Numerics for Queueing with Dispatcher Paper

Special Case: Exponential - Geometric - Exponential

Suppose the individual service times are exponential with parameter xi, then:

In[1]:=
```
xi[x_, ze_] := Assuming[ze > 0, ze / (ze + x)]
```

Suppose the batch sizes are geometric with parameter p, then:

In[2]:=
```
a[z_, p_] := Assuming[0 <= p && p <= 1, p * z / (1 - (1 - p) * z)]
```

Suppose the interarrival times are Erlang distributed with parameters lambda and r, then:

In[3]:=
```
psi[x_, lambda_] := Assuming[lambda > 0, lambda / (lambda + x)]
```

The joint transform of the arrivals with position-independent marking and the above assumptions is as follows.

In[4]:=
```
gamma[alpha_, theta_, z_, ze_, p_, lambda_] :=
 a[z * xi[alpha, ze], p] * psi[alpha + theta, lambda]
```

The Joint Functional -- Theorem 2.2

The joint functional Phi^d

In[5]:=
```
PhiD[alpha_, beta_, vartheta_, theta_, z_, ze_, p_, lambda_, T_] := InverseLaplaceTransform[
  (1 / (x + beta + theta)) * (1 - gamma[0, x + beta + theta, 1, ze, p, lambda]) /
   (1 - gamma[alpha + beta + x, vartheta + theta, z, ze, p, lambda]), x, T]
```

The joint functional Phi^s

In[6]:=
```
PhiS[alpha_, beta_, vartheta_, theta_, z_, ze_, p_, lambda_, T_] := InverseLaplaceTransform[
  (1 / x) * (gamma[beta, theta + x, z, ze, p, lambda] - gamma[beta + x, theta, z, ze, p, lambda]) /
   (1 - gamma[alpha + beta + x, vartheta + theta, z, ze, p, lambda]), x, T]
```

Let' s find the probability of the server being called back by the dispatcher for r=1.

In[7]:=
```
Simplify[PhiS[0, 0, 0, 0, 1, ze, p, lambda, T]]
```

Out[7]=

$$\frac{\text{lambda} - e^{-T\,(\text{lambda}+\text{p ze})}\,\text{lambda}}{\text{lambda} + \text{p ze}}$$

In[8]:=
```
Simplify[PhiD[0, 0, 0, 0, 1, ze, p, lambda, T]]
```

Out[8]=
$$\frac{e^{-T\,(\text{lambda}+p\,\text{ze})}\ \text{lambda}+p\,\text{ze}}{\text{lambda}+p\,\text{ze}}$$

The joint functional and primary result:

In[9]:=
```
Phi[alpha_, beta_, vartheta_, theta_, z_, ze_, p_, lambda_, T_] :=
 PhiD[alpha, beta, vartheta, theta, z, ze, p, lambda, T] +
  PhiS[alpha, beta, vartheta, theta, z, ze, p, lambda, T]
```

Special Case Results

Let' s find the mean queue length.

In[10]:=
```
Simplify[Assuming[ze > 0 && lambda > 0 && 0 <= p && p <= 1,
  Limit[D[Phi[0, 0, 0, 0, z, ze, p, lambda, T], {z, 1}], z → 1]]]
```

Out[10]=
$$\left(\text{lambda}\left(\text{lambda}-e^{-T\,(\text{lambda}+p\,\text{ze})}\ \text{lambda}+\text{lambda}\ p\ T\ \text{ze}+p^2\ T\ \text{ze}^2\right)\right)\Big/\left(p\left(\text{lambda}+p\,\text{ze}\right)^2\right)$$

In[11]:=
```
Qmean[ze_, p_, lambda_, T_] :=
 (lambda (lambda - e^(-T (lambda+p ze)) lambda + lambda p T ze + p^2 T ze^2)) / (p (lambda + p ze)^2)
```

Next, we will look for the variance, finding first first g''(1) and then Var(Q_nu).

In[12]:=
```
Simplify[Assuming[ze > 0 && lambda > 0 && 0 <= p && p <= 1,
  Limit[D[Phi[0, 0, 0, 0, z, ze, p, lambda, T], {z, 2}], z → 1]]]
```

Out[12]=
$$\frac{1}{p^2\left(\text{lambda}+p\,\text{ze}\right)^4}e^{-T\,(\text{lambda}+p\,\text{ze})}\ \text{lambda}\left(-2\,e^{T\,(\text{lambda}+p\,\text{ze})}\left(-1+p\right)p^4\,T\,\text{ze}^4+\text{lambda}\,p^2\,\text{ze}^2\left(-8+2\,p^2\,T\,\text{ze}+p\left(6-2\,T\,\text{ze}\right)+e^{T\,(\text{lambda}+p\,\text{ze})}\left(8+p^2\,T\,\text{ze}\,(-4+T\,\text{ze})+2\,p\left(-3+T\,\text{ze}\right)\right)\right)+\text{lambda}^3\left(2\left(-1+p+p^2\,T\,\text{ze}\right)+e^{T\,(\text{lambda}+p\,\text{ze})}\left(2+p^2\,T^2\,\text{ze}^2+2\,p\left(-1+T\,\text{ze}\right)\right)\right)+2\,\text{lambda}^2\,p\,\text{ze}\left(\left(-1+2\,p\right)\left(2+p\,T\,\text{ze}\right)+e^{T\,(\text{lambda}+p\,\text{ze})}\left(2+p\,(-4+T\,\text{ze})+p^2\,T\,\text{ze}\left(-1+T\,\text{ze}\right)\right)\right)\right)$$

In[13]:=
```
QsecondDerivative[ze_, p_, lambda_, T_] :=
 1/(p^2 (lambda + p ze)^4) e^(-T (lambda+p ze)) lambda (-2 e^(T (lambda+p ze)) (-1 + p) p^4 T ze^4 + lambda p^2 ze^2
    (-8 + 2 p^2 T ze + p (6 - 2 T ze) + e^(T (lambda+p ze)) (8 + p^2 T ze (-4 + T ze) + 2 p (-3 + T ze))) +
   lambda^3 (2 (-1 + p + p^2 T ze) + e^(T (lambda+p ze)) (2 + p^2 T^2 ze^2 + 2 p (-1 + T ze))) +
   2 lambda^2 p ze ((-1 + 2 p) (2 + p T ze) + e^(T (lambda+p ze)) (2 + p (-4 + T ze) + p^2 T ze (-1 + T ze))))
```

In[14]:=
```
Qvariance[ze_, p_, lambda_, T_] :=
 QsecondDerivative[ze, p, lambda, T] + Qmean[ze, p, lambda, T] - (Qmean[ze, p, lambda, T])^2
```

In[15]:=
```
Simplify[Qvariance[ze, p, lambda, T]]
```

Out[15]=
$$\frac{1}{p^2\left(\text{lambda}+p\,\text{ze}\right)^4}e^{-2\,T\,(\text{lambda}+p\,\text{ze})}\ \text{lambda}\left(-e^{2\,T\,(\text{lambda}+p\,\text{ze})}\left(-2+p\right)p^4\,T\,\text{ze}^4+e^{T\,(\text{lambda}+p\,\text{ze})}\,\text{lambda}^2\,p\,\text{ze}\left(-4+6\,p+4\,p^2\,T\,\text{ze}+e^{T\,(\text{lambda}+p\,\text{ze})}\left(4-6\,p+p^2\,T\,\text{ze}\right)\right)+\text{lambda}^3\left(-1+e^{2\,T\,(\text{lambda}+p\,\text{ze})}\left(1-p+p^2\,T\,\text{ze}\right)+e^{T\,(\text{lambda}+p\,\text{ze})}\,p\left(1+2\left(1+p\right)T\,\text{ze}\right)\right)-e^{T\,(\text{lambda}+p\,\text{ze})}\,\text{lambda}\,p^2\,\text{ze}^2\left(8-2\,p^2\,T\,\text{ze}+p\left(-5+2\,T\,\text{ze}\right)+e^{T\,(\text{lambda}+p\,\text{ze})}\left(-8+p^2\,T\,\text{ze}+p\left(5-2\,T\,\text{ze}\right)\right)\right)\right)$$

Next, we will find the variance of the return time tau_nu.

In[16]:=
```
(-1) * Simplify[Assuming[ze > 0 && lambda > 0 && 0 <= p && p <= 1,
   Limit[D[Phi[0, 0, 0, theta, 1, ze, p, lambda, T], {theta, 1}], theta → 0]]]
```

Out[16]=
$$-\frac{-p^2\,T\,ze^2+lambda\,\left(-1+e^{-T\,(lambda+p\,ze)}-p\,T\,ze\right)}{(lambda+p\,ze)^2}$$

In[17]:=
```
TauMean[ze_, p_, lambda_, T_] :=
 -((-p^2 T ze^2 + lambda (-1 + e^(-T (lambda+p ze)) - p T ze)) / (lambda + p ze)^2)
```

In[18]:=
```
TauMean[ze, p, lambda, T]
```

Out[18]=
$$-\frac{-p^2\,T\,ze^2+lambda\,\left(-1+e^{-T\,(lambda+p\,ze)}-p\,T\,ze\right)}{(lambda+p\,ze)^2}$$

In[19]:=
```
Simplify[Assuming[ze > 0 && lambda > 0 && 0 <= p && p <= 1,
  Limit[D[Phi[0, 0, 0, theta, 1, ze, p, lambda, T], {theta, 2}], theta → 0]]]
```

Out[19]=
$$\frac{1}{(lambda+p\,ze)^4}\left(-2\,e^{-T\,(lambda+p\,ze)}\,lambda^3\,T+p^4\,T^2\,ze^4+2\,lambda\,p\,ze\,\left(-2+2\,e^{-T\,(lambda+p\,ze)}+2\,p\,T\,ze+p^2\,T^2\,ze^2\right)+lambda^2\,\left(2+4\,p\,T\,ze+p^2\,T^2\,ze^2-2\,e^{-T\,(lambda+p\,ze)}\,(1+p\,T\,ze)\right)\right)$$

Next, we will find the variance of the return time tau_nu.

In[20]:=
```
TauSecondMean[ze_, p_, lambda_, T_] := 1/(lambda + p ze)^4
  (-2 e^(-T (lambda+p ze)) lambda^3 T + p^4 T^2 ze^4 + 2 lambda p ze (-2 + 2 e^(-T (lambda+p ze)) + 2 p T ze + p^2 T^2 ze^2) +
    lambda^2 (2 + 4 p T ze + p^2 T^2 ze^2 - 2 e^(-T (lambda+p ze)) (1 + p T ze)))
```

In[21]:=
```
TauSecondMean[ze, p, lambda, T]
```

Out[21]=
$$\frac{1}{(lambda+p\,ze)^4}\left(-2\,e^{-T\,(lambda+p\,ze)}\,lambda^3\,T+p^4\,T^2\,ze^4+2\,lambda\,p\,ze\,\left(-2+2\,e^{-T\,(lambda+p\,ze)}+2\,p\,T\,ze+p^2\,T^2\,ze^2\right)+lambda^2\,\left(2+4\,p\,T\,ze+p^2\,T^2\,ze^2-2\,e^{-T\,(lambda+p\,ze)}\,(1+p\,T\,ze)\right)\right)$$

In[22]:=
```
TauVariance[ze_, p_, lambda_, T_] :=
 TauSecondMean[ze, p, lambda, T] - TauMean[ze, p, lambda, T]^2
```

In[23]:=
```
Simplify[TauVariance[ze, p, lambda, T]]
```

Out[23]=
$$\frac{1}{(lambda+p\,ze)^4}e^{-2\,T\,(lambda+p\,ze)}\,lambda\left(-2\,e^{T\,(lambda+p\,ze)}\,lambda^2\,T+2\,e^{T\,(lambda+p\,ze)}\,p\,ze\,\left(2+p\,T\,ze+e^{T\,(lambda+p\,ze)}\,(-2+p\,T\,ze)\right)+lambda\,\left(-1+e^{2\,T\,(lambda+p\,ze)}\,(1+2\,p\,T\,ze)\right)\right)$$