BASIC COMBINATORICS
~ SOLUTIONS TO ALL EXERCISE ~
TAU SPRING 2022

Combined And Expanded Exercise Solution

Author

Saar Barak

LaTeX Documents Available on Git-Hub

Contents

1 Assignment 1

1.1 Problem 1.

Given $n \equiv 1(\text{mod } 8)$ we looking for the number of subsets n-element set, whose size is 0 (mod 4).
Let $S_n(a, b)$ be the sum of the binomial coefficients with $k \equiv a$ (mod b).

$$2^n = (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = S_n(1,2) + S_n(0,2)$$

$$(1+i)^n = \binom{n}{0}+\binom{n}{1}i-\binom{n}{2}-\binom{n}{3}i+\binom{n}{4}+\binom{n}{5}i-\cdots = S_n(0,4)+iS_n(1,4)-S_n(2,4)-iS_n(3,4)$$

$$(1-i)^n = \binom{n}{0}-\binom{n}{1}i-\binom{n}{2}+\binom{n}{3}i+\binom{n}{4}-\binom{n}{5}i-\cdots = S_n(0,4)-iS_n(1,4)-S_n(2,4)+iS_n(3,4)$$

we can notice that the amount of subsets of a n-element set, whose size is 0(mod4) when $n \equiv 1(\text{mod } 8)$

$$\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots = S_n(1,4)$$

and based on the symetric propertyy of the binomial its sufistace to add the binomal form of 0 :

$$0 = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = S_n(0,2) - S_n(1,2)$$

moreover we can see taht $S_n(0,2) = S_n(1,2) = 2^{n-1}$
while summing $2^n + (1+i)^n + (1-i)^n + 0$ them will leave us with

$$4\left(\binom{n}{0} + \binom{n}{4} + \binom{n}{8} \cdots\right) = 4S_n(1,4) = 2(S_n(0,2) + S_n(0,4) - S(2,4))$$

follow the property at Equating 2 from above $S_n(0,4) - S(2,4)$ can showed as the i part as $\mathbf{R}((i+1)^n)$ following:

$$1 + i = \sqrt{2}e\pi^{i/4} \Rightarrow \mathbf{R}((i+1)^n) = 2^{n/2}cos((\pi n)/4)$$

now for $n \equiv 1$ (mod 8) lets set in the last equating $1/\sqrt{2}$ summing all up together we get :

$$2S_n(1,4) = 2^{n-1} + 2^{(n-2)/2} * 2^{-1/2} = 2^{n-2} + 2^{(n-3)/2}$$

1.2 Problem 2.

Based on the following formula

$$\binom{n}{k} = \binom{n}{n-k} \Rightarrow \sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$$

acombinatorial proof of the identity above, lets say we have $2n$ student that want to study the "Basic Combinatoric" course, but for some reason the university decide to schedule the course in class with only n seats. so we need to choose group of n student that could have a seat in the class (all the other will watch from zoom), so in total we have

$$\binom{2n}{n}$$

on the other hand lets split all the student into 2 equal size group n, now we can choose k student from the first and $n-k$ from the second

$$\binom{n}{k}\binom{n}{n-k}$$

because we don't care about the order and want to cover all the sub-group sizes from each one of them, we will some up all the sub-group combination and get

$$\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}^2$$

1.3 Problem 3.

Combinatorial proof of the identity. suppose some guy lets call him Schrödinger want to place $r+1$ cats in $n+1$ boxes one in each box (otherwise they will fight).he can do so while assuming all of them identical ,in total of

$$\binom{n+1}{r+1}$$

different ways.
but for some reason our guy dont like to do things in the normal way ,he want to put the "boxes inside the cats" he claim that we can look at the first box if we decide to put cat inside we left with r cats to split in the rest n boxes, on the other hand if we decide to not put cat inside it, we will have $r+1$ cats to split in the rest boxes, hence

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$$

now lets look at the case he decide to leave the box empty and we still left with $r+1$ cats to drop in n box and follow the same proses.

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1} = \binom{n}{r} + \binom{n-1}{r} + \binom{n-1}{r+1}$$

we can now follow the same prosses until we have $r+1$ cats to place in $r+1$ box, witch is equal 1.

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{r+1}{r} + \binom{r+1}{r+1}$$

now for some m lets look at the first and secound binomial coefficients of it

$$\binom{m}{1} = 1, \binom{m}{2} = (m^2 - m)1/2$$

hence for $r = 2$ we will get the following equation

$$2\binom{n+1}{2+1} = 2\sum_{k=1}^{n-2}\binom{n-k}{2} = \sum_{k=1}^{n}(k^2 - k) = \sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k$$

$$\Rightarrow 2\binom{n+1}{2+1} + \frac{n^2+n}{2} = \frac{n+3n^2+2n^3}{6} = \sum_{k=1}^{n}k^2$$

For general k lets use again the idea described above (with the cats), at first we will the smallest coefficients(1) and the secound will be r and so one while $n > r + 1$, so for general k we will find linear formula that nullify the coefficients of the formula polynomial

1.4 Problem 4

1.4.1

claim 1.1. $c(p, k) \equiv 0$*(mod p) when* $0 < k < p$ *is prime*

Proof. first look at $c(n, k)$ binomial form

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} = p\frac{(p-1)!}{k!(p-k)!}$$

under the assumption that p is the grateset prime that divde the following equation, so in total we get

$$0 \text{ mod p}$$

□

and from the following claim we can immediately get

$$(1+x)^p = 1 + c(x, p-1) + x^p \equiv 1 + x^p(modp)$$

Theorem 1.2. *Fermat's Little Theorem (FLT).* $b^p \equiv b$ *(mod p) for any prime p and* $b \in F_p$

Proof. At the base case for $b = 0 \Rightarrow b^k = b$ witch hold $b^k \equiv b(\text{mod p})$
now using the claim above and our indication for some $b - 1$, we get

$$b^p = (1 + (b-1))^p \equiv 1 + (b-1)^p \text{MOD P} \Rightarrow b^p = 1 + (b-1)^p = 1 + b - 1 \equiv b \text{MOD P}$$

because $(b-1)^p \equiv b - 1$ mod p □

1.4.2

lets look at the following formula as the Multinomial theorem
"is a multinomial coefficient. The sum is taken over all combinations of nonnegative integer indices k1 through km such that the sum of all ki is n. That is, for each term in the expansion, the exponents of the xi must add up to n. Also, as with the binomial theorem, quantities of the form x0 that appear are taken to equal 1 ."
(from wikipedia)

$$\left(\sum_{i=1}^{m} x_i\right)^n = \sum_{\sum_{i=1}^{m} k_i=n} \binom{n}{k_1, k_2, k_3....., k_m} \prod_{i=1}^{m} x_i^{k_i}$$

Theorem 1.3. *Fermat's Little Theorem (FLT). second proof*

Proof. The summation above is summing over all sequences of nonnegative integers, lets express α such as $1 \leq \alpha \leq p-1$ as a sum of
1_s indicators to the power of $(1_1 + 1_2 + 1 + \dots 1_\alpha)^p$,we will get

$$\alpha^p = \sum_{k_1,k_2,k_3.....,k_\alpha} \binom{p}{k_1, k_2, k_3....., k_\alpha}$$

for prime p and $k_j \neq p$ for any j, we have

$$(\text{ MOD P }) \; 0 \equiv \sum_{k_1,k_2,k_3.....,k_\alpha} \binom{p}{k_1, k_2, k_3....., k_\alpha}$$

on the other hand for prime p and some $k_j = p$, we have

$$(\text{ MOD P }) \; 1 \equiv \sum_{k_1,k_2,k_3.....,k_\alpha} \binom{p}{k_1, k_2, k_3....., k_\alpha}$$

from the way we express α we know there is exactly α of this k_j witch hold the theorem □

1.4.3

Lets p be any prime $p \neq 2$,for round a carousel of p chairs we looking for the different colouring way using b colors , first if all the chairs apper in a row we looking in total of b^p different colouring. there are b ways of colouring with the same colour , so we can claim now there is $b^p - b$ ways to colouring chairs using at least 2 different colors. hence for prime p and some b using the "FLT" theorem we now that there is total p ways to route this carousel i.e $b^p - b \equiv 0$ mod p.
and in total including the ways of couriering with only one color we get total

$$b + \frac{b^p - b}{p}$$

distinct ways of painting the chairs.

1.5 Problem 5

For n integers $a_1, a_2, \dots an$, not necessarily distinct, lets look at the following n integers

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \cdots + a_n.$$

now lets devide the by n

$$\frac{a_1}{n}, \frac{a_1 + a_2}{n}, \frac{a_1 + a_2 + a_3}{n}, \dots, \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

lets look at the i reminisces for each one of them such as $0 \leq r_i \leq n - 1 \forall i \in n$ we can look at the following as

$$m_1 n + r_1, m_2 n + r_2, \dots, m_n n + r_n.$$

if one of the r_i is 0 we done.
otherwise according to the Pigeon-hole principle three is some $r_i = r_j$ lets say that $i < j$ so by reducing them we will get

$$\sum_{k=1}^{j} a_k - \sum_{k=1}^{i} a_k = n(m_j n + r - m_i n - r) = n(m_j n - m_i n + 0) = n(\frac{a_{i+1} + a_{i+2} \cdots + a_j}{n}) = \sum_{k=i+1}^{j} a_k$$

and we find him.

2 Assignment 2

2.1 Problem 1

claim. ***1*** *For any* $1 \leq k \leq n$ *and* $0 < x < 1$

$$\binom{n}{k} x^k \leq (1 + x)^n \leq e^{xn}$$

Proof. First lets notice that for $\forall x, k$ $x^k > 0$.now using the newton binomial we can get the following

$$(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \dots \underbrace{\binom{n}{k} x^k}_{\text{part of the sum}} + \binom{n}{k+1} x^{k+1} + \dots \binom{n}{n} x^n$$

because each one of the sum's element is non negtive the following hold

$$\binom{n}{k} x^k \leq (1 + x)^n$$

using Bernoulli's Inequality, for $\forall n \in N$ and $x > 0$

$$0 < 1 + x \leq \left(1 + \frac{x}{n}\right)^n \xrightarrow[n \to \infty]{} e^x$$

we can raise both side in power of n and we will get the complete formula

$$\binom{n}{k}x^k \leq (1+x)^n \leq e^{xn}$$

□

claim 2.1. *2 For any* $1 \leq k \leq n$ *and* $0 < x < 1$

$$\binom{n}{k} \leq (\frac{en}{k})^k$$

Proof. if k=n we Instantly get using the result of claim 1.

$$1 = \binom{n}{n} \leq (\frac{en}{n})^n = e^n$$

now for $k < n$ and using above inequality lets set $0 < x = \frac{k}{n} < 1$

$$\binom{n}{k}(\frac{k}{n})^k \leq e^{\frac{k}{n}n} \Rightarrow \binom{n}{k} \leq (\frac{n}{k})^k e^k = (\frac{en}{k})^k$$

□

claim 2.2. *3 For any* $1 \leq k \leq n$ *and* $0 < x < 1$

$$\sum_{i=0}^{k}\binom{n}{i} \leq (\frac{en}{k})^k$$

Proof. using the same way as above and claim 1 lets
$k < n$ set $0 < x = \frac{k}{n} < 1$

$$\sum_{i=1}^{n}\binom{n}{i}(\frac{k}{n})^i \leq (1+\frac{k}{n})^n = \sum_{i=0}^{n}\binom{n}{i}(\frac{k}{n})^i \leq e^{n\frac{k}{n}} = e^k$$

divide by $(\frac{k}{n})^k$

$$\sum_{i=0}^{n}\binom{n}{i} \leq \sum_{i=1}^{n}\binom{n}{i}(\frac{k}{n})^{i-k} \leq e^k(\frac{n}{k})^{-k} = (\frac{en}{k})^k$$

the first inequality holds because for $i \leq k$ we get $i - k < 0$ □

2.2 Problem 2.

Theorem 2.3. *For all* $n \geq 2$, $nlogn - n < log(n!) < nlogn$ *i.e* $\ln(n!) = \Theta(nln(n))$

Proof. First we note that $\ln(n!) = \sum_k^n lnk$ and using Riemann sum approximation, and the fact that ln is a non-decreasing function on $[1, \infty)$,for all $x \in [k, k+1)$ Integrating we get

$$\ln(k) \leq ln(x) \leq \ln(k+1)$$

$$\int_k^{k+1} \ln(k)dx \leq \int_k^{k+1} \ln(x)dx \leq \int_k^{k+1} \ln(k+1)dx.$$

Summing for k between 1 and n-1, we get

$$\sum_{k=1}^{n-1} \ln(k) \leq \sum_{k=1}^{n-1} \int_k^{k+1} ln(x)dx = \int_1^n \ln(x)dx \leq \sum_{k=1}^{n-1} \ln(k+1) = \sum_{k=2}^{n} \ln(k)$$

adding ln(1),ln(n)

$$\int_1^n \ln(x)dx + \ln(1) + ln(n) \leq \sum_{k=1}^{n-1} \ln(k) + \frac{ln(n)}{2} - \ln(1) \leq \int_1^n \ln(x)dx + \ln(n).$$

hence for

$$\int_1^n \ln x dx = x \ln x - x|_1^n = n \ln n - n + 1$$

we get

$$n \ln(n) + \frac{ln(n)}{2} - n + 1 \leq \sum_{k=1}^{n} \ln k \leq n \ln -n + \frac{3\ln(n)}{2} + 1$$

□

using the theorem above lets add to the of power e

$$\exp(n \ln n - n + 1 + \frac{\ln(n)}{2}) \leq \exp(\ln(n!)) \leq \exp(\frac{3\ln(n)}{2} + n \ln n - n + 1)$$

$$\Leftrightarrow n! = \Theta(\sqrt{n} e (\frac{n}{e})^n)$$

2.3 Problem 3.

2.3.1

Let q(n) denote the number of ordered sets of positive integers whose sum is n, lets define Q such that Q is sequence size n of 1's .

$$Q : 1\nabla 1\nabla 1\nabla 1\nabla \ldots \nabla 1$$

in total we looking at n times 1 and n-1 ∇. now lets say we have 2 operators $\{+, |\}$ we can replace each time ∇ with ine of them, if we choose the + we "merge" both of the sums, but if we choose | we "slice" the set.hence the sum of Q will always stay n and each unique decision of choosen operator in order will give us unique ordered sets of positive, we can choose 2 operators total n-1 times, hence the number of ordered sets is

$$q(n) = 2^{n-1}$$

2.3.2

using the group Q define above, to find all the ordered sets size k hows sum is n, we can say that now we must replace k-1 of ∇ with $|$, witch leaves us with total of k sets, and all the rest ∇ will get the + operator immediately. in total we have k-1 of $|$ to rplace k-1 operator of ∇ from total $n-1\nabla$,and by suuming all the option over k sizes of group we get.

$$\sum_{k=1}^{n-1}\binom{n-1}{k-1}=\sum_{k=0}^{n}\binom{n-1}{k}=2^{n-1}=q(n)$$

2.4 Problem 4

Let p(n) denote the number of unordered sets of positive integers whose sum is n. lets define $p_k(n)$ to be number of unordered sets of size k of positive integers whose sum is n.
using the result of problem 3 we know that for ordered set size k we have $\binom{n-1}{k-1}$ option.if we looking at k different element we will have total of.

$$p_k(n)=\frac{\binom{n-1}{k-1}}{k!}$$

but we might have some repeat numbers so its will be at most

$$p_k(n)\geq\frac{\binom{n-1}{k-1}}{k!}$$

since we define q(n) s.t $p(n)=\sum_k p_k(n)$ the following hold.

$$p(n)=\sum_k p_k(n)\geq \max_{1\geq k\geq n}\frac{\binom{n-1}{k-1}}{k!}$$

claim 2.4. *there is an absolute constant $c>0$ for which $p(n)\geq e^{c\sqrt{n}}$*

Proof.

$$p(n)=\geq \max_{1\geq k\geq n}\frac{\binom{n-1}{k-1}}{k!}\geq\frac{\binom{n-1}{k-1}}{k!}=\frac{1}{k!}\frac{k}{n}\binom{n}{k}$$

using (*)$\binom{n}{k}\geq(\frac{n}{k})^k$ (**)$k!\geq ek(\frac{n}{k})^k$,

$$\frac{1}{k!}\frac{k}{n}\binom{n}{k}\geq\underbrace{\frac{1}{ek(\frac{n}{k})^k}}_{**}\underbrace{(\frac{n}{k})^k}_{*}\frac{k}{n}$$

for $k=\sqrt{n}$

$$\frac{1}{e\sqrt{n}(\frac{n}{\sqrt{n}})^{\sqrt{n}}}(\frac{n}{\sqrt{n}})^{\sqrt{n}}\frac{\sqrt{n}}{n}=\frac{e^{\sqrt{n}}}{en}=\frac{e^{\sqrt{n}}}{e^{1+\ln(n)}}$$

using the detention of limit ,for some $1>\epsilon>0$

$$\frac{1+\ln(n)}{\sqrt{n}}\xrightarrow{n\to\infty}0\Rightarrow\frac{1+\ln(n)}{\sqrt{n}}<\epsilon\Rightarrow 1+\ln(n)>\epsilon\sqrt{n}$$

hence for $c = 1 - \epsilon$,$c > 0$

$$\frac{e^{\sqrt{n}}}{e^{1+\ln(n)}} \leq \frac{e^{\sqrt{n}}}{e^{\epsilon\sqrt{n}}} = e^{\sqrt{n}} - e^{\epsilon\sqrt{n}} = e^{c\sqrt{n}}$$

$\square$

2.5 Problem 5.

Let $\pi(m, n)$ denote the set of prime numbers in the interval $[m, n]$.

2.5.1

we can see the following $[m, 2m]$

$$\{m+1, m+2, ..., 2m\}$$

now lets partition it to prime and and non prime element s.t

$$\{m+1, m+2, ..., 2m\} \setminus \pi(m+1, 2m) = c(m+1, 2m)$$

$$n \in c(m+1, 2m) \leftrightarrow \{n \in [m, 2m] \vee n \text{ is not prime}\}$$

now lets look at $\binom{2m}{m}$

$$\binom{2m}{m} = \frac{2m(2m-1)...(m+1)}{m!} = \frac{1}{m!}\left(\prod_{p \in \pi(m+1,2m)} p\right)\left(\prod_{n \in c(m+1,2m)} n\right)$$

since for any $m + 1 \geq p \geq 2m, p \in \pi(m+1, 2m)$
i claim when $p > m$ thus

$$m! \nmid \left(\prod_{p \in \pi(m+1,2m)} p\right)$$

and we get

$$\binom{2m}{m} = \underbrace{\frac{\left(\prod_{n \in c(m+1,2m)} n\right)}{m!}}_{\geq 1}\left(\prod_{p \in \pi(m+1,2m)} p\right) \geq \left(\prod_{p \in \pi(m+1,2m)} p\right)$$

2.5.2

first lets notice that

$$4^n = 2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k} > \binom{2n}{n},$$

$$\text{and } 2 \cdot 2^{2n+1} > \binom{2n+1}{n} \Rightarrow \binom{2n+1}{n} \leq 2^{2n}$$

since its apper twice in the binomial coefficient, so both at scenario (even,odd) using the floor will give us bound for the given binom

$$\left(\prod_{p \in \pi(\lfloor m/2 \rfloor + 1, 2m)} p \right) \leq \binom{m}{\lfloor m/2 \rfloor} \leq 2^m$$

now lets use the floor function we can notice that

$$\lfloor \lceil m/2^{2k} \rceil / 2^k \rfloor = \lfloor m/2^{3k} \rfloor$$

hence for $2m, m, m/2$

$$\left(\prod_{p \in \pi(\lfloor m/4 \rfloor + 1, \lceil m/2 \rceil)} p \right) \left(\prod_{p \in \pi(\lceil m/2 \rceil + 1, m)} p \right) \leq \binom{m}{\lfloor m/2 \rfloor} \binom{\lceil m/2 \rceil}{\lfloor m/4 \rfloor} \leq 2^m 2^{\lfloor m/2 \rfloor}$$

we can apply it for all m,$\lceil m/2 \rceil$, $\lceil m/4 \rceil$...

$$\left(\prod_{p \in \pi(1,m)} p \right) = \left(\prod_{p \in \pi(0+1,1)} p \right) \cdots \left(\prod_{p \in \pi(\lceil m/2 \rceil + 1, m)} p \right) \leq \tag{1}$$

$$\binom{m}{\lfloor m/2 \rfloor} \binom{\lceil m/2 \rceil}{\lfloor m/4 \rfloor} \binom{\lceil m/4 \rceil}{\lfloor m/8 \rfloor} \cdots \leq 2^m \cdot 2^{\lfloor m/2 \rfloor} \cdot 2^{\lfloor m/4 \rfloor} \cdots \tag{2}$$

$$\leq 2^{m + \lfloor m/2 \rfloor + \lfloor m/4 \rfloor + \cdots} \leq 2^{m(1+1/2+1/4+\cdots)} \leq 2^{2m} = 4^m$$

2.5.3

using line (1),(2) we sow above

$$\log \left(\prod_{p \in \pi(1,n)} p \right) < \log(4^m) \Rightarrow O(2n \log 2)$$

$$\log \left(\prod_{p \in \pi(1,n)} p \right) = \sum_{I = (\lceil 2i/i \rceil + 1, i) i \in \pi(1,n)} \log \left(\prod_I p \right) \leq \sum_{I(\lceil 2i/i \rceil + 1, i) i \in \pi(1,n)} \log \underbrace{\binom{i}{\lfloor i/2 \rfloor}}_{2^n \leq \binom{n}{2n}}$$

since $2^n \leq \binom{n}{2n}$ and the result from sector 5.1 we can bound the following.

$$\prod_{I = (\lceil 2i/i \rceil + 1, i) i \in \pi(1,n)} \binom{i}{\lfloor i/2 \rfloor} < \left(\prod_{p \in \pi(1,n)} p \right) \leq (4^n)$$

log both sides

$$\log(\prod_{I=(\lceil 2i/i\rceil+1,i)i\in\pi(1,n)} \binom{i}{2i}) \leq |\pi(1,n)|\log(2^{\lg(n)\log 2}) < \log(4^n)$$

and we finality get

$$|\pi(1,n)|\lg(n)\log 2 < 2n\log 2 \Leftrightarrow |\pi(1,n)| = O(\frac{n}{\log(n)})$$

2.6 Problem 6.

claim 2.5. *Every tournament T of order $|V| = 2^k$ contains an undominated set of size $\leq k$.*

Proof. the base case of the induction is trivial for $k = 1, 2$ lets assume the hypothesis hold for some for 2^k, now lets look at $\hat{T}$ of order $|V| = 2^{k+1}$ lets look at the avarge $\deg_{out}$ i.e

$$\frac{|E|}{|V|} = \frac{2^{k+1}(2^{k+1}-1)}{2 * 2^{k+1}} = 2^k - \frac{1}{2}$$

hence exist some $v \in |V|$ such that $v_{\deg_{out}} \geq 2^k \Rightarrow v_{\deg_{in}} < 2^k$. now lets choose some $2^k = |S|, \{S : S \subseteq V\}$ such that v dominated by any $v_s \in S$. lets apply our induction assumption on sub-tournament S, since exist $\hat{S} \subseteq S$ size $|\hat{S}| \leq k$ that not dominated by any other vertex$\Rightarrow$ $|\hat{S} \cup v| \leq k+1$ sub-set size $k+1$ that not dominated in $|T| = 2^{k+1}$

□

now lets look at some random tournament T that any $e \in |E|$ have the same probability to be in each direction

$$\Pr(e : u \to v) = \Pr(e : v \to u) = 1/2$$

$\Rightarrow$ the probability that v is dominates on some u is 1/2
$\Rightarrow$ the probability v is dominates on $S \subseteq V$ size k is $1/2^k$
$\Rightarrow$ the probebilty that e dominated by some $|S| = k$ is $(1 - 1/2^k)$
$\Rightarrow$ the probebilty that $|T/S| = n-k$ dominated by some $|S| = k$
is $(1 - 1/2^k)^{n-k}$
the expected number of group size k can bound from above with $\binom{n}{k}$, hence when n holds

$$\binom{n}{k}(1 - 1/2^k)^{n-k} < 1$$

then there is an n-vertex tournament so that every set of k vertices is dominated.
now lets use the property proved in Q(1) and we can bound the following for any $k \geq 2$

$$\binom{n}{k}(1-1/2^k)^{n-k} \leq \underbrace{e^{-\frac{(n-k)}{2^k}}}_{Q1 \text{ and } 1-k\leq e^k} \underbrace{\left(\frac{en}{k}\right)^k}_{\leq\binom{n}{k}} < 1$$

hence for $n > k + 2^k \cdot k^2$ the following hold .

3 Assignment 3

3.1 Problem 1.

claim. *There is an integer n_0 such that for any $n \geq n_0$, in every 9-coloring of the integers $\{1, 2, 3, \ldots, n\}$, one of the 9 color classes contains 4 integers a, b, c, d such that $a+b+c = d$.*

Proof. based on Ramsay Theorem Let $n_0 = K(4, \ldots, 4)$, where 4 appears $k-1$ times. and lets c be r-colouring s.t:

$$c : \{1, \ldots, n\} \to \{1, \ldots, k\}$$

For graph K_n and labelling of its edge $\{1, \ldots, n\}$. we can colour any edge e_{ij} with $c(|i-j|)$. we got a $k-1$-colouring of K_n. then for n_0, we must have a K_4 with all edges different. for vertices $x \leq y \leq z \leq w$ then

$$a = y - x, b = z - y, c = w - z, d = w - x$$

Gives a solution □

3.2 Problem 2.

claim. *every tournament on n vertices, contains a transitive tournament on $\lfloor \log_2 n \rfloor$ vertices.*

Proof. Using induction for $n = 0, 1, 2$ its holds on empty. W.L.O.G[1] assume the claim holds for $n \leq 2^k$ now lets look at some tournament on 2^{k+1} vertices and we can pick any vertex v, and define:

$$v_{in} = \{u : \text{ exsit edege } v \leftarrow u\}, v_{out} = \{u : \text{ exsit edege } v \to u\}$$

Hence $|v_{in}| + |v_{out}| = 2^{k+1} - 1$ and one of them contain $|2^k|$ edges, lets assume its v_{in}[2] by our assumption its contains transitive tournament T_{in} size $|k|$. now $T_{in} \cup \{v\}$ is sub tournament and any edge points to v hence its transitive tournament on $|k+1|$ vertices. □

claim. *there exists a tournament on n vertices that does not contain a transitive tournament on $2 \log_2 n + 2$ vertices.*

Proof. The number of Tournament on n vertices is $2^{\binom{n}{2}}$.The number of tournaments of size k is $k!$, and there are $\binom{n}{k}$ sets of size k, and the number of ways to choose the edges outside the transitive tournament is $2^{\binom{n}{2}-\binom{k}{2}}$. hence if we show that

$$k!\binom{n}{k}2^{\binom{n}{2}-\binom{k}{2}} < 2^{\binom{n}{2}}$$

its yield that for some k the number of n-vertex tournaments with a transitive subtournament on k vertices is smaller than the total number of tournaments.

[1] we can modify any other tournament to to nearst power of 2 its will still hold for $\lfloor \log_2 n + 1 \rfloor$ see (10)

[2] its equivalence for v_{out}

$$
\begin{aligned}
2^{\binom{n}{2}} &> k!\binom{n}{k}2^{\binom{n}{2}-\binom{k}{2}} && (3)\\
2^{\binom{k}{2}} &> k!\binom{n}{k} && (4)\\
&> k!\frac{n!}{k!(n-k)!} && (5)\\
&> n(n-1)(n-2)\cdots(n-k+1) && (6)\\
&> n^k && (7)
\end{aligned}
$$

Taking $\log_2$ from (3)(7)

$$
\begin{aligned}
\binom{k}{2} &> k\log_2(n) && (8)\\
\frac{k!}{2(k-2)!} &> k\log_2(n) && (9)\\
\frac{k!}{k(k-2)!} &> 2\log_2(n) && (10)\\
k-1 &> 2\log_2(n) && (11)\\
k &> 2\log_2(n)+1 && (12)
\end{aligned}
$$

□

3.3 Problem 3

claim. *if an n-vertex graph $G=(V,E)$ has no copy of $K_{2,t}$*[3] *then*

$$|E| \le \frac{1}{2}(\sqrt{t-1}n^{\frac{3}{2}}+n)$$

Proof. W.L.O.G let $t \ge 1$. we can distinguish that any $e_1, e_2 \in E$ have at most[3] t neighbours. and each one of them can be part of pair. we can consider it as the number of path length 2 in G. Let $d(v_i)$ be the deg of $v_i \in G$ and we get that:

$$t\binom{n}{2} \ge \sum_{v\in V}\binom{d(v)}{2} \ge n\binom{2|E|/n}{2} \tag{13}$$

The right-hand side hold from Jensen's Inequality and since its minimized[4] the binomial when all the degrees are equal, $d_i = 2|E|/|V|$.

[3] I will use $t+1$ for the proof i.e $K_{2,t+1}$
[4] convex property

$$n\binom{2|E|/n}{2} = n\frac{(2|E|/n)(2|E|/n-1)}{2} \geq n\frac{(2|E|/n-1)^2}{2} \tag{14}$$

And

$$t\binom{n}{2} = t\frac{n^2-n}{2} \leq t\frac{n^2}{2} \tag{15}$$

.We conclude from (13)(14)(15) that

$$\begin{aligned}
n\frac{(2|E|/n-1)^2}{2} &\leq t\frac{n^2}{2} && (16)\\
(2|E|/n-1)^2 &\leq tn && (17)\\
2|E|/n &\leq \sqrt{tn}+1 && (18)\\
|E| &\leq^3 \frac{1}{2}(\sqrt{t}n^{\frac{3}{2}}+n) && (19)
\end{aligned}$$

□

3.4 Problem 4

claim. *Let $S_1, \ldots, S_n \in [n]$ such that $|S_i \cap S_j| \leq 1$ for all $1 \leq i < j \leq n$ then.*

$$\frac{1}{n}\sum_{i=1}^{n}|S_i| = O(\sqrt{n})$$

Proof. Let define $G = (V, E)$ such that

$$S = \{S_i : S_i \in [n]\}, U = \{i \in n\} \text{ and } E = \{e_{S_k,m} : m \in S_k\}, V = S \cup U$$

Its immediate $|V| = 2n$ and G is Bipartite since we can dived V into 2 disjoint independent sets S and U, that is any $e \in E$ connects a vertex in S to one in U. hence G has no copy of $K_{2,2}$, using **Problem 3** we can get that

$$\begin{aligned}
|E| &\leq \frac{1}{2}(\sqrt{2-1}(2n)^{\frac{3}{2}}+2n) && (20)\\
\sum_{i=1}^{n}|S_i| &\leq \sqrt{2}n^{\frac{3}{2}}+n && (21)\\
\frac{1}{n}\sum_{i=1}^{n}|S_i| &\leq \sqrt{2}\sqrt{n}+2 && (22)\\
\frac{1}{n}\sum_{i=1}^{n}|S_i| &= O(\sqrt{n}) && (23)
\end{aligned}$$

□

3.5 Problem 5

Theorem. *if* $G = (V, E)$ *has no copy of* K_{t+1} *then* $|E| \leq (1 - \frac{1}{t})\frac{n^2}{2}$.
(Turan's Theorem)

Proof. Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and f to be vector and weight function satisfying

$$\forall i\ 0 < x_i \leq 1, \sum_{i=1}^{n} x_i = 1, f(x) = \sum_{i,j \in E} x_i x_j$$

By taking $x = (\frac{1}{n}, \ldots, \frac{1}{n})$ we get

$$f(x) \geq \sum_{i,j \in E} \frac{1}{n^2} \geq \frac{|E|}{n^2} \tag{24}$$

The "weight shifting" method yield to shift the weight between any neighbours x_i, x_j if $e_{i,j} \notin E$.
We can notice that the sum is maximized when all the weight is concentrated on a clique. Since any shift is does not decrease the value of f we can repeat the processes. since $G = (V, E)$ has no copy of K_{t+1} we can have at most t size clique ,let name it $[T]$. we can get lower bound on f :

$$\begin{aligned}
f(x) &\leq \sum_{i,j \in [T]} x_i x_j = \sum_{i,j \in [T]} \frac{1}{t^2} && (25) \\
&\leq \frac{t(t-1)}{2} \frac{1}{t^2} && (26) \\
&\leq (1 - \frac{1}{t}) \frac{1}{2} && (27)
\end{aligned}$$

Combining (27) and (24) to finish the proof

$$\begin{aligned}
\frac{|E|}{n^2} &\leq (1 - \frac{1}{t}) \frac{1}{2} && (28) \\
|E| &\leq (1 - \frac{1}{t}) \frac{n^2}{2} && (29)
\end{aligned}$$

□

4 Assignment 4

4.1 Problem 1.

Proposition 1. *if* $T(n) = T(n/3) + T(2n/3) + n$ *then* $T(n) = O(n \log n)$

Proof. using induction, my induction hypothesis will be

$$T(n) \leq Cn \log n, \qquad \forall n < N, 0 < C$$

Now we get that

$$\begin{aligned}
T(n) &= T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n && (30)\\
&\leq C\frac{n}{3}\log\frac{n}{3} + C\frac{2n}{3}\log\frac{2n}{3} + n && (31)\\
&\leq C\frac{n}{3}\log n - C\frac{n}{3}\log 3 + C\frac{n}{3}\log n - C\frac{2n}{3}\log\frac{3}{2} + n && (32)\\
&\leq Cn\log n - C\frac{n}{3}\log 3 - C\frac{2n}{3}\log\frac{3}{2} + n && (33)\\
&\leq^{(*)} Cn\log n && (34)
\end{aligned}$$

(31) imply the induction hypothesis, and (34) will hold by choosing C s.t

$$n \leq C\frac{n}{3}\log 3 + C\frac{2n}{3}\log\frac{3}{2}$$

Hence for any $n \geq 1$ we can choose C such that

$$C \geq \frac{1}{\frac{1}{3}\log 3 + \frac{2}{3}\log\frac{3}{2}} \approx 1.578$$

We get that

$$T(n) = O(n \log n)$$

□

Proposition 2. *if* $T(n) = 2T(n/2) + n\log n$ *then* $T(n) = O(n\log^2 n)$*.*

Proof. using induction, my induction hypothesis will be

$$T(n) \leq Cn\log^2 n, \qquad \forall n < N, 0 < C$$

for $n = 2$

$$\begin{aligned}
T(2) &= 2T\left(\frac{2}{2}\right) + 2\log 2 && (35)\\
&\leq 2C + n\log n && (36)\\
&= O(n\log^2 n) && (37)
\end{aligned}$$

Hence for any $n \geq 2$

$$
\begin{aligned}
T(n) &= 2T(n) + n\log n && (38)\\
&\leq 2\log^2(\frac{n}{2})\frac{n}{2} + n\log n && (39)\\
&= O(n\log^2 n) && (40)
\end{aligned}
$$

(39) holds since $n > \frac{n+1}{2}$ $\square$

Proposition 3. *Let $c_1, \ldots, c_k$ be k positive reals satisfying $\sum_{i=1}^k c_i < 1$. if $T(n) = \sum_{i=1}^k T(c_i n) + n$ then $T(n) = O(n)$.*

Proof. by induction, my induction hypothesis will be

$$T(n) \leq Cn, \qquad \forall n < N, 0 < C$$

Its immediate that $T(0) \leq 0$ now lets

$$
\begin{aligned}
T(n) &= \sum_{i=1}^{k} T(c_i n) + n && (41)\\
&\leq \sum_{i=1}^{k} C(c_i n) + n && (42)\\
&\leq Cn\left(\sum_{i=1}^{k} c_i + \frac{1}{C}\right) && (43)\\
&\leq^{(*)} Cn && (44)
\end{aligned}
$$

(42) imply the induction hypothesis, and (44) will hold by choosing C s.t

$$C \geq \frac{1}{1 - \sum_{i=1}^k c_i}$$

Hence for any $n \geq 1$ (44) holds and we get that

$$T(n) = O(n)$$

$\square$

4.2 Problem 2.

Theorem. *Every tournament has a Hamilton path*

Proof 1. Let $T = (V, E)$ be tournament graph where $|V| = n$. using strong induction for $n \leq 2$ its immediate that hamiltonian path exists. now lets assume its hold for any $k < n$. lets choose some $v \in V$ and define 2 sets such that

$$V_{in} = \{u : \overrightarrow{(u, v)} \in E\}, V_{out} = \{u : \overrightarrow{(v, u)} \in E\}$$

Since $|V_{in}| < n, |V_{out}| < n$ by the induction hypothesis exists paths
$P_{in} \in V_{in}, P_{out} \in V_{in}$ such that P_{in} and P_{out} are hamiltonian paths. now the path $P_{in} \to v \to P_{out}$ is hamiltonian path for all vertices
$|V_{in}| \cup |v| \cup |V_{out}| = n$ □

Proof 2. First we can notice that $\chi(T) =$[5]n since any 2 vertex connected with an edge . Since $\chi(T) \leq |P|$ where P is the longest simple path in T. on the other hand its can be at most n since T have n vertex . we get that $|P| = n$.

Hence since P is simple path i.e its visit any vertex of the T exactly once, and P visit all the vertex or in other worlds P is Hamilton path in T □

4.3 Problem 3.

claim. *Any set X of $st + 1$ integers contains one of the following:*

- *A subset $T = \{x_1, \ldots, x_{t+1}\} \subseteq X$ of size $t + 1$ such that x_k divides x_{k+1} for every $1 \leq k \leq t$.*
- *A subset $S = \{x_1, \ldots, x_{s+1}\} \subseteq X$ of $s + 1$ integers such that x_i does not divides x_j for every $x_i, x_j \in S$.*

Proof. Consider the following Poset define by

$$\mathcal{P} =^{6} \{X, \langle x_1, x_2 \rangle : x_1 | x_2\}$$

Now lets look at $\mathcal{P}$ over X , first notice that when its have chain size $|t+1| \Rightarrow$ exists sequence of $x_1 \preceq \cdots \preceq x_{t+1}$ such that $x_1 | x_2 \ldots | x_{t+1} \Rightarrow$ exist $T \subseteq X$.
on the other hand if $\mathcal{P}$ over X , have anti-chain size $|s+1| \Rightarrow$ exist sequence of $x_1 \npreceq \cdots \npreceq x_{s+1}$ such that $x_1 \nmid x_2 \nmid \cdots \nmid x_{s+1} \Rightarrow$ exist $S \subseteq X$. Since

$$\omega(X)\alpha(X) \geq \omega(X)\frac{|X|}{\mathcal{X}(X)} \geq^{7} |X|$$

its following that splinting X into $\mathcal{X}(X)$ anti-chains, one of them will be at size $\frac{|X|}{\mathcal{X}(X)}$. if $\alpha(X) \geq s + 1$ then $S \subseteq X$, else $\alpha(X) \leq s$ and

$$\omega(X) \geq \frac{|X|}{\alpha(X)} \leq \frac{st+1}{s} = t + \frac{1}{s}$$

and we get that $\omega(X) \geq t + 1 \Rightarrow T \subseteq X$ □

[5] T is graph on n vertex hence its can be colored by at most n different colours
[6] by inculding $\langle 0, 0 \rangle$ as well
[7] Mirsky

4.4 Problem 4.

Consider the following Poset define by

$$\mathcal{P} = \{\mathscr{F}, \langle S_1, S_2 \rangle : S_1 \subseteq S_2\}$$

where $\mathscr{F}$ is collection $\mathscr{F} = \{S_1, \ldots, S_n\}$ of n sets.

Proposition 4. *both chain and anti-chain of $\mathcal{P}$ are union-free sets*

Proof. first by noticing that when its have chain $\Rightarrow$ exists sequence of $S_1 \preceq \cdots \preceq S_k$ such that $S_1 \subseteq S_2 \cdots \subseteq S_k$ let mark this set of element as $\mathcal{S}_{chain}$. lets assume that exsit some $S_i, S_j, S_k \in \mathcal{S}_{chain}$ s.t $S_i \cup S_j = S_k$,we can that hold only when $|S_i|, |S_j| \leq |S_k|$ but the Poset yield $S_i \preceq S_k$ and $S_j \preceq S_k$ hence $S_i = S_k$ or $S_j = S_k$ which lead to contradiction since $\mathcal{S}_{chain}$ is set.

Define $\mathcal{S}_{anti-chain}$ such that

$$\mathcal{S}_{anti-chain} = \{S_i, S_j : S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i \quad \text{s.t } S_i, S_j \in \mathscr{F} \quad \forall i, j\}$$

By assuming that exsit some $S_i, S_j, S_k \in \mathcal{S}_{anti-chain}$ s.t $S_i \cup S_j = S_k$. its followed that $S_i \subseteq S_k$ but $S_i \nsubseteq S_j$ and we get an contradiction. $\square$

claim. *every collection $\mathscr{F} = \{S_1, \ldots, S_n\}$ of n sets contains a sub-collection $S \subseteq \mathscr{F}$ of at least $\sqrt{n}$ sets which is union-free*

Proof. let $\alpha(\mathcal{P})$ be the longest $anti - chain$ of $\mathcal{P}$ over $\mathscr{F}$. If $\alpha(\mathcal{P}) \geq \sqrt{n}$ then exist such $S \in \mathscr{F}$. and if $\alpha(\mathcal{P}) < \sqrt{n}$

$$\frac{n}{\omega(\mathcal{P})} = \frac{|\mathscr{F}|}{\omega(\mathcal{P})} \leq \alpha(\mathcal{P}) < \sqrt{n}$$

$$\omega(\mathcal{P}) \geq \sqrt{n}$$

And again by proposition 4 we get that exist such $S \in \mathscr{F}$

$\square$

4.5 Problem 5

claim. *for a finite poset $\mathcal{P}$ and let x, y be two elements of $\mathcal{P}$ that are incomparable under $\mathcal{P}$. then $\mathcal{P}$ has a linear extension in which $x < y$.*

Proof. Let $\mathcal{P} = (X, \preceq)$ be a finite partial order in which $x, y \in X$ are incomparable. now lets define new post $\hat{\mathcal{P}} = (X, \hat{\preceq})$

$$\hat{\preceq} = \begin{cases} w \hat{\preceq} z & \text{if } w \preceq z \\ w \hat{\preceq} z & \text{if } z \preceq y \wedge x \preceq w \\ y \hat{\preceq} x & \end{cases}$$

- $\hat{\preceq}$ is reflexive since $\preceq$ is reflexive
- $\hat{\preceq}$ is Transitive since $\preceq$ is Transitive , and we apply apply only steps that respect the Transitive property of $\preceq$
- $\hat{\preceq}$ is Anti-symmetric. consider $w\hat{\preceq}z$ and $z\hat{\preceq}w$ and let assume that $w \neq z$. if $x = w$ or $y = w$ or $x = w, y = z$ its immediate lead to contradiction.
  and when $w\hat{\preceq}z \Rightarrow z \preceq y \wedge x \preceq w$ and $z\hat{\preceq}w \Rightarrow w \preceq y \wedge x \preceq z$ then $x \preceq z \preceq y$ contradiction to the fact that x, y are incomparable, hence $w\hat{\preceq}z$ and $z\hat{\preceq}w$ lead to $w = z$[8]

Hence $\hat{\mathcal{P}}$ is poset where $x\hat{\preceq}y$ and its contains less incomparable pairs than $\preceq$ does. If $\hat{\mathcal{P}}$ is linear then we done. otherwise exists some incomparable w, z and we can extend $\hat{\preceq}$ to $\hat{\preceq}_1$ and follow the proses until we cover all the chains or get some $\hat{\preceq}_k$ linear and respect $\mathcal{P}$ where $x < y$ □

4.6 Problem 6

claim. *in the setting of Arrow's Theorem, if the individuals have only two options, then they can come up with a non-dictator social choice function.*

Proof. Lets proof that the democracy/majority voting system model satisfies the 3 condition of the Arrow's Theorem when N voters choose from $|\{A, B\}| = 2$ choices.

$$F : \mathrm{S_2}^N \to \frac{\sum_i^N \mathbb{1}[S_i \text{ choose } A > B]}{N} \quad \text{i.e and indicator if } S_i \text{ prefer } A$$

if $F \geq \frac{1}{2}$ return (A, B) else (B, A)
Monotonicity For two preference profiles $R = (R_1, \ldots, R_N)$ and $S = (S_1, \ldots, S_N)$ such that both profiles prefer $A > B$ but

$$0.5 < \frac{\sum_i^N \mathbb{1}[S_i \text{ choose } A > B]}{N} < \frac{\sum_i^N \mathbb{1}[R_i \text{ choose } A > B]}{N}$$

more people support (A, B) and its yield that F is monotone[9]

Unanimity If alternative, $B < A$ for all orderings $R_1, \ldots, R_N$, $R_i = (A, B)\forall i$ then $F(R_1, R_2, \ldots, R_N) = (A, B)$ and A is ranked strictly higher than B by F. its immediate from the way we construct F

$$0.5 < \frac{\sum_i^N \mathbb{1}[R_i \text{ choose } A > B]}{N}$$

Non-dictatorship There is no individual, i whose strict preferences always prevail consider the profile define

$$R = (R_1, R_2, \ldots R_N) \quad \forall i R_i = (A, B)$$

[8]i actually miss an case but its kind of similar proof
[9]its the same idea for the monotone decrease case $B > A$

And

$$S = (S_1, S_2, \dots S_N) \quad \forall i S_i = (B, A)$$

For the 2 given profiles there is no individual who can change the result. $\square$

5 Assignment 5

5.1 Problem 1

claim. *the number of surjective mappings from [n] to [k] is given by*

$$\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n$$

Proof. denote

$$f_x = \{f : f^{-1}[C] \text{ s.t } [C] \subseteq [k], \quad |C| \leq |k - x|\}$$

to be the set of all function from $[n]$ to subset of $[k]$ where at least x element of k is not in the image of f_x. let look at f_1, we can choose 1 from k element to not be part of the image, it is $\binom{k}{1}$. now we have $k-1$ elements to choose from n items, i.e which item from $n_i \in n$ will map to $k_j \in k$. Hence we looking at total $\binom{k}{1}(k-1)$ functions. and for general x it is $|f_x| = \binom{k}{x}(k-x)^n$. now lets $f_0 = S$ to be the set of all function from $[n]$ to $[m]$, since $f_x \subseteq f_y$ for $0 \leq x \leq y \leq k$ then :

$$f_{onto} \in \bigcap_{i=1}^{k} \overline{f_i} \Rightarrow |\bigcap_{i=1}^{k} \overline{f_i}| = {}^{10}|S - \bigcup_{i=1}^{k} f_i|$$

using inclusion exclusion principle we get that.

$$\binom{k}{0}k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \cdots \pm \binom{k}{k-2}2^n \mp \binom{k}{k-1}1^n \pm \binom{k}{k}0^n$$

that is

$$\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n$$

$\square$

Proposition 5.

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i}(n-i)^n = n!$$

[10] De Morgan

Proof. $\Rightarrow$ using the result above, for $k = n$ its following that :

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^n = n!$$

$\Leftarrow$ the number of onto function from $[n]$ to $[n]$ is equivalence to to the number of ways to arrange n distinct elements in row , that is

$$n! = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^n$$

$\square$

Proposition 6.

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = 0 \quad \textit{when } k > n.$$

Proof. $\Rightarrow$ using the result above, for $k > n$ its following that :

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = 0$$

$\Leftarrow$ assume we have k pigeons, we need to find in how many ways we can place them all in n holes, when each one of them in different hole. after placing $n-k$ of them the all the holes are full and we left with $k-n > 0$ pigeons. following the Pigeonhole principle there are is-no option to do so, or equivalence to 0 ways. $\square$

Proposition 7.

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

where S(n, k) are the Stirling numbers of the second kind

Proof. $\Rightarrow$its immediate from the definition of $S(n,k)$:

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

$\Leftarrow$ we can consider the the set S_k:

$$S_k := \{\{f^{-1}(x)\}, \forall x \in k\}$$

we are looking at total of k non-empty sets. the amount of subjective function from $[n]$ to $[k]$ is number of ways to distribute the elements of n into these sets, let $S(n, S_k)$ be the number of ways to partition a set of n objects into $S_k = |k|$ non-empty subsets. now we can notice that any $k_i \in k$ can be associated with any of these sets i.e total of $k!$. and in total we get:

$$S(n,S_k)k! = S(n,k)k! = k!\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

$\square$

5.2 Problem 2

claim. *the number of ways of coloring the integers $\{1 \dots 2n\}$ using the colors red/blue in such a way that if i is colored red then so is $i-1$, is:*

$$\sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} 2^{2n-2k} = 2n+1$$

Proof. I will use counting in two ways method to deduce the identity
$\Rightarrow$ we can consider the problem as placing $2n$ items in a row and choose spot to place separator s.t any item to its left are red and all the other are blue. we looking at total of $2n-2$ in between any two adjacent numbers from 1 to $2n$. by including 2 more additional option that all of them red or blue, we get that the total of number of ways to place the separator is given by $2n+1$.
$\Leftarrow$ There are in total $S = 2^{2n}$ ways of coloring the integers. with same idea as above, we can consider the separator as choose pair of adjacent integers the first will be coloring with R and the second B and rest dont care, it is:

$$2^{2n-2} \binom{2n-1}{1}$$

now same idea for 2 paris

$$2^{2n-4} \binom{2n-1}{2}$$

and in general :

$$2^{2n-2i} \binom{2n-i}{i}$$

Using Inclusion exclusion principle we get that

$$2^{2n} - 2^{2n-2} \binom{2n-1}{1} + \dots \pm \binom{2n-n}{n} 2^{2n-2n}$$

that is :

$$\sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} 2^{2n-2k}$$

□

5.3 Problem 3

Proposition 8. *Let N be a set, then any $k \subseteq N$ have bijection such that $k \to \{0,1\}^{|N|}$*

let define the following mapping

$$f : \left\{ \; k \;\; \mapsto \;\; x \in N \mapsto \left\{ \begin{array}{ll} 0 & , \text{if } x \notin N \\ 1 & , \text{if } x \in N \end{array} \right. \right. \qquad f^{-1}\{\{0,1\}^N \mapsto \{x \in N \text{ s.t. } f(x) = 1\}$$

we can consider it as binary encode of the subset indicate $\mathbb{1}$ if the given integer in the subset and 0 otherwise

claim. *the number of subsets of size k of $\{1, \dots n\}$ which contain no pair of consecutive integers is given by* $\binom{n-k+1}{k}$

Proof. using Proposition 8. subset k can represented as some binary string of length n, its yield that if in some string have two consecutive appearances $\mathbb{1}$ then this subset contain pair of consecutive integers. moreover we can notice that if $n < 2k - 1$ then its can not contain pair of consecutive integers.

For given k let $f(k)$ define the bijection of subset k for some $n \geq 2k - 1$. if we assume its not have any consecutive numbers, then its have k $\mathbb{1}$'s and $n - k$ 0's. since we know $k - 1$ from the 0's must be followed by the first $k - 1$ of $\mathbb{1}$'s. hence the following problem becomes, how many ways could we distribute the remaining element i.e

$$n - (\underbrace{k}_{k \times \mathbb{1}'s} + \underbrace{(k-1)}_{(k-1) \times 0's}) = n - 2k + 1$$

it is $n - 2k + 1$ number of 0's in the $k + 1$ optinal positions and. that is "Stars and bars"[11] problem :

$$\binom{n - 2k + 1 - 1}{k + 1 - 1} = \binom{n - 2k}{k}$$

□

5.4 Problem 4

Lemma 5.1.

$$1 \geq m - \binom{m}{2} \quad m \geq 1, m \in \mathbb{N}$$

Proof.

$$1 \geq m - \binom{m}{2} \Leftrightarrow 1 \geq m - \frac{m^2 - m}{2}$$
$$m^2 - 3m + 2 \geq 0 \Leftrightarrow (m-1)(m-2) \geq 0$$

And the right hand size grater then zero for any $m \geq 2$ □

Lemma 5.2.

$$1 \leq m - \binom{m}{2} + \binom{m}{3} \quad m \geq 1, m \in \mathbb{N}$$

Proof.

$$\begin{aligned} 1 \leq m - \binom{m}{2} + \binom{m}{3} &\Leftrightarrow 1 \leq m - \frac{m^2 - m}{2} + \frac{m^3 - 2m^2 - 2m}{6} \\ &\Leftrightarrow 0 \leq m^3 - 6m^2 + 11m - 6 \\ &\Leftrightarrow 0 \leq (m-3)(m-2)(m-1) \end{aligned}$$

[11] Not sure if saw in class - *"Stars and Bars from Wikipedia"*

The right hand size grater then zero for any $m \geq 3$, and equal 0 for $m \in \{1,2\}$ since m is an integer. $\square$

Let $A_1, A_2 \ldots A_n$ be a family of n sets.

claim 5.3.

$$\left|\bigcup_{i=1}^{n} A_i\right| \geq \sum_{1\leq i\leq n} |A_i| - \sum_{1\leq i\leq j\leq n} |A_i \cap A_j|$$

Proof. to prove the following claim I will use "Donation to the Argument" [12] method. let assume that exists some $a \in A_i$. this a adding at most 1 to the left hand side. now consider a is part of some other $m \geq 1$ sets, then at the right hand side its count $\binom{m}{1}$ times at the first argument, and $\binom{m}{2}$ in the second. Hence using Lemma 5.1 the inequality hold for any $a \in A$. and that lead to finish the proof $\square$

claim 5.4.

$$\left|\bigcup_{i=1}^{n} A_i\right| \leq \sum_{1\leq i\leq n} |A_i| - \sum_{1\leq i\leq j\leq n} |A_i \cap A_j| + \sum_{1\leq i\leq j\leq k\leq n} |A_i \cap A_j \cap A_k|$$

Proof. using same idea described above, let $a \in A_i$ then a count once on the left hand-sid. At the right hand-side a count $\binom{m}{1}$ on the 1^{nd} term. $\binom{m}{2}$ on the 2^{nd} and $\binom{m}{3}$ at the 3^{nd} term. Hence using Lemma 5.3 the inequality hold for any $a \in A$. and that lead to finish the proof. $\square$

[12] To be honest I am not really sure what the name of this technique, Its kind of similar to "Counting derangements" I think