Homework Week 14

6. By looking at the objective function we know that

$$\alpha||\boldsymbol{\beta}||_0 + \frac{1}{2}||\boldsymbol{\beta} - \mathbf{x}||_2^2 = \sum_{j=1}^{p} \left\{ \alpha \mathbb{I}\{\beta_j \neq 0\} + \frac{1}{2}(\beta_j - x_j)^2 \right\}, \tag{1}$$

which implies the minimization problem associated with the proximal operator $\mathbf{HT}_\alpha(\mathbf{x})$ is *separable* in terms of $\boldsymbol{\beta}$, and we can solve the p sub-problems separably to obtain a global minimizer of the original problem. Below we focus on the jth sub-problem:

$$\min_{\beta_j} \quad \alpha \mathbb{I}\{\beta_j \neq 0\} + \frac{1}{2}(\beta_j - x_j)^2. \tag{2}$$

Now assume that β_j^* is the solution to (2). We consider the following two cases:

Case 1 ($x_j \neq 0$): In this case if $\beta_j^* = 0$, then the objective function becomes

$$\frac{1}{2}(\beta_j^* - x_j)^2 = \frac{1}{2}x_j^2. \tag{3}$$

On the other hand, if $\beta_j^* \neq 0$, the the objective function becomes

$$\alpha + \frac{1}{2}(\beta_j^* - x_j)^2 = \alpha \Rightarrow \beta_j^* = x_j. \tag{4}$$

since our goal is to to minimize the objective function. The above results (3) and (4) show that when $x_j \neq 0$, we have

$$\begin{aligned} \min_{\beta_j} \quad & \alpha \mathbb{I}\{\beta_j \neq 0\} + \frac{1}{2}(\beta_j - x_j)^2 \\ = \quad & \alpha \mathbb{I}\{\beta_j^* \neq 0\} + \frac{1}{2}(\beta_j^* - x_j)^2 \\ = \quad & \min\{\alpha, (1/2)x_j^2\}, \end{aligned}$$

which further implies that

$$\beta_j^* = \begin{cases} x_j & \text{if } |x_j| \geq \sqrt{2\alpha} \\ 0 & \text{otherwise} \end{cases} .$$

Case 2 ($x_j = 0$): In this case we simply let $\beta_j^* = x_j = 0$, and the objective function becomes

$$\alpha \mathbb{I}\{\beta_j \neq 0\} + \frac{1}{2}(\beta_j^* - x_j)^2 = 0.$$

n	n^{test}	$\tilde{\alpha}^*$	MSE	$\text{Err}^{\text{train}}$	Err^{test}
200	2000	0.0027	0.034	0.453	0.516

Table 1: Simulation results.

From the above two cases we conclude

$$[\text{HT}_\alpha(\mathbf{x})]_j = \beta_j^* = x_j \mathbb{I}\{|x_j| \geq \sqrt{2\alpha}\}.$$

7. Please see Figure 1 and Table 1.

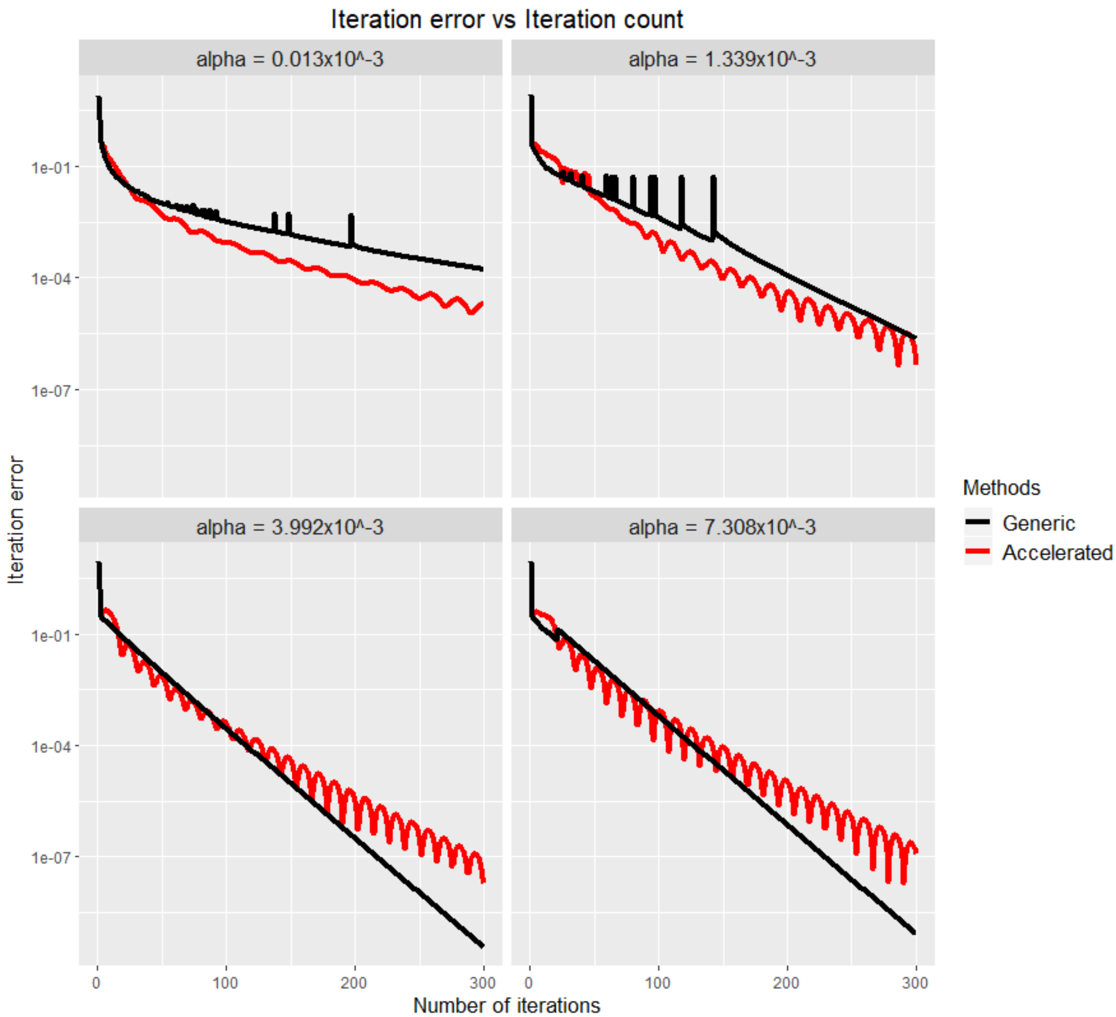

Figure 1: Simulation results.

Quizzes Week 14

pp. 14: By direct calculation, the proximal operator must satisfy the following equation:

$$-\frac{\alpha}{\theta} + (\theta - x) = 0 \Rightarrow \frac{x + \sqrt{x^2 + 4\alpha}}{2}.$$

pp. 30: Please refers to the solution to Question 6 of homework assignment Week 14.