Stochastic Simulation Applied to Elliptic Equations

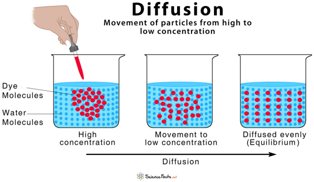
Midterm Presentation

Ruosen Gao

Imperial College London

June 24, 2021

Motivation



Source: https://www.sciencefacts.net/wp-content/uploads/2020/01/Diffusion.jpg

- Diffusion as a physical phenomenon
- Modelled as differential equations
  - Analytic methods, e.g. calculus of variations
  - Intractable solutions
- Modelled as Brownian motions
  - More recent
  - Solutions admit stochastic representations
  - Numerical results via simulations

Objective

1. Equation to be studied: find solutions $u : \overline{D} \to \mathbb{R}$ such that

$$\begin{cases} \Delta u + u^p = 0 & \text{in } D, \\ u > 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1}$$

   where $p > 1$ and D is a bounded regular open set in $\mathbb{R}^n$.

2. Stochastic representation: If u is a solution, then u also solves

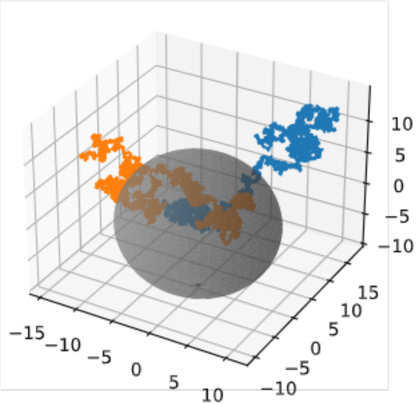
$$u(x) = \mathbb{E}_x \left[\int_0^{\tau_D} h(u(B_t))\, dt \right], \tag{2}$$

   where τ_D is the first exit time of B_t from D, and $h : \mathbb{R} \to \mathbb{R}$ is a generic continuous function.

3. Numerical results via simulations

Progression

1. Literature review
2. Appreciate the virtue of probabilistic approach
3. Bare-bones implementations



Outlook

1. Simulation in parallel
2. Advanced simulation methods
3. Simulation studies, e.g. error control, stability study
4. Compose the thesis

Some Reference

Karatzas, Ioannis., and Steven E. Shreve.
Brownian Motion and Stochastic Calculus . 2nd ed.
Springer-Verlag, 1991.

Mörters, Peter, et al.
Brownian Motion
Cambridge University Press, 2010.

Padilla, Pablo.
The Effect of the Shape of the Domain on the Existence of Solutions of an Equation Involving the Critical Sobolev Exponent.
Journal of Differential Equations, vol. 124, no. 2, Elsevier Inc, 1996, pp. 449–71, doi:10.1006/jdeq.1996.0019.