Finding Small Strategies in Game Trees

Lemke-Howson, Wilson's Improvement, Koller's Algorithm

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Finding Equilibria with Small Mixed Strategies

Motivation:

- Complexity depends on size of strategies.
  - (e.g. support size of a mixed strategy)
- Reduced strategy size $\implies$ reduced complexity

Idea:

- Most games are in extensive form.
- Many mixed strategies will have the same behavior.
- One "basic" mixed strategy can express the whole set.

Results:

- First algorithm finding equilibria in exponentitial time w.r.t. the size of the game tree.
  - Lemke-Howson (and others) are exponential in the number of pure strategies.

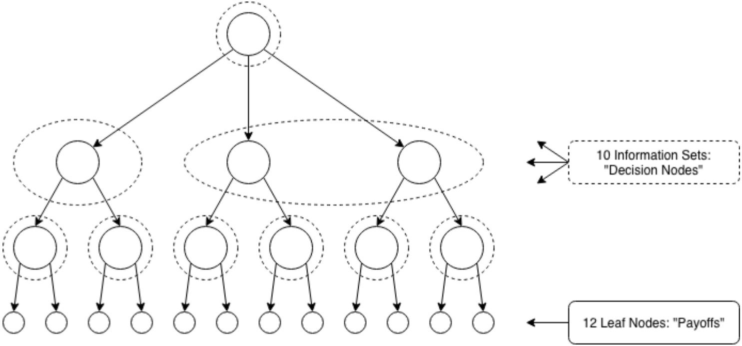
Background: Strategy Types

- **Pure Strategy**
  A pure strategy of player i is a function s_i that assigns to each information set h of player i a feasible action $s_i(h) \in A(h)$.
- **Mixed Strategy**
  A mixed strategy of player i is a probability distribution σ_i over i's pure strategies. $\sigma_i(s_i) \in [0, 1]$ is the prob assigned to pure strategy s_i . 'Global randomization' at the beginning of the game
- **Behavioral Strategy**
  A behavioral strategy of player i is a function b_i that assigns to each information set h of player i a probability distribution over the feasible actions $A(h)$. $b_i^h(a)$ is the prob of action $a \in A(h)$. 'Local randomization' as play proceeds.

Background: Game Tree

- **Player Partition of Moves**
  Assigns every move to some player
- **Information Set**
  Nodes in an information set of player i are 'indistinguishable' to player i; this requires, for instance, the same actions in each decision node of the information set. (next slide)
- **Chance Probabilities**
  A probability distribution on actions for each chance node
- **Utility Functions**
  Payoff for every game tree leaf for each player
- **Plays** are leaf nodes **Moves** are remaining nodes.
  Remaining nodes

Motivating Example



- There are 2^{10} pure strategies. Lemke-Howson is $O(2^{2^{10}})$.
- "Size" of tree is 12. New algorithm is $O(12 \cdot 2^{12 \cdot 10})$.

Lemke-Howson Review

Goal: Compute an equilibrium for a two-player game G.

- G has two *best-response* polytopes P_1 and P_2.
- Start with a pair of *labelled* vertices (v, w).
- Execute *pivot* operations until v and w are fully labelled.
  - Replace v (or w) with some adjacent vertex u.
  - Pick up new label from u.
  - Drop a label from v.
- Normalize the probabilities over the labels.

Problem: Each vertex has coordinates over all pure strategies.

Solution: Only maintain pure strategies probability > 0.

Assumption: Mixed strategies (vertices) are likely to be sparse.

Koller and Meggido's Solution

Don't assume mixed strategies are sparse, prove it!

Theorem

For any mixed strategy μ, there exists an equivalent strategy μ' whose size is at most $|T|$, the number of leaves in the game tree.

Definition

Two mixed strategies μ and μ' are said to be *equivalent* if for all payoff functions, the payoffs to all players under μ and μ' are identical.

Idea: Find a small mixed strategy in the equivalence class.

Koller and Meggido's Solution Continued

Definition

A pure strategy s *potentially reaches* node a if, for every information set on the path from root to a, the strategy s takes the decision leading to a.

Note: Consider some information set u. If a pure strategy s does not *potentially reach* any node in u, then the decision made by s at u is irrelevant.

Let the *realization weight* of mixed strategy μ on a be:

$$\mu[a] = \sum_{s \in R(a)} \mu(s)$$

Where $R(a)$ are strategies that *potentially reach* a.

Koller and Meggido's Solution Continued

Theorem

Two mixed strategies are equivalent if their realization weights are identical over all leaf nodes.

Both strategies have same probability of *potentially* reach leaf node a (payoff at a).

Therefore an equivalence class of mixed strategies is defined by $\mu[z]$ over all leaves z.

Koller and Meggido's Solution

Don't assume mixed strategies are small, prove it!

Theorem

For any mixed strategy μ, there exists an equivalent strategy μ' whose size is at most $|T|$, the number of leaves in the game tree.

Proof.

(Idea) In order for μ to be equivalent to μ' we must have $\mu[z] = \mu'[z]$ for all leaf nodes z. We can represent these constraints as a system of linear equations. Specifically, there are $|T|$ equations and $|T|$ unknowns and this solution describes a mixed strategy μ' with support $|T|$. $\square$

Kuhn's Theorems

Theorem

In a finite extensive form game with perfect recall:

1. *each behavioral strategy has an outcome-equivalent mixed strategy.*
2. *each mixed strategy has an outcome-equivalent behavioral strategy.*

Theorem

In a game with perfect recall the expected utility to each player $i \in P$ is the same whether they use a mixed strategy combination (σ_i) of the behavioral strategy combination (b_i), where b_i is the behavioral strategy associated with σ_i

Computing Best Responses on-the-fly

- Introduced by Wilson as part of his auxiliary procedure
- Computes best response pure strategies in behavioral form for player P
- Uses behavioral strategy corresponding to opponents mixed strategy
- Transforms the game tree into a decision tree for P
- crux of the procedure consists of lumping opponent together with chance according to the behavioral strategy

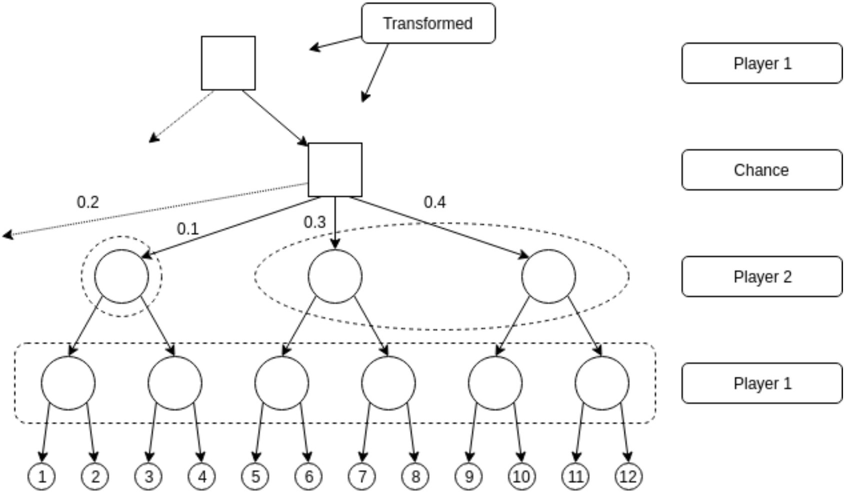
Theorem

The behavioral strategies associated with the mixed strategies of a player n's opponents induce a one-person perfect recall game for player n in which his opponents are identified with chance, and his information sets are singletons

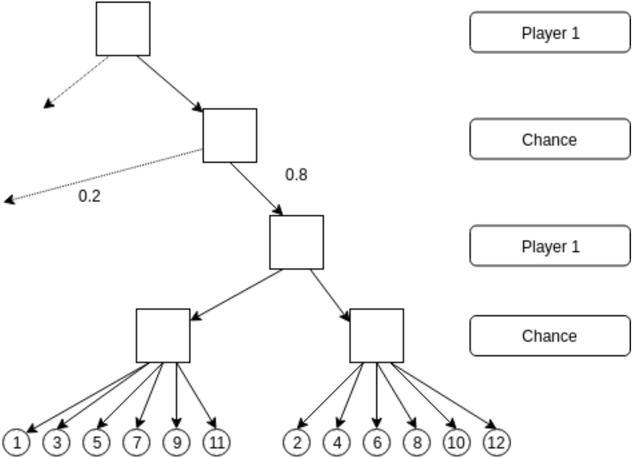
High Level Intuition

- Collapse information sets into single nodes.
- Collapse opponents and chance nodes into just chance nodes
- Reassign successors and predecessors to these collapsed nodes
- Iteratively remove violations (Multiple predecessors, Probability violations)
- Add dummy root node and dummy play nodes

Example Part 1



Example Part 2



Wilson's Auxiliary Procedure

Three step procedure:

1. **Opponent Behavioral Strategy**
   Compute the behavioral strategy associated with the opponent's mixed strategy
2. **Player's Decision Tree**
   Compute the one-person game decision tree for player p
3. **Best Response Computation**
   Expectimax over the game tree to obtain the pure strategy best response of player p. This can be done in a bottom-up manner via a dynamic program.

Summary

- Game Trees and Information Sets.
- Types of strategies, and equivalences between mixed and behavioral strategies.
- Wilson's Auxiliary Procedure: generating best responses as expectimax over a one-person collapsed game tree.