List of Probabilistic Solutions

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1 Feynman-Kac Formula, KaSh p268

Consider continuous functions $f : \mathbb{R}^d \to \mathbb{R}, k : \mathbb{R}^d \to [0, \infty)$, and $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}$. Suppose that v is a continuous, real-valued function on $[0, T] \times \mathbb{R}^d$, of class $C^{1,2}$ on $[0, T] \times \mathbb{R}^d$, and satisfies

$$
\begin{aligned}
-\frac{\partial v}{\partial t} + kv &= \frac{1}{2}\Delta v + g; \quad \text{on } [0, T) \times \mathbb{R}^d, \\
v(T, x) &= f(x); \quad x \in \mathbb{R}^d.
\end{aligned}
$$

Assume that

$$\max_{0 \le t \le T} |v(t, x)| + \max_{0 \le t \le T} |g(t, x)| \le K e^{a\|x\|^2}; \quad \forall x \in \mathbb{R}^d,$$

for some constants $K > 0$ and $0 < a < 1/(2Td)$. Then v admits the stochastic representation

$$
\begin{aligned}
v(t, x) = \mathbb{E}^x \Bigg[& f(W_{T-t}) \exp\left\{ -\int_0^{T-t} k(W_s)\, ds \right\} \\
& + \int_0^{T-t} g(t + \theta, W_\theta) \exp\left\{ -\int_0^{\theta} k(W_s)\, ds \right\} d\theta \Bigg]; \quad 0 \le t \le T, x \in \mathbb{R}^d.
\end{aligned}
$$

In particular, such a solution is unique.

2 Exercise 2.20, MoPe p63

Let $f : \mathbb{R}^d \to \mathbb{R}$ be C^2, $\{B(t) : t \ge 0\}$ a d-dimensional Brownian motion such that $\mathbb{E}_x \int_0^t e^{-\lambda s} |f(B(s))|\, ds < \infty$ and $\mathbb{E}_x \int_0^t e^{-\lambda s} |\Delta f(B(s))|\, ds < \infty$, for any $x \in \mathbb{R}^d$ and $t > 0$. Suppose U is a bounded open set, $\lambda \ge 0$, and $u : U \to \mathbb{R}$ is a bounded solution of

$$\frac{1}{2}\Delta u(x) = \lambda u(x), \quad \text{for } x \in U,$$

and $\lim_{x \to x_0} u(x) = f(x_0)$ for all $x_0 \in \partial U$. Then

$$u(x) = \mathbb{E}_x[f(B(\tau)) e^{-\lambda \tau}],$$

where $\tau = \inf\{t \ge 0 : B(t) \notin U\}$.

3 Dirichlet Problem, MoPe p69

Suppose $U \subset \mathbb{R}$ is a bounded domain such that every boundary point satisfies the Poincare cone condition, and suppose ϕ is a continuous function on ∂U. Let $\tau(\partial U) = \inf\{t > 0 : B(t) \in \partial U\}$, which is an almost surely finite stopping time. Then the function $u : \overline{U} \to \mathbb{R}$ given by

$$u(x) = \mathbb{E}_x[\phi(B(\tau(\partial U)))], \quad \text{for } x \in \overline{U},$$

is the unique continuous function harmonic on U with $u(x) = \phi(x)$ for all $x \in \partial U$.

Dirichlet Problem: given a bounded domain U and boundary data ϕ, find a continuous function $f : \overline{U} \to \mathbb{R}$ such that f is harmonic on U and $f = \phi$ on ∂U.

4 Theorem 7.43, MoPe p214

Suppose $V : \mathbb{R}^d \to \mathbb{R}$ is bounded. Then $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$u(t, x) = \mathbb{E}_x \left\{ \exp\left(\int_0^t V(B(r))\, dr \right) \right\},$$

solves the heat equation on $\mathbb{R}^d$ with dissipation rate V and initial condition one. i.e.

- $\lim_{\substack{x \to x_0 \\ t \to 0}} u(t, x) = 1$ for $x_0 \in \mathbb{R}^d$,
- $u_t = \frac{1}{2}\Delta_x u + Vu$ on $(0, \infty) \times \mathbb{R}^d$.

5 Theorem 7.44, MoPe p216

If u is a bounded, C^2 solution of the heat equation on the domain U, with zero dissipation rate and continuous initial condition f, then

$$u(t, x) = \mathbb{E}_x[f(B(t))\mathbb{I}_{t<\tau}],$$

where τ is the first exit time from the domain U. i.e. u solves

- $\lim_{\substack{x \to x_0 \\ t \to 0}} u(t, x) = f(x_0)$ if $x_0 \in U$,
- $\lim_{\substack{x \to x_0 \\ t \to 0}} u(t, x) = 0$ if $x_0 \in \partial U$,
- $u_t = \frac{1}{2}\Delta_x u$ on $(0, \infty) \times U$.

6 Theorem 7.46, MoPe p218

Let $d \geq 3$ and $V : \mathbb{R}^d \to [0, \infty)$ be bounded. Define

$$h(x) := \mathbb{E}_x \left[\exp\left(-\int_0^\infty V(B(t))\, dt \right) \right].$$

Then $h : \mathbb{R}^d \to [0, \infty)$ satisfies the equation

$$h(x) = 1 - \int G(x, y) V(y) h(y)\, dy \quad \text{for } x \in \mathbb{R}^d.$$

Where $G : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ is the Green kernel,

$$G(x, y) = \int_0^\infty \mathfrak{p}^*(t, x, y)\, dt,$$

and $\mathfrak{p}^* : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ is a transition density,

$$\mathbb{P}_x(B(t) \in A) = \int_A \mathfrak{p}^*(t, x, y)\, dy \quad \text{for } t > 0 \text{ and } A \subset \mathbb{R}^d \text{ Borel.}$$

Informally, h satisfies $\frac{1}{2}\Delta h = Vh$.