Stochastic Simulation applied to Elliptic equations

Dr. Ma. Elena Hernández-Hernández

Abstract

The relationship between probability theory and partial differential equations goes back to the pioneer works of Kakutani [5], [6] showing the connection between Brownian motion and harmonic functions. The existence of positive solutions to elliptic equations of the type

$$\left.\begin{aligned} \Delta u + u^p = 0, &\quad \mathit{in}\ \Omega, \\ u > 0, &\quad \text{in } \Omega, \\ u = 0, &\quad \text{in } \partial\Omega. \end{aligned}\right\} \tag{0.1}$$

have been widely studied from an analytical perspective by many authors including S. I. Pohozaev (1965) [8], C. Loewner and L. Nirenberg (1974) [7], Bahri and Coron (1988) [1], and Ding (1989) [2].

More recently, E.B. Dynkin (1991) [3] established connections between the theory of superprocesses and positive solutions of non-linear partial differential equations which include, as a particular case, the equation (0.1). However, one of the drawbacks of these approaches is that they are usually not applicable when the exponent p is greater than the critical Sobolev exponent $p^* := \frac{n+2}{n-2}$. Some preliminary findings in [4] state sufficient conditions for the existence of positive solutions to the equation (0.1) in terms of expected exit times and expected occupations times of the Brownian motion in the domain Ω.

This project requires:

- Reviewing the current state-of-the art in approximation methods for the solution to elliptic equations.
- Studying the existence of positive solutions to (0.1) (including the Sobolev equation) via the implementation of simulation methods.

Prerequisite/necessary skills:

1. Basic knowledge of Differential Equations.
2. Computational Stochastic Processes
3. Programming skills in R.

Stream Suitability: Theory and Methods.

References

[1] A. Bahri and J. Coron. On a nonlinear elliptic equations involving critical Sobolev exponent: the effect of the topology of the domain. *Comm. Pure Appl. Math.*, 41:253–294, 1988.

[2] W. Y. Ding. Positive solutions of $\delta + u^{\frac{n+2}{n-2}} = 0$ on contractible domains. *J. P. D. E.*, 2(4):83–88, 1989.

[3] E. B. Dynkin. A probabilistic approach to one class of nonlinear differential equations. *Probab. Theory Related Fields*, 89:89–115, 1991.

[4] M.E. Hernández-Hernández and P. Padilla-Longoria. Positive solutions to semilinear elliptic equations. a probabilistic approach. *(preprint in preparation)*, 2020.

[5] S. Kakutani. Two-dimensional brownian motion and harmonic functions. *Proc. Imp. Acad. Tokyo*, 20:706–714, 1944.

[6] S. Kakutani. Markoff process and the Dirichlet problem. *Proc. Japan Acad.*, 21:227–233, 1945.

[7] C. Loewner and L. Nirenberg. Partial differential equations invariant under conformal or projective transformations. In *Contributions to Analysis (A Collection of Paper Dedicated to Lipman Bers)*, pages 245–272. Academic Press, New York,, 1974.

[8] S. I. Pohozaev. On the eigenfunctions of the equation $\delta u + \lambda f(u) = 0$. *Dokl. Akad. Nauk SSSR*, 165:1408–1411, 1965.