Problem Set II Solutions

QF 430: Introduction to Derivatives
Due Friday, October 14

Please submit neatly handwritten or typed answers. You can turn in paper submissions in class or submit electronically a single pdf file through Canvas. Show your steps or reasoning. Do not round too much in intermediate calculation steps. Aim for accuracy of four decimal places in interest rates (0.0337 or 3.37%).

1 Interest Rates

Problem 1.1. The 6-month, 12-month. 18-month, and 24-month risk-free zero rates are 4%, 4.3%, 4.6%, and 4.9% with semiannual compounding.

(a) What are the rates with continuous compounding?

(b) What is the forward rate for the six-month period beginning in 18 months?

(c) What is two-year par yield?

(d) What is the value of an FRA where the holder pays LIBOR and receives 6.1% (semiannually compounded) on $40 million principal for a six-month period beginning in 18 months? The current forward LIBOR rate for the period is 5.9% (semiannually compounded).

Solution.

(a) Consider the 18-month zero rate as an example. The rate is 4.6% with semiannual compounding. This means that the rate is 4.6%/2 = 2.3% every six months (or 0.5 years). If the corresponding continuously compounded rate is r, then $e^{0.5r} = 1.023$. Solving this equation, we get $r = 2\ln(1.023) = 0.045479 = \boxed{\textbf{4.5479\%}}$ per annum continuously compounded. Other rates are in the following table.

Horizon (months)	6	12	18	24
Semiannually compounded zero rate (% pa)	4	4.3	4.6	4.9
Continuously compounded zero rate (% pa)	3.9605	4.2544	4.5479	4.8409

(b) The forward rate for the period 18 months from today to 24 months from today (expressed with continuous compounding) is

$$\frac{\text{2-year zero rate} \times \text{2 years} - \text{1.5-year zero rate} \times \text{1.5 years}}{\text{2 years} - \text{1.5 years}}$$
$$= \frac{4.8409\% \times 2 - 4.5479\% \times 1.5}{2 - 1.5} = 0.057201 = 5.7201\% \text{ pa cc.}$$

The rate expressed as semiannually compounded rate is $2(e^{0.057201\times 0.5}-1) = 0.058026 =$ $\boxed{\textbf{5.8026\% pa sa}}$.

(c) We have to find the yield of a bond that will be priced at par. We can assume that the face value of the bond is \$1 and the price is also \$1 (because the bond is priced at par). Let c be the coupon every six months. Then the bond pricing equation is:

$$\begin{aligned}1 &= ce^{-0.039605\times 0.5} + ce^{-0.042544\times 1} + ce^{-0.045479\times 1.5} + ce^{-0.048409\times 2} + 1\times e^{-0.048409\times 2}\\ &= c(e^{-0.039605\times 0.5} + e^{-0.042544\times 1} + e^{-0.045479\times 1.5} + e^{-0.048409\times 2}) + e^{-0.048409\times 2}\\ &= 3.780517c + 0.907721.\end{aligned}$$

Solving this equation, we get

$$c = \frac{1-0.907721}{3.780517} = 0.024409.$$

That is the bond pays 2.4409% coupon every six months. The par yield is $2\times 2.4409\% =$ $\boxed{\textbf{4.8818\% pa sc}}$.

Note that we could have solved this part using the original semiannually compounded zero rates as

$$\begin{aligned}1 &= c(\frac{1}{1.020^1} + \frac{1}{1.0215^2} + \frac{1}{1.023^3} + \frac{1}{1.0245^4}) + \frac{1}{1.0245^4}\\ &= 3.780517c + 0.907721.\end{aligned}$$

(d) The value of the FRA is

$$\begin{aligned}&\text{PV of (FRA period} \times \text{(rate received} - \text{forward rate)} \times \text{principal) at FRA maturity}\\ &= \frac{0.5\times(0.061-0.059)\times 40,000,000}{1.0245^4} = \boxed{\textbf{\$36,308.82}}.\end{aligned}$$

■

Problem 1.2. The following table gives the prices of Treasury bonds:

Bond Principal (\$)	Time to Maturity (years)	Annual Coupon Rate (%)*	Bond Price (\$)
100	0.5	0	97
100	1	0	94
1000	1.5	8	1020
100	2	4	95

*Half the stated coupon is paid every six months

(a) Calculate zero rates for maturities of 6 months, 12 months, 18 months, and 24 months.

(b) What are the forward rates for the periods: 6 months to 12 months, 12 months to 18 months, 18 months to 24 months?

(c) What are the 6-month, 12-month, 18-month, and 24-month par yields for bonds that provide semiannual coupon payments?

(d) Estimate the price and yield of a two-year bond providing a semiannual coupon of 5% per annum.

Solution.

(a) The zero rate for a maturity of six months, expressed with continuous compounding is $2\ln(1+2/97) = \boxed{\textbf{6.0918\% \ \ pa cc}}$. To see this, suppose the c.c. rate is r. Then the price of the 6-month bond is:

$$
\begin{aligned}
97 &= 100e^{-0.5r} \\
\implies r &= 2\ln(\frac{100}{97}) = 0.060918 = 6.0918\% \text{ pa cc.}
\end{aligned}
$$

Similarly, the zero rate for a maturity of one year is $\ln(100/94) = \boxed{\textbf{6.1875\% \ \ pa cc}}$.

Calculating the zero-rate for 18 months requires pricing of a bond with coupons. Suppose the c.c. rate for 18 months is r. Then the price of the 18-month bond is:

$$
\begin{aligned}
&1020 = 40e^{-0.5\times0.060918} + 40e^{-1\times0.061875} + 1040e^{-1.5r} \\
&\implies 1020 = 38.80 + 37.60 + 1040e^{-1.5r} \\
&\implies 1040e^{-1.5r} = 943.60 \\
&\implies r = \frac{1}{1.5}\ln\left(\frac{1040}{943.60}\right) = 0.064849 = \boxed{\textbf{6.4849\% \ \ pa cc}}.
\end{aligned}
$$

The previous calculation can be simplified a little by noting that the coupon of \$40 after six-months is 40% of \$100 cash flow of the first bond so the present value of the first coupon should be 40% of the price of the first bond, that is $0.40 \times \$97 = \$38.80$. Similarly, the present value of the second coupon can be calculated using the price of the second bond as $0.40 \times \$94 = \$37.60$.

Now suppose the c.c. rate for 24 months is r. Then the price of the 24-month bond is:

$$
\begin{aligned}
95 &= 2e^{-0.5\times0.060918} + 2e^{-1\times0.061875} + 2e^{-1.5\times0.064849} + 102e^{-2r} \\
&= 1.94 + 1.88 + 1.8146 + 102e^{-2r} \\
\implies r &= \frac{1}{2}\ln\left(\frac{102}{89.3654}\right) = 0.066120 = \boxed{\textbf{6.6120\% \ \ pa cc}}.
\end{aligned}
$$

(b) The forward rate for the period 6 months to 12 months is

$$
\frac{6.1875\% \times 1 - 6.0918\% \times 0.5}{1 - 0.5} = 0.062832 = \boxed{\textbf{6.2832\% \ \ pa cc}}.
$$

The forward rate for the period 12 months to 18 months is

$$
\frac{6.4849\% \times 1.5 - 6.1875\% \times 1}{1.5 - 1} = 0.070796 = \boxed{\textbf{7.0796\% \ \ pa cc}}.
$$

The forward rate for the period 18 months to 24 months is

$$
\frac{6.6120\% \times 2 - 6.4849\% \times 1.5}{2 - 1.5} = 0.069932 = \boxed{\textbf{6.9932\% \ \ pa cc}}.
$$

(c) The semiannually-compounded 6-months par yield is

$$2\frac{1-e^{-0.060918\times0.5}}{e^{-0.060918\times0.5}} = 0.061856 = \boxed{\mathbf{6.1856\%\ \ pa\ sc}}.$$

The semiannually-compounded 12-months par yield is

$$2\frac{1-e^{-0.061875\times1}}{e^{-0.060918\times0.5}+e^{-0.061875\times1}} = 0.062827 = \boxed{\mathbf{6.2827\%\ \ pa\ sc}}.$$

The semiannually-compounded 18-months par yield is

$$2\frac{1-e^{-0.064849\times1.5}}{e^{-0.060918\times0.5}+e^{-0.061875\times1}+e^{-0.064849\times1.5}} = 0.065802 = \boxed{\mathbf{6.5802\%\ \ pa\ sc}}.$$

The semiannually-compounded 24-months par yield is

$$2\frac{1-e^{-0.066120\times2}}{e^{-0.060918\times0.5}+e^{-0.061875\times1}+e^{-0.064849\times1.5}+e^{-0.066120\times2}} = 0.067075 = \boxed{\mathbf{6.6075\%\ \ pa\ sc}}.$$

The rates are summarized in the following table.

Maturity (yrs)	Zero Rate (% pa cc)	Forward Rate (% pa cc)	Par Yield (% pa sc)
0.5	6.0918	6.0918	6.1856
1	6.1875	6.2832	6.2827
1.5	6.4849	7.0796	6.5802
2	6.612	6.9932	6.7075

(d) The bond price is

$$2.5e^{-0.060918\times0.5}+2.5e^{-0.061875\times1}+2.5e^{-0.064849\times1.5}+102.5e^{-0.066120\times2} = \boxed{\mathbf{\$96.8467}}.$$

The yield is an interest rate such that the price of the bond equals the present value of all cash flows from the bond discounted using yield for all maturities. Let y be the continuously compounded yield. Then,

$$96.8467 = 2.5e^{-0.5y}+2.5e^{-y}+2.5e^{-1.5y}+102.5e^{-2y}.$$

The solution to this equation, obtained by trial-and-error, is $y = 0.066009$. The yield of the bond is $\boxed{\textbf{6.6009\% per annum continuously compounded}}$.

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2 Forward and Futures Prices

Problem 2.1. The risk-free rate of interest is 5% per annum with continuous compounding. A stock is trading at \$49. The stock pays dividend of \$0.60 every quarter. The next dividend will be paid after 2 months. What is the six-month futures price of the stock? Report the price up to one-tenth of a cent.

Solution. The six month futures price is

$$(49 - 0.60e^{-0.05\times 2/12} - 0.60e^{-0.05\times 5/12})e^{0.05\times 0.5} = \boxed{\mathbf{49.0279}}.$$

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Problem 2.2. The risk-free rate of interest is 5% per annum with continuous compounding, and the dividend yield on a stock index is 6% per annum. The current value of the index is 314. What is the six-month futures price (calculated at least to one-tenth of a cent)?

Solution. The six-month futures price is

$$314e^{(0.05-0.06)\times 6/12} = 312.434.$$

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Problem 2.3. The risk-free rate of interest is 5% per annum with continuous compounding, and the fixed dividend yield on a stock index is 7% per annum. The three-month stock index futures price is \$75 and the six-month stock index futures price is \$74. Describe an arbitrage strategy.

Solution. The ratio of the two futures prices is $F_0^{6M}/F_0^{3M} = 74/75 = 0.9867$. Based on interest rate and dividend yield, this ratio should be

$$\frac{F_0^{6M}}{F_0^{3M}} = \frac{S_0e^{(0.05-0.07)\times 0.5}}{S_0e^{(0.05-0.07)\times 0.25}} = e^{(0.05-0.07)\times 0.25} = 0.99501.$$

We conclude that the six-month futures price is too low relative to the three-month futures price (this may be because six-month futures price is too low while three-month price is correct or six-month futures price is correct while three-month futures price is too high or both prices are incorrect). The maxim of buy low, sell high suggests the following arbitrage strategy.

For some positive integer N, take a $\boxed{\textbf{short position in } N \textbf{ three-month futures}}$, take a $\boxed{\textbf{long position in } N \textbf{ six-month futures}}$, and lock in a $\boxed{\textbf{deposit of } \$75N}$ starting after three months and maturing six months from today. Use the short futures position to short (the stocks in) the stock index after three months for \$75N and deposit the proceeds. The interest on the deposit increases the balance at the rate of 5% per annum continuously compounded. However, you will use the balance to pay 7% per annum continuously dividends on the stock index to the party from which you borrowed (the stocks in) the stock index. Your balance will therefore, decline by 2% per annum continuously compounded. Six months from today, you will withdraw $\$75Ne^{-0.02\times 0.25} = \$74.6259N$. You will use \$74N out of this to buy N stock indices using the long futures position and close the short position. This will leave you a $\boxed{\textbf{net profit of } \$0.6259N \textbf{ after six months}}$, equivalent to a net profit of $\$0.6259Ne^{-0.05\times 0.5} = \boxed{\$0.6105N \textbf{ today}}$. ■

Problem 2.4. When a known future cash outflow in a foreign currency is hedged by a company using a long forward contract, there is no foreign exchange risk. When it is hedged using long futures contracts, the daily settlement process does leave the company exposed to some risk. Explain the nature of this risk.

Assume that the forward price equals the futures price. In each of the following cases, based on time value of money, explain which company is better off—company A taking a long position in the futures contract or company B taking a long position in the forward contract?

(a) The value of the foreign currency falls through the life of the contract

(b) The value of the foreign currency rises through the life of the contract

(c) The value of the foreign currency first rises and then falls back to its initial value

(d) The value of the foreign currency first falls and then rises back to its initial value

Solution. In total the gain or loss under a futures contract is equal to the gain or loss under the corresponding forward contract. However the timing of the cash flows is different. When the time value of money is taken into account, a futures contract may prove to be more valuable or less valuable than a forward contract. Usually, this is not known in advance. The long forward contract provides a perfect hedge. The long futures contract provides a slightly imperfect hedge.

(a) In this case, $\boxed{\textbf{company B is better off}}$. Both companies will experience the same total loss. Company B with a forward contract will realize the entire loss at the end. Company A with a futures contract will realize the loss day by day throughout the contract. On a present value basis, company B performs better.

(b) In this case, $\boxed{\textbf{company A is better off}}$. Both companies experience the same total gain. Company B will realize the entire gain at the end. Company A will realize the gain day by day throughout the life of the contract. On a present value basis, company A is better off.

(c) In this case, $\boxed{\textbf{company A is better off}}$. This is because it would realize positive cash flows early and negative cash flows later.

(d) In this case, $\boxed{\textbf{company B is better off}}$. This is because company A's early cash flows will be negative and the later cash flows will be positive.

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Problem 2.5. Gold price is \$1721 per ounce. The storage costs are \$3.60 per ounce per year payable *quarterly* at the *end* of each quarter. Assuming that interest rate is 5% per annum with continuous compounding for all maturities, calculate the futures price of gold for delivery in six months.

Solution. The present value of the storage costs for six months are

$$(3.60/4)e^{-0.05\times 0.25} + (3.60/4)e^{-0.05\times 0.5} = 1.7666 \text{ per ounce.}$$

The six-month futures price is

$$F_0 = (1721 + 1.7666)e^{0.05\times 0.5} = \boxed{\textbf{\$1766.379 \ per ounce}}.$$

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Problem 2.6. The current USD/Swiss franc exchange rate is 1.0207 dollars per Swiss franc. The one-year forward exchange rate is 1.0600 dollars per Swiss franc. The one-year USD interest rate is 3% per annum continuously compounded. Estimate the one-year Swiss franc interest rate.

Solution. If the one-year Swiss franc interest rate is r_f, then

$$1.0600 = 1.0207e^{(0.03-r_f)\times 1}$$

so that

$$0.03 - r_f = \ln(\frac{1.06}{1.0207})$$

or

$$r_f = 0.0103 - \ln(\frac{1.087}{1.07}) = -0.00778 = \boxed{\textbf{-0.778\% pacc}}.$$

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Problem 2.7. The six-month interest rates in the United Kingdom and the United States are 4.36% and 4.27% per annum, respectively, with continuous compounding. The spot price of the British pound sterling is \$1.1439. The futures price for a contract deliverable in six months is \$1.1431. What arbitrage opportunities does this create?

Solution. The theoretical futures price is

$$\$1.1439e^{(0.0427-0.0436)\times 6/12} = \$1.1434.$$

The actual futures price of \$1.1431 is too low. This suggests that an arbitrageur should sell British pound sterlings spot and go long British pound sterling futures. For example,

- $\boxed{\textbf{borrow £}e^{-0.0436\times 0.25}}$ (so that you have to return exactly £1 after 3 months),
- $\boxed{\textbf{convert to \$ and lend \$ } 1.1439e^{-0.0436\times 0.25}}$, and
- $\boxed{\textbf{take a long position in three-month futures contract on £1}}$.
- After three months, get back $\$e^{0.0427\times 0.25} \times 1.1439e^{-0.0436\times 0.25} = \$1.1434$.
- Use \$1.1431 to get £1 with the long futures position and repay the £ loan.
- Keep the $\boxed{\textbf{net profit of } 1.1434 - 1.1431 = \$0.0003}$.
- This strategy can be scaled up.

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