Dimensional Analysis 101

What are units and dimensions?

We all know that "60 miles-per-hour" is different from "60 gallons" and even from "60 cubic-feet"... But why?

The "60" in those expressions is simply a "numerical coefficient", much like the "60" in the algebraic expression "60x" or "60y". You can think of the numerical coefficient as simply (and implicitly) multiplying the thing it is followed by -- either "x" or "y" or even "miles-per-hour" or "gallons" or "cubic-feet".

The "miles-per-hour" or "gallon" or "cubic-feet", on the other hand, are the "units" associated with the numerical coefficient in those expressions. The units describe both the “scale” of the value as well as the “dimensions” of the value, giving it actual meaning in the real-world. Dimensions are things like “mass”, or “length”, or “time”, or even “area” or “volume” or “speed” or “acceleration”. Units are similar to dimensions, but more specific, like “kilogram”, or “foot”, or “second”, or even “acre” or “miles-per-hour” or “meters-per-second-squared”.

"A number is meaningless without units!" -- a mechanical engineering professor at University of Virginia

Intuitively, we can deduce that "miles-per-hour" is a unit of the dimension of “speed”, and we remember that speed is calculated as a distance divided by a time, and if we happened to travel a mile in an hour, we would not be surprised to find we were traveling at an average speed of "1 mile-per-hour"... Maybe we can even deduce that "60 miles-per-hour" is a speed that is 60 times faster than "1 mile-per-hour", so if we were traveling at an average speed of "60 miles-per-hour", we would not be surprised to find that we would travel 60 miles in an hour, or even 1 mile in a minute (since a minute happens to be 1/60th of an hour)...

Intuitively, we can deduce that a "gallon" is a unit of the dimension of “volume”, and we might remember that the volume of a rectangular prism is actually calculated as length times width times height, in 3-dimensions. But we probably only know gallons to be defined in terms of other units of volume -- like quarts, pints, cups, ounces, or even teaspoons -- "gallons" don't intuitively convert to something we can think of as length times width times height, but yet they are!

We can probably deduce that a "cubic-foot" is also a unit of the dimension of “volume” (like a gallon, but quite a bit larger) and it is equal to the volume of a cube whose sides are each a foot in length! And as such it is also equal to a length of 1 foot multiplied by a width of 1 foot multiplied by a height of 1 foot!

So, a "dimension" is represented by a set of units (like "miles-per-hour" or a "gallon" or a "cubic-foot", possibly including both a numerator and a denominator) that can be associated with a numerical coefficient to give an expression real-world meaning...

Why do we call them dimensions?

We have all heard the term "3-D" or "3-Dimensional" space -- it's the space we live in! In geometry, we often think of our 3 dimensions as the X, Y, and Z axes. If you remember, these axes are orthogonal (i.e., perpendicular to each other) and form a coordinate system. This means that the position on one axis is independent of the positions of the other two axes, and therefore to define a point in 3-Dimensional space requires knowing the X-coordinate value, the Y-coordinate value, and the Z-coordinate value.

So different "dimensions" are inherently orthogonal to each other, and can be measured or manipulated independently of each other!

You can think of two different dimensions (like "miles-per-hour" and "gallons") as "apples" and "oranges" -- you can't add them together because it has no meaning, just like you can't add the X value of a coordinate to the Y value of a coordinate -- because they are on different axes, moving in different directions!

How do units relate to dimensions?

If dimensions are the orthogonal axes in our real-world coordinate system, then units are like different points on those axes. For example, “time” is a dimension in our real-world coordinate. And “minutes” and “days” are different units of “time”. minutes are smaller than days, but they are both moving in the same direction on the same axes of the real-world coordinate system.

But if you want to add "1 minute” to "1 day”, you don't get "2" of anything, because minutes and days are different units (i.e., maybe “crabapples” and “apples”)... Instead, you'd first have to convert a day into minutes -- which is actually 1440 minutes -- and then you can add 1 more minute to that (i.e., “crabapples” and “crabapples”), and the result is 1441 minutes!

What are examples of dimensions or units I already know?

Let's look at other kinds of (base) dimensions or units you may already know:

base dimension:	*US Customary units:*	*System International (SI) units:*
mass	slug, ounce (ozm), pound (lbm)	gram (g), kilogram (kg), metric tons (Mg)
length	mil, inch (in) , foot (ft), yard, mile	millimeter (mm), meter (m), kilometer (km)
time	seconds (s), minutes (min), hours (hr), days	seconds (s), minutes (min), hours (hr), days
temperature	degrees F, R	degrees C, K
current	ampere (A)	ampere (A)
substance	moles (mol)	moles (mol)

Table 1

Base dimensions (above) are defined in terms of themselves -- they cannot be derived from other dimensions (much like the X, Y, and Z axes -- Z introduces a fundamentally new concept that cannot be derived from X and Y!

Derived dimensions or units can be derived (i.e., multiplied and divided, with "per" or "/" representing division) from base dimensions or units in common ways, such as:

dimension:	*derived as:*	*US Customary units:*	*System International (SI) units:*
area	length^2	square-inch, square-foot, acre	mm^2, m^2, km^2
volume	length^3	cup, gallon, cubic-foot (ft^3)	liter, mm^3, m^3, km^3
density	mass/volume	pound-per-cubic-foot (lbm/ft^3)	kg/liter, kg/m^3
speed	length/time	miles/hr (mph), feet/second (fps)	m/s, km/hr (kph)
acceleration	length/time^2	feet-per-second-squared (ft/s^2)	meters-per-second-squared (m/s^2)
force	mass*length/time^2	ounce (oz, ozf), pound (lb, lbf)	newton (N, kg*m/s^2)
energy	force*length	calorie, Btu, quad	joule (J), kilojoule (kJ), megajoule (MJ)
torque	force*length	foot-pound (ftlb)	newton-meter (Nm)
power	energy/time	horsepower (hp)	watt (W)
charge	current*time	coulomb (C)	coulomb (C)
voltage	energy/charge	volt (V)	volt (V)
capacitance	charge^2/energy	farad (F)	farad (F)
inductance	energy/current^2	henry (H)	henry (H)
pressure	force/area	pounds-per-square-inch (psi)	pascal (Pa), bar

Table 2

What are some other units? Basically, anything you can measure or calculate! Like dollars and cents (literally!) or bits and bytes (in computers) or even cans of beans or corn (in your pantry)! Units of light are their own animal -- you might hear of candela or lumens or even photons...

Ounces are the most confusing unit because they might be volume, mass, or force, depending on the context; pounds are nearly as confusing as they might be mass or force (technically, the US Customary unit for mass is the “slug”, but you have likely only seen those in the back yard under a leaf!).

What about powers and roots?

If volume is equal to length times width times height, where all three are dimensions of length where the three lengths axes are orthogonal to each other like the X, Y, and Z axes, then we can say *volume equals length cubed, or cubic length, or length^3!*

Likewise, we can say *area equals length squared, or square length, or length^2!*

Much more curious, though, is we can say that the square root of area is length! And the cube root of volume is length!

In fact, if we take the cube root of a gallon, we get about 6 inches; whereas, if we take the cube root of a cubic foot, we get of course 12 inches (since the “cube root” and “cubic” cancel each other out, and a foot is 12 inches!). This means a cubic foot is roughly 8 gallons in volume (since a cubic foot has twice the linear dimension of a gallon, and 2*2*2 = 8).

We also know speed is measured as distance divided by time... And we know acceleration is measured as speed divided by time. Consequently, acceleration is measured as distance divided by time squared! What does that even mean?

It can all get pretty abstract -- who would think that power is measured as “mass times distance squared divided by time cubed”?!? But so long as we refrain from trying to add (or subtract) “apples” and “oranges”, and we always remember to cancel units out between the numerator and denominator of the equation, the math “just works”, even for US Customary units (which are traditionally much harder to deal with than System International (SI) units).

Unit cancelations

Suppose I ask you to convert “60 miles-per-hour” into “feet-per-second” -- what would you do? If you remember that there are 5280 feet in a mile, and 60 seconds in a minute, and 60 minutes in an hour, the math is actually trivial:

$$\frac{\cancel{60}\ \cancel{miles}}{\cancel{hour}} \cdot \left(\frac{5280\ \boldsymbol{feet}}{\cancel{mile}}\right) \cdot \left(\frac{\cancel{hour}}{\cancel{60}\ \cancel{minutes}}\right) \cdot \left(\frac{\cancel{minute}}{60\ \boldsymbol{seconds}}\right) = \frac{5280\ \boldsymbol{feet}}{60\ \boldsymbol{seconds}} = 88\,\frac{\boldsymbol{feet}}{\boldsymbol{second}}$$

Let’s look at this closely... The first term, 60 miles/hour, is what we’re given. The last term, 88 feet/second, is what we are trying to compute.

In-between these we’re simply doing multiplications by “1” -- i.e., multiplying by “unity terms”, which is always legal -- and doing some simplification and cancelation! We know that 5280 feet equals a mile, so if we divide 5280 feet by a mile, the answer must be “1” (since the numerator and denominator are equal!). Likewise, we know that an hour equals 60 minutes and a minute equals 60 seconds! So all these are unity terms equal to 1, so multiplying by them has no effect on the resulting answer:

$$\left(\frac{5280\ feet}{mile}\right) = \left(\frac{hour}{60\ minutes}\right) = \left(\frac{minute}{60\ seconds}\right) = 1$$

So in the conversion above we simply placed the numerator and denominator of these “unity” terms in such a way as to cancel out the units we don’t want (i.e., miles/hour) and leave us with the units we do want (i.e., feet/second)! It “just works!”

Unit calculations (basic)

Suppose you can’t remember that speed times time equals distance? And what if I ask you to find how long in minutes it will take a car averaging 60 miles-per-hour to go 3 miles? We know we want to end up with minutes (so we need time in the numerator), and we know we’re given two “inputs” to the equation: 60 miles/hour and 3 miles... And we know there are 60 minutes in an hour. All we have to do is combine the terms in the numerator and denominator so that they cancel out, leaving time in the numerator, and we find:

$$\left(\frac{\cancel{hour}}{\cancel{60}\ \cancel{miles}}\right) \cdot (3\ \cancel{miles}) \cdot \left(\frac{\cancel{60}\ \boldsymbol{minute}}{\cancel{hour}}\right) = 3\ \boldsymbol{minutes}$$

Notice that we were given 60 miles/hour as an input, but for the calculation, we used hour/60 miles -- remember, if the numerator and denominator are equal, then when you divide, you get "1" -- and it doesn't matter if you flip the numerator and denominator, since the reciprocal of 1 is also 1!!!

The moral of this story is you can derive almost every physical equation of physics just by canceling out units! You rarely have to memorize an equation -- at most you have to memorize dimensionless factors like "2" or "1/2" or "1/3" or "pi"!

Unit calculations (advanced)

Let's get a little crazy here -- this is an example we first used in 1988! If you study fluid mechanics, you will learn that the drag force on a moving object (like your car) moving thru air can be expressed as:

$$F = \frac{1}{2} \cdot Cd \cdot \rho \cdot v^2 \cdot A$$

In this equation:

F = drag force (let's say we want to compute this in pounds-force)
Cd = dimensionless "coefficient of drag" (we'll assume 0.32 for a sports car)
ρ = density of air (we'll assume 1.293 kg/m^3)
v = speed of object (we'll assume 55 mph)
A = frontal cross section area of object (we'll assume 25 square-feet for a small car)

First let's look at the dimensions of this equation... On the left-hand-side we have "force", which we know from Tables 1 and 2 above, is "mass*length/time^2". On the right-hand-side we have two "dimensionless" numbers (1/2 and *Cd*) which don't contribute any dimensions, followed by "density times speed squared times area"... And we know that "density" is "mass/volume" which is also equal to "mass/length^3" and speed is "length/time" (notice we actually have speed *squared* in the equation above!) and area is "length^2".

That means the left- and right-hand-sides of the equation are of dimensions:

$$\frac{mass \cdot length}{time^2} = \frac{mass}{length^3} \cdot \left(\frac{length}{time}\right)^2 \cdot length^2$$

Simplifying this, we notice on the right-hand-side we have length^4/length^3, leaving us with length, and the left- and right-hand-sides of the equation are "dimensionally consistent", simplifying to:

$$\frac{mass \cdot length}{time^2} = \frac{mass \cdot length}{time^2}$$

This means the math will "just work" -- so long as we remember the dimensionless factors of 1/2 and *Cd* -- we don't have to remember the rest of the equation (except possibly that the speed term is squared), because we can derive it from the input parameters using "dimensional analysis" -- the math only works one way for the units to properly cancel out!

So let's keep going -- this will be hard once, and then I'll show you the easy way, I promise...

We're ready to break this into steps -- we'll do this in SI (*System International*) as the math is easier by hand...

First, let's convert mph (miles-per-hour) to meters-per-second (SI) using the unit cancelation method above:

$$55\,\frac{\cancel{miles}}{\cancel{hour}} \cdot \frac{1\,\cancel{hour}}{3600\,\boldsymbol{seconds}} \cdot \frac{5280\,\cancel{feet}}{1\,\cancel{mile}} \cdot \frac{12\,\cancel{inches}}{1\,\cancel{foot}} \cdot \frac{2.54\,\cancel{centimeter}}{\cancel{inch}} \cdot \frac{1\,\boldsymbol{meter}}{100\,\cancel{centimeter}} = 24.59\,\frac{\boldsymbol{meter}}{\boldsymbol{second}}$$

Next, let's convert the frontal cross section area from square feet to square meters (also SI):

$$25\,\cancel{feet^2} \cdot \left(\frac{12\,\cancel{inches}}{1\,\cancel{foot}}\right)^2 \cdot \left(\frac{2.54\,\cancel{centimeter}}{\cancel{inch}} \cdot \frac{1\,\boldsymbol{meter}}{100\,\cancel{centimeter}}\right)^2 = 2.323\,meter^2$$

Notice all we needed to know was 12 inches was equal to a foot (i.e., a “unity” term), and we could then square that unity term (since 1^2 still equals 1) to get the units to properly cancel out! Same for converting inches^2 to meters^2! I encourage you to run thru the math above in a calculator -- be sure you don’t forget the squared terms! Also notice the input term was 25 foot^2, not (25 foot)^2 (which would be more like a house than a car!)!

Then, let’s plug everything (including our dimensionless factors and the density of air we were given) into SI and get the result in newtons (which are the SI unit of force, and just a fancy name for kilogram-meters-per-second-squared):

$$\frac{1}{2} \cdot 0.32 \cdot 1.293 \frac{\boldsymbol{kg}}{\cancel{meter}^3} \cdot \left(24.59 \frac{\cancel{meter}}{\boldsymbol{second}}\right)^2 \cdot 2.323\, \boldsymbol{meter}^{\cancel{2}} = 290.6 \frac{\boldsymbol{kg \cdot meter}}{\boldsymbol{second}^2} = 290.6\, \boldsymbol{newtons}$$

Finally, let’s convert that back to pounds-force, like we were asked to calculate, using one conversion factor we have to look up (1 newton = 0.2248 pounds-force):

$$290.6\, \cancel{newtons} \cdot \frac{0.2248\, \boldsymbol{lbf}}{1\, \cancel{newton}} = 65.33\, \boldsymbol{lbf}$$

That was a lot of work, but it was not especially complicated -- we’re just canceling out units, multiplying by unity terms, with an occasional power, and we end up with the answer!

Finally, if we want to convert that force we just calculated to horsepower, we remember that:

$$P = F \cdot v$$

In this equation:

P = power (let’s say we want to compute this in horsepower)
F = force active on object (we’ll assume this is 65.33 lbf from the calculation above)
v = speed of object (we’ll again assume 55 mph)

We should be pros at this by now! We start with what we’re given, and we just make the units cancel until we end up with what we want! We just need to look up one more conversion factor (1 horsepower = 550 foot-pound-force/second):

$$65.33\, \cancel{lbf} \cdot \frac{55\, \cancel{miles}}{\cancel{hour}} \cdot \frac{1\, \cancel{hour}}{3600\, \cancel{seconds}} \cdot \frac{5280\, \cancel{feet}}{\cancel{mile}} \cdot \left(\frac{1\, \boldsymbol{horsepower}}{\left(\frac{550\, \cancel{foot}\, \cancel{lbf}}{\cancel{second}}\right)}\right) = 9.58\, \boldsymbol{horsepower}$$

Notice the last unity term -- it is a bit strange because we have a compound denominator, but it’s just a fraction -- it “just works” just like everything else in dimensional analysis -- since the numerator (1 horsepower) is equal to the denominator (550 foot-lbf-per-second), the result of the division is “1”!

Mechanical vs. electrical energy and work

So far, we’ve been talking about energy and power and force in a basically mechanical sense. We know that we can get newtons from kilograms and meters and seconds (we did it above).

And we might remember we can multiply force times length (like in meters) to get work or energy, and we can divide work or energy by time (like in seconds) to get power!

In fact, we just did that above as well -- we multiplied force times speed, right? And speed is length divided by time! So we multiplied force times length divided by time to get power!

But maybe you’re thinking about power in an electrical sense? Maybe you’ve heard voltage times current is power. How can that possibly be related to force times length divided by time???

Let me explain... Up to now, we've been talking about a real-world coordinate system with just 3 "base axes" -- mass, length, and time! We all intuitively understand them, even when we start squaring or cubing them.

But to understand electricity, we have to introduce one more "base axis" -- current. Current really represents the number of electrons passing a point in a circuit in a unit of time. Specifically, an ampere (A) means there are 6,241,509,074,000,000,000 electrons passing every second!

But what about voltage you might say? Why isn't that another "base axis"? Well, the reason is because we use mass, length, and time to define voltage (V) in terms of current! This allows electrical power and mechanical power to be mapped to one another, and measured with the exact same units!

The beautiful thing about electrical units is they are the same in SI and USCS unit systems! Hallelujah!

This gets really crazy now, but like I promised before and I have not forgotten, I'll show you the easy way very soon!

Let's start with the mechanical progression, starting with force, and then break into common and crazy electrical units:

dimension:	*units:*	*defined as:*	*base dimensions:*
force	newtons (N)	kg*m/s^2	mass*length*time^-2
energy	joules (J)	N*m	mass*length^2*time^-2
power	watts (W)	J/s	mass*length^2*time^-3
charge	coulomb (C)	ampere*s	time*current
voltage	volt (V)	J/coulomb	mass*length^2*time^-3*current^-1
capacitance	farad (F)	coulomb^2/J	mass^-1*length^-2*time^4*current^2
inductance	henry (H)	J/ampere^2	mass*length^2*time^-2*current^-2
magnetic flux	weber (Wb)	J/ampere	mass*length^2*time^-2*current^-1
magnetic flux density	tesla (T)	weber/m^2	mass*time^-2*current^-1

Table 3

If your head is not ready to explode something may be wrong! Managing all these units is nearly impossible, especially across engineering disciplines! And it's 10 times harder when you mix USCS and SI units for all of the common units (thankfully, electrical units are the same across the two measurement systems).

Calchemy™-- a units calculator!

This is the reason we wrote Calchemy™ -- a units calculator which allows you to perform dimensional arithmetic, analysis, and conversions using any standard units. Calchemy "understands" the relationships between these units and performs conversions for you automatically as needed. It also checks your equations to ensure that they are always dimensionally correct!

Calchemy literally does all the stuff we've been doing up to now -- and it does it seamlessly for you, right in the middle of your calculations! Calchemy is available from your web browser here:

`https://rtestardi.github.io/calchemy/calchemy.html`

For example, to calculate the horsepower from our advanced calculation above, you would simply enter the following expression, with the "?" operator indicating the units we would like the result expressed in (also notice we use speed^3 instead of speed^2, so we can go directly to horsepower, without an extra multiplication by speed like we did above):

0.5 * 0.32 * 1.293 kg/m^3 * (55 mph)^3 * 25 square feet ? hp

And Calchemy would respond with both its interpretation of the question, as well as the answer:

*> (0.5 * 0.32 * 1.293 kilo~gramm / (meter)^3) * (55 mph)^3 * 25 (foot)^2 ? horsepower*
= 9.5775 hp

Notice that Calchemy showed how it interpreted our expression above its answer, clarifying all abbreviations and numeric operator precedences, just to be safe.

And we can seamlessly interchange with electrical units as well! Once we have that answer, if we want to know how many amps it would take at 400 volts (DC) to create that power (assuming 100% efficiency), we could simply enter:

9.5775 hp; 400 volts ? amps

And Calchemy would respond:

> 9.5775 horsepower / (400 volt) ? ampere
= 17.8548 amps

Notice above we did not even tell Calchemy whether power or volts needed to be in the numerator or denominator -- we just used a ";" operator! The ";" operator tells Calchemy to figure out on its own whether the expressions on the left- and right-hand-sides of the ";" need to be in the numerator or denominator, to achieve the desired result (in this case, "amps") -- the interpretation shows that Calchemy placed the power in the numerator and the voltage in the denominator, since the math only works one way for the units to properly cancel out!

How does Calchemy work?

Glad you asked!

See: How does Calchemy work? · rtestardi/calchemy Wiki (github.com)

Examples

Below are some Calchemy examples -- just click the links below to enter them into Calchemy in your web browser:

0.5 * 0.32 * d_air * (60 mph)^2 * 25 square feet ? lbf # *air drag force on car*
77.7017 lbf * 60 mph ? hp # *power to overcome air drag force at speed*
2000 lbm * grav * 60 mph ? hp # *power to decelerate car at 1G*
1/2 * 2000 lbm * ((60 mph)^2 - (55 mph)^2) / (5 sec) ? hp # *measured power drag on car*
13.9795 hp; 55 mph; hcv_gasoline*20% ? mpg # *mpg estimation at 20% engine efficiency*
35.88 mpg ? km/l # *fuel efficiency*
1 cup * 4 gram/tsp ? lb # *weight of granulated sugar*
1/2 pi sqrt(23 millihenry * 10 microfarad) ? Hz # *frequency of LC oscillator*
60 PiB/month ? Gbps # *fast network bandwidth*
3.4 mega byte / (1200 baud) ? hour # *slow network time*
20000 btu/hour; 80% * 14 megajoules/day/meter^2 ? ft^2 # *solar collector size*
2 packs; 30 rolls/pack; 425 sheets/roll; 40 sheets/poop; 1 poops/person/day; 3 persons ? months
pi * (0.25 inch)^2; 60 ft; 5 gal/min ? sec # time for hot water to flow thru 1/2 inch pipe
40 gal; 120 degF - 45 degF; 1 cal/(cc deltaC); 40000 btu/hr ? min # *water heater recovery time*
10 mph * 30 feet * 10 gal/acre ? gal/min # *fertilizer flow rate on tractor swath*
30 ft * 20 meters ? acres # *lot size*
40 knot / (2500 rpm * 90%) ? inch/rev # *propeller pitch*
(3/4) cup * 1 cal/(cc deltaK) * (212 - 52) deltaF / (2 min + 4 sec) ? watt # *power of microwave oven*
(50 ft * 30 ft * 0.75 inches/week) / (5 gal/min) ? hours/week # *lawn sprinkler time*
40 gal * 1 cal/(cc deltaK) * (140-45) deltaF / (40000 btu/hour) ? hour # *water heater recovery time*
45 hp; 2 pi 8000 rpm ? ft lb # *convert power to torque*
(3/(4 pi) * 1 gal)^(1/3) ? inches # *radius of a gallon sphere*
(1 gallon)^(1/3) ? inches # *width of a gallon cube*

Conclusion

Unit arithmetic is easy if you understand the basics of dimensional analysis! It's so easy, in fact, that Calchemy can do it all for you! Calchemy can run on your computer, tablet, or even on your phone! See:

`https://rtestardi.github.io/calchemy/calchemy.html`

And if you're interested in how Calchemy actually works, see:

`https://github.com/rtestardi/calchemy/wiki/How-does-Calchemy-work%3F`

About the Author

Richard Testardi has a wife and 17yo daughter and lives in Colorado. He is grateful most of the time and Christian. He loves anything outdoors or math/science related. In the future, he hopes to be teaching high school students. He lives without a cell phone (well, except a sim-less phone for interoperability testing!)!