Forward and Futures Prices

Arbitrage Arguments for Pricing Forward and Futures

Two alternative strategies that achieve the same purpose (for example, both result in getting an asset after six months) must cost the same.

An arbitrage opportunity is a trading strategy that requires zero investment, earns positive profits with a positive probability, and has no possibility of a loss. In general, an arbitrage opportunity should not exist.

A typical arbitrage strategy consists of two trades with almost opposite cash flows. Often, we will compare trading through a forward contract with trading in the spot market.

This comparison is easier in case of an **investment asset**, an asset which is held solely for investment purpose by many traders.

A **consumption asset** on the other hand is held primarily for consumption.

Gold and silver can be considered investment assets while crude oil and corn are examples of consumption assets.

Traders may prefer getting consumption assets earlier because they derive some benefit from consuming these assets.

Storage costs have the opposite effect and reduce the value of getting an asset earlier.

Benefits and Costs of Holding Assets

If an asset is held from time 0 to time T, one benefit of the asset is the price change between time 0 and time T.

For assets that result in cash income, such as dividends or coupons, we calculate

- the **present value *I* of the income** stream or
- the **income yield *q***, the continuously compounded rate at which income is earned.

Similarly, we represent storage costs as

- the **present value *U* of all storage costs** between time 0 and time *T* or
- the **continuously compounded rate *u*** at which storage costs are incurred.

There may be other benefits of holding an asset that are difficult to estimate directly but impact prices.

We represent these benefits with **convenience yield *y***, the continuously compounded rate at which one benefits from holding an asset (in addition to any income from the asset).

Short Selling

A short seller borrows an asset from another investor, sells the borrowed asset, buys back the asset at a later date and then returns it to the investor from whom he/she borrowed the asset.

The short-seller of the security must also compensate the lender for any benefit from the security during the borrowing period.

For example, the short-seller pays to the lender any dividends or interest payments on the security.

Sometimes the borrowing demand for a security may exceed the supply of lenders and the lenders may be able to charge an extra fee. Our arbitrage arguments will assume ignore such fees.

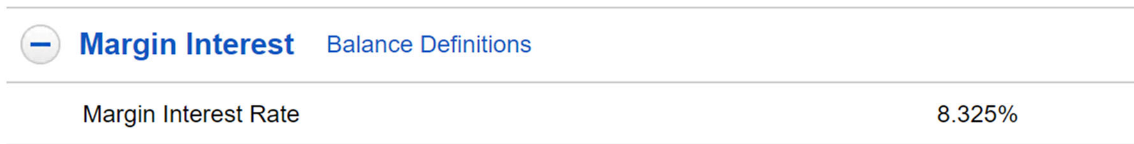
The cash flows of short selling a security are the opposite of buying the security.

A buyer pays initial price, gets income from security, and then gets the final price at selling.

A short seller gets initial price, pays income from security, and then pays the final price. Short selling can be considered as buying a negative amount of an asset.

[Video about short selling]

Margin interest rate in a retail brokerage account on March 1, 2021



Example of a hard-to-borrow stock quote

Forward Price for an Investment Asset

If S_0 is the spot price at time 0 of an investment asset, F_0 is the forward price at time 0 of a forward contract that matures at time T, and the risk-free rate is r, then

$$F_0 = S_0 e^{rT}$$

Notes:

1. We consider arbitrage strategies that compare spot and forward prices. The basic idea is that forward prices should be consistent with spot prices. However, we are not claiming that spot prices are correct. If investors consider the spot price itself be incorrect, they may trade to exploit this perceived mispricing, but usually such mispricing doesn't result in arbitrage opportunities.
2. Arbitrage arguments relying on short-selling also work without short-selling if there are some investors who hold the asset and are willing to sell.

Example 1: A zero-coupon bond trades at $900. The interest rate is 10% per annum with continuous compounding.

a. What is the forward price of a six-month forward contract?

$$F_0 = S_0 e^{rT} = 900e^{0.10\times 0.5} = \$946.144$$

b. If the forward price is $940, what arbitrage strategy exists?

Short sell spot, lend the resulting $900 at 10% pacc, and long forward. From the $946.144 received after six months, use $940 to take delivery of the bond. Return bond to close the short position. Keep $6.144 as profit.

c. If the forward price is $950, what arbitrage strategy exists?

Borrow $900 at 10% pacc to buy spot and short forward. After 6 months, deliver bond to get $950. Use $946.144 to repay the loan, keeping $3.856 as profit.

Forward Price for an Investment Asset with Known Income

If S_0 is the spot price at time 0 of an investment asset, F_0 is the forward price at time 0 of a forward contract that matures at time T, I is the present value of income from the asset between time 0 and T, and the risk-free rate is r, then

$$F_0 = (S_0 - I)e^{rT}$$

Example 2: A bond trades at $900. It will pay coupons of $10 each after two months and after five months. The interest rate is 10% per annum with continuous compounding.

a. What is the forward price of a six-month forward contract?

$$F_0 = (S_0 - I)e^{rT} = \left(900 - 10e^{-0.1\times 2/12} - 10e^{-0.1\times 5/12}\right)e^{0.10\times 0.5} = \$925.72$$

b. If the forward price is $920, what arbitrage strategy exists?

Long forward and short sell spot to get $900. Deposit $10e^{-0.1\times 2/12} + 10e^{-0.1\times 5/12}$ to meet coupon obligations on short position. Deposit the rest at 10% pacc to get back $925.72 after six months. Use $920 to take delivery of the bond. Return bond to close the short position. Keep $5.72 as profit.

c. If the forward price is $930, what arbitrage strategy exists?

Short forward. Borrow $900 at 10% pacc to buy spot. Use coupons to pay part of the loan. After 6 months, deliver bond to get $930. Use $925.72 to repay the remaining loan, keeping $4.28 as profit.

Forward Price for an Investment Asset with Known Income Yield

If S_0 is the spot price at time 0 of an investment asset, F_0 is the forward price at time 0 of a forward contract that matures at time T, q is the continuously compounded income yield from the asset between time 0 and T, and the risk-free rate is r, then

$$F_0 = S_0 e^{(r-q)T}$$

Example 3: A asset trades at $900. Over the next six months, it will earn income at a continuously compounded rate of 2%. The interest rate is 10% per annum with continuous compounding.

a. What is the forward price of a six-month forward contract?

$$F_0 = S_0 e^{(r-q)T} = 900e^{(0.10-0.02)\times 0.5} = \$936.73$$

b. If the forward price is $930, what arbitrage strategy exists?

Long forward and short sell spot. Deposit resulting $900. Earn 10% on deposit and pay 2% for income on short position for net return of 8%. Get back $936.73 after six months. Use $930 to take delivery of the bond. Return bond to close the short position. Keep $6.73 as profit.

c. If the forward price is $940, what arbitrage strategy exists?

Short forward. Borrow $900 at 10% pacc to buy spot. Earn 2% as income and pay 10% as interest for net interest expense of 8%. After 6 months, deliver bond to get $940. Use $936.73 to repay the remaining loan, keeping $3.27 as profit.

Valuation of Forward Contracts

When an investor enters into a forward contract, the forward price is determined so that the value of the contract is zero to both parties.

That is why, there is no cash flow exchange when a forward contract is initiated.

At a later date, the contract may have a positive or a negative value.

The contract can be valued by comparing it to new forward contracts with zero value but with same maturity date.

If f is the value of a forward contract with locked-in delivery price of K, F_0 is the forward price at time 0 of a forward contract that matures at time T, and r is the risk-free rate, then,

$$f = (F_0 - K)e^{-rT}$$

Note that the valuation formula doesn't directly depend on the spot price or other details of the asset.

All that matters is the difference in the locked-in price and the "fair" forward price that will lead to zero value.

Example 4: A asset trades at $900. Over the next six months, it will earn income at a continuously compounded rate of 2%. The interest rate is 10% per annum with continuous compounding. What is the value of a six-month forward contract with a delivery price of $930?

$$F_0 = S_0 e^{(r-q)T} = 900e^{(0.10-0.02)\times 0.5} = \$936.73$$

$$f = (F_0 - K)e^{-rT} = (936.73 - 930)e^{-0.10\times 0.5} = \$6.40$$

Forward Versus Futures

The equation for valuation of forward contracts shows that when forward price changes by $1, it changes the value of a forward contract by its present value.

This is because all cash flows from a forward contract are realized at maturity.

However, a change in futures price results in immediate cash flow due to daily settlement. Thus, gains and losses are realized earlier in a futures contract than in a forward contract.

Thus, the difference between gain/loss and the present value of gain/loss can cause values of forward and futures to differ even if they start with the same delivery price.

Gains and losses are expected to be zero so this difference can usually be ignored.

The difference can be significant when the futures price is correlated with risk-free rates so that gains and losses are discounted at different rates. This can happen, for example, when the underlying asset is the stock of a financial services firm.

Forward and Futures prices may also differ due to differences in credit risk. We ignore credit risk for most of this class.

Stock Index Futures

A stock index can be considered an investment asset with income from dividend.

If S_0 is the index value at time 0, F_0 is the forward price at time 0 of a forward contract that matures at time T, q is the continuously compounded dividend yield on the index between time 0 and T, and the risk-free rate is r, then

$$F_0 = S_0 e^{(r-q)T}$$

Example 5: Compare S&P 500 index value with futures prices on S&P 500 index.

Consider the following quotes on Oct 11, 2022 from https://www.cmegroup.com/markets/equities/sp/e-mini-sandp500.quotes.html:

E-MINI S&P 500 FUTURES - QUOTES

VENUE: GLOBEX

AUTO-REFRESH IS OFF   *Last Updated 22 Feb 2022 07:01:06 PM CT. Market data is delayed by at least 10 minutes.*

MONTH	OPTIONS	CHART	LAST	CHANGE	PRIOR SETTLE	OPEN	HIGH	LOW	VOLUME	UPDATED
MAR 2022 ESH2	OPT		4322.75	+22.75 (+0.53%)	4300.00	4310.75	4327.50	4309.00	17,187	18:51:05 CT 22 Feb 2022
JUN 2022 ESM2	OPT		4316.00	+23.00 (+0.54%)	4293.00	4293.75	4320.00	4293.75	59	18:50:08 CT 22 Feb 2022
SEP 2022 ESU2	OPT		4309.75	+16.25 (+0.38%)	4293.50	4310.50	4310.50	4309.75	3	18:32:13 CT 22 Feb 2022

Let S_0 be the value of S&P 500 index at the time these quotes are observed. Let t be the time to December 2022 futures contract maturity. Then, the prices for the three contracts are given by:

$$3605.00 = S_0 e^{(r-q)t}$$

$$3631.75 = S_0 e^{(r-q)(t+0.25)}$$

$$3650.00 = S_0 e^{(r-q)(t+0.5)}$$

If I know S_0, we can estimate *r-q* using any price. If we do not know S_0, we can take the ratio of two futures price to eliminate S_0. Taking the ratio of the prices for the March 2023 and the December 2022 contracts, we get

$$\frac{3631.75}{3605.00} = e^{0.25(r-q)}$$

which gives

$$r - q = 4ln\left(\frac{3631.75}{3605.00}\right) = 0.0296 = 2.96\%.$$

Taking the ratio of prices of June 2023 and March 2023 contracts, we get

$$r - q = 4ln\left(\frac{3650.00}{3631.75}\right) = 0.0201 = 2.01\%.$$

Thus, the first ratio of prices suggests that the the risk-free rate exceeds the expected dividend yield on S&P 500 by about 2.96%. The second ratio suggests that this difference is about 2.01%.

Two websites (https://ycharts.com/indicators/sp_500_dividend_yield and https://www.multpl.com/s-p-500-dividend-yield) report S&P 500 dividend yield to be about 1.8%. The 3-month LIBOR is about 3.9% and 6-month LIBOR is 4.4% (https://www.theice.com/marketdata/reports/170) (overnight SOFR is about 3% but 3-month and 6-month rates slightly higher). Thus, the risk-free rate is about 2.1% to 2.6% more than the dividend yield on S&P 500.

Why are our estimates of *r-q* slightly different?

The quotes are at different points of time

Quotes with lower volume are less reliable

Interest rates may change over time

Historical dividend yield differ from market's forward-looking expectation of dividend yield

Currency Forwards and Futures

A currency can be considered an investment asset with income from risk-free interest on that currency.

If S_0 is the price of a currency in dollars at time 0, F_0 is the forward price at time 0 of a forward contract on currency that matures at time T, r_f is the continuously compounded risk-free interest rate on the currency between time 0 and T, and r is the risk-free rate in dollars, then

$$F_0 = S_0 e^{(r-r_f)T}$$

Example 6: Compare risk-free rates in US, Japan, spot and futures exchange rates between US dollar and Japanese yen.

Here are the futures quotes in October 2022 from https://www.cmegroup.com/markets/fx/g10/japanese-yen.quotes.html:

JAPANESE YEN FUTURES - QUOTES

VENUE: GLOBEX

AUTO-REFRESH IS OFF *Last Updated 14 Oct 2022 06:44:31 PM CT. Market data is delayed by at least 10 minutes.*

MONTH	OPTIONS	CHART	LAST	CHANGE	PRIOR SETTLE	OPEN	HIGH	LOW	VOLUME	UPDATED
OCT 2022 6JV2	OPT		0.006729	-0.000068 (-1.00%)	0.006797	0.0067925	0.0067925	0.006721	138	16:37:38 CT 14 Oct 2022
NOV 2022 6JX2	OPT		0.0067375	-0.000079 (-1.16%)	0.0068165	0.006812	0.006812	0.0067375	90	16:37:54 CT 14 Oct 2022
DEC 2022 6JZ2	OPT		0.0067705	-0.000074 (-1.08%)	0.0068445	0.0068415	0.0068465	0.006765	174,310	16:38:25 CT 14 Oct 2022
JAN 2023 6JF3	OPT		-	-	0.0068745	-	-	-	0	16:37:55 CT 14 Oct 2022
FEB 2023 6JG3	OPT		-	-	0.000000	-	-	-	0	18:16:00 CT 13 Oct 2022
MAR 2023 6JH3	OPT		0.006857	-0.000076 (-1.10%)	0.006933	0.006927	0.006927	0.006857	58	16:37:28 CT 14 Oct 2022

The current exchange rate from https://www.dailyfx.com/usd-jpy is about JPY148.72/USD.

Let us use LIBOR rates as these are more readily available. From global-rates.com, USD 3-month rate is about 4.08% and USD 6-month rate is 4.54%. The JPY 3-month rate is -0.025% and JPY 6-month rate is 0.047%. Thus, r-r_f is approximately 4.10% for 3-months suggesting that Futures price of Yen should rise.

The predicted JPY Futures price in Dec 2022 is then

$$F_0 = \frac{1}{148.07} e^{(0.0408+0.00025)*(\frac{2}{12})} = 0.0068USD/JPY$$

The actual price is 0.0067705, which is about half a percent off. Using quotes data from two different sources can be problematic as the quotes may be at different times. Let us now compare two futures quotes 3-month apart. Suppose the time to maturity of Dec 2022 contract is t. Then, the ratio of Mar 2023 future price to Dec 2022 futures price is

$$\frac{S_0 e^{(r-r_f)(t+0.25)}}{S_0 e^{(r-r_f)t}} = e^{0.25(r-r_f)} = e^{0.25(0.0408+0.00025)} = 1.010315$$

The actual ratio is 0.006857/0.0067705 = 1.012776, different from what we predicted. How do we interpret the difference? The higher rate of increase of Yen futures price suggests that the difference between US rates and Japan rates may be higher than what we assumed. We already saw that the difference between 6-month US LIBOR and 6-month JPY LIBOR is 4.54% - 0.047% = 4.493%, higher than what we had assumed. Another possibility is that the quotes we are using are inconsistent. Note that the volume for Mar 2023 contract is low.

Forward/Futures Price for Consumption Assets

If

- S_0 is the spot price at time 0,
- F_0 is the forward price at time 0 of a forward contract that matures at time T,
- I is the present value of income from the asset between time 0 and T,
- U is the present value of storage costs of the asset between time 0 and T,
- y is the convenience yield from the asset between time 0 and T, and
- the risk-free rate is r, then

$$F_0 = (S_0 + U - I)e^{(r-y)T}$$

In practice, income and storage costs are easier to estimate than convenience yield.

The value of the convenience yield is inferred from prices using the above equation. The convenience yield is zero for an investment asset.

Income and storage costs are expressed as yields rather than present values. If

- S_0 is the spot price at time 0,
- F_0 is the forward price at time 0 of a forward contract that matures at time T,
- q is the income yield from the asset between time 0 and T,
- u is the storage cost rate on the asset between time 0 and T,
- y is the convenience yield from the asset between time 0 and T,
- and the risk-free rate is r, then

$$F_0 = S_0 e^{(r+u-q-y)T}$$

The **cost of carry** is the cost of holding an asset that consists of all costs net of all benefits (other than convenience yield). Thus, cost of carry $c = r + u - q$.

The futures price is

$$F_0 = S_0 e^{(c-y)T}$$

Delivery Options

Our formulas for forward and futures prices assume that the spot price and the time to maturity are known precisely.

If there is any ambiguity in the grade or quality of the asset or the time of delivery, the short party has the discretion.

The short party will choose the asset and the quality to deliver the asset at the lowest cost.

Futures prices are calculated based on this assumption.

Futures Price Versus Expected Spot Price

Suppose the one-year futures price of an investment asset is $100.

Investors must think that the spot price of the asset after one year can be less than $100, exactly $100, or more than $100 so a futures position can result in a gain or a loss relative to buying or selling the asset in the spot market.

But what about average profit or loss? How does the expected spot price after one year compare with the futures price of $100? The answer depends on the risk of the asset.

If the asset has positive systematic risk (that is the return on the asset is usually higher when the return on the market is higher), the asset earns higher return than the futures investor and the expected spot price after one year will be more than $100.

This situation is called **normal backwardation**.

If the asset has negative systematic risk (that is the return on the asset is usually lower when the return on the market is higher), the asset earns lower return than the futures investor and the expected spot price after one year will be less than $100. This situation is called **contango**.