. Payments are exchanged every year, with one exchange having just taken place. The swap will last two more years. What is the value of the swap to the financial institution? Assume all interest rates are continuously compounded.*

Solution. The financial institution is long a dollar bond and short a USD bond. The value of the dollar bond (in millions of dollars) is

$$13 \times 0.04e^{-0.05\times 1} + 13 \times 1.04e^{-0.05\times 2} = 12.728.$$

The value of the AUD bond (in millions of AUD) is

$$20 \times 0.05e^{-0.06\times 1} + 20 \times 1.05e^{-0.06\times 2} = 19.567.$$

The value of the swap (in millions of dollars) is therefore,

$$12.728 - 19.567 \times 0.65 = 0.00943$$

or $\boxed{\textbf{\$9,430}}$. As an alternative we can value the swap as a series of forward foreign exchange contracts. The one-year forward exchange rate is $0.65e^{-0.01} = 0.6435$. The two-year forward exchange rate is $0.65e^{-0.02} = 0.6371$. The value of the swap in millions of dollars is therefore,

$$(0.52 - 1 \times 0.6435)e^{-0.05\times 1} + (13.52 - 21 \times 0.6371)e^{-0.05\times 2} = 0.00943$$

which is in agreement with the first calculation. ■

Problem 3.4. *A company wants a swap where it receives semiannual payments at 6.5% per annum with semiannual compounding on a principal of \$5 million. The five-year swap rate with semiannual cash flows is 6% per annum with semiannual compounding. The OIS zero curve is flat at 5% per annum with continuous compounding. How much should a derivatives dealer charge the company?*

Solution. Since the swap rate is 6% per annum, a swap where the company receives 6% per annum and pays LIBOR is worth zero. The company wants to receive 6.5% per annum, that is an extra 0.5% per annum. The value of this swap is the present value of $0.5 \times (0.065 - 0.06) \times \$5,000,000 = \$12,500$ received every six months for five years. This is

$$\sum_{i=1}^{10} \$12,500 e^{-0.05\times 0.5i} = \boxed{\textbf{\$109,222.87}}.$$

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