Problem 450 – Hypocycloid and Lattice points

Ryan Dancy

1 July 2019

1 Introduction

The following is an explanation of the solution to Project Euler problem 450 "Hypocycloid and Lattice points" (`https://projecteuler.net/problem=450`), 100% difficulty, contained in `problem450.py`.

There are two classes of lattice points on hypocycloids that we must consider. The first occur when $t = \frac{n\pi}{2}, n \in \mathbb{Z}$; then $\sin t$ and $\cos t$ are equal to ± 1 or 0. We will call these points "normals". The second class of points have, in general, the following structure: $\sin t = \frac{a}{c}$ and $\cos t = \frac{b}{c}$ such that $a^2 + b^2 = c^2$ (i.e. a, b, and c form a Pythagorean triple). We will call this class of points "specials". Finding $T(N)$ is a matter of summing the calculated values for the normals and the specials.

We will now discuss the calculation of $T(N)$ for normals and specials separately.

2 Normals

For the purposes of the normals, it is simpler to transform the parametric equations of the hypocycloid as such:

$$
\begin{aligned}
x(t) &= (R-r)\cos t + r\cos\left(\frac{R}{r}t - t\right) \\
x(t) &= (R-r)\cos t + r\cos\frac{R}{r}t\cos t + r\sin\frac{R}{r}t\sin t \qquad (1)
\end{aligned}
$$

and

$$
y(t) = (R-r)\sin t - r\sin\left(\frac{R}{r}t - t\right)
$$

$$y(t) = (R-r)\sin t - r\sin\frac{R}{r}t\cos t + r\cos\frac{R}{r}t\sin t \tag{2}$$

Now, as previously mentioned, for normals,

$$t = \frac{n\pi}{2}$$

for some $n \in \mathbb{Z}$. As well, for $\sin\frac{R}{r}t$ and $\cos\frac{R}{r}t$ to be rational,

$$\frac{R}{r}t = \frac{Rn\pi}{2r}$$

must be of the form $\frac{m\pi}{2}$ for some $m \in \mathbb{Z}$. This occurs when r divides Rn. In other words, let R_r and r_r be coprime integers such that

$$\frac{R_r}{r_r} = \frac{R}{r},$$

i.e. R_r and r_r are respectively the numerator and denominator of the reduced fraction $\frac{R}{r}$. ($R_r = \frac{R}{\gcd(R,r)}$ and $r_r = \frac{r}{\gcd(R,r)}$.) Then $\frac{R}{r}t$ is of the form $\frac{m\pi}{2}$ when n is a multiple of r_r.

Consider the fact that for any angle θ, $\sin\theta = \sin(\theta+2\pi)$ and $\cos\theta = \cos(\theta+2\pi)$. Let $n = 4kr_r + mr_r$ for some $k \in \mathbb{Z}$ and $n \equiv m \pmod 4$, $0 \leq m \leq 3$. Then

$$\frac{n\pi}{2} = \frac{mr_r\pi}{2} + 2\pi kr_r$$

and

$$\frac{Rn\pi}{2r} = \frac{mR_r\pi}{2} + 2\pi kR_r.$$

Due to the identity previously mentioned, the $2\pi kr_r$ and $2\pi kR_r$ terms are irrelevant for the purpose of determining $x(t)$ and $y(t)$. Therefore, only the cases with $k = 0$, that is $n = 0$, r_r, $2r_r$, $3r_r$, must be considered.

If $n = 0$, then $t = 0$ and $\frac{R}{r}t = 0$. Then, by equations 1 and 2,

$$x(t) = (R-r) + r = R$$

and

$$y(t) = 0.$$

Thus, when $n = 0$,

$$|x| + |y| = R.$$

If $n = r_r$, then $t = \frac{r_r\pi}{2}$ and $\frac{R}{r}t = \frac{R_r\pi}{2}$. For the same reason that we only need to consider n between 0 and $3r_r$, we only need to consider R_r and r_r between 0 and 3 modulo 4. We obtain the following table. Note that since $2r < R$, $|2r - R| = R - 2r$.

$R_r \pmod 4$	$r_r \pmod 4$	x, y	$\lvert x\rvert + \lvert y\rvert$
0	0	R, 0	R
0	1	0, R	R
0	2	$-R$, 0	R
0	3	0, $-R$	R
1	0	$R-r$, $-r$	R
1	1	r, $R-r$	R
1	2	$r-R$, r	R
1	3	$-r$, $r-R$	R
2	0	$R-2r$, 0	$R-2r$
2	1	0, $R-2r$	$R-2r$
2	2	$2r-R$, 0	$R-2r$
2	3	0, $2r-R$	$R-2r$
3	0	$R-r$, $-r$	R
3	1	r, $R-r$	R
3	2	$r-R$, r	R
3	3	$-r$, $r-R$	R

If $n = 2r_r$, then $t = r_r\pi$ and $\frac{R}{r}t = R_r\pi$. Again, since the sine and cosine functions have period 2π, we need only consider R_r and r_r congruent to 0 and 1 modulo 2. We obtain the following table.

$R_r \pmod 2$	$r_r \pmod 2$	x, y	$\lvert x\rvert + \lvert y\rvert$
0	0	R, 0	R
0	1	$-R$, 0	R
1	0	$R-2r$, 0	$R-2r$
1	1	$2r-R$, 0	$R-2r$

If $n = 3r_r$, then $t = \frac{3r_r\pi}{2}$ and $\frac{R}{r}t = \frac{3R_r\pi}{2}$. We obtain the following table.

$R_r \pmod 4$	$r_r \pmod 4$	x, y	$\lvert x\rvert + \lvert y\rvert$
0	0	R, 0	R
0	1	0, $-R$	R
0	2	$-R$, 0	R
0	3	0, R	R
1	0	$R - r$, $-r$	R
1	1	$-r$, $r - R$	R
1	2	$r - R$, r	R
1	3	r, $R - r$	R
2	0	$R - 2r$, 0	$R - 2r$
2	1	0, $2r - R$	$R - 2r$
2	2	$2r - R$, 0	$R - 2r$
2	3	0, $R - 2r$	$R - 2r$
3	0	$R - r$, $-r$	R
3	1	$-r$, $r - R$	R
3	2	$r - R$, r	R
3	3	r, $R - r$	R

Note that for all four possible values of n, the value of r_r is irrelevant in determining $|x| + |y|$. We may then obtain the following summary table.

$R_r \pmod 4$	n	$\lvert x\rvert + \lvert y\rvert$	Sum of $\lvert x\rvert + \lvert y\rvert$ for all n
0	0	R	$4R$
	r_r	R	
	$2r_r$	R	
	$3r_r$	R	
1	0	R	$4R - 2r$
	r_r	R	
	$2r_r$	$R - 2r$	
	$3r_r$	R	
2	0	R	$4R - 4r$
	r_r	$R - 2r$	
	$2r_r$	R	
	$3r_r$	$R - 2r$	
3	0	R	$4R - 2r$
	r_r	R	
	$2r_r$	$R - 2r$	
	$3r_r$	R	

Thus, for any pair (R, r), we add $4R$ if $R_r \equiv 0 \pmod 4$, $4R-2r$ if $R_r \equiv 1 \pmod 2$ (i.e. R_r is odd), or $4R-4r$ if $R_r \equiv 2 \pmod 4$.

Consider the case where R is odd. Then, for any r, either $r \perp R$, in which case $R_r = R$ and so R_r is odd, or r will reduce R. In the latter case, R's prime factorization does not contain 2, and so that of $R_r = \frac{R}{\gcd(R,r)}$ cannot contain 2 either. Thus, R_r is odd in either case. Therefore, for each r, we add $4R-2r$ as dictated by the above table.

Let $U(R)$ denote the amount added to $T(N)$ from the normals for any given R, i.e.

$$U(R) = \sum_{r=1}^{\left\lfloor \frac{R-1}{2} \right\rfloor} S(R, r).$$

Since $2r < R$ and $r \geq 1$, there are $\left\lfloor \frac{R-1}{2} \right\rfloor$ values of r for any given R. For odd R, this simplifies to $\frac{R-1}{2}$. Since we add $4R-2r$ for each r, and the sum of the integers from 1 to m is $\frac{1}{2}m(m+1)$, this yields

$$U(R) = 4R\left(\frac{R-1}{2}\right) - 2\left(\frac{1}{2}\right)\left(\frac{R-1}{2}\right)\left(\frac{R-1}{2}+1\right) = \frac{1}{4}(R-1)(7R-1) \tag{3}$$

added for any given odd R.

For even R, it is more complicated. For a given even R, let k be a nonnegative integer such that 2^k is the largest power of 2 that divides R. Then, for each r, we add $4R-4r$ if 2^k divides r, $4R-2r$ if 2^{k-1} divides r but 2^k does not, and $4R$ if neither divides r.

Now, 2^k will divide r when $r = 2^k m$ for an arbitrary integer $m \geq 0$, and 2^{k-1}, but not 2^k, will divide r for $r = 2^{k-1} + 2^k m$. It also turns out that the number of values of r fulfilling these criteria for a given R is equal to

$$\frac{R-2^k}{2^{k+1}} \tag{4}$$

for both adding $4R-4r$ and adding $4R-2r$. This is because for some odd natural number x, $R = 2^k x$, and so

$$\frac{R-2^k}{2^{k+1}} = \frac{2^k x - 2^k}{2^{k+1}} = \frac{x-1}{2}.$$

Since x is odd, this is an integer, and since there are $\left\lfloor \frac{R-1}{2} \right\rfloor$ values of r, and each value fulfilling the criteria occurs once every 2^k values, this is the number of values fulfilling each criteria.

Now, we are interested in finding the sum of the values of r fulfilling each criterion so that we may compute $4R - 4r$ and $4R - 2r$ for all values of r without looping. Since the values of r fulfilling each criteria form arithmetic sequences, of which we wish to sum the first $\frac{R-2^k}{2^{k+1}}$ terms, this is simple:

- For the $4R-4r$ criterion, the arithmetic sequence is $r_n = 2^{k-1}+2^k(n-1)$, and so the sum of the first n terms is

  $$S_n = \frac{n}{2}[2(2^{k-1}) + (n-1)2^k] = \frac{n}{2}(2^k n) = 2^{k-1}n^2.$$

  Using expression 4 for the number of terms in which we are interested, this yields a sum of

  $$S = 2^{k-1}\left(\frac{R-2^k}{2^{k+1}}\right)$$

  which simplifies to

  $$S = \frac{(R-2^k)^2}{2^{k+3}} \tag{5}$$

- For the $4R-2r$ criterion, the arithmetic sequence is $r_n = 2^k+2^k(n-1)$, and so the sum of the first n terms is

  $$S_n = \frac{n}{2}[2(2^k) + (n-1)2^k] = \frac{n}{2}(2^k + 2^k n) = 2^{k-1}n(n+1)$$

  Using expression 4 for the number of terms in which we are interested, this yields a sum of

  $$S = 2^{k-1}\left(\frac{R-2^k}{2^{k+1}}\right)\left(\frac{R-2^k}{2^{k+1}}+1\right)$$

  which simplifies to

  $$S = \frac{R^2-4^k}{2^{k+3}} \tag{6}$$

To derive an equation for $U(R)$ for even R, we note that $4R$ is added for all values of r, of which there are $\left\lfloor\frac{R-1}{2}\right\rfloor$. Since R is even, this simplifies to $\frac{R-2}{2}$. Then, we note that we may simply subtract $2r$ and $4r$ when r fulfils the appropriate criterion. Using equations 5 and 6, this gives an equation for $U(R)$ of

$$U(R) = 4R\left(\frac{R-2}{2}\right) - 4\left[\frac{(R-2^k)^2}{2^{k+3}}\right] - 2\left(\frac{R^2-4^k}{2^{k+3}}\right)$$

$$
\begin{aligned}
&= 2R^2 - 4R - \frac{4(R-2^k)^2 + 2R^2 - 2(4^k)}{2^{k+3}} \\
&= 2R^2 - 4R - \frac{6R^2 - 8(2^k)R + 2(2^{2k})}{2^{k+3}} \\
&= 2R^2 - 4R - \frac{3R^2}{2^{k+2}} + R - 2^{k-2} \\
&= R^2\left(2 - \frac{3}{2^{k+2}}\right) - 2^{k-2} - 3R \\
&= \frac{R^2}{2^{k+3}}(2^{k+3} - 3) - 2^{k-2} - 3R \qquad (7)
\end{aligned}
$$

Note that this does not apply to cases with odd R since it assumes that $\left\lfloor \frac{R-1}{2} \right\rfloor = \frac{R-2}{2}$, which is not true for odd R.

Thus, combining equations 3 and 7 into a single expression,

$$
U(R) = \begin{cases} \frac{1}{4}(R-1)(7R-1) & \text{if } R \text{ is odd} \\ \frac{R^2}{2^{k+3}}(2^{k+3}-3) - 2^{k-2} - 3R & \text{if } R \text{ is even} \end{cases} \qquad (8)
$$

Let $T_n(N)$ denote the contribution to $T(N)$ from the normals. Then

$$
T_n(N) = \sum_{R=3}^{N} U(R).
$$

We will now find an explicit expression for $T_n(N)$. Once again, it is easier to consider the odd and even values of R separately.

For odd R, consider that there are $\left\lfloor \frac{N-1}{2} \right\rfloor$ odd values of R such that $3 \leq R \leq N$. Let $R = 2n+1$ for some positive integer $n \geq 1$. Then, using equation 8,

$$
\begin{aligned}
\sum_{R=3,\text{ odd}}^{N} U(R) &= \sum_{n=1}^{\left\lfloor \frac{N-1}{2} \right\rfloor} U(2n+1) \\
&= \sum_{n=1}^{\left\lfloor \frac{N-1}{2} \right\rfloor} \left[\frac{1}{4}(2n+1-1)(7(2n+1)-1)\right] \\
&= \sum_{n=1}^{\left\lfloor \frac{N-1}{2} \right\rfloor} (7n^2 + 3n)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{7}{6}\left\lfloor \frac{N-1}{2} \right\rfloor \left(\left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \left(2\left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \\
&\quad + \frac{3}{2}\left\lfloor \frac{N-1}{2} \right\rfloor \left(\left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \\
&= \frac{1}{3}\left\lfloor \frac{N-1}{2} \right\rfloor \left(\left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \left(7\left\lfloor \frac{N-1}{2} \right\rfloor + 8 \right) \qquad (9)
\end{aligned}
$$

For even R, we may treat R as if $1 \leq R \leq N$ rather than $3 \leq R \leq N$ since the only even R between 1 and 3 is 2, which has $k = 1$, yielding $U(2) = 0$ and so the $R = 2$ case has no effect on the result. Now, the number of values of R with a certain value of $k \geq 1$ less than N is

$$\left\lfloor \frac{N + 2^k}{2^{k+1}} \right\rfloor,$$

and the maximum value of k for any $R \leq N$ is $\lfloor \log_2 N \rfloor$. Note that if 2^k divides R, then $\frac{R}{2^k}$ is odd. Thus, let $R = 2^k(2n-1)$ for some integer $n \geq 1$. Then

$$\sum_{R=3,\text{ even}}^{N} U(R) = \sum_{k=1}^{\lfloor \log_2 N \rfloor} \sum_{n=1}^{\left\lfloor \frac{N+2^k}{2^{k+1}} \right\rfloor} U(2^k(2n-1))$$

Now, using equation 8,

$$
\begin{aligned}
U(2^k(2n-1)) &= \frac{[2^k(2n-1)]^2}{2^{k+3}}(2^{k+3}-3) - 2^{k-2} - 3(2^k)(2n-1) \\
&= 2^{k-2}(2n-1)^2(2^{k+3}-3) - 2^{k-2} - 3(2^k)(2n-1) \\
&= 4n^2(2^{2k+1}) - 4n(2^{2k+1}) + 2^{2k+1} - 12n^2(2^{k-2}) + 12n(2^{k-2}) \\
&\qquad - 3(2^{k-2}) - 2^{k-2} - 3n(2^{k+1}) + 3(2^k) \\
&= 8n^2(2^{2k}) - 8n(2^{2k}) + 2(2^{2k}) - 3n^2(2^k) - 3n(2^k) + 2(2^k) \\
&= 2^{2k+1}(4n^2 - 4n + 1) - 2^k(3n^2 + 3n - 2) \\
&= 2^{2k+1}(2n-1)^2 - 2^k(3n^2 + 3n - 2) \qquad (10)
\end{aligned}
$$

Then the inner sum expands as follows:

$$\sum_{n=1}^{\left\lfloor \frac{N+2^k}{2^{k+1}} \right\rfloor} U(2^k(2n-1)) = \sum_{n=1}^{\left\lfloor \frac{N+2^k}{2^{k+1}} \right\rfloor} \left[2^{2k+1}(2n-1)^2 - 2^k(3n^2+3n-2) \right]$$

$$= \frac{2^k}{3}\left\lfloor\frac{N+2^k}{2^{k+1}}\right\rfloor\left(\left\lfloor\frac{N+2^k}{2^{k+1}}\right\rfloor^2(2^{k+3}-3)-2^{k+1}-9\left\lfloor\frac{N+2^k}{2^{k+1}}\right\rfloor\right) \tag{11}$$

Then, in total,

$$\begin{aligned} T_n(N) &= \frac{1}{3}\left\lfloor\frac{N-1}{2}\right\rfloor\left(\left\lfloor\frac{N-1}{2}\right\rfloor+1\right)\left(7\left\lfloor\frac{N-1}{2}\right\rfloor+8\right) \\ &+ \sum_{k=1}^{\lfloor\log_2 N\rfloor}\frac{2^k}{3}\left\lfloor\frac{N+2^k}{2^{k+1}}\right\rfloor\left(\left\lfloor\frac{N+2^k}{2^{k+1}}\right\rfloor^2(2^{k+3}-3)-2^{k+1}-9\left\lfloor\frac{N+2^k}{2^{k+1}}\right\rfloor\right) \end{aligned} \tag{12}$$

This may be computed easily. In `problem450.py`, this is computed in the `normals()` function.

3 Specials

For the purpose of the specials, let $Q = \frac{R-r}{r}$. Then, $Qr = R - r$ and $R = (Q+1)r$. We will consider the parametric equations of the hypoycloid in the following form using Q:

$$\begin{aligned} x(t) &= Qr\cos t + r\cos Qt \\ &= r(Q\cos t + \cos Qt) \end{aligned}$$

and

$$\begin{aligned} y(t) &= Qr\sin t - r\sin Qt \\ &= r(Q\sin t - \sin Qt). \end{aligned}$$

Then

$$|x| + |y| = r(|Q\cos t + \cos Qt| + |Q\sin t - \sin Qt|).$$

It follows from $1 \leq r < \frac{R}{2}$ that $1 < Q \leq R$. Consider whole-numbered Q. Since by the Chebyshev method any $\sin Qt$ or $\cos Qt$ can be expressed in terms of $\sin t$ and $\cos t$ for whole Q, $\sin Qt$ and $\cos Qt$ are rational iff $\sin t$ and $\cos t$ are rational. Furthermore, if c is the denominator of $\sin t$ and $\cos t$ (which must have the same denominator because their numerators and the common denominator form a Pythagorean triple), then c^Q is the denominator of $\sin Qt$ and $\cos Qt$, with the numerators forming the legs of another Pythagorean triple. Because of this, for x and y to be integers, r must be divisible by c^Q, and so R must be divisible by $(Q+1)c^Q$.

For non-integral Q, consider that if $\sin(\frac{1}{n}t)$ and $\cos(\frac{1}{n}t)$ are rational for some whole number n, then $\sin(\frac{m}{n}t)$ and $\cos(\frac{m}{n}t)$ are rational for any whole number m since it may be reduced via the Chebyshev method. As well, for positive integers a_1, c_1, a_2, c_2, and n,

$$\sin\left(n \arcsin\frac{a_1}{c_1}\right) = \frac{a_2}{c_2} \rightarrow \sin\left(\frac{1}{n}\arcsin\frac{a_2}{c_2}\right) = \frac{a_1}{c_1}$$

and

$$\cos\left(n \arccos\frac{a_1}{c_1}\right) = \frac{a_2}{c_2} \rightarrow \cos\left(\frac{1}{n}\arccos\frac{a_2}{c_2}\right) = \frac{a_1}{c_1}.$$

Fractional values of Q will be discussed in greater detail later.

Now, consider that for integral Q, since using the Chebyshev method $\sin Qt$ and $\cos Qt$ will always reduce to an expression containing $\sin t$ and $\cos t$, the values of $\sin t$ and $\cos t$ determine the values of $\sin Qt$ and $\cos Qt$ for a given Q; that is, there is no way to have the same values of $\sin t$, $\cos t$, and Q but different values of $\sin Qt$ and $\cos Qt$. Thus, if for any Pythagorean triple (a,b,c) we can enumerate all possible combinations of $\sin t$ and $\cos t$, we also have all possible values of $\sin Qt$ and $\cos Qt$. This may be done by adding and subtracting π and 2π to produce various angles around the unit circle, all of which lead to distinct $\sin t$ and $\cos t$. However, the parity of Q will lead to duplication if not considered. Thus, we obtain all combinations through evaluating $\sin t$, $\cos t$, $\sin Qt$, and $\cos Qt$ for t equal to the following for a given Pythagorean triple (a,b,c):

$$\begin{aligned}
t &= \arccos\frac{a}{c} & t &= \arccos\frac{b}{c} \\
t &= \frac{m\pi}{2} - \arccos\frac{a}{c} & t &= \frac{m\pi}{2} - \arccos\frac{b}{c} \\
t &= \frac{m\pi}{2} + \arccos\frac{a}{c} & t &= \frac{m\pi}{2} + \arccos\frac{b}{c} \\
t &= m\pi - \arccos\frac{a}{c} & t &= m\pi - \arccos\frac{b}{c}
\end{aligned}$$

where $m = 2$ if Q is even and $m = 3$ if Q is odd. This generates the following values for $(\sin t, \cos t)$, though not necessarily in this order:

$$\left(\frac{b}{c}, \frac{a}{c}\right), \left(\frac{b}{c}, -\frac{a}{c}\right), \left(-\frac{b}{c}, -\frac{a}{c}\right), \left(-\frac{b}{c}, \frac{a}{c}\right),$$

$$\left(\frac{a}{c}, \frac{b}{c}\right), \left(\frac{a}{c}, -\frac{b}{c}\right), \left(-\frac{a}{c}, -\frac{b}{c}\right), \left(-\frac{a}{c}, \frac{b}{c}\right)$$

Corresponding $\sin Qt$ and $\cos Qt$ values are generated as well, although they are not necessarily unique. This produces a map of $(\sin t, \cos t)$ values to $(\sin Qt, \cos Qt)$ values, which is generated by the `pattern()` function in `problem450.py`.

What is especially interesting with regards to this map is that $\sin \frac{1}{n}t$ and $\cos \frac{1}{n}t$ may be computed by using the map for $Q = n$ in reverse, yielding the following method of computing $\sin Qt$ and $\cos Qt$ for fractional $Q = \frac{m}{n}$:

1. Obtain the map for $Q = n$ and perform a reverse lookup of t.
2. Use the result as the key for the map for $Q = m$, yielding the values of $\sin \frac{m}{n}t$ and $\cos \frac{m}{n}t$.

The last piece of knowledge necessary for a specials-determining algorithm is a function for the sum of all values of r having specials for a given Pythagorean triple (a, b, c) and a given value of Q for any value of N. Let this function be denoted $\rho(N)$. Now, it was previously determined that r must be divisible by c^Q, and so R must be divisible by $(Q+1)c^Q$.

For integer Q, it is clear that the values of r will increase successively by c^Q. Therefore $\rho(N)$ will equal c^Q multiplied by the sum of 1 to n, where n is the number of valid values of r for a given N, which is the same as the number of values of $R \leq N$. Since $(Q+1)c^Q$ divides R, this is given by

$$\left\lfloor \frac{N}{(Q+1)c^Q} \right\rfloor,$$

and so for integer Q,

$$\rho(N) = \frac{c^Q}{2} \left\lfloor \frac{N}{(Q+1)c^Q} \right\rfloor \left(\left\lfloor \frac{N}{(Q+1)c^Q} \right\rfloor + 1 \right). \tag{13}$$

For non-integer $Q = \frac{m}{n}$, it is slightly more complicated, since using the formula for integer Q counts cases where R is not an integer. For example, if $Q = \frac{3}{2}$ and $c = 25$, the above formula would count $(R, r) = (312.5, 125)$, which is obviously not valid. However, complicating this is the fact that if c and n (the denominator of Q) are non-coprime, multiplication by c^Q is enough to make some otherwise invalid values of r valid. A working formula for $Q = \frac{m}{n}$ is

$$\rho(N) = \frac{c^m}{2} \left\lfloor \frac{n}{\gcd(c, n)} \right\rfloor M(M+1) \tag{14}$$

where

$$\begin{aligned} M &= \left\lfloor \frac{N}{c^m \left(\frac{m}{\gcd(c,n)} + \frac{n}{\gcd(c,n)} \right)} \right\rfloor \\ &= \left\lfloor \frac{N \gcd(c, n)}{c^m (m+n)} \right\rfloor. \end{aligned}$$

The following algorithm then suffices to calculate the specials' contribution to $T(N)$.

1. For every Pythagorean triple (a, b, c) such that $3c^2 \leq N$ (i.e. $Q = 2$, the lowest integral value of Q, has at least one r), we loop over integral values of Q starting at 2 until $(Q+1)c^Q > N$.

2. We obtain the map of $\sin t$ and $\cos t$ values to $\sin Qt$ and $\cos Qt$ values for this triple and value of Q.

3. For each $(\sin t, \cos t, \sin Qt, \cos Qt)$ 4-tuple, we add

   $$\rho(N)(|Q \cos t + \cos Qt| + |Q \sin t - \sin Qt|)$$

   to the total, obtaining $\rho(N)$ using equation 13. (We compute $\rho(N)$ once.)

4. We then consider fractional values of Q with the old integral Q as the denominator. Let n be the old value of Q. We loop over integral m such that $m > n$ (i.e. $Q > 1$) and $m \perp n$, as otherwise n would not be the denominator of Q. We stop when

   $$\left(\frac{m}{n} + 1 \right) (c^n)^{\frac{m}{n}} = \left(\frac{m}{n} + 1 \right) c^m > N.$$

5. We then obtain the map of $\sin t$ and $\cos t$ values to $\sin Qt$ and $\cos Qt$ values for this triple and $Q = m$. We also calculate $\rho(N)$ using equation 14.

6. We then loop over the *original* map for the original integral Q, but in reverse, assigning $\sin t$ and $\cos t$ to the values of the map originally known as $\sin Qt$ and $\cos Qt$. We plug the keys of the map with denominator c into the map for $Q = m$ to obtain the new $\sin Qt$ and $\cos Qt$ for $Q = \frac{m}{n}$.

7. For each of these combinations, we add

$$\rho(N)(|Q \cos t + \cos Qt| + |Q \sin t - \sin Qt|).$$

Let the contribution to $T(N)$ obtained via this algorithm be $T_s(N)$. Then, using the equation for the normals' contribution, $T_n(N)$, from equation 12,

$$T(N) = T_n(N) + T_s(N).$$

4 Appendix

We calculate $\sin Qt$ and $\cos Qt$ for integer Q using a recursive algorithm based on the Chebyshev method of finding the nth multiple-angle formulae for the sine and cosine functions. It states that for integral $n > 2$,

$$\sin n\theta = 2 \cos \theta \sin((n-1)\theta) - \sin((n-2)\theta)$$

and

$$\cos n\theta = 2 \cos \theta \cos((n-1)\theta) - \cos((n-2)\theta).$$

We use the standard double-angle formulae $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\cos 2\theta = 2 \cos^2 \theta - 1$ for the base case. Our algorithm calculates the exact rational value of

$$\sin\left(x\pi \pm \arccos \frac{b}{c}\right) \qquad \text{and} \qquad \cos\left(x\pi \pm \arccos \frac{b}{c}\right)$$

for x equal to an integer or a half-integer.

The algorithm is implemented in `problem450.py` in the `chebyshev_sin()` and `chebyshev_cos()` functions.

As well, the Pythagorean triples are generated using F. J. M. Barning's matrix method, by which a tree of primitive Pythagorean triples is generated by multiplying the three matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$$

on the right by a column vector forming a Pythagorean triple, starting with $(3, 4, 5)^T$. Numpy is used to represent the matrices. Since in the tree formed by this method each 'child' Pythagorean triple has a larger c than its parent, all Pythagorean triples such that $3c^2 \leq N$ may be easily generated by stopping when $3c^2 > N$.