Basic Combinatorics - Spring, Home Assignment 4

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Problem 1.

Proposition 1. *if* $T(n) = T(n/3) + T(2n/3) + n$ *then* $T(n) = O(n \log n)$

Proof. using induction, my induction hypothesis will be

$$T(n) \leq Cn \log n, \qquad \forall n < N, 0 < C$$

Now we get that

$$\begin{aligned}
T(n) &= T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n && (1)\\
&\leq C\frac{n}{3}\log\frac{n}{3} + C\frac{2n}{3}\log\frac{2n}{3} + n && (2)\\
&\leq C\frac{n}{3}\log n - C\frac{n}{3}\log 3 + C\frac{n}{3}\log n - C\frac{2n}{3}\log\frac{3}{2} + n && (3)\\
&\leq Cn\log n - C\frac{n}{3}\log 3 - C\frac{2n}{3}\log\frac{3}{2} + n && (4)\\
&\leq^{(*)} Cn\log n && (5)
\end{aligned}$$

(2) imply the induction hypothesis, and (5) will hold by choosing C s.t

$$n \leq C\frac{n}{3}\log 3 + C\frac{2n}{3}\log\frac{3}{2}$$

Hence for any $n \geq 1$ we can choose C such that

$$C \geq \frac{1}{\frac{1}{3}\log 3 + \frac{2}{3}\log\frac{3}{2}} \approx 1.578$$

We get that

$$T(n) = O(n\log n)$$

□

Proposition 2. *if* $T(n) = 2T(n/2) + n \log n$ *then* $T(n) = O(n \log^2 n)$.

Proof. using induction, my induction hypothesis will be

$$T(n) \leq Cn \log^2 n, \qquad \forall n < N, 0 < C$$

for $n = 2$

$$\begin{aligned} T(2) &= 2T\left(\frac{2}{2}\right) + 2\log 2 && (6)\\ &\leq 2C + n \log n && (7)\\ &= O(n \log^2 n) && (8) \end{aligned}$$

Hence for any $n \geq 2$

$$\begin{aligned} T(n) &= 2T(n) + n \log n && (9)\\ &\leq 2\log^2(\frac{n}{2})\frac{n}{2} + n \log n && (10)\\ &= O(n \log^2 n) && (11) \end{aligned}$$

(10) holds since $n > \frac{n+1}{2}$ $\square$

Proposition 3. *Let* $c_1, \ldots, c_k$ *be* k *positive reals satisfying* $\sum_{i=1}^{k} c_i < 1$. *if* $T(n) = \sum_{i=1}^{k} T(c_i n) + n$ *then* $T(n) = O(n)$.

Proof. by induction, my induction hypothesis will be

$$T(n) \leq Cn, \qquad \forall n < N, 0 < C$$

Its immediate that $T(0) \leq 0$ now lets

$$\begin{aligned} T(n) &= \sum_{i=1}^{k} T(c_i n) + n && (12)\\ &\leq \sum_{i=1}^{k} C(c_i n) + n && (13)\\ &\leq Cn\left(\sum_{i=1}^{k} c_i + \frac{1}{C}\right) && (14)\\ &\leq^{(*)} Cn && (15) \end{aligned}$$

(13) imply the induction hypothesis, and (15) will hold by choosing C s.t

$$C \geq \frac{1}{1 - \sum_{i=1}^{k} c_i}$$

Hence for any $n \geq 1$ (15) holds and we get that

$$T(n) = O(n)$$

$\square$

Problem 2.

Theorem. *Every tournament has a Hamilton path*

Proof 1. Let $T = (V, E)$ be tournament graph where $|V| = n$. using strong induction for $n \leq 2$ its immediate that hamiltonian path exists. now lets assume its hold for any $k < n$. lets choose some $v \in V$ and define 2 sets such that

$$V_{in} = \{u : \overrightarrow{(u, v)} \in E\}, V_{out} = \{u : \overrightarrow{(v, u)} \in E\}$$

Since $|V_{in}| < n, |V_{out}| < n$ by the induction hypothesis exists paths
$P_{in} \in V_{in}, P_{out} \in V_{in}$ such that P_{in} and P_{out} are hamiltonian paths. now the path $P_{in} \rightarrow v \rightarrow P_{out}$ is hamiltonian path for all vertices
$|V_{in}| \cup |v| \cup |V_{out}| = n$ $\square$

Proof 2. First we can notice that $\chi(T) =$[1]n since any 2 vertex connected with an edge . Since $\chi(T) \leq |P|$ where P is the longest simple path in T. on the other hand its can be at most n since T have n vertex . we get that $|P| = n$.

Hence since P is simple path i.e its visit any vertex of the T exactly once, and P visit all the vertex or in other worlds P is Hamilton path in T $\square$

[1]T is graph on n vertex hence its can be colored by at most n different colours

Problem 3.

claim. *Any set X of $st+1$ integers contains one of the following:*

- *A subset $T = \{x_1, \dots, x_{t+1}\} \subseteq X$ of size $t+1$ such that x_k divides x_{k+1} for every $1 \leq k \leq t$.*
- *A subset $S = \{x_1, \dots, x_{s+1}\} \subseteq X$ of $s+1$ integers such that x_i does not divides x_j for every $x_i, x_j \in S$.*

Proof. Consider the following Poset define by

$$\mathcal{P} = {}^{2}\{X, \langle x_1, x_2 \rangle : x_1 | x_2\}$$

Now lets look at $\mathcal{P}$ over X , first notice that when its have chain size $|t+1| \Rightarrow$ exists sequence of $x_1 \preceq \dots \preceq x_{t+1}$ such that $x_1 | x_2 \dots | x_{t+1} \Rightarrow$ exist $T \subseteq X$. on the other hand if $\mathcal{P}$ over X , have anti-chain size $|s+1| \Rightarrow$ exist sequence of $x_1 \npreceq \dots \npreceq x_{s+1}$ such that $x_1 \nmid x_2 \nmid \dots \nmid x_{s+1} \Rightarrow$ exist $S \subseteq X$. Since

$$\omega(X)\alpha(X) \geq \omega(X)\frac{|X|}{\mathcal{X}(X)} \geq {}^{3}|X|$$

its following that splinting X into $\mathcal{X}(X)$ anti-chains, one of them will be at size $\frac{|X|}{\mathcal{X}(X)}$. if $\alpha(X) \geq s+1$ then $S \subseteq X$, else $\alpha(X) \leq s$ and

$$\omega(X) \geq \frac{|X|}{\alpha(X)} \leq \frac{st+1}{s} = t + \frac{1}{s}$$

and we get that $\omega(X) \geq t+1 \Rightarrow T \subseteq X$ $\square$

Problem 4.

Consider the following Poset define by

$$\mathcal{P} = \{\mathscr{F}, \langle S_1, S_2 \rangle : S_1 \subseteq S_2\}$$

where $\mathscr{F}$ is collection $\mathscr{F} = \{S_1, \dots, S_n\}$ of n sets.

[2] by inculding $\langle 0, 0 \rangle$ as well
[3] Mirsky

Proposition 4. *both chain and anti-chain of $\mathcal{P}$ are union-free sets*

Proof. first by noticing that when its have chain $\Rightarrow$ exists sequence of $S_1 \preceq \cdots \preceq S_k$ such that $S_1 \subseteq S_2 \cdots \subseteq S_k$ let mark this set of element as $\mathcal{S}_{chain}$. lets assume that exsit some $S_i, S_j, S_k \in \mathcal{S}_{chain}$ s.t $S_i \cup S_j = S_k$,we can that hold only when $|S_i|, |S_j| \leq |S_k|$ but the Poset yield $S_i \preceq S_k$ and $S_j \preceq S_k$ hence $S_i = S_k$ or $S_j = S_k$ which lead to contradiction since $\mathcal{S}_{chain}$ is set.

Define $\mathcal{S}_{anti-chain}$ such that

$$\mathcal{S}_{anti-chain} = \{S_i, S_j : S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i \quad \text{s.t } S_i, S_j \in \mathscr{F} \quad \forall i, j\}$$

By assuming that exsit some $S_i, S_j, S_k \in \mathcal{S}_{anti-chain}$ s.t $S_i \cup S_j = S_k$. its followed that $S_i \subseteq S_k$ but $S_i \nsubseteq S_j$ and we get an contradiction. □

claim. *every collection $\mathscr{F} = \{S_1, \ldots, S_n\}$ of n sets contains a sub-collection $S \subseteq \mathscr{F}$ of at least $\sqrt{n}$ sets which is union-free*

Proof. let $\alpha(\mathcal{P})$ be the longest $anti-chain$ of $\mathcal{P}$ over $\mathscr{F}$. If $\alpha(\mathcal{P}) \geq \sqrt{n}$ then exist such $S \in \mathscr{F}$. and if $\alpha(\mathcal{P}) < \sqrt{n}$

$$\frac{n}{\omega(\mathcal{P})} = \frac{|\mathscr{F}|}{\omega(\mathcal{P})} \leq \alpha(\mathcal{P}) < \sqrt{n}$$

$$\omega(\mathcal{P}) \geq \sqrt{n}$$

And again by proposition 4 we get that exist such $S \in \mathscr{F}$

□

Problem 5.

claim. *for a finite poset $\mathcal{P}$ and let x, y be two elements of $\mathcal{P}$ that are incomparable under $\mathcal{P}$. then $\mathcal{P}$ has a linear extension in which $x < y$.*

Proof. Let $\mathcal{P} = (X, \preceq)$ be a finite partial order in which $x, y \in X$ are incomparable. now lets define new post $\hat{\mathcal{P}} = (X, \hat{\preceq})$

$$\hat{\preceq} = \begin{cases} w \hat{\preceq} z & \text{if } w \preceq z \\ w \hat{\preceq} z & \text{if } z \preceq y \wedge x \preceq w \\ y \hat{\preceq} x & \end{cases}$$

- $\hat{\preceq}$ is reflexive since $\preceq$ is reflexive
- $\hat{\preceq}$ is Transitive since $\preceq$ is Transitive , and we apply apply only steps that respect the Transitive property of $\preceq$
- $\hat{\preceq}$ is Anti-symmetric. consider $w\hat{\preceq}z$ and $z\hat{\preceq}w$ and let assume that $w \neq z$. if $x = w$ or $y = w$ or $x = w, y = z$ its immediate lead to contradiction.
  and when $w\hat{\preceq}z \Rightarrow z \preceq y \wedge x \preceq w$ and $z\hat{\preceq}w \Rightarrow w \preceq y \wedge x \preceq z$ then $x \preceq z \preceq y$ contradiction to the fact that x, y are incomparable, hence $w\hat{\preceq}z$ and $z\hat{\preceq}w$ lead to $w = z$[4]

Hence $\hat{\mathcal{P}}$ is poset where $x\hat{\preceq}y$ and its contains less incomparable pairs than $\preceq$ does. If $\hat{\mathcal{P}}$ is linear then we done. otherwise exists some incomparable w, z and we can extend $\hat{\preceq}$ to $\hat{\preceq}_1$ and follow the proses until we cover all the chains or get some $\hat{\preceq}_k$ linear and respect $\mathcal{P}$ where $x < y$ □

Problem 6.

claim. *in the setting of Arrow's Theorem, if the individuals have only two options, then they can come up with a non-dictator social choice function.*

Proof. Lets proof that the democracy/majority voting system model satisfies the 3 condition of the Arrow's Theorem when N voters choose from $|\{A, B\}| = 2$ choices.

$$F : \mathrm{S_2}^N \to \frac{\sum_i^N \mathbb{1}[S_i \text{ choose } A > B]}{N} \quad \text{i.e and indicator if } S_i \text{ prefer } A$$

if $F \geq \frac{1}{2}$ return (A, B) else (B, A)
Monotonicity For two preference profiles $R = (R_1, \ldots, R_N)$ and $S = (S_1, \ldots, S_N)$ such that both profiles prefer $A > B$ but

$$0.5 < \frac{\sum_i^N \mathbb{1}[S_i \text{ choose } A > B]}{N} < \frac{\sum_i^N \mathbb{1}[R_i \text{ choose } A > B]}{N}$$

more people support (A, B) and its yield that F is monotone[5]

[4] i actually miss an case but its kind of similar proof
[5] its the same idea for the monotone decrease case $B > A$

Unanimity If alternative, $B < A$ for all orderings $R_1, \dots, R_N$, $R_i = (A, B) \forall i$ then $F(R_1, R_2, \dots, R_N) = (A, B)$ and A is ranked strictly higher than B by F. its immediate from the way we construct F

$$0.5 < \frac{\sum_i^N \mathbb{1}[R_i \text{ choose } A > B]}{N}$$

Non-dictatorship There is no individual, i whose strict preferences always prevail consider the profile define

$$R = (R_1, R_2, \dots R_N) \quad \forall i R_i = (A, B)$$

And

$$S = (S_1, S_2, \dots S_N) \quad \forall i S_i = (B, A)$$

For the 2 given profiles there is no individual who can change the result. $\square$

(*)

thanks for the tips! just saw your notes on assignment 3 and I will apply them from now on :)