Note on Morters and Peres' Book

Ruosen Gao

1 Chapter 1

- Time inversion.
- Theorem 1.14: almost surely,

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \frac{|B(t+h)-B(t)|}{\sqrt{2h\log(1/h)}} = 1.$$

- Almost surely, Brownian motion is locally α-Holder continuous for $\alpha < 1/2$.
- Almost surely, Brownian motion is nowhere differentiable.

2 Chapter 2

- $\{B(t+s)-B(s) : t \ge 0\}$ is independent of $\mathcal{F}^0(s) := \sigma(\{B(r) : 0 \le r \le s\})$.
- $\{B(t+s)-B(s) : t \ge 0\}$ is independent of $\mathcal{F}^+(s) := \bigcap_{t>s} \mathcal{F}^0(t)$.
- Blumenthal's and Kolmogorov's 0-1 Law
- Strong Markov Property
- a.s., Zero set of a 1-D Brownian motion is closed with no isolated points, and further uncountable.
- Transition kernel vs transition matrix (for Markov chains).
- Intuitively, for a martingale, the current is a best prediction for the further.
- Optional stopping theorem.
- Wald's lemma and Wald's second lemma can be used to obtain the expected exit times for a 1-D Brownian motion.
- Theorem 2.47 on constructing a martingale via Laplacian from a Brownian motion.

3 Chapter 3

- Maximum principle for harmonic functions.
- Theorem 3.8 relates first hitting time with harmonic functions.
- Theorem 3.19: a Brownian motion being point recurrent, neighbourhood recurrent, transient depends on dimension d.
- Theorem 3.25: $d = 1$, occupation measure defined via Lebesgue measure.
- Theorem 3.36: $d \geq 2$, occupation time depends on transience or recurrence.
- Definition 3.27: transient Brownian motion (up to a stopping time T).
- Definition 3.30: Green's function (kernel) is the density of the expected occupation measure for the transient Brownian motion.

4 Chapter 4

- Hausdorff dimension improves over Minkowski dimension.
- Hausdorff dimnesion is defined via the Hausdorff outer measure.
- Theorem 4.29: dimensions of graphs and ranges of Brownian motions.

5 Chapter 5

- Theorem 5.1: asymptotic envelopes for a standard linear Brownian motion.
- Theorem 5.4: asymptotic envelopes for a simple random walk.
- Embedding technique to derive theorem 5.4 from theorem 5.1.
- Theorem 5.14 a.s. Brownian motion has no local points of increase.
- Skorokhod embedding theorem (from BM to RW)
- Donsker invariance principle (from discrete to BM)
- Arcsine laws for Brownian motion and random walk.

6 Chapter 6

- Don't understand.

7 Chapter 7

- Theorem 7.6: stochastic integral.
- Definition 7.8: stochastic integral up to t.
- Ito's formula serves as FTC for stochastic calculus.
- Theorem 7.17 relates local martingale, stopping time, and harmonic function.
- Proposition 7.46: expected occupation time of the unit ball by a 3-D BM.
- Theorem 7.47 (Ciesielski-Taylor identity): relate first exit time by 1-D BM to total occupation time of 3-D BM.

8 Chapter 8

- Theorem 8.3 regular points and Poincare cone condition.
- Theorem 8.5 solution to Dirichlet problem via probabilistic representation.
- Remark 8.7: probabilistic approach to Poisson's problem.
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