Basic Combinatorics - Spring, Home Assignment 3

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Problem 1.

claim. *There is an integer n_0 such that for any $n \geq n_0$, in every 9-coloring of the integers $\{1, 2, 3, \ldots, n\}$, one of the 9 color classes contains 4 integers a, b, c, d such that $a + b + c = d$.*

Proof. based on Ramsay Theorem Let $n_0 = K(4, \ldots, 4)$, where 4 appears $k - 1$ times. and lets c be r-colouring s.t:

$$c : \{1, \ldots, n\} \to \{1, \ldots, k\}$$

For graph K_n and labelling of its edge $\{1, \ldots, n\}$. we can colour any edge e_{ij} with $c(|i - j|)$. we got a $k - 1$-colouring of K_n. then for n_0, we must have a K_4 with all edges different. for vertices $x \leq y \leq z \leq w$ then

$$a = y - x, b = z - y, c = w - z, d = w - x$$

Gives a solution □

Problem 2.

claim. *every tournament on n vertices, contains a transitive tournament on $\lfloor \log_2 n \rfloor$ vertices.*

Proof. Using induction for $n = 0, 1, 2$ its holds on empty. W.L.O.G[1] assume the claim holds for $n \leq 2^k$ now lets look at some tournament on 2^{k+1} vertices and we can pick any vertex v, and define:

$$v_{in} = \{u : \text{ exsit edege } v \leftarrow u\}, v_{out} = \{u : \text{ exsit edege } v \to u\}$$

Hence $|v_{in}| + |v_{out}| = 2^{k+1} - 1$ and one of them contain $|2^k|$ edges, lets assume its v_{in}[2] by our assumption its contains transitive tournament T_{in} size $|k|$. now $T_{in} \cup \{v\}$ is sub tournament and any edge points to v hence its transitive tournament on $|k + 1|$ vertices. □

[1]we can modify any other tournament to to nearst power of 2 its will still hold for $\lfloor \log_2 n + 1 \rfloor$ see (10)

[2]its equivalence for v_{out}

claim. *there exists a tournament on n vertices that does not contain a transitive tournament on* $2\log_2 n + 2$ *vertices.*

Proof. The number of Tournament on n vertices is $2^{\binom{n}{2}}$.The number of tournaments of size k is $k!$, and there are $\binom{n}{k}$ sets of size k, and the number of ways to choose the edges outside the transitive tournament is $2^{\binom{n}{2}-\binom{k}{2}}$. hence if we show that

$$k!\binom{n}{k}2^{\binom{n}{2}-\binom{k}{2}} < 2^{\binom{n}{2}}$$

its yield that for some k the number of n-vertex tournaments with a transitive subtournament on k vertices is smaller than the total number of tournaments.

$$\begin{aligned}
2^{\binom{n}{2}} &> k!\binom{n}{k}2^{\binom{n}{2}-\binom{k}{2}} && (1)\\
2^{\binom{k}{2}} &> k!\binom{n}{k} && (2)\\
&> k!\frac{n!}{k!(n-k)!} && (3)\\
&> n(n-1)(n-2)\cdots(n-k+1) && (4)\\
&> n^k && (5)
\end{aligned}$$

Taking $\log_2$ from (1)(5)

$$\begin{aligned}
\binom{k}{2} &> k\log_2(n) && (6)\\
\frac{k!}{2(k-2)!} &> k\log_2(n) && (7)\\
\frac{k!}{k(k-2)!} &> 2\log_2(n) && (8)\\
k-1 &> 2\log_2(n) && (9)\\
k &> 2\log_2(n)+1 && (10)
\end{aligned}$$

□

Problem 3.

claim. *if an n-vertex graph $G=(V,E)$ has no copy of $K_{2,t}$*[3] *then*

$$|E| \leq \frac{1}{2}(\sqrt{t-1}n^{\frac{3}{2}}+n)$$

Proof. W.L.O.G let $t \geq 1$. we can distinguish that any $e_1, e_2 \in E$ have at most[3] t neighbours. and each one of them can be part of pair. we can consider it as the number of path length 2 in G. Let $d(v_i)$ be the deg of $v_i \in G$ and we get that:

$$t\binom{n}{2} \geq \sum_{v \in V}\binom{d(v)}{2} \geq n\binom{2|E|/n}{2} \tag{11}$$

The right-hand side hold from Jensen's Inequality and since its minimized[4] the binomial when all the degrees are equal, $d_i = 2|E|/|V|$.

$$n\binom{2|E|/n}{2} = n\frac{(2|E|/n)(2|E|/n-1)}{2} \geq n\frac{(2|E|/n-1)^2}{2} \tag{12}$$

And

$$t\binom{n}{2} = t\frac{n^2-n}{2} \leq t\frac{n^2}{2} \tag{13}$$

.We conclude from (11)(12)(13) that

$$\begin{aligned} n\frac{(2|E|/n-1)^2}{2} &\leq t\frac{n^2}{2} && (14)\\ (2|E|/n-1)^2 &\leq tn && (15)\\ 2|E|/n &\leq \sqrt{tn}+1 && (16)\\ |E| &\leq^3 \frac{1}{2}(\sqrt{t}n^{\frac{3}{2}}+n) && (17) \end{aligned}$$

□

[3] I will use $t+1$ for the proof i.e $K_{2,t+1}$

[4] convex property

Problem 4.

claim. *Let* $S_1, \dots, S_n \in [n]$ *such that* $|S_i \cap S_j| \leq 1$ *for all* $1 \leq i < j \leq n$ *then.*

$$\frac{1}{n}\sum_{i=1}^{n} |S_i| = O(\sqrt{n})$$

Proof. Let define $G = (V, E)$ such that

$$S = \{S_i : S_i \in [n]\}, U = \{i \in n\} \text{ and } E = \{e_{S_k,m} : m \in S_k\}, V = S \cup U$$

Its immediate $|V| = 2n$ and G is Bipartite since we can dived V into 2 disjoint independent sets S and U, that is any $e \in E$ connects a vertex in S to one in U. hence G has no copy of $K_{2,2}$, using **Problem 3** we can get that

$$|E| \leq \frac{1}{2}(\sqrt{2-1}(2n)^{\frac{3}{2}} + 2n) \tag{18}$$

$$\sum_{i=1}^{n} |S_i| \leq \sqrt{2}n^{\frac{3}{2}} + n \tag{19}$$

$$\frac{1}{n}\sum_{i=1}^{n} |S_i| \leq \sqrt{2}\sqrt{n} + 2 \tag{20}$$

$$\frac{1}{n}\sum_{i=1}^{n} |S_i| = O(\sqrt{n}) \tag{21}$$

□

Problem 5.

Theorem. *if $G = (V, E)$ has no copy of K_{t+1} then $|E| \leq (1 - \frac{1}{t})\frac{n^2}{2}$.*
(Turan's Theorem)

Proof. Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and f to be vector and weight function satisfying

$$\forall i \ 0 < x_i \leq 1, \sum_{i=1}^{n} x_i = 1, f(x) = \sum_{i,j \in E} x_i x_j$$

By taking $x = (\frac{1}{n}, \ldots, \frac{1}{n})$ we get

$$f(x) \geq \sum_{i,j \in E} \frac{1}{n^2} \geq \frac{|E|}{n^2} \tag{22}$$

The "weight shifting" method yield to shift the weight between any neighbours x_i, x_j if $e_{i,j} \notin E$.
We can notice that the sum is maximized when all the weight is concentrated on a clique. Since any shift is does not decrease the value of f we can repeat the processes. since $G = (V, E)$ has no copy of K_{t+1} we can have at most t size clique ,let name it $[T]$. we can get lower bound on f :

$$\begin{aligned}
f(x) &\leq \sum_{i,j \in [T]} x_i x_j = \sum_{i,j \in [T]} \frac{1}{t^2} && (23) \\
&\leq \frac{t(t-1)}{2} \frac{1}{t^2} && (24) \\
&\leq (1 - \frac{1}{t})\frac{1}{2} && (25)
\end{aligned}$$

Combining (25) and (22) to finish the proof

$$\begin{aligned}
\frac{|E|}{n^2} &\leq (1 - \frac{1}{t})\frac{1}{2} && (26) \\
|E| &\leq (1 - \frac{1}{t})\frac{n^2}{2} && (27)
\end{aligned}$$

□