Basic Combinatorics - Spring
∼ Home Assignment 5 ∼

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Problem 1

claim. *the number of surjective mappings from [n] to [k] is given by*

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

Proof. denote

$$f_x = \{f : f^{-1}[C] \text{ s.t } [C] \subseteq [k], \quad |C| \leq |k-x|\}$$

to be the set of all function from $[n]$ to subset of $[k]$ where at least x element of k is not in the image of f_x. let look at f_1, we can choose 1 from k element to not be part of the image, it is $\binom{k}{1}$. now we have $k-1$ elements to choose from n items, i.e which item from $n_i \in n$ will map to $k_j \in k$. Hence we looking at total $\binom{k}{1}(k-1)$ functions. and for general x it is $|f_x| = \binom{k}{x}(k-x)^n$. now lets $f_0 = S$ to be the set of all function from $[n]$ to $[m]$, since $f_x \subseteq f_y$ for $0 \leq x \leq y \leq k$ then :

$$f_{onto} \in \bigcap_{i=1}^{k}\overline{f_i} \Rightarrow |\bigcap_{i=1}^{k}\overline{f_i}| = {}^1|S - \bigcup_{i=1}^{k} f_i|$$

using inclusion exclusion principle we get that.

$$\binom{k}{0}k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \cdots \pm \binom{k}{k-2}2^n \mp \binom{k}{k-1}1^n \pm \binom{k}{k}0^n$$

that is

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

□

[1] De Morgan

Proposition 1.
$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^n = n!$$

Proof. $\Rightarrow$ using the result above, for $k = n$ its following that :

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^n = n!$$

$\Leftarrow$ the number of onto function from $[n]$ to $[n]$ is equivalence to to the number of ways to arrange n distinct elements in row , that is

$$n! = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^n$$

$\square$

Proposition 2.

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = 0 \quad \textit{when } k > n.$$

Proof. $\Rightarrow$ using the result above, for $k > n$ its following that :

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = 0$$

$\Leftarrow$ assume we have k pigeons, we need to find in how many ways we can place them all in n holes, when each one of them in different hole. after placing $n-k$ of them the all the holes are full and we left with $k-n > 0$ pigeons. following the Pigeonhole principle there are is-no option to do so, or equivalence to 0 ways. $\square$

Proposition 3.

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

where S(n, k) are the Stirling numbers of the second kind

Proof. $\Rightarrow$its immediate from the definition of $S(n,k)$:

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

$\Leftarrow$ we can consider the the set S_k:

$$S_k := \{\{f^{-1}(x)\}, \forall x \in k\}$$

we are looking at total of k non-empty sets. the amount of subjective function from $[n]$ to $[k]$ is number of ways to distribute the elements of n into these sets, let $S(n, S_k)$ be the number of ways to partition a set of n objects into $S_k = |k|$ non-empty subsets. now we can notice that any $k_i \in k$ can be associated with any of these sets i.e total of $k!$. and in total we get:

$$S(n,S_k)k! = S(n,k)k! = k!\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

$\square$

Problem 2

claim. *the number of ways of coloring the integers $\{1\dots 2n\}$ using the colors red/blue in such a way that if i is colored red then so is $i-1$, is:*

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}2^{2n-2k} = 2n+1$$

Proof. I will use counting in two ways method to deduce the identity
$\Rightarrow$ we can consider the problem as placing $2n$ items in a row and choose spot to place separator s.t any item to its left are red and all the other are blue. we looking at total of $2n-2$ in between any two adjacent numbers from 1 to $2n$. by including 2 more additional option that all of them red or blue, we get that the total of number of ways to place the separator is given by $2n+1$.
$\Leftarrow$ There are in total $S = 2^{2n}$ ways of coloring the integers. with same idea as above, we can consider the separator as choose pair of adjacent integers the first will be coloring with R and the second B and rest dont care, it is:

$$2^{2n-2}\binom{2n-1}{1}$$

now same idea for 2 paris

$$2^{2n-4}\binom{2n-1}{2}$$

and in general :

$$2^{2n-2i}\binom{2n-i}{i}$$

Using Inclusion exclusion principle we get that

$$2^{2n} - 2^{2n-2}\binom{2n-1}{1} + \cdots \pm \binom{2n-n}{n}2^{2n-2n}$$

that is :

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}2^{2n-2k}$$

□

Problem 3

Proposition 4. *Let N be a set, then any $k \subseteq N$ have bijection such that $k \to \{0,1\}^{|N|}$*

let define the following mapping

$$f : \left\{ k \quad \mapsto \quad x \in N \mapsto \left\{ \begin{array}{ll} 0 & , \text{if } x \notin N \\ 1 & , \text{if } x \in N \end{array} \right. \right. \qquad f^{-1}\{\{0,1\}^N \mapsto \{x \in N \text{ s.t. } f(x) = 1\}$$

we can consider it as binary encode of the subset indicate $\mathbb{1}$ if the given integer in the subset and 0 otherwise

claim. *the number of subsets of size k of $\{1, \ldots n\}$ which contain no pair of consecutive integers is given by $\binom{n-k+1}{k}$*

Proof. using Proposition 4. subset k can represented as some binary string of length n, its yield that if in some string have two consecutive appearances $\mathbb{1}$ then this subset contain pair of consecutive integers. moreover we can notice that if $n < 2k-1$ then its can not contain pair of consecutive integers.

For given k let $f(k)$ define the bijection of subset k for some $n \geq 2k-1$. if we assume its not have any consecutive numbers, then its have k $\mathbb{1}$'s and

$n-k$ 0's. since we know $k-1$ from the 0's must be followed by the first $k-1$ of $\mathbb{1}$'s. hence the following problem becomes, how many ways could we distribute the remaining element i.e

$$n-(\underbrace{k}_{k\times\mathbb{1}'s}+\underbrace{(k-1)}_{(k-1)\times 0's})=n-2k+1$$

it is $n-2k+1$ number of 0's in the $k+1$ optinal positions and. that is "Stars and bars"[2] problem :

$$\binom{n-2k+1-1}{k+1-1}=\binom{n-2k}{k}$$

□

Problem 4

Lemma 4.1.

$$1\geq m-\binom{m}{2}\quad m\geq 1, m\in\mathbb{N}$$

Proof.

$$1\geq m-\binom{m}{2}\Leftrightarrow 1\geq m-\frac{m^2-m}{2}$$
$$m^2-3m+2\geq 0\Leftrightarrow (m-1)(m-2)\geq 0$$

And the right hand size grater then zero for any $m\geq 2$ □

Lemma 4.2.

$$1\leq m-\binom{m}{2}+\binom{m}{3}\quad m\geq 1, m\in\mathbb{N}$$

Proof.

$$\begin{aligned}1\leq m-\binom{m}{2}+\binom{m}{3} &\Leftrightarrow 1\leq m-\frac{m^2-m}{2}+\frac{m^3-2m^2-2m}{6}\\ &\Leftrightarrow 0\leq m^3-6m^2+11m-6\\ &\Leftrightarrow 0\leq (m-3)(m-2)(m-1)\end{aligned}$$

[2] Not sure if saw in class -*"Stars and Bars from Wikipedia"*

The right hand size grater then zero for any $m \geq 3$, and equal 0 for $m \in \{1, 2\}$ since m is an integer. $\square$

Let $A_1, A_2 \ldots A_n$ be a family of n sets.

claim 4.3.

$$\left| \bigcup_{i=1}^{n} A_i \right| \geq \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j|$$

Proof. to prove the following claim I will use "Donation to the Argument" [3] method. let assume that exists some $a \in A_i$. this a adding at most 1 to the left hand side. now consider a is part of some other $m \geq 1$ sets, then at the right hand side its count $\binom{m}{1}$ times at the first argument, and $\binom{m}{2}$ in the second. Hence using Lemma 4.2 the inequality hold for any $a \in A$. and that lead to finish the proof $\square$

claim 4.4.

$$\left| \bigcup_{i=1}^{n} A_i \right| \leq \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j| + \sum_{1 \leq i \leq j \leq k \leq n} |A_i \cap A_j \cap A_k|$$

Proof. using same idea described above, let $a \in A_i$ then a count once on the left hand-sid. At the right hand-side a count $\binom{m}{1}$ on the 1^{nd} term. $\binom{m}{2}$ on the 2^{nd} and $\binom{m}{3}$ at the 3^{nd} term. Hence using Lemma 4.2 the inequality hold for any $a \in A$. and that lead to finish the proof. $\square$

[3] To be honest I am not really sure what the name of this technique, Its kind of similar to "Counting derangements" I think