Basic Combinatorics - Spring, Home Assignment 2

Problem 1.

claim. *1 For any* $1 \leq k \leq n$ *and* $0 < x < 1$

$$\binom{n}{k} x^k \leq (1+x)^n \leq e^{xn}$$

Proof. First lets notice that for $\forall x, k$ $x^k > 0$.now using the newton binomial we can get the following

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \ldots \underbrace{\binom{n}{k} x^k}_{\text{part of the sum}} + \binom{n}{k+1} x^{k+1} + \ldots \binom{n}{n} x^n$$

because each one of the sum's element is non negtive the following hold

$$\binom{n}{k} x^k \leq (1+x)^n$$

using Bernoulli's Inequality, for $\forall n \in N$ and $x > 0$

$$0 < 1 + x \leq \left(1 + \frac{x}{n}\right)^n \xrightarrow[n \to \infty]{} e^x$$

we can raise both side in power of n and we will get the complete formula

$$\binom{n}{k} x^k \leq (1+x)^n \leq e^{xn}$$

$\square$

claim. *2 For any* $1 \leq k \leq n$ *and* $0 < x < 1$

$$\binom{n}{k} \leq (\frac{en}{k})^k$$

Proof. if k=n we Instantly get using the result of claim 1.

$$1 = \binom{n}{n} \leq (\frac{en}{n})^n = e^n$$

now for $k < n$ and using above inequality lets set $0 < x = \frac{k}{n} < 1$

$$\binom{n}{k}(\frac{k}{n})^k \leq e^{\frac{k}{n}n} \Rightarrow \binom{n}{k} \leq (\frac{n}{k})^k e^k = (\frac{en}{k})^k$$

$\square$

claim. *3 For any* $1 \leq k \leq n$ *and* $0 < x < 1$

$$\sum_{i=0}^{k} \binom{n}{i} \leq (\frac{en}{k})^k$$

Proof. using the same way as above and claim 1 lets
$k < n$ set $0 < x = \frac{k}{n} < 1$

$$\sum_{i=1}^{n} \binom{n}{i}(\frac{k}{n})^i \leq (1 + \frac{k}{n})^n = \sum_{i=0}^{n} \binom{n}{i}(\frac{k}{n})^i \leq e^{n\frac{k}{n}} = e^k$$

divide by $(\frac{k}{n})^k$

$$\sum_{i=0}^{n} \binom{n}{i} \leq \sum_{i=1}^{n} \binom{n}{i}(\frac{k}{n})^{i-k} \leq e^k(\frac{n}{k})^{-k} = (\frac{en}{k})^k$$

the first inequality holds because for $i \leq k$ we get $i - k < 0$ $\square$

Problem 2.

Theorem. *For all* $n \geq 2$, $nlogn - n < log(n!) < nlogn$ *i.e* $\ln(n!) = \Theta(nln(n))$

Proof. First we note that $\ln(n!) = \sum_k^n lnk$ and using Riemann sum approximation, and the fact that ln is a non-decreasing function on $[1, \infty)$,for all $x \in [k, k+1)$ Integrating we get

$$\ln(k) \leq ln(x) \leq \ln(k+1)$$

$$\int_k^{k+1} \ln(k)dx \leq \int_k^{k+1} \ln(x)dx \leq \int_k^{k+1} \ln(k+1)dx.$$

Summing for k between 1 and n-1, we get

$$\sum_{k=1}^{n-1} \ln(k) \leq \sum_{k=1}^{n-1} \int_k^{k+1} ln(x)dx = \int_1^n \ln(x)dx \leq \sum_{k=1}^{n-1} \ln(k+1) = \sum_{k=2}^{n} \ln(k)$$

adding ln(1),ln(n)

$$\int_1^n \ln(x)dx + \ln(1) + ln(n) \leq \sum_{k=1}^{n-1} \ln(k) + \frac{ln(n)}{2} - \ln(1) \leq \int_1^n \ln(x)dx + \ln(n).$$

hence for

$$\int_1^n \ln x dx = x \ln x - x|_1^n = n \ln n - n + 1$$

we get

$$n \ln(n) + \frac{ln(n)}{2} - n + 1 \leq \sum_{k=1}^{n} \ln k \leq n \ln -n + \frac{3 \ln(n)}{2} + 1$$

$\square$

using the theorem above lets add to the of power e

$$\exp(n \ln n - n + 1 + \frac{\ln(n)}{2}) \leq \exp(\ln(n!)) \leq \exp(\frac{3 \ln(n)}{2} + n \ln n - n + 1)$$

$$\Leftrightarrow n! = \Theta(\sqrt{n} e (\frac{n}{e})^n)$$

Problem 3.

(3.1)

Let q(n) denote the number of ordered sets of positive integers whose sum is n, lets define Q such that Q is sequence size n of 1's .

$$Q : 1 \nabla 1 \nabla 1 \nabla 1 \nabla \ldots \nabla 1$$

in total we looking at n times 1 and n-1 ∇. now lets say we have 2 operators $\{+, |\}$ we can replace each time ∇ with ine of them, if we choose the + we "merge" both of the sums, but if we choose | we "slice" the set.hence the sum of Q will always stay n and each unique decision of choosen operator in order will give us unique ordered sets of positive, we can choose 2 operators total n-1 times, hence the number of ordered sets is

$$q(n) = 2^{n-1}$$

(3.2)

using the group Q define above, to find all the ordered sets size k hows sum is n, we can say that now we must replace k-1 of ∇ with $|$, witch leaves us with total of k sets, and all the rest ∇ will get the + operator immediately. in total we have k-1 of $|$ to rplace k-1 operator of ∇ from total $n-1\nabla$,and by suuming all the option over k sizes of group we get.

$$\sum_{k=1}^{n-1}\binom{n-1}{k-1}=\sum_{k=0}^{n}\binom{n-1}{k}=2^{n-1}=q(n)$$

Problem 4.

Let p(n) denote the number of unordered sets of positive integers whose sum is n. lets define $p_k(n)$ to be number of unordered sets of size k of positive integers whose sum is n.
using the result of problem 3 we know that for ordered set size k we have $\binom{n-1}{k-1}$ option.if we looking at k different element we will have total of.

$$p_k(n)=\frac{\binom{n-1}{k-1}}{k!}$$

but we might have some repeat numbers so its will be at most

$$p_k(n)\geq\frac{\binom{n-1}{k-1}}{k!}$$

since we define q(n) s.t $p(n)=\sum_k p_k(n)$ the following hold.

$$p(n)=\sum_k p_k(n)\geq \max_{1\geq k\geq n}\frac{\binom{n-1}{k-1}}{k!}$$

claim. *there is an absolute constant $c>0$ for which $p(n)\geq e^{c\sqrt{n}}$*

Proof.

$$p(n)=\geq \max_{1\geq k\geq n}\frac{\binom{n-1}{k-1}}{k!}\geq\frac{\binom{n-1}{k-1}}{k!}=\frac{1}{k!}\frac{k}{n}\binom{n}{k}$$

using $(*)\binom{n}{k}\geq(\frac{n}{k})^k$ $(**)k!\geq ek(\frac{n}{k})^k$,

$$\frac{1}{k!}\frac{k}{n}\binom{n}{k}\geq\underbrace{\frac{1}{ek(\frac{n}{k})^k}}_{**}\underbrace{(\frac{n}{k})^k}_{*}\frac{k}{n}$$

for $k = \sqrt{n}$

$$\frac{1}{e\sqrt{n}(\frac{n}{\sqrt{n}})^{\sqrt{n}}}(\frac{n}{\sqrt{n}})^{\sqrt{n}}\frac{\sqrt{n}}{n} = \frac{e^{\sqrt{n}}}{en} = \frac{e^{\sqrt{n}}}{e^{1+\ln(n)}}$$

using the detention of limit ,for some $1 > \epsilon > 0$

$$\frac{1+\ln(n)}{\sqrt{n}} \xrightarrow{n \to \infty} 0 \Rightarrow \frac{1+\ln(n)}{\sqrt{n}} < \epsilon \Rightarrow 1 + \ln(n) > \epsilon\sqrt{n}$$

hence for $c = 1 - \epsilon$,$c > 0$

$$\frac{e^{\sqrt{n}}}{e^{1+\ln(n)}} \leq \frac{e^{\sqrt{n}}}{e^{\epsilon\sqrt{n}}} = e^{\sqrt{n}} - e^{\epsilon\sqrt{n}} = e^{c\sqrt{n}}$$

$\square$

Problem 5.

Let $\pi(m, n)$ denote the set of prime numbers in the interval $[m, n]$.

(5.1)

we can see the following $[m, 2m]$

$$\{m + 1, m + 2, ..., 2m\}$$

now lets partition it to prime and and non prime element s.t

$$\{m + 1, m + 2, ..., 2m\} \setminus \pi(m + 1, 2m) = c(m + 1, 2m)$$

$$n \in c(m + 1, 2m) \leftrightarrow \{n \in [m, 2m] \vee n \text{ is not prime}\}$$

now lets look at $\binom{2m}{m}$

$$\binom{2m}{m} = \frac{2m(2m-1)...(m+1)}{m!} = \frac{1}{m!}\left(\prod_{p\in\pi(m+1,2m)} p\right)\left(\prod_{n\in c(m+1,2m)} n\right)$$

since for any $m + 1 \geq p \geq 2m, p \in \pi(m + 1, 2m)$
i claim when $p > m$ thus

$$m! \nmid \left(\prod_{p\in\pi(m+1,2m)} p\right)$$

and we get

$$\binom{2m}{m} = \underbrace{\frac{\left(\prod_{n\in c(m+1,2m)} n\right)}{m!}}_{\geq 1} \left(\prod_{p\in\pi(m+1,2m)} p\right) \geq \left(\prod_{p\in\pi(m+1,2m)} p\right)$$

(5.2)

first lets notice that

$$4^n = 2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} > \binom{2n}{n},$$

$$\text{and } 2\cdot 2^{2n+1} > \binom{2n+1}{n} \Rightarrow \binom{2n+1}{n} \leq 2^{2n}$$

since its apper twice in the binomial coefficient, so both at scenario (even,odd) using the floor will give us bound for the given binom

$$\left(\prod_{p\in\pi(\lfloor m/2\rfloor+1,2m)} p\right) \leq \binom{m}{\lfloor m/2\rfloor} \leq 2^m$$

now lets use the floor function we can notice that

$$\lfloor \lceil m/2^{2k}\rceil/2^k\rfloor = \lfloor m/2^{3k}\rfloor$$

hence for $2m, m, m/2$

$$\left(\prod_{p\in\pi(\lfloor m/4\rfloor+1,\lceil m/2\rceil)} p\right)\left(\prod_{p\in\pi(\lceil m/2\rceil+1,m)} p\right) \leq \binom{m}{\lfloor m/2\rfloor}\binom{\lceil m/2\rceil}{\lfloor m/4\rfloor} \leq 2^m 2^{\lfloor m/2\rfloor}$$

we can apply it for all m,$\lceil m/2\rceil$, $\lceil m/4\rceil$...

$$\left(\prod_{p\in\pi(1,m)} p\right) = \left(\prod_{p\in\pi(0+1,1)} p\right)\cdots\left(\prod_{p\in\pi(\lceil m/2\rceil+1,m)} p\right) \leq \tag{1}$$

$$\binom{m}{\lfloor m/2\rfloor}\binom{\lceil m/2\rceil}{\lfloor m/4\rfloor}\binom{\lceil m/4\rceil}{\lfloor m/8\rfloor}\cdots \leq 2^m\cdot 2^{\lfloor m/2\rfloor}\cdot 2^{\lfloor m/4\rfloor}\cdots \tag{2}$$

$$\leq 2^{m+\lfloor m/2\rfloor+\lfloor m/4\rfloor+\cdots} \leq 2^{m(1+1/2+1/4+\cdots)} \leq 2^{2m} = 4^m$$

(5.3)

using line (1),(2) we sow above

$$\log\left(\prod_{p\in\pi(1,n)} p\right) < \log(4^m) \Rightarrow O(2n\log 2)$$

$$\log\left(\prod_{p\in\pi(1,n)} p\right) = \sum_{I=(\lceil 2i/i\rceil+1,i) i\in\pi(1,n)} \log\left(\prod_I p\right) \le \sum_{I(\lceil 2i/i\rceil+1,i) i\in\pi(1,n)} \log \underbrace{\binom{i}{\lfloor i/2\rfloor}}_{2^n\le\binom{n}{2n}}$$

since $2^n \le \binom{n}{2n}$ and the result from sector 5.1 we can bound the following.

$$\prod_{I=(\lceil 2i/i\rceil+1,i) i\in\pi(1,n)} \binom{i}{\lfloor i/2\rfloor} < \left(\prod_{p\in\pi(1,n)} p\right) \le (4^n)$$

log both sides

$$\log(\prod_{I=(\lceil 2i/i\rceil+1,i) i\in\pi(1,n)} \binom{i}{2i}) \le |\pi(1,n)|\log(2^{\lg(n)\log 2}) < \log(4^n)$$

and we finality get

$$|\pi(1,n)|\lg(n)\log 2 < 2n\log 2 \Leftrightarrow |\pi(1,n)| = O(\frac{n}{\log(n)})$$

Problem 6.

claim. *Every tournament T of order $|V| = 2^k$ contains an undominated set of size $\le k$.*

Proof. the base case of the induction is trivial for $k = 1, 2$ lets assume the hypothesis hold for some for 2^k, now lets look at $\hat{T}$ of order $|V| = 2^{k+1}$ lets look at the avarge $\deg_{out}$ i.e

$$\frac{|E|}{|V|} = \frac{2^{k+1}(2^{k+1}-1)}{2 * 2^{k+1}} = 2^k - \frac{1}{2}$$

hence exist some $v \in |V|$ such that $v_{\deg_{out}} \geq 2^k \Rightarrow v_{\deg_{in}} < 2^k$. now lets choose some $2^k = |S|, \{S : S \subseteq V\}$ such that v dominated by any $v_s \in S$. lets apply our induction assumption on sub-tournament S, since exist $\hat{S} \subseteq S$ size $|\hat{S}| \leq k$ that not dominated by any other vertex$\Rightarrow |\hat{S} \cup v| \leq k+1$ sub-set size $k+1$ that not dominated in $|T| = 2^{k+1}$

$\square$

now lets look at some random tournament T that any $e \in |E|$ have the same probability to be in each direction

$$\Pr(e : u \to v) = \Pr(e : v \to u) = 1/2$$

$\Rightarrow$ the probability that v is dominates on some u is 1/2
$\Rightarrow$ the probability v is dominates on $S \subseteq V$ size k is $1/2^k$
$\Rightarrow$ the probebilty that e dominated by some $|S| = k$ is $(1 - 1/2^k)$
$\Rightarrow$ the probebilty that $|T/S| = n - k$ dominated by some $|S| = k$
is $(1 - 1/2^k)^{n-k}$
the expected number of group size k can bound from above with $\binom{n}{k}$, hence when n holds

$$\binom{n}{k}(1 - 1/2^k)^{n-k} < 1$$

then there is an n-vertex tournament so that every set of k vertices is dominated.
now lets use the property proved in Q(1) and we can bound the following for any $k \geq 2$

$$\binom{n}{k}(1 - 1/2^k)^{n-k} \leq \underbrace{e^{-\frac{(n-k)}{2^k}}}_{Q1 \text{ and } 1-k \leq e^k} \underbrace{\left(\frac{en}{k}\right)^k}_{\leq \binom{n}{k}} < 1$$

hence for $n > k + 2^k \cdot k^2$ the following hold .