Problem Set III Solutions

QF 430: Introduction to Derivatives
Due Monday, October 24

Please submit neatly handwritten or typed answers. You can turn in paper submissions in class or submit electronically a single pdf file through Canvas. Show your steps or reasoning. Do not round too much in intermediate calculation steps. Aim for accuracy of four decimal places in interest rates (0.0337 or 3.37%).

1 Interest Rate Derivatives

Problem 1.1. The Wall Street Journal's quoted ask price on October 13, 2022 for a 2.875% U.S. government treasury note maturing on May 15, 2032 is 91.1240. This quote means that the ask quote is \$91 and 12.5 32nds for every \$100 of face value. Assume that the note pays semiannual coupons. Treasury bond transactions settle the next business day so this is the quote for making the payment and getting the bond on October 14, 2022.

(a) What is the cash price? You can optionally verify your answer to within a cent using the reported (semiannually compounded) yield of 3.963%. See Treasury Bond Pricing Example on Canvas.

(b) How does your answer change if the bond is a corporate bond?

Reference: Slides Examples 1-2

Solution.

(a) The actual number of days between the last coupon date of May 15, 2022 and October 14, 2022 is 152. The number of days between the last coupon date of May 15, 2022 and the next coupon date of November 15, 2022 is 184. The accrued interest for the government bond is $2.875/2 \times 152/184 = 1.1875$. The cash price of the bond is $91\frac{12.5}{32} + 1.1875 = \boxed{\mathbf{92.5781}}$. To verify, we calculate bond price using the yield of 3.963%. The next coupon is in 32 days and there are 19 more coupons after that.

$$\frac{1}{(1+\frac{0.03963}{2})^{32/184}}\left\{\frac{2.875}{2}+\frac{2.875/2}{0.03963/2}\left(1-\frac{1}{(1+\frac{0.03963}{2})^{19}}\right)+\frac{100}{(1+\frac{0.03963}{2})^{19}}\right\}$$
$$= 92.5774$$

which matches the cash price to within a cent.

(b) Using a 30/360 day count, the number of days between the last coupon date of May 15, 2022 and October 14, 2022 is 149 and the number of days between the last coupon date of May 15, 2022 and the next coupon date of November 15, 2022 is 180. The accrued interest for the corporate bond is $2.875/2 \times 149/180 = 1.1899$. The cash price of the bond is $91\frac{12.5}{32} + 1.1899 = \boxed{\mathbf{92.5806}}$.

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Problem 1.2. The price of a 90-day Treasury bill is quoted as 4.40. What continuously compounded return (on an actual/365 basis) does an investor earn on the Treasury bill for the 90-day period?

Reference: Slides Example 4

Solution. The cash price of the Treasury bill (assuming face value of \$100) is

$$\$100 - \frac{90}{360} \times \$4.4 = \$98.90.$$

The annualized continuously compounded return is

$$\frac{365}{90} \ln(\frac{100}{98.90}) = 0.044858 = \boxed{\mathbf{4.4858\% \quad pacc}}.$$

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Problem 1.3. Download the treasury conversion factors spreadsheet from CME group website at https://www.cmegroup.com/trading/interest-rates/treasury-conversion-factors.html. Open the tab with conversion factors and scroll down to the table titled "U.S. TREASURY BOND FUTURES CONTRACTS." Each row provides conversion factors for a specific bond. Use the *second* row from the table (we will ignore the first row because it has only one conversion factor). Find the conversion factors for the futures contract with the two earliest delivery months. Fill in the bond information, the delivery months, and the conversion factors below. Next, perform and show your conversion factor calculations to verify both conversion factor values.

Coupon	
Maturity Date	
CUSIP	
Delivery Month 1	
Conversion Factor 1	
Delivery Month 2	
Conversion Factor 2	

Reference: Slides Example 6

Solution. The details of the specified conversion factors from the spreadsheet follow.

Coupon	4.5
Maturity Date	5/15/38
CUSIP	912810PX0
Delivery Month 1	December 2022
Conversion Factor 1	0.8514
Delivery Month 2	March 2023
Conversion Factor 2	0.853

For the December 2022 delivery month, the maturity of the bond is rounded down from May 15, 2038 to the nearest multiple of three-months from December 1, 2022. The standardized bond matures on March 1, 2038, has face value of \$100 and pays coupons of \$2.25 on March 1 and September 1 every year. The next coupon is paid on March 1, 2023 and there are 30 coupon payments after that. Assuming yield to be 6% pasc, the quoted bond price immediately after the coupon payment on March 1, 2023 is

$$\frac{2.25}{0.03}\left(1-\frac{1}{1.03^{30}}\right)+\frac{100}{1.03^{30}}=\$85.2997.$$

Adding the coupon payable on March 1, 2023 and discounting back, the cash price on December 1, 2022 is

$$\frac{85.2997+2.25}{1.03^{0.5}}=\$86.2653.$$

Subtracting the accrued coupon, the quoted price is

$$86.2653-\frac{2.25}{2}=\$85.1403.$$

The conversion factor is $85.1403/100=\boxed{\mathbf{0.8514}}$.

For the March 2023 delivery month, the maturity of the bond is rounded down from May 15, 2038 to the nearest multiple of three-months from June 1, 2023. The standardized bond matures on March 1, 2038, has face value of \$100 and pays coupons of \$2.25 on March 1 and September 1 every year. There are 30 coupons payments with the next coupon payable six months after March 1, 2023. Assuming yield to be 6% pasc, the cash price (and quoted price) of the bond on March 1, 2023 is

$$\frac{2.25}{0.03}\left(1-\frac{1}{1.03^{30}}\right)+\frac{100}{1.03^{30}}=\$85.2997.$$

The conversion factor is $85.2997/100=\boxed{\mathbf{0.8530}}$. ■

Problem 1.4. Suppose that the Treasury bond futures price is 102-11. Which of the following four bonds is cheapest to deliver?

Bond	Quoted Price	Conversion Factor
1	91-19	0.8913
2	97-28	0.9535
3	107-10	1.0441
4	116-18	1.1349

Reference: Slides Example 6

Solution. The cheapest-to-deliver bond is the one for which

$$\text{Quoted Price} - \text{Futures Price} \times \text{Conversion Factor}$$

is least. Calculating this cost for each of the 4 bonds, we get

$$\begin{aligned}
&\text{Bond 1: } 91\tfrac{19}{32} - 102\tfrac{11}{32} \times 0.8913 && = 91.59375 - 102.34375 \times 0.8913 && = \boxed{\mathbf{0.0684}} \\
&\text{Bond 2: } 97\tfrac{28}{32} - 102\tfrac{11}{32} \times 0.9535 && = 97.875 - 102.34375 \times 0.9535 && = \boxed{\mathbf{-0.0375}} \\
&\text{Bond 3: } 107\tfrac{10}{32} - 102\tfrac{11}{32} \times 1.0441 && = 107.3125 - 102.34375 \times 1.0441 && = \boxed{\mathbf{0.0965}} \\
&\text{Bond 4: } 116\tfrac{18}{32} - 102\tfrac{11}{32} \times 1.1349 && = 116.5625 - 102.34375 \times 1.1349 && = \boxed{\mathbf{0.0225}}.
\end{aligned}$$

$\boxed{\textbf{Bond 2}}$ is the cheapest to deliver. ■

Problem 1.5. It is October 14, 2022. The cheapest-to-deliver bond in an April 2023 Treasury bond futures contract is a 3% coupon bond, and delivery is expected to be made on April 30, 2023. Coupon payments on the bond are made on March 1 and September 1 each year. The rate of interest with continuous compounding is 4% per annum for all maturities. The conversion factor for the bond is 0.6403. The current quoted bond price is \$85.31. Calculate the quoted futures price for the contract.

Reference: Slides Example 7

Solution. There have been 43 days since the previous coupon date of September 1, 2022. There are 181 days between the previous coupon date of September 1, 2022 and the next coupon date of March 1, 2023. Therefore, the cash bond price is $S_0 = 85.31 + 43/181 \times 1.5 =$ \$85.6664.
The only coupon on the bond before delivery of the futures contract is a coupon of \$1.50 to be received after 181-43=138 days. The present value of the coupon is $1.5e^{-0.04\times 138/365} = 1.4775$.

The delivery on the futures contract will happen after 198 days. The cash futures price if it were written on the 3% bond would therefore, be

$$(85.6664 - 1.4775)e^{0.04\times\frac{198}{365}} = \$86.0356.$$

At delivery, there are 60 days of accrued interest and the number of days from the previous coupon date of March 1, 2023 to the next coupon date of September 1, 2023 is 184. The quoted futures price if the contract were written on the 3% bond would therefore, be

$$86.0356 - 1.5 \times \frac{60}{184} = 85.5465.$$

The quoted price is

$$85.5465/0.6403 = \boxed{\mathbf{133.6038}}.$$

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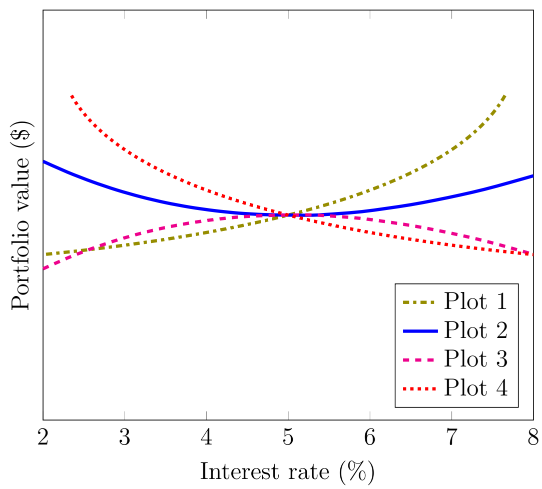
Problem 1.6. Suppose the yield curve is flat at 5% per annum continuously compounded. Moreover, yield curve can only have parallel shifts. That is, interest rates can change but interest rate stays same for all maturities.

(a) How much should you invest in a 5-year zero coupon bond such that an increase in (continuously compounded) interest rate of 0.01% results in a loss of 10 cents? How much does this investment pay after 5 years?

(b) After the investment you calculated in the 5-year bond, how much of a 10-year zero coupon bond should you short to immunize yourself against small changes in interest rate? That is, such that the gain in your short position on an interest rate increase of 0.01% is 10 cents. How much do you need to repay after 10 years on the short position?

(c) Duration matching does not protect against large interest rate changes because bond prices are convex in interest rates (prices of longer-term bonds are more convex). Which of the following plots shows how your portfolio value changes with interest rates? Why?

Reference: Duration



Solution.

(a) Since the duration of the 5-year ZCB is 5 years, A 0.01% increase in interest rate will lower bond price by $5 \times 0.01\% = 0.05\%$. The loss, 0.05% of the investment equals ten cents so the investment must be $\$0.10/0.0005 = \boxed{\mathbf{\$200}}$. The investment will grow to $200e^{0.05\times5} = \boxed{\mathbf{\$256.81}}$.

(b) A portfolio is immunized against interest rate risk if its duration is 0. Let x be the amount for which the 10-year bond is shorted. Then, the duration the portfolio is

$$\frac{200 \times 5 - x \times 10}{200 - x}.$$

This equals zero if $x = 100$. You should short $\boxed{\mathbf{\$100}}$ of the 10-year bond. You need to repay $100e^{0.05\times10} = \boxed{\mathbf{\$164.87}}$.

(c) The portfolio is immunized against small interest rate changes so the plot must be horizontal at 5%. This rules out Plot 1 and Plot 4. A bond's value is convex in interest rate so for large increase or decrease in interest rate the bond price is higher than what duration suggests. The long position in 5-year ZCB would result in a convex value (U-shaped). The short position in 10-year ZCB would result in a concave value (inverted U-shaped). The second effect dominates as the 10-year bond is much more sensitive to interest rates than the 5-year bond. Therefore, the portfolio value is represented by $\boxed{\textbf{Plot 3}}$. You can verify this with actually calculating portfolio value as $256.81e^{-5r} - 164.87e^{-10r}$ for different values of r.

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Problem 1.7. A portfolio manager wants to hedge a bond portfolio against interest rate risk over the next three months. The manager plans to hedge using a Treasury bond futures contract maturing in four months. The portfolio is worth \$38 million and will have a duration of 11.2 years in four months. The futures price is 113, and each futures contract is on \$100,000 of bonds. The bond that is expected to be cheapest to deliver will have a duration of 17.5 years at the maturity of the futures contract.

(a) What position in futures contracts is required?

(b) Suppose that all rates increase over three months, but longer-term rates increase less than shorter-term rates. What is the effect of this on the performance of the hedge?

Reference: Slides Example 9

Solution.

(a) The number of short futures contracts required is

$$\frac{38,000,000 \times 11.2}{113,000 \times 17.5} = 215.22$$

Rounding to the nearest whole number, $\boxed{\textbf{215 contracts should be shorted}}$.

(b) In this case $\boxed{\textbf{the gain on the short futures position is likely to be more than the loss on the bond portfolio}}$. This is because the gain on the short futures position depends on the size of the movement in long-term rates and the loss on the bond portfolio depends on the size of the movement in medium-term rates. Duration-based hedging assumes that the movements in the two rates are the same.

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2 Swaps

Problem 2.1. A U.S. company wants to borrow Australian dollars (AUD) at a fixed rate of interest. An Australian company wants to borrow U.S. dollars at a fixed rate of interest. They have been quoted the following interest rates (per annum):

	US Dollars	AUD
US company	7.8%	8.4%
Australian company	8.2%	8.0%

Design a swap that will net a bank, acting as intermediary, 10 basis points per annum and that will produce a gain of 35 basis points per annum for each of the two companies.

Reference: Slides/Book - Currency Swaps Comparative Advantage

Solution. The US company has an advantage over the Australian company of 40 basis points in USD market because it can raise USD at 7.8%, 0.4% less than the Australian company. The US company has an advantage over the Australian company of -40 basis points (or disadvantahe of 40 basis points) in AUD market because it can raise AUD at 8.4%, 0.4% more than the Australian company. The total benefit to all parties from the swap is therefore, 40-(-40)=80 basis points.
It is therefore, possible to design a swap which will earn 10 basis points for the bank while making each company 35 basis points better off than they would be by going directly to financial markets. One possible swap is shown in Figure 1.

Figure 1: One possible swap

The US company borrows at an effective rate of 8.05% per annum in AUD. The Australian company borrows at an effective rate of 7.85% per annum in US dollars. Each pays 0.35% lower rate than what it would have paid by directly borrowing in that currency. The bank earns 5 basis points in AUD and another 5 basis points in USD for a net 10 basis-point spread. The way in which currency swaps such as this operate is as follows. Principal amounts in USD and AUD that are roughly equivalent are chosen. These principal amounts flow in the opposite direction to the arrows at the time the swap is initiated. Interest payments then flow in the same direction as the arrows during the life of the swap and the principal amounts flow in the same direction as the arrows at the end of the life of the swap. The swap can be structured in different ways by increasing or decreasing all the rates that the financial institution pays or receives by the same percentage points.
Note that the bank is exposed to some exchange rate risk in the swap. It earns 5 basis points in USD and 5 basis points in AUD. This exchange rate risk could be hedged using forward contracts. ■

Problem 2.2. A financial institution has agreed to pay three-month LIBOR and to receive 7% per annum in return in an interest rate swap. The notional principal is $50 million and payments are exchanged every three months. The swap has a remaining life of 11 months. Three-month forward LIBOR for all maturities is currently 6.6% per annum. The three-month LIBOR rate one month ago was 6.8% per annum. OIS rates for all maturities are currently 6% with continuous compounding. All other rates are compounded quarterly. What is the value of the swap?

Reference: Slides Example 3

Solution. We can value the swap as a series of forward rate agreements (FRAs). In each FRA, the fixed rate is 7% paqc. The floating rate is 6.8% paqc for the first FRA and the forward rate of 6.6% paqc for subsequent FRAs. The cash flows are discounted using OIS rates. The value of the swap is

$$50,000,000 \times (7\% - 6.8\%) \times 0.25e^{-0.06\times 2/12} + 50,000,000 \times (7\% - 6.6)\% \times 0.25e^{-0.06\times 5/12} +$$
$$50,000,000 \times (7\% - 6.6\%) \times 0.25e^{-0.06\times 8/12} + 50,000,000 \times (7\% - 6.6\%) \times 0.25e^{-0.06\times 11/12}$$
$$= \boxed{\mathbf{\$168{,}880}}$$

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Problem 2.3. Consider fixed-for-floating swaps with *annual exchanges* of cash flows and one-year LIBOR as the floating rate. The two-year swap rate and the three-year swap rate are 4% per annum and 4.1% per annum, respectively, both annually compounded. The risk-free zero interest rates are 3% for one year, 3.1% for two years, and 3.2% for three years, each continuously compounded. What is the forward LIBOR rate for the period between 2 years and 3 years? (Hint: Both 2-year and 3-year swaps have value zero so the differences in their cash flows also have value zero. You can assume the principal amount to be $100 million but it does not matter.)

Reference: Bootstrapping - Book Example 7.2, Slides Example 4

Solution. Consider a position in which you pay fixed and receive floating in the 2-year swap and pay floating and receive fixed in the 3-year swap. Assume that the notional principal is the same for both swaps. The value of this position is zero because each swap is valued at zero. Then, the sum of values of the net cash flows on the three dates must equal zero. The floating payments on two swaps cancel out in the first two payments. The net cash flow is receiving $4.1\% - 4.0\% = 0.1\%$. Suppose the required forward LIBOR rate is F. We get the equation:

$$0.1\%e^{-0.03} + 0.1\%e^{-0.031\times 2} + (4.1\% - F)e^{-0.032\times 3} = 0.$$

This simplifies to

$$(4.1\% - F)e^{-0.032\times 3} = -0.1910\%.$$

which gives $F = \boxed{\mathbf{4.3103\%\ pa}}$. ■

Problem 2.4. A company wants a floating-for-fixed swap where it receives semiannual payments at 5.2% per annum with semiannual compounding on a principal of $20 million for three years. The three-year swap rate with semiannual cash flows is 5.0% per annum with semiannual compounding. The OIS zero curve is flat at 4% per annum with continuous compounding. How much should a derivatives dealer charge the company?

Reference: Valuation of Swaps

Solution. Since the swap rate is 5.0% per annum, a swap where the company receives 5.0% per annum and pays LIBOR is worth zero. The company wants to receive 5.2% per annum, an extra 0.2% per annum. The value of this swap is the present value of $0.5 \times (0.052 - 0.05) \times \$20,000,000 = \$20,000$ received every six months for three years. This is

$$\sum_{i=1}^{6} \$20,000e^{-0.04\times 0.5i} = \boxed{\mathbf{\$111{,}952.54}}.$$

To avoid calculating 6 present values, use annuity formula with quarterly interest rate of $e^{0.04\times 0.5} - 1$:

$$\frac{\$20,000}{e^{0.04\times 0.5} - 1}\left(1 - \frac{1}{e^{0.04\times 0.5\times 6}}\right) = \$111,952.54.$$

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Problem 2.5. Suppose that the term structure of risk-free interest rates is flat in the United States and Britain. The USD interest rate is 5.1% per annum continuously compounded and the British rate is 4.8% per annum continuously compounded. The current value of a pound (GBP) is 1.12 USD. Under the terms of a swap agreement, a financial institution pays 6% per annum in USD and receives 6% per annum in GBP. The principals in the two currencies are 12 million USD and 10 million GBP. Payments are exchanged every year, with one exchange having just taken place. The swap will last two more years. What is the value of the swap to the financial institution?

Reference: Slides Example 6

Solution. The financial institution is short a USD bond and long a GBP bond. The value of the USD bond (in millions of USD) is

$$12 \times 0.06e^{-0.051\times 1} + 12 \times 1.06e^{-0.051\times 2} = 12.1707.$$

The value of the GBP bond (in millions of GBP) is

$$10 \times 0.06e^{-0.048\times 1} + 10 \times 1.06e^{-0.048\times 2} = 10.2016.$$

The value of the swap (in millions of dollars) is therefore,

$$10.2016 \times 1.12 - 12.1707 = -0.74494584$$

or $\boxed{\textbf{-\$744,945.84}}$. As an alternative we can value the swap as a series of forward foreign exchange contracts. The one-year forward exchange rate is $1.12e^{0.051-0.048} = 1.1234$. The two-year forward exchange rate is $1.50e^{2(0.051-0.048)} = 1.1267$. The value of the swap in millions of dollars is therefore,

$$(10\times 0.06\times 1.1234-12\times 0.06)e^{-0.051\times 1}+(10\times 1.06\times 1.1267-12\times 1.06)e^{-0.051\times 2} = -0.74494584$$

which is in agreement with the first calculation. ■