QF 430 Final Review

Stevens Institute of Technology

December 14, 2022

Options Markets

Call Option and Put Option

A call option is the right without the obligation to buy an asset at a predetermined price in a certain time period.

A put option is the right without the obligation to sell an asset at a predetermined price in a certain time period.

Terminology

1. **Underlying asset**: This is the asset one can buy or sell using option. Underlying asset is standardized for exchange traded options. The underlying asset can be a physical commodity like wheat or iron, a financial security such as stock or bond, or a cash amount whose value depends on a future outcome such as interest rate or the amount of snowfall.
2. **Exercise/strike price (X or K)**: This is the predetermined price at which the option owner can buy or sell the underlying asset.
3. **Exercising the option**: An option owner exercises the option when he/she exercising the right granted by the option to buy or sell the underlying asset at the exercise price.
4. **Expiration date**: The last date by which option must be exercised.
5. **Maturity**: This is the time remaining before the expiration date.

6. **European option**: European options can be exercised only on the expiration date.
7. **American option**: American options can be exercised at any time on or before expiration date.
8. **Spot price (S)**: The spot price is the market price of the underlying asset.
9. **Moneyness**: An option is **in-the-money** if exercising the option results in a profit. A call option is in-the-money if S>K while a put option is in-the-money if S<K. An option is **out-of-money** if exercising the option results in a loss. A call option is out-of-money if S<K while a put option is out-of-money if S>K. An option is **at-the-money** if exercising the option results in zero profit or loss. Call and put options are at-the-money if S=K.
10. **Intrinsic value**: Intrinsic value of an option is the profit from exercising the option if the option is in-the-money and zero otherwise.
11. **Writing an option**: Writing an option is the same as selling an option.
12. **Option price / premium (C, P, c, p)**: This is the price an investor pays to buy an option or the price an investor gets by writing an option.

Use of Options

- An option owner can profit in some future situations without incurring any future losses.
- This means that an option has positive value and the option owner pays a price to acquire the option.
- Buying futures and forwards does not require initial payments because the possibility of profit is offset by the possibility of loss.

Trading

- Exchanges list options. They do *not* create options.
- Once an option has been listed on an exchange, investors can trade in the option. A trade occurs when one party agrees to buy and another agrees to sell.
- The number of long positions equals the number of short positions. This common number is called the **open interest**.
- Open interest increases when an investor buys an option to increase his long position in the option and another investor sells the option to increase his short position in the option.
- Open interest can decline if investors trade to close their prior positions.
- It is also important to realize that corporations have no special role in the options written on their stock.
- For example, Microsoft is not involved in creation or trading of options on Microsoft stock. Market makers in an option post bid-ask quotes for that option.

Specification

- There are often multiple options on an underlying asset.
- These options may differ in the months in which they expire.
- The expiration date in a given month is determined by the convention used by the exchange that lists the option.
- Even for a given expiration, there may be multiple options with different strike prices.
- For stocks, strike prices are usually near current stock price.
- If price moves significantly, new strikes may be introduced.

Contract Specification Example

- PHLX World Currency Options® Euro Product Specifications
- **Description**: PHLX U.S. dollar-settled Euro currency options are quoted in terms of U.S. dollars per unit of the underlying currency (Euro) and premium is paid and received in U.S. dollars.
- **Contract Size**: 10,000 Euros
- Trading Symbol: XDE
- Settlement Value Symbol: EDY
- CUSIP® Number: 69336Y
- Exercise Style: European
- **Expiration Date**: Saturday following the third Friday of the expiration month.
- **Expiration Cycle**: Quarterly on the March cycle plus two additional near-term months (six months at all times).
- **Settlement**: U.S. dollars
- **Settlement Value for Expiring Contracts**: Based on the 12:00 Noon (Eastern Time) Buying Rate, as determined by the Federal Reserve Bank of New York on the last trading day prior to expiration (usually a Friday). If the Noon Buying Rate is not announced …
- **Last Trading Day for Expiring Contracts**: The third Friday of the expiration month.
- **Contract Point Value**: $100 (i.e., .01 x 10,000)
- **Exercise (Strike) Price Intervals**: The Exchange shall determine fixed-point intervals of exercise prices. Generally, the exercise (strike) price interval shall be set at half-cent intervals.

- **Premium Quotation**: One point = $100. Thus a premium quote of 2.13 is $213. The minimum change in a premium is .01= $1.00.
- **Position Limits**: 200,000 contracts on the same side of the market. Hedge exemptions are available.
- **Trading Hours**: 9:30 a.m. to 4:00 p.m. (Eastern Time)
- **Issuer and Guarantor**: The Options Clearing Corporation (OCC)

Clearing

Option traders may need to post margin depending on their position and portfolio.

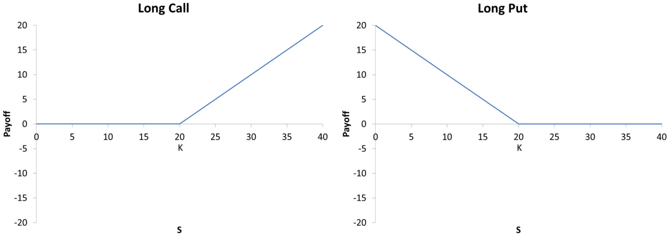
Options Clearing Corporation (OCC) clears all option positions.

That is, OCC acts as a counterparty to each (exchange traded) option trade.

Option Payoffs

Option payoff at expiration depends on spot price S.

Long Call: max(S-K,0) Long Put: max(K-S,0)



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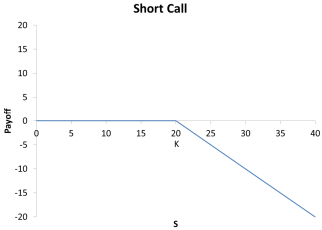
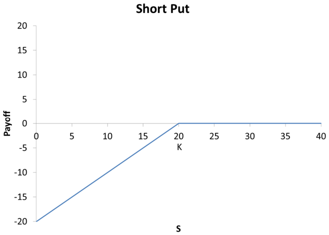
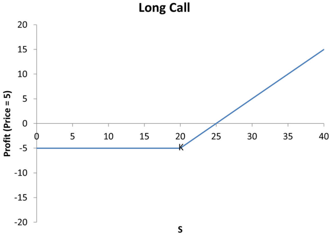
Short Call: -max(S-K,0)   Short Put: -max(K-S,0)



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Profit (Price = 5)
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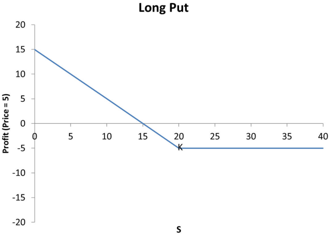
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Profit (Price = 5)
K
S

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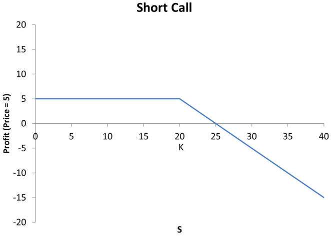
Short Call
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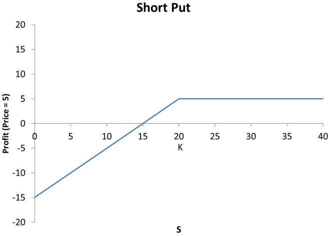
Short Put
Profit (Price = 5)
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Underlying Assets

- Options on several thousand stocks are traded on the CBOE, NYSE Euronext, International Securities Exchange, and Boston Options Exchange.
- Each contract is for 100 shares of a stock.
- CBOE also trades options on Exchange Traded Products such as Exchange Traded Funds (ETFs).
- ETFs track performance of an equity index of a bond index.
- Options on foreign currency options are mostly traded in over-the-counter market but are also traded on some exchanges such as Philadelphia Stock Exchange.
- These are European-style contracts.
- Options on indices are also very popular.
- CBOE trades options on S&P 500 index, NASDAQ-100 index, and Down Jones Index. These options are also usually European.
- Futures options are options on prices of futures contracts.

Contract Specification

- Expiration Dates are usually monthly.
- Stock options in US are typically traded for first two months, and then three and six months from current month.
- LEAPS are longer-term equity options with initial maturities in several years.
- Strike prices for stock options in US are typically $2.50, $5, or $10 apart.
- Usually exchanges introduce contracts with strike prices close to the current price.
- All options of the same type (Call or put) on a stock are referred to as an option class.
- All options in an option class with the same expiration date and strike price form an option series.

The direction of impact of these variables on call and put option prices is described in the following table. A '+' means that the option price moves in the same direction as the variable and a '-' means that the option price moves in an opposite direction to that of the variable. European and American call option prices are denoted by c and C, respectively. European and American put option prices are denoted by p and P, respectively.

Variable	Effect on c	Effect on C	Effect on p	Effect on P
S_0	+	+	-	-
K	-	-	+	+
T	?	+	?	+
σ	+	+	+	+
r	+	+	-	-
D	-	-	+	+

Bounds on Prices for Options on Non-Dividend-Paying Stock

- European call

$$max(S_0 - PV\ of\ K, 0) \leq c \leq S_0$$

- European put

$$max(PV\ of\ K - S_0, 0) \leq p \leq PV\ of\ K$$

- American call

$$max(S_0 - K, S_0 - PV\ of\ K, 0) \leq C \leq S_0$$

- American put

$$max(K - S_0, PV\ of\ K - S_0, 0) \leq P \leq K$$

Example 1: An American put option is written on a non-dividend-paying stock. The stock price is \$50, the exercise price is \$66, expiration is after one year, and the risk-free rate is 10% per year.

a. What is the no-arbitrage price range for the put option?
b. Suggest an arbitrage strategy if the put price is \$9.
c. Suggest an arbitrage strategy if the put price is \$70.

a. Using the bounds for the price of American put option:

$$P \geq max(K - S_0, PV\ of\ K - S_0, 0) = max\left(66 - 50, \frac{66}{1.1} - 50, 0\right) = max(16,10,0) = 16$$

and

$$P \leq K = 66$$

The no-arbitrage price range is \$16 to \$66.

b. At \$9, put is underpriced as it is less than the lower bound of \$16 obtained from $K - S_0$, the profit from exercising the option immediately. This suggests the arbitrage strategy of buying the put at \$9 and immediately exercising to make \$16, resulting in a net profit of \$7.

c. At \$70, put is overpriced as its price is above the upper bound of \$66 obtained from K, the exercise price. This suggests an arbitrage strategy in which you sell the put at \$70 and pay \$66. Specifically, sell put option for \$70, keep \$66 aside to pay to the put buyer if the put buyer chooses to exercise the option at any time. The remaining $4 is your profit. In addition, if the put buyer exercises, you will get a stock. If the put buyer doesn't exercise, you will keep $66 too.

Bounds on Prices for Options on Dividend-Paying Stock

If the dividends to be paid on the stock during the life of the option are known and have present value D, then

- European call

$$max(S_0 - D - PV\ of\ K, 0) \leq c \leq S_0$$

- European put

$$max(PV\ of\ K + D - S_0, 0) \leq p \leq PV\ of\ K$$

- American call

$$max(S_0 - K, S_0 - D - PV\ of\ K, 0) \leq C \leq S_0$$

- American put

$$max(K - S_0, PV\ of\ K + D - S_0, 0) \leq P \leq K$$

Put Call Parity

$$S_0 + p = PV\ of\ K + c$$

$$S_0 = c - p + PV\ of\ K$$

This equation represents put-call parity.

The left-hand-side of the equation is the price of the **real stock**.

The right-hand-side is the price of a **synthetic stock** created from a long call, a short put, and a long bond.

The real stock and the synthetic stock will have the same value at maturity so they must cost the same today.

If this equation is violated, investors can buy the cheaper of the two and simultaneously sell the costlier of the two to earn a positive profit without any future risk.

Put-call parity holds for European options on non-dividend-paying stocks.

Put Call Parity with Dividends

When the underlying stock pays dividends, put-call parity is modified as follows:

$$S_0 = c - p + PV\ of\ K + D$$

where D is the present value of dividends on the underlying stock before option maturity.

Put Call Parity with American Options

With American options, put-call parity does not hold as equality but results in inequalities.

$$S_0 - D - K \leq C - P \leq S_0 - PV\ of\ K$$

Early Exercise

American calls on non-dividend-paying stocks should not be exercised early (before maturity). Why?

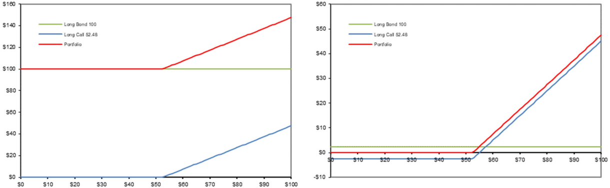
1. There is a possibility that the stock price declines a lot after early exercise. The resulting losses could have been reduced by not exercising the option.
2. It is better to pay exercise price later rather than earlier.

Since American call options on non-dividend-paying stocks should not be exercised early, they offer no advantage over European call options and must have the same value. That is, $C = c$ for non-dividend-paying stock.

Financial Engineering with Options

There are several trading strategies that use options to meet certain investor needs. We will consider some of these.

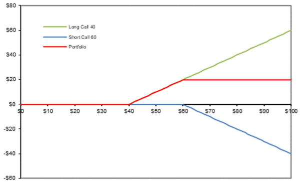
1. **Principal Protected Note**: This is a hybrid of a bond and a stock with some upside of stock but no downside. How is this possible? The note can be constructed using call option on stock rather than the stock itself. But bondholders usually do not want to pay separately for the call option. Instead, the call option is bundled with the bond by increasing the price of the bond to include the price of the call option. An important lesson from the financial crisis of 2007-2008 is that principal protected notes, like some other derivatives, provide protection against market risk but are still vulnerable to **credit risk**. That is, all the protection promised is worthless if the seller of the principal protected notes is unable to meet its promise.

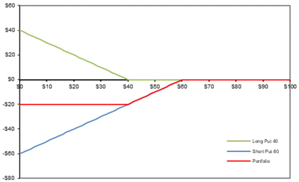


Principal protected note payoff without and with accounting for initial cash flow (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

2. **Bull Spread**: This is created by going long in a call with a lower strike price (K_1) and going short in a call with a higher strike price (K_2). Alternatively, it can be created by going long in a put with a lower strike price and going short in a put with a higher strike price. A bull spread allows investor to bet on stock price increase while limiting the investor's upside as well as downside.

$$\text{Value of bull spread using calls} = C(K_1) - C(K_2)$$
$$\text{Value of bull spread using puts} = P(K_1) - P(K_2)$$





Bull spread with call options and with put options (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

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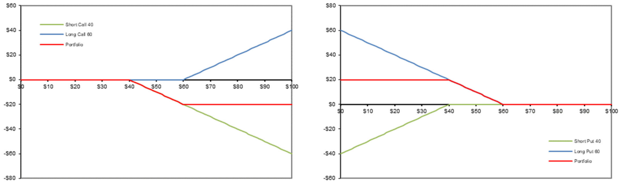
3. **Bear Spread**: This is created by going short in a call with a lower strike price (K_1) and going long in a call with a higher strike price (K_2). Alternatively, it can be created by going short in a put with a lower strike price and going long in a put with a higher strike price. A bear spread allows investor to bet on stock price decline while limiting the investor's upside as well as downside.

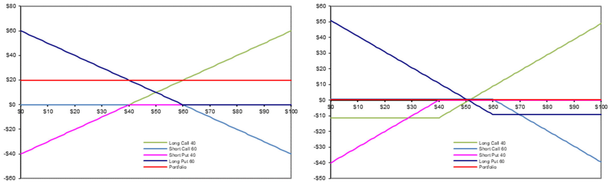
$$Value\ of\ bear\ spread\ using\ calls = C(K_2) - C(K_1)$$
$$Value\ of\ bear\ spread\ using\ puts = P(K_2) - P(K_1)$$



Bear spread payoff with call options and with put options (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

4. **Box Spread**: This consists of a bull spread created using calls and a bear spread created using puts with the same exercise prices K_1 and K_2. The payoff from the box spread is fixed at $K_2 - K_1$.

$$\begin{aligned} Value\ of\ box\ spread &= C(K_1) - C(K_2) - P(K_1) + P(K_2) \\ &= \{C(K_1) - P(K_1)\} - \{C(K_2) - P(K_2)\} \\ &= \{S_0 - PV\ of\ K_1\} - \{S_0 - PV\ of\ K_2\} \\ &= PV\ of\ (K_2 - K_1) \end{aligned}$$



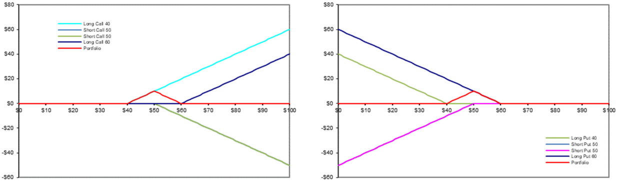
Box spread payoff without and with accounting for initial cash flow (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

5. **Butterfly Spread**: This can be used by an investor who expects volatility to be low in the future. It can be created by going long in an option with a low strike price (K_1) and another option with a high strike price (K_3) and simultaneously taking short position in two options with a middle strike price (K_2). All of the options should be calls or all should be puts.

$$Value\ of\ butterfly\ spread = C(K_1) - 2C(K_2) + C(K_3)$$

or

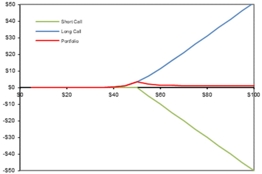
$$Value\ of\ butterfly\ spread = P(K_1) - 2P(K_2) + P(K_3)$$

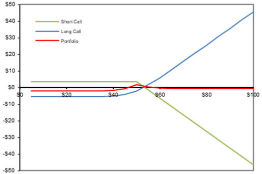


Butterfly spread payoff with call options and with put options (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

6. **Calendar Spread**: A calendar spread can also be used to bet on low volatility. It is created by being long in a longer maturity (T_2) call and being short in a shorter maturity (T_1) call with the same strike price.

$$Value\ of\ calendar\ spread = C_{T_2}(K) - C_{T_1}(K)$$

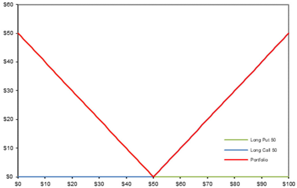


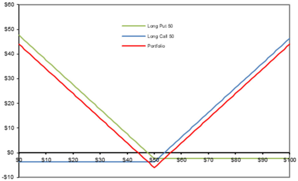


Calendar spread payoff without and with accounting for initial cash flow (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

7. **Straddle**: It is a combination of a long position in a call and a long position in a put, both with same strike and maturity. A straddle allows an investor to bet on high volatility.

$$Value\ of\ straddle = C(K) + P(K)$$

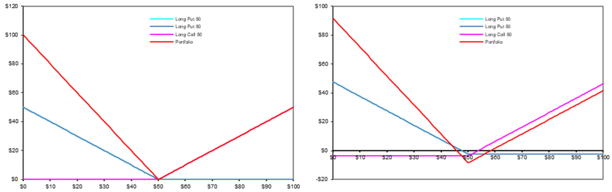




Straddle payoff without and with accounting for initial cash flow (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

8. **Strip**: It is a combination of a long position in one call and a long position in two puts with the same strike and maturity.

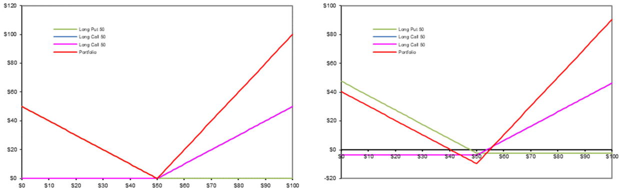
$$Value\ of\ strip = C(K) + 2P(K)$$



Strip payoff without and with accounting for initial cash flow (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

9. **Strap**: It is a combination of a long position in two calls and a long position in one put with the same strike and maturity.

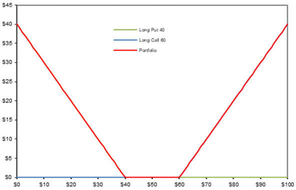
$$Value\ of\ strap = 2C(K) + P(K)$$

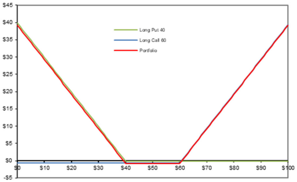


Strap payoff without and with accounting for initial cash flow (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

10.**Strangle**: It is a combination of a long position in one call and a long position in one put with the same maturity but with call strike greater than put strike.

$$Value\ of\ strangle = P(K_1) + C(K_2)$$





Strangle payoff without and with accounting for initial cash flow (horizontal axis is stock price at expiration, vertical axis is portfolio cash flow)

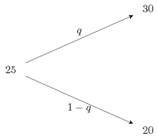
Option Valuation Using Binomial Trees

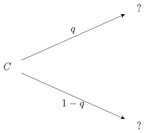
Riskless Portfolio Approach

- In this valuation method, we combine the underlying asset and the option to create a riskless portfolio.
- This means that the cash flow of the portfolio in the future is known with certainty.
- Any riskless security earns risk-free rate so this portfolio's price can be calculated by discounting the portfolio's future cash flow at risk-free rate.

Example 3: Suppose a stock is worth \$25 today and will either be worth 20% more or 20% less in one year. The risk-free rate is 5% per annum continuously compounded. What is the value of an at-the-money European call option that matures after one year?

The binomial trees for the prices of the stock and the call option are shown below:





We solve these valuation problems backward by first determining the value of the option at maturity. If stock price is \$30, the call option is worth $max(30-25,0) = 5$. If stock price is \$20, the call option is worth $max(20-25,0) = 0$.

Create a riskless portfolio using stocks and call options. Suppose we choose x stocks and y options. Then the cash flows in future will be $30x + 5y$ and $20x$. This portfolio will be riskless if the two cash flows are identical:

$$30x + 5y = 20x$$

This means $\frac{x}{y} = -0.5$. That is, for each long option, we must short 0.5 shares of stock or for each short option, we must long 0.5 shares of stock. The number 0.5 is the option's **delta** after one year. This value can be calculated as:

$$Option\ delta = \frac{Difference\ in\ Option\ Cash\ Flows}{Difference\ in\ Stock\ Cash\ Flows}$$

We can verify that $(5 - 0)/(30 - 20) = 0.5$. There are many values of x and y which will work. Let is assume we have one call option and we short 0.5 shares of stock That is, $y = 1, x = -0.5$. The final payoff of this portfolio is sure to be -10. So the current value of this portfolio can be calculated by discounting at risk-free rate of 5%.

Current value of portfolio = $-10e^{-0.05\times 1} = -\$9.5123$.

But since the portfolio consists of 1 call and -0.5 shares of stock, its value equals $C - 0.5 \times 25 = C - 12.5$. So we can solve

$$C - 12.5 = -9.5123$$

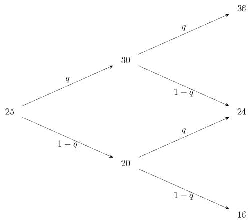
to get $C = 2.9877$.

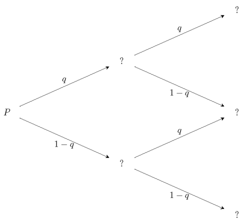
Multiple Periods

- The above approach can be used for binomial trees with multiple levels.
- Proceed backwards from maturity and moving left, repeatedly determine the option price one level earlier.
- By increasing the number of periods, we can
    1. value options with longer maturity, or
    2. value a fixed maturity option with greater accuracy.

Example 4: Suppose a stock is worth \$25 today and each year its price increases by 20% or decreases by 20%. The stock does not pay any dividends. The risk-free rate is 5% per annum continuously compounded. What is the value of a European put option with exercise price of \$26 that matures after two years?

The binomial trees for the prices of the stock and the put option are shown below:





Start by first determining the value of the option at maturity. If stock price is \$36, the put option is worth $max(26 - 36,0) = 0$. If stock price is \$24, the put option is worth $max(26 - 24,0) = 2$. If stock price is \$16, the put option is worth $max(26 - 16,0) = 10$.

Consider the node after one year at which the stock price is \$30. How many shares of stock must be combined with an option to create a riskless portfolio?

We first calculate the option delta for the following year as $(0 - 2)/(36 - 24) = -1/6$. So a portfolio of 1 put option and 1/6 share will be a riskless portfolio.

You can verify that the value of this portfolio at maturity is fixed at \$6. So the value of the portfolio when it is created at the time stock price is \$30 should be $6e^{-0.05\times 1} = \$5.7074$.

Since this value must equal the value of a put option and the price of 1/6 share of stock, we have $P_1 + \frac{30}{6} = 5.7074$ which gives $P_1 = 0.7074$.

Next, consider the node after one year at which the stock price is \$20. Now the option delta for the following year is $(2 - 10)/(24 - 16) = -1$.

So a portfolio of 1 put option and 1 share will be a riskless portfolio. You can verify that the value of this portfolio at maturity is fixed at \$26.

The value of the portfolio when it is created at the time stock price is \$20 should be $26e^{-0.05\times 1} = \$24.7320$.

Since this value must equal the value of a put option and the price of 1 share of stock, we have $P_2 + 20 = 24.7320$ which gives $P_2 = 4.7320$.

Finally, consider the initial node at which the stock price is \$25. The option delta for the following year is $(0.7074 - 4.7320)/(30 - 20) = -0.4025$.

So a portfolio of 1 put option and 0.4025 share will be a riskless portfolio. You can verify that the value of this portfolio after one year is fixed at \$12.7811.

So the initial value of the portfolio should be $12.7811e^{-0.05\times 1} = \$12.1578$. Since this value must equal the value of a put option and the price of 0.4025 share of stock, we have $P + 0.4025 \times 25 = 12.1578$ which gives $P = 2.0963$.

Risk Neutral Probability Approach

- There are two difficulties in discounted cash flow valuation of options:
    1. The probability distribution of future value of the underlying asset is not known
    2. The appropriate discount rate is not known
- **Risk neutral valuation** allows us to overcome both problems.
- Insight behind risk neutral valuation: We can create a combination of the derivative and the underlying asset in which the price movements exactly offset each other so that the portfolio's cash flow is riskless.
- We considered examples of this earlier in the riskless portfolio approach.
- No discount rate other than risk-free rate is used so investor beliefs or risk aversion aren't necessary if the price of the underlying asset is known.
- That means that even if investors in the world were risk neutral, they should arrive at the same price of the derivative as the real investors, which may be risk-averse.
- We pretend that investors are risk-neutral to solve the above two problems.
- Since risk-neutral investors demand risk-free rate from all assets, the future value of the underlying asset must grow at the risk-free rate.
- Moreover, we can use risk-free rate to discount option cash flows.
- Suppose the underlying asset's price changes from S_0 to S_h or S_l in time Δt.
- In a risk-neutral world, investors assign probabilities to these two future prices so that the expected return on the underlying asset equals the risk-free rate r.

- Let q be the **risk-neutral probability** assigned to price S_h. Then,

$$S_0 e^{r\times\Delta t} = qS_h + (1-q)S_l,$$

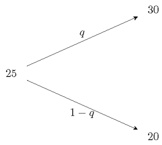
or,

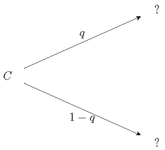
$$q = \frac{S_0 e^{r\times\Delta t} - S_l}{S_h - S_l}.$$

- The risk-neutral probabilities depend only on the prices chosen in the binomial tree of the underlying asset and on the risk-free rate.
- They do not depend on the derivative. This means that risk-neutral probabilities need to be calculated once and can be used to value multiple derivatives.
- It is important to understand that even though risk-neutral probabilities can be used to value derivatives, these are not real-world probabilities because in the real world few assets grow at risk-free rate.

Example 5: Suppose a stock is worth \$25 today and will either be worth 20% more or 20% less in one year. The risk-free rate is 5% per annum continuously compounded. What is the value of an at-the-money European call option that matures after one year?

The binomial trees for the prices of the stock and the call option are shown below:





We first determine the risk-neutral probability that the stock price will increase to \$30. It is given by

$$q = \frac{S_0 e^{r\times\Delta t} - S_l}{S_h - S_l} = \frac{25e^{0.05\times 1} - 20}{30 - 20} = 0.6282.$$

At maturity, the call option is worth $max(30 - 25,0) = 5$ if the stock price is \$30 and $max(20 - 25,0) = 0$ if the stock price is \$20. We determine the current value call price by calculating expected value at maturity using risk-neutral probabilities and then discounting at risk-free rate:

$$C = \{0.6282 \times 5 + (1 - 0.6282) \times 0\}e^{-0.05\times 1} = \$2.9877.$$

Multiple Periods

- The risk-neutral probability approach can be used for binomial trees with multiple levels by proceeding backwards from maturity and moving left, repeatedly determining the option price one level earlier.
- In general, risk-neutral probabilities may differ at different nodes and need to be calculated separately for each node.
- However, if we create the binomial tree in a particular way, the same risk-neutral probabilities can be used at all nodes. This requires that
    1. The underlying asset does not pay any dividend.
    2. In each step, the stock price changes from S to Su or Sd where $d = 1/u$.
- For such a tree, the risk-neutral probability that the stock price will change from S to Su is given by

$$q = \frac{e^{r\times\Delta t} - d}{u - d}.$$

Example 6: Suppose a stock is worth \$25 today and each year its price increases by 20% or decreases by 20%. The stock does not pay any dividends. The risk-free rate is 5% per annum continuously compounded. What is the value of a European put option with exercise price of \$26 that matures after two years?

The risk-neutral probability that the stock price increases by 20% in a year is given by

$$q = \frac{e^{r\times\Delta t} - d}{u - d} = \frac{e^{0.05\times1} - 0.8}{1.2 - 0.8} = 0.6282.$$

At maturity, if stock price is \$36, the put option is worth $max(26 - 36,0) = 0$, if stock price is \$24, the put option is worth $max(26 - 24,0) = 2$, and if stock price is \$16, the put option is worth $max(26 - 16,0) = 10$. The put option value after one year, when the stock price is \$30, is calculated as

$$P_1 = \{0.6282 \times 0 + (1 - 0.6282) \times 2\}e^{-0.05\times1} = \$0.7074.$$

The put option value after one year, when the stock price is \$20, is calculated as

$$P_1 = \{0.6282 \times 2 + (1 - 0.6282) \times 10\}e^{-0.05\times1} = \$4.7320.$$

The initial put option value is calculated as

$$P = \{0.6282 \times 0.7074 + (1 - 0.6282) \times 4.7320\}e^{-0.05\times1} = \$2.0963.$$

Valuation of American Options

- American options can be easily valued using binomial tree method by checking at each node whether it is optimal to exercise the option and get intrinsic value or to hold the option to realize higher expected value from possibility of future exercise.
- The difference from European options is that each time we move back in time, the earlier option value calculated using riskless portfolio approach or the risk-neutral probability approach must be compared with intrinsic value of the option and the higher of the two values is the option value.

Example 7: Suppose a stock is worth \$25 today and each year its price increases by 20% or decreases by 20%. The stock does not pay any dividends. The risk-free rate is 5% per annum continuously compounded. What is the value of an American put option with exercise price of \$26 that matures after two years?

The difference from Example 6 is that now we must also consider the possibility of early exercise at nodes before maturity. From example 6, if the stock price is \$30 after one year, and the put option is not exercised, then its value is

$$\{0.6282 \times 0 + (1 - 0.6282) \times 2\}e^{-0.05\times 1} = \$0.7074.$$

Exercising the option is unprofitable because the option is out-of-money so it is optimal to hold on to the option and its value is $P_1 = \$0.7074$. If the stock price is \$20 after one year and the put option is not exercised, its value is

$$\{0.6282 \times 2 + (1 - 0.6282) \times 10\}e^{-0.05\times 1} = \$4.7320.$$

However, exercising the option results in $max(26 - 20{,}0) = \$6$ which is more than $\$4.7320$ so it is optimal to exercise the option early and its value is $P_2 = \$6$. The initial put option value, if it is not exercised, is

$$P = \{0.6282 \times 0.7074 + (1 - 0.6282) \times 6\}e^{-0.05\times 1} = \$2.5448.$$

Exercising the option results in $max(26-25,0)=\$1$ which is less than $\$2.5448$ so it is optimal to hold the option and option value is $\$2.5448$. The difference of $\$1.5448$ from the intrinsic value of $1 is sometimes called the **time value of the option**.

Definition 1

A continuous-time stochastic process $\{B_t : 0 \leq T\}$ is called a **Standard Brownian Motion** on $[0, T]$ if it has the following four properties:

1. $B_0 = 0$
2. The increments of B_t are independent; that is, for any finite set of times $0 \leq t_1 < t_2 < \ldots < t_n < T$, the random variables

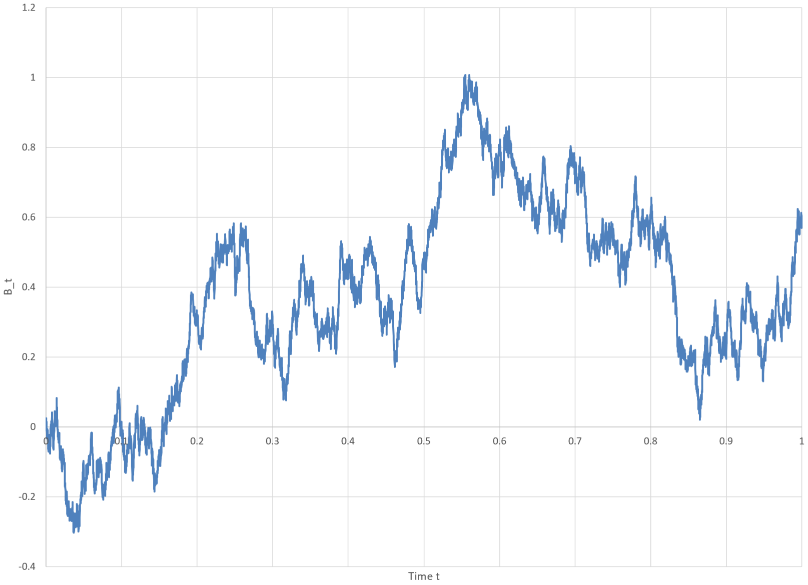
   $$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}$$

   are independent.
3. For any $0 \leq s \leq t < T$, the increment $B_t - B_s$ has the Gaussian (normal) distribution with mean 0 and variance $t - s$.
4. For all ω in a set of probability one, $B_t(\omega)$ is a continuous function of t.

Brownian Motion

1. We can think of a standard Brownian motion as an infinite collection of paths.
2. Each path is associated with a different random event ω in the sample space Ω. That is, each time a standard Brownian motion is realized, a particular ω is picked from Ω and the path corresponding to that ω is realized.
3. The change in value of Brownian motion over time t is a normally distributed random variable with zero mean and variance t.

Sample Brownian motion path - Time 0 to 1



Martingales

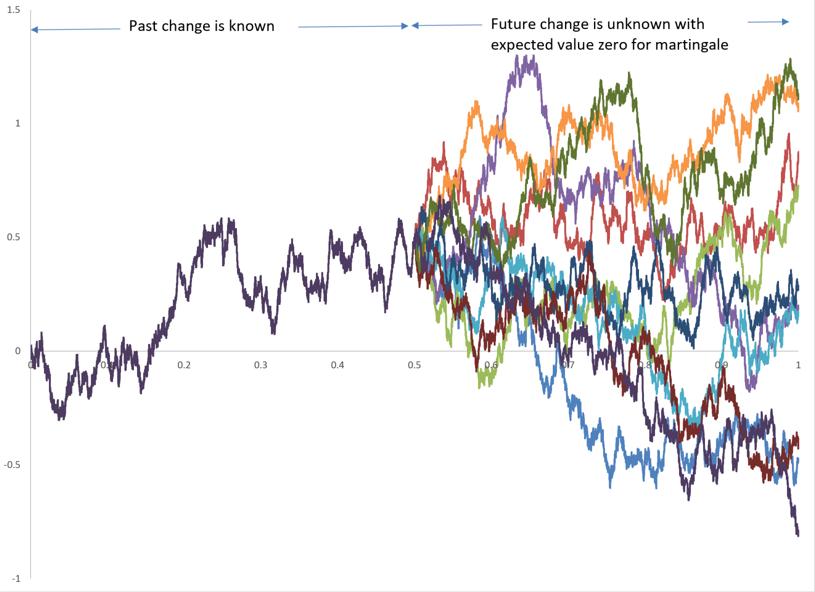
1. Even after observing the path up to some time t, one cannot distinguish between that path from another path that happens to be identical up to time t but diverges after that.
2. Definition 1 requires that future changes in a standard Brownian motion are unrelated to and cannot be predicted based on past changes. Furthermore, all changes in standard Brownian motion are normally distributed.

Definition 2

A stochastic process is a **martingale** if expected value of B_τ at time $t < \tau$ is B_t.

3. A standard Brownian motion is a martingale because future changes are on average zero.

Brownian motion is Martingale



Stochastic Differential Equations

- If the price of the underlying stock follows a stochastic process, so does the price of a derivative on the stock.
- To price a derivative, we create portfolios of underlying stock and derivative and analyze the returns.
- Financial principles such as law of one price or absence of arbitrage impose conditions on the return on the underlying stock and the return on the derivative.
- These restrictions take the form of an equation relating small changes in stock price, small changes in derivative price, and small changes in time.
- Equations relating small changes in quantities are called differential equations.

- When the changes are in stochastic quantities (stock price or derivative price in our case), these equations are called **stochastic differential equations**.
- Stochastic differential equations can also be expressed as **stochastic integrals** which express the change in a stochastic quantity by integrating small changes in that quantity.
- An **Itô integral** is a stochastic integral that integrates with respect to a Brownian motion.

Itô integral

Definition 3

An Itô integral is an integral of the form

$$I(f)(\omega) = \int_0^T f(\omega, t) \mathrm{d}B_t. \tag{1}$$

Properties of Stochastic Differentials

- The following properties will be very useful in our analysis of stochastic differential equations and stochastic integrals:

$$
\begin{aligned}
\mathrm{d}B_t^2 &= \mathrm{d}t \\
\mathrm{d}B_t \mathrm{d}t &= 0 \\
\mathrm{d}t^2 &= 0
\end{aligned} \tag{2}
$$

- Here $\mathrm{d}t$ represents an infinitesimal time interval and $\mathrm{d}B_t$ the change in the Brownian motion during that time interval.
- The approximations in the above equation are at the scale of $\mathrm{d}t$. That is, as $\mathrm{d}t$ is made smaller and approaches zero, quantities that approach zero even faster than $\mathrm{d}t$ are assumed to be zero.

Itô's Formula

- The following important formula states that an Itô integral can be evaluated using only a few partial derivatives, ignoring the partial derivatives that are zero based on the previous equation.

Theorem 1

(Itô's Formula) For a function $f(t, x)$ differentiable in t and twice differentiable in x,

$$f(t, B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\mathrm{d}B_s + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\mathrm{d}s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\mathrm{d}s. \tag{3}$$

Itô's Formula

- The notation $\frac{\partial f}{\partial x}$ represents the partial derivative of f with respect to x, the ratio of the change in f to the change in x for a small change in x, keeping all other inputs to f fixed.
- The formula states that for a function of time and Brownian motion, if its value is available at one point, its value can be obtained at any other point by starting from the original value and integrating the effect of changes in the Brownian motion, integrating the effect of changes in time, and integrating the effect of squared changes in the Brownian motion (which equal change in time).
- The theorem differs from the corresponding results in non-stochastic calculus in having the third term on the right, a term not required when the integration is with respect to a deterministic process.

Itô's Formula

- Itô's formula can also be expressed in differential form as

$$\mathrm{d}f(t, B_t) = \frac{\partial f}{\partial x}(t, B_t)\mathrm{d}B_t + \frac{\partial f}{\partial t}(t, B_t)\mathrm{d}t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)\mathrm{d}t. \tag{4}$$

- Notice that in the last term, $\mathrm{d}B_t \cdot \mathrm{d}B_t$ has been substituted with $\mathrm{d}t$ following the properties of stochastic differentials mentioned earlier.
- If f depends on time and another stochastic quantity (that depends on Brownian motion, instead of being Brownian motion as above), the corresponding formula is

$$\mathrm{d}f(t, S_t) = \frac{\partial f}{\partial x}(t, S_t)\mathrm{d}S_t + \frac{\partial f}{\partial t}(t, S_t)\mathrm{d}t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t)\mathrm{d}S_t \cdot \mathrm{d}S_t. \tag{5}$$

Arithmetic Brownian Motion

- Standard Brownian motions can be used to create other continuous stochastic processes.
- An **arithmetic Brownian motion** (ABM) is a process X_t which follows

$$\mathrm{d}X_t = \alpha \mathrm{d}t + \sigma \mathrm{d}B_t \tag{6}$$

- An ABM is a generalization of the standard Brownian motion.
- The first term captures time trend and α is called the drift.
- The second term, the stochastic component with no time trend, is called diffusion and σ is the volatility.
- A standard Brownian motion is a special arithmetic Brownian motion with drift $\alpha = 0$ and volatility $\sigma = 1$.
- Like standard Brownian motion, changes in the value of an ABM over different time intervals are independent and normally distributed. Unlike standard Brownian motion, the mean and the variance do not necessary equal zero and the time interval, respectively.

Arithmetic Brownian Motion Example

Example 1

- *Consider an arithmetic Brownian motion:*

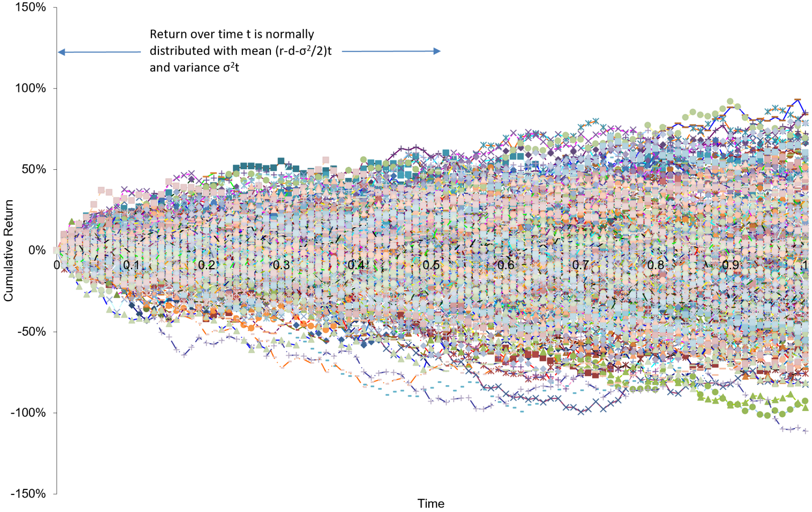
$$\mathrm{d}X_t = 0.05\mathrm{d}t + 0.4\mathrm{d}B_t \tag{7}$$

- *What is the drift of X_t?*
- *What is the volatility of X_t?*
- *Suppose $X_0 = 10$. What is the expected value of X_2?*
- *What is the variance of X_2?*
- *What is the standard deviation of X_2?*

Arithmetic Brownian Motion

- Logarithmic stock return is random and can be approximated as normally distributed but not necessarily zero on average.
- Stock returns are therefore, not modeled as standard Brownian motion which has zero drift.
- It is common to model a stock's return as an arithmetic Brownian motion with α the expected return on the stock and σ the volatility of the stock's return.
- Since an arithmetic return is not bounded and can reach arbitrarily high or arbitrarily low value, it is not a good choice for modeling stock prices which cannot be negative.
- However, the log return of the stock price can be used to recover the stock price.
- We model a stock's price as the process whose log is an ABM. Such a process is called a geometric Brownian motion.

Return as Arithmetic Brownian Motion



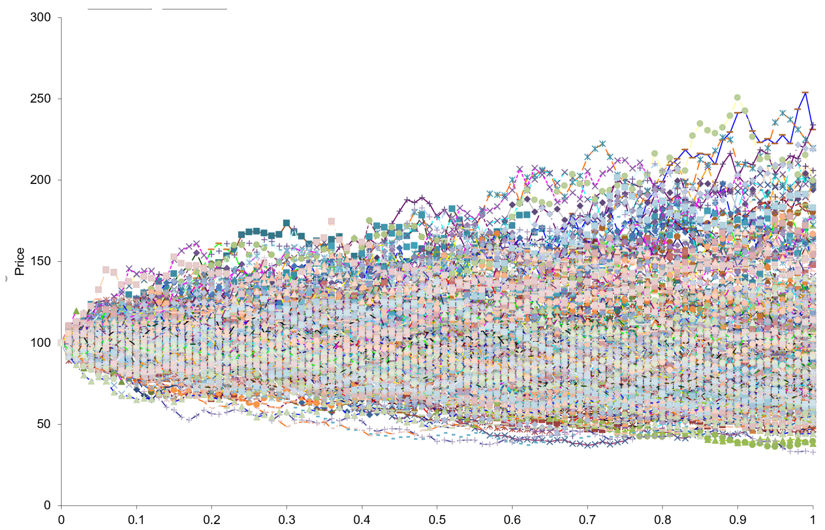
Geometric Brownian Motion

- A **geometric Brownian motion** (GBM) is a process X_t which follows

$$\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t \tag{8}$$

- Compare the stochastic differential equation of a GBM to that of an ABM. The right side is multiplied by stock price in the equation for GBM.
- Thus, changes in an ABM do not depend on current level while changes in a GBM are proportional to the current level.
- The log of a GBM is an ABM.
- Properties of a GBM make it a popular choice for modeling a stock's price.
- A GBM that starts as a positive value remains positive.
- Moreover, the return on a GBM is an ABM so if stock price is modeled as a GBM, stock returns over different time intervals are independent (market efficiency) and are normally distributed.

Price as Geometric Brownian Motion



Mean Reverting Process

- A **mean reverting process** is of the form

$$\mathrm{d}X_t = \kappa(\mu - X_t)\mathrm{d}t + \sigma X_t^\gamma \mathrm{d}B_t \tag{9}$$

- While stock prices do not exhibit mean reversion, a mean-reverting process can be a good modeling choice for quantities which exhibit a trend to revert to mean.

GBM and Probability Distribution

- We now consider the distribution of stock price changes if the stock price is modeled as a geometric Brownian motion.
- Suppose a stock's price at time t, S_t, is S_0 at time 0 and follows a geometric Brownian motion with drift μ and volatility σ.
- Then, the log of stock price $\ln(S_t)$ starts at $\ln(S_0)$ and follows an arithmetic Brownian motion with drift $\mu - \sigma^2/2$ and volatility σ.
- The log of the stock price $\ln(S_T)$ is normally distributed.
- This is the same as saying that the stock price S_T is lognormally distributed.
- The exact distribution for the change in log of stock price is:

$$\ln(S_T) - \ln(S_0) \sim \phi\left[(\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right]. \tag{10}$$

GBM and Probability Distribution of Log Price

- The change in log stock price, $\ln(S_T) - \ln(S_0)$ is normally distributed with mean $(\mu - \frac{\sigma^2}{2})T$ and variance $\sigma^2 T$ or standard deviation $\sigma\sqrt{T}$.
- The above equation can be rewritten as

$$\ln\left(\frac{S_T}{S_0}\right) \sim \phi\left[(\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right]. \tag{11}$$

- The log of future stock price is then normally distributed as follows:

$$\ln(S_T) \sim \phi\left[\ln(S_0) + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right]. \tag{12}$$

- The normal distribution is symmetric so the mean and median are the same. Thus, mean or median value of $\ln(S_T)$ is $\ln(S_0) + (\mu - \frac{\sigma^2}{2})T$.

GBM and Probability Distribution of Price

- Percentiles of stock price can be obtained readily by taking exponential of the percentiles of the log of stock price.
- The p-th percentile value of stock price is the exponential of the p-th percentile value of log of stock price.
- In particular, the median value of stock price is the exponential of the median value of the log of stock price.
- However, the mean of stock price is not the exponential of the mean of log of stock price.
- The reason is that log is not a linear function. The stock price increase from an increase λ in the log of stock price is higher than the stock price decrease from a decrease of λ in the log of stock price.
- The mean and variance of the stock price are given by

$$\mathbb{E}(S_T) = S_0 e^{\mu T} \tag{13}$$

and

$$\operatorname{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1). \tag{14}$$

GBM and Probability Distribution Example I

Example 2

- *Consider a stock with initial price of \$50, (continuously compounded) expected return of 6.5% per annum and volatility of 30%.*
- *The log of stock price after one year will be normally distributed with mean and median of* $\ln(50) + (0.065 - 0.3^2/2) * 1$ *and volatility* 0.3.
- *The expected stock price after one year is* $50e^{0.065*1}$.

Example 2

- *The following figure shows the probability distributions of log of stock price and of stock price.*
- *The horizontal axis is the probability percentile, the left vertical axis is for log of stock price plotted as dashed line and the right vertical axis is for stock price, plotted as solid line.*
- *The vertical line in the middle is at 50th percentile and intersects the two plots at their median values.*
- *The mean for the log of stock price is the same as median because its distribution is symmetric.*
- *However, the mean of the stock price exceeds the median because the stock price rise to the right is much steeper than the stock price fall to the left.*

Black-Scholes Model

- We will now model a continuous-time arbitrage argument to derive Black-Scholes formula for pricing European call options.
- The most important assumptions behind this model:
  - stock price follows geometric Brownian motion
  - all assets can be traded in any amount with no transaction costs
  - investors agree on stock and bond prices.
- Let S_t denote the price of a non-dividend-paying stock at time t and β_t denote the price of a riskless bond at time t. The time dynamics of these processes are modeled as:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \tag{15}$$

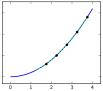
and

$$\mathrm{d}\beta_t = r\beta_t \mathrm{d}t. \tag{16}$$

- Note that we assume that the stock price follows a geometric Brownian motion and the bond price follows a deterministic process with exponential growth at risk-free rate.
- The arguments in the following derivation apply to any general European derivative, not just a call option.
- Suppose the payoff of the derivative at expiration time T is given by $h(S_T)$.
- For a European call option with strike price K, $h(S_T) = (S_t - K)^+ \equiv \max(S_T - K, 0)$. To price this derivative, we consider a self-financing replicating portfolio.

Self-Financing Replicating Portfolio

- By the absence of arbitrage argument, derivative price must equal the cost of setting up a replicating portfolio whose payoff equals the derivative payoff when the derivative expires.
- A static portfolio of stocks and bonds cannot replicate derivative payoff, which is non-linear in stock and bond prices.



- Since a nonlinear relation can be treated as linear for small changes, a portfolio of stocks and bonds can replicate change in derivative value over a small time interval.
- The replicating portfolio is rebalanced continuously by buying and selling securities so that it continues to replicate the derivative payoff over the next short time interval.
- Since a European derivative has no cash flow before maturity, the trades for the dynamic rebalancing of the replicating portfolio should not yield net positive or negative cash flow.
- Any cash required for buying securities to the replicating portfolio must be obtained by selling some other securities. Such a portfolio is called a self-financing portfolio.

Change in Value of Replicating Portfolio

- Suppose the replicating portfolio consists of a_t units of stock and b_t units of bond at time t. The portfolio value at time t is
$$V_t = a_t S_t + b_t \beta_t. \tag{17}$$
- The boundary condition at expiration is that the replicating portfolio replicate the option payoff
$$V_T = h(S_T). \tag{18}$$
- The change in the value (17) of the portfolio during a small time interval $\mathrm{d}t$ is $S_t \mathrm{d}a_t + a_t \mathrm{d}S_t + \beta_t \mathrm{d}b_t + b_t \mathrm{d}\beta_t$.
- Note that we assume that the stock does not pay dividends.
- The self-financing condition that no cash be invested or withdrawn from the portfolio for trades implies $S_t \mathrm{d}a_t + \beta_t \mathrm{d}b_t = 0$.
- Then, the change in value of the portfolio arises only from the changes in stock price and bond price:
$$\mathrm{d}V_t = a_t \mathrm{d}S_t + b_t \mathrm{d}\beta_t. \tag{19}$$

Stochastic Process for Replicating Portfolio - I

- We will arrive at two forms of the stochastic differential equation that governs changes in the value of the replicating portfolio. We will then use Itô calculus and match coefficients of differential equations.
- We use our expression for portfolio value as a function of stock price and time: $V_t = f(t, S_t)$. Substituting (15) and (16) in (19), we get

$$\begin{aligned} \mathrm{d}V_t &= a_t(\mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t) + b_t(r\beta_t \mathrm{d}t) \\ &= (a_t \mu S_t + r b_t \beta_t)\mathrm{d}t + a_t \sigma S_t \mathrm{d}B_t. \end{aligned} \tag{20}$$

- Applying Itô's formula (5) to the stochastic process $V_t = f(t, S_t)$ and then substituting (15), we get another expression for $\mathrm{d}V_t$:

$$\begin{aligned}\mathrm{d}V_t &= f_t(t, S_t)\mathrm{d}t + \frac{1}{2} f_{xx}(t, S_t)\mathrm{d}S_t \cdot \mathrm{d}S_t + f_x(t, S_t)\mathrm{d}S_t \\ &= \left\{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t)\sigma^2 S_t^2 + f_x(t, S_t)\mu S_t \right\} \mathrm{d}t \\ &+ f_x(t, S_t)\sigma S_t \mathrm{d}B_t. \qquad (21)\end{aligned}$$

- In the above equation, the symbol x in the partial derivatives f_x and f_{xx} stands for stock price and $f_x \equiv \frac{\partial f}{\partial x}$, $f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}$, and $f_t \equiv \frac{\partial f}{\partial t}$.

Number of Stocks in Replicating Portfolio

- Matching the coefficients of $\mathrm{d}B_t$ in (20) and (21), we get

$$a_t = f_x(t, S_t). \tag{22}$$

- To interpret this equation, note that the left side is the number of stocks in the replicating portfolio.
- The right side is the derivative of the option price with respect to stock price or the sensitivity of the derivative price to the stock price.
- Recall that we call this quantity **delta** of the derivative.
- The above equation states that the number of stocks held in a replicating portfolio at any time equals the delta of the derivative at that time.

Number of Bonds in Replicating Portfolio

- Next, we match the coefficients of $\mathrm{d}t$ in (20) and (21) to get

$$f_x(t, S_t)\mu S_t + rb_t\beta_t = f_t(t, S_t) + \frac{1}{2}f_{xx}(t, S_t)\sigma^2 S_t^2 + f_x(t, S_t)\mu S_t. \tag{23}$$

- After canceling $f_x(t, S_t)\mu S_t$ from both sides, we can solve for b_t:

$$b_t = \frac{1}{r\beta_t}\left\{f_t(t, S_t) + \frac{1}{2}f_{xx}(t, S_t)\sigma^2 S_t^2\right\}. \tag{24}$$

- The above equation gives the number of bonds in the replicating portfolio at any point of time. Since we didn't specify the face value of bonds, a more meaningful quantity is the investment $b_t\beta_t$ in bonds:

$$\frac{1}{r}\left\{f_t(t, S_t) + \frac{1}{2}f_{xx}(t, S_t)\sigma^2 S_t^2\right\}. \tag{25}$$

Black-Scholes Partial Differential Equation

- We can now write the value of the portfolio as the sum of the value of the stocks and the bonds in the portfolio:

$$\begin{aligned} f(t, S_t) &= V_t = a_t S_t + b_t \beta_t \\ &= f_x(t, S_t) S_t + \frac{1}{r} \left\{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S_t^2 \right\}. \end{aligned} \tag{26}$$

- Using the symbol x for stock price S_t, the above equation can be written as Black-Scholes Partial Differential Equation (PDE):

$$f_t(t, x) = -\frac{1}{2} \sigma^2 x^2 f_{xx}(t, x) - r x f_x(t, x) + r f(t, x) \tag{27}$$

with the terminal boundary condition

$$f(T, x) = h(x) \text{ for all } x. \tag{28}$$

Black-Scholes Partial Differential Equation

- There are four terms in the Black-Scholes PDE (27), representing the derivative value and its partial derivatives.
- The term $f(t, x)$ is the value of the derivative. The term $f_t(t, x)$ is the rate of change of value of the derivative with the passage of time and is called the **theta** (Θ) of the derivative.
- The term $f_x(t, x)$ is the sensitivity of the derivative price to the price of the underlying stock and is called the **delta** (Δ) of the derivative.
- Finally, the term $f_{xx}(t, x)$ is the second derivative of the derivative price with respect to the stock price or the rate of change of the derivative's delta with respect to the stock price and is called the derivative's **gamma** (Γ).
- Using Π to indicate the derivative value and S to indicate stock price, the Black-Scholes PDE can be written as

$$\Theta = -\frac{1}{2}\sigma^2 S^2 \Gamma - rS\Delta + r\Pi. \tag{29}$$

Black-Scholes Formula for Call

- The solution of the Black-Scholes PDE (27) with call option payoff in the boundary condition (28) decomposes into two integrals, each of which can be converted to an integral of the normal probability density (of the form e^{-x^2}).
- The result is the Black-Scholes formula for the price of a European call option:

$$
\begin{aligned}
f(t, S_t) = & S_t \Phi\left(\frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
& - K e^{-r\tau} \Phi\left(\frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \qquad (30)
\end{aligned}
$$

- Here, Φ is the cumulative normal distribution function. Sometimes the symbol N is used instead of the symbol Φ.

Black-Scholes Formula for Put

- The put-call parity or an alternative boundary condition can be used to derive the formula for the price of a European put option:

$$Ke^{-r\tau}\Phi\left(-\frac{\log(S_t/K)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - S_t\Phi\left(-\frac{\log(S_t/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \tag{31}$$

Dividends

- If the underlying stock has a known dividend yield q, these formulas can be adjusted to get European call price

$$S_t e^{-q\tau} \Phi\left(\frac{\log(S_t/K) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau} \Phi\left(\frac{\log(S_t/K) + (r - q - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \quad (32)$$

- The European put price is

$$Ke^{-r\tau} \Phi\left(-\frac{\log(S_t/K) + (r - q - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - S_t e^{-q\tau} \Phi\left(-\frac{\log(S_t/K) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \quad (33)$$

Options on Stock Indices

- Index options can be priced just like options on individual stocks by using the dividend yield on the index in place of stock dividend yield.
- As with all derivative pricing, the dividend yield must be based on expected dividends whose ex-dividend dates lie between pricing date and the expiration of the option.
- Note that index volatility is usually less than the average volatility of individual stocks in the index.
- Option on an index is therefore, cheaper than options on individual stocks.
- Often, the process of calculating index option price from inputs is reversed.
- That is, the prices of index options are used to infer the market's expectations of the dividend yield on the index.

Currency Options

- Currency options are mostly traded in over-the-counter market. A call or a put allows the option owner to buy or sell a specified amount of a foreign currency at a pre-specified exchange rate.
- These options can also be priced like options on a stock with the modification that the interest rate on the foreign currency replaces the dividend yield on the underlying stock.
- Again, the task of pricing can be reversed to infer implied volatility about the exchange rate from the option price.

Futures Options

- A futures option is an option with a futures contract as the underlying asset.
- That is, a call (put) gives the option holder the right to buy (sell) a futures contract at a fixed price. These options are naturally related to the options on the asset underlying the futures contract.
- For example, a call on a futures contract on stock XYZ is related to a call on stock XYZ.
- Traders may sometimes prefer to use futures options rather than options directly on underlying assets if the futures market is more liquid and allows transactions with lower transaction costs.

Futures Options

- The price c of a European call option on futures and the price p of a European put option on futures are given by

$$c = e^{-rT}\left[F_0 N(d_1) - KN(d_2)\right], \tag{34}$$

$$p = e^{-rT}\left[KN(-d_2) - F_0 N(-d_1),\right] \tag{35}$$

where

$$d_1 = \frac{\ln F_0/K + \sigma^2 T/2}{\sigma\sqrt{T}}, d_2 = \frac{\ln F_0/K - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

- σ is the volatility of the futures price, T is the time to maturity of the option, and N (same as Φ) is the cumulative normal distribution function.

Risk-Neutral Pricing

- A powerful technique for valuing derivatives is risk-neutral pricing. The technique can be used when the payoffs of the assets being priced can be replicated by dynamically trading other assets with known prices.
- The technique can be used when there are enough assets with related payoffs to pin down payoff of the asset being priced.
- For example, a stock and a bond can be traded to pin down price of a call or put option on the stock. In this case, risk neutral pricing can be used.
- If other assets cannot completely capture the risk of the asset being priced, risk-neutral pricing cannot be used.
- The risk-neutral technique assumes that investors do not assign any risk premium to the risk of the underlying stock.
- That means that the stock, any derivatives on the stock, and riskless bonds (or even risky bonds), all earn expected return of risk-free rate. This assumption simplifies math.

Risk-Neutral Pricing

- A risk-neutral investor can use risk neutral pricing with two features:
  1. the stock value grows at the risk-free rate on average and
  2. all cash flows are discounted at the risk-free rate.
- The first feature allows the investor to predict stock price probability distribution at option expiration.
- This distribution is used to calculate the derivative cash flow distribution at expiration and therefore, the expected value of the derivative cash flow at expiration.
- The second feature allows the investor to discount the derivative's expected cash flow to determine derivative price today.

Risk-Neutral Pricing

- One application of risk neutral pricing is the construction of multiple step binomial trees for numerical pricing of derivatives.
- The probabilities for up and down movement of stock price are calculated using risk-neutral pricing assumption.
- We now apply this principle derivative pricing under Black-Scholes assumption about stock price movement.

Risk-Neutral Pricing

- Consider the price of a derivative security at time t with stock price x.
- Let the derivative expire at time $T = t + \tau$ and let h be the derivative's payoff at expiration.
- Using the lognormal distribution (12) for log of stock price at expiration, $y = \ln(S_T)$, and replacing drift μ with risk-free rate r, the derivative price is

$$f(t,x) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} h(e^y) e^{-\frac{1}{2}\left(\frac{y-\ln(x)-(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)^2} \mathrm{d}y. \quad (36)$$

- Substituting $z = \frac{y-\ln(x)-(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}$, we get

$$f(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[e^{-r\tau} h(xe^{z\sigma\sqrt{\tau}+(r-\sigma^2/2)\tau})\right] e^{-\frac{z^2}{2}} \mathrm{d}z. \quad (37)$$

Risk-Neutral Pricing

- For a European call option with strike K, substituting $h(x) = \max(x - K, 0)$, the price is given by

$$c(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\max(x e^{z\sigma\sqrt{\tau} - \sigma^2 \tau/2} - K e^{-r\tau}, 0) \right] e^{-\frac{z^2}{2}} \mathrm{d}z. \tag{38}$$

- This expression provides Black-Scholes formula with some algebra

Option Risk Measures

Risk Measures

- The risks of an option or a portfolio in general are the factors on which the value of the option or the portfolio depends.
- For call or put options on stocks, option value depends on the price of the underlying stock, the time to maturity, the risk-free rate, and the volatility of the underlying stock.
- Investors may want to
  - identify risks
  - decide how much risk they are willing to take
  - reduce the risks
- A risk measure of an option (or a portfolio) with respect to a parameter x is the ratio of the change in the value of the option (or portfolio) to a small change in x. The risk measures for options are traditionally represented by Greek letters.

Applications of Risk Measures

- If an investor has a position in an option and wants to neutralize its risk, the investor can take another position in some other assets with opposite risk measure.
- An investor can create an option position synthetically without actually buying the option by creating a portfolio using other assets with the same risk measures as the desired option.
- We saw in our discussion of put-call parity that a synthetic stock can be created using calls, puts, and bonds.

Covered and Naked Positions

- Consider an investor who has written a call option on a stock.
- The investor's position is covered if the investor owns the stock that can be delivered if the call is exercised.
- The investor's position is naked if the investor does not hold the stock.
- A covered position hedges the investor's exposure if the call is exercised.
- A naked position covers the investor if the call is not exercised.
- However, the investor cannot be sure if the call will be exercised or not so both naked and covered positions are extreme positions and provide imperfect hedge.

Example: A stock is trading at $25. After one period, the stock price will increase to $30 or decrease to $20. Consider a short position in a call option with strike price of $25. What will be the future values of your portfolio if your portfolio consists of

a) Naked call
b) Covered call

If the call is naked, the portfolio value equals the value of the short call, and it will be worth -$5 if the stock price rises to $30 and worth $0 if the stock price falls to $20.

If the call is covered, the portfolio value equals the value of the short call and the stock. It will be worth -$5 + $30 = $25 if the stock price rises to $30 and worth $0 + $20 = $20 if the stock price falls to $20.

Thus, a naked short call position results in a loss if the stock price rises, while a covered short call position results in a loss if the stock price falls. None is riskless.

Stop-Loss Strategy

- Both naked call and covered call are imperfect hedges whose effectiveness depends on the moneyness of the option.
- A dynamic strategy changes portfolio as the moneyness of the option changes.
- Simplest dynamic strategy:
  - covered position when the option is in the money
  - naked position when the option is out-of-money
  - to hedge a short position in a call, buy the underlying stock when its price rises above the exercise price and sell when its price falls below the exercise price
  - this is called stop-loss strategy.
- The stop-loss strategy is not very effective in practice because it is difficult to predict if the stock price will increase or decrease relative to exercise price.
- This means that stock will be purchased when the price is already more than exercise price and stock will be sold at prices below exercise price. This will lead to loss on average. This is not a problem as gain/loss in trading offset gain/loss on short call position.
- The problem is that offsetting is not perfect and total gain/loss is uncertain.
- This uncertainty is undesirable as a hedging strategy should reduce risk.

Example: See Excel spreadsheet for simulation of stop-loss strategy.

Delta

- The Delta (Δ) of an option measures the sensitivity of the option with respect to the stock price change.
- It measures the ratio of the change in the value of the option to the change in stock price when the change in stock price is infinitesimal and other parameters are kept unchanged.
- Recall that the value of a European call option is given by the Black-Scholes formula:

$$d_1 = \frac{ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

$$c = SN(d_1) - Xe^{-rT}N(d_2)$$

$$p = Xe^{-rT}N(-d_2) - SN(-d_1)$$

where $N()$ is cumulative normal distribution.

- Taking derivative of call price with respect to stock price S, we get

$$\Delta = N(d_1)$$

- For a European put option,

$$\Delta = -N(-d_1) = N(d_1) - 1$$

Notes:

1. The expressions provided for option risk measures assume Black-Scholes pricing formula.
2. The formulas provided assume that the stock does not pay dividends. These can be adjusted if the stock pays dividend.
3. If an option does not follow Black-Scholes assumptions or if the option is not American, explicit formulas for option risk measures may not exist. In such cases, numerical methods such as binomial tree pricing can be used to estimate option risk measures.

Dynamic Delta Hedging

- To hedge an option exposure against stock price movements, a position in stocks is held and adjusted dynamically as the delta of the option changes.
- The idea in hedging is to make the delta of the portfolio zero.
- The underlying stock has a delta of +1.
- If there is a long position in a call, hedging it requires shorting delta number of stocks.
- To synthetically create a call position, one must buy delta number of stocks.
- As option delta changes with time, the number of stocks held or shorted must also change.
- With frequent trading, most of the risk associated with stock price can be mitigated.
- However, trading may involve transaction costs.
- In practice, the frequency of trading is based on a compromise between the risk of position and the transaction costs.

Example: A European call option on a non-dividend paying stock has a delta of 0.6. If you have shorted 50 contracts (each contract is for 100 shares), what position in stocks would you take to Delta hedge your portfolio? Alternatively, what position in corresponding puts can you take? If the stock price increases, will you buy stock or sell stock?

Delta of the short position in 50 contracts = -0.6 × 50 × 100 = -3000

To make portfolio Delta-neutral, we need a position with Delta = 3000. Since each share of stock has a Delta of +1, we need to buy 3000 shares of stock.

Alternatively, we could hedge using put options. Delta of a put option = Delta of a call option – 1 = 0.6 – 1 = -0.4. Number of put options needed to make the portfolio Delta-neutral is 3000/(-0.4) = -7500. That is, 7,500 put options should be shorted. If each put contract is for 100 shares, we need to short 75 put contracts.

Theta

- The Theta (Θ) of an option measures the sensitivity of the option with respect to the passage of time.
- Theta of options is usually negative.
- For a European call option on a non-dividend paying stock:

$$\Theta = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

and for the corresponding put,

$$\Theta = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(d_2)$$

where $N()$ is cumulative normal distribution and $N'()$ is Normal distribution density.

Gamma

- Delta hedging is perfect only if it is carried out continuously.
- In practice, if hedging position is updated at an interval, the delta of the position changes in that interval making hedge imperfect.
- Gamma (Γ) of an option measures the sensitivity of the delta of the option with respect to the stock price.
- It is a measure of the convexity of the option price as a function of stock price.
- For a European call or put option on a non-dividend paying stock:

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

- The positive value of gamma shows that delta of an option increases as the underlying stock price increases.
- That is why a Delta hedged portfolio must be dynamically adjusted.
- Gamma and Theta of an option are related so that the change in the value of a delta-neutral portfolio is approximately:

$$\Delta \Pi = \Theta \Delta t + \frac{1}{2} \Gamma \Delta S^2$$

Vega

- The Vega of a portfolio is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset.
- The Vega of the underlying asset itself is zero.
- For a European call or put option, Vega is given by

$$\mathrm{V} = S_0\sqrt{T}N'(d_1)$$

- The volatility is forward-looking volatility.
- A popular measure of market's estimate of future volatility is the volatility index **VIX** which estimates 30-day volatility based on prices of options on S&P 500 index.
- The VIX index is often referred to as the "Fear Index."

Rho

- The rho of a portfolio is the rate of change of the value of the portfolio with respect to the interest rate.
- For a European call option on a non-dividend paying stock:

$$\rho = KTe^{-rT}N(d_2)$$

- For a European put option on a non-dividend paying stock:

$$\rho = -KTe^{-rT}N(-d_2)$$

CREDIT DERIVATIVES

- Derivatives where the payoff depends on the credit quality of a company or sovereign entity
- The market started to grow fast in the late 1990s but has declined somewhat since the 2007-2009 crisis

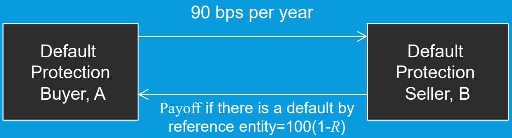
1

CREDIT DEFAULT SWAPS

- Buyer of the instrument acquires protection from the seller against a default by a particular company or country (the reference entity)
- Example: Buyer pays a premium of 90 bps per year for $100 million of 5-year protection against company X
- Premium is known as the *credit default spread*. It is paid for life of contract or until default
- If there is a default, the buyer has the right to sell bonds with a face value of $100 million issued by company X for $100 million (Several bonds may be deliverable)

2

CDS STRUCTURE



Recovery rate, R, is the ratio of the value of the bond issued by reference entity immediately after default to the face value of the bond

3

OTHER DETAILS

- Payments are usually made quarterly in arrears
- In the event of default there is a final accrual payment by the buyer
- Settlement can be specified as delivery of the bonds or (more usually) a cash equivalent amount
- An auction process usually determines a cash payout
- Suppose payments are made quarterly in the example just considered. What are the cash flows if there is a default after 3 years and 1 month and recovery rate is 40%?

4

CDSS AND BONDS

- A 5-year bond plus a 5-year CDS produces a portfolio that is (approximately) risk-free
- This shows that bond yield spreads should be close to CDS spreads
- The CDS-bond basis is the excess of CDS spreads over the corresponding bond yield spreads. (Negative during the credit crisis)

5

THE PAYOFF

- Usually there are a number of bonds that can be delivered in the event of a default
- The protection buyer can choose to deliver the bond with the lowest price
- But in practice an auction process is usually used to determine a cash payoff

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ATTRACTIONS OF THE CDS MARKET

- Allows credit risks to be traded in the same way as market risks
- Can be used to transfer credit risks to a third party
- Can be used to diversify credit risks

7

HAZARD RATES

- A hazard rate of $h(t)$ at time t means that there is a probability of $h(t)\Delta t$ of a default between times t and $t+\Delta t$ conditional on no earlier default
- The survival probability to time t is $e^{-\bar{h}t}$ where $\bar{h}$ is the average hazard rate up to time t

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CDS VALUATION

- Hazard rate for reference entity is 2%.
- Assume payments are made annually in arrears, that defaults always happen halfway through a year, and that the expected recovery rate is 40%.
- The risk-free rate is 5% per annum continuously compounded.
- Let the breakeven CDS rate be s per dollar of notional principal

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UNCONDITIONAL DEFAULT AND SURVIVAL PROBABILITIES

Time (years)	Survival Probability	Default Probability
1	0.9802	0.0198
2	0.9608	0.0194
3	0.9418	0.0190
4	0.9231	0.0186
5	0.9048	0.0183

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CALCULATION OF PV OF PAYMENTS

Time (yrs)	Survival Probability	Expected Payment	Discount Factor	PV of Expected Payment
1	0.9802	0.9802s	0.9512	0.9324s
2	0.9608	0.9608s	0.9048	0.8694s
3	0.9418	0.9418s	0.8607	0.8106s
4	0.9231	0.9231s	0.8187	0.7558s
5	0.9048	0.9048s	0.7788	0.7047s
Total				4.0728s

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PRESENT VALUE OF EXPECTED PAYOFF (PRINCIPAL = $1)

Time (yrs)	Default Probability	Recovery Rate	Expected Payoff	Discount Factor	PV of Expected Payoff
0.5	0.0198	0.4	0.0119	0.9753	0.0116
1.5	0.0194	0.4	0.0116	0.9277	0.0108
2.5	0.0190	0.4	0.0114	0.8825	0.0101
3.5	0.0186	0.4	0.0112	0.8395	0.0094
4.5	0.0183	0.4	0.0110	0.7985	0.0088
Total					0.0506

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PV OF ACCRUAL PAYMENT MADE IN EVENT OF A DEFAULT (PRINCIPAL = $1)

Time	Default Probability	Expected Accrued Payment	Discount Factor	PV of Payment
0.5	0.0198	$0.0099s$	0.9753	$0.0097s$
1.5	0.0194	$0.0097s$	0.9277	$0.0090s$
2.5	0.0190	$0.0095s$	0.8825	$0.0084s$
3.5	0.0186	$0.0093s$	0.8395	$0.0078s$
4.5	0.0183	$0.0091s$	0.7985	$0.0073s$
Total				$0.0422s$

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PUTTING IT ALL TOGETHER

- PV of expected payments is $4.0728s + 0.0422s = 4.1150s$
- The breakeven CDS spread is given by

  $4.1150s = 0.0506$ or $s = 0.0123$ (123 bps)

- The value of a swap negotiated some time ago with a CDS spread of 150bps would be $4.1150\times0.0150-0.0506 = 0.0111$ per dollar of the principal.

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CREDIT INDICES

- CDX NA IG tracks the average CDS spread for a portfolio of 125 investment-grade (rated BBB or above) North American companies
- iTraxx Europe tracks the average CDS sppread for a portfolio of 125 investment-grade European companies

STANDARD TRANCHES ARE CREATED FROM STANDARD PORTFOLIOS

- CDX NA IG:
  - Tranches: 0-3%, 3-7%, 7-10%, 10-15%, 15-30%, 30-100%
- iTraxx Europe:
  - Tranches: 0-3%, 3-6%, 6-9%, 9-12%, 12-22%, 22-100%

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