QF 430: Introduction to Derivatives

Old Final Exam Solution

- Write your name here:______________________________
- The exam duration is 75 minutes.
- You are allowed two sheets of notes (each two-sided) and a calculator.
- Do not use other books, notes, or study material.
- Do not use phones or other electronics devices.
- Check that your exam has pages 1-9.

(1) It is sometimes optimal to exercise an American call option at expiration or on a day immediately preceding an ex-dividend date of the underlying stock. Precisely explain why the option should not be exercised at any other date.

5 points

Consider a date t before expiration that does not precede an ex-dividend date. Then, the stock does not pay dividend between dates t and $t+1$. Exercising the call on date $t+1$ is preferable to exercising the call on date t because both provide the same stock but exercising later requires payment of strike price at a later date and costs less in present value terms.

(2) An investor wants a call option on stock X but options do not trade on stock X. A market maker sells the desired option to the investor. Precisely explain how the market maker can hedge her exposure. 5 points

The market maker can create a synthetic option by investing the price of the option in a portfolio that consists of Δ stocks and the rest of the investment in bonds. The instantaneous loss or gain on this portfolio will equal the instantaneous loss or gain on the call option. The market maker will have to continuously adjust the portfolio as option Δ changes by shifting cash between stocks and bonds.

(3) A bank entered into a currency swap a few years ago. The swap expires in 18 months. The swap consists of annual exchanges of cash flows. The bank pays 3% per annum annually compounded on US dollars and receives 3% per annum annually compounded on Pound sterling. The principals are \$40 million and £30 million. The current exchange rate is \$1 = £0.76. Assume that the term structure of interest rates is flat with a rate of 2.5% per annum continuously compounded in US dollars and 1% per annum continuously compounded in Pound sterling. What is the value of the swap to the bank? 10 points

The present value of US dollar payments paid is

$$40,000,000 \times 0.03 \times e^{-0.025\times 0.5} + 40,000,000 \times 1.03 \times e^{-0.025\times 1.5} = \$40,868,703.$$

The present value of Pound sterling payments received is

$$30,000,000 \times 0.03 \times e^{-0.01\times 0.5} + 30,000,000 \times 1.03 \times e^{-0.01\times 1.5} = \pounds 31,335,470.$$

The value of swap is

$$\pounds 31,335,470 - \$40,868,703 = \$31,335,470/0.76 - \$40,868,703 = \$362,178.4$$

or

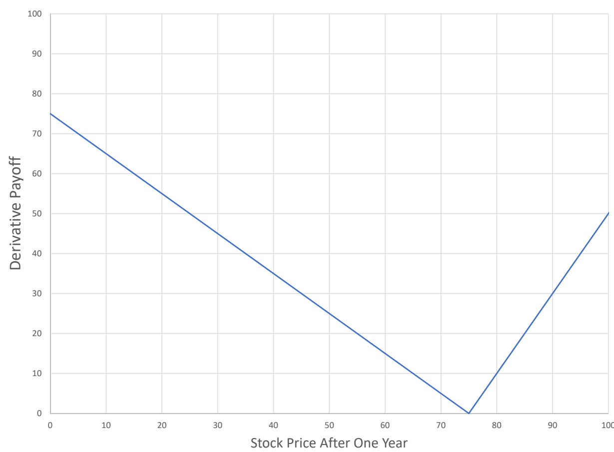
$$\pounds 31,335,470 - \pounds 40,868,703 \times 0.76 = \pounds 275,255.6$$

(4) A stock that does not pay dividend is trading for \$50. A riskless bond that will pay \$100 after a year is trading for \$96. A European put option on the stock with strike price of \$60 and one year to maturity is trading for \$7. Propose an arbitrage strategy and prove that it is an arbitrage strategy.

10 points

Buy put option and stock by borrowing \$57. After one year, exercise the put option to sell the stock for \$60. Repay $\$57/0.96 = \$59.375$ and keep \$0.625 as arbitrage profit.

(5) A stock is priced at \$75 and has volatility of 40% per annum. The risk-free rate is 4% per annum compounded continuously. Price a European derivative that matures in one year and will pay \$1 for every \$1 decrease in stock price below \$75 and \$2 for every \$1 increase in stock price above \$75 (see the following figure). 10 points



The derivative is a combination of one put and two calls, both with strike price of \$75 and maturity one year.

$$d_1 = \frac{\ln(75/75) + (0.04 + 0.4^2/2) \times 1}{0.4\sqrt{1}} = 0.3$$

$$d_2 = \frac{\ln(75/75) + (0.04 - 0.4^2/2) \times 1}{0.4\sqrt{1}} = -0.1$$

$$C = 75N(0.3) - 75e^{-.04\times 1}N(-0.1) = \$13.18$$

$$P = 75e^{-.04\times 1}N(0.1) - 75N(-0.3) = \$10.24$$

The derivative price is therefore, $10.24 + 2 \times 13.18 = \$36.61$.

(6) A European derivative will pay a dollar amount after one year equal to the square root of a stock's price in dollars at that time (\$2 if the stock price is \$4, \$3, if the stock price is \$9, etc.). The stock is currently trading at \$20. After one year, its price will be either \$16 or \$25. Value the derivative using the risk-neutral method. The risk-free rate is 5% per annum compounded continuously. 10 points

The risk-neutral probability that the stock price will rise to \$25 is

$$\frac{20e^{0.05} - 16}{25 - 16} = 0.5584.$$

The price of the derivative is

$$e^{-0.05}\left(0.5584 \times \sqrt{25} + (1 - 0.5584) \times \sqrt{16}\right) = \$4.336.$$

(7) The price $f(t,x)$ of a European derivative, at time t when the underlying stock price is x, satisfies the Black-Scholes Partial Differential Equation (PDE) given by

$$f_t(t,x) = -\frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) - rxf_x(t,x) + rf(t,x),$$

where σ is stock volatility and r is the risk-free rate. This equation relates derivative price to derivative Greeks. A stock's price is \$60 and volatility is 30% per annum. The risk-free rate is 5%. A derivative on the stock has delta of 0.6, gamma (sensitivity of delta to stock price) of 0.05, and theta (sensitivity of option price to time passage) of -9. What is the price of the derivative? 10 points

The PDE can be rewritten as

$$\Theta = -\frac{1}{2}\sigma^2 x^2 \Gamma - rx\Delta + r\Pi$$

where Π is the derivative price. Rearranging,

$$\begin{aligned}\Pi &= \frac{\Theta + \frac{1}{2}\sigma^2 x^2 \Gamma + rx\Delta}{r} \\ &= \frac{-9 + \frac{1}{2}0.3^2 \times 60^2 \times 0.05 + 0.05 \times 60 \times 0.6}{0.05} = \$18.\end{aligned}$$

Use the following table for cumulative normal probability distribution. For *d*<0, use N(*d*)=1-N(-*d*). For example, N(-0.05) = 1 – N(0.05) = 1 – 0.5199 = 0.4801.

d	**N(d)**	**d**	**N(d)**	**d**	**N(d)**	**d**	**N(d)**	**d**	**N(d)**	**d**	**N(d)**
0.00	0.5000	0.40	0.6554	0.80	0.7881	1.20	0.8849	1.60	0.9452	2.00	0.9772
0.01	0.5040	0.41	0.6591	0.81	0.7910	1.21	0.8869	1.61	0.9463	2.05	0.9798
0.02	0.5080	0.42	0.6628	0.82	0.7939	1.22	0.8888	1.62	0.9474	2.10	0.9821
0.03	0.5120	0.43	0.6664	0.83	0.7967	1.23	0.8907	1.63	0.9484	2.15	0.9842
0.04	0.5160	0.44	0.6700	0.84	0.7995	1.24	0.8925	1.64	0.9495	2.20	0.9861
0.05	0.5199	0.45	0.6736	0.85	0.8023	1.25	0.8944	1.65	0.9505	2.25	0.9878
0.06	0.5239	0.46	0.6772	0.86	0.8051	1.26	0.8962	1.66	0.9515	2.30	0.9893
0.07	0.5279	0.47	0.6808	0.87	0.8078	1.27	0.8980	1.67	0.9525	2.35	0.9906
0.08	0.5319	0.48	0.6844	0.88	0.8106	1.28	0.8997	1.68	0.9535	2.40	0.9918
0.09	0.5359	0.49	0.6879	0.89	0.8133	1.29	0.9015	1.69	0.9545	2.45	0.9929
0.10	0.5398	0.50	0.6915	0.90	0.8159	1.30	0.9032	1.70	0.9554	2.50	0.9938
0.11	0.5438	0.51	0.6950	0.91	0.8186	1.31	0.9049	1.71	0.9564	2.55	0.9946
0.12	0.5478	0.52	0.6985	0.92	0.8212	1.32	0.9066	1.72	0.9573	2.60	0.9953
0.13	0.5517	0.53	0.7019	0.93	0.8238	1.33	0.9082	1.73	0.9582	2.65	0.9960
0.14	0.5557	0.54	0.7054	0.94	0.8264	1.34	0.9099	1.74	0.9591	2.70	0.9965
0.15	0.5596	0.55	0.7088	0.95	0.8289	1.35	0.9115	1.75	0.9599	2.75	0.9970
0.16	0.5636	0.56	0.7123	0.96	0.8315	1.36	0.9131	1.76	0.9608	2.80	0.9974
0.17	0.5675	0.57	0.7157	0.97	0.8340	1.37	0.9147	1.77	0.9616	2.85	0.9978
0.18	0.5714	0.58	0.7190	0.98	0.8365	1.38	0.9162	1.78	0.9625	2.90	0.9981
0.19	0.5753	0.59	0.7224	0.99	0.8389	1.39	0.9177	1.79	0.9633	2.95	0.9984
0.20	0.5793	0.60	0.7257	1.00	0.8413	1.40	0.9192	1.80	0.9641	3.00	0.9987
0.21	0.5832	0.61	0.7291	1.01	0.8438	1.41	0.9207	1.81	0.9649	3.05	0.9989
0.22	0.5871	0.62	0.7324	1.02	0.8461	1.42	0.9222	1.82	0.9656	3.10	0.9990
0.23	0.5910	0.63	0.7357	1.03	0.8485	1.43	0.9236	1.83	0.9664	3.15	0.9992
0.24	0.5948	0.64	0.7389	1.04	0.8508	1.44	0.9251	1.84	0.9671	3.20	0.9993
0.25	0.5987	0.65	0.7422	1.05	0.8531	1.45	0.9265	1.85	0.9678	3.25	0.9994
0.26	0.6026	0.66	0.7454	1.06	0.8554	1.46	0.9279	1.86	0.9686	3.30	0.9995
0.27	0.6064	0.67	0.7486	1.07	0.8577	1.47	0.9292	1.87	0.9693	3.35	0.9996
0.28	0.6103	0.68	0.7517	1.08	0.8599	1.48	0.9306	1.88	0.9699	3.40	0.9997
0.29	0.6141	0.69	0.7549	1.09	0.8621	1.49	0.9319	1.89	0.9706	3.45	0.9997
0.30	0.6179	0.70	0.7580	1.10	0.8643	1.50	0.9332	1.90	0.9713	3.50	0.9998
0.31	0.6217	0.71	0.7611	1.11	0.8665	1.51	0.9345	1.91	0.9719	3.55	0.9998
0.32	0.6255	0.72	0.7642	1.12	0.8686	1.52	0.9357	1.92	0.9726	3.60	0.9998
0.33	0.6293	0.73	0.7673	1.13	0.8708	1.53	0.9370	1.93	0.9732	3.65	0.9999
0.34	0.6331	0.74	0.7704	1.14	0.8729	1.54	0.9382	1.94	0.9738	3.70	0.9999
0.35	0.6368	0.75	0.7734	1.15	0.8749	1.55	0.9394	1.95	0.9744	3.75	0.9999
0.36	0.6406	0.76	0.7764	1.16	0.8770	1.56	0.9406	1.96	0.9750	3.80	0.9999
0.37	0.6443	0.77	0.7794	1.17	0.8790	1.57	0.9418	1.97	0.9756	3.85	0.9999
0.38	0.6480	0.78	0.7823	1.18	0.8810	1.58	0.9429	1.98	0.9761	3.90	1.0000
0.39	0.6517	0.79	0.7852	1.19	0.8830	1.59	0.9441	1.99	0.9767	3.95	1.0000