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Thesis for the Master degree in Mathematics

Inner model theory

– an introduction

Revised edition

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Abstract

We introduce the field of inner model theory, starting from extenders and potential premice. We prove the basic properties of mice such as comparison, the Dodd-Jensen lemma, condensation and solidity. We construct mice via the Jensen-Steel method of using robust K^c-constructions and then use these mice to construct the core model K under the assumption that there is no proper class model with a Woodin cardinal – this is proven without the assumption that there exists a measurable cardinal in V. This construction of K is due to Steel and Jensen in their 2013 article "K without the measurable".

Preface

We believe the reader to be well acquainted with Gödel's constructible universe L (including the basic things on rudimentary functions from (Jensen, 1972)), measures and ultrapowers, as well as the basic theory of forcing.

Most of my notation will be standard, but as some parts of notation don't have a standard variant (yet!), some explanation is still required. I will denote the class of all ordinals by On and the direct limit of a sequence of models $\langle \mathcal{M}_\alpha \mid \alpha < \theta \rangle$ by $\varinjlim_\alpha \mathcal{M}_\alpha$. If $\pi : \mathcal{M} \to \mathcal{N}$ is Γ-preserving for some set of formulas Γ, then I'll write $\pi : \mathcal{M} \to_\Gamma \mathcal{N}$. I won't syntactically distinguish between a model and its universe, so I'll write things such as $x \in \mathcal{M}$ for $\mathcal{M} = \langle M, \in, \ldots \rangle$, which frees up the notation $|\mathcal{M}|$ for the cardinality of (the universe of) $\mathcal{M}$. Tuples and sequences will be written with angled brackets $\langle x_1, \ldots, x_n \rangle$ and (closed, open) intervals will be written with (square, round) brackets. I use $\lightning$ to denote a contradiction.

I would like to thank David Schrittesser for the helpful input he's given me in our discussions, and I would also thank Asger Törnquist for being the closest I've had to a mentor during my studies.

Revised version update: In this version I have fixed several errors and provided extra clarifications throughout the thesis. If you find any errors, please do let me know.

Contents

Introduction

In Gödel's 1947 paper "What is Cantor's continuum problem" he proposed a program, the goal of which was to "decide interesting mathematical propositions independent of ZFC in well-justified extensions of ZFC."

Today, we have found a great many extensions of ZFC in the hierarchy of large cardinals, pre-wellordered by consistency strength. A notable phenomenon is that for "natural" theories T and U, if T has smaller consistency strength than U then the Σ^0_ω consequences of T are also Σ^0_ω consequences of U – so climbing this hierarchy we in fact uncover more truths about the natural numbers. Moving on the the reals, these *also* attain this monotone behaviour, as long as one has moved sufficiently far up the hierarchy, namely past the existence of infinitely many so-called Woodin cardinals. This phenomenon also occurs for sets of reals.

So what makes these ZFC extensions well-justified? This is where inner model theory comes into the mix, as it provides canonical models exhibiting these large cardinals. The large amount of structure present in such a canonical inner model makes it possible to analyse these large cardinals thoroughly, and thereby also providing empirical evidence of the consistency of the existence of the given large cardinal (by Gödel's Second Incompleteness Theorem, we cannot hope to prove the *actual* consistency of the existence of any large cardinal, so solid empirical evidence is the next best thing).

Some of these canonical inner models have the special status of being a *core model*, which is the unique proper class inner model K which, among other things, is absolutely definable and "close to V". Below $0^\sharp$, the core model is simply Gödel's constructible universe L. In this case, L being close to V manifests itself in the *strong covering property*, saying that every set of ordinals lies inside a constructible set of ordinals of the same cardinality. When we move higher up the ladder of large cardinals, we cannot have this strong covering property for K anymore, so we have to resort to the *weak covering property*, which is to say that $\kappa^{+V} = \kappa^{+K}$ holds for every singular cardinal κ.

The construction of these core models started in (Dodd & R.B., 1981), (Dodd & R.B., 1982a) and (Dodd & R.B., 1982b), constructing K below $0^\dagger$. In this case $0^\sharp \in K$, so that K properly extends L. Shortly thereafter Mitchell improved this result by constructing K below a sharp for a proper class model of ZFC with a measurable cardinal κ of order κ^{++} in (Mitchell, 1984) and (Mitchell, 1982). Steel then further improved this to up below a Woodin cardinal in (Steel, 1996), but he required the extra hypothesis that there existed a measurable in V. A proof that Steel's K has the weak covering property was then proven in (Mitchell, Schimmerling, & Steel, 1997). This weak covering result was then used in (Jensen & Steel, 2013) to construct Steel's K without the assumption of the measurable, using the *robustness* concept introduced in (Jensen, 2003).

All these core models are based on extensions of L, witnessing the existence of certain elementary embeddings. At the measurable level, this is simply models of the form $L[U]$ with U a κ-complete non-principal ultrafilter (also known as a measure) on some measurable cardinal κ. This was extended to *coherent sequences* of measures to accomodate the existence of multiple measurables, then to *extenders* to witness a strong cardinal and finally to *fine extender sequences*.

In this thesis we will develop the theory of fine extender sequences, provide the construction of mice with these sequences on them and then use these mice to construct Steel's K below a Woodin, without the assumption of the measurable cardinal in V. This will include the details in (Steel & Mitchell, 1994) and (Steel, 1996), with the modern approach in (Steel, 2010) and with the corrections and additions provided by (R.-D. Schindler, Steel, & Zeman, 2002) and (Jensen & Steel, 2013). A number of illustrations and motivation will be provided as well, inspired by (Steel, 2010), (Löwe & Steel, 1997), (Steel, 2012) and (Schimmerling, 2001).

Woodin has shown that if it's possible to build an inner model with a so-called supercompact cardinal then this would enable the construction of an inner model for all other known large cardinals as well, so the main focus right now is to reach the supercompact. This is the current goal of *descriptive inner model theory*, where methods of descriptive set theory become intertwined with inner model theory.

1 Extenders and potential premice

Inner model theory is a field concerned about constructing canonical models of set theory admitting large cardinals. These canonical models can then be used in various different ways. They can shed more light on the behaviour of the large cardinals, provide lower consistency bounds on axioms of interest and also provide more knowledge on the structure of V by studying *core models*. We will be building a specific core model in the very last chapter.

As large cardinal axioms are closely related to the existence of elementary embeddings, our models should somehow "know" that these embeddings exist. At the level of measurable cardinals[1] we're putting *measures*[2] into our models. If we want to admit stronger large cardinals we have to "strengthen" our measures somehow. A way to do that is by using *extenders*.

1.1 Extenders

Extenders will be a sequence of measures built in such a way that the ultrapowers of the measures will give rise to a "limit" ultrapower of the extender.

Definition 1.1. Let $\kappa < \lambda$ and assume $\mathcal{M}$ is transitive rud closed[3]. Then E is a (short) (κ, λ)-**extender** over $\mathcal{M}$ if there exists a Σ_0-embedding $j : \mathcal{M} \to \mathcal{N}$ to a transitive rud closed $\mathcal{N}$, such that $\kappa = \mathrm{crit}(j)$, $\lambda \leqslant j(\kappa)$ and

$$E = \bigcup_{n<\omega} \{(a, x) \in [\lambda]^n \times \mathcal{P}^{\mathcal{M}}([\kappa]^n) \mid a \in j(x)\}.$$

[1] Recall that a cardinal κ is *measurable* if there exists an elementary embedding $j : V \to L$ with $\kappa = \mathrm{crit}\, j$.

[2] I.e. κ-complete non-principal ultrafilters.

[3] See (Jensen, 1972) or (R. Schindler & Zeman, 2010) for the definitions and results on rud functions and rud closed structures.

If we relax the condition that $\mathcal{N}$ is transitive to $\lambda + 1 \subseteq \text{wfp}(\mathcal{N})$ then we call E a **pre-extender**. We call κ the *critical point* of E, $\kappa = \text{crit}\, E$, and λ the *length* of E, $\lambda = \text{lh}\, E$. $\circ$

The "shortness" of the extender is the condition that $\lambda \leqslant j(\kappa)$. If this condition is removed, these *long* extenders will still have a lot of the same properties as the short ones, but will make the arguments more complex. By default "pre-extender" and "extender" will refer to the short variants, and long extenders won't be needed until the last chapters 7 and 8. Now, given a (κ, λ) pre-extender E, we have for each $a \in [\lambda]^{<\omega}$ the cross-section

$$E_a := E \cap (\{a\} \times \mathcal{M}) = \{x \in \mathcal{P}^{\mathcal{M}}([\kappa]^{|a|}) \mid a \in j(x)\}.$$

Proposition 1.2. *For each $a \in [\lambda]^{<\omega}$, E_a is a κ-complete measure over $[\kappa]^{|a|}$ with critical point κ, which is principal iff $a \subseteq \kappa$.*

Proof. By shortness of E we get that $a \in [\lambda]^{|a|} \subseteq [j(\kappa)]^{|a|} = j([\kappa]^{|a|})$, so $[\kappa]^{|a|} \in E_a$. As j is Σ_0-elementary it preserves both finite intersections and the subset relation, which implies that E_a is a filter. If $a \notin j(x)$ then $a \in j(x)^c = j(x^c)$, with $(-)^c$ denoting the relative complement, so it's also an ultrafilter. As the critical point is κ, j also preserves unions of $< \kappa$ many sets, making E_a κ-complete.

Finally, assume that E_a is principal, so that $\{x\} \in E_a$ for some $x \in [\kappa]^{|a|}$. Then $a \in j(\{x\}) = \{j(x)\}$, meaning $a = j(x)$. But x is below the critical point, so $j(x) = x$, meaning $a \subseteq \kappa$. Conversely if $a \subseteq \kappa$, then we show that $\bigcap E_a \in E_a$. As a is below the critical point, $a = j(a)$, implying that $j(a) = a \in j(\bigcap E_a)$ holds iff $a \in \bigcap E_a$ holds, of which the latter is true since $a \in j(x) \cap [\kappa]^{<\omega} = x$ for every $x \in E_a$. ■

This proposition thus justifies our intuitive notion of an extender being a sequence of measures. We will borrow the following convenient notation from measure theory.

Definition 1.3. Let E be a pre-extender over some $\mathcal{M}$, $a \in [\text{lh}\, E]^{<\omega}$ and $\varphi(u)$ a predicate. Then we say that φ **holds for E_a-a.e.** u if $\{u \mid \mathcal{M} \models \varphi[u]\} \in E_a$. We will also write $\forall^{E_a} u\varphi$, and we will leave out E_a when it is understood. $\circ$

Just as with usual measures, it's also possible to define the ultrapower of any (κ, λ)-pre-extender E. This is going to be a direct limit of the cross-section ultrapowers $\mathrm{Ult}(\mathcal{M}, E_a)$, so we first need to show how these section ultrapowers form a directed system.[4]

Let $a, b \in [\lambda]^{<\omega}$ satisfy $a \subseteq b$. Identifying such finite subsets of λ with their corresponding increasing enumerations and letting s be the increasing enumeration of $\{i < \omega \mid b(i) \in a\}$, we can define $x_{ab} \in \mathcal{P}^{\mathcal{M}}([\kappa]^{|b|})$ for $x \in \mathcal{P}^{\mathcal{M}}([\kappa]^{|a|})$ as $x_{ab} := \{u \in [\kappa]^{|b|} \mid u \circ s \in x\}$. We can think of x_{ab} as x with "added dummy variables".

Proposition 1.4 (Coherence). *For any $x \in \mathcal{P}^{\mathcal{M}}([\kappa]^{|a|})$ and $a, b \in [\lambda]^{<\omega}$ such that $a \subseteq b$, it holds that $x \in E_a$ iff $x_{ab} \in E_b$.*

Proof. If $x \in E_a$ then $a \in j(x)$. But as $b \circ s = a$ by definition, $b \in j(x_{ab})$ and thus $x_{ab} \in E_b$. Conversely, if $x_{ab} \in E_b$ then $b \in j(x_{ab})$, so $a = b \circ s \in x$ and $x \in E_a$. ■

Now, defining $f^{ab} : [\kappa]^{|b|} \to \mathcal{M}$ as $f^{ab}(u) := f(u \circ s)$ for $f : [\kappa]^{|a|} \to \mathcal{M}$, set $\Theta_{ab} : \mathrm{Ult}(\mathcal{M}, E_a) \to \mathrm{Ult}(\mathcal{M}, E_b)$ to be $\Theta_{ab}([f]_{E_a}) := [f^{ab}]_{E_b}$. To see that Θ_{ab} is well-defined, assume that $f^{ab}(u) = g^{ab}(u)$ for E_b-many u. This E_b set then looks like

$$X := \{u \in [\kappa]^{|b|} \mid f(u \circ s) = g(u \circ s)\}.$$

But note that $X = x_{ab}$, where $x := \{u \in [\kappa]^{|a|} \mid f(u) = g(u)\}$, so since $X \in E_b$, $x \in E_a$ by the above proposition and $f(u) = g(u)$ for E_a-many u. To show that $\{\Theta_{ab}\}_{a,b}$ form a directed set, just note that for any $a, b \in [\lambda]^{<\omega}$ we have maps $\Theta_{a,a\cup b}$ and $\Theta_{b,a\cup b}$. We can thus use these maps to define the ultrapower of a pre-extender.

Definition 1.5. The **ultrapower** of a (κ, λ)-pre-extender E is given by

$$\mathrm{Ult}(\mathcal{M}, E) := \varinjlim_{a \in [\lambda]^{<\omega}} \mathrm{Ult}(\mathcal{M}, E_a),$$

[4] For the definition of directed system and direct limits of structures, see (Kanamori, 2009).

where the direct limit is over the system $\{\Theta_{ab}\}_{a,b\in[\lambda]^{<\omega}}$ defined above. ○

Somewhat more concretely, we can write $[a, f]$ as the image of $[f] \in \mathrm{Ult}(\mathcal{M}, E_a)$ under the direct limit map, so that the direct limit is isomorphic to

$$\{(a, f) \mid a \in [\lambda]^{<\omega} \wedge f : [\kappa]^{|a|} \to \mathcal{M} \wedge f \in \mathcal{M}\}/\sim_E,$$

where $(a, f) \sim_E (b, g)$ iff for $E_{a\cup b}$-many u, $f^{a,a\cup b}(u) = g^{b,a\cup b}(u)$. As usual, $[a, f] \in_E [b, g]$ holds iff $f^{a,a\cup b}(u) \in g^{b,a\cup b}(u)$ for $E_{a\cup b}$-many u.

Just as ultrapowers of measures, we get a version of Łoś' Theorem.

Theorem 1.6 (Łoś). *Let $\mathcal{M}$ be a transitive and rud closed structure with $\mathcal{M} \models \mathsf{AC}$. Let E be a (κ, λ)-pre-extender over $\mathcal{M}$ and set $\mathrm{Ult} := \mathrm{Ult}(\mathcal{M}, E)$. Then given any Σ_0-formula $\varphi(v_1, \dots, v_n)$ in the language of $\mathcal{M}$ and elements $[a_i, f_i] \in \mathrm{Ult}$ for $1 \leqslant i \leqslant n$, setting $c := \bigcup_i a_i$ we have that*

$$\mathrm{Ult} \models \varphi[[a_1, f_1], \dots, [a_n, f_n]] \quad \textit{iff} \quad \forall^{E_c} u : \mathcal{M} \models \varphi[f_1^{a_1,c}(u), \dots, f_n^{a_n,c}(u)].$$

Proof. Induction on φ. If φ is atomic then it's by definition of equality and $\in_E$ in the ultrapower. The conjunction and negation steps are trivial. If φ is $\exists w \in v_1 \psi$ and $\mathrm{Ult}(\mathcal{M}, E) \models \varphi[[a_1, f_1], \dots, [a_n, f_n]]$ there exists some $[a, f] \in_E [a_1, f_1]$ such that $\mathrm{Ult}(\mathcal{M}, E) \models \psi[[a, f], [a_1, f_1], \dots, [a_n, f_n]]$. This implies that for $E_{c\cup a}$-many u, $\mathcal{M} \models \psi[f^{a,c\cup a}(u), f_1^{a_1,c\cup a}(u), \dots, f_n^{a_n,c\cup a}(u)]$, which by coherence implies that $\mathcal{M} \models \exists w \in f_1^{a_1,c}(u)\psi[w, f_1^{a_1,c}(u), \dots, f_n^{a_n,c}(u)]$ for E_c-many u.

If we conversely assume this latter statement, use AC in $\mathcal{M}$ to fix a wellordering $<^* \in \mathcal{M}$ on $\operatorname{ran} f_1$ and let $h : [\kappa]^{|c|} \to \operatorname{ran} f_1$ be defined as $h(u) :=$ the $<^*$-smallest $x \in \operatorname{ran} f_1$ satisfying that $\mathcal{M} \models \psi[x, f_1^{a_1,c}(u), \dots, f_n^{a_n,c}(u)]$ if such an x exists and $h(u) := \varnothing$ otherwise. This condition is $\mathbf{\Sigma}_0^{\mathcal{M}}$ and hence rud, so as $\mathcal{M}$ is rud closed, $h \in \mathcal{M}$. But then for E_c-many u,

Is this true? The fact that Σ_0-definability is Σ_1 seems to ruin it?

$$\mathcal{M} \models h(u) \in f_1^{a_1,c}(u) \wedge \psi[h(u), f_1^{a_1,c}(u), \dots, f_n^{a_n,c}(u)],$$

so $\mathrm{Ult}(\mathcal{M}, E) \models [c, h] \in [a_1, f_1] \wedge \psi[[c, h], [a_1, f_1], \dots, [a_n, f_n]]$, concluding that $\mathrm{Ult}(\mathcal{M}, E) \models \varphi[[a_1, f_1], \dots, [a_n, f_n]]$. ■

Fix now a transitive and rud closed $\mathcal{M}$ for the remainder of this section. We have an ultrapower map $i_E : \mathcal{M} \to \mathrm{Ult}(\mathcal{M}, E)$ given as $i_E(x) := [\{\kappa\}, c_x]$ with c_x the constant function on x, which is a Σ_0-embedding by Łoś' theorem.

Proposition 1.7. *The ultrapower embedding $i_E : \mathcal{M} \to \mathrm{Ult}(\mathcal{M}, E)$ is cofinal*[5]*, and thus a Σ_1-embedding.*

Proof. Let $[a, f] \in \mathrm{Ult}(\mathcal{M}, E)$ and $m := \operatorname{ran} f \in \mathcal{M}$. Then $f^{a, a \cup \{\kappa\}}(u) \in m$ for a.e. u, so $[a, f] \in i_E(m)$, making i_E cofinal. If $\mathrm{Ult}(\mathcal{M}, E) \models \exists x \varphi$ with φ Σ_0, then we can just find some $m \in \mathcal{M}$ such that $\mathrm{Ult}(\mathcal{M}, E) \models \exists x \in i_E(m) \varphi$, and then $\mathcal{M} \models \exists x \in m \varphi$. ■

Now, there are a few notions concerning extenders which will be useful.

Definition 1.8. Let E be a (κ, λ)-pre-extender over $\mathcal{M}$. Then for $\xi \leqslant \lambda$ we set $E \upharpoonright \xi := \{(a, x) \in E \mid a \subseteq \xi\}$. We then have an embedding

$$\sigma_\xi : \mathrm{Ult}(\mathcal{M}, E \upharpoonright \xi) \to \mathrm{Ult}(\mathcal{M}, E)$$

given by $\sigma_\xi [a, f]_{E \upharpoonright \xi} := [a, f]_E$. A **generator** of E is an ordinal $\xi \leqslant \lambda$ such that $\xi = \mathrm{crit}(\sigma_\xi)$, i.e. that for every $a \in [\xi]^{<\omega}$ and f, $\xi \neq [a, f]_E$, so ξ cannot be "approximated from below". The **natural length** of E is then defined as

$$\nu_E := \sup(\kappa^{+\mathcal{M}} \cup \{\xi + 1 \mid \xi \text{ is a generator of } E\}). \qquad \circ$$

A generator thus marks the "steps" of an ultrapower, in that if ξ is a generator then $E \upharpoonright \xi + 1$ contains strictly more information that $E \upharpoonright \xi$. Note that κ is the smallest generator and every other generator is $> \kappa^{+\mathcal{M}}$. To see this latter statement note that as generators ξ are critical points they in particular are cardinals of $\mathrm{Ult}(\mathcal{M}, E \upharpoonright \xi)$, and since $\kappa^{+\mathcal{M}} = \kappa^{+\mathrm{Ult}(\mathcal{M}, E \upharpoonright \xi)}$ the next generator after κ is $\geqslant \kappa^{+\mathcal{M}}$. To see

[5] I.e. that given any $y \in \mathrm{Ult}(\mathcal{M}, E)$ we can find an $x \in \mathcal{M}$ such that $y \in i_E(x)$.

that $\kappa^{+\mathcal{M}}$ is *not* a generator, note that $\kappa^{+\mathcal{M}} = [\{\kappa\}, f]$ with $f(\{\xi\}) = \xi^+$, as $\kappa^{+\,\mathrm{Ult}} = \kappa^{+\mathcal{M}}$.

Definition 1.9. Two pre-extenders on the same critical point are **equivalent** if their ultrapowers are isomorphic. ○

Note that any extender E is equivalent to $E \upharpoonright \nu_E$ since every $\gamma \geqslant \nu_E$ can be written as $[a, f]$ for some $a \subseteq \nu_E$ by definition of ν_E, so we can always assume that $a \subseteq \nu_E$ whenever $\langle a, x\rangle \in E$. The connection between measures and extenders is then the following result.

Proposition 1.10. *Measures are in a bijective correspondence to extenders with a single generator, up to equivalence.*

Proof. For U a measure just take E with $E_{\{\kappa\}} = U$ and $E_a = \varnothing$ otherwise. We will show that every extender with a single generator is equivalent to one of the form $E_{\{\kappa\}}$. Let thus E be a (κ, λ)-extender with a single generator. Then $\mathrm{Ult}(\mathcal{M}, E) = \varinjlim_{a \in [\lambda]^{<\omega}} \mathrm{Ult}(\mathcal{M}, E_a)$, so it suffices to show that if $a \in [\lambda]^{<\omega}$ with $a \nsubseteq \kappa$ and $\xi \in a - \kappa$ then $\mathrm{Ult}(\mathcal{M}, E_{\{\xi\}}) \cong \mathrm{Ult}(\mathcal{M}, E_a)$. Since E only has one generator, κ, we can assume that $a \in [\kappa+1]^{<\omega}$, so that $\xi = \kappa$.

Define the map φ as taking $[f]_{E_{\{\xi\}}}$ to $[f^{\{\xi\},a}]_{E_a}$, which is a well-defined homomorphism as $\{\xi\} \subseteq a$. For injectivity, assume $f^{\{\xi\},a} \sim_{E_a} g^{\{\xi\},a}$, so that $j(f^{\{\xi\},a})(a) = j(g^{\{\xi\},a})(a)$. But by definition of $f^{\{\xi\},a}$, $j(f^{\{\xi\},a})(a) = j(f)(\{\xi\})$ and likewise $j(g^{\{\xi\},a})(a) = j(g)(\{\xi\})$, so $f \sim_{E_{\{\xi\}}} g$.

For surjectivity, let $[f]_{E_a}$ be given and define $\tilde{f} : [\kappa]^{|a|} \to \mathcal{M}$ as $\tilde{f}(u) = f(\{a_0, \dots, a_{|a|-2}, u_{|a|-1}\})$ with $u \in [\kappa]^{|a|}$. Then $\tilde{f}$ is clearly in the image of φ, so we need to show that $f \sim_{E_a} \tilde{f}$, i.e. $j(f)(a) = j(\tilde{f})(a)$. But since $a_i < \kappa$ for $i < |a| - 1$, $j(a_i) = a_i$, $j(\tilde{f})(u) = j(f)(\{a_0, \dots, a_{|a|-2}, u_{|a|-1}\})$ and thus $j(f)(a) = j(\tilde{f})(a)$. ■

Thus, extenders generalise measures. Moreover, an extender encodes more information about the elementary embedding j than the derived measure of j does[6]. To

[6]The derived measure of j is $\{x \in \mathcal{P}^{\mathcal{M}}(\kappa) \mid \kappa \in j(x)\}$ – see (Kanamori, 2009).

see this, first note that just as with measures we have a factorisation

$$\begin{tikzcd} \mathcal{M} \arrow[r, "j"] \arrow[dr, "i_E"'] & \mathcal{N} \\ & \mathrm{Ult}(\mathcal{M}, E) \arrow[u, "k"'] \end{tikzcd}$$

where $k[a, f] := j(f)(a)$, because we have that

$$(k \circ i_E)(x) = k[\{\kappa\}, c_x] = j(c_x)(\{\kappa\}) = c_{j(x)}(\{\kappa\}) = j(x).$$

When we form ultrapowers with measures we can only expect that $k \upharpoonright \kappa = \mathrm{id}$, but in this case with extenders we get more. Towards showing this, we first show an important property for pre-extenders called normality.

Definition 1.11. Define $\mathrm{pr} : V \to V$ to be the union function: $\mathrm{pr}(u) := \bigcup u$.[7] ○

Lemma 1.12 (Normality). *Let E be a (κ, λ)-pre-extender over $\mathcal{M}$. Let $a \in [\lambda]^{<\omega}$ and $f \in \mathcal{M}$ such that $f : [\kappa]^{|a|} \to \kappa$. If $f(u) < \max(u)$ for E_a-many u then there is some $\beta < \max(a)$ such that $f^{a, a\cup\{\beta\}}(u) = \mathrm{pr}^{\{\beta\}, a\cup\{\beta\}}(u)$ holds for $E_{a\cup\{\beta\}}$-many u.*

Proof. Let j be the embedding derived from E and set $\beta := j(f)(a)$. Since $f(u) < \max(u)$ for E_a-many u, $\beta = j(f)(a) < \max(a)$. By the choice of β we get that $f^{a,a\cup\{\beta\}}(u) \in u$ for a.e. u, so that $f^{a,a\cup\{\beta\}}(u) = \mathrm{pr}^{\{\gamma\},a\cup\{\beta\}}(u)$ for a.e. u for some $\gamma \in a \cup \{\beta\}$ since $E_{a\cup\{\beta\}}$ is an ultrafilter. But then

$$\beta = j(f)(a) = \mathrm{pr}^{\{\gamma\},a\cup\{\beta\}}(a \cup \{\beta\}) = \gamma$$

and we're done. ■

One can show that a sequence $E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ of measures is a pre-extender iff it satisfies both coherence and normality, which can be used to define pre-extenderhood in a first-order fashion. For a proof this fact, see (Steel, 2010,

[7] This notation is from (R. Schindler & Zeman, 2010).

Section 2.1). Now, we can finally show that extenders encode more information than measures – they encode information up to their length.

Proposition 1.13. *Let E be a (κ, λ)-pre-extender over $\mathcal{M}$. Then the following holds:*

(i) *If $\alpha < \lambda$ and $[a, f] \in_E [\{\alpha\}, \mathrm{pr}]$ then $[a, f] = [\{\beta\}, \mathrm{pr}]$ for some $\beta < \alpha$;*

(ii) $\mathrm{Ult}(\mathcal{M}, E) \models [\{\alpha\}, \mathrm{pr}] = \alpha$ *for every* $\alpha < \lambda$*;*

(iii) $\mathrm{Ult}(\mathcal{M}, E) \models [a, \mathrm{id}]_E = a$ *for every* $a \in [\lambda]^{<\omega}$*;*

(iv) $k \restriction \lambda = \mathrm{id}$.

Proof. (i): Let $[a, f] \in_E [\{\alpha\}, \mathrm{pr}]$ and set $b := a \cup \{\alpha\}$. By Łoś, $f^{a,b}(u) \in \mathrm{pr}^{\{\alpha\},b}(u)$ for E_b-many u, so that $f(u) < \max(u)$. By normality, there is some $\beta < \alpha$ such that setting $c := b \cup \{\beta\}$, $f^{a,c}(u) = \mathrm{pr}^{\{\beta\},c}(u)$ for E_c-many u. But then by Łoś again, $[a, f] = [\{\beta\}, \mathrm{pr}]$. (ii) is easily proved by induction on $\alpha < \lambda$, using (i).

(iii): If $[b, f] \in_E [a, \mathrm{id}]$ then setting $c := a \cup b$ we have that $f^{b,c}(u) \in \mathrm{id}^{a,c}(u)$ for E_c-many u. But since E_c is an ultrafilter there is some $\alpha \in a$ such that $f^{b,c}(u) = \mathrm{id}^{\{\alpha\},c}(u)$ for E_c-many u, so by (ii), $[b, f] = \alpha$. Conversely, if $\alpha \in a$ then $j(\mathrm{pr})(\{\alpha\}) = \alpha \in a = \mathrm{id}(a)$, so that $\mathrm{pr}^{\{\alpha\},a}(u) \in \mathrm{id}(u)$ for E_a-many u, meaning that $\alpha = [\{\alpha\}, \mathrm{pr}] \in_E [a, \mathrm{id}]$. Thus $[a, \mathrm{id}] = a$.

(iv): If $\alpha < \lambda$ then $k(\alpha) = k[\{\alpha\}, \mathrm{pr}] = j(\mathrm{pr})(\{\alpha\}) = \alpha$, using (ii). ■

1.2 Sequences of extenders

We are going to work with not only extenders, but *sequences* of extenders. This is needed to make sure that our models will be able to contain several large cardinals, and also because some large cardinals require the existence of several embeddings.

One of the properties that canonical inner models such as L and $L[U]$ for U a measure satisfy is GCH. A related property we will require of our models is the following.

Definition 1.14. A set A is **acceptable at** α if given any $\beta < \alpha$ and any κ,

$$J^A_{\beta+1} \models \exists x(x \subseteq \kappa \wedge x \notin J^A_\beta) \to |J^A_\beta| \leqslant \kappa.$$

Said in another way, if there is a new subset of κ in $J^A_{\beta+1}$, then there is a surjection from κ onto J^A_β inside $J^A_{\beta+1}$ as well. $\circ$

Note that if A is acceptable at α and $J^A_\alpha \models \kappa^+$ exists, then $J^A_\alpha \models \mathsf{GCH}$. We will need the following technical definition as well.

Definition 1.15. Let E be a (κ, λ) pre-extender over $\mathcal{M}$, where $\mathcal{M} \models \kappa^+$ exists. Let $\eta := (\nu_E)^{+\,\mathrm{Ult}(\mathcal{M},E)}$. Then the (κ, η)-pre-extender derived from E is called the **trivial completion** of E, denoted by E^*. More concretely,

$$E^* := \bigcup_{n<\omega} \{(a, x) \mid a \in [\eta]^n \wedge x \subseteq \mathcal{P}^{\mathcal{M}}([\kappa]^n) \wedge a \in i_E(x)\} \qquad \circ$$

Note that $\nu_E = \nu_{E^*}$ and $E \upharpoonright \nu_E = E^* \upharpoonright \nu_{E^*}$, so that E and E^* are equivalent and there's thus nothing harmful in working with E^* instead of E. An even more technical property that we'll need is the following.

Definition 1.16. Let E be an pre-extender over $\mathcal{M}$. Then E is of **type** Z[8] if there exists some limit ordinal δ of $\mathcal{M}$ such that $\nu_E = \delta + 1$, $\delta = \nu_{E \upharpoonright \delta}$ and $\delta^{+\,\mathrm{Ult}(\mathcal{M},E)} = \delta^{+\,\mathrm{Ult}(\mathcal{M},E \upharpoonright \delta)}$. This implies that δ is the top generator which is itself a limit of generators (since $\delta = \nu_{E \upharpoonright \delta}$), and such that $E^* = (E \upharpoonright \delta)^*$. $\circ$

As we plan on indexing our extenders on the lengths of their trivial completions, we would index E and $E \upharpoonright \delta$ at the same spot since $E^* = (E \upharpoonright \delta)^*$. To avoid this, we make sure that type Z extenders won't appear on our sequences. We now arrive at one of the most important definitions in this thesis.

Definition 1.17. A **fine extender sequence** is a sequence $\vec{E}$ such that

(i) $\vec{E}$ is acceptable at every $\alpha \in \operatorname{dom} \vec{E}$;
(ii) Either $E_\alpha = \varnothing$ or E_α is a (κ, α) pre-extender over $J^{\vec{E}}_\alpha$ for some κ such that $J^{\vec{E}}_\alpha \models \kappa^+$ exists;
(iii) $E_\alpha = (E_\alpha \upharpoonright \nu_{E_\alpha})^*$ and E_α is not of type Z;
(iv) (Coherence) Letting $i : J^{\vec{E}}_\alpha \to \mathrm{Ult}(J^{\vec{E}}_\alpha, E_\alpha)$ be the ultrapower embedding, $i(\vec{E} \upharpoonright \kappa) \upharpoonright \alpha = \vec{E} \upharpoonright \alpha$ and $i(\vec{E} \upharpoonright \kappa)_\alpha = \varnothing$;

[8] This was introduced in (R.-D. Schindler et al., 2002) to fix errors with the so-called “good” extender sequences in (Steel & Mitchell, 1994).

(v) (Initial segment condition) For any η satisfying $\kappa^{+J^{\vec{E}}_\alpha} \leqslant \eta < \nu_{E_\alpha}$, $\eta = \nu_{E_\alpha \upharpoonright \eta}$ and $E_\alpha \upharpoonright \eta$ is not of type Z, one of the following holds:

(a) There exists some $\gamma < \alpha$ such that $(E_\alpha \upharpoonright \eta)^* = E_\gamma$;

(b) $E_\eta \neq \varnothing$ and if $j : J^{\vec{E}}_\eta \to \mathrm{Ult}(J^{\vec{E}}_\eta, E_\eta)$ is the ultrapower embedding then there is $\gamma < \alpha$ such that $(E_\alpha \upharpoonright \eta)^* = j(\vec{E} \upharpoonright \operatorname{crit} j)_\gamma$. ○

A naive definition of our models would then be structures $\langle J^{\vec{E}}_\alpha, \in, \vec{E} \upharpoonright \alpha, F\rangle$ with $\vec{E}$ being a fine extender sequence and F being "the top extender" with length α. The problem with this approach is that it will *not* give us an amenable structure[9], so we lose out on important properties such as Łoś' Theorem 1.6. A way we can fix this is by not using F as the predicate, but *encoding* F in such a way that we can recover F from the code and using the code as the amenable predicate.[10]

Lemma 1.18 (Amenability Lemma). *Let $\vec{E}$ be a fine extender sequence and let $\alpha \in \operatorname{dom} \vec{E}$ be such that $E_\alpha \neq \varnothing$. Set $\kappa := \operatorname{crit} E_\alpha$ and $\nu := \nu_{E_\alpha}$. Then for every $\eta < \alpha$ and $\xi < \kappa^{+J^{\vec{E}}_\alpha}$ it holds that $E_{\eta,\xi} := E_\alpha \cap ([\eta]^{<\omega} \times J^{\vec{E}}_\xi) \in J^{\vec{E}}_\alpha$. Moreover, if we define γ_ξ to be the least $\gamma < \alpha$ such that $E_{\nu,\xi} \in J^{\vec{E}}_\gamma$, where $\xi < \kappa^{+J^{\vec{E}}_\alpha}$, then these γ_ξ are cofinal in α.*

Proof. Fix $\eta < \alpha$ and $\xi < \kappa^{+J^{\vec{E}}_\alpha}$. Since $\bigcup_{n<\omega} \mathcal{P}([\kappa]^n) \cap J^{\vec{E}}_\xi$ has cardinality κ in $J^{\vec{E}}_\alpha$, let $\langle x_\beta \mid \beta < \kappa\rangle \in J^{\vec{E}}_\alpha$ be an enumeration of it. Let $i : J^{\vec{E}}_\alpha \to \mathrm{Ult}(J^{\vec{E}}_\alpha, E_\alpha)$ be the ultrapower embedding – write $\mathrm{Ult} := \mathrm{Ult}(J^{\vec{E}}_\alpha, E_\alpha)$. Since

$$i(\langle x_\beta \mid \beta < \kappa\rangle) \upharpoonright \kappa = \langle i(x_\beta) \mid \beta < \kappa\rangle,$$

the latter function, call it f, is an element in the ultrapower. But now

$$E_{\eta,\xi} = \{(a, x_\beta) \mid a \in [\eta]^{<\omega} \wedge a \in i(x_\beta)\},$$

which is a subset of $[\eta]^{<\omega} \times \operatorname{ran} f \in \mathrm{Ult}$, so comprehension implies that $E_{\eta,\xi} \in \mathrm{Ult}$. But since α is a cardinal in the ultrapower ($\alpha = \nu^{+\,\mathrm{Ult}}$), acceptability at α implies that $E_{\eta,\xi} \in J^{i(\vec{E} \upharpoonright \alpha)}_\alpha$. But $J^{i(\vec{E} \upharpoonright \alpha)}_\alpha = J^{\vec{E}}_\alpha$ by coherence, so $E_{\eta,\xi} \in J^{\vec{E}}_\alpha$, as wanted.

[9] Recall that $\mathcal{M} = \langle M, \in, A_0, \dots, A_n\rangle$ is **amenable** if $x \cap A_i \in M$ for every $x \in M$ and $i \leqslant n$. $\mathcal{M}$ is rud closed iff M is rud closed and $\mathcal{M}$ is amenable (Jensen, 1972, Corollary 1.4).

[10] This amenable coding was noted in (Steel & Mitchell, 1994), but not used. It is introduced in (Steel, 2010) and the following lemma is from that article.

For the "moreover" part, it suffices to show that whenever $A \subseteq \nu$ and $A \in \mathrm{Ult}$ then $A \in J_{\gamma_\xi+1}$, for some ξ, because $\mathcal{P}^{\mathrm{Ult}}(\nu)$ is cofinal in $J^{i(\vec{E}\restriction\alpha)}_{\nu^{+\mathrm{Ult}}} = J^{\vec{E}}_\alpha$, using acceptability and coherence again. Fix thus $A \subseteq \nu$ and $A \in \mathrm{Ult}$ – write $A = [a, f]$ with $a \subseteq \nu$.

Claim 1.18.1. We can assume that $f : J^{\vec{E}}_\kappa \to J^{\vec{E}}_\kappa$ and $\exists\xi < \kappa^{+J^{\vec{E}}_\alpha} : f \in J^{\vec{E}}_\xi$.

Proof of claim. Since $\operatorname{dom} f = [\kappa]^{|a|}$ and thus also $J^{\vec{E}}_\alpha \models |\operatorname{dom} f| = \kappa$, we can assume that $\operatorname{dom} f = J^{\vec{E}}_\kappa$. Furthermore, as $f^{a,a\cup\{\nu\}}(u) \subseteq \mathrm{pr}^{\{\nu\},a\cup\{\nu\}}(u)$ for $(E_\alpha)_{a\cup\{\nu\}}$-many u and that $\mathrm{pr}(u) < \kappa$, we can assume that $\operatorname{ran} f \subseteq J^{\vec{E}}_\alpha$. But then we also have that $f \subseteq J^{\vec{E}}_\kappa \times J^{\vec{E}}_\kappa$, so acceptability of α implies that $J^{\vec{E}}_\alpha \models f \in J^{\vec{E}}_{\kappa^+}$ and thus $f \in J^{\vec{E}}_\xi$ for some $\xi < \kappa^{+J^{\vec{E}}_\alpha}$. ⊣

Now for $\eta < \nu$, it holds that $\eta \in A$ iff

$$A_\eta := \{u \in [\kappa]^{|a\cup\{\eta\}|} \mid \mathrm{pr}^{\{\eta\},a\cup\{\eta\}}(u) \in f(u)\} \in (E_\alpha)_{a\cup\{\eta\}},$$

but since $f \in J^{\vec{E}}_\xi$ by the above claim, $A_\eta \in J^{\vec{E}}_\xi$ as well. But then A can be computed from $E_{\nu,\xi}$, which is an element of $J^{\vec{E}}_{\gamma_\xi}$, so $A \in J^{\vec{E}}_{\gamma_\xi+1}$. ■

Besides laying down the groundwork for our amenable encoding of the top extenders, it in fact also proves the non-amenability of the models – namely, since the γ_ξ's are cofinal in α, we get that $E_\alpha \cap ([\nu]^{<\omega} \times J^{\vec{E}}_{\kappa^{+J^{\vec{E}}_\alpha}}) \notin J^{\vec{E}}_\alpha$.

Before we introduce the amenable encoding, we will need the following general facts on fine extender sequences, which will often be used throughout the thesis.

Proposition 1.19. *Let $\vec{E}$ be a fine extender sequence. Then the following holds:*

(i) There are no cardinals strictly above ν_{E_α} in $J^{\vec{E}}_\alpha$;

(ii) If ν_{E_α} is a limit ordinal, then it's a cardinal in both $J^{\vec{E}}_\alpha$ and $\mathrm{Ult}(J^{\vec{E}\restriction\alpha}_\alpha, E_\alpha)$;

(iii) For every $\alpha \in \operatorname{dom}\vec{E}$, α is not a cardinal in $J^{\vec{E}}_{\alpha+1}$.

Proof. (i): Let $i : J^{\vec{E}}_\alpha \to \mathrm{Ult} := \mathrm{Ult}(J^{\vec{E}}_\alpha, E_\alpha)$ be the ultrapower embedding. Since $\alpha = \nu^{+\mathrm{Ult}}_{E_\alpha}$, $\mathrm{Ult} \models$ there are no cardinals $> \nu_{E_\alpha}$ in $J^{i(\vec{E}\restriction\alpha)}_\alpha$. But since $i(\vec{E}\restriction\alpha)$ is acceptable at every $\beta < \sup_{\gamma<\alpha} i(\gamma)$ as acceptability at a limit α is

Π_1-definable, we also have that $J_\alpha^{i(\vec{E}\upharpoonright\alpha)} \models$ there are no cardinals $> \nu_{E_\alpha}$. But since $J_\alpha^{\vec{E}} = J_\alpha^{i(\vec{E}\upharpoonright\alpha)}$ by coherence, this also holds in $J_\alpha^{\vec{E}}$.

(ii): Suppose not, and let $f : \eta \to \nu(E_\alpha)$ be a surjection such that $f \in J_\alpha^{\vec{E}}$ or equivalently $f \in \operatorname{Ult}(J_\alpha^{\vec{E}}, E_\alpha)$ by coherence. Write $f = [a, g]$ for $a \in [\nu(E_\alpha)]^{<\omega}$ and fix a generator $\xi \in (\max a \cup \{\eta\}, \nu(E_\alpha))$. Note then that $f \in \operatorname{ran}\sigma_\xi$, so that ξ isn't a cardinal in $\operatorname{Ult}(J_\alpha^{\vec{E}}, E_\alpha \upharpoonright \xi)$, contrary to ξ being a generator.

(iii): Writing $\nu := \nu_{E_\alpha}$, define the function $\varphi : [\nu]^{<\omega} \times (\mathcal{P}(\kappa) \cap J_\nu^{\vec{E}}) \to \alpha$ as $\varphi(a, x) = \beta$ iff $\operatorname{Ult}(J_\alpha^{\vec{E}}, E_\alpha) \models [a, f_x] = \beta$, where $f_x : [\kappa]^{|a|} \to \kappa$ corresponds to $x \in \mathcal{P}(\kappa)$. Since $J_\alpha^{\vec{E}} \models \mathcal{P}(\kappa) = \mathcal{P}(\kappa) \cap J_{\kappa^+}^{\vec{E}}$ and $\kappa^{+J_\alpha^{\vec{E}}} \leqslant \nu$, we have that φ is surjective. As it's furthermore definable with parameters over $J_\alpha^{\vec{E}}$, using that the amenable encoding for $(E_\alpha)_a$ is in $J_\alpha^{\vec{E}}$, we have that $\varphi \in J_{\alpha+1}^{\vec{E}}$. Since we also have a definable surjection from ν onto $\operatorname{dom}\varphi$, we get that α is not a cardinal in $J_{\alpha+1}^{\vec{E}}$. ■

Definition 1.20. Let $\vec{E}$ be a fine extender sequence, $\kappa := \operatorname{crit} E_\alpha$ and $\nu := \nu_{E_\alpha}$. The **amenable encoding** of E_α is then the set E_α^c defined as

$$\begin{aligned} \langle\gamma, \xi, a, x\rangle \in E_\alpha^c \quad &\Leftrightarrow \quad \gamma \in (\nu, \alpha) \wedge \xi \in (\kappa, \kappa^{+J_\alpha^{\vec{E}}}) \\ &\qquad \wedge \langle a, x\rangle \in E_{\gamma,\xi} \wedge E_{\nu,\xi} \in J_\gamma^{\vec{E}} \end{aligned}$$ ○

Proposition 1.21. *For any fine extender sequence $\vec{E}$ with $E_\alpha \neq \varnothing$, the structure $\langle J_\alpha^{\vec{E}}, \in, \vec{E} \upharpoonright \alpha, E_\alpha^c\rangle$ is amenable.*

Proof. Let $x \in J_\alpha^{\vec{E}}$. First of all $x \cap \vec{E} \in J_\alpha^{\vec{E}}$ by definition of $J_\alpha^{\vec{E}}$, so we need to show that $x \cap E_\alpha^c \in J_\alpha^{\vec{E}}$. But then there exists $\beta < \alpha$ and $\delta < \kappa^{+J_\alpha^{\vec{E}}}$ such that

$$x \cap E_\alpha^c = \{\langle\gamma, \xi, a, x\rangle \mid \langle\gamma, \xi\rangle \in (\nu, \beta) \times (\kappa, \delta) \wedge \langle a, x\rangle \in E_{\gamma,\xi} \wedge E_{\nu,\xi} \in J_\gamma^{\vec{E}}\},$$

so since β, δ and $E_{\gamma,\xi}$ are all elements of $J_\alpha^{\vec{E}}$, the latter by the amenability lemma, $x \cap E_\alpha^c \in J_\alpha^{\vec{E}}$ as well. ■

Note that from an amenable encoding E^c of an extender E, we can recover the extender itself - this should be fairly obvious using the Amenability Lemma 1.18. Due to this, if $F := E^c$ we will sometimes write things like $\langle a, x\rangle \in F$; consider this merely an innocent rephrasing of $\langle a, x\rangle$ being an element of the extender that F is encoding (namely E in this case).

The language that our models are built on will then contain symbols for the predicates $\in$, $\vec{E}$ and the top extender F. We will furthermore also put in some constant symbols for convenience, to make sure that they're put into all our hulls. Finally, we will include countably many predicate symbols $\dot{T}_n$'s, whose interpretation will be defined in the next chapter.

Definition 1.22. Define the language $\mathcal{L} := \{\in, \dot{E}, \dot{F}, \dot{\kappa}, \dot{\nu}, \dot{\gamma}\} \cup \{\dot{T}_n \mid 1 \leqslant n < \omega\}$, with $\dot{E}$ a 4-ary relation symbol, $\dot{F}$ a unary relation symbol, $\dot{\kappa}$, $\dot{\nu}$ and $\dot{\gamma}$ constant symbols and the $\dot{T}_n$'s trinary relation symbols. ○

We've thus arrived at our definition of our models, which will be set approximations to the corresponding proper class models. These approximations will (eventually) be called *mice*, but since we need to add some more properties to these structures, we will start off with the following *potential premice*.

Definition 1.23. A (fine structural) **potential premouse**[11] (ppm) is an $\mathcal{L}$-structure $\mathcal{M}$ of the form $\langle J^{\vec{E}}_\alpha, \vec{E} \restriction \alpha, (E_\alpha \restriction \alpha)^c, \dot{\kappa}^{\mathcal{M}}, \dot{\nu}^{\mathcal{M}}, \dot{\gamma}^{\mathcal{M}}, \dot{T}^{\mathcal{M}}_n\rangle$ with $\vec{E}$ being a fine extender sequence, satisfying the following:

- If ν_{E_α} is a limit ordinal strictly greater than $(\text{crit}\, E_\alpha)^{+\mathcal{M}}$ then $\alpha = \nu_{E_\alpha}$, and otherwise $\alpha = (\nu_{E_\alpha})^{+\,\text{Ult}(\mathcal{M}, E_\alpha \restriction \nu_{E_\alpha})}$;
- If $E_\alpha \neq \varnothing$ then $\dot{\kappa}^{\mathcal{M}} = \text{crit}\, E_\alpha$ and $\dot{\nu}^{\mathcal{M}} = \nu_{E_\alpha}$; otherwise $\dot{\kappa}^{\mathcal{M}} = \dot{\nu}^{\mathcal{M}} = 0$;
- If ν_{E_α} is a successor ordinal and $E_\alpha \neq \varnothing$ then there exists a longest non-type-Z proper initial segment F of E_α containing properly less information than E_α itself. Namely, $F := (E_\alpha \restriction \dot{\nu}^{\mathcal{M}} - 1)^*$ if it's not of type Z, and $F := (E_\alpha \restriction \nu_{E_\alpha \restriction \dot{\nu}^{\mathcal{M}} - 1} - 1)^*$ otherwise.

  Then $\dot{\gamma}^{\mathcal{M}}$ is the place on which F appears on $\dot{E}^{\mathcal{M}}$ or an ultrapower of $\dot{E}^{\mathcal{M}}$; i.e. that $\dot{\gamma}^{\mathcal{M}}$ is the unique $\xi \in \text{dom}\, \vec{E}$ such that $F = E_\xi$ if such a ξ exists, and otherwise the intial segment condition implies that F is on the extender sequence of $\text{Ult}(J^{\vec{E}}_\eta, E_\eta)$ where $\eta = \nu_G$ and $\dot{\gamma}^{\mathcal{M}} = \langle \eta, a, f\rangle$ where $F = [a, f]^{J^{\vec{E}}_\eta}_{E_\eta}$.[12]

Argue that the extenders and ultrapowers make sense in the type III case.

[11] This definition is called a coded ppm in (Steel, 2010). As we're only going to work with this code, we prefer to just call it a ppm.
[12] $\dot{\gamma}^{\mathcal{M}}$ is used to describe ppm-ness – we'll get back to this in the following chapter.

If $E_\alpha = \emptyset$ or if ν_{E_α} is not a successor ordinal, $\dot\gamma^{\mathcal{M}} = 0$. ∘

As mentioned earlier, we will postpone the interpretation of the $\dot T_n$'s until next chapter.

Definition 1.24. A ppm $\mathcal{M}$ is **active** if $\dot F^{\mathcal{M}} \neq \emptyset$ and **passive** otherwise. ∘

Definition 1.25. An active ppm $\mathcal{M}$ is said to be of **type I** if $\dot\nu^{\mathcal{M}} = (\dot\kappa^{\mathcal{M}})^{+\mathcal{M}}$, **type II** if $\dot\nu^{\mathcal{M}}$ is a successor ordinal, and **type III** if $\dot\nu^{\mathcal{M}}$ is a limit ordinal $> (\dot\kappa^{\mathcal{M}})^{+\mathcal{M}}$. ∘

The reason why we treated the type III case in the definition of fine extender sequence differently, is that if we didn't "cut down" the ppm to the natural length of the top extender then the ultrapower of that ppm using the top extender wouldn't be a ppm anymore[13].

Definition 1.26. Let $\mathcal{M}$ be a ppm. Then we call $\dot F^{\mathcal{M}}$ the **last extender** of $\mathcal{M}$ and we say that an extender E **is on the $\mathcal{M}$-sequence** if either $E = \dot E^{\mathcal{M}}_\gamma$ for some $\gamma \in \operatorname{dom} \dot E^{\mathcal{M}}$ or that $E = (\dot F^{\mathcal{M}})^*$. ∘

[13]The initial segment condition is what would break down, as we wouldn't necessarily have that $\sup i_F``\nu_F = i_F(\nu_F)$. See (Steel & Mitchell, 1994, Lemma 9.1).

2 Fine structure and premice

We will need to provide a detailed analysis of the structure of our ppms, analogous to the fine structural analysis of L in (Jensen, 1972). We cannot simply work with the usual Levy hierarchy Σ_n though, as it's possible to encode arbitrary information as Σ_2 predicates[1], which conflicts with our goal of trying to construct *canonical* models. Therefore, we will zoom in on the gap between Σ_1 and Σ_2 formulas.

2.1 Restricted formulas and hulls

Definition 2.1. Let φ be an $\mathcal{L}$-formula. Then

- φ is $r\Sigma_i$ if φ is Σ_i for $i = 0, 1$;
- φ is $r\Sigma_{n+1}$ for $n \geqslant 1$ if there is a Σ_1 formula $\psi(\eta, q, y, \vec{v})$ of the language $\mathcal{L} - \{\dot{T}_n \mid 1 \leqslant n < \omega\}$ such that

$$\varphi \equiv \exists\eta\exists q\exists y(\dot{T}_n(\eta, q, y) \wedge \psi(\eta, q, y, \vec{v})). \qquad \circ$$

The "r" stands for "restricted", as an $r\Sigma_n$ formula has a lot less expressive power than proper Σ_n formulae for $n \geqslant 2$.

Definition 2.2. A formula φ is Σ_n **over** $r\Sigma_k$ if there is an $r\Sigma_k$ formula ψ such that

$$\varphi \equiv \exists x_1 \forall x_2 \cdots Q x_n \psi,$$

where Q is either $\exists$ or $\forall$. The definition of Π_n over $r\Sigma_k$ is analogous.[2] $\circ$

Note that $r\Sigma_k$ definability is $r\Sigma_k$ definable, the proof being analogous to Σ_k definability being Σ_k definable (Devlin, 1984). Furthermore also note that for a ppm $\mathcal{M}$

[1] This example is due to Mitchell, and uses terminology which is to be defined in the upcoming chapters. Let $\langle\kappa_i \mid i < \omega\rangle$ be an increasing sequence of measurable cardinals of $\mathcal{N}$ with $\rho_1(\mathcal{N}) \leqslant \kappa_0$ and suppose that $\mathcal{N}$ is 1-sound and iterable. Let $a \subseteq \omega$ be *any* subset (this is the arbitrary information), and let $\mathcal{M}$ be the result from iterating $\mathcal{N}$ by hitting normal measures with critical point κ_i iff $i \in a$. Then a is $\Sigma_2^{\mathcal{M}}$ since $i \in a$ iff κ_i isn't $\Sigma_1^{\mathcal{M}}(\kappa_i \cup p_i(\mathcal{M}))$.

[2] Note here that Σ_1 over $r\Sigma_k$ is just $r\Sigma_k$, by definition of $r\Sigma_k$ formulas.

and $k \geqslant 1$, there exist $r\Sigma_k$ Skolem functions over $\mathcal{M}$, by just picking the $<_{\mathcal{M}}$-least witness – note here that $<_{\mathcal{M}}$ is Σ_1-definable, just as for L.

Definition 2.3. Let $\mathcal{M}$ be any $\mathcal{L}$-structure and let $x \subseteq \mathcal{M}$. Then the **n-theory** is

$$\mathrm{Th}_n^{\mathcal{M}}(x) := \{\langle k, p\rangle \in \omega \times x^{<\omega} \mid \mathcal{M} \models \varphi_k[p]\},$$

where $\langle \varphi_k \mid k < \omega\rangle$ is a recursive enumeration of all $r\Sigma_n$-formulas. ○

We can now define the interpretation of the $\dot{T}_n$'s in our ppms.

Definition 2.4. For a ppm $\mathcal{M}$, $\dot{T}_n^{\mathcal{M}}(\eta, q, y)$ holds iff $\mathrm{Th}_n^{\mathcal{M}}(\eta \cup \{q\}) = y$. ○

It perhaps looks like we've encountered a vicious circle, as it seems like the $\dot{T}_n$'s are using the definition of $r\Sigma_n$ formulae which again require the use of the $\dot{T}_n$'s. But keep in mind that in our definition of $r\Sigma_n$ formulae we're only using the *syntactical* symbol $\dot{T}_n$, where we only use the $r\Sigma_n$ formulae in our definition of the interpretation $\dot{T}_n^{\mathcal{M}}$ in a ppm $\mathcal{M}$.

Definition 2.5. Let $\mathcal{M}$ be any $\mathcal{L}$-structure, $X \subseteq \mathcal{M}$ and $n \leqslant \omega$. Then the n**'th hull of** X, $\mathrm{Hull}_n^{\mathcal{M}}(X)$, is the substructure of $\mathcal{M}$ whose universe consists of all $x \in \mathcal{M}$ such that $\{x\}$ is $r\Sigma_n^{\mathcal{M}}$ definable from parameters in X. The n**'th collapsed hull of** X, $\mathrm{cHull}_n^{\mathcal{M}}(X)$, is then the transitive collapse of $\mathrm{Hull}_n^{\mathcal{M}}(X)$. If $n = \omega$ we will omit the subscript. ○

Hulls will play an important role in the upcoming chapters. In numerous occations we might want to shift focus to another model by "going up" or "going down", analogous to shifting attention to a subset or a superset. Hulls provide the means for going down, and in the next chapter we will define ultrapowers which is a way of "going up". We need to make sure that we preserve ppmness when going up and down however, and the key notion here is the following.

Definition 2.6. A Q**-formula** $\psi(\vec{v})$ is an $\mathcal{L}$-formula of the form

$$\forall x \forall \theta < \dot{\kappa}^+ \exists y \supseteq x \exists \nu \in [\theta, \dot{\kappa}^+)(\varphi(y, \nu, \vec{v}) \wedge \forall a \in x \exists b \in y \chi(a, b, \vec{v}))$$

where φ is Σ_1 and does not have x or θ free, and χ is Σ_0 and does not have x or y free.[3] ○

It's easy to see that Q-formulae are preserved downwards by Σ_1-embeddings and upwards by cofinal Σ_0-embeddings, where $\pi : \mathcal{M} \to \mathcal{N}$ is **cofinal** if

- For every $y \in \mathcal{N}$ there exists $x \in \mathcal{M}$ with $y \in \pi(x)$;
- π"$\dot{\kappa}^{+\mathcal{M}}$ is a cofinal subset of $\pi(\dot{\kappa}^{+\mathcal{M}})$.

Since the hull embeddings $\mathrm{cHull}_n^{\mathcal{M}}(X) \to \mathcal{M}$ are always Σ_1-preserving as long as $n \geqslant 1$, we see that if we can define ppmness in a Q-fashion then the hulls would also be ppms. This is indeed the case.

Proposition 2.7. *There are Q-sentences σ_1, σ_2, σ_3 and σ_4 such that if $\mathcal{M}$ is a transitive $\mathcal{L}$-structure, then*

- *$\mathcal{M} \models \sigma_1$ iff $\mathcal{M}$ is a passive ppm;*
- *$\mathcal{M} \models \sigma_2$ iff $\mathcal{M}$ is of type I;*
- *$\mathcal{M} \models \sigma_3$ iff $\mathcal{M}$ is of type II;*
- *If $\mathcal{M}$ is a type III ppm then $\mathcal{M} \models \sigma_4$;*
- *If $\mathcal{M} \models \sigma_4$ and $(\mathrm{On}^{\mathcal{M}})^{+\,\mathrm{Ult}(\mathcal{M},\dot{F}^{\mathcal{M}})} + 1 \subseteq \mathrm{wfp}(\mathrm{Ult}(\mathcal{M}, \dot{F}^{\mathcal{M}}))$ then either $\mathcal{M}$ is a type III ppm or $\mathcal{M} \models \dot{\kappa}$ is a Shelah limit of Shelah cardinals.*[4]

Proof. See Lemmata 2.5 and 3.3 in (Steel & Mitchell, 1994). This is where we need $\dot{\gamma}^{\mathcal{M}}$, as it makes us able to describe the initial segment condition for the "last" initial segment of the models last extender. ■

The reason why the type III case has the extra complications is because determining whether or not $\mathcal{M}$ is type III requires taking an ultrapower to check whether or not $\mathrm{On}^{\mathcal{M}}$ is the natural length of the last extender of $\mathcal{M}$.

We can be in the frustrating case that taking such an ultrapower doesn't help, in that $i(\dot{\kappa}^{\mathcal{M}}) = \nu(\dot{F}^{\mathcal{M}})^{+\,\mathrm{Ult}(\mathcal{M},\dot{F}^{\mathcal{M}})}$ where $i : \mathcal{M} \to \mathrm{Ult}(\mathcal{M}, \dot{F}^{\mathcal{M}})$ is the ul-

[3] This definition deviates slightly both from the notion of Q-formula in (R. Schindler & Zeman, 2010) and the notion of rQ- and P-formula in (Steel & Mitchell, 1994), but all of these should be special cases of this notion.

[4] A Shelah cardinal is a certain large cardinal which is of stronger consistency strength than a Woodin cardinal.

trapower map (recall that $\operatorname{lh}\dot{F}^{\mathcal{M}} = \nu(\dot{F}^{\mathcal{M}})$ in this type III case). This property will however entail that $\mathcal{M}$ has a Shelah limit of Shelah cardinals, and since such cardinals are stronger than Woodins, it doesn't affect the inner model theory below a Woodin. We will thus from now on assume that

There is no proper class inner model with a Shelah limit of Shelahs.

Corollary 2.8. *Let $\mathcal{M}$ be a ppm.*

(i) If $\pi : \mathcal{H} \to_{\Sigma_1} \mathcal{M}$ then $\mathcal{H}$ is a ppm of the same type as $\mathcal{M}$;

(ii) If $\pi : \mathcal{M} \to_Q \mathcal{P}$ then $\mathcal{P}$ is a ppm of the same type as $\mathcal{M}$.

Proof. This is clear by Proposition 2.7 and the definition of pre-extender, noting that $\operatorname{lh}\dot{F}^{\mathcal{M}} = \mathrm{On}^{\mathcal{M}}$. ■

Corollary 2.9. *If $\mathcal{M}$ is a ppm and $X \subseteq \mathcal{M}$ then $\mathrm{cHull}_n^{\mathcal{M}}(X)$ is a ppm of the same type.* ■

2.2 Projecta, parameters and cores

We will now begin the fine structure of the $r\Sigma_n$ hierarchy, and in that regard we will define the *n'th projectum* $\rho_n(\mathcal{M})$, the *n'th standard parameter* $p_n(\mathcal{M})$ and the *n'th core* $\mathfrak{C}_n(\mathcal{M})$ of a ppm $\mathcal{M}$. These will be tools making it possible to examine the fine behaviour of our ppms. The idea behind the projectum and the standard parameter is the same as in (Jensen, 1972).

As all these definitions will depend upon eachother, we could have chosen to define them in the formally correct way by defining them all at once. We choose to be slightly less formal and define them one by one, with the caveat that the definitions will refer to each other. No vicious circle will occur however, as the definitions at the $(n+1)$'st level will only refer to the definitions at the n'th level. We start with the projectum.

Definition 2.10. For a ppm $\mathcal{M}$ and $n < \omega$, set $\rho_0^{\mathcal{M}} := \mathrm{On}^{\mathcal{M}}$ and $\rho_{n+1}^{\mathcal{M}} :=$ the least $\rho \leqslant \mathrm{On}^{\mathcal{M}}$ such that $\mathcal{P}(\rho) \cap \mathbf{r\Sigma}_{n+1}^{\mathcal{M}} \nsubseteq \mathcal{M}$. Then the **$n$'th projectum** of $\mathcal{M}$ is $\rho_n(\mathcal{M}) := \rho_n^{\mathfrak{C}_{n-1}(\mathcal{M})}$, and the **projectum** is then $\rho(\mathcal{M}) := \lim_n \rho_n(\mathcal{M})$.[5] ∘

[5] This is the notation used in (Sargsyan, 2014), which deviates from the notation in (R. Schindler & Zeman, 2010), where their $\rho(\mathcal{M})$ is our $\rho_1(\mathcal{M})$ for any ppm $\mathcal{M}$. Our notation corresponds to the

If $\mathcal{M} \models \mathsf{ZF}^-$ then $\rho(\mathcal{M}) = \mathrm{On}^{\mathcal{M}}$ by using Replacement.

Definition 2.11. Let $\mathcal{M}$ be a ppm, $k \leqslant \omega$ and κ some $\mathcal{M}$-cardinal. Then we say that $\mathcal{M}$ **k-projects to κ** if $\rho_k(\mathcal{M}) = \kappa$ and **k-projects across κ** if $\rho_k(\mathcal{M}) \leqslant \kappa$. We also define **projects to κ** and **projects across κ** if it holds for some $k \leqslant \omega$. ○

Note that the "new set" witnessed by the projectum is *boldface* definable, so it's with a parameter $x \in \mathcal{M}$. To provide a better analysis of this parameter, we note that we have a Σ_1-definable surjection from all finite sequences of ordinals in $\mathcal{M}$ to $\mathcal{M}$ itself[6], so we can assume our parameters are of that form.

Definition 2.12. A **parameter** is a finite strictly decreasing ordinal sequence. ○

We could then define the standard parameter as the least parameter, but it turns out that we need some more structure than that. We shall need the following technical definition.

Definition 2.13. Let $\mathcal{M}$ be a ppm and $p = \langle \alpha_1, \dots, \alpha_k \rangle$ a parameter in $\mathcal{M}$. Then p is **n-solid in $\mathcal{M}$** if $\mathcal{P}(\alpha_i) \cap r\Sigma_n^{\mathcal{M}}(p \upharpoonright i) \subseteq \mathcal{M}$ for every $i \leqslant k$. ○

Note that $p = \langle \alpha_1, \dots, \alpha_k \rangle$ is n-solid iff $\mathrm{Th}_n^{\mathcal{M}}(\alpha_i \cup p \upharpoonright i) \in \mathcal{M}$ for every $i \leqslant k$, so we define the **n-solidity witness for $p = \langle \alpha_1, \dots, \alpha_k \rangle$ in $\mathcal{M}$** as

$$w_n(p, \mathcal{M}) := \{\mathrm{Th}_n^{\mathcal{M}}(\alpha_i \cup p \upharpoonright i) \mid i \leqslant k\},$$

so that p is n-solid in $\mathcal{M}$ iff $w_n(p, \mathcal{M}) \in \mathcal{M}$.

Definition 2.14. Let $k < \omega$. Then the **k'th standard parameter** $p_k(\mathcal{M})$ of a ppm $\mathcal{M}$ is a sequence $\vec{q} := \langle (u_1, p_1), \dots, (u_k, p_k) \rangle$, where p_{i+1} is the lexicographically least parameter of $\mathfrak{C}_i(\mathcal{M})$ such that

$$\mathcal{P}(\rho_{i+1}(\mathcal{M})) \cap r\Sigma_{i+1}^{\mathfrak{C}_i(\mathcal{M})}(\{q_{i+1}\}) \nsubseteq \mathfrak{C}_i(\mathcal{M})$$

notation of (Steel & Mitchell, 1994) and (Steel, 2010), with the slight difference that our $\rho(\mathcal{M})$ being their $\rho_\omega(\mathcal{M})$.

[6]For a proof of this, see (R. Schindler & Zeman, 2010, Lemma 1.17).

and $u_i := w_i(p_i, \mathfrak{C}_i(\mathcal{M}))$ if p_i is i-solid and otherwise $u_i := \varnothing$. Set $p_0(\mathcal{M}) := \varnothing$. If $u_i \neq \varnothing$ for every i then we say $p_k(\mathcal{M})$ is **solid**. ○

Definition 2.15. Let $\mathcal{M}$ be a ppm and $n < \omega$. Then the n**'th core** of $\mathcal{M}$ is defined recursively as $\mathfrak{C}_0(\mathcal{M}) := \mathcal{M}$ and

$$\mathfrak{C}_{n+1}(\mathcal{M}) := \mathrm{cHull}_{n+1}^{\mathfrak{C}_n(\mathcal{M})}(\rho_{n+1}(\mathcal{M}) \cup p_{n+1}(\mathcal{M})).$$

For convenience we also set $\mathfrak{C}_{-1}(\mathcal{M}) := \mathcal{M}$. The **core** of $\mathcal{M}$ is then defined as $\mathfrak{C}(\mathcal{M}) := \lim_n \mathfrak{C}_n(\mathcal{M})$. ○

Corollary 2.16. *If $\mathcal{M}$ is a ppm then so is $\mathfrak{C}_n(\mathcal{M})$ for any $n \leqslant \omega$, of the same type.*

Proof. Directly from Corollary 2.9. ■

This finishes the "circular" definitions of projectum, standard parameter and core.

2.3 Premice

Another property that our standard parameter can have is the following.

Definition 2.17. Let $\mathcal{M}$ be a ppm. Then $p_{k+1}(\mathcal{M})$ is **universal** if

$$\mathcal{P}^{\mathfrak{C}_k(\mathcal{M})}(\rho_{k+1}(\mathcal{M})) \subseteq \mathfrak{C}_{k+1}(\mathcal{M}).$$ ○

Definition 2.18. A ppm $\mathcal{M}$ is

- n-**solid** if $p_n(\mathcal{M})$ is solid and universal;
- **solid** if it's n-solid for every $n < \omega$;
- n-**sound** if it's n-solid and $\mathfrak{C}_n(\mathcal{M}) = \mathcal{M}$;
- **sound** if it's n-sound for every $n < \omega$. ○

We can now use this soundness definition to show what solidity and universality is used for. The reason why solidity is useful is that it provides a lower bound for being a standard parameter, making it easier to show that the standard parameter is preserved under certain embeddings.

Proposition 2.19. *Let $\mathcal{M}$ be a k-sound ppm, $p = \langle \alpha_0, \dots, \alpha_k \rangle$ the last parameter of $p_{k+1}(\mathcal{M})$ and $q = \langle \beta_0, \dots, \beta_n \rangle$ a k-solid parameter of $\mathcal{M}$. Then $q \leqslant_{lex} p$.*

Proof. Assume that $p <_{\text{lex}} q$ and let $A \subseteq \rho_k(\mathcal{M})$ be $r\Sigma_k^{\mathcal{M}}(\{p\})$-definable such that $A \notin \mathcal{M}$ – this exists since $\mathcal{M} = \mathfrak{C}_k(\mathcal{M})$ by k-soundness. Let $t \leqslant \min\{k, n\}$ be least such that $\alpha_t \neq \beta_t$, so that $\alpha_t < \beta_t$. Solidity implies that

$$\mathcal{P}(\beta_t) \cap r\Sigma_k^{\mathcal{M}}(\{q \upharpoonright t\}) \subseteq \mathcal{M} .$$

But consider $B := A \cup \{\alpha_t, \dots, \alpha_k\}$. Since B is both $r\Sigma_k^{\mathcal{M}}(\{q \upharpoonright t\})$ and a subset of β_t, we have that $B \in \mathcal{M}$. But then $A \in \mathcal{M}$ as well, $\lightning$. ■

The usefulness of universality is the following lemma and corollary.

Lemma 2.20. *Let $k < \omega$, $\mathcal{M}$ a k-sound ppm, $p \in \mathcal{H}$ and $\pi : \mathcal{H} \to \mathcal{M}$ an $r\Sigma_{k+1}$-elementary embedding. Assume that $\rho_{k+1}(\mathcal{M}) \subseteq \mathrm{On}^{\mathcal{H}}$, $\pi \upharpoonright \rho_{k+1}(\mathcal{M}) = \mathrm{id}$, $\pi(p) = p_{k+1}(\mathcal{M})$ and $p_{k+1}(\mathcal{M})$ is universal. Then it holds that*

(i) $\rho_{k+1}(\mathcal{H}) = \rho_{k+1}(\mathcal{M})$;

(ii) $p = p_{k+1}(\mathcal{H})$;

(iii) $p_{k+1}(\mathcal{H})$ *is universal.*

Proof. (i): To show $\rho_{k+1}(\mathcal{H}) \geqslant \rho_{k+1}(\mathcal{M})$, let $\alpha < \rho_{k+1}(\mathcal{M})$ and let $A \subseteq \alpha$ be $\mathbf{r\Sigma}_{k+1}^{\mathcal{H}}$. Then $A = \pi"A$ is $\mathbf{r\Sigma}_{k+1}^{\mathcal{M}}$ by $r\Sigma_{k+1}$-elementarity, so $A \in \mathcal{M}$ by definition of $\rho_{k+1}(\mathcal{M})$. By acceptability we thus get that $A \in \mathcal{M}\,|\rho_{k+1}(\mathcal{M}) = \mathcal{H}|\rho_{k+1}(\mathcal{M})$, which shows that $\rho_{k+1}(\mathcal{M}) \leqslant \rho_{k+1}(\mathcal{H})$. As for $\rho_{k+1}(\mathcal{H}) \leqslant \rho_{k+1}(\mathcal{M})$, let $A \subseteq \rho_{k+1}(\mathcal{M})$ be $r\Sigma_{k+1}^{\mathcal{H}}(p)$-definable and assume that $A \in \mathcal{H}$. Then $A = \pi(A) \cap \rho_{k+1}(\mathcal{M})$ is $r\Sigma_{k+1}^{\mathcal{M}}(p_{k+1}(\mathcal{M}))$-definable and $A \in \mathcal{M}$. Contraposing we get that letting $A \subseteq \rho_{k+1}(\mathcal{M})$ be $r\Sigma_{k+1}^{\mathcal{M}}(p_{k+1}(\mathcal{M}))$-definable such that $A \notin \mathcal{M}$, A is $r\Sigma^{\mathcal{H}}(p)$-definable and $A \notin \mathcal{H}$. Thus $\rho_{k+1}(\mathcal{H}) = \rho_{k+1}(\mathcal{M})$.

(ii): By the argument above, we have an $r\Sigma_{k+1}^{\mathcal{H}}(p)$-definable $A \subseteq \rho_{k+1}(\mathcal{H})$ such that $A \notin \mathcal{H}$, so that $p_{k+1}(\mathcal{H}) \leqslant_{\text{lex}} p$. Assume that $p_{k+1}(\mathcal{H}) <_{\text{lex}} p$, so that $\pi(p_{k+1}(\mathcal{H})) <_{\text{lex}} \pi(p) = p_{k+1}(\mathcal{M})$. Let $B \subseteq \rho_{k+1}(\mathcal{M})$ be $r\Sigma_{k+1}^{\mathcal{M}}(\pi(p_{k+1}(\mathcal{H})))$, so that minimality of $p_{k+1}(\mathcal{M})$ implies that $B \in \mathcal{M}$, so that $B \in \mathcal{H}$ by universality of $p_{k+1}(\mathcal{H})$. But then B is an $r\Sigma_{k+1}^{\mathcal{H}}(p_{k+1}(\mathcal{H}))$-definable subset of $\rho_{k+1}(\mathcal{H})$ in $\mathcal{H}$, so that since B was arbitrary, $p_{k+1}(\mathcal{H})$ is not the $k+1$'th standard parameter of $\mathcal{H}$, $\lightning$.

(iii): Let $A \subseteq \rho_{k+1}(\mathcal{H})$ with $A \in \mathcal{H}$. Then $A = \pi(A) \cap \rho_{k+1}(M) \in \mathcal{M}$, so A is $r\Sigma_{k+1}^{\mathcal{M}}(\rho_{k+1}(\mathcal{M}) \cup p_{k+1}(\mathcal{M}))$-definable since $\rho_{k+1}(\mathcal{M}) = \rho_{k+1}(\mathcal{H})$ and universality of $p_{k+1}(\mathcal{M})$, and thus also $r\Sigma_{k+1}^{\mathcal{H}}(\rho_{k+1}(\mathcal{H}) \cup p_{k+1}(\mathcal{H}))$ by pulling the definition back via π, using (i) and (ii). Thus $p_{k+1}(\mathcal{H})$ is universal. ■

Corollary 2.21. *Let $k < \omega$, $\mathcal{M}$ a k-sound ppm and assume that $p_{k+1}(\mathcal{M})$ is universal. Then $\rho_{k+1}(\mathfrak{C}_{k+1}(\mathcal{M})) = \rho_{k+1}(\mathcal{M})$, $p_{k+1}(\mathfrak{C}_{k+1}(\mathcal{M})) = p_{k+1}(\mathcal{M})$ and $p_{k+1}(\mathfrak{C}_{k+1}(\mathcal{M}))$ is universal. Furthermore, if $\mathcal{M}$ is $(k+1)$-solid then $\mathfrak{C}_{k+1}(\mathcal{M})$ is $(k+1)$-sound.*

Proof. The first statement is directly from Lemma 2.20. As for the last part we clearly have that $\mathfrak{C}_{k+1}(\mathfrak{C}_{k+1}(\mathcal{M})) = \mathfrak{C}_{k+1}(\mathcal{M})$, and solidity holds as we've put the solidity witnesses into $p_{k+1}(\mathcal{M})$. ■

We're interested in embeddings preserving all the fine structure up to a given level n – these will be called *n-embeddings*.

Definition 2.22. Let $j : \mathcal{M} \to \mathcal{N}$ be a map between ppms and let $n < \omega$. Then j is a **near n-embedding** if

(i) $\mathcal{M}$ and $\mathcal{N}$ are n-sound;
(ii) j is $r\Sigma_{n+1}$ elementary;
(iii) $j(p_i(\mathcal{M})) = p_i(\mathcal{N})$ for all $i \leqslant n$;
(iv) $j(\rho_i(\mathcal{M})) = \rho_i(\mathcal{N})$ for all $i < n$, and $\sup j``\rho_n(\mathcal{M}) \leqslant \rho_n(\mathcal{N})$.

If the $\leqslant$ in (iv) is an equality then j is an **n-embedding**. ○

We then see by Corollary 2.21 that if $\mathfrak{C}_k(\mathcal{M})$ is k-sound then the **core embedding** $\mathfrak{C}_{k+1}(\mathcal{M}) \to \mathfrak{C}_k(\mathcal{M})$, i.e. the uncollapse, is a k-embedding. As for whether or not the k'th core is in fact k-sound, see the discussion at the end of this chapter.

We would prefer to be able to work with only sound ppms, but these turn out to not be preserved by ultrapowers, as we will see in the next chapter. The property that all *initial segments* of ppms are sound *is* preserved though, which is the next

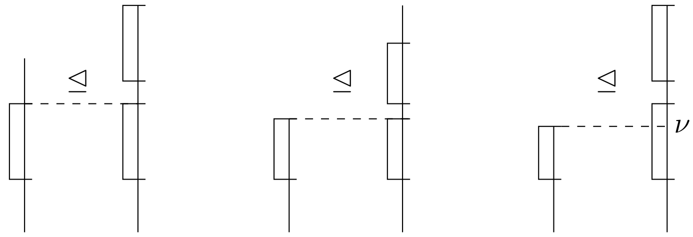


Figure 2.1: A passive, type I/II and type III initial segment.

best thing. We have to be a bit careful in defining initial segments though, as we have to ensure that initial segments are in fact ppms as well, which isn't trivial if the initial segment is of type III.

Definition 2.23. If $\mathcal{M}$ is a ppm and $\xi < \mathrm{On}^{\mathcal{M}}$, then define the $\mathcal{L}$-structure

$$\mathcal{M}\,|\xi := \begin{cases} \langle J_\xi^{\dot{E}^{\mathcal{M}}}, \dot{E}^{\mathcal{M}} \restriction \xi, (\dot{E}_\xi^{\mathcal{M}})^c \rangle & \text{, if this is of type I or II} \\ \langle J_\nu^{\dot{E}^{\mathcal{M}}}, \dot{E}^{\mathcal{M}} \restriction \nu, (\dot{E}_\xi^{\mathcal{M}} \restriction \nu)^c \rangle & \text{, otherwise, where } \nu := \nu(\dot{E}_\xi^{\mathcal{M}}). \end{cases}$$

where the constant symbols $\dot{\kappa}$, $\dot{\nu}$ and $\dot{\gamma}$ are interpreted as usual. If $\xi = \mathrm{On}^{\mathcal{M}}$ we also set $\mathcal{M}\,|\xi := \mathcal{M}$. ○

Definition 2.24. If $\mathcal{M}$ is a ppm then an **initial segment** of $\mathcal{M}$ is a ppm $\mathcal{N}$ of the form $\mathcal{M}\,|\xi$ for some $\xi \leqslant \mathrm{On}^{\mathcal{M}}$ and we write $\mathcal{N} \trianglelefteq \mathcal{M}$. If $\xi < \mathrm{On}^{\mathcal{M}}$ then $\mathcal{N}$ is a **proper initial segment** of $\mathcal{M}$ and we write $\mathcal{N} \triangleleft \mathcal{M}$ in this case.[7] ○

Definition 2.25. A **premouse** is a ppm all of whose proper initial segments are sound. ○

When building mice, we thus want to ensure that every initial segment is sound. To ensure this, we will build our mice bottom up, taking cores along the way,

[7]This is where we diverge a bit from (Steel & Mitchell, 1994) due to our "type III" clause in the definition of $\mathcal{M}\,|\xi$. They only have the naive version of initial segment, but on the other hand they have to "squash" their type III ppms whenever they take ultrapowers of them (where all our mice are "squashed" from the beginning). It is to note that using our method, we avoid certain anomalous cases for type III mice described in (Steel, 2010). A downside is possibly that player I has fewer options when picking extenders in the iteration game defined in chapter 4, so iteration strategies might then be slightly weaker.

so we would like that $\mathfrak{C}(\mathcal{M})$ is sound. If $\mathcal{M}$ is 1-solid then the above Corollary 2.21 implies that $\mathfrak{C}_1(\mathcal{M})$ is 1-sound. If we could then show that $\mathfrak{C}_1(\mathcal{M})$ is 2-solid (equivalently, that $\mathcal{M}$ is 2-solid), then $\mathfrak{C}_2(\mathcal{M})$ would be 2-sound, and so on.

We thus seem to require that $\mathcal{M}$ has this special property that $\mathfrak{C}_k(\mathcal{M})$ is $(k+1)$-solid for every $k < \omega$. In chapter 6 we will produce certain ppms $\mathcal{M}$ such that $\mathfrak{C}_k(\mathcal{M})$ is *fully k-iterable* for every $k < \omega$ (iterability will be introduced in chapter 4) and in chapter 5 we show that this implies that $\mathfrak{C}_k(\mathcal{M})$ is in fact $(k+1)$-solid. Inductively we then get that $\mathfrak{C}(\mathcal{M})$ is sound.

3 Fine ultrapowers

We're going to strengthen the ultrapower defined in 1.5 to accomodate for the fine structure from the previous chapter. We'd like to make sure that our ultrapowers preserve the fine structure, meaning that the ultrapower embeddings should be n-embeddings for a suitable n. We will therefore define n-ultrapowers and show that these actually satisfy this property.

3.1 Łoś' Theorem

Definition 3.1. Let $n < \omega$, $\mathcal{M}$ an n-sound ppm and let E be a (κ, λ)-pre-extender over $\mathcal{M}$ such that $\kappa < \rho_n(\mathcal{M})$. Then the n-**ultrapower** is defined as $\mathrm{Ult}_0(\mathcal{M}, E) := \mathrm{Ult}(\mathcal{M}, E)$ and for $n > 0$ it is

$$\mathrm{Ult}_n(\mathcal{M}, E) := \{(a, f) \mid a \in [\lambda]^{<\omega} \wedge f : [\kappa]^{|a|} \to \mathcal{M} \wedge f \in \mathbf{r\Sigma}_n^{\mathcal{M}}\}/\sim_E$$

with $\sim_E$ and $\in_E$ defined as usual. For $\mathcal{N} := \mathrm{Ult}_n(\mathcal{M}, E)$ we furthermore define

$$\begin{aligned} [a, f] \in \dot{E}^{\mathcal{N}} \quad &\text{iff} \quad \forall^{E_a} u : f(u) \in \dot{E}^{\mathcal{M}} \\ [a, f] \in \dot{F}^{\mathcal{N}} \quad &\text{iff} \quad \forall^{E_a} u : f(u) \in \dot{F}^{\mathcal{M}}, \end{aligned}$$

and $\dot{\kappa}^N$, $\dot{\nu}^{\mathcal{N}}$, $\dot{\gamma}^{\mathcal{N}}$, $\dot{T}_n^{\mathcal{N}}$ are defined as usual. ○

Whenever $\mathrm{Ult}_n(\mathcal{M}, E)$ is wellfounded we will identify it with its transitive collapse. The reason why we require that $\kappa < \rho_n(\mathcal{M})$ is because we want the ultrapower embedding to preserve the fine structure up to level n, so that if $\rho_n(\mathcal{M}) \leqslant \kappa$ then $\rho_n(\mathrm{Ult}) \leqslant \kappa$ and then by n-soundness κ would be definable in the ultrapower, a contradiction.

We proceed now to a version of Łoś' Theorem for our fine ultrapowers. This will ensure that the ultrapower embedding into the n-ultrapower is $r\Sigma_n$-elementary.

Theorem 3.2 (Łoś). *Let $n < \omega$, $\mathcal{M}$ an n-sound ppm and E a (κ, λ)-pre-extender over $\mathcal{M}$ where $\kappa < \rho_n(\mathcal{M})$. Write* $\mathrm{Ult} := \mathrm{Ult}_n(\mathcal{M}, E)$. *Then given $[a_i, f_i] \in \mathrm{Ult}$ for $i \leqslant k$ it holds that, setting $b := \bigcup_{i \leqslant k} a_i$,*

$$\mathrm{Ult} \models \varphi[[a_0, f_0], \dots, [a_k, f_k]] \quad \textit{iff} \quad \forall^{E_b} u : \mathcal{M} \models \varphi[f_0^{a_0,b}(u), \dots, f_k^{a_k,b}(u)]$$

for any $r\Sigma_n$ formula φ.

Proof. Since $\mathcal{M}$ is transitive and rud closed we get the result for $n = 0$ by Łoś' Theorem 1.6. Assume thus that $n > 0$. We will show that the result holds for $r\Sigma_i$ formulas, where $i \leqslant n$.

If $i = 0$ the usual argument for Σ_0 formulas goes through, see the proof of Łoś' Theorem 1.6. Assume $i = 1$ and φ be Σ_1, say $\varphi \equiv \exists x \psi$ for $\psi \in \Sigma_0$. Assume that $\mathrm{Ult} \models \psi[[a, f], [a_0, f_0], \dots, [a_k, f_k]]$ for some $[a, f] \in \mathrm{Ult}$. Then by the $i = 0$ case it holds that

$$\mathcal{M} \models \psi[f^{a, a \cup b}(u), f_0^{a_0, a \cup b}(u), \dots, f_k^{a_k, a \cup b}(u)]$$

for $E_{a \cup b}$-many u. Say $\theta \in \mathbf{r\Sigma}_n$ defines f in $\mathcal{M}$. Then

$$\mathcal{M} \models \exists x(\theta[u, x] \wedge \psi[x, f_0^{a_0, a \cup b}(u), \dots, f_k^{a_k, a \cup b}(u)])$$

for $E_{a \cup b}$-many u, and thus also for E_b-many u by coherence. In particular we get that $\mathcal{M} \models \varphi[f_0^{a_0,b}(u), \dots, f_k^{a_k,b}(u)]$ for E_b-many u, as wanted. Assume now this latter fact. Let τ be a Σ_1 Skolem function for φ, so that

$$\mathcal{M} \models \psi[\tau[f_0^{a_0,b}(u), \dots, f_k^{a_k,b}(u)], f_0^{a_0,b}(u), \dots, f_k^{a_k,b}(u)]$$

for E_b-many u. Note that $h := \tau \circ (f_0^{a_0,b}, \dots, f_k^{a_k,b})$ is $\mathbf{r\Sigma}_n$ definable over $\mathcal{M}$, representing an element $[b, h] \in \mathrm{Ult}$, so that

$$\mathrm{Ult} \models \psi[[b, h], [a_0, f_0], \dots, [a_k, f_k]]$$

and thus also $\mathrm{Ult} \models \varphi[[a_0, f_0], \dots, [a_k, f_k]]$, which is what we wanted to show.

Assume now that it holds for i – we'll show it holds for $i + 1$. Let $\varphi(\vec{v}) \in r\Sigma_{i+1}$; write $\varphi \equiv \exists \eta \exists q \exists y(\dot{T}_i(\eta, q, y) \wedge \psi(\eta, q, y, \vec{v}))$ with ψ being Σ_1. Assume first that Ult $\models \varphi[[a_0, f_0], \ldots, [a_k, f_k]]$ and write $\mathrm{Th}_i^{\mathrm{Ult}}(\eta \cup \{q\}) = [a, f]$, $\eta = [c_1, g_1], q = [c_2, g_2] \in \mathrm{Ult}$. Then

$$\mathrm{Ult} \models \psi[[c_1, g_1], [c_2, g_2], [a, f], [a_0, f_0], \ldots, [a_k, f_k]]$$

and thus $\mathcal{M} \models \psi[g_1^{a_1,d}(u), g_2^{a_2,d}(u), f^{a,d}(u), f_0^{a_0,d}(u), \ldots, f_k^{a_k,d}(u)]$ for E_d-many u, where $d := a_1 \cup a_2 \cup a \cup b$.

Claim 3.2.1. $a = \mathrm{Th}_j^{\mathcal{N}}(b)$ is uniformly Π_1 over $r\Sigma_j$ definable.

Proof of claim. Recall that

$$\mathrm{Th}_j^{\mathcal{N}} = \{\langle l, s\rangle \mid l < \omega \wedge s \in [b]^{<\omega} \wedge \mathcal{N} \models \varphi_l[s]\}$$

where $\langle \varphi_l \mid l < \omega\rangle$ is a recursive enumeration of all $r\Sigma_j$ formulae. Then $a = \mathrm{Th}_j^{\mathcal{N}}$ iff the following formula holds:

$$\forall x[x \in a \leftrightarrow (\mathsf{fst}(x) \in \omega \wedge \mathsf{snd}(x) \in [b]^{<\omega} \wedge \mathcal{N} \models \varphi_l[\mathsf{snd}(x)])]$$

Since $r\Sigma_j$ satisfaction is $r\Sigma_j$ definable, $a = \mathrm{Th}_j^{\mathcal{N}}(b)$ is thus uniformly Π_1 over $r\Sigma_j$ definable. $\dashv$

The claim, induction hypothesis and coherence then implies that

$$\begin{aligned} &\mathrm{Ult} \models [a, f] = r\Sigma_i([c_1, g_1] \cup \{[c_2, g_2]\}) \\ &\quad \text{iff} \quad \forall^{E_e} u : \mathcal{M} \models f^{a,e}(u) = r\Sigma_i(g_1^{c_1,e}(u) \cup \{g_2^{c_2,e}(u)\}) \end{aligned} \tag{1}$$

where $e := a \cup c_1 \cup c_2$. Here the last equivalence is due to coherence.

Claim 3.2.2. $\rho_i(\mathrm{Ult}) = j(\rho_i(\mathcal{M}))$ if $\rho_i(\mathcal{M}) < \mathrm{On}^{\mathcal{M}}$ and $\rho_i(\mathrm{Ult}) = \mathrm{On}^{\mathrm{Ult}}$ otherwise, with $j : \mathcal{M} \to \mathrm{Ult}$ being the ultrapower embedding.

Proof of claim. Assume first $\rho_i(\mathcal{M}) < \mathrm{On}^{\mathcal{M}}$. To show $\rho_i(\mathrm{Ult}) \leqslant j(\rho_i(\mathcal{M}))$ pick $q \in \mathcal{M}$ such that $\mathrm{Th}_i^{\mathcal{M}}(\rho_i(\mathcal{M}) \cup \{q\}) \notin \mathcal{M}$, whence Claim 3.2.1 ensures $\mathrm{Th}_i^{\mathrm{Ult}}(j(\rho_i(\mathcal{M})) \cup \{j(q)\}) \notin \mathrm{Ult}$, so that $\rho_i(\mathrm{Ult}) \leqslant j(\rho_i(\mathcal{M}))$.

Conversely, let $\alpha = [a, f] < j(\rho_i(\mathcal{M}))$ and let $q = [a, g]$. Without loss of generality $f(u) < \rho_i(\mathcal{M})$ for every u. Define $h : [\kappa]^{|a|} \to \mathcal{M}$ as given by $h(u) := \mathrm{Th}_i^{\mathcal{M}}(f(u) \cup \{g(u)\})$, so that h is $\mathbf{r\Sigma}_n^{\mathcal{M}}$ definable and then by Claim 3.2.1 we get that $\mathrm{Th}_i^{\mathrm{Ult}}(\alpha \cup \{q\}) = [a, h] \in \mathrm{Ult}$. Since α was arbitrary, $j(\rho_i(\mathcal{M})) \leqslant \rho_i(\mathrm{Ult})$.

If $\rho_i(\mathcal{M}) = \mathrm{On}^{\mathcal{M}}$ then repeat the argument in the previous paragraph to show that given any $\alpha \in \mathrm{On}^{\mathrm{Ult}}$ and $q \in \mathrm{Ult}$, $\mathrm{Th}_i^{\mathrm{Ult}}(\alpha \cup \{q\}) \in \mathrm{Ult}$, so that $\rho_i(\mathrm{Ult}) = \mathrm{On}^{\mathrm{Ult}}$, as wanted. ⊣

Now this claim, (1) and coherence implies that

$$\begin{aligned} &\mathrm{Ult} \models \dot{T}_i([c_1, g_1], [c_2, g_2], [a, f]) \\ &\quad \text{iff} \quad \forall^{E_e} u : \mathcal{M} \models \dot{T}_i(g_1^{c_1,e}(u), g_2^{c_2,e}(u), f^{a,e}(u)). \end{aligned} \tag{2}$$

But then, for E_e-many u it holds that

$$\mathcal{M} \models \exists\eta\exists q\exists y(\dot{T}_i[\eta, q, y] \wedge \psi[\eta, q, y, f_0^{a_0,e}(u), \dots, f_k^{a_k,e}(u)])$$

and thus also for E_b-many u by coherence again, which is what we wanted to show. Now assume this latter fact. Let τ_1, τ_2, τ_3 be $r\Sigma_n$ Skolem functions such that

$$\mathcal{M} \models \dot{T}_i(\tau_1[u], \tau_2[u], \tau_3[u]) \wedge \psi[\tau_1[u], \tau_2[u], \tau_3[u], f_0^{a_0,b}(u), \dots, f_k^{a_k,b}(u)]$$

so that (2) and the $i = 1$ case implies that Ult satisfies

$$\dot{T}_i([b, \tau_1], [b, \tau_2], [b, \tau_3]) \wedge \psi[[b, \tau_1], [b, \tau_2], [b, \tau_3], [a_0, f_0], \dots, [a_k, f_k]]$$

concluding that $\mathrm{Ult} \models \varphi[[a_0, f_0], \dots, [a_k, f_k]]$ holds as well. ■

As usual, Łoś' Theorem implies that the n-ultrapower embedding is $r\Sigma_n$-elementary. We can use this to prove that ppmness is preserved under taking ultrapowers.

Proposition 3.3. *Let $n < \omega$, $\mathcal{M}$ an n-sound ppm and E a (κ, λ)-extender over $\mathcal{M}$ such that $\kappa < \rho_n(\mathcal{M})$. Then $\mathrm{Ult} := \mathrm{Ult}_n(\mathcal{M}, E)$ is a ppm as well, of the same kind as $\mathcal{M}$.*

Proof. We just have to check that the ultrapower embedding $j : \mathcal{M} \to \mathrm{Ult}$ is Q-preserving by Corollary 2.8, so it's enough to check that it's cofinal Σ_0 preserving. It's clearly Σ_0-preserving by Łoś' Theorem 3.2, so we show that it's cofinal. Let $[a, f] \in \mathrm{Ult}$. Then $f(u) \in \operatorname{ran} f$ for a.e. u, so $[a, f] \in j(\operatorname{ran} f)$. Furthermore we have that $\kappa^{+\mathcal{M}} = \kappa^{+\mathrm{Ult}}$, so that $\sup j"\kappa^{+\mathcal{M}} = \kappa^{+\mathrm{Ult}} = j(\kappa^{+\mathcal{M}})$, making it cofinal. ■

3.2 The ultrapower embedding is an n-embedding

We will work towards proving that the ultrapower embedding associated to an n-ultrapower is not only just $r\Sigma_n$-elementary, but also an n-embedding. We first focus on the preservation of the projecta.

Lemma 3.4. *Let $n < \omega$, $\mathcal{M}$ an n-sound ppm and E a (κ, λ)-pre-extender over $\mathcal{M}$ such that $\kappa < \rho_n(\mathcal{M})$. Letting $\mathrm{Ult} := \mathrm{Ult}_n(\mathcal{M}, E)$, $j : \mathcal{M} \to \mathrm{Ult}$ be the ultrapower embedding and $i < n$, it holds that*

(i) $\rho_i(\mathcal{M}) < \mathrm{On}^{\mathcal{M}}$ iff $\rho_i(\mathrm{Ult}) < \mathrm{On}^{\mathrm{Ult}}$;

(ii) $\rho_i(\mathrm{Ult}) = j(\rho_i(\mathcal{M}))$ if $\rho_i(\mathcal{M}) < \mathrm{On}^{\mathcal{M}}$.

Moreover, $\rho_n(\mathrm{Ult}) = \sup j"\rho_n(\mathcal{M})$.

Proof. (i) and (ii) is by Claim 3.2.2 in the proof of Łoś' Theorem 3.2, so we just have to prove that $\rho_n(\mathrm{Ult}) = \sup j"\rho_n(\mathcal{M})$. We start by proving $\leqslant$. Here it suffices to show that

$$\mathrm{Hull}_n^{\mathrm{Ult}}(\sup j"\rho_n(\mathcal{M}) \cup \{j(p_n(\mathcal{M}))\}) = \mathrm{Ult} \tag{1}$$

since $\rho_n(\mathrm{Ult})$ is least with this property. Let thus $[a, f] \in \mathrm{Ult}$, where $f \in \mathbf{r\Sigma}_n^{\mathcal{M}}$, φ is $r\Sigma_n$ and $q \in [\rho_n(\mathcal{M}) \cup \{p_n(\mathcal{M})\}]^{<\omega}$ be such that

$$f(u) = y \quad \text{iff} \quad \mathcal{M} \models \varphi[u, y, q],$$

using that $\mathcal{M}$ is n-sound. But then by Łoś' Theorem 3.2 we get that

$$\begin{aligned}[a, f] = y \quad &\text{iff} \quad \mathrm{Ult} \models \varphi[[a, \mathrm{id}], y, [\{\kappa\}, c_q]] \\ &\text{iff} \quad \mathrm{Ult} \models \varphi[a, y, j(q)],\end{aligned}$$

where the last bi-implication is by Proposition 1.13. But since

$$j(q) \in [\sup j\text{"}\rho_n(\mathcal{M}) \cup \{j(p_n(\mathcal{M}))\}]^{<\omega}$$

and $a \in [\lambda]^{<\omega} \subseteq [j(\kappa)]^{<\omega} \subseteq [\sup j\text{"}\rho_n(\mathcal{M})]^{<\omega}$ by shortness and assumption,

$$[a, f] \in \mathrm{Hull}_n^{\mathrm{Ult}}(\sup j\text{"}\rho_n(\mathcal{M}) \cup \{j(p_n(\mathcal{M}))\})$$

and we've shown (1) and thus also $\leqslant$. For $\geqslant$, Łoś' Theorem 3.2 gives us that

$$\mathrm{Th}_n^{\mathcal{M}}(a) = b \quad \text{iff} \quad \mathrm{Th}_n^{\mathrm{Ult}}(j(a)) = j(b),$$

which entails that

$$\mathrm{Th}_n^{\mathrm{Ult}}(\gamma \cup \{j(p)\}) \in \mathrm{Ult} \ \text{ for every } \gamma < \sup j\text{"}\rho_n(\mathcal{M}) \text{ and } p \in \mathcal{M} \tag{2}$$

Let $\alpha < \sup j\text{"}\rho_n(\mathcal{M})$ and $q \in \mathrm{Ult}$ (so that now our parameter q is arbitrary, where in (2) we required that it was in $\operatorname{ran} j$); it suffices to show that

$$\mathrm{Th}_n^{\mathrm{Ult}}(\alpha \cup \{q\}) \in \mathrm{Ult}\,. \tag{3}$$

Let γ be such that $\gamma \in (\alpha, \sup j\text{"}\rho_n(\mathcal{M}))$ and that $q \in \mathrm{Hull}_n^{\mathrm{Ult}}(\gamma \cup \{j(p_n(\mathcal{M}))\})$, which is possible by (1). Then $\mathrm{Th}_n^{\mathrm{Ult}}(\gamma \cup \{j(p_n(\mathcal{M}))\}) \in \mathrm{Ult}$ by (2), so since $\alpha < \gamma$ and $q \in \mathrm{Hull}_n^{\mathrm{Ult}}(\gamma \cup \{j(p_n(\mathcal{M}))\})$, (3) holds as well. We conclude that $\rho_n(\mathrm{Ult}) = \sup j\text{"}\rho_n(\mathcal{M})$. ■

Lemma 3.5. *Let $n < \omega$, $\mathcal{M}$ an n-sound ppm and E a (κ, λ)-pre-extender over $\mathcal{M}$ such that $\kappa < \rho_n(\mathcal{M})$. Then the ultrapower embedding $j : \mathcal{M} \to \mathrm{Ult}_n(\mathcal{M}, E)$ is $r\Sigma_{n+1}$ elementary.*

Proof. Write $\mathrm{Ult} := \mathrm{Ult}_n(\mathcal{M}, E)$. Let $\alpha < \rho_n(\mathrm{Ult})$ and $q = [a, f] \in \mathrm{Ult}$. The substantial part of the proof will be to show that for every $[b, g] \in \mathrm{Ult}$,

$$\dot{T}_n^{\mathrm{Ult}}(\alpha, q, [b, g]) \quad \text{iff} \quad \forall^* u : \dot{T}_n^{\mathcal{M}}(\mathrm{pr}(u), f(u), g(u)). \tag{1}$$

The $\Leftarrow$ direction is directly by Łoś' Theorem 3.2 as $\dot{T}_n^{\mathcal{N}}(\gamma, r, y)$ is $r\Sigma_n$ definable. Assume thus the left-hand side of (1). In the proof of Lemma 3.4 we showed that

$$\mathrm{Ult} = \mathrm{Hull}_n^{\mathrm{Ult}}(\rho_n(\mathrm{Ult}) \cup \{j(p_n(\mathcal{M}))\}),$$

so fix some $r\Sigma_n$ formula θ such that $\mathrm{Ult} \models \theta[x, j(p_n(\mathcal{M})), s]$ iff $x = q$, where $s \in [\rho_n(\mathrm{Ult})]^{<\omega}$. Pick $\gamma < \rho_n(\mathcal{M})$ such that $\max(s \cup \{\alpha\}) < j(\gamma)$, which is possible as $\rho_n(\mathrm{Ult}) = \sup j\text{"}\rho_n(\mathcal{M})$ by Lemma 3.4. Set $b := \mathrm{Th}_n^{\mathcal{M}}(\gamma \cup \{p_n(\mathcal{M})\})$, so that $j(b) = \mathrm{Th}_n^{\mathrm{Ult}}(j(\gamma) \cup \{j(p_n(\mathcal{M}))\})$. Now define the map

$$\pi : \mathrm{Th}_n^{\mathrm{Ult}}(\alpha \cup \{q\}) \to \mathrm{Th}_n^{\mathrm{Ult}}(j(\gamma) \cup \{j(p_n(\mathcal{M}))\})$$

given as $\langle i, t\rangle \mapsto \langle \hat{i}, \hat{t}\rangle$, where $\varphi_{\hat{i}}$ is the formula obtained from φ_i by replacing all occurences of q with the corresponding use of θ. As θ uniquely defines q in Ult, π is $\Delta_1^{\mathrm{Ult}}(\{\alpha, q, s\})$. Now it holds that

$$\langle i, t\rangle \in [b, g] \quad \text{iff} \quad \varphi_i \in r\Sigma_n \wedge t \in [\alpha \cup \{q\}]^{<\omega} \wedge \langle \hat{i}, \hat{t}\rangle \in j(b). \tag{2}$$

Writing $s = [d, h]$, (2) is a Π_1 statement over Ult with parameters in $[\{\alpha\}, \mathrm{pr}]$, $[a, f]$, $[b, g]$ and $[d, h]$. By Łoś' Theorem 3.2 we get that

$$\langle i, t\rangle \in g(u) \quad \text{iff} \quad \varphi_i \in r\Sigma_n \wedge t \in [\mathrm{pr}(u) \cup \{f(u)\}]^{<\omega} \wedge \langle \tilde{i}, \tilde{t}\rangle \in b,$$

where $\langle \tilde{i}, \tilde{t}\rangle$ is the analogoue of $\langle \hat{i}, \hat{t}\rangle$ in Ult, replacing occurences of $f(u)$ in φ_i with the corresponding use of θ with parameters in $h(u)$ and $p_n(\mathcal{M})$. Since $f(u) = y$ iff $\mathcal{M} \models \theta[y, p_n(\mathcal{M}), h(u)]$ for a.e. u, $g(u) = \mathrm{Th}_n^{\mathcal{M}}(\mathrm{pr}(u) \cup \{f(u)\})$ holds for a.e. u as well. As $\mathrm{pr}(u) < \rho_n(\mathcal{M})$ holds for a.e. u because $\alpha < \rho_n(\mathrm{Ult}) = \sup j\text{"}\rho_n(\mathcal{M})$, we get that $\dot{T}_n(\mathrm{pr}(u), f(u), g(u))$ holds for a.e. u, showing (1).

Now let $\varphi(\vec{v})$ be an $r\Sigma_{n+1}$ formula for $n \geqslant 1$. If $\mathcal{M} \models \varphi[\vec{x}]$ then pick η, q and y in $\mathcal{M}$ such that $\mathcal{M} \models \dot{T}_n[\eta, q, y] \wedge \psi[\eta, q, y, \vec{x}]$ and then

$$\mathrm{Ult} \models \dot{T}_n[j(\eta), j(q), j(y)] \wedge \psi[j(\eta), j(q), j(y), j(\vec{x})]$$

by (1) and that j is Σ_1 preserving. Conversely, if $\mathrm{Ult} \models \varphi[j(\vec{x})]$ then we get $\alpha < \rho_n(\mathrm{Ult})$ and a, f, b, g such that

$$\mathrm{Ult} \models \dot{T}_n[[\{\alpha\}, \mathrm{pr}], [a,f], [b,g]] \wedge \psi[[\{\alpha\}, \mathrm{pr}], [a,f], [b,g], j(\vec{x})].$$

By (1) again we get that

$$\mathcal{M} \models \dot{T}_n[\mathrm{pr}(u), f(u), g(u)] \wedge \psi[\mathrm{pr}(u), f(u), g(u), \vec{x}]$$

for a.e. u, so $\mathcal{M} \models \varphi[\vec{x}]$. ■

Theorem 3.6. *Let $n < \omega$, $\mathcal{M}$ an n-sound ppm and E a (κ, λ)-extender over $\mathcal{M}$ such that $\kappa < \rho_n(\mathcal{M})$. Then the ultrapower embedding $j : \mathcal{M} \to \mathrm{Ult}_n(\mathcal{M}, E)$ is an n-embedding.*

Proof. Write $\mathrm{Ult} := \mathrm{Ult}_n(\mathcal{M}, E)$. By Lemmata 3.4 and 3.5, we need to show that, for every $i \leqslant n$,

(i) $p_i(\mathrm{Ult}) = j(p_i(\mathcal{M}))$;
(ii) $p_i(\mathrm{Ult})$ is solid and universal;
(iii) $\mathfrak{C}_i(\mathrm{Ult}) = \mathrm{Ult}$.

We prove (i)-(iii) by induction on i. If $i = 0$ there's nothing to show, so assume it holds for some $i \in (0, n)$ – we'll show it holds for $i+1$. Since $\mathcal{M}$ is $(i+1)$-sound, $p_{i+1}(\mathcal{M})$ is both solid and universal. Let $p := \langle \alpha_0, \dots, \alpha_l \rangle$ be the last parameter of $p_{i+1}(\mathcal{M})$.

Assume first that s is the last parameter of $p_{i+1}(\mathrm{Ult})$ and that $s <_{\mathrm{lex}} j(p)$. Let $A \subseteq \rho_{i+1}(\mathrm{Ult})$ be such that $A \in r\Sigma_{i+1}^{\mathrm{Ult}}(\{p_{i+1}(\mathrm{Ult})\})$ and $A \notin \mathrm{Ult}$. Write $s = [a, f]$, so that for some $r\Sigma_{i+1}$ formula θ,

$$\begin{aligned} [a,g] \in A \quad &\text{iff} \quad \mathrm{Ult} \models \theta[[a,g], p_i(\mathrm{Ult}), s] \\ &\text{iff} \quad \mathcal{M} \models \theta[g(u), p_i(\mathcal{M}), f(u)] \text{ for a.e. } u. \end{aligned} \tag{1}$$

But (1) defines sets $B_u \subseteq \rho_{i+1}(\mathcal{M})$ such that $B_u \in r\Sigma_{i+1}^{\mathcal{M}}(\{p_i(\mathcal{M}), f(u)\})$, where $f(u) <_{\text{lex}} p$, so that $B_u \in \mathfrak{C}_i(\mathcal{M}) = \mathcal{M}$ by solidity of $p_{i+1}(\mathcal{M})$. Then

$$\mathcal{M} \models \forall x(x \in B_u \leftrightarrow \theta[x, p_i(\mathcal{M}), f(u)])$$

for a.e. u, so

$$\text{Ult} \models \forall x(x \in [a, \lambda u.B_u] \leftrightarrow \theta[x, p_i(\text{Ult}), s]),$$

meaning that $A = [a, \lambda u.B_u] \in \text{Ult}$, $\lightning$. This means that $j(p) \leqslant_{\text{lex}} s$. To show the converse, we show that

$$\text{Hull}_{i+1}^{\text{Ult}}(\rho_{i+1}(\text{Ult}) \cup \{j(p_{i+1}(\mathcal{M}))\}) = \text{Ult}\,. \tag{2}$$

If $i = n$ then this was proven in the proof of Lemma 3.4, so assume that $i < n$ and let $[a, f] \in \text{Ult}$ be any element. Let now $\theta(x, y, q)$ be the formula

$$\begin{aligned} y = {<_L}\text{-least } \langle k, t\rangle \text{ such that } \varphi_k \text{ is } \Sigma_{i+1},\ t \in [\rho_{i+1}(\mathcal{M})]^{<\omega} \\ \text{and } z = x \text{ iff } \mathcal{M} \models \varphi_k[t, q, z], \end{aligned}$$

which can be seen to be Σ_1 over $r\Sigma_{i+1}$, i.e. $r\Sigma_{i+1}$. Let $\psi(v_0, v_1, v_2)$ be the $r\Sigma_n$ formula and $q \in \mathcal{M}$ such that $f(u) = x$ iff $\mathcal{M} \models \psi[u, x, q]$. Consider now

$$\chi(u) :\equiv \exists y \exists x(\psi(u, x, q) \wedge \theta(x, y, p_{i+1}(\mathcal{M}))),$$

which is Σ_1 over $r\Sigma_n$ with parameters q and $p_{i+1}(\mathcal{M})$. Let h be an $r\Sigma_n$ Skolem function for χ, so that

$$\begin{aligned} [a, h] = {<_L}\text{-least } \langle k, t\rangle \text{ such that } \varphi_k \text{ is } r\Sigma_{i+1},\ t \in [\rho_{i+1}(\text{Ult})]^{<\omega} \\ \text{and } z = [a, f] \text{ iff Ult} \models \varphi_k[t, j(p_{i+1}(\mathcal{M})), z], \end{aligned}$$

where we used Lemma 3.4 which can be used as $i + 1 < n$, so

$$[a, f] \in \text{Hull}_{i+1}^{\text{Ult}}(\rho_{i+1}(\text{Ult}) \cup \{j(p_{i+1}(\mathcal{M}))\}),$$

concluding (2) and thus also that $p_{i+1}(\text{Ult}) = j(p_{i+1}(\mathcal{M}))$ and $\mathfrak{C}_{i+1}(\text{Ult}) = \text{Ult}$.

It remains to show that $p_{i+1}(\mathrm{Ult})$ is universal and solid. Universality follows from $\mathfrak{C}_{i+1}(\mathrm{Ult}) = \mathrm{Ult}$, so assume that $p_{i+1}(\mathrm{Ult})$ isn't solid and let $s \leqslant l$ be least such that there is a set $A \in r\Sigma_{i+1}^{\mathrm{Ult}}(\{j(\alpha_0), \dots, j(\alpha_{s-1})\})$ satisfying that $A \cap j(\alpha_s) \notin \mathrm{Ult}$. Say $\varphi(x)$ is $r\Sigma_{i+1}$ defining A. Then

$$\begin{aligned} [a, g] \in A \cap j(\alpha_s) \quad &\text{iff} \quad \mathrm{Ult} \models [a, g] \in j(\alpha_s) \wedge \varphi[[a, g], j(\alpha_0), \dots, j(\alpha_{s-1})] \\ &\text{iff} \quad \mathcal{M} \models g(u) \in \alpha_s \wedge \varphi[f(u), \alpha_0, \dots, \alpha_{s-1}] \text{ for a.e. } u. \end{aligned}$$

Define $B_u := \{z \in \alpha_s \mid \mathcal{M} \models \varphi[z, \alpha_0, \dots, \alpha_{s-1}]\}$. Then $B_u \in \mathcal{M}$ by definition of $p_{i+1}(\mathcal{M})$, so just like before we get that $A = [a, \lambda u.B_u] \in \mathrm{Ult}$, $\lightning$. Thus $p_{i+1}(\mathcal{M})$ is indeed solid. ■

3.3 Preserving degree $n + 1$ fine structure

That the ultrapower embedding is an n-embedding means that it preserves the fine structure up to degree n. If we require some more of the extender we can ensure that the degree $n + 1$ fine structure is preserved as well.

Definition 3.7. Let $\mathcal{M}$ be a ppm and E a (κ, λ)-extender over $\mathcal{M}$. Then E is **close to** $\mathcal{M}$ if for every $a \in [\lambda]^{<\omega}$ it holds that

(i) E_a is $\mathbf{\Sigma}_1^{\mathcal{M}}$;
(ii) $E_a \cap A \in \mathcal{M}$ for every $A \in \mathcal{M}$ such that $\mathcal{M} \models |A| \leqslant \kappa$. ○

We start again with preservation of the projectum.

Lemma 3.8. *Let $n < \omega$, $\mathcal{M}$ an n-sound ppm and E a (κ, λ)-extender which is close to $\mathcal{M}$ such that $\kappa < \rho_n(\mathcal{M})$. Set $\mathrm{Ult} := \mathrm{Ult}_n(\mathcal{M}, E)$. Then $\mathcal{P}^{\mathcal{M}}(\kappa) = \mathcal{P}^{\mathrm{Ult}}(\kappa)$. If furthermore $\rho_{n+1}(\mathcal{M}) \leqslant \kappa$ then $\rho_{n+1}(\mathcal{M}) = \rho_{n+1}(\mathrm{Ult})$.*

Proof. Let $j : \mathcal{M} \to \mathrm{Ult}$ be the ultrapower embedding. Since $\mathcal{P}^{\mathcal{M}}(\kappa) \subseteq \mathcal{P}^{\mathrm{Ult}}(\kappa)$ always holds as $A = j(A) \cap \kappa$ for every $A \in \mathcal{P}^{\mathcal{M}}(\kappa)$, let $[a, f] \subseteq \kappa$ where we without loss of generality can assume that $f : [\kappa]^{|a|} \to \mathcal{P}(\kappa)$. Since f is $\mathbf{r\Sigma}_n^{\mathcal{M}}$ and can be coded as a subset of $\kappa < \rho_n$, it holds that $f \in \mathcal{M}$. For $\alpha < \kappa$ define

$$A_\alpha := \{u \mid \alpha \in f(u)\},$$

so that $\alpha \in [a, f]$ iff $A_\alpha \in E_{a\cup\{\kappa\}}$. Since E is close to $\mathcal{M}$, $E_{a\cup\{\kappa\}} \cap A_\alpha \in \mathcal{M}$ for every α, so that $[a, f] \in \mathcal{M}$ as it can be computed from the A_α's.

Now assume that $\rho_{n+1}(\mathcal{M}) \leqslant \kappa$. To show that $\rho_{n+1}(\mathcal{M}) \geqslant \rho_{n+1}(\mathrm{Ult})$, let $A \subseteq \rho_{n+1}(\mathcal{M})$ be $\mathbf{r\Sigma}^{\mathcal{M}}_{n+1}$ such that $A \notin \mathcal{M}$. We show that $A \notin \mathrm{Ult}$. Since j is $r\Sigma_{n+1}$-elementary by Lemma 3.5, A is $\mathbf{r\Sigma}^{\mathrm{Ult}}_{n+1}$ as well. But $\rho_{n+1}(\mathcal{M}) \leqslant \kappa$ and we just showed that $\mathcal{P}^{\mathcal{M}}(\kappa) = \mathcal{P}^{\mathrm{Ult}}(\kappa)$, so $A \notin \mathrm{Ult}$. Hence $\rho_{n+1}(\mathcal{M}) \geqslant \rho_{n+1}(\mathrm{Ult})$.

Now to show that $\rho_{n+1}(\mathcal{M}) \leqslant \rho_{n+1}(\mathrm{Ult})$. Let $\alpha < \rho_{n+1}(\mathcal{M})$ and $B \subseteq \alpha$ be $\mathbf{r\Sigma}^{\mathrm{Ult}}_{n+1}$; we show that $B \in \mathrm{Ult}$. It suffices to show that B is $\mathbf{r\Sigma}^{\mathcal{M}}_{n+1}$, because then $B \in \mathcal{M}$ as $\alpha < \rho_{n+1}(\mathcal{M})$, and as $\mathcal{P}^{\mathcal{M}}(\kappa) = \mathcal{P}^{\mathrm{Ult}}(\kappa)$ and $\rho_{n+1}(\mathcal{M}) \leqslant \kappa$, $B \in \mathrm{Ult}$ as well.

Let ψ be $r\Sigma_{n+1}$ and $q = [a, f] \in \mathrm{Ult}$ be such that ψ and q defines B. Then for every $\gamma \in \alpha$ it holds that

$$\begin{aligned} \gamma \in B \quad &\text{iff} \quad \mathrm{Ult} \models \psi[\gamma, q] \\ &\text{iff} \quad \mathcal{M} \models \psi[\gamma, f(u)] \text{ for a.e. } u \\ &\text{iff} \quad \mathcal{M} \models \theta[\{u \mid \mathcal{M} \models \psi[\gamma, f(u)]\}, r], \end{aligned}$$

where θ is Σ_1 and $r \in \mathcal{M}$ are such that $x \in E_a$ iff $\mathcal{M} \models \theta[x, r]$, which is possible since E is close to $\mathcal{M}$. This latter formula is Σ_1 over $r\Sigma_{n+1}$ with parameters from $\mathcal{M}$, so B is $\mathbf{r\Sigma}^{\mathcal{M}}_{n+1}$ and we're done. ■

Next, we deal with the standard parameter.

Lemma 3.9. *Let $n < \omega$, $\mathcal{M}$ an n-sound ppm and E a (κ, λ)-extender which is close to $\mathcal{M}$ such that $\kappa \in [\rho_{n+1}(\mathcal{M}), \rho_n(\mathcal{M}))$. Assume that $p_{n+1}(\mathcal{M})$ is solid and let $j : \mathcal{M} \to \mathrm{Ult} := \mathrm{Ult}_n(\mathcal{M}, E)$ be the ultrapower embedding. Then $p_{n+1}(\mathrm{Ult})$ is solid as well and $j(p_{n+1}(\mathcal{M})) = p_{n+1}(\mathrm{Ult})$.*

Proof. Since $j(p_n(\mathcal{M})) = p_n(\mathrm{Ult})$ by Theorem 3.6 as $\mathcal{M}$ is n-sound, we just need to show that $j(p) = q$, where p and q are the last parameters of $p_{n+1}(\mathcal{M})$

and $p_{n+1}(\mathrm{Ult})$, respectively. To see that $j(p) \geqslant_{\mathrm{lex}} q$, let $A \subseteq \rho_{n+1}(\mathcal{M})$ be such that $A \in r\Sigma_{n+1}^{\mathcal{M}}(\{p\})$ and $A \notin \mathcal{M}$. By Lemma 3.8 $\rho_{n+1}(\mathrm{Ult}) = \rho_{n+1}(\mathcal{M})$, so $A \subseteq \rho_{n+1}(\mathrm{Ult})$ as well. Let ψ witness that $A \in r\Sigma_{n+1}^{\mathcal{M}}(\{p\})$, so that

$$x \in A \quad \text{iff} \quad \mathcal{M} \models \psi[x, p] \quad \text{iff} \quad \mathrm{Ult} \models \psi[x, j(p)],$$

where we used that $\rho_{n+1}(\mathcal{M}) \subseteq \kappa$. Thus A is also $r\Sigma_{n+1}^{\mathrm{Ult}}(\{j(p)\})$ and since $A \notin \mathcal{M}$ we have $A \notin \mathrm{Ult}$, concluding that $j(p) \geqslant_{\mathrm{lex}} q$. If we can show that $j(p)$ is $(n+1)$-solid then Proposition 2.19 implies that $j(p) \leqslant_{\mathrm{lex}} q$, so write $p = \langle \alpha_0, \ldots, \alpha_l \rangle$ and let $i \leqslant l$. Then we need to show that

$$\mathcal{P}(j(\alpha_i)) \cap r\Sigma_n^{\mathrm{Ult}}(\{j(\alpha_0), \ldots, j(\alpha_{i-1})\}) \subseteq \mathrm{Ult}\,. \tag{1}$$

Let $A \subseteq j(\alpha_i)$ and let $\psi \in r\Sigma_n$ be such that, for all $\gamma = [a, f] \in j(\alpha_i)$,

$$\begin{aligned} \gamma \in A \quad &\text{iff} \quad \mathrm{Ult} \models \psi[\gamma, j(\alpha_0), \ldots, j(\alpha_{i-1})] \\ &\text{iff} \quad \mathcal{M} \models \psi[f(u), \alpha_0, \ldots, \alpha_{i-1}] \text{ for a.e. } u. \end{aligned}$$

Define $B := \{x \in \alpha_i \mid \mathcal{M} \models \psi[x, \alpha_0, \ldots, \alpha_{i-1}]\}$, a $r\Sigma_n^{\mathcal{M}}(\{\alpha_0, \ldots, \alpha_{i-1}\})$-definable set, so $(n+1)$-solidity of p implies that $B \in \mathcal{M}$. Then

$$\gamma \in A \quad \text{iff} \quad \mathcal{M} \models f(u) \in c_B(u) \text{ for a.e. } u,$$

meaning that $A = [a \cup \{\kappa\}, c_B] \in \mathrm{Ult}$, showing (1). We conclude that $p_{n+1}(\mathrm{Ult})$ is solid and that $j(p_{n+1}(\mathcal{M})) = p_{n+1}(\mathrm{Ult})$. ■

The following theorem summarises the previous lemmata as well as treating general iterations of ppms.

Theorem 3.10. *Let $n \leqslant \omega$, $\mathcal{M}$ an n-sound ppm and $\theta \in \mathrm{On}$. Assume that we have a linear directed system $\langle \mathcal{M}_\alpha, i_{\alpha,\beta} \mid \alpha, \beta \leqslant \theta \rangle$ of wellfounded ppms with*

- *$\mathcal{M}_0 = \mathcal{M}$;*
- *$\mathcal{M}_{\alpha+1} = \mathrm{Ult}_n(\mathcal{M}_\alpha, E_\alpha)$ with E_α some extender close to $\mathcal{M}_\alpha$;*
- *$\mathcal{M}_\lambda = \varinjlim_{\alpha<\lambda} \mathcal{M}_\alpha$ for λ limit;*
- *$i_{\alpha\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is the canonical map.*

Assume that $\operatorname{crit} i_{0\theta} < \rho_n(\mathcal{M})$. *Then* $i_{0\theta}$ *is an* n*-embedding. If furthermore* $\mathcal{M}$ *is* $(n+1)$*-solid and* $\rho_{n+1}(\mathcal{M}) \leqslant \operatorname{crit} i_{0\theta}$ *then*

(i) $\rho_{n+1}(\mathcal{M}) = \rho_{n+1}(\mathcal{M}_\theta)$*;*

(ii) $i_{0\theta}(p_{n+1}(\mathcal{M})) = p_{n+1}(\mathcal{M}_\theta)$*;*

(iii) $\mathcal{M}_\theta$ *is* $(n+1)$*-solid,* $\mathfrak{C}_{n+1}(\mathcal{M}_\theta) = \mathfrak{C}_{n+1}(\mathcal{M})$ *and* $i_{0\theta} \upharpoonright \mathfrak{C}_{n+1}(\mathcal{M}) = \sigma$*, where* $\sigma : \mathfrak{C}_{n+1}(\mathcal{M}_\theta) \to \mathfrak{C}_n(\mathcal{M}_\theta)$ *is the uncollapse.*

Proof. To show that $i_{0\theta}$ is an n-embedding follows from Theorem 3.6 in the case where θ is a successor and only an exercise in direct limits of models if θ is a limit, so we omit it. Let thus $n < \omega$ be given. If θ is a successor then (i)-(iii) is directly by Lemmata 3.8 and 3.9, so let θ be a limit. In this case (i) is the same argument as in showing $i_{0\theta}$ is an n-embedding.

Note that $p_{n+1}(\mathcal{M}_\theta) \leqslant_{\text{lex}} i_{0\theta}(p_{n+1}(\mathcal{M}))$ since if it was the case that every $r\Sigma_{n+1}^{\mathcal{M}_\theta}(\{i_{0\theta}(p_{n+1}(\mathcal{M}))\})$ subset $A \subseteq \rho_{n+1}(\mathcal{M}_\theta)$ was an element of $\mathcal{M}_\theta$, it would be of the form $i_{\alpha\theta}(B)$ for some $\alpha < \theta$ and $B \subseteq \rho_{n+1}(\mathcal{M}_\alpha)$ by (i). But as $i_{\alpha\theta}$ is an n-embedding, B is $r\Sigma_{n+1}^{\mathcal{M}_\alpha}(\{i_{0\alpha}(p_{n+1}(\mathcal{M}))\}) = r\Sigma_{n+1}^{\mathcal{M}_\alpha}(\{p_{n+1}(\mathcal{M}_\alpha)\})$ as well. But then every such B is in $\mathcal{M}_\alpha$, contradicting the definition of $p_{n+1}(\mathcal{M}_\alpha)$.

To show that $i_{0\theta}(p_{n+1}(\mathcal{M})) \leqslant_{\text{lex}} p_{n+1}(\mathcal{M}_\theta)$ it suffices by Proposition 2.19 to show that $i_{0\theta}(p_{n+1}(\mathcal{M}))$ is solid. Let thus $p = \langle \alpha_0, \dots, \alpha_l \rangle$ be the last parameter of $p_{n+1}(\mathcal{M})$ and $t \leqslant l$. We have to show that any $A \subseteq i_{0\theta}(\alpha_t)$ which is $r\Sigma_{n+1}^{\mathcal{M}_\theta}(\{i_{0\theta}(p_n(\mathcal{M})), i_{0\theta}(p) \upharpoonright t\})$ is a member of $\mathcal{M}_\theta$.

Define $A_\alpha := i_{\alpha\theta}^{-1}{}''A$, so that $A_\alpha \subseteq A_\beta$ for every $\alpha \leqslant \beta < \theta$ as we're working in a directed system. By solidity of the $p_{n+1}(\mathcal{M}_\alpha)$'s we get that $A_\alpha \in \mathcal{M}_\alpha$, so $i_{\alpha\theta}(A_\alpha) \in \mathcal{M}_\theta$ for every $\alpha < \theta$. As $A = \bigcup_{\alpha<\theta} i_{\alpha\theta}{}''A_\alpha$ by definition of direct limit, we then have $A \in \mathcal{M}_\theta$, showing solidity and therefore also (ii).

Points (i) and (ii) readily implies that $\mathfrak{C}_{n+1}(\mathcal{M}_\theta) = \mathfrak{C}_{n+1}(\mathcal{M})$. Finally $i_{0\theta}$ being an n-embedding entails $i_{0\theta} \upharpoonright \mathfrak{C}_{n+1}(\mathcal{M}) = \sigma$, by just moving $r\Sigma_{n+1}$ definitions around between $\mathfrak{C}_{n+1}(\mathcal{M})$, $\mathfrak{C}_{n+1}(\mathcal{M}_\theta)$ and $\mathfrak{C}_n(\mathcal{M}_\theta) = \mathcal{M}_\theta$. ■

4 Iterability and mice

A key property we want our structures to have is *comparison*. That is, given two premice $\mathcal{M}$ and $\mathcal{N}$, we want to determine which one is "largest". As the extenders on the two premice can be completely different this seems to be a daunting task – the key here is to use iterations of ultrapowers. Historically this started with linear iterations, but problems arose when the extenders used in the iteration overlapped. This motivated the study of iteration *trees*, where the branches correspond to the linear iterations, and whenever we have an overlap, we just make sure that the overlap doesn't happen on the same branch.

4.1 Iteration trees and mice

For technical reasons, we will not merely deal with iteration trees on premice, but sequences of premice, called *phalanxes*.

Definition 4.1. Two premice $\mathcal{M}$ and $\mathcal{N}$ **agree below** γ if $\mathcal{M}|\beta = \mathcal{N}|\beta$ for every $\beta < \gamma$. ○

Definition 4.2. Let $\vec{\lambda} = \langle \lambda_\alpha \mid \alpha < \gamma \rangle$ be a strictly increasing sequence of ordinals and $\vec{\mathcal{Q}} = \langle \mathcal{Q}_\alpha \mid \alpha \leqslant \gamma \rangle$ a sequence of premice. Then $(\vec{\mathcal{Q}}, \vec{\lambda})$ is a **phalanx of length** $\gamma + 1$ if $\mathcal{Q}_\alpha$ agrees with $\mathcal{Q}_\beta$ below λ_α and λ_α is a $\mathcal{Q}_\beta$-cardinal, for every $\alpha < \beta \leqslant \gamma$. ○

Definition 4.3. A (rooted) **tree order** on $\alpha \in \mathrm{On}$ is a strict partial ordering $<_T$ of α with a function $\mathrm{root} : \alpha \to \alpha$ satisfying that, for every $\beta, \gamma \in \alpha$,

(i) $\mathrm{root}\,\gamma \leqslant_T \gamma$;
(ii) $\beta \leqslant_T \mathrm{root}\,\gamma$ iff $\beta = \mathrm{root}\,\gamma$;
(iii) If $\beta <_T \gamma$ then $\beta < \gamma$ (i.e. that $<_T \subseteq \in \restriction \alpha$);
(iv) $[\mathrm{root}\,\gamma, \gamma)_T$ is wellordered by $<_T$;[1]

[1] Here $[\gamma, \beta)_T := \{\xi \in \alpha \mid \gamma \leqslant_T \xi <_T \beta\}$. The intervals $[\gamma, \beta]_T$, $(\gamma, \beta)_T$ and $(\gamma, \beta]_T$ are defined similarly.

(v) γ is a successor ordinal iff γ is a $<_T$-successor;

(vi) If γ is a limit ordinal then $[\operatorname{root}\gamma, \gamma)_T$ is $\in$-cofinal in γ. ○

We will now describe the **iteration game** $\mathcal{G}_k(\Phi, \theta)$ for $k \leqslant \omega$, $\theta \in \mathrm{On}$ and a phalanx $\Phi = (\vec{\mathcal{Q}}, \vec{\lambda})$ with $\xi + 1 := \operatorname{lh}\Phi \leqslant \theta$. The game will have length θ, and plays will result in certain trees $\mathcal{T}$. At turn $\alpha + 1 < \theta$ player I will play a sextuple

$$\langle \mathcal{M}^{\mathcal{T}}_{\alpha+1}, E^{\mathcal{T}}_{\alpha}, \lambda^{\mathcal{T}}_{\alpha}, <_T \restriction \alpha + 2, D^{\mathcal{T}} \cap \alpha + 2, \{i^{\mathcal{T}}_{\beta,\alpha+1}\}\rangle,$$

and at limit turns $\lambda < \theta$ player II will play a quadruple

$$\langle \mathcal{M}^{\mathcal{T}}_{\lambda}, <_T \restriction \lambda + 1, D^{\mathcal{T}} \cap \lambda + 1, \{i^{\mathcal{T}}_{\alpha,\lambda}\}\rangle,$$

where

- $\mathcal{M}^{\mathcal{T}}_{\alpha}$ is a premouse;
- $E^{\mathcal{T}}_{\alpha}$ is an extender from the $\mathcal{M}^{\mathcal{T}}_{\alpha}$ sequence;
- $\lambda^{\mathcal{T}}_{\alpha}$ is an ordinal;
- $<_T \restriction \alpha$ is a tree order on α such that $<_T \restriction \gamma \subseteq <_T \restriction \alpha$ for $\gamma < \alpha$;
- $D^{\mathcal{T}} \cap \alpha \subseteq \alpha$ is a set of *drops* such that $D \cap \gamma \subseteq D \cap \alpha$ for $\gamma < \alpha$;
- $i^{\mathcal{T}}_{\beta,\alpha} : \mathcal{M}_\beta \to \mathcal{M}_\alpha$ are l-embeddings for some $l \leqslant k$.

We will usually leave out the superscript $\mathcal{T}$ when it is understood. The rules of the game will ensure that

(G1) If $\beta \leqslant \alpha$ then $\mathcal{M}_\beta$ agrees with $\mathcal{M}_\alpha$ below λ_β;

(G2) If $\beta < \alpha$ then λ_β is a cardinal of $\mathcal{M}_\alpha$.

The game is played as follows. On the first ξ turns (recall that $\operatorname{lh}\Phi = \xi + 1$) player I plays

$$\mathcal{M}^{\mathcal{T}}_{\alpha} = \mathcal{Q}_\alpha\,, \quad \lambda^{\mathcal{T}}_{\alpha} = \lambda_\alpha \quad \text{and} \quad <_T \restriction \alpha + 1 = D^{\mathcal{T}} \cap \alpha + 1 = \{i^{\mathcal{T}}_{\alpha,\lambda}\} = \varnothing.$$

On turn $\alpha + 1 \geqslant \xi + 1$, player I has to pick an extender E_α from the $\mathcal{M}_\alpha$ sequence satisfying $\lambda_\gamma < \operatorname{lh} E_\alpha$ for every $\gamma < \alpha$. If he cannot do this, he loses the game. Set $\lambda_\alpha := \operatorname{lh} E_\alpha$ and let $\eta \leqslant \alpha$ be least such that $\operatorname{crit} E_\alpha < \lambda_\eta$, and set $\mathcal{M}^*_{\alpha+1} := \mathcal{M}_\eta\,|\gamma$, where γ is largest such that E_α is a pre-extender over $\mathcal{M}_\eta\,|\gamma$.

We claim that γ exists and $\lambda_\eta \leqslant \gamma$. If $\eta = \alpha$ then it's trivial, so assume that $\eta < \alpha$. Writing $\kappa := \operatorname{crit} E_\alpha$, (G1) and (G2) implies that

$$\mathcal{P}(\kappa) \cap \mathcal{M}_\eta \,|\lambda_\eta = \mathcal{P}(\kappa) \cap \mathcal{M}_\alpha \,|\lambda_\eta = \mathcal{P}(\kappa) \cap \mathcal{M}_\alpha,$$

so E_α is a pre-extender over $\mathcal{M}_\eta \,|\lambda_\eta$. Thus γ exists and $\gamma \geqslant \lambda_\eta$. Now define $T \restriction \alpha + 2 := T \restriction \alpha + 1 \cup \{\langle \eta, \alpha + 1\rangle\}$ and

$$D \cap \alpha + 2 := \begin{cases} D \cap \alpha + 1 \cup \{\alpha + 1\} & \text{if } \mathcal{M}^*_{\alpha+1} \lhd \mathcal{M}_\eta \\ D \cap \alpha + 1 & \text{otherwise.} \end{cases}$$

Let $n \leqslant \omega$ be largest satisfying both that $\operatorname{crit} E_\alpha < \rho_n(\mathcal{M}^*_{\alpha+1})$ and $D \cap [\operatorname{root} \alpha + 1, \alpha + 1]_T = \varnothing \Rightarrow n \leqslant k$. Then set

$$\mathcal{M}_{\alpha+1} := \operatorname{Ult}_n(\mathcal{M}^*_{\alpha+1}, E_\alpha).$$

If this ultrapower is illfounded, player II loses the game.[2]

Proposition 4.4. *On turn $\alpha + 1 < \theta$, (G1) and (G2) are satisfied.*

Proof. For $\alpha < \xi$ both (G1) and (G2) holds by definition of phalanx.[3] Assume thus $\alpha \geqslant \xi$. Set $\kappa := \operatorname{crit} E_\alpha$ and let

$$\begin{aligned} i &: \mathcal{M}^*_{\alpha+1} \to \mathcal{M}_{\alpha+1} = \operatorname{Ult}_n(\mathcal{M}^*_{\alpha+1}, E_\alpha) \\ j &: \mathcal{M}^*_{\alpha+1} \to \mathcal{P} := \operatorname{Ult}_0(\mathcal{M}^*_{\alpha+1}, E_\alpha) \\ h &: \mathcal{M}_\alpha \,|\lambda_\alpha \to \mathcal{Q} := \operatorname{Ult}_0(\mathcal{M}_\alpha \,|\lambda_\alpha, E_\alpha) \end{aligned}$$

be the ultrapower embeddings. Since $\kappa < \lambda_\eta$ we have that $\mathcal{M}_\alpha \,|\lambda_\alpha$ and $\mathcal{M}^*_{\alpha+1}$ agree below their common value of κ^+; denote this value by λ. This entails that $\delta := h(\lambda) = j(\lambda)$, and we claim that we also have that $\delta = i(\lambda)$. Indeed, as every $r\Sigma_n^{\mathcal{M}^*_{\alpha+1}}$-definable function from κ to κ is in $\mathcal{M}^*_{\alpha+1}$ because $\kappa < \rho_n(\mathcal{M}^*_{\alpha+1})$ holds

[2] In (Steel, 2010) there's an anomalous case where $\mathrm{On}^{\mathcal{M}^*_{\alpha+1}} = \operatorname{crit} E_\alpha$, so that the above ultrapower wouldn't make sense. In our setup however, this case is impossible as we're never doing any "squashing" – see (Steel & Mitchell, 1994, Chapter 3).

[3] This was the sole reason for that part of the definition.

by assumption, the n-ultrapower $\mathcal{M}_{\alpha+1}$ and the 0-ultrapower $\mathcal{P}$ agree below δ and we thus have $\delta = i(\lambda)$.

As $\mathcal{M}_\alpha\,|\lambda_\alpha$ agrees with $\mathcal{M}^*_{\alpha+1}$ below λ, $\mathcal{Q}$ and $\mathcal{P}$ agree below δ. Also, by definition of a fine extender sequence, $\mathcal{M}_\alpha$ agrees with $\mathcal{Q}$ below λ_α. We can summarise this in the following diagram, where we write $\mathcal{M} \overset{<\xi}{\rightsquigarrow} \mathcal{N}$ if $\mathcal{M}$ and $\mathcal{N}$ agree below ξ.

$$\begin{tikzcd}
& \mathcal{M}_\alpha\,|\lambda_\alpha \arrow[dl, hook'] \arrow[d, "h"] \arrow[r, squiggly, dash, "<\lambda"] & \mathcal{M}^*_{\alpha+1} \arrow[d, "j"] \arrow[dr, "i"] & \\
\mathcal{M}_\alpha \arrow[r, squiggly, dash, "<\lambda_\alpha"'] & \mathcal{Q} \arrow[r, squiggly, dash, "<\delta"'] & \mathcal{P} \arrow[r, squiggly, dash, "<\delta"'] & \mathcal{M}_{\alpha+1}
\end{tikzcd}$$

We can therefore conclude that $\mathcal{M}_\alpha \overset{<\lambda_\alpha}{\rightsquigarrow} \mathcal{M}_{\alpha+1}$ since $\lambda_\alpha < h(\lambda) = \delta$. ■

On a limit turn $\lambda < \theta$, player II picks a branch in $\mathcal{T}$ which is $\in$-cofinal in λ such that

(i) The drops in b below λ, i.e. $D^{\mathcal{T}} \cap b \cap \lambda$, are bounded;

(ii) $\varinjlim_{\alpha\in b-\sup(D\cap b)} \mathcal{M}_\alpha$ is wellfounded.

We will say that b is **wellfounded** if the above conditions (i)-(ii) apply. If player II cannot find such a branch, she loses the game. Otherwise set

- $\mathcal{M}_\lambda := \varinjlim_{\alpha\in b-\sup(D\cap b)} \mathcal{M}_\alpha$;
- $D \cap \lambda + 1 := \bigcup_{\alpha<\lambda} D \cap \alpha$;
- $<_T \restriction \lambda + 1 = \bigcup_{\alpha<\lambda} <_T \restriction \alpha \cup \{\langle\alpha, \lambda\rangle \mid \alpha \in b\}$;
- $i_{\alpha,\lambda} : \mathcal{M}_\alpha \to \mathcal{M}_\lambda$ is the direct limit embedding, for $\alpha \in b - \sup(D \cap b)$.

We again need to make sure that (G1) and (G2) are still satisfied at turn λ, but this is clear just by the definition of a direct limit of structures. This finishes the definition of the game. If no one has lost after θ many turns, player II wins. In this special case where Φ is just a single premouse $\mathcal{M}$, we denote the iteration game by $\mathcal{G}_k(\mathcal{M}, \theta)$.

Definition 4.5. A **putative (normal)** k**-iteration tree on** Φ is a partial play of $\mathcal{G}_k(\Phi, \theta)$, i.e. a tree $\mathcal{T}$ of pairs $\langle\mathcal{M}_\alpha, E_\alpha\rangle$ which are connected according to the tree order $<_T$, along with embeddings $i_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ following the rules above. A

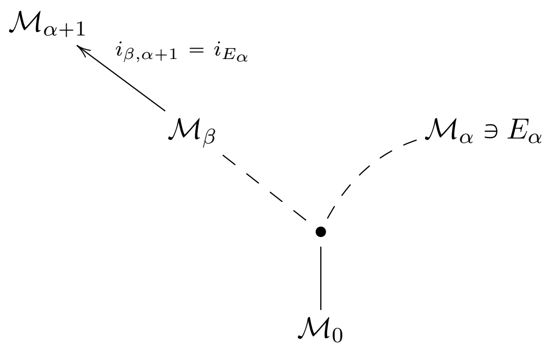


Figure 4.1: An iteration tree.

(normal) k-iteration tree on Φ is a putative k-iteration tree on Φ in which neither player has lost. ○

Here **normality** is referring to the condition that $\lambda_\alpha < \lambda_\beta$ for every $\alpha < \beta$. See Figure 4.1 for an illustration of an iteration tree.

Definition 4.6. Let $\mathcal{T}$ be an iteration tree with models $\mathcal{M}_\alpha$, extenders E_α and $\alpha + 1 < \operatorname{lh}\mathcal{T}$. Then the **degree of** $\alpha + 1$, $\deg^{\mathcal{T}}(\alpha+1)$, is the $n \leqslant \omega$ such that $\mathcal{M}_{\alpha+1} = \operatorname{Ult}_n(\mathcal{M}^*_{\alpha+1}, E_\alpha)$. We also write $i^*_{\alpha+1}$ for the ultrapower embedding from $\mathcal{M}^*_{\alpha+1}$ into this ultrapower. ○

The degree of α shows how much fine structure $\mathcal{M}_\alpha$ has. A k-iteration tree starts off with fine structure up to degree k, and the amount of fine structure preserved throughout the branches can drop.

We can now prove the previously mentioned fact that we don't have overlapping extenders along branches of iteration trees.

Proposition 4.7. *Let $\mathcal{T}$ be a k-iteration tree for some $k \leqslant \omega$, and let b be a branch of $\mathcal{T}$. Then for every $\alpha, \beta \in b$ with $\alpha <_T \beta$, it holds that $\nu_E \leqslant \operatorname{crit} F$ with E being the extender used at α and F the extender used at β in $\mathcal{T}$. We say that **generators aren't moved along branches of iteration trees**.*

Proof. Without loss of generality we can assume that $\alpha = \mathrm{pred}_T\,\beta$. Say $E = E^{\mathcal{T}}_\eta$ and $F = E^{\mathcal{T}}_\xi$, so that β is least such that $\operatorname{crit} E^{\mathcal{T}}_\xi < \lambda_\beta$. Since $\eta < \beta$ (actually $\beta = \eta + 1$), minimality implies that $\lambda_\eta \leqslant \operatorname{crit} E^{\mathcal{T}}_\xi$ so in particular $\nu_E \leqslant \operatorname{crit} F$. ■

Definition 4.8. Let E and F be pre-extenders and set $\nu := \min\{\nu_E, \nu_F\}$. Then E and F are **compatible** if $E \restriction \nu = F \restriction \nu$. ○

Note that if E and F are compatible they have the same critical point and measure the same subsets. Compatibility is weaker than equivalence, since if $\nu_E = \nu_F$ and E is compatible with F, then they are also equivalent.

Proposition 4.9. *Let E_α and E_β be extenders used in a k-iteration tree $\mathcal{T}$. Then E_α and E_β are not compatible.*

Proof. Assume not. Write $E := E_\alpha$ and $F := E_\beta$ and assume without loss of generality that $\nu_E < \nu_F$, so that $E = (F \restriction \nu_E)^*$. Then the initial segment condition implies that either E is on the $\mathcal{M}$-sequence, in which case $E \in \mathcal{M}$ by amenability, or on the $\mathrm{Ult}(\mathcal{M}\,|\nu_E, E_\eta)$-sequence below $\mathrm{On}^{\mathcal{M}}$ in which case it's definable over $\mathcal{M}\,|\nu_E$, again implying that $E \in \mathcal{M}$. But then we also have a surjection from ν_E onto $\operatorname{lh} E$ inside $\mathcal{M}_\beta$, contradicting that $\operatorname{lh} E$ is a cardinal of $\mathcal{M}_\beta$ by (G2), $\lightning$. ■

The definition of $\mathcal{G}_k(\Phi, \theta)$ yields the following very useful property.

Proposition 4.10. *Let $\mathcal{T}$ be an iteration tree on a phalanx Φ and let $\alpha + 1 \in [\operatorname{lh}\Phi, \operatorname{lh}\mathcal{T})$. Then E_α is close to $\mathcal{M}^*_{\alpha+1}$.*

Proof. Induction on α. Let $\eta := \mathrm{pred}_T(\alpha + 1)$. If $\alpha = \eta$ then $\mathcal{M}^*_{\alpha+1} = \mathcal{M}_\alpha\,|\gamma$ for some $\gamma \geqslant \operatorname{lh} E_\alpha$ and E_α is on the $\mathcal{M}_\alpha$ sequence. Since $\gamma \geqslant \operatorname{lh} E_\alpha$, E_α is also on the $\mathcal{M}^*_{\alpha+1}$ sequence. But then E_α is quite trivially close to $\mathcal{M}^*_{\alpha+1}$ as well.

Assume thus that $\eta < \alpha$. Fix some $a \in [\operatorname{lh} E_\alpha]^{<\omega}$. We will start by showing the second condition of closeness. Set $\kappa := \operatorname{crit} E_\alpha$. Since $\eta = \mathrm{pred}_T(\alpha + 1)$ we have $\kappa < \lambda_\eta$ and since λ_η is a cardinal of $\mathcal{M}_\alpha$, $\kappa^{+\mathcal{M}_\alpha} \leqslant \lambda_\eta$.

Let $\mathcal{A} \in \mathcal{M}^*_{\alpha+1}$ and assume without loss of generality that $\mathcal{A} \subseteq \mathcal{P}([\kappa]^{|a|})$. We want to show that $(E_\alpha)_a \cap \mathcal{A} \in \mathcal{M}^*_{\alpha+1}$. Since $\mathcal{P}^{\mathcal{M}_\alpha}(\kappa) = \mathcal{P}^{\mathcal{M}^*_{\alpha+1}}(\kappa)$, $\mathcal{A} \in \mathcal{M}_\alpha$ as well, so $\mathcal{A} \cap (E_\alpha)_a \in \mathcal{M}_\alpha$ by amenability, and thus $\mathcal{A} \cap (E_\alpha)_a \in \mathcal{M}^*_{\alpha+1}$ as well. Note that we didn't need that $\mathcal{M}^*_{\alpha+1} \models |\mathcal{A}| \leqslant \kappa$.

We then want to show the first condition of closeness, that $(E_\alpha)_a$ is $\mathbf{\Sigma}_1^{\mathcal{M}^*_{\alpha+1}}$. Note that $\mathcal{M}_\eta \,|\lambda_\eta \trianglelefteq \mathcal{M}^*_{\alpha+1}$. We have the following claim.

Claim 4.10.1. If $A \in \mathcal{P}^{\mathcal{M}_\xi}(\lambda_\eta)$ for some $\xi > \eta$ then A is $\mathbf{\Sigma}_1$ over $\mathcal{M}_\eta \,|\lambda_\eta$.

Proof of claim. Since $A \subseteq \lambda_\eta$ and λ_η is a cardinal in $\mathcal{M}_{\eta+1}$, we have that $A \in \mathcal{M}_{\eta+1}$ by acceptability. Write $A = [a, f] \in \mathrm{Ult}_n(\mathcal{M}^*_{\eta+1}, E_\eta)$ for some $n \leqslant \omega$ and set $\mu := \operatorname{crit} E_\eta$. Then since $A \subseteq \lambda_\eta$, we can assume that $f(u) < \mu$ for every $u \in [\mu]^{|a|}$ by shortness of E_η. Since f is $\mathbf{r\Sigma}_n^{\mathcal{M}^*_{\eta+1}}$ and $\mu < \rho_n(\mathcal{M}^*_{\eta+1})$, we get that $f \in \mathcal{M}^*_{\eta+1}$.

We also have that $\mathcal{M}_\eta$ agrees with $\mathcal{M}^*_{\eta+1}$ below $\lambda :=$ their common value of μ^+ and $f \in \mathcal{M}^*_{\eta+1} \,|\lambda = \mathcal{M}_\eta \,|\lambda$, so that $A = [a, f] \in \mathrm{Ult}_n(\mathcal{M}_\eta \,|\lambda, E_\eta)$. Then we have that, for $\beta \in \lambda$,

$$\beta \in A \quad \text{iff} \quad \{u \in [\mu]^{a \cup \{\beta\}} \mid \mathrm{pr}^{\{\beta\}, a\cup\{\beta\}}(u) \in f^{a, a\cup\{\beta\}}(u)\} \in (E_\eta)_{a \cup \{\beta\}}$$

and since $(E_\eta)_{a\cup\{\beta\}} \in \mathcal{M}_\alpha$ is $\mathbf{\Sigma}_1^{\mathcal{M}_\eta \,|\lambda}$, A is $\mathbf{\Sigma}_1^{\mathcal{M}_\eta \,|\lambda}$ and thus $\mathbf{\Sigma}_1^{\mathcal{M}_\eta \,|\lambda_\eta}$ as well. ⊣

Now, if $(E_\alpha)_a \in \mathcal{M}_\alpha$ then since $(E_\alpha)_a$ can be coded as a subset of λ, then Claim 4.10.1 implies that $(E_\alpha)_a$ is $\mathbf{\Sigma}_1^{\mathcal{M}_\eta \,|\lambda}$ and thus $\mathbf{\Sigma}_1^{\mathcal{M}^*_{\eta+1}}$, as wanted. But we do indeed have that the amenable encoding of $(E_\alpha)_a$ is in $\mathcal{M}_\alpha$ by amenability, so since we can Σ_1-define $(E_\alpha)_a$ with the amenable code as parameter, we get the wanted. ■

Definition 4.11. Let $\mathcal{T}$ be an iteration tree and b a branch of $\mathcal{T}$. Then b **drops in model** if $D \cap b \neq \varnothing$ and b **drops in degree** if $\deg(b) < \deg(\operatorname{root} b)$. ○

Insert intuition about drops.

Theorem 4.12. *Let $\mathcal{T}$ be a k-iteration tree on a k-sound premouse $\mathcal{M}_0$, with models $\mathcal{M}_\alpha$ and embeddings $i_{\alpha,\beta}$. Let $\alpha+1 <_T \beta$ and $D \cap (\alpha+1,\beta]_T = \varnothing$. Then*

(i) $\deg(\alpha+1) \geqslant \deg(\xi+1)$ *for every* $\xi+1 \in (\alpha+1,\beta]_T$*;*

(ii) If $\deg(\alpha+1) = \deg(\xi+1) = n$ *for every* $\xi+1 \in (\alpha+1,\beta]$ *then*

$$i_{\alpha+1,\beta} \circ i^*_{\alpha+1} : \mathcal{M}^*_{\alpha+1} \to \mathcal{M}_\beta$$

is an n-embedding. Moreover, if $[0,\alpha]_T$ drops in model or degree then

(a) $\rho_{n+1}(\mathcal{M}^*_{\alpha+1}) = \rho_{n+1}(\mathcal{M}_\beta) \leqslant \operatorname{crit}(i_{\alpha+1,\beta} \circ i^*_{\alpha+1})$*;*

(b) $(i_{\alpha+1,\beta} \circ i^*_{\alpha+1})(p_{n+1}(\mathcal{M}^*_{\alpha+1})) = p_{n+1}(\mathcal{M}_\beta)$*;*

(c) $\mathfrak{C}_{n+1}(\mathcal{M}^*_{\alpha+1}) = \mathfrak{C}_{n+1}(\mathcal{M}_\beta)$*.*

(d) $(i_{\alpha+1,\beta} \circ i^*_{\alpha+1}) \restriction \mathfrak{C}_{n+1}(\mathcal{M}^*_{\alpha+1}) = \sigma$*, where* $\sigma : \mathfrak{C}_{n+1}(\mathcal{M}_\beta) \to \mathfrak{C}_n(\mathcal{M}_\beta)$ *is the core embedding.* ■

Proof. For (i), note that closeness ensures that

$$\mathcal{P}^{\mathcal{M}^*_{\alpha+1}}(\operatorname{crit} E_\alpha) = \mathcal{P}^{\mathcal{M}_{\alpha+1}}(\operatorname{crit} E_\alpha),$$

so that if $\rho_n(\mathcal{M}^*_{\alpha+1}) \leqslant \operatorname{crit} E_\alpha$ then $\rho_n(\mathcal{M}_{\alpha+1}) \leqslant \operatorname{crit} E_\alpha$ as well – this shows that $\deg(\alpha+1) \geqslant \deg(\operatorname{succ}_T(\alpha+1))$. We can then use our assumption that $D \cap (\alpha+1,\beta]_T = \varnothing$, so that we can continue this argument ad infinitum.

For $\lambda \in (\alpha+1,\beta]_T$ limit with $\lambda = \operatorname{pred}_T(\xi+1)$, our inductive assumption ensures that $\deg\lambda := \inf_{\gamma<\lambda}(\deg\gamma)$ makes sense, so that $\deg(\alpha+1) \geqslant \deg\lambda$ as well. This entails that $\deg(\alpha+1) \geqslant \deg(\xi+1)$ by repeating the above argument.

(ii): Now assume that $n := \deg(\alpha+1) = \deg(\xi+1)$ for every $\xi+1 \in (\alpha+1,\beta]$. That is, no drops occur on the $(\alpha+1,\beta]$ branch. Then

$$\operatorname{crit}(i_{\alpha+1,\beta} \circ i^*_{\alpha+1}) < \rho_n(\mathcal{M}^*_{\alpha+1}),$$

so that Theorem 3.10 gives us that $i_{\alpha+1,\beta} \circ i^*_{\alpha+1}$ is in fact an n-embedding. Assume now that $[0,\alpha]_T$ drops in model or degree – we need to show (a)-(d). It suffices to show that $\mathcal{M}^*_{\alpha+1}$ is $(n+1)$-solid, since we also have that

$$\rho_{n+1}(\mathcal{M}_\beta) \leqslant \operatorname{crit}(i_{\alpha+1,\beta} \circ i^*_{\alpha+1})$$

by definition of n, so that Theorem 3.10 gives us (a)-(d).

To show that $\mathcal{M}^*_{\alpha+1}$ is $(n+1)$-solid note first that it's k-sound and there's a drop on the $\mathcal{M}$-to-$\mathcal{M}^*_{\alpha+1}$ branch. If there's a drop in degree then the previous models are $n+1 \leqslant k$ sound, so that in particular $(n+1)$-solid. Then Theorem 3.10 implies that $\mathcal{M}^*_{\alpha+1}$ is $(n+1)$-solid as well. If there's a drop in model, then at the drop we get a sound structure as we're working with premice. Thus in particular every future structure will be $(k+1)$-solid, so again by Theorem 3.10 we get that $\mathcal{M}^*_{\alpha+1}$ is $(k+1)$-solid as well, so in particular $(n+1)$-solid. ■

Definition 4.13. A (k,θ)**-iteration strategy** for a phalanx Φ is a winning strategy for player II in $\mathcal{G}_k(\Phi,\theta)$, and Φ is (k,θ)**-iterable** if such a strategy exists. ○

Definition 4.14. A (k,θ)**-mouse** is a premouse which is (k,θ)-iterable. ○

4.2 Comparison

As mentioned above, comparison of mice is the main use of iterability.

Theorem 4.15 (Comparison). *Let $\theta \in$ On and Φ,Ψ be phalanxes of k-sound (k,θ^++1)-mice of size $\leqslant \theta$. Fix (k,θ^++1)-iteration strategies Σ,Γ for Φ,Ψ. Then there are iteration trees $\mathcal{T},\mathcal{U}$ on Φ,Ψ by Σ,Γ with last models $\mathcal{P},\mathcal{Q}$ such that one of the two following holds:*

- *The Φ-to-$\mathcal{P}$ branch in $\mathcal{T}$ does not drop in model or degree, and $\mathcal{P} \trianglelefteq \mathcal{Q}$;*
- *The Ψ-to-$\mathcal{Q}$ branch in $\mathcal{U}$ does not drop in model or degree, and $\mathcal{Q} \trianglelefteq \mathcal{P}$.*

Proof. We will build $\mathcal{T}$ and $\mathcal{U}$ by a recursive process called "iterating away the least disagreement". The idea is exactly that. Namely, if we've gotten to some stage $\alpha+1 < \theta^+$ with approximations $\mathcal{T}_\alpha,\mathcal{U}_\alpha$ of $\mathcal{T},\mathcal{U}$ such that the last models of $\mathcal{T}_\alpha$ and $\mathcal{U}_\alpha$ disagree, then we "push" this disagreement up the ladder of ordinals. That this actually works is the main part of the proof. But first, let's do the construction of $\mathcal{T}_\alpha$ and $\mathcal{U}_\alpha$.

First, let $\mathcal{T}_0$ and $\mathcal{U}_0$ just be Φ and Ψ, respectively. Assume now that $\mathcal{T}_\alpha$ and $\mathcal{U}_\alpha$ have been constructed, with last models $\mathcal{P}$ and $\mathcal{Q}$. If $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq \mathcal{P}$ then stop the construction. Otherwise define λ to be least such that $\mathcal{P}|\lambda \neq \mathcal{Q}|\lambda$, where

$\mathcal{P}|\lambda \neq \mathcal{Q}|\lambda$ means that $\dot{F}^{\mathcal{P}|\lambda} \neq \dot{F}^{\mathcal{Q}|\lambda}$. If $\mathcal{P}|\lambda$ is active then set $\beta + 1 := \operatorname{lh} \mathcal{T}_\alpha$ and $E_\beta^{\mathcal{T}_{\alpha+1}} := \dot{F}^{\mathcal{P}|\lambda}$ and then the rules of the iteration game provides a unique one-model extension of $\mathcal{T}_\alpha$; call this extension $\mathcal{T}_{\alpha+1}$. If $\mathcal{P}|\lambda$ is passive then set $\mathcal{T}_{\alpha+1} := \mathcal{T}_\alpha$. Define $\mathcal{U}_{\alpha+1}$ completely analogously.

Note that by this construction, the last models of $\mathcal{T}_{\alpha+1}$ and $\mathcal{U}_{\alpha+1}$ will agree below $\lambda + 1$, so the next extender will necessarily have length $> \lambda$, so that the iteration is normal. If δ is a limit ordinal then we define $\mathcal{T}_\delta := \bigcup_{\alpha<\delta} \mathcal{T}_\alpha$ if the $\mathcal{T}_\alpha$'s become constant as $\alpha \to \delta$. Otherwise set $\mathcal{T}_\delta$ to be the one-model extension of $\bigcup_{\alpha<\delta} \mathcal{T}_\alpha$ determined by the cofinal wellfounded branch of $\bigcup_{\alpha<\delta} \mathcal{T}_\alpha$ given to us by Σ. Define $\mathcal{U}_\delta$ analogously.

Now, we need to show that this construction stops at some stage $\alpha < \theta^+$. Assume not, so that we have trees $\mathcal{T} := \mathcal{T}_{\theta^+}$ and $\mathcal{U} := \mathcal{U}_{\theta^+}$. We claim that $\mathcal{T}$ and $\mathcal{U}$ have length $\theta^+ + 1$. Assume first $\operatorname{lh} \mathcal{T} < \theta^+$, which means that there must be θ^+ many distinct branches out of $\mathcal{M}$ for some $\mathcal{M}$ on Φ, by regularity of θ^+ (or that the construction stopped, but we're assuming that's not the case).

But then the θ^+ many extenders for these branches have to have strictly increasing lengths by normality, but $|\mathcal{M}| \leqslant \theta$, so this is impossible. But then since the $\mathcal{T}_\alpha$'s don't get constant as $\alpha \to \theta^+$ because $\operatorname{lh} \mathcal{T}_\alpha < \theta^+$ for $\alpha < \theta^+$, we get that $\operatorname{lh} \mathcal{T}_{\theta^+} = \theta^+ + 1$. Analogously $\operatorname{lh} \mathcal{U} = \theta^+ + 1$.

Claim 4.15.1. For any $\alpha, \beta < \theta^+$, $E_\alpha^{\mathcal{T}}$ is incompatible with $E_\beta^{\mathcal{U}}$.

Proof of claim. Write $E := E_\alpha^{\mathcal{T}}$ and $F := E_\beta^{\mathcal{U}}$ and assume they're compatible; say without loss of generality that $\nu_E < \nu_F$, so that $E \restriction \nu_E = F \restriction \nu_E$. Let ξ be such that E is the extender used to go from $\mathcal{T}_\xi$ to $\mathcal{T}_{\xi+1}$, and γ such that F is the one used in the passage from $\mathcal{U}_\gamma$ to $\mathcal{U}_{\gamma+1}$, so that $\xi < \gamma$ since we're iterating away least disagreements and $\nu_E < \nu_F$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be the last models of $\mathcal{T}_\gamma$ and $\mathcal{U}_\gamma$, respectively. Since $\lambda_\alpha^{\mathcal{T}}$ is a cardinal of $\mathcal{P}$ by (G1) and that $\mathcal{P}$ agrees with $\mathcal{Q}$ below $\lambda_\beta^{\mathcal{U}}$, $\lambda_\alpha^{\mathcal{T}}$ is also a cardinal of $\mathcal{Q}|\lambda_\beta^{\mathcal{U}}$. But the initial segment condition implies that $E \in \mathcal{Q}|\lambda_\beta^{\mathcal{U}}$ (in both its cases), so that $\lambda_\alpha^{\mathcal{T}}$ is *not* a cardinal of $\mathcal{Q}|\lambda_\beta^{\mathcal{U}}$, $\lightning$. $\dashv$

The strategy now is then to find compatible extenders in $\mathcal{T}$ and $\mathcal{U}$ to reach our desired contradiction. Let $\pi : \operatorname{cHull}^{V_\eta}(\theta \cup \{\theta, \mathcal{T}, \mathcal{U}\}) \to V_\eta$ be the uncollapse for

η sufficiently large and set $\bar{x} := \pi^{-1}(x)$ for every $x \in \operatorname{ran}\pi$. Setting $\alpha := \operatorname{crit}\pi$, note that $\theta < \alpha$. Define furthermore $\operatorname{root}\mathcal{T} := \operatorname{root}^{\mathcal{T}}\theta^+$ and $\operatorname{root}\mathcal{U} := \operatorname{root}^{\mathcal{U}}\theta^+$.

Claim 4.15.2. $\bar{\mathcal{T}} = \mathcal{T} \upharpoonright \alpha + 1$ and $\bar{\mathcal{U}} = \mathcal{U} \upharpoonright \alpha + 1$.

Proof of claim. As all mice on Φ and Ψ have size $\leqslant \theta < \alpha$, $\bar{\mathcal{T}}$ and $\bar{\mathcal{U}}$ are trees on Φ and Ψ, respectively. In the same manner, $\bar{\mathcal{T}} \upharpoonright \alpha = \mathcal{T} \upharpoonright \alpha$ and $\bar{\mathcal{U}} \upharpoonright \alpha = \mathcal{U} \upharpoonright \alpha$. Furthermore $[\operatorname{root}\mathcal{T}, \alpha]_{\bar{T}} = [\operatorname{root}\mathcal{T}, \theta^+]_T \cap \alpha$ and $[\operatorname{root}\mathcal{U}, \alpha]_{\bar{U}} = [\operatorname{root}\mathcal{U}, \theta^+]_U \cap \alpha$. This means that $[\operatorname{root}\mathcal{T}, \alpha]_{\bar{T}}$ has limit order type, and any branch of an iteration tree must be closed below its supremum by definition of tree order, so $\alpha \in [\operatorname{root}\mathcal{T}, \theta^+]_T$ and $\alpha \in [\operatorname{root}\mathcal{U}, \theta^+]_U$, implying that $[\operatorname{root}\mathcal{T}, \alpha]_{\bar{T}} = [\operatorname{root}\mathcal{T}, \alpha]_T$ and $[\operatorname{root}\mathcal{U}, \alpha]_{\bar{U}} = [\operatorname{root}\mathcal{U}, \alpha]_U$. We can then conclude that $\bar{\mathcal{T}} = \mathcal{T} \upharpoonright \alpha + 1$ and $\bar{\mathcal{U}} = \mathcal{U} \upharpoonright \alpha + 1$ since $\theta^{+\mathcal{H}} = \alpha$ and the direct limit construction is absolute to $\mathcal{H}$. $\dashv$

Let $\gamma \in [\operatorname{root}\mathcal{T}, \alpha]_T$ be such that $D^{\mathcal{T}} \cap [\operatorname{root}\mathcal{T}, \alpha]_T \subseteq \gamma$. Then by the above Claim 4.15.2 and by using π, $D^{\mathcal{T}} \cap [\operatorname{root}\mathcal{T}, \theta^+]_T \subseteq \gamma$, so that $i^{\mathcal{T}}_{\alpha,\theta^+}$ is defined. But we can get even more information on $i^{\mathcal{T}}_{\alpha,\theta^+}$: if $x \in \mathcal{M}^{\mathcal{T}}_\alpha$ then letting γ and $\bar{x} \in \mathcal{M}^{\mathcal{T}}_\gamma$ be such that $x = i^{\mathcal{T}}_{\gamma,\alpha}(\bar{x}) = i^{\bar{\mathcal{T}}}_{\gamma,\alpha}(\bar{x})$, we get that

$$\pi(x) = i^{\mathcal{T}}_{\gamma,\theta^+}(\bar{x}) = (i^{\mathcal{T}}_{\alpha,\theta^+} \circ i^{\mathcal{T}}_{\gamma,\alpha})(\bar{x}) = i^{\mathcal{T}}_{\alpha,\theta^+}(x),$$

so that $i^{\mathcal{T}}_{\alpha,\theta^+} = \pi \upharpoonright \mathcal{M}^{\mathcal{T}}_\alpha$ and analogously $i^{\mathcal{U}}_{\alpha,\theta^+} = \pi \upharpoonright \mathcal{M}^{\mathcal{U}}_\alpha$, meaning that $i^{\mathcal{T}}_{\alpha,\theta^+}$ and $i^{\mathcal{U}}_{\alpha,\theta^+}$ agree when both are defined. This leads to

$$\mathcal{P}^{\mathcal{M}^{\mathcal{T}}_\alpha}(\alpha) = \mathcal{P}^{\mathcal{M}^{\mathcal{T}}_{\theta^+}}(\alpha) = \mathcal{P}^{\mathcal{M}^{\mathcal{U}}_{\theta^+}}(\alpha) = \mathcal{P}^{\mathcal{M}^{\mathcal{U}}_\alpha}(\alpha),$$

where the first and last equality is due to $\operatorname{crit} i^{\mathcal{T}}_{\alpha,\theta^+} = \operatorname{crit} i^{\mathcal{U}}_{\alpha,\theta^+} = \alpha$ and the middle is because $\mathcal{M}^{\mathcal{T}}_{\theta^+}$ and $\mathcal{M}^{\mathcal{U}}_{\theta^+}$ agree below θ^+. This entails that $i^{\mathcal{T}}_{\alpha,\theta^+}$ and $i^{\mathcal{U}}_{\alpha,\theta^+}$ agree on the same subsets of α. Let now $\xi + 1 \in [\operatorname{root}\mathcal{T}, \theta^+]_T$, $\gamma + 1 \in [\operatorname{root}\mathcal{U}, \theta^+]_U$ be the T- and U-successor of α, respectively. Write $\nu := \inf(\nu_{E^{\mathcal{T}}_\xi}, \nu_{E^{\mathcal{U}}_\gamma})$. Then given any $a \in [\nu]^{<\omega}$ and $B \in \mathcal{M}_\alpha^{\mathcal{T}} \cap \mathcal{M}_\alpha^{\mathcal{U}}$ we get

$$\begin{aligned} B \in (E_\xi^{\mathcal{T}})_a \quad &\text{iff} \quad a \in i^{\mathcal{T}}_{\alpha,\xi+1}(B) \\ &\text{iff} \quad a \in i^{\mathcal{T}}_{\alpha,\theta^+}(B) \\ &\text{iff} \quad a \in i^{\mathcal{U}}_{\alpha,\theta^+}(B) \\ &\text{iff} \quad a \in i^{\mathcal{U}}_{\alpha,\gamma+1}(B) \\ &\text{iff} \quad B \in (E_\gamma^{\mathcal{U}})_a, \end{aligned}$$

so that $E_\xi^{\mathcal{T}} \restriction \nu = E_\gamma^{\mathcal{U}} \restriction \nu$, so that $E_\xi^{\mathcal{T}}$ and $E_\gamma^{\mathcal{U}}$ are compatible, contradicting Claim 4.15.1. This shows that there is some $\alpha < \theta^+$ such that, letting $\mathcal{P}$ and $\mathcal{Q}$ be the last models of $\mathcal{T}_\alpha$ and $\mathcal{U}_\alpha$, respectively, either $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq \mathcal{P}$.

It remains to show that no drops in model or degree occur on the $\mathcal{M}$-to-$\mathcal{P}$ branch if $\mathcal{P} \trianglelefteq \mathcal{Q}$ and vice versa with the $\mathcal{M}$-to-$\mathcal{Q}$ branch if $\mathcal{Q} \trianglelefteq \mathcal{P}$. If $\mathcal{P} \triangleleft \mathcal{Q}$ and the $\mathcal{M}$-to-$\mathcal{P}$ branch dropped in model or degree then Theorem 4.12 implies that $\mathcal{P}$ isn't sound, but this contradicts that $\mathcal{Q}$ is a premouse. We get the same conclusion if we assume $\mathcal{Q} \triangleleft \mathcal{P}$, so we can assume that $\mathcal{P} = \mathcal{Q}$.

Assume that both the $\mathcal{M}$-to-$\mathcal{P}$ and $\mathcal{M}$-to-$\mathcal{Q}$ branch drops in model or degree. Say $\operatorname{lh}\mathcal{T} = \beta+1$ and $\operatorname{lh}\mathcal{U} = \gamma+1$. Then by Theorem 4.12, $\deg^{\mathcal{T}}(\beta) = \deg^{\mathcal{U}}(\gamma) = n$, where n is largest such that $\mathcal{P} = \mathcal{Q}$ is n-sound. But then pick $\xi+1$ and $\eta+1$ to be the last drop of $\mathcal{T}$ and $\mathcal{U}$, respectively. We'll then have that $\mathfrak{C}_{n+1}(\mathcal{M}^*_{\xi+1}) = \mathcal{M}^*_{\xi+1}$ as it's a drop, so that Theorem 4.12 implies that the map $i^{\mathcal{T}}_{\xi+1,\beta} \circ i^{*\mathcal{T}}_{\xi+1}$ is the core embedding $\mathfrak{C}_{n+1}(\mathcal{P}) \to \mathfrak{C}_n(\mathcal{P})$. This is completely analogous for $\eta+1$, so that

$$\begin{aligned} i^{\mathcal{T}}_{\xi+1,\beta} \circ i^{*\mathcal{T}}_{\xi+1} &= \text{core embedding } \mathfrak{C}_{n+1}(\mathcal{P}) \to \mathfrak{C}_n(\mathcal{P}) \\ &= \text{core embedding } \mathfrak{C}_{n+1}(\mathcal{Q}) \to \mathfrak{C}_n(\mathcal{Q}) \\ &= i^{\mathcal{U}}_{\eta+1,\gamma} \circ i^{*\mathcal{U}}_{\eta+1}. \end{aligned}$$

But then, analogously to what we did in Claim 4.15.1, we see that the extender used in $i^{*\mathcal{T}}_{\xi+1}$ is compatible with the extender used in $i^{*\mathcal{U}}_{\gamma+1}$, contradicting the same Claim 4.15.1. We conclude thus that no drops in model or degree occur in the two branches. ■

Comparison can also be used to define a pre-wellordering on all mice, the so-called *mouse order*, defined in the obvious way. We will not be using this fact, however. An interesting consequence of comparison is that for sufficiently nice mice it implies an actual direct comparison of the mice.

Corollary 4.16. *Let $\mathcal{M}$ and $\mathcal{N}$ be sound $(\omega, \omega_1 + 1)$-mice ω-projecting to ω. Then either $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$.*

Proof. As both mice are sound and project to ω they are both countable, so we have enough iterability to compare them by Comparison Theorem 4.15. Assume thus without loss of generality that we have the comparison

$$\begin{tikzcd} \mathcal{P} & \trianglelefteq & \mathcal{Q} \\ \mathcal{M} \arrow[u] & & \mathcal{N} \arrow[u, dash] \end{tikzcd}$$

and that the $\mathcal{M}$-to-$\mathcal{P}$ branch doesn't drop in model or degree. If the $\mathcal{M}$-to-$\mathcal{P}$ branch had length > 0 then the first extender E used along the branch would satisfy $\operatorname{crit} E < \rho(\mathcal{M}) = \omega$, $\lightning$. Thus $\mathcal{M} = \mathcal{P}$.

Assume without loss of generality that the $\mathcal{N}$-to-$\mathcal{Q}$ branch has length > 0, because otherwise we're done. As $\rho(\mathcal{N}) = \omega$ the rules of the iteration game ensures that $\mathcal{Q}$ is not sound, meaning then that $\mathcal{P} \triangleleft \mathcal{Q}$. Since $\rho(\mathcal{M}) = \omega$, $\mathcal{M}$ is countable in $\mathcal{Q}$. This means that $\mathcal{M} \trianglelefteq \mathcal{N}$ since ultrapowers can't add any reals. ■

4.3 Copying construction

These remaining two sections will provide tools needed in comparison arguments. We start off with the *copying construction*, making it possible to "lift" embeddings between phalanxes to iteration trees on them. The construction of the "copied tree" will consist of successive applications of the following *shift lemma*.

Lemma 4.17 (Shift lemma). *Let $\bar{\mathcal{N}}, \bar{\mathcal{M}}, \mathcal{N}, \mathcal{M}$ be premice, $n \leqslant \omega$, $\psi : \bar{\mathcal{N}} \to \mathcal{N}$ a (near) 0-embedding and $\pi : \bar{\mathcal{M}} \to \mathcal{M}$ a (near) n-embedding. Writing $\bar{\kappa} :=$ $\operatorname{crit} \dot{F}^{\bar{\mathcal{N}}}$, assume that*

- *$\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ agree below $\bar{\kappa}^{+\bar{\mathcal{M}}}$;*

- $\bar{\kappa}^{+\bar{\mathcal{M}}} \leqslant \bar{\kappa}^{+\bar{\mathcal{N}}}$;
- $\pi \restriction \bar{\kappa}^{+\bar{\mathcal{M}}} = \psi \restriction \bar{\kappa}^{+\bar{\mathcal{M}}}$;
- $\bar{\kappa} < \rho_n(\bar{\mathcal{M}})$ *and* $\mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}})$ *is wellfounded.*

Then there is a unique (near) n-embedding $\sigma : \mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}}) \to \mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}})$ *satisfying that*

(i) $\mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}})$ *and* $\bar{\mathcal{N}}$ *agree below* $\mathrm{On}^{\bar{\mathcal{N}}}$;
(ii) $\sigma \restriction \mathrm{On}^{\bar{\mathcal{N}}} = \psi \restriction \mathrm{On}^{\bar{\mathcal{N}}}$;
(iii) The following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}}) \arrow[r, "\sigma"] & \mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}}) \\
\bar{\mathcal{M}} \arrow[u, "i"] \arrow[r, "\pi"] & \mathcal{M} \arrow[u, "j"']
\end{tikzcd}$$

Proof. We start by showing uniqueness. Let σ satisfy all the above properties. By (ii), we get for every $a \in [\mathrm{lh}\, \dot{F}^{\bar{\mathcal{N}}}]^{<\omega}$ that

$$\sigma[a, \mathrm{id}] = \sigma(a) = \psi(a) = [\psi(a), \mathrm{id}] = [\psi(a), \pi(\mathrm{id})]. \tag{1}$$

By (iii) we also get that $\sigma i = j\pi$, so that for any $\bar{x} \in \bar{\mathcal{M}}$ it holds that

$$\begin{aligned} \sigma[\{\kappa\}, c_{\bar{x}}] &= (\sigma i)(\bar{x}) = (j\pi)(\bar{x}) = [\{\kappa\}, c_{\pi(\bar{x})}] \\ &= [\{\kappa\}, \pi(c_{\bar{x}})] = [\psi(\{\kappa\}), \pi(c_{\bar{x}})]. \end{aligned} \tag{2}$$

Now let $[a, f] \in \mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}})$ be arbitrary. Then $[a, f]$ is $\mathbf{r\Sigma}_{n+1}$-definable with parameters from a and $\mathrm{ran}\, i$, since if $<_{\bar{\mathcal{M}}}$ is the (Σ_1-definable) constructibility order on $\mathcal{M}$, $i``<_{\bar{\mathcal{M}}}$ is the corresponding (Σ_1-definable) order on the ultrapower. As σ is $r\Sigma_{n+1}$ elementary by assumption, $\sigma[a, f]$ is also definable by the same definition, so σ is uniquely characterized by what it does on $\mathrm{lh}\, \dot{F}^{\bar{\mathcal{N}}}$ and $\mathrm{ran}\, i$. Letting

$$\sigma[a, f] := [\psi(a), \pi(f)], \tag{3}$$

it has both properties (1) and (2), so it is the unique embedding with these properties by the above argument.

As for the existence, we have to check that σ defined as in (3) is actually an n-embedding satisfying (i)-(iii). To show that it's an n-embedding, let φ be $r\Sigma_{n+1}$ and $[a_i, f_i] \in \mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}})$ for $i = 1, \dots, l$. Then, setting $b := \bigcup_i a_i$,

$$\begin{aligned}\mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}}) \models \varphi[[a_i, f_i]] \quad &\text{iff} \quad \forall^{E_b} u : \bar{\mathcal{M}} \models \varphi[f_i^{a_i,b}(u)] \\ &\text{iff} \quad \forall^{E_b} u : \mathcal{M} \models \varphi[\pi(f_i)^{a_i,b}(u)] \\ &\text{iff} \quad \forall^{E_{\psi(b)}} u : \mathcal{M} \models \varphi[\pi(f_i)^{\psi(a_i),\psi(b)}(u)] \\ &\text{iff} \quad \mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}}) \models \varphi[\sigma[a_i, f_i]],\end{aligned}$$

so σ is $r\Sigma_{n+1}$-elementary. Both ultrapowers are n-sound as π is an n-embedding. As for the preservation of the standard parameter we have that, for $i \leqslant n$,

$$\sigma(p_i(\mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}}))) = (\sigma i)(p_i(\bar{\mathcal{M}})) = (j\pi)(p_i(\bar{\mathcal{M}})) = p_i(\mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}})),$$

and for the projectum the $i < n$ case is analogous to standard parameter case. As for ρ_n we split into the case where π is a near n-embedding and an n-embedding. If we're in the near case then we have that

$$\begin{aligned}\rho_n(\mathrm{Ult}_n(\mathcal{M}, \dot{F}^{\mathcal{N}})) &\leqslant \sup (j\pi)\text{"}\rho_n(\bar{\mathcal{M}}) \\ &= \sup (\sigma i)\text{"}\rho_n(\bar{\mathcal{M}}) \\ &= \sup \sigma\text{"}\rho_n(\mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}})),\end{aligned}$$

making σ a near n-embedding as well. If π was an n-embedding then the above inequality would be an equality, making σ an n-embedding as well. Now, (ii) and (iii) holds by construction of σ, so we only need to check (i). Setting $\lambda := \bar{\kappa}^{+\bar{\mathcal{M}}}$, we have by assumption that $\mathcal{M}|\lambda = \bar{\mathcal{N}}|\lambda$, so that

$$\mathrm{Ult}_n(\bar{\mathcal{M}}|\lambda, \dot{F}^{\bar{\mathcal{N}}}) = \mathrm{Ult}_n(\bar{\mathcal{N}}|\lambda, \dot{F}^{\bar{\mathcal{N}}})$$

as well. This means that $\mathrm{Ult}_n(\bar{\mathcal{M}}, \dot{F}^{\bar{\mathcal{N}}})$ and $\mathrm{Ult}_n(\bar{\mathcal{N}}, \dot{F}^{\bar{\mathcal{N}}})$ agree below $i(\lambda)$, so they also agree below $\mathrm{On}^{\bar{\mathcal{N}}}$ as $\mathrm{On}^{\bar{\mathcal{N}}} = \mathrm{lh}\,\dot{F}^{\bar{\mathcal{N}}}$ and $\dot{F}^{\bar{\mathcal{N}}}$ is short. Since $\bar{\mathcal{N}}$ also agrees with $\mathrm{Ult}_n(\bar{\mathcal{N}}, \dot{F}^{\bar{\mathcal{N}}})$ below $\mathrm{lh}\,\dot{F}^{\bar{\mathcal{N}}} = \mathrm{On}^{\bar{\mathcal{N}}}$ by coherence, we get (i). ■

In our formulation of the copying construction we will need the notions of (near) k-embeddings between phalanxes and also between iteration trees.

Definition 4.18. Let $k \leqslant \omega$ and let Φ and Ψ be phalanxes. Then a **(near) k-embedding between phalanxes** $\pi : \Phi \to \Psi$ is a sequence $\pi := \langle \pi_\alpha \mid \alpha < \operatorname{lh} \Phi \rangle$ such that

(i) $\pi_\alpha : \mathcal{M}^\Phi_\alpha \to \mathcal{M}^\Psi_\alpha$ is a (near) k-embedding;
(ii) If $\alpha \leqslant \beta < \operatorname{lh} \Phi$ then $\pi_\alpha \restriction \lambda^\Phi_\alpha = \pi_\beta \restriction \lambda^\Phi_\alpha$. ○

Definition 4.19. Let $k \leqslant \omega$ and $\mathcal{T}, \mathcal{U}$ be k-iteration trees on phalanxes Φ, Ψ, respectively. Then a **(near) k-embedding between k-iteration trees** $\theta : \mathcal{T} \to \mathcal{U}$ is a pair $\theta = \langle \vec{\pi}, \varphi \rangle$, where $\vec{\pi} := \langle \pi_\alpha \mid \alpha < \operatorname{lh} \mathcal{T} \rangle$ is a sequence and $\varphi : \operatorname{lh} \mathcal{T} \to \operatorname{lh} \mathcal{U}$ is injective and tree order preserving such that

(i) $\langle \pi_\alpha \mid \alpha < \operatorname{lh} \Phi \rangle$ is a (near) k-embedding $\Phi \to \Psi$;
(ii) $\varphi \restriction \operatorname{lh} \Phi = \mathrm{id}$;
(iii) $\pi_\alpha : \mathcal{M}^\mathcal{T}_\alpha \to \mathcal{M}^\mathcal{U}_{\varphi(\alpha)}$ is a (near) $\deg^\mathcal{T}(\alpha)$-embedding;
(iv) If $\beta < \alpha$ and E_β is the last extender of the initial segment $\mathcal{P}$ of $\mathcal{M}_\beta$ then $\pi_\beta \restriction \mathrm{On}^\mathcal{P} = \pi_\alpha \restriction \mathrm{On}^\mathcal{P}$;
(v) If $\beta <_T \alpha$ and $(\beta, \alpha]_T \cap D^\mathcal{T} = \varnothing$ then

$$\begin{tikzcd}
\mathcal{M}^\mathcal{T}_\alpha \arrow[r, "\pi_\alpha"] & \mathcal{M}^\mathcal{U}_{\varphi(\alpha)} \\
\mathcal{M}^\mathcal{T}_\beta \arrow[r, "\pi_\beta"] \arrow[u, "i^\mathcal{T}_{\beta,\alpha}"] & \mathcal{M}^\mathcal{U}_{\varphi(\beta)} \arrow[u, "i^\mathcal{U}_{\varphi(\beta),\varphi(\alpha)}"']
\end{tikzcd}$$

commutes. ○

Lemma 4.20 (Copying construction). *Let $k \leqslant \omega$, $\pi : \Phi \to \Psi$ a (near) k-embedding between phalanxes of the same length and $\mathcal{T}$ a k-iteration tree on Ψ. Then there is a k-iteration tree $\pi\mathcal{T}$ on Φ and a (near) k-embedding $\vec{\pi} : \mathcal{T} \to \pi\mathcal{T}$ between k-iteration trees.*

Proof. Let $\mathcal{M}_\alpha$ be the models and $i_{\alpha,\beta}$ the embeddings of $\mathcal{T}$. We will define $\pi\mathcal{T} \restriction \alpha$ and π_α recursively, where we write $\mathcal{N}_\alpha$ and $j_{\alpha,\beta}$ for the models and embeddings of $\pi\mathcal{T}$, and inductively verify (iii)-(v) in the definition of a (near) k-embedding between k-iteration trees, where we set $\varphi := \mathrm{id}$.

For every $\alpha < \mathrm{lh}\,\Phi$ set $\mathcal{N}_\alpha := \mathcal{M}_\alpha^\Psi$ and π_α the α'th component of π, so that (i) and (ii) holds. For $\lambda \in [\mathrm{lh}\,\Phi, \mathrm{lh}\,\mathcal{T})$ limit, define

$$\mathcal{N}_\lambda := \varinjlim_{\alpha \in b - \sup(b \cap D^{\mathcal{T}})} \mathcal{N}_\alpha$$

where $b := [\mathrm{root}\,\lambda, \lambda)_T$, and let $\pi_\lambda : \mathcal{M}_\lambda \to \mathcal{N}_\lambda$ be $\pi_\lambda(j_{\alpha,\lambda}(x)) := i_{\alpha,\lambda}(\pi_\alpha(x))$. Then (v) holds by definition of π_λ and both (iii) and (iv) hold by just using the definition of direct limit.

Assume now that $\alpha = \beta + 1$ and that $\pi\mathcal{T} \upharpoonright \alpha$ and π_γ are defined for $\gamma < \alpha$. We then want to apply the Shift Lemma 4.17, so with the notation as in that lemma, set:

- $\bar{\mathcal{N}}$ is the initial segment of $\mathcal{M}_\beta$ whose last extender is E_β;
- $\mathcal{N}$ is $\pi_\beta(\bar{\mathcal{N}})$ if $\bar{\mathcal{N}} \lhd \mathcal{M}_\beta$ and $\mathcal{N}_\beta$ otherwise;
- $\psi := \pi_\beta \upharpoonright \bar{\mathcal{N}} : \bar{\mathcal{N}} \to \mathcal{N}$;
- $F_\beta := \dot{F}^{\mathcal{N}}$;
- $\bar{\mathcal{M}} := \mathcal{M}_\alpha^* = \mathcal{M}_\eta|\gamma$ with $\eta := \mathrm{pred}_T(\alpha)$;
- $\mathcal{M}$ is $\mathcal{N}_\eta|\pi_{\eta(\gamma)}$ if $\gamma \in \mathcal{M}_\eta$ and $\mathcal{N}_\eta$ otherwise;
- $\pi := \pi_\eta \upharpoonright \bar{\mathcal{M}} : \bar{\mathcal{M}} \to \mathcal{M}$.

We have to verify the assumptions in the Shift Lemma. Firstly, $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ should agree below $\bar{\kappa}^{+\bar{\mathcal{M}}}$, where $\bar{\kappa} := \mathrm{crit}\,\dot{F}^{\bar{\mathcal{N}}}$, but this amounts to showing that $\mathcal{M}_\alpha^*$ agrees with $\mathcal{M}_\beta$ below $\mathrm{crit}\,E_\beta^{+\bar{\mathcal{N}}}$, but that's true by definition of $\mathcal{M}_\alpha^*$.

Secondly, we have to show that $\bar{\kappa}^{+\bar{\mathcal{M}}} \leqslant \bar{\kappa}^{+\bar{\mathcal{N}}}$, which amounts to showing that $\mathrm{crit}\,E_\beta^{+\mathcal{M}_\alpha^*} \leqslant \mathrm{crit}\,E_\beta^{+\mathcal{M}_\beta}$, which again holds by definition of $\mathcal{M}_\alpha^*$. Thirdly we're to show that π and ψ agree below $\bar{\kappa}^{+\bar{\mathcal{M}}}$, i.e. that π_β and π_η agree below $\mathrm{crit}\,E_\beta^{+\mathcal{M}_\alpha^*}$. But $\mathrm{crit}\,E_\beta^{+\mathcal{M}_\alpha^*} < \mathrm{lh}\,E_\eta$ and π_β agrees with π_η below $\mathrm{lh}\,E_\eta$ by (iv). Penultimately we're to show that $\mathrm{crit}\,E_\beta < \rho_{\deg^{\mathcal{T}}(\alpha)}(\mathcal{M}_\alpha^*)$, but this holds by definition of $\mathcal{M}_\alpha^*$. Lastly, π_α should be a (near) $\deg^{\mathcal{T}}(\alpha)$-embedding, but π_η is a (near) $\deg^{\mathcal{T}}(\eta)$-embedding and $\deg^{\mathcal{T}}(\eta) \geqslant \deg^{\mathcal{T}}(\alpha)$ by Theorem 4.12. Thus, the

Shift Lemma 4.17 grants us with a (near) $\deg^{\mathcal{T}}(\alpha)$-embedding

$$\sigma : \mathcal{M}_\alpha = \mathrm{Ult}_{\deg^{\mathcal{T}}(\alpha)}(\mathcal{M}^*_\alpha, E_\beta) \to \mathrm{Ult}_{\deg^{\mathcal{T}}(\alpha)}(\mathcal{N}^*_\alpha, F_\beta)$$

Let $n := \deg^{\mathcal{T}}(\alpha)$ and $m := \deg^{\pi\mathcal{T}}(\alpha)$, where m's value is given by the rules of the game. We claim that $n \leqslant m$. Indeed, if $m < n$ then $\mathrm{crit}\, E_\beta < \rho_n(\mathcal{M}^*_\alpha)$, so by n-soundness of $\mathcal{M}^*_\alpha$ it holds that $\rho_n(\mathcal{M}^*_\alpha)$ is an $\mathcal{M}^*_\alpha$-cardinal, so that $\mathrm{crit}\, E_\beta < \zeta$ for some $\zeta < \rho_n(\mathcal{M}^*_\alpha)$. Then

$$\mathrm{crit}\, F_\beta = \pi_\eta(\mathrm{crit}\, E_\beta) < \pi_\eta(\zeta) \leqslant \sup \pi\text{"} \rho_n(\mathcal{M}^*_\alpha) \leqslant \rho_n(\mathcal{N}^*_\alpha),$$

contradicting maximality of m. If $\mathrm{Ult}_m(\mathcal{N}, F_\alpha)$ is illfounded set $\pi\mathcal{T} := \pi\mathcal{T} \upharpoonright \alpha+1$. Otherwise let π_α be the composition

$$\mathcal{M}_\alpha \xrightarrow{\sigma} \mathrm{Ult}_n(\mathcal{N}^*_\alpha, F_\beta) \xrightarrow{\tau} \mathrm{Ult}_m(\mathcal{N}^*_\alpha, F_\beta) = \mathcal{N}_\alpha,$$

with τ being the natural map and σ the shift map. It remains to show that $\pi\mathcal{T} \upharpoonright \alpha+2$ and $\pi_{\alpha+1}$ satisfy (iii)-(v). But we checked (iii) above and (iv)-(v) are directly by the Shift Lemma 4.17. ■

4.4 Dodd-Jensen Lemma

The second tool we will need is the Dodd-Jensen Lemma. Intuitively it says that iteration maps are "minimal" in some sense. Before we state the Dodd-Jensen Lemma, we need to introduce some notions. The first one is a generalisation of the iteration game, in which there are several *rounds*.

Let $k < \omega$, $\theta, \alpha \in \mathrm{On}$ and $\mathcal{M}$ a k-sound premouse. Then the **iteration game with rounds**, written $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$, is played almost in the same way as the iteration game, the difference being that instead of the two players playing iteration trees, they play sequences $\vec{T} = \langle \mathcal{T}_\gamma \mid \gamma < \alpha \rangle$ of iteration trees, which we will call **stacks**. We will also denote the $\mathcal{T}_\gamma$'s in a given stack as the **rounds** in the game.

The first round is just a playing $\mathcal{G}_k(\mathcal{M}, \theta)$. In the $(\beta+1)$'st round, let first $\mathcal{Q}$ be the last model of $\mathcal{T}_\beta$ and $q \leqslant \omega$ the degree of $\mathcal{Q}$ in $\mathcal{T}_\beta$. Then player I picks an initial segment $\mathcal{P} \trianglelefteq \mathcal{Q}$ and $i \leqslant \omega$ such that $i \leqslant q$ if $\mathcal{P} = \mathcal{Q}$. Then the $(\beta+1)$'st round is a play of $\mathcal{G}_i(\mathcal{P}, \theta)$, but where player I is allowed to skip to the next round if he's suddenly in trouble.

If $\lambda > 0$ is a limit, then in the λ'th round let $\mathcal{Q}$ be the direct limit along the unique cofinal branch in $\bigoplus_{\gamma<\lambda} \mathcal{T}_\gamma$. If this direct limit is illfounded, however, player I wins. Then $q \leqslant \omega$ is defined as the eventual values of the degrees from the previous rounds, and again player I picks $\mathcal{P}$ and i and the λ'th round is then a play of $\mathcal{G}_i(\mathcal{P}, \theta)$ once again, where we again allow player I to skip to the next round.

Definition 4.21. Let $k \leqslant \omega$ and $\alpha, \theta \in \mathrm{On}$. Then a (k, α, θ)-**iteration strategy** for $\mathcal{M}$ is a winning strategy for player II in $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$. $\mathcal{M}$ is called (k, α, θ)-**iterable** and a (k, α, θ)-**mouse**, if there exists a (k, α, θ)-iteration strategy for $\mathcal{M}$. Furthermore we say $\mathcal{M}$ is **fully** k**-iterable** if it's $(k, \mathrm{On}, \mathrm{On})$-iterable, **fully iterable** if it's $(\omega, \mathrm{On}, \mathrm{On})$-iterable and $\mathcal{M}$ is a **mouse** if it's a fully iterable premouse. ○

Definition 4.22. Let $\pi : \mathcal{M} \to \mathcal{N}$ be a near k-embedding and Σ a (k, α, θ)-iteration strategy for $\mathcal{N}$. Then the **pullback strategy of** Σ **under** π is the strategy Σ^π on $\mathcal{M}$ such that for any stack $\vec{\mathcal{T}}$, $\vec{\mathcal{T}}$ follows Σ^π iff $\pi\mathcal{T}$ follows Σ. ○

Theorem 4.23. *Let $\mathcal{N}$ be a (k, α, θ)-mouse and let $\pi : \mathcal{M} \to \mathcal{N}$ be a near k-embedding. Then the pullback strategy along π is a (k, α, θ)-iteration strategy on $\mathcal{M}$, so that $\mathcal{M}$ in particular is a (k, α, θ)-mouse as well.*

Proof. Let Σ be the iteration strategy for $\mathcal{N}$ and Σ^π the pullback strategy for $\mathcal{M}$. Assume Σ^π is not winning for player II, so that we have an iteration tree $\mathcal{T}$ on $\mathcal{M}$ following Σ^π of limit length, where $\Sigma^\pi(\mathcal{T})$ is an illfounded cofinal branch of $\mathcal{T}$. But then $\pi\mathcal{T}$ follows Σ, making $\Sigma(\pi\mathcal{T})$ illfounded as well, $\lightning$. ■

Definition 4.24. A stack $\vec{\mathcal{T}}$ is (k, λ, θ)-**unambiguous** if whenever Σ is a (k, λ, θ)-iteration strategy and α is any limit ordinal, $\Sigma(\vec{\mathcal{T}} \upharpoonright \alpha)$ is the unique cofinal branch b of $\vec{\mathcal{T}} \upharpoonright \alpha$ such that $\mathcal{M}_b^{\vec{\mathcal{T}}}$ is $(\deg b, \lambda, \theta)$-iterable. ○

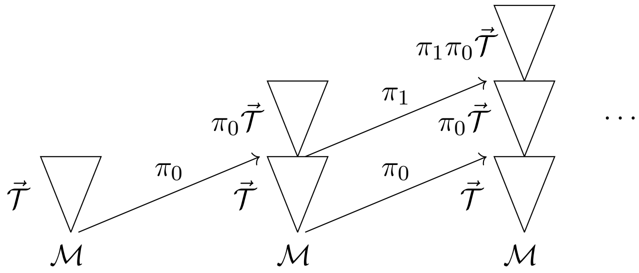


Figure 4.2: The construction of $\vec{\mathcal{T}}_\omega$

Recall that an ordinal λ is **additively closed** if $\gamma + \beta < \lambda$ for every $\gamma, \beta < \lambda$. Examples of additively closed ordinals include all cardinals and also ordinals of the form δ^ε with δ a limit ordinal and $\varepsilon > 1$.

Theorem 4.25 (Dodd-Jensen). *Let λ be additively closed, $\mathcal{M}$ a (k, λ, θ)-mouse and $\vec{\mathcal{T}}$ a (k, λ, θ)-unambiguous stack of length $\alpha + 1$ with last model $\mathcal{P}$. Assume $\deg^{\vec{\mathcal{T}}}(\alpha) = k$ and let $\pi : \mathcal{M} \to \mathcal{N}$ be a near k-embedding with $\mathcal{N} \trianglelefteq \mathcal{P}$. Then*

Don't we only need $\operatorname{lh}\vec{\mathcal{T}} < \lambda$ and $\deg^{\vec{\mathcal{T}}} = k$?

(i) $\mathcal{N} = \mathcal{P}$;

(ii) the $\mathcal{M}$-to-$\mathcal{P}$ branch of $\vec{\mathcal{T}}$ does not drop in model or degree;

(iii) $i^{\vec{\mathcal{T}}}_{0,\alpha}(x) \leqslant_{\mathcal{P}} \pi(x)$ for every $x \in \mathcal{M}$.

Proof. Let Σ be a (k, λ, θ)-iteration strategy for $\mathcal{M}$. For (i), assume for a contradiction that $\mathcal{N} \triangleleft \mathcal{P}$. We will construct a play of $\mathcal{G}_k(\mathcal{M}, \lambda, \theta)$ which is according to Σ but still losing for player II, given us our desired contradiction. First, we recursively define stacks $\vec{\mathcal{T}}_n$, premice $\mathcal{M}_n$ and embeddings $\pi_n : \mathcal{M}_n \to \mathcal{M}_{n+1}$ for $n < \omega$.

Set $\vec{\mathcal{T}}_0 := \vec{\mathcal{T}}$, $\mathcal{M}_0 := \mathcal{M}$, $\mathcal{M}_1 := \mathcal{N}$ and $\pi_0 := \pi$. For successors $n + 1$, firstly let $\vec{\mathcal{T}}_{n+1} := \pi_n\vec{\mathcal{T}}_n$ and let $\mathcal{P}$ and $\mathcal{Q}$ be the last models of $\vec{\mathcal{T}}_n$ and $\vec{\mathcal{T}}_{n+1}$, respectively. Then the copying construction 4.20 gives us an embedding $\sigma : \mathcal{P} \to \mathcal{Q}$. Assuming inductively that $\mathcal{M}_{n+1} \triangleleft \mathcal{P}$, we get that $\mathcal{M}_{n+1} \in \mathcal{P}$, so we can then set $\mathcal{M}_{n+2} := \sigma(\mathcal{M}_{n+1})$ and $\pi_{n+1} := \sigma \restriction \mathcal{M}_{n+1}$.

Now set $\vec{\mathcal{T}}_\omega := \bigoplus_{n<\omega} \vec{\mathcal{T}}_n$, see Figure 4.2. Since $\operatorname{lh}\vec{\mathcal{T}}_n = \operatorname{lh}\vec{\mathcal{T}} = \alpha + 1 < \lambda$ and λ is additively closed, $\bigoplus_{i=0}^n \vec{\mathcal{T}}_i$ has length $< \lambda$ for every $n < \omega$, so that $\operatorname{lh}\vec{\mathcal{T}}_\omega \leqslant \lambda$, making $\vec{\mathcal{T}}_\omega$ indeed a play of $\mathcal{G}_k(\mathcal{M}, \lambda, \theta)$. But now $\vec{\mathcal{T}}_\omega$ has cofinally many drops (one at the beginning of each $\vec{\mathcal{T}}_n$), so $\vec{\mathcal{T}}_\omega$ is a losing play for player II. All that remains to show is that $\vec{\mathcal{T}}_\omega$ is still played according to Σ.

To show this, we inductively show that $\bigoplus_{i=0}^n \vec{\mathcal{T}}_i$ is played according to Σ for every $n < \omega$. The $n = 0$ case is trivial. To show that $\bigoplus_{i=0}^{n+1} \vec{\mathcal{T}}_i$ is according to Σ it's enough to show that given any branch in $\bigoplus_{i=0}^n \vec{\mathcal{T}}_i$ according to Σ, $\vec{\mathcal{T}}_{n+1}$ should be played according to the induced strategy Σ_b. But this is the case if and only if $\vec{\mathcal{T}}$ is played according to $(\Sigma_b)^{\pi_n \circ \cdots \circ \pi_0}$ since $\vec{\mathcal{T}}_{n+1} = (\pi_n \circ \cdots \circ \pi_0)\vec{\mathcal{T}}$, and because $\vec{\mathcal{T}}$ is (k, λ, θ)-unambiguous and $(\Sigma_b)^{\pi_n \circ \cdots \circ \pi_0}$ is a (k, λ, θ)-iteration strategy, this is indeed the case. Thus $\vec{\mathcal{T}}_\omega$ is according to Σ as well, $\lightning$.

To show (ii), we construct $\mathcal{M}_n$, $\mathcal{T}_n$ and π_n as above, except (i) now implies that $\mathcal{M}_{n+1}$ is the last model of $\mathcal{T}_n$. The $\mathcal{M}$-to-$\mathcal{N}$ branch does not drop in degree because $\deg^{\vec{\mathcal{T}}}(\alpha) = k = \deg^{\vec{\mathcal{T}}}(0)$ by assumption, so assume that it drops in model. Then the $\mathcal{M}_n$-to-$\mathcal{M}_{n+1}$ branch in $\vec{\mathcal{T}}_\omega$ drops for every $n < \omega$, making $\vec{\mathcal{T}}_\omega$ a losing play for player II again, giving the same contradiction as above, showing (ii).

Lastly, assume that (iii) fails, so that we have some $x_0 \in \mathcal{M}_0$ satisfying that $\pi_0(x_0) <_{\mathcal{P}} i_0(x_0)$. Recursively define $x_{n+1} \in \mathcal{M}_{n+1}$ as $x_{n+1} := \pi_n(x_n)$. We claim that $x_{n+1} <_{\mathcal{P}} i_n(x_n)$ for every $n < \omega$. It holds for $n = 0$ by assumption. Assuming $x_{n+1} <_{\mathcal{P}} i_n(x_n)$, we have that

$$x_{n+2} = \pi_{n+1}(x_{n+1}) <_{\mathcal{P}} (\pi_{n+1} \circ i_n)(x_n) = (i_{n+1} \circ \pi_n)(x_n) = i_{n+1}(x_{n+1}),$$

where we used that $\pi_{n+1} \circ i_n = i_{n+1} \circ \pi_n$ by the commutativity of the copy maps from the copy construction 4.20, so the claim is shown. This entails once again that $\vec{\mathcal{T}}_\omega$ is a losing play for player II, $\lightning$. ■

The assumption that $\vec{\mathcal{T}}$ is unambiguous is too strong for some applications, so we will need a weaker version of the Dodd-Jensen Lemma. Towards this, we will need the following notion.

Definition 4.26. Let $\mathcal{M}$ and $\mathcal{P}$ be premice. Then $\mathcal{P}$ is $(\mathcal{M}, k)$-**large** if there is an initial segment $\mathcal{Q} \trianglelefteq \mathcal{P}$ and a near k-embedding $j : \mathcal{M} \to \mathcal{Q}$. ○

We can think of $\mathcal{P}$ being $(\mathcal{M}, k)$-large if it wins the comparison with $\mathcal{M}$, thus making it "larger" than $\mathcal{M}$. Of course, the comparison need not exist, as we only require $\mathcal{M}$ and $\mathcal{P}$ to be premice.

The idea of the iteration embedding being minimal in the Dodd-Jensen Lemma also applies to the weaker version of the lemma. Minimality will be the following notion.

Definition 4.27. Let $\mathcal{M}$ be a countable premouse, $\mathcal{P}$ any premouse and $\vec{e}$ some enumeration of $\mathcal{M}$ in order-type ω. Define an ordering $<_{\vec{e}}$ on $\mathcal{M}$ as $e_i <_{\vec{e}} e_j$ iff $i < j$. Then $j : \mathcal{M} \to \mathcal{P}$ is an **$\vec{e}$-minimal near k-embedding** if it's a near k-embedding which is $<_{\vec{e}}$-minimal. That is, whenever $\mathcal{Q} \trianglelefteq \mathcal{P}$ and $\pi : \mathcal{M} \to \mathcal{Q}$ is a near k-embedding, then $\mathcal{Q} = \mathcal{P}$ and letting $<^*_{\vec{e}}$ be the lexicographical wellordering on $\mathcal{M}^\omega$ induced by $<_{\vec{e}}$, $\langle j(e_i) \mid i < \omega \rangle <^*_{\vec{e}} \langle \pi(e_i) \mid i < \omega \rangle$. $\circ$

Note that if $\mathcal{P}$ is $(\mathcal{M}, k)$-large and no $\mathcal{Q} \triangleleft \mathcal{P}$ is $(\mathcal{M}, k)$-large, there is an $\vec{e}$-minimal near k-embedding $j : \mathcal{M} \to \mathcal{P}$.

Theorem 4.28 (Weak Dodd-Jensen). *Let $\mathcal{M}$ be a countable (k, ω_1, θ)-mouse with an enumeration $\vec{e}$ in order-type ω. Then there is a (k, ω_1, θ)-iteration strategy Σ for $\mathcal{M}$ such that whenever $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}$ according to Σ and $\mathcal{P}$ is an $(\mathcal{M}, k)$-large Σ-iterate of $\mathcal{M}$, the iteration embedding $i : \mathcal{M} \to \mathcal{P}$ exists and is an $\vec{e}$-minimal near k-embedding.*

Proof. Let Γ be any (k, ω_1, θ)-iteration strategy for $\mathcal{M}$. We will construct a stack $\vec{\mathcal{T}}$ according to Γ with last model $\mathcal{P}$ and an $\vec{e}$-minimal near k-embedding $\pi : \mathcal{M} \to \mathcal{P}$ satisfying the following property called **strong $\vec{e}$-minimality**:

> Let u be the $\mathcal{M}$-to-$\mathcal{Q}$ branch of $\vec{\mathcal{T}}$. Then for any $(\mathcal{M}, k)$-large Γ_u-iterate $\mathcal{R}$, it holds that there's no drop in the $\mathcal{Q}$-to-$\mathcal{R}$ branch and $i \circ \pi$ is $\vec{e}$-minimal with $i : \mathcal{Q} \to \mathcal{R}$ the iteration map.

Then we will show that $\Sigma := (\Gamma_u)^\pi$ has the wanted property. Say that a stack $\vec{\mathcal{T}}$ is **suitable** if the last iteration tree in $\vec{\mathcal{T}}$ has a single model $\mathcal{R}$ which is $(\mathcal{M}, k)$-large but where every proper initial segment of $\mathcal{R}$ is not $(\mathcal{M}, k)$-large. To obtain our $\vec{\mathcal{T}}$ and π, we will recursively construct suitable stacks $\vec{\mathcal{T}}^n$ for every $n < \omega$.

Set $\vec{\mathcal{T}}^0$ to be a single iteration tree with $\mathcal{M}$ as the only model. Assume now that $\vec{\mathcal{T}}^n$ is constructed and that $\vec{\mathcal{T}}^{k+1}$ extends $\vec{\mathcal{T}}^k$ for every $k < n$. Let $\mathcal{P}_k$ be the last model of $\vec{\mathcal{T}}^k$ for every $k \leqslant n$.

> **Case 1.** There is a suitable $\vec{\mathcal{U}}$ extending $\vec{\mathcal{T}}^n$ with last model $\mathcal{P}$ such that the $\mathcal{P}_n$-to-$\mathcal{P}$ branch has a drop.

In this case set $\vec{\mathcal{T}}^{n+1} := \vec{\mathcal{U}}$. If this is not the case, let $\tau : \mathcal{M} \to \mathcal{P}_n$ be an $\vec{e}$-minimal near k-embedding.

> **Case 2.** τ is not strongly $\vec{e}$-minimal, i.e. that there is a suitable $\vec{\mathcal{U}}$ with last model $\mathcal{Q}$ extending $\vec{\mathcal{T}}^n$ such that $i \circ \tau : \mathcal{M} \to \mathcal{P}$ is not $\vec{e}$-minimal, where $i : \mathcal{P}_n \to \mathcal{Q}$ is the iteration map.

In this case let $m < \omega$ be least such that for some suitable $\vec{\mathcal{U}}$ with last model $\mathcal{Q}$ and iteration map $j : \mathcal{P}_n \to \mathcal{Q}$, $\sigma(e_m) \neq (j \circ \tau)(e_m)$ with $\sigma : \mathcal{M} \to \mathcal{Q}$ being $\vec{e}$-minimal. Let $\vec{\mathcal{T}}^{n+1} := \vec{\mathcal{U}}$.

> **Case 3.** Otherwise.

In this case τ is strongly $\vec{e}$-minimal, and we take $\vec{\mathcal{T}} := \vec{\mathcal{T}}^n$, $\pi := \tau$ and stop the construction. This finishes the construction of the $\vec{\mathcal{T}}^n$'s. We claim that the construction stops after finitely many steps. Suppose it doesn't. Note that the first case can only happen finitely many times as otherwise we would have an iteration tree with infinitely many drops.

Assume thus that we're only in the second case after some $n_0 < \omega$. This means that for every $n, m \geqslant n_0$ satisfying that $n \leqslant m$, we have a k-embedding $i_{n,m} : \mathcal{P}_n \to \mathcal{P}_m$. For $n \geqslant n_0$ set $\pi_n : \mathcal{M} \to \mathcal{P}_n$ to be $\vec{e}$-minimal, so that $\pi_n \leqslant^*_{\vec{e}} i_{m,n} \circ \pi_m$ for every $m, n \geqslant n_0$ such that $m < n$.

Define $\mathcal{P} := \varinjlim_n \mathcal{P}_n$. We claim that for any $j < \omega$, $(i_{n,m} \circ \pi_n)(e_j) = \pi_m(e_j)$ for sufficiently large n and m. Indeed, if it wasn't the case then we would have that $(i_{m,\infty} \circ \pi_m)(e_j) <^*_{\vec{e}} (i_{n,\infty} \circ \pi_n)(e_j)$ for every $m, n \geqslant n_0$ such that $n < m$. But then $<^*_{\vec{e}}$ is illfounded, $\lightning$.

Let $\vec{\mathcal{T}}$ be the union of all the $\vec{\mathcal{T}}^n$'s, and with a single last iteration tree with $\mathcal{P}$ as the only model. We claim that $\vec{\mathcal{T}}$ is suitable. Define $\pi : \mathcal{M} \to \mathcal{P}$ as $\pi(e_j)$

being the eventual value of $(i_{n,\infty} \circ \pi_n)(e_j)$ as $n \to \infty$. Then π is clearly a near k-embedding, so $\mathcal{P}$ is $(\mathcal{M}, k)$-large. Say now that there was an $(\mathcal{M}, k)$-large proper initial segment $\mathcal{R} \lhd \mathcal{P}$. Then the union of all the $\vec{\mathcal{T}}^n$'s with a single last iteration tree with $\mathcal{R}$ being the only model, could be used as a witness for the first case at any stage $n > n_0$, $\lightning$.

It remains to show that π is strongly $\vec{e}$-minimal. To show $\vec{e}$-minimality, say it's not the case and let $\sigma : \mathcal{M} \to \mathcal{P}$ be an $\vec{e}$-minimal near k-embedding such that $\sigma <^*_{\vec{e}} \pi$ and let $m_0 < \omega$ be least such that $\sigma(e_{m_0}) \neq \pi(e_{m_0})$ and let $l < \omega$ be large enough such that $l > n_0$ and $\pi(e_j) = (i_{l,\infty} \circ \pi_l)(e_j)$ for every $j \leqslant m_0$, so that $m_0 < m$, where m is as in the second case at stage l – this is because $\pi_{l+1}(e_j) = (i_{l,l+1} \circ \pi_l)(e_j)$ for every $j \leqslant m_0$ and π_{l+1} is $\vec{e}$-minimal. But we also have that $\mathcal{P}$ and σ can serve as the witnesses for the second case at stage l, so that $m \leqslant m_0$, $\lightning$. Thus π is $\vec{e}$-minimal.

Now to show that π is in fact strongly $\vec{e}$-minimal. Let $\mathcal{R}$ be an $(\mathcal{M}, k)$-large Γ_u-iterate with u the $\mathcal{M}$-to-$\mathcal{P}$ branch. Then there is a stack $\vec{\mathcal{U}}$ extending $\vec{\mathcal{T}}$ such that the last tree in $\vec{\mathcal{U}}$ has the single model $\mathcal{R}$. If there was a drop on the $\mathcal{P}$-to-$\mathcal{R}$ branch then we could, just as above, show that this gives a witness to the first case above n_0, $\lightning$. If $i : \mathcal{P} \to \mathcal{R}$ is the iteration map such that $i \circ \pi$ is *not* $\vec{e}$-minimal, then this entails the same "second case" contradiction as above. Thus π is strongly $\vec{e}$-minimal.

We now want to show that $\Sigma := (\Gamma_u)^\pi$ has the property in the theorem. Let thus $\vec{\mathcal{U}}$ be a stack on $\mathcal{M}$ played according to Σ. Then $\pi\vec{\mathcal{U}}$ is a stack on $\mathcal{P}$ played according to Γ_u. Let $\mathcal{R}$ be an $(\mathcal{M}, k)$-large Σ-iterate of $\mathcal{M}$ and let $\pi\mathcal{R}$ be the copied version of $\mathcal{R}$ on $\pi\vec{\mathcal{U}}$. Then $\pi\mathcal{R}$ is an $(\mathcal{M}, k)$-large Γ_u-iterate, so by strongness there's no drop in the $\mathcal{P}$-to-$\pi\mathcal{R}$ branch, implying that $i : \mathcal{P} \to \pi\mathcal{R}$ exists and $i \circ \pi$ is $\vec{e}$-minimal. But then the iteration map $j : \mathcal{M} \to \mathcal{R}$ exists as well as $\vec{\mathcal{U}}$ and $\pi\vec{\mathcal{U}}$ have the same drop structure, and is $\vec{e}$-minimal, as we wanted to show. ■

We call the property Σ has in the conclusion of Theorem 4.28 the **$\vec{e}$-weak Dodd-Jensen property**. Note that the main difference between weak and strong Dodd-Jensen is that the weak version only implies the *existence* of a well-behaved strategy, where the strong version says that *every* strategy is well-behaved. Another thing to note is that the minimality in the weak version is only lexicographically, where it's pointwise in the strong version.

$$
\begin{tikzcd}
\mathcal{P} \arrow[r, phantom, "\trianglelefteq"] & \mathcal{Q} \\
\mathcal{P}^* \arrow[u, "i^*"'] \arrow[u, dash, dotted, "\hat{\mathcal{U}}"] \arrow[r, phantom, "\neq"] & \mathcal{Q}^* \arrow[u, "j^*"] \arrow[u, dash, dotted, "\hat{\mathcal{V}}"'] \\
\mathcal{M} \arrow[u, dash, dotted, "\mathcal{U}^*"] \arrow[uu, bend left=60, "i"] \arrow[r, phantom, "="] & \mathcal{M} \arrow[u, dash, dotted, "\mathcal{V}^*"'] \arrow[uu, bend right=60, "j"']
\end{tikzcd}
$$

Figure 4.3: The situation in the proof of Theorem 4.29.

One thing to note is that the existence of the well-behaved strategy is accompanied by a uniqueness result. We won't be needing this fact however, but we include it here for independent interest.

Theorem 4.29. *Let $\vec{e}$ be an enumeration of a countable k-sound premouse $\mathcal{M}$ in order-type ω. Then there exists at most one $(k, \omega_1 + 1)$-iteration strategy for $\mathcal{M}$ with the $\vec{e}$-weak Dodd-Jensen property.*

Proof. Assume that Σ and Γ are distinct such strategies. Let $\mathcal{T}$ be an iteration tree played according to both Σ and Γ of limit length λ, such that $\Sigma(\mathcal{T}) \neq \Gamma(\mathcal{T})$. Set $\mathcal{U}^*$ and $\mathcal{V}^*$ to be iteration trees extending $\mathcal{T}$ with $\Sigma(\mathcal{T})$ and $\Gamma(\mathcal{T})$, respectively. Let $\mathcal{P}^*$ and $\mathcal{Q}^*$ be the last models of $\mathcal{U}^*$ and $\mathcal{V}^*$, respectively. Now assume without loss of generality that $\mathcal{Q}^*$ iterates past $\mathcal{P}^*$ in their coiteration, see Figure 4.3.

Set $\mathcal{U} := \mathcal{U}^* \oplus \hat{\mathcal{U}}$ and $\mathcal{V} := \mathcal{V}^* \oplus \hat{\mathcal{V}}$. The Comparison Theorem 4.15 implies that the $\mathcal{M}$-to-$\mathcal{P}$ branch doesn't drop, so we get iteration maps $i^* : \mathcal{P}^* \to_k \mathcal{P}$ and $i : \mathcal{M} \to_k \mathcal{P}$. This also means that $\mathcal{Q}$ is $(\mathcal{M}, k)$-large, so since Γ has the $\vec{e}$-weak Dodd-Jensen property, we also get iteration maps $j^* : \mathcal{Q}^* \to_k \mathcal{Q}$ and $j : \mathcal{M} \to \mathcal{Q}$, the latter of which is $\vec{e}$-minimal.

But then no proper initial segment of $\mathcal{Q}$ is $(\mathcal{M}, k)$-large, so that $\mathcal{P} = \mathcal{Q}$. As Σ also has the $\vec{e}$-weak Dodd-Jensen property, i is also $\vec{e}$-minimal, so that $i = j$. Let now γ be the largest ordinal such that $\gamma \in [0, \alpha]_U \cap [0, \beta]_V$, which exists because branches in iteration trees are closed below their sups. Note that $\gamma \leqslant \lambda$ (recall that $\lambda = \operatorname{lh} \mathcal{T}$), because $\mathcal{P}^* \neq \mathcal{Q}^*$, so we can define $\nu := \sup\{\nu_{E_\xi^{\mathcal{T}}} \mid \xi <_T \gamma\}$.

Since every member of $\mathcal{R} := \mathcal{M}_\gamma^\mathcal{T}$ is of the form $\tau^\mathcal{R}[a]$ for a Skolem term τ of some $r\Sigma_n$-formula and $a \in [\nu]^{<\omega}$, and since crit i^*, crit $j^* \geqslant \nu$ because generators aren't moved along branches of iteration trees, we get that

$$\begin{aligned} i^*(\tau^\mathcal{R}[a]) &= i(\tau^\mathcal{M})[i^*(a)] \\ &= i(\tau^\mathcal{M})[a] \\ &= j(\tau^\mathcal{M})[a] \\ &= j(\tau^\mathcal{M})[j^*(a)] \\ &= j^*(\tau^\mathcal{R}[a]), \end{aligned}$$

so that $i^* = j^*$. Define now $\xi + 1 \in (\gamma, \alpha]_U$ and $\sigma + 1 \in (\gamma, \beta]_V$ to be the immediate successors of γ in $\mathcal{U}$ and $\mathcal{V}$. Since $i^* = j^*$, the extenders $E_\xi^\mathcal{U}$ and $E_\sigma^\mathcal{V}$ are compatible. If $\xi < \lambda$ or $\sigma < \lambda$ then this is a contradiction as no two extenders in the same iteration tree can be compatible by Proposition 4.9. Otherwise, if $\xi, \sigma \geqslant \lambda$, we have a contradiction as well, since no two extenders used in a coiteration can be compatible by Claim 4.15.1. Thus Σ and Γ cannot be distinct. ■

5 Condensation and solidity

Given the theory on iterability from the previous chapter, we're able to prove two key results on mice, condensation and that k-sound mice are $(k+1)$-solid. The style of the argument is quite typical in inner model theory and involves coiterating a suitable pair of mice.

5.1 Condensation

Theorem 5.1. *Let $k \leqslant \omega$, $\mathcal{M}$ a k-sound $(k, \omega_1, \omega_1+1)$-mouse and let $\pi : \mathcal{H} \to \mathcal{M}$ be a near k-embedding such that $\operatorname{crit} \pi \geqslant \rho_k(\mathcal{H})$. Assume that $k < \omega$ only if $\mathcal{M}$ has a unique $(k, \omega_1, \omega_1+1)$-strategy. Then either*

- *(i) $\mathcal{H} \lhd \mathcal{M}$, or*
- *(ii) $\mathcal{H} \lhd \mathrm{Ult}_0(\mathcal{M}, E)$ for some extender E on the $\mathcal{M}$-sequence of length $\rho_k(\mathcal{H})$.*

Proof. Note first that $\operatorname{crit} \pi > \rho_k(\mathcal{H})$ is impossible, since then $\rho_k(\mathcal{H}) = \rho_k(\mathcal{M})$ as π is a near k-embedding, so k-soundness of $\mathcal{M}$ implies that $\operatorname{crit} \pi$ is $\mathbf{r\Sigma}_k$-definable with parameters from $\operatorname{ran} \pi$, $\lightning$. Thus $\operatorname{crit} \pi = \rho_k(\mathcal{H})$.

Assume both (i) and (ii) fails for some $\mathcal{H}$ and $\mathcal{M}$. We can firstly without loss of generality assume that $\mathcal{M}$ is countable. Indeed, if it was not then fix some sufficiently large limit ordinal η such that $\mathcal{H}, \mathcal{M} \in V_\eta$ and let

$$\pi : \mathrm{cHull}^{V_\eta}(\{\mathcal{H}, \mathcal{M}\}) \to V_\eta$$

be the uncollapse. It is then not too hard to see that $\pi^{-1}(\mathcal{H})$ and $\pi^{-1}(\mathcal{M})$ also witness a counterexample to the theorem, so we could just as well have started with $\pi^{-1}(\mathcal{M})$ instead of $\mathcal{M}$.

Assume thus $\mathcal{M}$ is countable and let $\vec{e}$ be an enumeration of $\mathcal{M}$ in order-type ω. By the Weak Dodd-Jensen Theorem 4.28 we can fix a $(k, \omega_1, \omega_1+1)$-iteration strategy Σ with the $\vec{e}$-weak Dodd-Jensen property for $\mathcal{M}$. Our strategy now is to compare $\mathcal{M}$ with $\mathcal{H}$, but due to some complications we will see later in the proof, we will have to modify this plan slightly.

$$
\begin{tikzcd}
\mathcal{R} & \mathcal{P} \arrow[l, "\tau"'] & \mathcal{Q} \\
\Psi \arrow[u, "\vec{\pi}\mathcal{T}"'] & \Phi \arrow[l, "\vec{\pi}"] \arrow[u, "i \; \mathcal{T}"'] & \Psi \arrow[u, dash, "\mathcal{U}"']
\end{tikzcd}
$$

Figure 5.1: The conclusion of Claim 5.1.1.

Define the phalanx $\Phi := (\langle\mathcal{M}, \mathcal{H}\rangle, \langle\rho_k(\mathcal{H})\rangle)$, which is really a phalanx because $\rho_k(\mathcal{H})$ is an $\mathcal{H}$-cardinal and $\operatorname{crit}\pi \geqslant \rho_k(\mathcal{H})$ implies that $\mathcal{H}$ and $\mathcal{M}$ agree below $\rho_k(\mathcal{H})$. Furthermore define the phalanx $\Psi := (\langle\mathcal{M}, \mathcal{M}\rangle, \langle\omega\rangle)$ and the near k-embedding $\vec{\pi} := \langle\mathrm{id}_\mathcal{M}, \pi\rangle : \Phi \to \Psi$.

We can then form the coiteration of Φ and Ψ, forming k-iteration trees $\mathcal{T}$ and $\mathcal{U}$ with last models $\mathcal{P}$ and $\mathcal{Q}$ on Φ and $\mathcal{M}$ by $\Sigma^{\vec{\pi}}$ and Σ, respectively. Then the copying construction 4.20 grants us with a fully elementary $\tau : \mathcal{P} \to \mathcal{R}$, where $\mathcal{R}$ is the last model of the copied tree $\vec{\pi}\mathcal{T}$ on Ψ by Σ.

Claim 5.1.1. The Φ-to-$\mathcal{P}$ branch doesn't drop (see Figure 5.1).

Proof of claim. If it dropped, then the $\mathcal{M}$-to-$\mathcal{Q}$ branch does not drop, so we get an iteration map $j : \mathcal{M} \to \mathcal{Q}$. Then $\mathcal{Q} \neq \mathcal{P}$ as $\mathcal{Q}$ is k-sound and $\mathcal{P}$ isn't, and $\mathcal{P}$ isn't an initial segment of $\mathcal{Q}$ as $\mathcal{P}$ isn't sound, so $\mathcal{Q} \lhd \mathcal{P}$. But now $\tau \circ j$ maps $\mathcal{M}$ to a proper initial segment of $\mathcal{R}$, contradicting the $\vec{e}$-weak Dodd-Jensen property of Σ. Thus the Φ-to-$\mathcal{P}$ branch doesn't drop. $\dashv$

We then get an iteration map i from the root of the Φ-to-$\mathcal{P}$ branch to $\mathcal{P}$. Since the Φ-to-$\mathcal{P}$ branch doesn't drop, we have that $\operatorname{crit} i < \rho_k(\mathcal{H})$.

Claim 5.1.2. $\mathcal{P}$ lies above $\mathcal{H}$ in $\mathcal{T}$.

Proof of claim. Assume it's not the case, so that $\mathcal{P}$ lies above $\mathcal{M}$. Assume furthermore that the $\mathcal{M}$-to-$\mathcal{Q}$ branch drops – we'll show that this implies that $\mathcal{P} \lhd \mathcal{Q}$. If $k = \omega$ then it is simply because $\mathcal{T}$ and $\mathcal{U}$ are ω-iteration trees on sound mice. If $k < \omega$ we have that $\mathcal{M}$ has a unique $(k, \omega_1, \omega_1 + 1)$-iteration strategy by assumption, so we've got the Dodd-Jensen Theorem 4.25 at our disposal, which implies that the $\mathcal{M}$-to-$\mathcal{Q}$ branch doesn't drop, $\lightning$. Thus $\mathcal{P} \lhd \mathcal{Q}$.

Then since $\mathcal{T}$ and $\mathcal{U}$ are k-iteration trees on k-sound mice we get that $\mathcal{P} \lhd \mathcal{Q}$. But now i maps $\mathcal{M}$ to a proper initial segment of a Σ-iterate of $\mathcal{M}$, contradicting the $\vec{e}$-weak Dodd-Jensen property of Σ. Thus the $\mathcal{M}$-to-$\mathcal{Q}$ branch doesn't drop and we get the iteration map $j : \mathcal{M} \to \mathcal{Q}$. The situation now looks as follows:

$$\begin{tikzcd}
\mathcal{R} & \mathcal{P} \arrow[l, "\tau"'] & \mathcal{Q} \arrow[l, phantom, "="] \\
\mathcal{M} \arrow[u, "\vec{\pi}\mathcal{T}"'] & \mathcal{M} \arrow[l, "\mathrm{id}"] \arrow[u, "i"', "\mathcal{T}"] & \mathcal{M} \arrow[u, "j"', "\mathcal{U}"]
\end{tikzcd}$$

Here $\mathcal{P} = \mathcal{Q}$ as if $\mathcal{P} \lhd \mathcal{Q}$, i maps $\mathcal{M}$ into a proper initial segment of a Σ-iterate of $\mathcal{M}$, $\lightning$, and if $\mathcal{Q} \lhd \mathcal{P}$ then $\tau \circ j$ maps $\mathcal{M}$ into a proper initial segment of a Σ-iterate of $\mathcal{M}$, $\lightning$.

We next claim that $i = j$, so assume that it's not the case. Let x be the $<_{\vec{e}}$-least element of $\mathcal{M}$ such that $i(x) \neq j(x)$. If $i(x) <_{\mathcal{P}} j(x)$ then j is an iteration map given by Σ which isn't $\vec{e}$-minimal, contradicting the $\vec{e}$-weak Dodd-Jensen property of Σ. So $j(x) <_{\mathcal{P}} i(x)$. As the $\mathcal{M}$-to-$\mathcal{P}$ branch in $\mathcal{T}$ doesn't drop, the $\mathcal{M}$-to-$\mathcal{R}$ branch in $\vec{\pi}\mathcal{T}$ doesn't drop either, so we get an iteration map $k : \mathcal{M} \to \mathcal{R}$. Now we get that

$$(\tau \circ j)(x) <_{\mathcal{P}} (\tau \circ i)(x) = k(x),$$

where we used that the copying construction gives us that $\tau \circ i = k \circ \mathrm{id}_{\mathcal{M}} = k$. Thus $\tau \circ j$ witnesses that k is not $\vec{e}$-minimal, $\lightning$. Thus $i = j$. Letting $\alpha + 1$ be the T-successor of 0, $\beta + 1$ the U-successor of 0, $\nu := \inf(\nu_{E^{\mathcal{T}}_\alpha}, \nu_{E^{\mathcal{U}}_\beta})$, $a \in [\nu]^{<\omega}$ and $B \in \mathcal{M}$,

$$\begin{aligned}
B \in (E^{\mathcal{T}}_\alpha)_a \quad & \text{iff} \quad a \in i_{0,\alpha+1}(B) \\
& \text{iff} \quad a \in i(B) \\
& \text{iff} \quad a \in j(B) \\
& \text{iff} \quad a \in j_{0,\beta+1}(B) \\
& \text{iff} \quad B \in (E^{\mathcal{U}}_\beta)_a,
\end{aligned}$$

so that $E^{\mathcal{T}}_\alpha$ is compatible with $E^{\mathcal{U}}_\beta$, contradicting Claim 4.15.1. Thus $\mathcal{P}$ is indeed above $\mathcal{H}$ in $\mathcal{T}$. $\dashv$

Now, if $i : \mathcal{H} \to \mathcal{P}$ is not the identity, our rules for $\mathcal{T}$ ensure the $\operatorname{crit} i \geqslant \rho_k(\mathcal{H})$, so that the $\mathcal{H}$-to-$\mathcal{P}$ branch would have to drop, $\lightning$. Thus i is the identity and $\mathcal{H} = \mathcal{P}$.

Furthermore $\mathcal{Q}$ cannot be a proper initial segment of $\mathcal{H}$, as otherwise the $\mathcal{M}$-to-$\mathcal{Q}$ branch doesn't drop and we get an iteration map $j : \mathcal{M} \to \mathcal{Q}$. Then $\tau \circ j$ maps $\mathcal{M}$ to a proper initial segment of itself, $\lightning$. Thus $\mathcal{H} \trianglelefteq \mathcal{Q}$. We cannot have that $\mathcal{Q} = \mathcal{H}$ either, because otherwise we get $j : \mathcal{M} \to \mathcal{Q}$ again and then it holds that

$$\rho_k(\mathcal{H}) < \pi(\rho_k(\mathcal{H})) = \rho_k(\mathcal{M}) \leqslant j(\rho_k(\mathcal{M})) = \rho_k(\mathcal{Q}) = \rho_k(\mathcal{H}),$$

a contradiction. The current scenario thus looks like the following, where the initial segment is proper in the case where $k = \omega$ and $\operatorname{crit} \pi = \rho(\mathcal{H})$.

$$\begin{tikzcd} \mathcal{H} \arrow[r, phantom, "\lhd"] & \mathcal{Q} \\ \mathcal{H} \arrow[u, "\mathrm{id}"', "\mathcal{T}"] & \mathcal{M} \arrow[u, dash, "\mathcal{U}"'] \end{tikzcd}$$

Now suppose that $\mathcal{H}$ is not an intial segment of $\mathcal{M}$, so that $\mathcal{U}$ uses at least one extender $E := E^{\mathcal{U}}_0$. As $\mathcal{H}$ agrees with $\mathcal{M}$ below $\rho_k(\mathcal{H})$, we must have that $\rho_k(\mathcal{H}) \leqslant \operatorname{lh} E$. Furthermore we also have that $\operatorname{lh} E \leqslant \mathrm{On}^{\mathcal{H}}$ as $\mathcal{H}$ isn't an initial segment of $\mathcal{M}$, so that a disagreement occurs below $\mathrm{On}^{\mathcal{H}} + 1$. But as there's a surjection $\rho_k(\mathcal{H}) \to \mathrm{On}^{\mathcal{H}}$ in $\mathcal{Q}$ by k-soundness of $\mathcal{H}$, we have that $|\,\mathrm{On}^{\mathcal{H}}\,| \leqslant \rho_k(\mathcal{H})$ in $\mathcal{Q}$, and since $\operatorname{lh} E$ is a cardinal in $\mathcal{Q}$, $\operatorname{lh} E \leqslant |\,\mathrm{On}^{\mathcal{H}}\,|$. Thus $\operatorname{lh} E = \rho_k(\mathcal{H})$.

If $E^{\mathcal{U}}_1$ exists then $|\,\mathrm{On}^{\mathcal{H}}\,| = \operatorname{lh} E < \operatorname{lh} E^{\mathcal{U}}_1$, and since $\operatorname{lh} E^{\mathcal{U}}_1$ is a cardinal in $\mathcal{Q}$, we also get that $\mathrm{On}^{\mathcal{H}} < \operatorname{lh} E^{\mathcal{U}}_1$, so that $E^{\mathcal{U}}_1$ does not exist. This then means that $\mathcal{Q} = \mathrm{Ult}_k(\mathcal{M}, E)$ for some $k \leqslant \omega$. Since $\mathrm{Ult}_0(\mathcal{M}, E)$ and $\mathrm{Ult}_k(\mathcal{M}, E)$ agree below their common value of $(\operatorname{lh} E)^+$ and $(\operatorname{lh} E)^{+\,\mathrm{Ult}} = \rho(\mathcal{H})^{+\,\mathrm{Ult}} > \mathrm{On}^{\mathcal{H}}$, we get that $\mathcal{H} \lhd \mathrm{Ult}_0(\mathcal{M}, E)$. ■

Note that if $\rho_k(\mathcal{H})$ is an $\mathcal{M}$-cardinal then part (ii) of the conclusion in the Condensation Theorem isn't possible, since $\operatorname{lh} E$ is not a cardinal in $\mathcal{M}$. We will isolate a part of the proof of the above theorem for future use.

$$\begin{tikzcd} \mathcal{P} \arrow[r, phantom, "\trianglelefteq"] & \mathcal{Q} \\ \mathcal{H} \arrow[u, "i", "\mathcal{T}"'] & \mathcal{M} \arrow[u, "\mathcal{U}"'] \end{tikzcd}$$

Figure 5.2: The conclusion of the Dodd-Jensen trick.

Lemma 5.2 (The Dodd-Jensen trick). *Let $\mathcal{M}$ be a countable k-sound $(k, \omega_1, \omega_1+1)$-mouse with $\vec{e}$ enumerating its universe and $\pi : \mathcal{H} \to \mathcal{M}$ a near k-embedding, where $\mathcal{H}$ agrees with $\mathcal{M}$ below some $\mathcal{H}$-cardinal κ. Then there are k-iteration trees $\mathcal{T}, \mathcal{U}$ on $\mathcal{H}, \mathcal{M}$ with last models $\mathcal{P}, \mathcal{Q}$ by Σ^π, Σ such that Σ has the $\vec{e}$-weak Dodd-Jensen property, $\mathcal{P} \trianglelefteq \mathcal{Q}$ and $\operatorname{crit} i^{\mathcal{T}} \geqslant \kappa$.*

Proof. Just as in the proof of the Condensation Theorem 5.1. ■

5.2 Solidity

Theorem 5.3. *Let $k < \omega$ and $\mathcal{M}$ a k-sound $(k, \omega_1, \omega_1 + 1)$-mouse. Then $\mathcal{M}$ is $(k+1)$-solid and $\mathfrak{C}_{k+1}(\mathcal{M})$ agrees with $\mathcal{M}$ below every γ of $\mathcal{M}$-cardinality $\rho_{k+1}(\mathcal{M})$.*

Proof. Let $\mathcal{M}$ be a counter-example to the theorem and let $p := p_{k+1}(\mathcal{M})$ and $\rho := \rho_{k+1}(\mathcal{M})$. We first claim that we without loss of generality can assume that $\mathcal{M}$ is countable. Otherwise we can fix a sufficiently large limit ordinal η such that $p, \mathcal{M} \in V_\eta$, let $\mathcal{H} := \mathrm{cHull}^{V_\eta}(\{p, \mathcal{M}\})$ and $\pi : \mathcal{H} \to V_\eta$ the uncollapse map. Since $\mathcal{H}$ inherits the $(k, \omega_1, \omega_1 + 1)$-iterability from $\mathcal{M}$ and the failure of the theorem is expressible as a first-order sentence, $\pi^{-1}(p)$ and $\pi^{-1}(\mathcal{M})$ is still a counter-example to the theorem.

Assume thus that $\mathcal{M}$ is countable. Write $\langle \alpha_0, \dots, \alpha_n \rangle$ for the last parameter of p, and let $\vec{e}$ be an enumeration of $\mathcal{M}$ in order-type ω, satisfying that $e_i = \alpha_i$ for every $i \leqslant n$ and $e_{n+1} = \rho$. Fix a $(k, \omega_1, \omega_1 + 1)$-iteration strategy Σ for $\mathcal{M}$ with the $\vec{e}$-weak Dodd-Jensen property.

Let $i \leqslant n+1$ be least such that there exists an $r\Sigma_{k+1}^{\mathcal{M}}(e_i \cup \{p \restriction n, e_0, \dots, e_{i-1}\})$ set $A \notin \mathcal{M}$, which exists as $\mathcal{M}$ is k-sound, so $i = n$ would always satisfy this. Set

$\mathcal{H} := \mathrm{Hull}^{\mathcal{M}}_{k+1}(e_i \cup \{p \restriction n, e_0, \ldots, e_{i-1}\})$. Since e_i is an $\mathcal{H}$-cardinal and $\mathcal{M}$ agrees with $\mathcal{H}$ below e_i, the Dodd-Jensen trick gives us the coiteration

$$\begin{tikzcd} \mathcal{P} \arrow[r, phantom, "\trianglelefteq"] & \mathcal{Q} \\ \mathcal{H} \arrow[u, "i"', "\mathcal{T}"] & \mathcal{M} \arrow[u, "\mathcal{U}"] \end{tikzcd}$$

where $\operatorname{crit} i \geqslant e_i$. Since $A \notin \mathcal{Q}$ we also get that $\mathcal{P}$ is not a proper initial segment of $\mathcal{Q}$ as A is definable over $\mathcal{P}$, so $\mathcal{P} = \mathcal{Q}$. This also entails that the $\mathcal{M}$-to-$\mathcal{Q}$ branch doesn't drop as $\mathcal{P}$ is k-sound. We thus get an embedding $j : \mathcal{M} \to \mathcal{Q}$. Let $\bar{x} := \pi^{-1}(x)$ for $x \in \operatorname{ran} \pi$.

Claim 5.3.1. $i(\bar{\alpha}_s) = j(\alpha_s)$ for every $s < i$.

Proof of claim. For $\geqslant$ this follows as in Proposition 2.19, where the parameter $\langle i(\bar{\alpha}_0), \ldots, i(\bar{\alpha}_{i-1})\rangle$ plays the role as the standard parameter in this case, using our set A. For $\leqslant$, if $s < i$ is least such that $j(\alpha_s) <_{\mathcal{P}} i(\bar{\alpha}_s)$, we get that

$$(\tau \circ j)(\alpha_s) <_{\mathcal{R}} (\tau \circ i)(\alpha_s) = k(\alpha_s),$$

with $\tau : \mathcal{P} \to \mathcal{R}$ being the copy map to the iteration tree $(\mathrm{id}_{\mathcal{M}}, \pi)\mathcal{T}$ on the phalanx $(\langle \mathcal{M}, \mathcal{M}\rangle, \langle \omega \rangle)$ with last model $\mathcal{R}$. This contradicts the $\vec{e}$-minimality of k. Note that we used here our specific choice of enumeration $\vec{e}$, so that the least disagreement between $\tau \circ j$ and k *is* actually one of the α_s's. $\dashv$

Claim 5.3.2. $\operatorname{crit} j \geqslant e_i$.

Proof of claim. Let $\kappa := \operatorname{crit} j$ and assume that $\kappa < e_i$. Then

$$S := \mathrm{Th}^{\mathcal{M}}_{k+1}(\kappa \cup \{p \restriction n, \alpha_0, \ldots, \alpha_{i-1}\}) \in \mathcal{M},$$

so $j(S) \in \mathcal{Q}$, and from $j(S)$ one can compute[1]

$$\mathrm{Th}^{\mathcal{Q}}_{k+1}(j(\kappa) \cup \{j(p \restriction n), j(\alpha_0), \dots, j(\alpha_{i-1})\}).$$

Note also that since $\mathcal{H}$ and $\mathcal{M}$ agree below e_i, the first extender used along the $\mathcal{M}$-to-$\mathcal{Q}$ branch must have length $\geqslant e_i$, so that $e_i < j(\kappa)$. Using this fact together with $\mathcal{P} = \mathcal{Q}$ and the previous claim gives us that

$$\mathrm{Th}^{\mathcal{P}}_{k+1}(e_i \cup \{i(\bar{p} \restriction n), i(\bar{\alpha}_0), \dots, i(\bar{\alpha}_{i-1})\}) \in \mathcal{P}$$

entailing $A \in \mathcal{M}$ since $\operatorname{crit} i, \operatorname{crit} \pi \geqslant e_i$, $\lightning$. $\dashv$

Now, assuming that p isn't solid, we have that $i \leqslant n$. Let $B \subseteq \rho$ be $\mathbf{r\Sigma}^{\mathcal{M}}_{k+1}$ such that $B \notin \mathcal{M}$. Then B is $\mathbf{r\Sigma}^{\mathcal{Q}}_{k+1}$ as well, thus $\mathbf{r\Sigma}^{\mathcal{P}}_{k+1}$ and finally also $\mathbf{r\Sigma}^{\mathcal{H}}_{k+1}$. But this means that B is definable with parameters from $\alpha_i \cup \{p \restriction n, \alpha_0, \dots, \alpha_{i-1}\}$, which then contradicts the minimality of p. Thus $i = n + 1$ and p is solid.

To show universality of p, note that the above scenario with $e_i = \rho$ and thus also $\mathcal{H} = \mathfrak{C}_{k+1}(\mathcal{M})$ says that $\operatorname{crit} i, \operatorname{crit} j \geqslant \rho$. This means that we have that

$$\mathcal{P}^{\mathcal{H}}(\rho) = \mathcal{P}^{\mathcal{P}}(\rho) = \mathcal{P}^{\mathcal{Q}}(\rho) = \mathcal{P}^{\mathcal{M}}(\rho),$$

so that $\mathcal{P}^{\mathcal{M}}(\rho) \subseteq \mathcal{H}$, which is exactly universality. Finally, for any γ such that $\mathcal{M} \models |\gamma| = \rho$ we have that $\mathcal{H}$ agrees with $\mathcal{P}$ below γ and $\mathcal{Q}$ agrees with $\mathcal{M}$ below γ, concluding that $\mathcal{H}$ agrees with $\mathcal{M}$ below γ as well as $\mathcal{P} = \mathcal{Q}$. ■

[1] This is not entirely trivial, as we can't just apply j to $\mathcal{M}$ in "$\mathcal{M} \models \varphi[x]$", so we have to apply it to initial segments of $\mathcal{M}$ and consider the limit of these images. For details see (Steel & Mitchell, 1994, Lemma 4.6).

6 | Robust K^c-constructions

So far we have only been working with mice abstractly, but we've yet to actually construct any. As mentioned in the discussion at the end of chapter 2, our goal is to produce ppms $\mathcal{M}$ such that $\mathfrak{C}(\mathcal{M})$ is sound, so that these can then constitute legal initial segments of some premouse. To prove this we mentioned in the same discussion that it suffices to show that $\mathfrak{C}_k(\mathcal{M})$ is fully k-iterable, since we now have access to Theorem 5.3.

This was done in (Steel, 1996) using something called *background certified extenders*[1], but unfortunately this construction was done inside V_κ where κ was a measurable cardinal. This assumption was removed in (Jensen & Steel, 2013), replacing these certified extenders with *robust* extenders, a notion which is due to Jensen, introduced in (Jensen, 2003). We will solely be working with robust extenders.

6.1 Robustness

Definition 6.1. Let u be any set. Then the (strong) **Chang model relative to** u is the class $C_\infty(u)$ defined recursively as follows:

- $C_0(u) := \mathrm{trcl}(\{u\})$;
- $C_{\xi+1}(u) := \mathrm{Def}(C_\xi(u)) \cup [\xi]^\omega$;
- $C_\lambda(u) := \bigcup_{\xi<\lambda} C_\xi(u)$ for λ limit;
- $C_\infty(u) := \bigcup_{\xi\in\mathrm{On}} C_\xi(u)$. ∘

The Chang model relative to u is the smallest inner model containing u and all countable sets of ordinals. We need the following variant of this model.

Definition 6.2. Define the language $\mathcal{L}_0 := \{\in, A\}$ with A a predicate symbol. Then setting $\bar{C}^E_{\tau,\eta} := C_\eta(\langle J^E_\tau, E\cap J^E_\tau\rangle)$ for any predicate E, define the $\mathcal{L}_0$-structure $C^E_{\tau,\eta} := \langle \bar{C}^E_{\tau,\eta}, \in, \langle \bar{C}^E_{\tau,\xi} \mid \xi<\eta\rangle\rangle$. ∘

[1] This is where the superscript "c" comes from in K^c.

Definition 6.3. Let $\mathcal{M}$ be a ppm and $\mathcal{U} \subseteq \mathcal{M}$. Then set

$$\mathrm{Sat}(\mathcal{U}) := \{\langle k, x\rangle \mid x \in \mathcal{U}^\omega \wedge C_{\tau,\infty} \models \varphi_k[x]\},$$

where $\tau := \sup(\mathcal{U} \cap \mathrm{On}^{\mathcal{M}})$ and $\langle \varphi_k \mid k < \omega\rangle$ recursively enumerates the Σ_1 formulas in $\mathcal{L}_0$. If furthermore $\psi : \mathcal{U} \to \mathcal{M}$ then set

$$\mathrm{Sat}(U, \psi) := \{\langle k, x\rangle \mid x \in \mathcal{U}^\omega \wedge C_{\bar\tau,\infty} \models \varphi[\psi \circ x]\},$$

where $\bar\tau := \sup(\operatorname{ran}\psi \cap \mathrm{On}^{\mathcal{M}})$. ○

Definition 6.4. Let $\mathcal{M}$ be an active premouse with last extender E, $\kappa := \operatorname{crit} E$ and $\lambda := \operatorname{lh} E$. Then E is **robust with respect to** $\mathcal{M}$ if whenever $\mathcal{U} \subseteq \mathcal{M}$ is countable then there is a function $\pi : \mathcal{U} \to \mathcal{M}\,|\kappa$ such that the following holds.[2]

(i) $\pi \restriction \mathcal{U} \cap \kappa = \mathrm{id}$;
(ii) $\mathrm{Sat}(\mathcal{U}) = \mathrm{Sat}(\mathcal{U}, \pi)$;
(iii) For every $a \in [\mathcal{U} \cap \lambda]^{<\omega}$ and $x \in \mathcal{U} \cap \mathcal{P}([\kappa]^{|a|})$, $\pi(a) \in x$ iff $x \in E_a$. ○

Robustness can be seen as $\mathcal{M}$ "reflects" Σ_1 statements involving countable sets of ordinals down to $\mathcal{M}\,|\operatorname{crit} E$, while "retaining knowledge" of E, really making $\mathcal{M}\,|\operatorname{crit} E$ the "robust part".

Definition 6.5. A **robust K^c-construction** is a sequence $\langle \mathcal{N}_\alpha \mid \alpha < \theta\rangle$ of premice such that

(i) $\mathcal{N}_0 := \langle V_\omega, \in, \varnothing, \varnothing\rangle$;
(ii) If $\alpha + 1 < \theta$ then $\mathcal{N}_\alpha$ is solid and either
  (a) $\mathfrak{C}(\mathcal{N}_\alpha)$ is passive and $\mathcal{N}_{\alpha+1}$ is $\mathfrak{C}(\mathcal{N}_\alpha)$ with a robust top extender, or
  (b) If $\mathfrak{C}(\mathcal{N}_\alpha) = \langle J^{\vec E}_\gamma, \in, \vec E, F\rangle$ then $\mathcal{N}_{\alpha+1} = \langle J^{\vec E}_{\gamma+1}, \in, \vec E \oplus F, \varnothing\rangle$;
(iii) If $\lambda < \theta$ is a limit ordinal then $\mathcal{N}_\lambda$ is the unique passive premouse such that
  (a) $\omega\beta < \mathrm{On}^{\mathcal{N}_\lambda}$ iff $\mathcal{N}_\alpha\,|\beta$ is defined and eventually constant as $\alpha \to \lambda$;
  (b) $\mathcal{N}_\lambda\,|\beta =$ eventual value of $\mathcal{N}_\alpha\,|\beta$ as $\alpha \to \lambda$, for β with $\omega\beta < \mathrm{On}^{\mathcal{N}_\lambda}$.○

[2]This is not the definition used in (Jensen, 2003), but is an equivalent formulation used in (Jensen & Steel, 2013).

As we're taking cores in our definition of robust K^c-constructions, it's not at all clear that the construction would wind up in a proper class model after On-many steps. This *is* actually the case though, as the following proposition shows.

Proposition 6.6. *Let κ be an uncountable regular cardinal or $\kappa = \mathrm{On}$, and let $\langle \mathcal{N}_\alpha \mid \alpha < \kappa \rangle$ be a robust K^c-construction. Then there exists a unique premouse $\mathcal{N}_\kappa$ of ordinal height κ such that $\langle \mathcal{N}_\alpha \mid \alpha \leqslant \kappa \rangle$ is a robust K^c-construction as well.*

Proof. It is clear that there is in fact a unique premouse $\mathcal{N}_\kappa$ satisfying that $\langle \mathcal{N}_\alpha \mid \alpha \leqslant \kappa \rangle$ is a robust K^c-construction, just by letting it be defined as in clause (iii) in Definition 6.5 – note that this makes sense because if $\mathcal{N}_\alpha | \beta$ is defined and eventually constant as $\alpha \to \lambda$ then this also holds for any $\gamma < \beta$. We thus only need to check that $|\mathcal{N}_\kappa| = \kappa$. Since $|\mathcal{N}_\alpha| < \kappa$ for every $\alpha < \kappa$ we get that $|\mathcal{N}_\kappa| \leqslant \kappa$. For $\gamma < \kappa$ define

$$\theta_\gamma := \inf\{\rho(\mathcal{N}_\alpha) \mid \alpha \in [\gamma, \kappa)\},$$

(so that $\theta_0 = \omega$ and the θ_γ's are nondecreasing. Since $\mathcal{N}_\alpha$ agrees with $\mathfrak{C}(\mathcal{N}_\alpha)$ below any ordinal of cardinality $\rho(\mathcal{N}_\alpha)$ by Theorem 5.3, we get that $\mathcal{N}_\gamma$ agrees with $\mathcal{N}_\alpha$ below θ_γ for every $\alpha \in [\gamma, \kappa)$, so if κ is the sup of the θ_γ's then $|\mathcal{N}_\kappa| \geqslant \kappa$ by definition of $\mathcal{N}_\kappa$ and we're done.

If κ is not the sup of the θ_γ's then the θ_γ's have to be eventually constant by regularity of κ, so say $\rho := \theta_\gamma$ for every $\gamma \in [\eta, \kappa)$, for some η. This means that $\rho(\mathcal{N}_\gamma) = \rho$ for cofinally many $\gamma < \kappa$ and thus also for club many $\gamma < \kappa$ by picking out the limit points.

Because case (ii)(b) in the definition of robust K^c-constructions happens cofinally often, and thus also club often, we can intersect these two clubs to get club many $\gamma < \kappa$ in case (ii)(b) such that $\rho(\mathcal{N}_\gamma) = \rho$. For these γ it holds that $\mathfrak{C}(\mathcal{N}_\gamma) \lhd \mathcal{N}_{\gamma+1}$ and thus $\mathfrak{C}(\mathcal{N}_\gamma) \in \mathcal{N}_{\gamma+1}$ as well by definability of the core.

Since $\mathcal{N}_{\gamma+1} \models |\mathfrak{C}(\mathcal{N}_\gamma)| = \rho$ we have that $\mathfrak{C}(\mathcal{N}_\gamma) \trianglelefteq \mathcal{N}_\alpha$ for every $\alpha \in [\gamma, \kappa)$. As ρ was the infimum of the projecta, we then get that $\mathfrak{C}(\mathcal{N}_\gamma) \trianglelefteq \mathfrak{C}(\mathcal{N}_\alpha)$ for all $\alpha \in [\gamma, \kappa)$ as well. Note that these are strict initial segments. Indeed, the subset $x \subseteq \rho$ witnessing that $\rho = \rho(\mathcal{N}_\gamma)$ is definable over $\mathcal{N}_\gamma$, so that $x \in \mathcal{N}_{\gamma+1}$.

This means that the new subset $y \subseteq \rho(\mathcal{N}_{\gamma+1})$ has to be defined with at least one parameter p from $\mathcal{N}_{\gamma+1} - \mathcal{N}_\gamma$, so that $p \in \mathfrak{C}(\mathcal{N}_{\gamma+1}) - \mathfrak{C}(\mathcal{N}_\gamma)$.

As there are cofinally many such γ, we get that $|\mathcal{N}_\kappa| = \kappa$. ■

Even though a K^c-construction provides us with a proper class model after On-many steps, it's not necessarily the case that we actually *can* continue for that many steps, as we require that $\mathcal{N}_\alpha$ is solid. This is the goal of the remaining part of this chapter.

6.2 Iterability of robust K^c

Having defined our proposed mice, we move towards showing the iterability of these structures. Iterability arguments usually are split up into two parts: branch existence and uniqueness. These will then entail that not only is the given structure iterable, but is *uniquely iterable*. This property is very useful, as it allows us to use the (strong) Dodd-Jensen Theorem 4.25, since every tree according to the strategy will then be unambiguous.

Branch existence itself is split up into a countable case and an uncountable case, where the uncountable case involves a reflection argument. We won't provide a proof of the countable case of branch existence. This is the main theorem in (Jensen, 2003) and requires introducing even more new terminology. As this argument itself isn't essential to the rest of the theory, we will leave it out.

Theorem 6.7 (Branch existence). *Let $\langle \mathcal{N}_\alpha \mid \alpha \leqslant \theta \rangle$ be a robust K^c-construction, $\pi : \mathcal{M} \to \mathfrak{C}_k(\mathcal{N}_\theta)$ a near k-embedding with $\mathcal{M}$ countable and $\mathcal{T}$ a countable putative k-iteration tree on $\mathcal{M}$. Then either there is a maximal branch of $\mathcal{T}$ with limit model $\mathcal{P}$ or $\mathcal{T}$ has a last model $\mathcal{P}$, such that, letting n be the degree of $\mathcal{P}$,*

- *(i) There are no drops along the $\mathcal{M}$-to-$\mathcal{P}$ branch, $n = k$ and there is a near n-embedding $\sigma : \mathcal{P} \to \mathfrak{C}_n(\mathcal{N}_\theta)$ such that, letting $i : \mathcal{M} \to \mathcal{P}$ be the iteration embedding, $\sigma \circ i = \pi$, or*
- *(ii) There is a drop along the $\mathcal{M}$-to-$\mathcal{P}$ branch or $n < k$, and there is a near n-embedding $\sigma : \mathcal{P} \to \mathfrak{C}_n(\mathcal{N}_\alpha)$ for some $\alpha \leqslant \theta$ where $\alpha < \theta$ if a drop occured along the $\mathcal{M}$-to-$\mathcal{P}$ branch.*

"Proof". See (Jensen, 2003). ■

We will now introduce a new game, the **weak iteration game** $\mathcal{W}_k(\mathcal{M}, \omega)$, which will be a helpful gadget in showing the iterability of our K^c-constructions in the next section. There are ω many rounds, played as follows:

$$\begin{array}{lccccccc} \text{I} & \mathcal{T}_0 & & \mathcal{P}_1, i_i, \mathcal{T}_1 & & \mathcal{P}_2, i_2, \mathcal{T}_2 & & \cdots \\ \text{II} & & b_0 & & b_1 & & b_2 & & \cdots \end{array}$$

The game starts with player I playing a countable putative k-iteration tree $\mathcal{T}_0$ on $\mathcal{M}$. Then player II plays b_0, which is either "accept" or a maximal wellfounded branch of $\mathcal{T}_0$, but only allowing her to accept if $\mathcal{T}_0$ has a last model. Let $\mathcal{Q}_1$ be the last model of $\mathcal{T}_0$ if she accepts, and $\mathcal{M}_{b_0}^{\mathcal{T}_0}$ otherwise. Set k_1 to be the degree of $\mathcal{Q}_1$.

Player I then picks an initial segment $\mathcal{P}_1 \trianglelefteq \mathcal{Q}_1$ and $i_1 \leqslant \omega$ such that $i_1 \leqslant k_1$ if $\mathcal{P}_1 = \mathcal{Q}_1$. He also plays a countable putative i_1-iteration tree $\mathcal{T}_1$ on $\mathcal{P}_1$. Then player II either accepts or plays a maximal wellfounded branch of $\mathcal{T}_1$, and the game continues in this fashion. Player II then wins if the cofinal branch through the composition of all the $\mathcal{T}_i$'s is wellfounded.

Definition 6.8. A **weak (k, ω)-iteration strategy for $\mathcal{M}$** is a winning strategy for player II in $\mathcal{W}_k(\mathcal{M}, \omega)$. $\mathcal{M}$ is **weakly (k, ω)-iterable** if such a strategy exists. ○

Corollary 6.9. *Let $\langle \mathcal{N}_\alpha \mid \alpha \leqslant \theta \rangle$ be a robust K^c-construction, $\pi : \mathcal{M} \to \mathfrak{C}_k(\mathcal{N}_\theta)$ a near k-embedding with $\mathcal{M}$ countable. Then $\mathcal{M}$ is weakly (k, ω)-iterable.*

Proof. Let Σ be the weak (k, ω)-iteration strategy given by accepting every tree with a last model, which is legal as the Branch Existence Theorem 6.7 ensures that this last model is wellfounded, and given any tree of limit length, play a maximal wellfounded branch as given by the Branch Existence Theorem 6.7.

We need to show that the strategy doesn't break down. The only way this could happen was if composed branch through $\bigoplus_n \mathcal{T}_n$ was illfounded. But say $\mathcal{P}_i = \mathcal{Q}_i$ for all $i \geqslant N$ (this will happen for some N). Then we get an embedding $i : \mathcal{P}_N \to \varinjlim_n \mathcal{P}_n$, so since $\mathcal{P}_N$ is wellfounded, so is the direct limit. ■

Definition 6.10. Let $\mathcal{T}$ be a k-iteration tree on a premouse $\mathcal{M}$ of limit length λ. Then define

- $\delta(\mathcal{T}) := \sup\{\mathrm{lh}\, E^{\mathcal{T}}_\alpha \mid \alpha < \lambda\}$;
- $E(\mathcal{T}) := \bigcup_{\alpha<\lambda} \dot{E}^{\mathcal{M}^{\mathcal{T}}_\alpha} \upharpoonright \mathrm{lh}\, E^{\mathcal{T}}_\alpha$;
- $\mathcal{M}(\mathcal{T}) := J^{E(\mathcal{T})}_{\delta(\mathcal{T})} = \varinjlim_{\alpha<\lambda} \mathcal{M}^{\mathcal{T}}_\alpha \,|\, \mathrm{lh}\, E^{\mathcal{T}}_\alpha$. ○

We will now introduce the most important large cardinal notion in this thesis, the Woodin cardinal.

Definition 6.11. Let $\kappa < \delta$ and $A \subseteq V_\delta$. Then κ is **A-reflecting in δ** if for every $\lambda < \delta$ there is a V-extender E with $\mathrm{crit}\, E = \kappa$, $i_E(\kappa) > \lambda$ and $i_E(A) \cap V_\lambda = A \cap V_\lambda$. ○

Definition 6.12. A cardinal δ is a **Woodin cardinal** if for every $A \subseteq \delta$ there is a $\kappa < \delta$ which is A-reflecting in δ. ○

Woodin has noted that the argument of (Mitchell, 1974, Theorem 4.1) implies that the extenders witnessing that δ is Woodin can be taken to lie in V_δ, so that Woodinness is a $\mathbf{\Pi}^{V_{\delta+1}}_1$-property. This means that for a ppm $\mathcal{M}$ with $\delta^{+\mathcal{M}} \in \mathcal{M}$, $\mathcal{M} \models$ "δ is Woodin" implies that every initial segment $\mathcal{N} \trianglelefteq \mathcal{M}$ with $\delta^{+\mathcal{M}} \in \mathcal{N}$ also thinks that δ is Woodin.

Theorem 6.13 (Branch uniqueness). *Let b and c be distinct cofinal branches of a k-iteration tree $\mathcal{T}$, $\delta := \delta(\mathcal{T})$ and let $A \subseteq \delta$ be such that $\delta, A \in \mathrm{wfp}(\mathcal{M}^{\mathcal{T}}_b) \cap \mathrm{wfp}(\mathcal{M}^{\mathcal{T}}_c)$. Then it holds that*

$$\mathcal{M}^{\mathcal{T}}_b \models \exists\kappa < \delta : \kappa \text{ is } A\text{-reflecting in } \delta.$$

Proof. We first claim that the extenders used on b and c have an overlapping pattern, as pictured in Figure 6.1. To see this, pick any successor ordinal $\alpha_0 + 1 \in b - c$ and then recursively define

$$\beta_n + 1 := \min\{\gamma \in c \mid \gamma > \alpha_n + 1\}$$
$$\alpha_{n+1} + 1 := \min\{\eta \in b \mid \eta > \beta_n + 1\}.$$

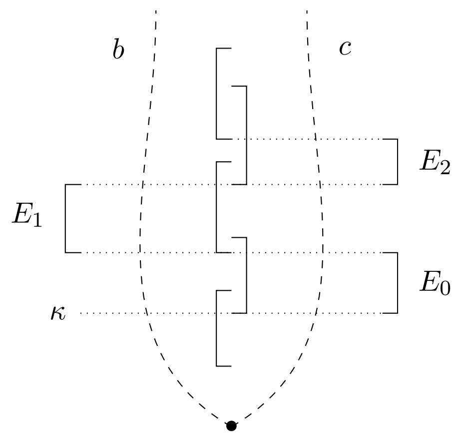


Figure 6.1: The overlapping pattern in the proof of Theorem 6.13.

Now, given any $n < \omega$, the T-predecessor of $\beta_n + 1$ is on c and is $\leqslant \alpha_n + 1$ by definition of $\beta_n + 1$. As the predecessor doesn't lie on b, we get that it is $\leqslant \alpha_n$. The rules of the iteration game then implies that $\operatorname{crit} F_{\beta_n} < \nu_{F_{\alpha_n}}$. Completely analogous we also get that $\operatorname{crit} F_{\alpha_{n+1}} < \nu_{F_{\alpha_n}}$. Since generators aren't moved along branches of iteration trees by Proposition 4.7, we get that

$$\begin{aligned}\operatorname{crit} F_{\beta_n} < \nu_{F_{\alpha_n}} &\leqslant \operatorname{crit} F_{\alpha_{n+1}} \\ < \nu_{F_{\beta_n}} &\leqslant \operatorname{crit} F_{\beta_{n+1}} \\ < \nu_{F_{\alpha_{n+1}}} &\leqslant \operatorname{crit} F_{\alpha_{n+2}} \\ &\vdots\end{aligned}$$

which is exactly the overlapping pattern pictured above. We also have that the supremum of the α_n's and the supremum of the β_n's agree and is equal to $\operatorname{lh} \mathcal{T}$, as branches of iteration trees are closed below their sups.

Assume that α_0 was picked large enough so that setting $\xi := \operatorname{pred}_T(\beta_0 + 1)$ and $\eta := \operatorname{pred}_T(\alpha_1 + 1)$, we have that $A = i_{\xi,c}(A^*) = i_{\eta,b}(A^{**})$ for some A^* and A^{**}. Set

$$\kappa := \operatorname{crit} F_{\beta_0} = \operatorname{crit} i_{\xi,c},$$

where the equality is by definition of T-predecessors in the iteration game. We will show that κ is A-reflecting in δ in the model $\mathcal{M}_b$. Define $E_0 := F_{\beta_0} \upharpoonright \operatorname{crit} F_{\alpha_1}$, so that the overlapping pattern implies that E_0 is a proper initial segment of F_{β_0}, so that the initial segment condition then entails that $E_0 \in \mathcal{M}$. The agreement between models of an iteration tree then furthermore implies that $E_0 \in \mathcal{M}_b$ as well.

If $j : \mathcal{M}_b \to \operatorname{Ult}(\mathcal{M}_b, E_0)$ is the ultrapower embedding, then since A agrees with A^* below κ, $j(A)$ agrees with $i_{\xi,c}(A^*)$ below $\operatorname{lh} E_0 = \operatorname{crit} F_{\alpha_1}$ since E_0 is an initial segment of F_{α_1}. As $i_{\xi,c}(A^*) = A$, $j(A)$ agrees with A below $\operatorname{lh} E_0$, so that E_0 witnesses that κ is A-reflecting up to $\operatorname{lh} E_0$ in $\mathcal{M}_b$.

To take this all the way up to δ, recursively define

$$\begin{aligned} E_{2n} &:= F_{\beta_n} \upharpoonright \operatorname{crit} F_{\alpha_{n+1}} \\ E_{2n+1} &:= F_{\alpha_{n+1}} \upharpoonright \operatorname{crit} F_{\beta_{n+1}}, \end{aligned}$$

which are also pictured on Figure 6.1. Just as with E_0 we get that $E_n \in \mathcal{M}_b$ for every $n < \omega$. By "composing" the ultrapower embeddings as follows

$$\mathcal{M}_b \xrightarrow{i_0} \operatorname{Ult}(\mathcal{M}_b, E_0) \xrightarrow{i_1} \operatorname{Ult}(\operatorname{Ult}(\mathcal{M}_b, E_0), i_0(E_1)) \xrightarrow{i_2} \cdots,$$

we get that the extender E derived from composing the first $2n$ ultrapower embeddings lie in $\mathcal{M}_b$ as well since all the information lie in the E_i's. But then by an analogous argument as above, $E \in \mathcal{M}_b$ and E witnesses that κ is A-reflecting up to $\operatorname{lh} E_n$ in the model $\mathcal{M}_b$. Since $\operatorname{lh} E_n \to \delta$ as $n \to \omega$, κ is A-reflecting in δ inside the model $\mathcal{M}_b$. ■

We will be needing a fine structural version of the uniqueness result, in which we will be needing the following *$\mathcal{Q}$-structures*.

Definition 6.14. Let $\mathcal{T}$ be a k-iteration tree on $\mathcal{M}$ of limit length and let b be a cofinal branch of $\mathcal{T}$. Let $\gamma \in \mathrm{On}$ be least such that either

- $\omega\gamma < \mathrm{On}^{\mathcal{M}_b}$ and $\mathcal{M}_b | \gamma + 1 \models \delta(\mathcal{T})$ is not Woodin, or
- $\omega\gamma = \mathrm{On}^{\mathcal{M}_b}$ and $\rho_{n+1}(\mathcal{M}_b) < \delta(\mathcal{T})$ for some $n < \omega$ such that $n + 1 \leqslant k$ if there are no drops along b.

Then set $\mathcal{Q}(b,T) := \mathcal{M}_b|\gamma$ if there is such a γ, and otherwise undefined. ○

Theorem 6.15. *Let $\mathcal{T}$ be a k-iteration tree of limit length and let b, c be distinct cofinal wellfounded branches of $\mathcal{T}$ such that $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{Q}(c,\mathcal{T})$ exists. Then neither is an initial segment of the other.*

Proof. Assume that one of them is an initial segment of the other. Then as they're both minimal with respect to the same first-order property, they're equal. If $\mathcal{Q}(b,\mathcal{T}) \in \mathcal{M}_b$ then this forces $\mathcal{Q}(c,\mathcal{T}) \in \mathcal{M}_c$ as well, because otherwise $\mathcal{M}_c \in \mathcal{M}_b$, contradicting that c is cofinal. Then $\mathcal{Q}(b,\mathcal{T}) \in \mathcal{M}_b \cap \mathcal{M}_c$ and then by the definition of $\mathcal{Q}$-structure there is an $A \subseteq \delta$, $A \in \mathcal{M}_b \cap \mathcal{M}_c$, satisfying that

$$\mathcal{M}_b \models \neg\exists\kappa < \delta(\mathcal{T}) : \kappa \text{ is } A\text{-reflecting in } \delta,$$

so that $b = c$ by the Uniqueness Theorem 6.13, $\lightning$. Thus $\mathcal{Q}(b,\mathcal{T}) = \mathcal{M}_b$. Analogously $\mathcal{Q}(c,\mathcal{T}) = \mathcal{M}_c$, so that $\mathcal{M}_b = \mathcal{M}_c$ as well. By definition of $\mathcal{Q}$-structure we then have that $\rho_{l+1}(\mathcal{M}_b) < \delta(\mathcal{T})$ for some $l < \omega$ such that $l+1 \leqslant k$ if b has no drops. Let $n < \omega$ be least such that $\rho_{n+1}(\mathcal{M}_b) < \delta(\mathcal{T})$. This means that $\deg b, \deg c = n$, so that there are cofinally many extenders used along b and c which have critical points $\geqslant \rho_{n+1}(\mathcal{M}_b)$.

This means that letting $\eta \in b$ and $\xi \in c$ be the last drops along the branches, which exist as both branches are wellfounded, we have that $\mathcal{M}_\eta$ and $\mathcal{M}_\xi$ are $(n+1)$-sound and furthermore

$$\mathcal{M}^*_\eta = \mathfrak{C}_{n+1}(\mathcal{M}^*_\eta) = \mathfrak{C}_{n+1}(\mathcal{M}_b) = \mathfrak{C}_{n+1}(\mathcal{M}_c) = \mathfrak{C}_{n+1}(\mathcal{M}^*_\xi) = \mathcal{M}^*_\xi$$

and that $i_{\eta,b} \circ i^*_\eta : \mathcal{M}^*_\eta \to \mathcal{M}_b$ and $i_{\xi,c} \circ i^*_\xi : \mathcal{M}^*_\xi \to \mathcal{M}_c$ exists and are n-embeddings by Theorem 4.12, and our choice of η and ξ furthermore guarantees that their critical points are $\geqslant \rho_{n+1}(\mathcal{M}^*_\eta)$. But since Theorem 4.12 also gives us that $\rho_{n+1}(\mathcal{M}^*_\eta) = \rho_{n+1}(\mathcal{M}_b) = \rho_{n+1}(\mathcal{M}^*_\xi)$ and

$$(i_{\eta,b} \circ i^*_\eta)(p_{n+1}(\mathcal{M}^*_\eta)) = p_{n+1}(\mathcal{M}_b) = (i_{\xi,c} \circ i^*_\xi)(p_{n+1}(\mathcal{M}^*_\xi)),$$

we get that $i_{\eta,b}\circ i^*_\eta$ and $i_{\xi,c}\circ i^*_\xi$ are both equal to the core embedding

$$\pi:\mathfrak{C}_{n+1}(\mathcal{M}_b)\to\mathfrak{C}_n(\mathcal{M}_b)$$

and are therefore equal. But then the extender applied to $\mathcal{M}^*_\eta$ in b is compatible with the extender applied to $\mathcal{M}^*_\xi$ in c, so that $\eta=\xi$. This means that $\eta\in b\cap c$, so let $\alpha\in b\cap c$ be largest. As we picked η and ξ such that $\mathcal{M}_\eta$ and $\mathcal{M}_\xi$ were $(n+1)$-sound, the argument above ensures that $\mathcal{M}_{\operatorname{succ}_T(\eta)}=\mathcal{M}_{\operatorname{succ}_T(\xi)}$, so $\alpha>\eta$.

Now, defining $\nu:=\sup\{\nu_{E_\beta}\mid\beta<_T\alpha\}$ we get that

$$\mathcal{M}_\alpha=\{(i_{\eta,\alpha}\circ i^*_\eta)(f)(a)\mid f\in\mathbf{r\Sigma}_n^{\mathcal{M}^*_\eta}\wedge a\in[\nu]^{<\omega}\}. \tag{1}$$

Since $i_{\alpha,b}$ and $i_{\alpha,c}$ are the identity on ν (by the agreement between models in iterations trees and that $\nu<\sup\{\operatorname{lh}E_\beta\mid\beta<_T\alpha\}$), are n-embeddings and also agree on $\operatorname{ran}(i_{\eta,\alpha}\circ i^*_\eta)$ since

$$\begin{aligned}
i_{\alpha,b}(i_{\eta,\alpha}(i^*_\eta(x))) &= i_{\eta,b}(i^*_\eta(x))\\
&= i_{\xi,c}(i^*_\xi(x))\\
&= i_{\alpha,c}(i_{\xi,\alpha}(i^*_\xi(x)))\\
&= i_{\alpha,c}(i_{\eta,\alpha}(i^*_\eta(x))),
\end{aligned}$$

we get that $i_{\alpha,b}=i_{\alpha,c}$ by (1). But then the extender applied to $\mathcal{M}_\alpha$ in b is compatible with the extender applied to $\mathcal{M}_\alpha$ in c, contradicting the maximality of α. ■

Definition 6.16. Let $\mathcal{M}$ be a premouse. Then η is a **cutpoint of $\mathcal{M}$** if it doesn't overlap any extenders on the $\mathcal{M}$-sequence. That is, there is no extender E on the $\mathcal{M}$-sequence satisfying that $\eta\in[\operatorname{crit}E,\operatorname{lh}E]$. ○

Definition 6.17. A premouse $\mathcal{M}$ is **1-small** if there is no Woodin cardinal in $\mathcal{M}|\kappa$ whenever κ is the critical point of an extender on the $\mathcal{M}$-sequence. If furthermore

$$\mathcal{M}\models\text{there is no Woodin cardinal + there are finally many cardinals}$$

then $\mathcal{M}$ is **properly 1-small**. ○

Definition 6.18. A premouse $\mathcal{N}$ is **countably** (k, α, θ)**-iterable** if whenever $\mathcal{M}$ is a countable premouse and $\pi : \mathcal{M} \to \mathcal{N}$ is a near k-embedding then $\mathcal{M}$ is (k, α, θ)-iterable. ○

Theorem 6.19. *If $\mathcal{N}$ is weakly (k, ω)-iterable and properly 1-small then it's also countably (k, ω_1, ω_1)-iterable.*

Proof. Let $\pi : \mathcal{M} \to \mathcal{N}$ be a near k-embedding with $\mathcal{M}$ countable. As a weak (k, ω)-iteration strategy for $\mathcal{N}$ induces one for $\mathcal{M}$, let Σ be the weak (k, ω)-iteration strategy for $\mathcal{M}$. If we can show that every countable putative k-iteration tree $\mathcal{T}$ on $\mathcal{M}$ of limit length have at most one maximal wellfounded branch, then Σ induces a (k, ω_1, ω_1)-iteration strategy Γ on $\mathcal{M}$ by just picking the unique branch as given by Σ.

Assume that it's not the case and let $\mathcal{T}$ be the shortest tree with two distinct maximal wellfounded branches b and c. As we've picked unique branches at earlier stages, the branches have to be cofinal in $\mathcal{T}$.

Claim 6.19.1. $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$ exist.

Proof of claim. We show that $\mathcal{Q} := \mathcal{Q}(b, \mathcal{T})$ exists, as it's completely analogous for c. If b drops then $\mathcal{Q}(b, \mathcal{T})$ exists as then $\rho_{n+1}(\mathcal{M}_b) < \delta(\mathcal{T})$ for some $n < \omega$, so the second clause of the definition of $\mathcal{Q}$-construction would always hold. Assume thus that b doesn't drop, so that we get the k-embedding $i : \mathcal{M} \to \mathcal{M}_b$.

Note that since $\mathcal{M}$ has finally many cardinals, $\mathcal{M}_b$ has as well, since this is a $\neg Q$-property and every k-embedding preserves these. But then $\delta(\mathcal{T}) \in \mathcal{M}_b$, because if $\delta(\mathcal{T}) = \mathrm{On}^{\mathcal{M}_b}$ then this would exactly entail that there were cofinally many cardinals in $\mathcal{M}_b$ (recall that the lengths of the extenders are all cardinals in $\mathcal{M}_b$). As $\mathcal{M}_b$ has no Woodins, there has to be a largest initial segment of $\mathcal{M}_b$ thinking that $\delta(\mathcal{T})$ is Woodin, which is precisely $\mathcal{Q}$. $\dashv$

Claim 6.19.2. $\delta(\mathcal{T})$ is a cutpoint of both $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$.

Proof of claim. We show that $\delta := \delta(\mathcal{T})$ is a cutpoint of $\mathcal{Q} := \mathcal{Q}(b, \mathcal{T})$, the argument being analogous for $\mathcal{Q}(c, \mathcal{T})$. Assume δ overlaps the (κ, λ)-extender E on the $\mathcal{Q}$-sequence. Form the ultrapower $\mathrm{Ult} := \mathrm{Ult}(\mathcal{Q}, E)$ and note that δ is still Woodin in Ult, as $\mathcal{P}^{\mathcal{M}}(\delta) = \mathcal{P}^{\mathrm{Ult}}(\delta)$ ["$\subseteq$" is by coherence. "$\supseteq$" is seen using that $\mathrm{lh}\, E$ is a cardinal in the ultrapower, so acceptability implies that subsets of δ in Ult lies below δ and thus in $\mathcal{Q}$ by coherence, using that $\delta < \lambda$]. This means that we have that $\mathrm{Ult}\,|i_E(\kappa) \models$ there is a Woodin, so that $\mathcal{Q}\,|\kappa \models$ there is a Woodin as well, meaning that $\mathcal{Q}\,|\kappa = \mathcal{M}_b\,|\kappa$ witnesses that $\mathcal{M}_b$ is not 1-small. But $\mathcal{M}_b$ is 1-small since $\mathcal{M}$ is, $\lightning$. $\dashv$

Now, since both $\mathcal{Q}$-structures think that δ is Woodin, $\mathrm{On}^{\mathcal{Q}}$ has to be strictly below the critical points of the extenders lying above $\delta(\mathcal{T})$ in $\mathcal{M}_b$ for both $\mathcal{Q}$-structures $\mathcal{Q}$, as otherwise we would contradict 1-smallness. But since the $\mathcal{Q}$-structures agree below δ by the agreement of models in iteration trees, one of the $\mathcal{Q}$-structures is an initial segment of the other, contradicting Theorem 6.15. This shows the above-mentioned uniqueness property for Σ, so $\mathcal{M}$ is (k, ω_1, ω_1)-iterable. ∎

Corollary 6.20. *Let $\langle \mathcal{N}_\alpha \mid \alpha \leqslant \theta \rangle$ be a robust K^c-construction with $\mathcal{N}_\theta$ properly 1-small. Then $\mathfrak{C}_k(\mathcal{N}_\theta)$ is countably (k, ω_1, ω_1)-iterable.*

Proof. This follows directly from Corollary 6.9 and Theorem 6.19. ∎

The following theorem summarises the previous results and involves the reflection argument to show the uncountable case of branch existence.

Theorem 6.21. *Assume there is no proper class model with a Woodin cardinal and let $\mathcal{N}$ be the output of a robust K^c-construction. Then $\mathcal{M} := \mathfrak{C}_k(\mathcal{N})$ is fully k-iterable.*

Proof. If $\mathcal{M}$ wasn't 1-small then we could linearly iterate $\mathcal{M}$ On-many times using the least extender on $\mathcal{M}$ witnessing the non-1-smallness, so that we would wind up with a proper class premouse with a Woodin, $\lightning$. So we can assume that $\mathcal{M}$ is 1-small.

If $\mathcal{M}$ wasn't properly 1-small then we add ordinals of top of $\mathcal{M}$ until it thinks that are no Woodins, which happens as otherwise we would reach a proper class premouse with a Woodin, $\lightning$. Making sure that the cardinals in the resulting model are final, we get a properly 1-small premouse. Note that this premouse also lies on a K^c construction, and any iteration strategy for it will induce an iteration strategy for $\mathcal{M}$, so we can assume that $\mathcal{M}$ is properly 1-small.

By adding ordinals on top of $\mathcal{M}(\mathcal{T})$, we get a least premouse $\mathcal{Q} = \mathcal{J}_\gamma^{\vec{E}(\mathcal{T})}$ such that either $\mathcal{J}_{\gamma+1}^{\vec{E}(\mathcal{T})} \models \delta(\mathcal{T})$ is not Woodin, or $\mathcal{Q} = \mathcal{M}(\mathcal{T})$ and $\rho_{n+1}(\mathcal{Q}) < \delta(\mathcal{T})$. We know that $\mathcal{Q}$ exists as otherwise there would be a proper class premouse with a Woodin, $\lightning$.

Now let $\mathcal{H} := \mathrm{cHull}^{V_\eta}(\{\mathcal{M}, \mathcal{M}(\mathcal{T}), \mathcal{T}, \mathcal{Q}\})$ for η sufficiently large. Since $\mathcal{H}$ is countable and the uncollapse embedding $\pi \restriction \bar{\mathcal{M}} : \bar{\mathcal{M}} \to \mathcal{M}$ is elementary, $\bar{\mathcal{M}}$ is (k, ω_1, ω_1)-iterable by Corollary 6.20, so $\bar{\mathcal{T}}$ has a unique cofinal wellfounded branch b. To show that $\mathcal{T}$ also have such a branch, we have to show that $b \in \mathcal{H}$.

Note that $\mathcal{Q}(b, \bar{\mathcal{T}})$ exists by our proper 1-smallness assumption on $\mathcal{M}$, so that $\bar{\mathcal{Q}} \trianglelefteq \mathcal{Q}(b, \bar{\mathcal{T}})$ by elementarity of π. Then b is the *unique* cofinal wellfounded branch of $\bar{\mathcal{T}}$ such that $\bar{\mathcal{Q}} \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}}$. Indeed, if c was another such branch, $\mathcal{Q}(c, \bar{\mathcal{T}})$ would then exist as well and $\bar{\mathcal{Q}} \trianglelefteq \mathcal{Q}(c, \bar{\mathcal{T}})$. But since $\mathcal{Q}(b, \bar{\mathcal{T}})$ and $\mathcal{Q}(c, \bar{\mathcal{T}})$ cannot be compared by Theorem 6.15, we must have that $\bar{\mathcal{Q}}$ is a proper initial segment of both. But $\bar{\mathcal{Q}}$ codes up a failure of Woodinness of $\delta(\bar{\mathcal{T}})$, contradicting the Uniqueness Theorem 6.13.

If we then let G be $\mathrm{Col}(\omega, |\bar{\mathcal{Q}}|)$-generic, we have that $\mathcal{H}[G] \models \varphi_b[\bar{\mathcal{T}}, \bar{\mathcal{Q}}]$, where φ_b is the Σ^1_1-formula

$$\exists x \in \omega^\omega : x \text{ codes } b \wedge \bar{\mathcal{Q}} \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}} .$$

To see this, first assume that we chose our $\mathcal{H}$ such that letting $\theta := \mathrm{On}^{\mathcal{H}}$ we had $b, \bar{\mathcal{T}}, \bar{\mathcal{Q}} \in V_\theta[G]$ (which is possible as $\bar{\mathcal{Q}}$ is countable in the generic extension). Then $V_\theta[G] \models \varphi_b[\bar{\mathcal{T}}, \bar{\mathcal{Q}}]$, so by Shoenfield's Absoluteness Lemma[3] we get that $\mathcal{H}[G]$ also satisfies this, as $\mathcal{H}[G]$ and $V_\theta[G]$ has the same ordinals and both con-

[3]Shoenfield's Lemma says that Σ^1_2 formulas are absolute between transitive models with the same ordinals and sharing the same parameters used in the formula. See (Shoenfield, 2003).

tain $\bar{\mathcal{T}}$ and $\bar{\mathcal{Q}}$. This means that $b \in \mathcal{H}[G]$, so by homogeneity[4] of $\mathrm{Col}(\omega, |\bar{\mathcal{Q}}|)$ we then get that $b \in \mathcal{H}$ and we're done. ■

We can now (finally!) prove that our structures in fact satisfy that their cores are sound, so that the K^c-construction doesn't halt at any point, resulting in a proper class model.

Corollary 6.22. *Assume there is no proper class with a Woodin cardinal and let $\mathcal{N}$ be the output of a robust K^c-construction. Then $\mathfrak{C}(\mathcal{N})$ is sound.*

Proof. This is by the argument mentioned in the discussion at the end of chapter 2, but we will recall it here. $\mathfrak{C}_0(\mathcal{N}) = \mathcal{N}$ is trivially 0-sound. Assuming $\mathfrak{C}_k(\mathcal{N})$ is k-sound, we get that it's $(k+1)$-solid by Theorem 5.3 since it's k-iterable by Theorem 6.21, so that by Corollary 2.21 we get that it's $(k+1)$-sound. Thus $\mathfrak{C}_k(\mathcal{N})$ is k-sound for every $k < \omega$, so that $\mathfrak{C}(\mathcal{N})$ is sound. ■

[4]This essentially means that if an element in the generic extension witnesses an existential formula with parameters in the ground model, the element also lies in the ground model. See (Jech, 2013).

7 Pseudo-K

From this point onwards we strengthen our global assumption from chapter 2, that there is no proper class inner model with a Shelah limit of Shelahs, to the following.

There is no proper class inner model with a Woodin cardinal.

Note that under this global assumption, any premouse is countably (k, ω_1, ω_1)-iterable iff it's fully k-iterable by the argument in Theorem 6.21. Also, this global assumption ensures that the K^c-constructions from the previous chapter actually produce mice.

We now want to use these mice to define the *core model* K, which is in some sense the unique canonical inner model that is close to V. To be able to define K we will need an intermediate model called *pseudo-K* approximating K, and towards this fix the following.

- τ is a regular cardinal $\geqslant \omega_3$;
- Ω is a regular cardinal such that $2^{<\tau} < \Omega$, $\tau^{++} < \Omega$ and $\forall\alpha < \Omega : \alpha^\omega < \Omega$.

Pseudo-K will then be a structure $\tilde{K}(\tau, \Omega)$ depending solely on τ and Ω. In the next chapter we will show that this model will have ordinal height $\geqslant \tau$ and we will show that specific choices of τ and Ω will make sure that these pseudo-K's converge into a unique model, which will be our K.

7.1 Weasels and stability

Definition 7.1. A **preweasel** is a premouse of ordinal height Ω. A **weasel** is a fully iterable preweasel. ○

Definition 7.2. Let $\mathcal{M}$ be a (pre)mouse. Then

- $\mathcal{M}$ is a **mini-universe** if it's a (pre)weasel and $\mathcal{M} \models$ there are cofinally many cardinals;

- $\mathcal{M}$ is **collapsing** if it's a (pre)weasel and $\mathcal{M} \models$ there is a largest cardinal. In this case let $\gamma^{\mathcal{M}}$ be that largest cardinal of $\mathcal{M}$ and $\eta^{\mathcal{M}} := \operatorname{cof}^{\mathcal{M}} \gamma$;
- $\mathcal{M}$ is **stable** if it's a mini-universe, or that it's collapsing and $\eta^{\mathcal{M}}$ is not the critical point of a total extender on the $\mathcal{M}$-sequence, or $\mathrm{On}^{\mathcal{M}} < \Omega$;
- $\mathcal{M}$ is **universal** if it's fully iterable and it wins the coiteration with any stable mouse. $\circ$

We will by default be working with mice and weasels, and will specifically point out when premice and preweasels are meant. We will call mice **stably universal** if they're both stable and universal.

Proposition 7.3.

(i) *If $\mathcal{W}$ is unstable collapsing then $\mathrm{Ult}(\mathcal{W}, U)|\Omega$ is stable collapsing, where U is the order zero measure*[1] *of $\mathcal{W}$ on $\eta^{\mathcal{W}}$;*

(ii) *Any stable collapsing weasel is stably universal;*

(iii) *If there exists a collapsing weasel then there's no universal mini-universe;*

(iv) *If $\mathcal{W}$ and $\mathcal{R}$ are collapsing then $\operatorname{cof}^V \gamma^{\mathcal{W}} = \operatorname{cof}^V \gamma^{\mathcal{R}}$.*

Proof. (i): Since U is the order zero measure on $\eta^{\mathcal{W}}$, $\eta^{\mathcal{W}}$ isn't measurable in $\mathrm{Ult}(\mathcal{W}, U)$. As $\gamma^{\mathcal{W}}$ is still the largest cardinal below Ω of the ultrapower, we get that $\eta^{\mathcal{W}} = \eta^{\mathrm{Ult}(\mathcal{W},U)|\Omega}$, so that the $\mathrm{Ult}(\mathcal{W}, U)|\Omega$ is stable.

Why's that?

(ii): Let $\mathcal{W}$ be stable collapsing, and $\mathcal{M}$ any stable mouse. Assume towards a contradiction that $\mathcal{M}$ iterates strictly past $\mathcal{W}$, so that we have that

$$\begin{tikzcd} \mathcal{P} \arrow[r, phantom, "\lhd"] & \mathcal{Q} \\ \mathcal{W} \arrow[u, "i", "\mathcal{T}"'] & \mathcal{M} \arrow[u, dash, "\mathcal{U}"] \end{tikzcd}$$

where there are no drops on the $\mathcal{W}$-to-$\mathcal{P}$ branch. But now the lengths of all the extenders used along the $\mathcal{M}$-to-$\mathcal{Q}$ branch leaves cofinally many $\mathcal{Q}$-cardinals below Ω, so that $\mathcal{P}$ is a mini-universe, $\lightning$.

(iii): Assume $\mathcal{W}$ is collapsing and $\mathcal{R}$ is a universal mini-universe. By (i) we can assume that $\mathcal{W}$ is stable, so that it's universal by (ii). Then in the comparison of

[1] This is by the Mitchell order, defined as $U \lhd U'$ iff $U \in \mathrm{Ult}(\mathcal{W}, U')$. It can be shown that this is in fact a wellfounded preorder, see (Kanamori, 2009).

$\mathcal{W}$ and $\mathcal{R}$, they both iterate to the same $\mathcal{P}$ with the iteration maps existing on both sides, so that by elementarity $\mathcal{W}$ is a mini-universe, $\lightning$.

(iv): Assume not and without loss of generality that $\mathcal{W}$ and $\mathcal{R}$ are both stable and $\operatorname{cof}\gamma^{\mathcal{W}} \leqslant \operatorname{cof}\gamma^{\mathcal{R}}$. Then they're both universal by (ii) and thus iterate to some $\mathcal{P}$ via iteration maps $i : \mathcal{W} \to \mathcal{P}$ and $j : \mathcal{R} \to \mathcal{P}$. By elementarity $\mathcal{P}$ has a largest cardinal $\gamma^{\mathcal{P}}$ such that $i(\gamma^{\mathcal{W}}) = \gamma^{\mathcal{P}} = j(\gamma^{\mathcal{R}})$, so that $\operatorname{cof}\gamma^{\mathcal{P}} \leqslant \operatorname{cof}\gamma^{\mathcal{W}} \leqslant \operatorname{cof}\gamma^{\mathcal{R}}$. But no new sequences appear along iteration trees, so that we also get that $\operatorname{cof}\gamma^{\mathcal{R}} \leqslant \operatorname{cof}\gamma^{\mathcal{P}}$ Why's that?
and thus the wanted equality as well. ■

Definition 7.4. A phalanx Φ is **stable** if

(i) every $\mathcal{M}^{\Phi}_{\xi}$ is stable;

(ii) whenever $\xi + 1 < \operatorname{lh}\Phi$ and $\mathcal{M}^{\Phi}_{\xi}$ is collapsing such that $(\eta^{\mathcal{M}^{\Phi}_{\xi}})^{+\mathcal{M}^{\Phi}_{\xi}} \leqslant \lambda^{\Phi}_{\xi}$, it holds that $\mathcal{M}^{\Phi}_{\beta} \models \eta^{\mathcal{M}^{\Phi}_{\xi}}$ is not measurable, for every $\beta \geqslant \xi$. ○

Definition 7.5. Let $\mathcal{T}$ be an iteration tree on some phalanx Ψ. Then the **induced phalanx by** $\mathcal{T}$ is the phalanx Φ defined as $\mathcal{M}^{\Phi}_{\alpha} := \mathcal{M}^{\mathcal{T}}_{\alpha}$ and λ^{Φ}_{α} being λ^{Ψ}_{α} if $\alpha < \operatorname{lh}\Psi$ and $\operatorname{lh} E_{\alpha}$ otherwise. We denote this induced phalanx by $\Phi(\mathcal{T})$. ○

The primary use of stability of phalanxes is the following result that it's preserved under iterations.

Lemma 7.6. *Let Φ be a stable phalanx and $\mathcal{T}$ be an iteration tree on Φ with a last model and $\operatorname{lh}\mathcal{T} < \Omega$. Then $\Phi(\mathcal{T})$ is stable. In particular, every model of $\mathcal{T}$ have ordinal height $\leqslant \Omega$.*

Proof. Write $\Phi(\mathcal{T}) = (\langle \mathcal{M}_{\xi} \mid \xi \leqslant \gamma\rangle, \langle \lambda_{\xi} \mid \xi < \gamma\rangle)$. We show it by induction on $\gamma < \Omega$. The base case is if $\Phi(\mathcal{T}) = \Phi$, in which case it's true by assumption. Assume thus that $\Phi(\mathcal{T} \upharpoonright \gamma)$ is stable; we'll show that $\Phi(\mathcal{T})$ is then stable as well.

We first show the second stability clause. It's trivial if γ is a limit ordinal, so assume that $\gamma = \xi + 1$, $\mathcal{M}_{\xi}$ is collapsing, $(\eta^{\mathcal{M}_{\xi}})^{+\mathcal{M}_{\xi}} \leqslant \lambda_{\xi}$ and for a contradiction that $\eta^{\mathcal{M}_{\xi}}$ is measurable in $\mathcal{M}_{\gamma}$, witnessed by a measure $U \in \mathcal{M}_{\gamma}$. But then since $(\eta^{+\mathcal{M}_{\xi}})^{+\mathcal{M}_{\xi}} \leqslant \lambda_{\xi}$ we have that

$$U \in \mathcal{M}_{\gamma} | (\eta^{\mathcal{M}_{\xi}})^{+\mathcal{M}_{\xi}} = \mathcal{M}_{\xi} | (\eta^{\mathcal{M}_{\xi}})^{+\mathcal{M}_{\xi}} \subseteq \mathcal{M}_{\xi},$$

contradicting the stability of $\mathcal{M}_\xi$ by the induction assumption.

For the first stability clause we have to show that $\mathcal{M}_\gamma$ is stable. If $\mathcal{M}_\gamma$ is a mini-universe or satisfies that $\mathrm{On}^{\mathcal{M}_\gamma} < \Omega$ it's trivial, so assume that it's collapsing. Assume furthermore that $\eta^{\mathcal{M}_\gamma}$ is the critical point of an extender E on the $\mathcal{M}_\gamma$-sequence which is total on $\mathcal{M}_\gamma$, and assume that γ is a successor ordinal as the limit case is clear by elementarity. Set $\xi := \mathrm{pred}_T\,\gamma$. If $\gamma \notin D^{\mathcal{T}}$ then it's clear by elementarity again, and if $\gamma \in D^{\mathcal{T}}$ then $\mathrm{On}^{\mathcal{M}_\gamma} < \Omega$ as we have that $\mathrm{On}^{\mathcal{M}^*_\gamma} < \Omega$, $\lightning$. The last case is if $\mathrm{On}^{\mathcal{M}_\gamma} > \Omega$, and we'll show that this is impossible.

Assume $\mathrm{On}^{\mathcal{M}_\gamma} > \Omega$, and furthermore assume that γ is a limit ordinal. Then by definition of direct limits we have that $\Omega = i^{\mathcal{T}}_{\eta,\gamma}(\mu)$ for some $\eta < \gamma$. By our induction hypothesis, $i_{\eta,\nu}(\mu) < \Omega$ for every $\nu <_T \gamma$. Define for these ν the sets

$$X_\nu := i_{\nu,\gamma}\text{"}\, i_{\eta,\nu}(\mu),$$

so that $\Omega = \bigcup\{X_\nu \mid \nu \in (\eta,\gamma)_T\}$. But then Ω is a union of $\gamma < \Omega$ sets of size $< \Omega$, contradicting the regularity of Ω.

Assume now that $\gamma = \beta + 1$, and let $\xi := \mathrm{pred}_T\,\gamma$ and $\mathcal{M}_\gamma = \mathrm{Ult}_k(Q, E)$, where $Q \trianglelefteq \mathcal{M}_\xi$ and $E = E^{\mathcal{T}}_\beta$. Then $\mathrm{On}^{\mathcal{M}_\gamma} > \Omega$ forces $\mathcal{M}_\xi$ to be a collapsing
Why? weasel and $\eta^{\mathcal{M}_\xi} = \mathrm{crit}\,E$. Because $\mathcal{M}_\beta$ is stable we have that $\xi < \beta$. Since
Why? $\beta + 1 \notin D^{\mathcal{T}}$ we get that $(\eta^{\mathcal{M}_\xi})^{+\mathcal{M}_\xi} \leqslant \lambda_\xi$, but $\eta^{\mathcal{M}_\xi}$ is measurable in $\mathcal{M}_\beta$, contradicting the fact that $\Phi(\mathcal{T} \restriction \gamma)$ satisfies the second stability clause. ■

We also have the following useful properties regarding coiterations of stable mice, which will be used in the next section.

Proposition 7.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be stable mice, and assume that $\mathcal{N}$ iterates past $\mathcal{M}$. Then the iteration map $i : \mathcal{M} \to \mathcal{P}$ satisfies that $i\text{"}\Omega \subseteq \Omega$. Furthermore, the coiteration ends in $\leqslant \Omega + 1$ steps.*

Proof. In the coiteration we get that $i^{\mathcal{T}}_\alpha\text{"}\Omega \subseteq \Omega$ for every $\alpha < \Omega$ by Lemma 7.6. At the Ω'th stage we have a direct limit, and if κ is sent to Ω in this direct limit then by stability there's a cardinal $\mu > \kappa$ which then has to be sent to something strictly above Ω. But then there had to be a point along the way to the Ω'th stage where some element was sent to something $\geqslant \Omega$, $\lightning$.

By the above, there are no extenders above Ω and since the models agree below Ω, they agree with each other. Note that Ω can't be the length of an extender since then it wouldn't be a V-cardinal. ■ Pretty messy proof.

Lemma 7.8 (Stacking lemma). *Let $\mathcal{W}$ be a mini-universe and let $\mathcal{M}$ be a k-sound mouse such that $k < \omega$, $\mathcal{W} \trianglelefteq \mathcal{M}$ and $\rho_k(\mathcal{M}) = \Omega$. Then*

(i) $\rho(\mathcal{M}) = \Omega$;

(ii) given any other l-sound mouse $\mathcal{N}$ satisfying that $\rho_l(N) = \Omega$, then either $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$.[2]

Proof. (i): Suppose $\rho(\mathcal{M}) < \Omega$ and let $A \subseteq \rho(\mathcal{M})$ be definable such that $A \notin \mathcal{M}$. Fix some sufficiently large θ such that $\mathcal{M}, \Omega \in V_\theta$ and $\rho(\mathcal{M}) \subseteq V_\theta$ and let

$$\pi : \mathrm{cHull}^{V_\theta}(\rho(\mathcal{M}) \cup \{p_k(\mathcal{M}), \mathcal{M}, \Omega\}) \to V_\theta$$

be the uncollapse map. Then the Condensation Theorem 5.1 implies that either $\bar{\mathcal{M}} \triangleleft \mathcal{M}$ or $\bar{\mathcal{M}} \triangleleft \mathrm{Ult}_0(\mathcal{M}, E)$ with $\mathrm{lh}\, E = \rho(\mathcal{M})$. Since $\rho(\mathcal{M})$ is an $\mathcal{M}$-cardinal and $\mathrm{lh}\, E$ isn't, we have that $\bar{\mathcal{M}} \triangleleft \mathcal{M}$.

We claim now that $\bar{\mathcal{M}} \triangleleft \mathcal{W}$. To see this it suffices to show that $\mathrm{On}^{\bar{\mathcal{M}}} < \mathrm{On}^{\mathcal{W}}$. But since $\rho_k(\mathcal{M}) = \Omega$ we have that any $r\Sigma_k^{\bar{\mathcal{M}}}(p_k(\bar{\mathcal{M}}))$-definable $B \subseteq \rho_k(\bar{\mathcal{M}})$ lies in $\mathcal{M}$ by soundness, so it lies in $\mathcal{M} \,|\rho_k(\bar{\mathcal{M}})^{+\mathcal{M}} = \mathcal{M} \,|\rho_k(\bar{\mathcal{M}})^{+\mathcal{W}}$ by acceptability. By definition of $\rho_k(\bar{\mathcal{M}})$ we then have that $\mathrm{On}^{\bar{\mathcal{M}}} \leqslant \rho_k(\bar{\mathcal{M}})^{+\mathcal{W}} < \Omega$, using that $\mathcal{W}$ is a mini-universe. Thus $\bar{\mathcal{M}} \triangleleft \mathcal{W}$. But now A is definable over $\bar{\mathcal{M}}$ by elementarity, so $A \in \mathcal{W} \,|\, \mathrm{On}^{\bar{\mathcal{M}}} + 1$, ↯.

(ii): Assume not and let $\mathcal{M}$ and $\mathcal{N}$ witness this fact. Then just as in (i) we get initial segments $\bar{\mathcal{M}}, \bar{\mathcal{N}} \triangleleft \mathcal{W}$ which cannot be compared, ↯. ■

Definition 7.9. Let $\mathcal{W}$ be a mini-universe. Then the **mouse stack over $\mathcal{W}$** is the unique premouse $S(\mathcal{W})$ such that $\mathcal{N} \trianglelefteq S(\mathcal{W})$ iff $\mathcal{N} \trianglelefteq \mathcal{M}$ for some sound mouse $\mathcal{M}$ extending $\mathcal{W}$ and projecting to Ω. If $\mathcal{W}$ is collapsing then $S(\mathcal{W}) := \mathcal{W}$. ○

[2]Note the similarity between this lemma and Corollary 4.16.

One can show that $S(\mathcal{W})$ is in fact a mouse, has no last mouse and has Ω as largest cardinal (Jensen, Schimmerling, Schindler, & Steel, 2009, Lemma 3.3). The following fact will be useful.

Proposition 7.10. *Let $\mathcal{W}$ be a mini-universe and $\mathcal{M}$ a k-sound premouse extending $\mathcal{W}$ and k-projecting to Ω. Then the following are equivalent.*

(i) $\mathcal{M} \trianglelefteq S(\mathcal{W})$;

(ii) $\mathrm{cHull}_k^{\mathcal{M}}(\alpha \cup p_k(\mathcal{M})) \trianglelefteq \mathcal{W}$ *for club many* $\alpha < \Omega$;

(iii) $\mathrm{cHull}_k^{\mathcal{M}}(\alpha \cup p_k(\mathcal{M})) \trianglelefteq \mathcal{W}$ *for cofinally many* $\alpha < \Omega$.

Proof. For $(i) \Rightarrow (ii)$, let $\alpha < \Omega$ be a cardinal of $\mathcal{W}$. Write

$$\mathcal{H} := \mathrm{cHull}^{\mathcal{M}}(\alpha \cup p_k(\mathcal{M}))$$

and note that $\mathrm{Hull}^{\mathcal{H}}(\alpha \cup \bar{p}) = \mathcal{H}$, where $\bar{p}$ corresponds to $p_k(\mathcal{M})$ via the collapse. This means that $\rho_k(\mathcal{H}) \leqslant \alpha$. As for the other inclusion, if $\beta < \alpha$ and $A \subseteq \beta$ is $\mathbf{r\Sigma}_k^{\mathcal{H}}$-definable then it's also $\mathbf{r\Sigma}_k^{\mathcal{M}}$-definable and since $\beta < \Omega = \rho_k(\mathcal{M})$, $A \in \mathcal{M}$. Acceptability then implies that $A \in \mathcal{M}|\beta^{+\mathcal{M}} = \mathcal{M}|\beta^{+\mathcal{W}} \subseteq \mathcal{M}|\alpha$, since α is a $\mathcal{W}$-cardinal. Then $A \in \mathcal{H}$ as well. Thus $\alpha = \rho_k(\mathcal{H})$.

Let $\pi : \mathcal{H} \to \mathcal{M}$ be the uncollapse and note that $\mathrm{crit}\,\pi = \alpha = \rho(\mathcal{H})$. Thus Condensation 5.1 implies that $\mathcal{H} \triangleleft \mathcal{M}$, as the ultrapower clause of condensation isn't possible as $\rho_k(\mathcal{H}) = \alpha$ is an $\mathcal{M}$-cardinal. Since $\alpha < \Omega$ we get that $\mathcal{H} \trianglelefteq \mathcal{W}$. Since $\mathcal{W}$ is a mini-universe we then get (ii).

$(ii) \Rightarrow (iii)$ is obvious. For $(iii) \Rightarrow (i)$, we just have to show that $\mathcal{M}$ is countably iterable, by definition of $S(\mathcal{W})$. But given any near k-embedding $\pi : \mathcal{H} \to \mathcal{M}$ with $\mathcal{H}$ being countable, $\mathrm{ran}\,\pi \subseteq \mathrm{cHull}_k^{\mathcal{M}}(\alpha \cup p_k(\mathcal{M}))$ holds for some $\alpha < \Omega$ because $\mathrm{ran}\,\pi$ is countable and $\mathrm{Hull}_k^{\mathcal{M}}(\Omega \cup p_k(\mathcal{M})) = \mathcal{M}$ by k-soundness. As cofinally many of these are initial segments of $\mathcal{W}$ by (iii), $\mathcal{H}$ is iterable since $\mathcal{W}$ is iterable and thus also countably iterable. ■

Lemma 7.11. *Let $\mathcal{W}$ be stably universal and $\mathcal{M}$ a sound mouse satisfying that $S(\mathcal{W}) \trianglelefteq^* \mathcal{M}$, i.e. that $S(\mathcal{W}) \trianglelefteq \mathcal{M}$ and $\mathrm{On}^{S(\mathcal{W})}$ is a cutpoint of $\mathcal{M}$. Then $\rho(\mathcal{M}) \geqslant \mathrm{On}^{S(\mathcal{W})}$.*

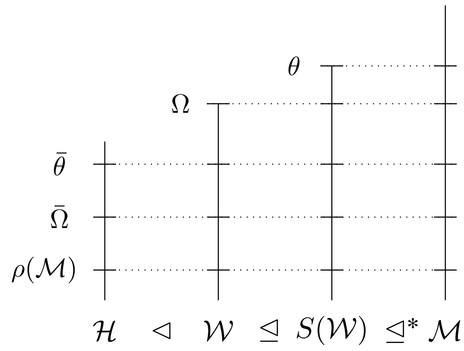
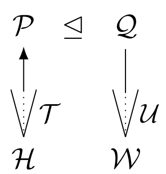


Figure 7.1: The situation in the proof of Lemma 7.11

Proof. If $\mathcal{W}$ is stable collapsing and $\rho(\mathcal{M}) < \mathrm{On}^{S(\mathcal{W})} = \Omega$ then Ω wouldn't be a cardinal in V by soundness of $\mathcal{M}$, $\lightning$. Assume thus that $\mathcal{W}$ is a mini-universe. Let $\mathcal{M}$ be a minimal counter-example to the lemma. Since Ω is the largest cardinal of $S(\mathcal{W})$, $\rho(\mathcal{M}) \leqslant \Omega$. If $\rho(\mathcal{M}) = \Omega$ then $\mathcal{M}$ is one of the mice stacked in $S(\mathcal{W})$, meaning that $S(\mathcal{W}) = \mathcal{M}$ – but the stack never has a last mouse, $\lightning$.

Thus $\rho(\mathcal{M}) < \Omega$. Fix a cardinal $\bar{\Omega}$ of $\mathcal{W}$ such that $\rho(\mathcal{M}) < \bar{\Omega} < \Omega$, which exists as $\mathcal{W}$ is a mini-universe. Let now $\pi : \mathcal{H} \cong X \prec \mathcal{M}$ be the uncollapsing map with $X \cap \Omega = \bar{\Omega}$ (see Figure 7.1). Then $\mathcal{H}$ agrees with $\mathcal{W}$ up to $\bar{\Omega}$. Let now $\mathcal{N} \in \mathcal{M}$ be a mouse in the stack $S(\mathcal{W})$, and assume that $\mathcal{N} \in X$. Let $\bar{\mathcal{N}} \in \mathcal{H}$ be the image of $\mathcal{N}$ inside the hull. Then Condensation 5.1 implies that $\bar{\mathcal{N}} \lhd \mathcal{M}$, so that $\bar{\mathcal{N}} \unlhd \mathcal{W}$. As these stacking mice have heights cofinal in $\theta := \mathrm{On}^{S(\mathcal{W})}$, the corresponding images have heights cofinal in $\bar{\theta} := \pi^{-1}(\theta)$, so that $\mathcal{H}$ and $\mathcal{W}$ agree below $\bar{\theta}$.

As both $\mathcal{H}$ and $\mathcal{W}$ are iterable and $\mathcal{H}$ is stable, $\mathcal{W}$ will win the comparison with $\mathcal{M}$ as $\mathcal{W}$ is universal; we therefore have the following coiteration.



Since $\mathcal{H}$ agrees with $\mathcal{W}$ below $\bar{\theta}$, the least agreement between the two mice is then $\geqslant \bar{\theta}$. Since θ is a cutpoint in $\mathcal{M}$ by assumption, $\bar{\theta}$ is a cutpoint in $\mathcal{H}$ by elementarity. Therefore any extender used on $\mathcal{T}$ will have critical point $> \bar{\theta}$. But

$\rho(\mathcal{H}) \leqslant \rho(\mathcal{M}) < \bar{\theta}$, so if $\mathcal{T}$ is non-trivial it will have to drop. Thus, $\mathcal{T}$ is trivial and $\mathcal{H} \trianglelefteq \mathcal{Q}$. But then $\mathcal{H} \lhd \mathcal{W}$ as $\mathrm{On}^{\mathcal{H}} < \Omega$, so the new subset of $\rho(\mathcal{M})$ is then definable over $\mathcal{H}$ by elementarity, making it an element of $\mathcal{M}$, $\lightning$. ■

Corollary 7.12. *Let $\mathcal{W}$ be stably universal. Then $L[S(\mathcal{W})] \models \mathrm{On}^{S(\mathcal{W})}$ is a cardinal.*

Proof. Write $\theta := \mathrm{On}^{S(\mathcal{W})}$. Assume the statement is false, so that we have a surjection $f : \kappa \to \theta$ lying inside $L[S(\mathcal{W})]$. Let $\alpha + 1$ be least such that $f \in J_{\alpha+1}[S(\mathcal{W})]$. Then $\alpha + 1 \in (\theta, \theta^{+L[S(\mathcal{W})]})$, so that $\rho(J_{\alpha+1}[S(\mathcal{W})]) \leqslant \theta$. But since $J_{\alpha+1}[S(\mathcal{W})]$ is a sound mouse with $S(\mathcal{W})$ as a cutpoint initial segment, $\rho(J_{\alpha+1}[S(\mathcal{W})]) \geqslant \theta$ by Lemma 7.11, so equality holds. But then θ is a $J_{\alpha+1}[S(\mathcal{W})]$-cardinal, so an $L[S(\mathcal{W})]$-cardinal as well, $\lightning$. ■

7.2 Thickness

Pseudo-K will eventually be a certain collapsed hull of our K^c-constructions. These will be defined using *thick sets*, a notion introduced by Mitchell. Recall that a μ-**club** $C \subseteq X$ is an unbounded set of X which is closed under suprema of chains of order type μ.

Definition 7.13. Let $\mathcal{W}$ be a weasel and let $C \subseteq \mathrm{On}^{S(\mathcal{W})}$. Then C is **strongly $\mathcal{W}$-thick** if

(i) $\mathrm{cof}(\mathrm{On}^{S(\mathcal{W})}) \geqslant \Omega$;
(ii) C is τ-club in $\mathrm{On}^{S(\mathcal{W})}$;
(iii) $\mathrm{cof}^{S(\mathcal{W})} \eta$ is not the critical point of a total extender over $\mathcal{W}$ from the $\mathcal{W}$-sequence, for any $\eta \in C$.

We also say that a set $\Gamma \subseteq S(\mathcal{W})$ is **$\mathcal{W}$-thick** if Γ has a strongly $\mathcal{W}$-thick subset. ○

Proposition 7.14. *Let $\mathcal{W}$ be a weasel.*

(i) The intersection of $< \Omega$-many strongly $\mathcal{W}$-thick sets is again strongly $\mathcal{W}$-thick;
(ii) If $S(\mathcal{W})$ is $\mathcal{W}$-thick then the set of all $\mathcal{W}$-thick sets is an Ω-complete filter.

Proof. This is straightforward. ■

The reason why the cofinality clause (i) was included in the thickness definition is that it turns out to be equivalent to universality of $\mathcal{W}$. To be able to prove this fact we first need to show that iterations on stable phalanxes can be "lifted" to iterations on the corresponding stacks, which is also of independent interest.

We will here be using *long* extenders E, i.e. extenders that allows $i_E(\operatorname{crit} E) < \operatorname{lh} E$. Note that all results from chapter 1 still goes through for these long extenders *except* Proposition 1.2, where we used shortness to show that E_a measures $[\operatorname{crit} E]^{|a|}$. But if we just allow these long extenders to have a *sequence* of critical points $\langle \kappa_a \mid a \in [\operatorname{lh} E]^{<\omega} \rangle$ with κ_a least such that $i_{E_a}(\kappa_a) > \operatorname{lh} E$ and E_a a measure on κ_a, this goes through without changing the validity of the rest of the "long versions" of the results in chapter 1.

Lemma 7.15. *Let $\mathcal{W}$ be a weasel in a stable phalanx Φ and $i : \mathcal{W} \to \mathcal{R}$ an iteration map from an iteration tree $\mathcal{U}$ of length $\leqslant \Omega+1$ on Φ, such that i"$\Omega \subseteq \Omega$. Let E be the long extender over $\mathcal{W}$ of length Ω from i. Then*

$$\operatorname{Ult}(S(\mathcal{W}), E) = S(\mathcal{R}).$$

Proof. If $\mathcal{W}$ is collapsing then we have to show that $\operatorname{Ult}(\mathcal{W}, E) = \mathcal{R}$, since $\mathcal{R}$ is then collapsing as well. As i is cofinal in Ω we get $\operatorname{On}^{\operatorname{Ult}(\mathcal{W},E)} = \Omega$, so since $\operatorname{lh} E = \Omega$ the factor map $k : \operatorname{Ult}(\mathcal{W}, E) \to \mathcal{R}$ is the identity, so if $\operatorname{On}^{\mathcal{R}} = \Omega$ we have $\operatorname{Ult}(\mathcal{W}, E) = \mathcal{R}$. But if $\operatorname{On}^{\mathcal{R}} > \Omega$ then $\gamma^{\mathcal{R}} \geqslant \Omega$ as Ω is a V-cardinal, so that $i(\gamma^{\mathcal{W}}) = \gamma^{\mathcal{R}}$, but i"$\Omega \subseteq \Omega$, $\lightning$.

Assume thus that $\mathcal{W}$ is a mini-universe for which the lemma fails. Let θ be sufficiently large and let $\pi : \mathcal{H} \to V_\theta$ be the uncollapse, where

$$\mathcal{H} := \operatorname{cHull}^{V_\theta}(\kappa \cup \{\Omega, \mathcal{U}, S(\mathcal{W}), \mathcal{N}, E\}).$$

where $\kappa < \Omega$ is some $\mathcal{W}$-cardinal and $\mathcal{N} \trianglelefteq S(\mathcal{R})$ is the first mouse stacked on $\mathcal{R}$ above $\operatorname{Ult}(S(\mathcal{W}), E)$. Since $\mathcal{N}$ is sound, $\operatorname{crit} \pi = \kappa = \bar{\Omega} = \rho(\bar{\mathcal{N}})$ and $\bar{\Omega} = \kappa$ is a $\mathcal{W}$-cardinal, Condensation 5.1 implies that $\bar{\mathcal{N}} \triangleleft \mathcal{N}$, so $\bar{\mathcal{N}} \trianglelefteq \mathcal{R}$ as $\operatorname{On}^{\bar{\mathcal{N}}} \leqslant \operatorname{On}^{\mathcal{R}}$. Then $\bar{\mathcal{N}} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\bar{\Omega}}$ as $\mathcal{R} | \bar{\Omega}^{+\mathcal{R}} = \mathcal{M}^{\mathcal{U}}_{\bar{\Omega}} | \bar{\Omega}^{+\mathcal{M}^{\mathcal{U}}_{\bar{\Omega}}}$ by the argument in the proof of the

Comparison Theorem 4.15 since $\bar{\Omega}$ is the critical point of the iteration $\mathcal{M}_\Omega^\mathcal{U} \to \mathcal{R}$ (which is just π by that same argument), so that $\mathcal{N} = \pi(\bar{\mathcal{N}}) \trianglelefteq \mathcal{M}_\Omega^\mathcal{U} = \mathcal{R}$ by elementarity, $\lightning$. ■

Theorem 7.16 (Stacked iterations). *Let $\mathcal{W}$ be a model of a stable phalanx Φ such that $S(\mathcal{W})$ is $\mathcal{W}$-thick and $i : \mathcal{W} \to \mathcal{R}$ an iteration map from an iteration tree of length $\leqslant \Omega + 1$ on Φ, satisfying that $i``\Omega \subseteq \Omega$. Let E be the long extender over $\mathcal{W}$ of length Ω from i. Let $i^* : S(\mathcal{W}) \to \mathrm{Ult}(S(\mathcal{W}), E)$ be the ultrapower embedding. Then*

(i) $\mathrm{Ult}(S(\mathcal{W}), E) = S(\mathcal{R})$*;*

(ii) $\{\alpha \mid i^*$ *is continuous at* $\alpha\}$ *is $\mathcal{W}$-thick;*

(iii) $\mathrm{ran}\, i^*$ *is $\mathcal{R}$-thick.*

Proof. (i) is directly by Lemma 7.15. Let $C \subseteq S(\mathcal{W})$ be strongly $\mathcal{W}$-thick, existing by assumption, and write $\theta := \mathrm{On}^{S(\mathcal{W})}$. To show (ii), let $D \subseteq \theta$ be the set of continuity points of i^* and set $\bar{D} := C \cap D$. It's clear that $\bar{D}$ satisfies every clause in the definition of strong $\mathcal{W}$-thickness, except maybe being τ-club. But note that $\alpha \in D$ if $\mathrm{cof}\, \alpha < \mathrm{crit}\, i^*$, so D is cofinal in θ. By intersecting $\bar{D}$ with the set of limit points in D we then get a τ-club.

To show (iii), define $\bar{C} := i^*``\bar{D}$, which we will show is strongly $\mathcal{W}$-thick. Again we only need to worry about τ-clubness, where τ-closedness is directly by definition of C and D and it's cofinal in $\mathrm{On}^{S(\mathcal{R})}$ by (i) and as i^* is cofinal. ■

Theorem 7.17. *For a mini-universe $\mathcal{W}$, $\mathcal{W}$ is universal iff* $\mathrm{cof}\, \mathrm{On}^{S(\mathcal{W})} \geqslant \Omega$.

Proof. Assume first that $\mathcal{W}$ is *not* universal and let $\mathcal{M}$ be a stable mouse witnessing this, so that we have the coiteration:

$$\begin{tikzcd} \mathcal{P} \arrow[r, phantom, "\lhd"] & \mathcal{Q} \\ \mathcal{W} \arrow[u, "i"] \arrow[u, phantom, "\mathcal{T}"'] & \mathcal{M} \arrow[u, dash, "\mathcal{U}"] \end{tikzcd}$$

Let $i^* : S(\mathcal{W}) \to S(\mathcal{P})$ be the extension of i as in Theorem 7.16.

Claim 7.17.1. $S(\mathcal{P}) \trianglelefteq \mathcal{Q}$.

Proof of claim. Assume it's not the case, so that we get some sound $\mathcal{N}$ such that $\mathcal{P} \trianglelefteq \mathcal{N}$ and $\rho(\mathcal{N}) = \Omega$, which isn't an initial segment of $\mathcal{Q}$. Let $\pi : \mathcal{H} \to V_\theta$ be the uncollapse with $\mathcal{H} := \mathrm{cHull}^{V_\theta}(\kappa \cup \{\Omega, \mathcal{T}, \mathcal{U}, \mathcal{N}\})$ for some $\mathcal{N}$-cardinal $\kappa < \Omega$ and some sufficiently large θ. Assume for notational simplicity that $\mathcal{T}$ has been padded, so that $\mathrm{lh}\,\mathcal{T} = \mathrm{lh}\,\mathcal{U} = \Omega + 1$. Here $\mathrm{lh}\,\mathcal{U} = \Omega + 1$ since otherwise we wouldn't have that $\mathrm{On}^{\mathcal{Q}} > \Omega$, by stability of $\mathcal{M}$ – of course, here we use that $\mathrm{lh}\,\mathcal{U} \leqslant \Omega + 1$ as it's a coiteration of stable mice, see Proposition 7.7. Then as in the proof of the Comparison Theorem 4.15 we get that

$$\mathcal{M}^{\mathcal{U}}_{\bar{\Omega}} | \bar{\Omega}^{+\mathcal{M}^{\mathcal{U}}_{\bar{\Omega}}} = \mathcal{M}^{\mathcal{T}}_{\bar{\Omega}} | \bar{\Omega}^{+\mathcal{M}^{\mathcal{T}}_{\bar{\Omega}}} = \mathcal{P} | \bar{\Omega}^{+\mathcal{P}}.$$

But then $\bar{\mathcal{N}} \lhd \mathcal{N}$ by Condensation 5.1, using that $\mathcal{N}$ is sound and $\rho(\bar{\mathcal{N}}) = \kappa$ is an $\mathcal{N}$-cardinal, and thus also that $\bar{\mathcal{N}} \trianglelefteq \mathcal{P}$. As $\mathrm{On}^{\bar{\mathcal{N}}} < \bar{\Omega}^{+\mathcal{P}}$, the previous equalities entail that $\bar{\mathcal{N}} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\bar{\Omega}}$. But then $\mathcal{N} = \pi(\bar{\mathcal{N}}) \trianglelefteq \pi(\mathcal{M}^{\mathcal{U}}_{\bar{\Omega}}) = \mathcal{M}^{\mathcal{U}}_{\Omega} = \mathcal{Q}$, contradicting the choice of $\mathcal{N}$. $\dashv$

We now also claim that $S(\mathcal{P}) = \mathcal{Q} | \Omega^{+\mathcal{Q}}$. Indeed, if it weren't the case then $S(\mathcal{P}) \lhd \mathcal{Q} | \Omega^{+\mathcal{Q}}$ and since $\mathcal{Q} | \Omega^{+\mathcal{Q}}$ projects to Ω and is sound, it would contradict the maximality of the stack $S(\mathcal{P})$.

Why? This is probably false..

Now let $\alpha < \Omega$ be large enough so that $i^{\mathcal{U}}_{\alpha}$ is defined and let $\mu \in \mathrm{On}^{\mathcal{M}^{\mathcal{U}}_{\alpha}}$ be such that $i^{\mathcal{U}}_{\alpha}(\mu) = \Omega$, which exists as the critical points converge to Ω and we're in the direct limit $\mathcal{Q}$. Then $i^{\mathcal{U}}_{\alpha}$ maps $\mu^{+\mathcal{M}^{\mathcal{U}}_{\alpha}}$ cofinally into $\Omega^{+\mathcal{Q}} = \mathrm{On}^{S(\mathcal{P})}$, so that $\mathrm{cof}\,\mathrm{On}^{S(\mathcal{P})} < \Omega$. But i^* maps $\mathrm{On}^{S(\mathcal{W})}$ cofinally into $\mathrm{On}^{S(\mathcal{P})}$, so

$$\mathrm{cof}\,\mathrm{On}^{S(\mathcal{W})} \leqslant \mathrm{cof}\,\mathrm{On}^{S(\mathcal{P})} < \Omega.$$

Assume conversely that $\mathcal{W}$ is a universal mini-universe. By the $\Leftarrow$ direction we just proved, along with Theorem 7.22 we get that K^c_τ is universal. Furthermore, by Theorem 7.16 and the Stacking Lemma 7.8, the coiteration between $\mathcal{W}$ and K^c_τ is really also a coiteration between $S(\mathcal{W})$ and $S(K^c_\tau)$, so by universality we get the following comparison:

$$\begin{tikzcd} S(\mathcal{Q}) \arrow[r, phantom, "="] & S(\mathcal{Q}) \\ S(\mathcal{W}) \arrow[u, "i^*"', "\mathcal{T}"'] & S(K^c_\tau) \arrow[u, "j^*"', "\mathcal{U}"'] \end{tikzcd}$$

By continuity of i^* and j^* at $\mathrm{On}^{S(\mathcal{W})}$ and $\mathrm{On}^{S(K^c_\tau)}$ we then get that

$$\Omega \leqslant \mathrm{cof}\ \mathrm{On}^{S(K^c_\tau)} = \mathrm{cof}\ \mathrm{On}^{S(\mathcal{Q})} = \mathrm{cof}\ \mathrm{On}^{S(\mathcal{W})},$$

where $\mathrm{cof}\ \mathrm{On}^{S(\mathcal{Q})} \leqslant \mathrm{cof}\ \mathrm{On}^{S(K^c_\tau)}$ and $\mathrm{cof}\ \mathrm{On}^{S(\mathcal{Q})} \leqslant \mathrm{cof}\ \mathrm{On}^{S(\mathcal{W})}$ holds because if $f : \zeta \to \mathrm{On}^{S(\mathcal{W})}$ is cofinal then $i^{*}``f : i^{*}``\zeta \to \mathrm{On}^{S(\mathcal{Q})}$ is cofinal, so $\mathrm{cof}\ \mathrm{On}^{S(\mathcal{Q})} \leqslant \zeta$ as witnessed by both f and $i^{*}``f$, and analogously for K^c_τ. ■

Not entirely satisfied with this explanation.

Definition 7.18. A robust K^c-construction is **maximal** if it's defined by clause (iia) in Definition 6.5 whenever possible. That is, we put robust extenders on top of our construction whenever we can. ○

Definition 7.19. Let X be any set. Then a robust K^c-construction **forbids critical points of cofinality in** X if $\mathrm{cof}^V(\dot\kappa^{\mathcal{N}_\gamma}) \notin X$ for every $\mathcal{N}_\gamma$ in the construction. ○

We write $K^c(X)$ for the last model of a maximal K^c-construction forbidding critical points of cofinality in X, of longest length possible. Also, for readibility we set $K^c := K^c(\varnothing)$ and $K^c_{x_1,\dots,x_n} := K^c(\{x_1,\dots,x_n\})$.

We now want to use these K^c-constructions to ensure that there exists a weasel $\mathcal{W}$ such that $S(\mathcal{W})$ is $\mathcal{W}$-thick. We have two cases, whether or not K^c_τ is a mini-universe or collapsing. Starting with the mini-universe case, we'll need the following definition.

Definition 7.20. Let $\mathcal{M}$ be an active premouse. Then $\mathcal{M}$ is **hull-certified by** π if

(i) $\pi : \mathcal{H} \to H_\xi$ is fully elementary, $\mathcal{H}$ is transitive, $\mathcal{H}$ and H_ξ are both closed under ω-sequences and $\mathcal{M}|\kappa^{+\mathcal{M}} \in \mathcal{H}$;

(ii) $(\dot F^{\mathcal{M}} \restriction \dot\nu^{\mathcal{M}})^* = E_\pi \cap \mathcal{M}$. ○

Lemma 7.21. *Any hull-certified $\mathcal{M}$ is robust.*

"Proof". See Lemma 2.6 in (Jensen et al., 2009). ■

Theorem 7.22. *If K^c_τ is a mini-universe then $S(K^c_\tau)$ is K^c_τ-thick.*

Proof sketch. Let $\theta := \mathrm{On}^{S(K^c_\tau)}$ and define $C := \{\alpha < \theta \mid \operatorname{cof}\alpha = \tau\}$; we claim that C is strongly K^c_τ-thick. As the K^c-construction forbids critical points of cofinality τ, clause (iii) of strong K^c_τ-thickness is satisfied. As C clearly is τ-club, it remains to show that $\operatorname{cof}\theta \geqslant \Omega$. This is proven as in Theorem 3.4 of (Jensen et al., 2009), replacing their "certified by a collapse" with our "hull-certified" everywhere. This is where we use the assumption that $\tau^{++} < \Omega$, so that we have two regular cardinals that *are* allowed as cofinalities of critical points in the K^c-construction. ■

We now prove the collapsing case.

Theorem 7.23. *Assume there exists a collapsing weasel $\mathcal{W}$ and set $\eta := \operatorname{cof}^V \gamma^{\mathcal{W}}$. Then $K^c_{\tau,\eta}$ is stable collapsing and Ω is $K^c_{\tau,\eta}$-thick.*

Proof. We can without loss of generality assume that $\mathcal{W}$ is stable. Indeed, if it weren't then $\eta^{\mathcal{W}}$ is measurable in $\mathcal{W}$, so letting $\mathcal{W}^* := \mathrm{Ult}(\mathcal{W},\mu)|\Omega$ with μ the order zero measure on $\eta^{\mathcal{W}}$, $\mathcal{W}^*$ is a stable collapsing weasel such that $\operatorname{cof}^V \gamma^{\mathcal{W}^*} = \operatorname{cof}^V \gamma^{\mathcal{W}}$ by Proposition 7.3, so we can take $\mathcal{W}$ to be $\mathcal{W}^*$ in this case.

If $K^c_{\tau,\eta}$ is a mini-universe then it has to be universal by Theorems 7.22 and 7.17, contradicting Proposition 7.3. Thus $K^c_{\tau,\eta}$ is collapsing and it's also stable since $\operatorname{cof}^V \gamma^{K^c_{\tau,\eta}} = \eta$ by Proposition 7.3 and critical points of V-cofinality η isn't allowed in the construction of $K^c_{\tau,\eta}$. Let now

$$C := \{\alpha < \mathrm{On}^{S(K^c_{\tau,\eta})} \mid \operatorname{cof}\alpha = \tau\},$$

which is clearly strongly $K^c_{\tau,\eta}$-thick, using that $S(K^c_{\tau,\eta}) = K^c_{\tau,\eta}$ and Ω is regular. ■

Corollary 7.24. *There is a stably universal weasel $\mathcal{W}$ such that $S(\mathcal{W})$ is $\mathcal{W}$-thick.*

Proof. Directly by Theorems 7.22 and 7.23. ■

7.3 Definition of Pseudo-K

Definition 7.25. Assume $S(\mathcal{W})$ is $\mathcal{W}$-thick. Then define

$$\mathrm{Def}^{\mathcal{W}} := \bigcap\{\mathrm{Hull}^{S(\mathcal{W})}(\Gamma) \mid \Gamma \text{ is } \mathcal{W}\text{-thick}\}. \qquad \circ$$

Lemma 7.26. *If $S(\mathcal{W})$ is $\mathcal{W}$-thick and $S(\mathcal{R})$ is $\mathcal{R}$-thick,* $(\mathrm{Def}^{\mathcal{W}}, \in) \cong (\mathrm{Def}^{\mathcal{R}}, \in)$.

Proof. Compare $\mathcal{W}$ with $\mathcal{R}$, granting iteration trees on $\mathcal{W}, \mathcal{R}$ with last models $\mathcal{P}, \mathcal{Q}$. As both $\mathcal{W}$ and $\mathcal{R}$ are universal, $\mathcal{P} = \mathcal{Q}$ and neither branch drops, giving us iteration maps $i : \mathcal{W} \to \mathcal{P}$ and $j : \mathcal{R} \to \mathcal{P}$ and thus also the canonical extensions $i^* : S(\mathcal{W}) \to S(\mathcal{P})$ and $j^* : S(\mathcal{R}) \to S(\mathcal{P})$ by Theorem 7.16. Now, given any $\mathcal{P}$-thick Γ, $\Gamma \cap \operatorname{ran} i^* \cap \operatorname{ran} j^*$ is $\mathcal{P}$-thick as well by Theorem 7.16 again, so we get that $i^{*}\text{"}\,\mathrm{Def}^{\mathcal{W}} = \mathrm{Def}^{\mathcal{Q}} = j^{*}\text{"}\,\mathrm{Def}^{\mathcal{R}}$ ■

Definition 7.27. **Pseudo-K**, $\tilde{K}(\tau, \Omega)$, is the common transitive collapse of all $\mathrm{Def}^{\mathcal{W}}$, for $\mathcal{W}$ such that $S(\mathcal{W})$ is $\mathcal{W}$-thick. ○

8 K below a Woodin

Having defined pseudo-K we want to find a way to stitch together various choices of pseudo-K to form a proper class premouse K, satisfying various canonicity properties as well as being "close to V". More specifically, we will have the following theorem.

Theorem 8.1. *There are Σ_2 formulae $\psi_K(v)$ and $\psi_\Sigma(v)$ such that*

(i) $K = \{v \mid \psi_K[v]\}$ *is a transitive proper class premouse satisfying* ZFC*;*

(ii) $\{v \mid \psi_\Sigma[v]\}$ *is the unique iteration strategy for K acting on set-sized iteration trees;*

(iii) (Generic absoluteness) $\psi_K^V = \psi_K^{V[G]}$ and $\psi_\Sigma^V = \psi_\Sigma^{V[G]} \cap V$ for any V-generic G over a set-sized poset;

(iv) (Inductive definition) $K|\omega_1^V$ is Σ_1-definable over $J_{\omega_1}(\mathbb{R})$;

(v) (Weak covering) For any $\lambda \geqslant \omega_2^V$ which is a successor cardinal of K, $\operatorname{cof}\lambda \geqslant |\lambda|$*. Thus $\alpha^{+K} = \alpha^+$ whenever α is a singular cardinal of V.*

8.1 The hull property

In our definition of K we will need a certain property called *the hull property.* We first show that thick hulls of weasels can be lifted to their stacks, in the same manner as iterations could be lifted to their stacks as well.

Lemma 8.2 (Stacked hulls). *Let $\mathcal{W}$ be a weasel, Γ be $\mathcal{W}$-thick, $\mathcal{H} := \operatorname{cHull}^{S(\mathcal{W})}(\Gamma)$ and $\pi : \mathcal{H} \to S(\mathcal{W})$ the uncollapse. Then*

(i) $\operatorname{Hull}^{S(\mathcal{W})}(\Gamma)$ *is cofinal in Ω;*

(ii) $\mathcal{H} = S(\mathcal{H}|\Omega)$*;*

(iii) $\{\alpha < \mathrm{On}^{\mathcal{H}} \mid \pi$ *is continuous at* $\alpha\}$ *is* $\mathcal{H}|\Omega$*-thick;*

(iv) $\mathcal{H}|\Omega$ *is universal.*

PROOF. (i): This is clear if $\mathcal{W}$ is collapsing, so assume it's a mini-universe. Assuming $\mathrm{Hull}^{S(\mathcal{W})}(\Gamma)$ is bounded in Ω, $\mathcal{H}|\Omega$ is a collapsing weasel as otherwise there would be $S(\mathcal{W})$-cardinals above Ω, $\lightning$.. But $\mathcal{W}$ is a universal mini-universe by Theorem 7.17, contradicting Proposition 7.3.

(ii): We first show that $\mathcal{H} \trianglelefteq S(\mathcal{H}|\Omega)$. Let $\mathcal{N} \lhd \mathcal{H}$ be such that $\mathcal{H}|\Omega \trianglelefteq \mathcal{N}$. To show that $\mathcal{H} \trianglelefteq S(\mathcal{H}|\Omega)$ it suffices to show that $\mathcal{N}$ is sound and projects to Ω. It is sound simply because $\mathcal{H}$ is a premouse, so assume that $\mathcal{N}$ is least such that it doesn't project to Ω. This makes $\mathcal{N}$ definable, so that $\mathcal{N} \in \mathcal{H}$. Then $\pi(\mathcal{N}) \lhd S(\mathcal{W})$, making $\pi(\mathcal{N})$ a sound premouse projecting to Ω. Since $\pi(\Omega) = \Omega$ as $\mathrm{Hull}^{S(\mathcal{W})}(\Gamma)$ is cofinal in Ω, we get by elementarity that $\mathcal{N}$ projects to Ω, $\lightning$. Thus $\mathcal{H} \trianglelefteq S(\mathcal{H}|\Omega)$.

For the converse, assume that $\mathcal{H} \lhd S(\mathcal{H}|\Omega)$ and let $\mathcal{P}$ be the least mouse stacked onto $\mathcal{H}|\Omega$ satisfying that $\mathcal{P}$ is not an initial segment of $\mathcal{H}$. Say $\mathcal{P}$ is k-projecting to Ω and set

$$\mathcal{Q} := \mathrm{Ult}_k(\mathcal{P}, E_\pi \restriction \Omega)$$

and note that we have $\rho_k(\mathcal{Q}) = \Omega$ since $\pi(\Omega) = \Omega$, and $S(\mathcal{W}) \lhd \mathcal{Q}$ because Γ is cofinal in $\mathrm{On}^{S(\mathcal{W})}$ (as then any $\alpha > \mathrm{On}^{\mathcal{H}}$ in $\mathcal{P}$ has to be sent to something $> \mathrm{On}^{S(\mathcal{W})}$ by monotonicity). But now $\mathrm{cHull}_k^{\mathcal{P}}(\alpha \cup p_k(\mathcal{P})) \trianglelefteq \mathcal{H}|\Omega$ holds for club many $\alpha < \Omega$ by Proposition 7.10, so intersecting this club with the fixpoints of π we get club many $\alpha < \Omega$ satisfying that $\mathrm{cHull}_k^{\mathcal{Q}}(\alpha \cup p_k(\mathcal{Q})) \trianglelefteq \mathcal{W}$, so $\mathcal{Q} \trianglelefteq S(\mathcal{W})$ by Proposition 7.10, $\lightning$.

(iii) is clear and (iv) is by (iii) and Theorem 7.17. ■

DEFINITION 8.3. Let $S(\mathcal{W})$ be $\mathcal{W}$-thick and let $\alpha < \Omega$. Then $\mathcal{W}$ has the **hull property at** α if for every $\mathcal{W}$-thick Γ it holds that $\mathcal{P}^{\mathcal{W}}(\alpha) \subseteq \mathrm{cHull}^{S(\mathcal{W})}(\Gamma \cup \alpha)$. ∘

Note that the hull property is trivial if $\mathcal{W}$ is collapsing, since $\mathrm{cHull}^{S(\mathcal{W})}(\Gamma \cup \alpha) = \mathcal{W}$ as $|\Gamma| = \Omega = \mathrm{On}^{\mathcal{W}}$ in this case.

Lemma 8.4. *If $S(\mathcal{W})$ is $\mathcal{W}$-thick then $\mathcal{W}$ has the hull property at club many $\alpha < \Omega$.*

Proof. This is trivial if $\mathcal{W}$ is collapsing, so assume that it's a mini-universe. Since Ω isn't Woodin in $L[S(\mathcal{W})]$ by our global assumption, we can pick $A \in S(\mathcal{W})$ least such that no $\kappa < \Omega$ is A-reflecting in Ω inside $L[S(\mathcal{W})]$.

This means that given any $\kappa < \Omega$ there is some least $\lambda_\kappa < \Omega$ such that for any $L[S(\mathcal{W})]$-extender E with $\operatorname{crit} E = \kappa$ and $i_E(\kappa) > \lambda_\kappa$ it holds that $i_E(A) \cap \lambda_\kappa \neq A \cap \lambda_\kappa$. Since extenders on the $\mathcal{W}$-sequence are in particular $L[S(\mathcal{W})]$-extenders, we will restrict ourselves to these. We claim that

$$\Lambda := \{\alpha < \Omega \mid \forall \kappa < \alpha : \text{ if } E \text{ is on the } \mathcal{W}\text{-sequence with} \operatorname{crit} E = \kappa \text{ and } \alpha < i_E(\kappa) \text{ then } i_E(A) \cap \alpha \neq A \cap \alpha\}$$

is unbounded in Ω. To see this, let $\eta_0 < \Omega$ be arbitrary and for $n+1 < \omega$ set $\eta_{n+1} := \sup_{\kappa < \eta_n} \lambda_\kappa$, which is below Ω by regularity. Then $\sup_{n<\omega} \eta_n \in \Lambda$, so Λ is unbounded. Intersecting Λ with its limit points we wind up with a club, and intersecting this club with all $\mathcal{W}$-cardinals we get a club $C \subseteq \Lambda$, using that $\mathcal{W}$ is a mini-universe.

We claim that $\mathcal{W}$ has the hull property at every $\alpha \in C$. Let thus Γ be $\mathcal{W}$-thick – we want to show that $\mathcal{P}^{\mathcal{W}}(\alpha) \subseteq \operatorname{cHull}^{S(\mathcal{W})}(\Gamma \cup \alpha)$. Let $\mathcal{H}$ be such that $\operatorname{cHull}^{S(\mathcal{W})}(\Gamma \cup \alpha) = S(\mathcal{H})$, which exists by Lemma 8.2. Note that A is definable over $S(\mathcal{W})$, so that $A \in \operatorname{ran} \pi$ where $\pi : S(\mathcal{H}) \to S(\mathcal{W})$ is the uncollapse.

If $\alpha \in \operatorname{Hull}^{S(\mathcal{W})}(\Gamma \cup \alpha)$ then α is an $\mathcal{H}$-cardinal by elementarity, and otherwise it's still an $\mathcal{H}$-cardinal, being the critical point of π. As we furthermore have that $\mathcal{W}$ agrees with $\mathcal{H}$ below α, $\Phi := (\langle \mathcal{W}, \mathcal{H} \rangle, \langle \alpha \rangle)$ is a phalanx. Comparing Φ with $\mathcal{W}$ we get iteration trees $\mathcal{T}, \mathcal{U}$ with last models $\mathcal{P}$, $\mathcal{Q}$.

We now want to do the Dodd-Jensen trick 5.2, but this requires $\mathcal{W}$ to be countable. But under our global assumption we get that $\mathcal{W}$ has a unique iteration strategy, so that any tree according to the strategy would be unambiguous. This means that we can execute the Dodd-Jensen trick 5.2 using the "strong" Dodd-Jensen 4.25 and get that $\mathcal{P}$ lies above $\mathcal{H}$ in $\mathcal{T}$, the $\mathcal{H}$-to-$\mathcal{P}$ branch doesn't drop giving an iteration map $i : \mathcal{H} \to \mathcal{P}$ with $\operatorname{crit} i \geqslant \alpha$, and lastly that $\mathcal{P} \leqslant \mathcal{Q}$. The coiteration thus looks like the following:

$$\begin{tikzcd} \mathcal{P} \arrow[r, phantom, "\trianglelefteq"] & \mathcal{Q} \arrow[d, "\mathcal{U}"] \\ \mathcal{H} \arrow[u, "i"'] \arrow[u, dashed, "\mathcal{T}"] & \mathcal{W} \end{tikzcd}$$

But as $\mathcal{H}$ is universal by Lemma 8.2 we get that $\mathcal{P} = \mathcal{Q}$, so that the $\mathcal{W}$-to-$\mathcal{Q}$ branch doesn't drop either and we get an iteration map $j : \mathcal{W} \to \mathcal{Q}$. Extend now i and j to $i^* : S(\mathcal{H}) \to S(\mathcal{P})$ and $j^* : S(\mathcal{W}) \to S(\mathcal{P})$ by using Theorem 7.16.

If $\operatorname{crit} j^* < \alpha$ then the first extender E used by j^* witnesses that $\operatorname{crit} j^*$ is A-reflecting up to α in $\mathcal{W}$ since $\operatorname{lh} E \geqslant \alpha$ and that E is short, $\lightning$. So $\operatorname{crit} j^* \geqslant \alpha$, but then $\mathcal{P}^{\mathcal{W}}(\alpha) = \mathcal{P}^{\mathcal{P}}(\alpha) = \mathcal{P}^{\mathcal{H}}(\alpha) \subseteq S(\mathcal{H})$, as wanted. ∎

It should be noted that in the above proof we made essential use of our global assumption that there's no proper class inner model with a Woodin cardinal.

8.2 The thick trick

In the following sections we will need an analogue of the Dodd-Jensen trick 5.2 in comparison arguments, when we don't necessarily have an embedding between our premice in question available. This will result in "the thick trick", involving universality and thickness arguments, which is taken from (Steel, 1996). A key property of pseudo-K that we'll need to execute this trick is the following.

Lemma 8.5. *$\tilde{K}(\tau, \Omega)$ has ordinal height $\geqslant \tau$.*

Proof. Note that by Proposition 7.3, either every $\mathcal{W}$ used in $\tilde{K}(\tau, \Omega)$ is collapsing or else every $\mathcal{W}$ used in $\tilde{K}(\tau, \Omega)$ is a mini-universe, since they're always universal by Theorem 7.17. Assume first that we're in the collapsing case and let $\mathcal{W}$ be such that $S(\mathcal{W})$ is $\mathcal{W}$-thick.

For every $\xi < \gamma^{\mathcal{W}}$ let Γ_ξ be strongly $\mathcal{W}$-thick such that $\xi \notin \operatorname{Hull}^{\mathcal{W}}(\Gamma_\xi)$ if such a Γ_ξ exists, and otherwise let $\Gamma_\xi := \Omega$. Set $\Gamma := \bigcap_{\xi < \gamma^{\mathcal{W}}} \Gamma_\xi$, so that Γ is strongly $\mathcal{W}$-thick and that our construction of the Γ_ξ ensures that

$$\operatorname{Hull}^{\mathcal{W}}(\Gamma) \cap \gamma^{\mathcal{W}} = \operatorname{Def}^{\mathcal{W}} \cap \gamma^{\mathcal{W}}. \tag{1}$$

We then claim that $\mathrm{Hull}^{\mathcal{W}}(\Gamma) = \mathrm{Def}^{\mathcal{W}}$. To show this we need to show that $\mathrm{Hull}^{\mathcal{W}}(\Gamma) \subseteq \mathrm{Hull}^{\mathcal{W}}(\Lambda)$ for every strongly $\mathcal{W}$-thick $\Lambda \subseteq \Gamma$. Let thus $\xi \in \mathrm{Hull}^{\mathcal{W}}(\Gamma)$. We can then find a function $f \in \mathrm{Hull}^{\mathcal{W}}(\Lambda)$ with $\operatorname{dom} f = \gamma^{\mathcal{W}}$ such that $\xi \in \operatorname{ran} f$ (as otherwise $\mathcal{W}$ would have a cardinal $> \gamma^{\mathcal{W}}$), so that $\xi = f(\mu)$ with $\mu \in \mathrm{Hull}^{\mathcal{W}}(\Gamma) \cap \gamma^{\mathcal{W}} = \mathrm{Def}^{\mathcal{W}} \cap \gamma^{\mathcal{W}}$ using (1), implying that $\mu \in \mathrm{Hull}^{\mathcal{W}}(\Lambda)$ and then $\xi \in \mathrm{Hull}^{\mathcal{W}}(\Lambda)$ as well, showing $\mathrm{Hull}^{\mathcal{W}}(\Gamma) = \mathrm{Def}^{\mathcal{W}}$. This means that $\Omega \subseteq \tilde{K}(\tau, \Omega)$, which is more than we claimed.

Assume now that we're in the mini-universe case. In particular $\mathcal{W} := K^c_\tau$ is a mini-universe as well by Theorem 7.22. Assume towards a contradiction that

$$\mathrm{Def}^{\mathcal{W}} \cap \Omega \text{ has order-type } \beta < \tau. \tag{2}$$

As before, we can construct a strongly $\mathcal{W}$-thick set Γ_0 such that $\mathrm{Hull}^{S(\mathcal{W})}(\Gamma_0) \cap \beta = \mathrm{Def}^{\mathcal{W}} \cap \beta$. Let b_0 be the least element of the set $(\mathrm{Hull}^{S(\mathcal{W})}(\Gamma_0) \cap \Omega) - \mathrm{Def}^{\mathcal{W}}$. Now pick a decreasing sequence $\langle \Gamma_\xi \mid \xi < \Omega \rangle$ of strongly $\mathcal{W}$-thick sets such that letting b_ξ be the least ordinal in $\mathrm{Hull}^{S(\mathcal{W})}(\Gamma_\xi) - \mathrm{Def}^{\mathcal{W}}$ for every $\xi < \Omega$, we have that $\xi < \gamma \Rightarrow b_\xi < b_\gamma$ for every $\xi, \gamma < \Omega$. Such a sequence can be constructed by setting $\Gamma_{\xi+1} := \Gamma_\xi - \{b_\xi\}$. Indeed, if $b_\xi \geqslant \tau$ then $\Gamma_{\xi+1}$ is still strongly $\mathcal{W}$-thick as then b_ξ has to be a successor since it otherwise wouldn't be least, and if $b_\xi < \tau$ then it's okay to remove it, even if it's a limit (as we're only working with τ-clubs).

Claim 8.5.1 (Mitchell). There's no $\xi < \Omega$ such that $b_\gamma < \xi$ for every $\gamma < \xi$ and $\xi \in \mathrm{Hull}^{S(\mathcal{W})}(\xi \cup \Gamma_{\xi+1})$.

Proof of claim. Assume ξ is an ordinal with these properties. Then pick $c \in \xi$, $d \in \Gamma_{\xi+1}^{<\omega}$ and a Skolem term σ such that $\xi = \sigma^{\mathcal{W}}[c, d]$. Since ξ is closed under the b_γ's by assumption, we can find some $\gamma < \xi$ such that $b_\gamma < \xi$ and $c < b_\gamma$, so that

$$\mathrm{Hull}^{S(\mathcal{W})}(\Gamma_\gamma) \models \exists c < b_\gamma : \sigma[c, d] \in (b_\gamma, b_{\xi+1}), \tag{3}$$

noting that $\Gamma_{\xi+1} \subseteq \Gamma_\gamma$, so that $b_\gamma, b_{\xi+1} \in \mathrm{Hull}^{S(\mathcal{W})}(\Gamma_\gamma)$. But the witness e to the existential quantifier in (3) is then in $\mathrm{Hull}^{S(\mathcal{W})}(\Gamma_\gamma) \cap b_\gamma$ and thus in $\mathrm{Def}^{\mathcal{W}}$ by definition of b_γ. This means in particular that $\sigma^{\mathcal{W}}[e, d] \in \mathrm{Hull}^{S(\mathcal{W})}(\Gamma_{\xi+1})$. (3) also implies that $\sigma^{\mathcal{W}}[e, d] > b_\gamma$ and $\sigma^{\mathcal{W}}[e, d] \in \mathrm{Hull}^{S(\mathcal{W})}(\Gamma_\gamma)$, which means by definition of b_γ that $\sigma^{\mathcal{W}}[e,d] \notin \mathrm{Def}^{\mathcal{W}}$. But then $\sigma^{\mathcal{W}}[e,d]$ contradicts the minimality of $b_{\xi+1}$. ⊣

Define now the following sets:

$$C_1 := \{\nu < \Omega \mid \mathcal{W} \text{ has the hull property at } \nu\};$$
$$C_2 := \{\nu < \Omega \mid \operatorname{cof} \nu = \tau^+\};$$
$$C_3 := \{\nu < \Omega \mid \forall \xi < \nu : b_\xi < \nu \wedge \nu \notin \mathrm{Hull}^{S(\mathcal{W})}(\nu \cup \Gamma_{\nu+1}\}.$$

It's clear that C_1 is club and C_2 is τ^+-club. To show that C_3 is club, the above claim implies that we only need to show that $\forall \xi < \nu : b_\xi < \nu$ holds for club many $\nu < \Omega$. But given any $\eta < \Omega$ pick some $\nu_0 := b_\gamma > \eta$. Then recursively set $\nu_{n+1} := b_{\nu_n}$, so that $\nu := \sup_n \nu_n$ has the property that $b_\nu = \nu$, so that $\nu \in C_3$. As it's clear that C_3 is closed, it's club.

Define then the τ^+-club $C := C_1 \cap C_2 \cap C_3$. For $\nu \in C$, let

$$\sigma_\nu : \mathcal{N}_\nu := \mathrm{cHull}^{S(\mathcal{W})}(\nu \cup \Gamma_{\nu+1}) \to \mathcal{W}$$

be the uncollapse and let E_ν be the $(\nu, \sigma_\nu(\nu))$-extender derived from σ_ν, where $\nu = \operatorname{crit} \sigma_\nu$ as $\nu \notin \mathrm{Hull}^{S(\mathcal{W})}(\nu \cup \Gamma_{\nu+1})$. Note that since $\mathcal{W}$ has the hull property at ν, E_ν measures all subsets of ν in $\mathcal{W}$.

Here E_ν is on the $\mathcal{W}$-sequence *if* it's robust because $\mathcal{W} = K^c_\tau$ is maximal, the next extender is unique (why is it unique? Couldn't the bicephalus argument fail if E_ν and $\dot{E}^{\mathcal{W}}_\nu$ are of type I/III and II?) and $\operatorname{cof}^V \nu = \tau^+ \neq \tau$, so it's legal, and E_ν coheres with $\mathcal{W}$.

Claim 8.5.2. For every $\nu \in C$ there is a $\beta \leqslant \sigma_\nu(\nu)$ such that $E_\nu \upharpoonright \beta$ is not of type Z and isn't robust.

Proof of claim. E_ν cannot be on the $\mathcal{W}$-sequence because then there's a Shelah limit of Shelahs in $\mathcal{W}$ as $\mathrm{lh}\, E_\nu = \sigma_\nu(\nu)$. The only way this can happen is if E_ν isn't robust, so there is a least $\beta \leqslant \mathrm{lh}\, E_\nu$ such that $E_\nu \upharpoonright \beta$ isn't type Z and isn't robust with respect to $\mathcal{W} | \beta$. ⊣

Let β_ν be the least such β witnessing the claim for ν. For every $\nu \in C$, pick a witness $\mathcal{U}_\nu$ to the non-robustness of $F_\nu \upharpoonright \beta_\nu$ with respect to $\mathcal{W} | \beta_\nu$. This means exactly that $\mathcal{U}_\nu$ is a countable subset of $\mathcal{W} | \beta_\nu$ and there exists no $\pi : \mathcal{U}_\nu \to \mathcal{W} | \nu$ such that, setting $\beta := \sup(\mathcal{U}_\nu \cap \beta_\nu)$ and $\bar{\beta} := \sup \pi``\beta$, we have that

(i) $\pi \restriction \mathcal{U}_\nu \cap \nu = \mathrm{id}$;

(ii) $\mathrm{Sat}(\mathcal{U}_\nu) = \mathrm{Sat}(\mathcal{U}_\nu, \pi)$;

(iii) For every $a \in [\mathcal{U}_\nu \cap \beta_\nu]^{<\omega}$ and $x \subseteq [\nu]^{|a|}$ with $x \in \mathcal{U}_\nu$, it holds that $a \in \sigma_\nu(x)$ iff $\pi(a) \in x$.

Since $\forall \alpha < \Omega : \alpha^\omega < \Omega$, we can simultaneously fix ω many regressive $f : \Omega \to \Omega$ on a τ^+-stationary set. For every $\nu \in C$ fix an enumeration Why does this hold?

$$\nu \cap \mathcal{U}_\nu = \langle \xi_\nu(n) \mid n < \omega \rangle.$$

Define then the functions $f_n : C \to \Omega$ given as $f_n(\nu) := \xi_\nu(n)$ and note that every f_n is regressive, so we then get a τ^+-stationary set $S_0 \subseteq C$ such that for some y_n, $n < \omega$, it holds that $\nu \cap \mathcal{U}_\nu = \{y_n \mid n < \omega\}$ for every $\nu \in S_0$. Pick now enumerations

$$\begin{aligned} \mathcal{U}_\nu &= \langle z_n^\nu \mid n < \omega \rangle \\ [\mathcal{U}_\nu \cap \beta_\nu]^{<\omega} &= \langle a_n^\nu \mid n < \omega \rangle \\ \bigcup_{n<\omega} \mathcal{P}^{\mathcal{U}_\nu}([\nu]^n) &= \langle x_n^\nu \mid n < \omega \rangle \end{aligned}$$

and set $\gamma_\nu := \sup(\mathcal{U}_\nu \cap \beta_\nu)$. Expand $\mathcal{L}_0$ to

$$\mathcal{L}_1 := \mathcal{L}_0 \cup \{\dot{z}_n, \dot{a}_n, \dot{x}_n, \dot{y}_n \mid n < \omega\} \cup \{\dot{f} \mid f \in \omega^\omega\},$$

and let $\mathcal{U}_\nu^*$ be the expansion of $C_{\beta_\nu,\Omega}$ to an $\mathcal{L}_1$-structure with the $\dot{f}$'s interpreted as $\dot{f}^{\mathcal{U}_\nu^*}(n) := z_{f(n)}^\nu$ and the constant symbols having the obvious interpretations. Thin out S_0 to a τ^+-stationary $S_1 \subseteq S_0$ such that the first-order theory of $\mathcal{U}_\nu^*$ is constant on S_1, which can be done as $\omega^\omega < \Omega$.

Now let $\xi, \nu \in S_1$ be such that $\beta_\xi < \nu$ (this will ensure that $\mathcal{U}_\xi \subseteq \mathcal{W}|\nu$). We have a bijection $\pi : \mathcal{U}_\nu \to \mathcal{U}_\xi$ given by $\pi(z_n^\nu) := z_n^\xi$. We want to show that this violates the non-robustness of $\mathcal{U}_\nu$. Since $\mathcal{U}_\nu^*$ is elementarily equivalent to $\mathcal{U}_\xi^*$ by definition of S_1, we get that $\mathrm{Sat}(\mathcal{U}_\nu) = \mathrm{Sat}(\mathcal{U}_\nu, \pi)$. Also, since $\pi(y_n) = y_n$ for every $n < \omega$ by definition of S_0, $\pi \restriction \mathcal{U}_\nu \cap \nu = \mathrm{id}$. Thus (i) and (ii) of the second property of $\mathcal{U}_\nu$ is satisfied by π.

To show (iii), we will thin out S_1 some more to be able to find ν and ξ satisfying (iii). Note that we showed that *any* choice of $\nu, \xi \in S_1$ satisfies (i) and (ii). For $\nu \in S_1$ and $n < \omega$, write

$$\sigma_\nu(x_n^\nu) = \tau_n^\nu[\alpha_n^\nu, d_n^\nu],$$

where τ_n^ν is a Skolem term, $\alpha_n^\nu < \nu$ and $d_n^\nu \in \Gamma_{\nu+1}^{<\omega}$. Then we can thin out S_1 to a τ^+-stationary $S_2 \subseteq S_1$ such that for one of the Skolem terms τ and for some of the α's, $\tau_n = \tau_n^\nu$ and $\alpha_n^\nu = \alpha_n$ for every $\nu \in S_2$. Now, for $\nu \in S_2$, let

$$g(\nu) := \{\langle n, k\rangle \mid a_n^\nu \in \sigma_\nu(x_k^\nu)\}$$

and thin out S_2 to a τ^+-stationary S_3 such that g is constant on S_3. For $\nu \in S_3$ put

$$R^\nu := \{\langle n, \theta, \mu\rangle \mid \theta \in \tau_n^{\mathcal{W}}[\mu, d_n^\nu] \wedge \theta, \mu \in \mathrm{Def}^{\mathcal{W}}\}.$$

Here recall that we assumed $\tilde{K}(\tau, \Omega)$ has ordinal height $< \tau$ in (2), meaning that R^ν can be encoded as an ordinal below $2^{<\tau}$. As $2^{<\tau} < \Omega$, we can thin out S_3 to a τ^+-stationary $S_4 \subseteq S_3$ on which R^ν is constant.

This completes the thinning. Now let $\xi, \nu \in S_4$ be such that $\sigma_\xi(\xi) < \nu$. For this choice of ξ and ν, let $\pi : \mathcal{U}_\nu \to \mathcal{U}_\xi$ be as before: $\pi(z_n^\nu) := z_n^\xi$. We will show that π satisfies (iii); that is, for every n, k it holds that

$$a_n^\nu \in \sigma_\nu(x_k^\nu) \quad \text{iff} \quad a_n^\xi \in x_k^\nu.$$

As we're in S_3 we get that $a_n^\nu \in \sigma_\nu(x_k^\nu)$ holds iff $a_n^\xi \in \sigma_\xi(x_k^\xi)$ holds, so it's enough to show that $\sigma_\xi(x_k^\xi) = x_k^\nu \cap [\sigma_\xi(\xi)]^{<\omega}$ for every $k < \omega$. Suppose it fails for k. Note that $\sigma_\xi(\xi) \leqslant b_{\xi+1}$ since $\xi \leqslant b_{\xi+1}$ and $\sigma_\xi(b_{\xi+1}) = b_{\xi+1}$ as $b_{\xi+1} \in \mathrm{Hull}^{S(\mathcal{W})}(\Gamma_{\xi+1})$. As we've assumed that $\sigma_\xi(\xi) < \nu$ we get that $[\sigma_\xi(\xi)]^{<\omega}$, meaning that

$$\sigma_\nu(x_k^\nu) \cap [\sigma_\xi(\xi)]^{<\omega} = x_k^\nu \cap [\sigma_\xi(\xi)]^{<\omega} \tag{1}$$

and as $\sigma_\nu(x_k^\nu) = \tau_k[\alpha_k, d_k^\nu]$, the assumption that $x_k^\nu \cap [\sigma_\xi(\xi)]^{<\omega} \neq \sigma_\xi(x_k^\xi)$ along with (1) means that there is some θ such that $\theta \in \tau_k[\alpha_k, d_k^\xi]$ iff $\theta \notin \tau_k[\alpha_k, d_k^\nu]$. We thus have that

$$\mathcal{W} \models \exists \theta, \mu < b_{\xi+1} : \theta \in \tau_k[\mu, d_k^\xi] \leftrightarrow \theta \notin \tau_k[\mu, d_k^\nu],$$

and as the above formula is a formula about elements of $\mathrm{Hull}^{S(\mathcal{W})}(\Gamma_{\xi+1})$, we can find witnesses θ, μ to it inside $\mathrm{Hull}^{S(\mathcal{W})}(\Gamma_{\xi+1})$ as well, by elementarity. As $\theta, \mu < b_{\xi+1}$ we get that $\theta, \mu \in \mathrm{Def}^{\mathcal{W}}$ by definition of $\Gamma_{\xi+1}$. But this then implies that $R^\xi \neq R^\nu$, $\lightning$. Thus ν and ξ satisfy (a), (b) and (c), a contradiction to the choice of $\mathcal{U}_\nu$. This finishes the proof. ∎

Note that using this lemma, we can then always pick a weasel $\mathcal{W}$ such that $S(\mathcal{W})$ is $\mathcal{W}$-thick and $\tau \subseteq \mathrm{Def}^{\mathcal{W}}$. We'll now describe the construction of such a weasel. Let $\mathcal{R}$ be any weasel such that $S(\mathcal{R})$ is $\mathcal{R}$-thick and let $\pi : \tilde{K}(\tau, \Omega) \to \mathcal{R}$ be the uncollapse. Set $\theta := \sup \pi``\tau$ and note that $\theta < \Omega$ by regularity of Ω. For every $\alpha \in \theta - \mathrm{ran}\,\pi$ pick some $\mathcal{R}$-thick Γ_α such that $\alpha \notin \mathrm{Hull}^{\mathcal{R}}(\Gamma_\alpha)$. Then set

$$\Gamma := \bigcap \{\Gamma_\alpha \mid \alpha \in \theta - \mathrm{ran}\,\pi\}$$

and note that Γ is $\mathcal{R}$-thick since $\theta < \Omega$. Then $\mathrm{Def}^{\mathcal{R}} \subseteq \mathrm{Hull}^{\mathcal{R}}(\Gamma)$ and $\mathrm{Def}^{\mathcal{R}} \cap \theta = \mathrm{Hull}^{\mathcal{R}}(\Gamma) \cap \theta$ by construction. Set now $\mathcal{W} := \mathrm{cHull}^{\mathcal{R}}(\Gamma$, which is then a weasel with $S(\mathcal{W})$ being $\mathcal{W}$-thick and $\tau \subseteq \mathcal{W}$. This fact will be needed to be able to apply the following.

Lemma 8.6 (The thick trick). *Let $\mathcal{W}$ be a weasel such that $S(\mathcal{W})$ is $\mathcal{W}$-thick and assume $\Phi := (\langle \mathcal{W}, \mathcal{M}\rangle, \langle\kappa\rangle)$ is a stable iterable phalanx, where κ is a $\mathcal{W}$-cardinal. Assume furthermore that $\kappa \subseteq \mathrm{Def}^{\mathcal{W}}$. Let $\mathcal{T}, \mathcal{U}$ be the iteration trees in the coiteration of $\mathcal{W}, \Phi$ with last models $\mathcal{P}, \mathcal{Q}$. Then $\mathcal{M}$ is below $\mathcal{Q}$ in $\mathcal{U}$, $\mathcal{W}$ wins the comparison and* $\mathrm{crit}\, j \geqslant \kappa$.

Maybe we require κ to be a limit cardinal – but then the trick can't be used in the inductive definition of K?

Proof. Note that $\kappa \subseteq \mathrm{Def}^{\mathcal{W}}$ implies that $\mathcal{W}$ has the hull property at every $\alpha < \kappa$, using acceptability and that κ is a $\mathcal{W}$-cardinal. Assume towards a contradiction that $\mathcal{W}$ is below $\mathcal{Q}$, so that universality of $\mathcal{W}$ implies that $\mathcal{P} = \mathcal{Q}$ and that both iteration maps $i : \mathcal{W} \to \mathcal{P}$ and $j : \mathcal{W} \to \mathcal{Q}$ exist. Using Theorem 7.16 on stacked iterations we then have a coiteration:

$$\begin{tikzcd}
S(\mathcal{P}) \arrow[r, phantom, "="] & S(\mathcal{Q}) \\
S(\mathcal{W}) \arrow[u, "i^*"'] \arrow[u, phantom, "\mathcal{T}^*"] & S(\mathcal{W}) \arrow[u, "j^*"'] \arrow[u, phantom, "\mathcal{U}^*"]
\end{tikzcd}$$

By definition of $\mathcal{U}$ we get that $\mu := \operatorname{crit} j^* = \operatorname{crit} j < \kappa$.

Claim 8.6.1. $\mathcal{Q}$ does not have the hull property at $\mu^{+\mathcal{W}} = \mu^{+\mathcal{Q}}$.

Proof of claim. Let E be the first extender used along the $\mathcal{W}$-to-$\mathcal{Q}$ branch of $\mathcal{U}$, so that $\operatorname{crit} E = \mu$ and as $\mathcal{M}$ agrees with $\mathcal{W}$ below κ by assumption, $\operatorname{lh} E \geqslant \kappa$. Define $\mathcal{N} := \operatorname{Ult}(\mathcal{W}, E)$, which satisfies that $S(\mathcal{N})$ is $\mathcal{N}$-thick by Theorem 7.16, and also set $\Gamma := \operatorname{ran} i^*_E$ with $i^*_E : S(\mathcal{W}) \to S(\mathcal{N})$ the induced map, which is $\mathcal{N}$-thick by Theorem 7.16 again. Let ξ be the first generator of E strictly above μ and factor i_E as

Why does this exist? It does if we required κ to be a limit cardinal.

$$\begin{tikzcd}
S(\mathcal{W}) \arrow[r, "i^*_E"] \arrow[dr, "j^*_E"'] & S(\mathcal{N}) \\
& S(\operatorname{Ult}(\mathcal{W}, E \restriction \xi)) \arrow[u, "k^*_E"']
\end{tikzcd}$$

Here $k_E[a, f] := i_E(f)(a)$. Then $\operatorname{crit} k^*_E = \operatorname{crit} k_E = \xi$ as ξ is a generator, and $\operatorname{ran} k^*_E = \operatorname{Hull}^{S(\mathcal{N})}(\xi \cup \Gamma)$. By coherence and the initial segment condition we also have that $\dot{E}^{\mathcal{N}}_\xi = \dot{E}^{\mathcal{W}}_\xi = E \restriction \xi$, so that $E \restriction \xi \in \mathcal{N}$. Since $E \restriction \xi$ can be coded as a subset of $\mu^{+\mathcal{N}}$ and $E \restriction \xi \notin \operatorname{Ult}(\mathcal{W}, E \restriction \xi)$, $E \restriction \xi$ witnesses that $\mathcal{P}^{\mathcal{N}}(\xi) \nsubseteq \operatorname{Hull}^{S(\mathcal{N})}(\xi \cup \Gamma)$ for every $\mathcal{N}$-thick Γ. Since ξ is strictly below all other critical points of extenders on the $\mathcal{W}$-to-$\mathcal{Q}$ branch because generators aren't moved along branches of iteration trees by Proposition 4.7, we get that $\mathcal{Q}$ doesn't have the hull property at $\mu^{+\mathcal{W}}$ either. $\dashv$

Thus $\mathcal{Q}$ doesn't have the hull property at $\mu^{+\mathcal{W}} = \mu^{+\mathcal{M}}$ but $\mathcal{W}$ and hence $\mathcal{Q}$ has the hull property at every $\alpha < \mu^{+\mathcal{W}}$ since $\mu^{+\mathcal{W}} \leqslant \kappa$ and $\mathcal{P}^{\mathcal{W}}(\mu) = \mathcal{P}^{\mathcal{Q}}(\mu)$. If $\operatorname{crit} i \geqslant \mu^{+\mathcal{W}}$ then $\mathcal{P} = \mathcal{Q}$ has the hull property at $\mu^{+\mathcal{W}}$ as $\mathcal{P}^{\mathcal{P}}(\operatorname{crit} i) = \mathcal{P}^{\mathcal{W}}(\operatorname{crit} i)$, $\lightning$. Thus $\operatorname{crit} i \leqslant \mu$, so that $\operatorname{crit} i = \mu$.

Now let $A \in \mathcal{P}^{\mathcal{W}}(\mu)$ and $\Lambda := \{\alpha \in \mathrm{On}^{S(\mathcal{W})} \mid i^*(\alpha) = j^*(\alpha) = \alpha\}$, which can be seen to be $\mathcal{W}$-thick. Since $\mathcal{W}$ has the hull property at μ, we can find a Skolem term τ such that $A = \tau^{\mathcal{W}}[s,t] \cap \mu$, where $s \in \mu^{<\omega}$ and $c \in \Lambda^{<\omega}$. We then get that

$$j(A) = \tau^{\mathcal{Q}}[s,t] \cap j(\mu),$$

using that s lies below the critical point of j, and every element of Λ is fixed by j by definition of Λ. But then $A = j(A) \cap \mu = \tau^{\mathcal{Q}}[s,t] \cap \mu = \tau^{\mathcal{P}}[s,t] \cap \mu$, so that

$$i(A) = \tau^{\mathcal{P}}[s,t] \cap i(\mu).$$

It then follows that the first extenders used along the $\mathcal{W}$-to-$\mathcal{P}$ and $\mathcal{W}$-to-$\mathcal{Q}$ branches are compatible, contradicting Claim 4.15.1. Thus $\mathcal{M}$ lies below $\mathcal{Q}$ in $\mathcal{U}$, and by universality of $\mathcal{W}$ and stability of Φ we get that $\mathcal{W}$ wins the comparison. Finally, $\operatorname{crit} j \geqslant \kappa$ holds by the rules of the iteration game. ■

8.3 Weak covering

The following weak covering property will make sure that the construction of pseudo-K will be independent of τ and Ω, in some sense. This will then make the construction of a *canonical* premouse K possible.

Theorem 8.7 (Weak covering). *Let κ be a singular strong limit cardinal. Then there is a mouse $\mathcal{M}$ such that $\kappa^{+\mathcal{M}} = \kappa^+$.*

The entire section will be devoted to proving this theorem. To get an overview of the proof, we supply here the essential steps:

(i) Fix a weasel $\mathcal{W}_0$ such that $S(\mathcal{W}_0)$ is $\mathcal{W}_0$-thick and $\kappa^+ \subseteq \mathrm{Def}^{\mathcal{W}_0}$ and assume towards a contradiction that $\lambda := \kappa^{+\mathcal{W}_0} < \kappa^+$;
(ii) Replace $\mathcal{W}_0$ by another weasel $\mathcal{W}$ with $\kappa^{+\mathcal{W}} = \lambda$, fixing a technicality;
(iii) Use that $\lambda < \kappa^+$ to come up with a new premouse $\mathcal{S}$ with $\kappa^{+\mathcal{S}} = \lambda$;
(iv) Show that the phalanx $(\langle \mathcal{W}, \mathcal{S} \rangle, \langle \kappa \rangle)$ is iterable;

(v) Derive a contradiction after comparing $(\langle\mathcal{W},\mathcal{S}\rangle,\langle\kappa\rangle)$ with $\mathcal{W}$, using that the phalanx-side doesn't move.

Parts of the proof will be omitted with a reference to the source. We're only going to omit proofs which don't depend crucially upon our results from the previous sections, so the ideas hopefully still shine through. In step (iii), to get an idea of how $\mathcal{S}$ is constructed we expand this step into the following substeps:

(a) Collapse everything in sight with the Mostowski collapse π and use that $\lambda<\kappa^+$ to ensure that π is cofinal in λ;
(b) Iterate $\bar{\mathcal{W}}$ to some $\mathcal{P}$ such that $\bar{\mathcal{W}}$ agrees with $\mathcal{P}$ below $\bar{\lambda}$;
(c) Set $\mathcal{R}$ to be the $E_\pi\restriction\kappa$-ultrapower of $\mathcal{P}$, so that $\mathcal{W}$ agrees with $\mathcal{R}$ below λ since π was cofinal in λ, so that $\kappa^{+\mathcal{R}}=\lambda$;
(d) $\mathcal{R}$ is not necessarily a premouse, so "approximate" $\mathcal{R}$ by a premouse $\mathcal{S}$, still such that $\kappa^{+\mathcal{S}}=\lambda$.

We now commence with the proof. Let thus κ be a singular strong limit cardinal and fix Ω sufficiently large such that $\tilde{K}(\kappa^+,\Omega)$ has ordinal height $\geqslant\kappa^+$. Fix $\mathcal{W}_0$ such that $S(\mathcal{W}_0)$ is $\mathcal{W}_0$-thick and $\kappa^+\subseteq\mathrm{Def}^{\mathcal{W}_0}$. Assume that $\lambda:=\kappa^{+\mathcal{W}_0}<\kappa^+$.

If $\mathcal{W}_0$ is either a mini-universe or is collapsing with $\nu:=\mathrm{cof}^V\gamma^{\mathcal{W}_0}\geqslant\kappa$ then set $\mathcal{W}:=\mathcal{W}_0$. But if $\mathcal{W}_0$ is collapsing with $\nu<\kappa$ we call $\mathcal{W}_0$ *phalanx-unstable*, as this sort of $\mathcal{W}_0$ will make it possible for some of our phalanxes that we'll introduce later to become unstable. To fix this, we need to get rid of all measurable cardinals inside $\mathcal{W}_0$ below κ^+ with V-cofinality ν.

To remove these measurables, we linearly iterate $\mathcal{W}_0$ by normal measures: letting $\mathcal{W}_\alpha$ be the α'th model of the iteration, set $\mathcal{W}_{\alpha+1}:=\mathrm{Ult}(\mathcal{W}_\alpha,U)$, where U is the order zero measure on the least measurable cardinal in $\mathcal{W}_\alpha$ below κ^+ and with V-cofinality ν. The iteration stops if there is no such measurable, and the iteration will thus stop after $\leqslant\kappa^+$ steps. Since the critical points in the iteration are increasing, it is normal. Let $\mathcal{W}$ be the final model of the iteration.

We now move towards our goal of defining $\mathcal{S}$. First let $\pi:\mathcal{H}\to V_{\Omega+\omega}$ be the collapse with $\mathcal{H}$ transitive, $|\mathcal{H}|<\kappa$, $\mathrm{ran}\,\pi$ cofinal in λ (which can be done as $\lambda<\kappa^+$), everything of interest being inside $\mathrm{ran}\,\pi$ and $\mathcal{H}$ being closed under ω-sequences. If $\mathcal{W}_0$ is collapsing such that $\nu<\kappa$ then we also require that $\mathcal{H}$ is closed under ν-sequences. Let

$$\kappa_\alpha := \alpha\text{'th infinite cardinal of } \bar{\mathcal{W}} = \aleph_\alpha^{\bar{\mathcal{W}}}.$$

Fix θ such that $\pi(\kappa_\theta) = \kappa^+$. Note now that $\kappa_{\bar\kappa} = \bar\kappa$, $\kappa_{\bar\kappa+1} = \bar\lambda$ and $\kappa_{\bar\kappa+2} \leqslant \theta$.

Lemma 8.8. *There is a normal iteration tree on $\mathcal{W}$ with last model $\mathcal{N}$ such that $\bar{\mathcal{W}}$ agrees with $\mathcal{N}$ below $\bar\lambda + 1$.*

Proof. See section 5 in (Jensen & Steel, 2013). To get an idea of how it's shown, see the comment in the proof of Lemma 8.9. ■

Let $\mathcal{T}$ be an iteration tree of minimal length witnessing the above Lemma 8.8, with last model $\mathcal{N}$. Define $\eta < \operatorname{lh}\mathcal{T} - 1$ to be least such that $\nu_{E_\eta^\mathcal{T}} > \bar\kappa$ if it exists and otherwise $\operatorname{lh}\mathcal{T} - 1$, and let γ be least such that $\rho(\mathcal{M}_\eta^\mathcal{T}|\gamma) < \bar\lambda$ if it exists and otherwise $\mathrm{On}^{\mathcal{M}_\eta^\mathcal{T}}$. This leads us to

$$\mathcal{P} := \mathcal{M}_\eta^\mathcal{T}|\gamma.$$

Note that $\mathcal{P}$ agrees with $\bar{\mathcal{W}}$ below $\bar\lambda$. Indeed, since $\mathcal{N}$ satisfies this property we only have to show that $\bar\lambda \leqslant \operatorname{lh} E_\eta$. But $\bar\kappa < \nu(E_\eta)$, $\bar\lambda = \bar\kappa^{+\bar{\mathcal{W}}}$ and since $\mathcal{N}$ agrees with $\bar{\mathcal{W}}$ below $\bar\lambda + 1$, $\bar\lambda = \bar\kappa^{+\mathcal{N}}$ as well. As $\operatorname{lh} E_\eta$ is a cardinal of $\mathcal{N}$, we get that $\bar\lambda \leqslant \operatorname{lh} E_\eta$. If $\mathcal{M}_\eta = \mathcal{N}$ then it's direct since $\mathcal{N}$ agrees with $\bar{\mathcal{W}}$ below $\bar\lambda + 1$.

Now, let $m \leqslant \omega$ be the largest such that $\bar\lambda < \rho_m(\mathcal{P})$; then set

$$\mathcal{R} := \mathrm{Ult}_m(\mathcal{P}, E_\pi \upharpoonright \kappa),$$

and note that $\mathcal{R}$ agrees with $\mathcal{W}$ below λ since $\mathcal{P}$ agrees with $\bar{\mathcal{W}}$ below $\bar\lambda$, $\pi(\bar\lambda) = \lambda$ and that $\sup \pi``\bar\lambda = \pi(\bar\lambda) = \lambda$ since π is cofinal in λ. We might have that the active extender of $\mathcal{R}$ isn't total so that it isn't a premouse[1]. This is solved in (Mitchell et al., 1997, p. 234) by introducing a premouse $\mathcal{S}$ such that $\mathcal{S}$ agrees with $\mathcal{R}$, and thus also $\mathcal{W}$, below λ. At the same time another premouse $\mathcal{Q}$ is introduced, replacing $\mathcal{P}$ in the same way, and such that $\mathcal{S} = \mathrm{Ult}(\mathcal{Q}, E_\pi \upharpoonright \kappa)$.

[1]This breed of mice is called *protomice* in (Mitchell et al., 1997).

Lemma 8.9. $\Phi := (\langle \mathcal{W}, \mathcal{S} \rangle, \langle \kappa \rangle)$ *is a stable iterable phalanx.*

Proof. See section 5 in (Jensen & Steel, 2013). The proof is shown in a helix-like fashion, building approximations $\mathcal{P}_\alpha$, $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$ to $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{S}$, and inductively showing Lemma 8.8 as well as iterability of everything in sight at that stage, which is then used to prove Lemma 8.8 for the next stage, and so on. ■

We now compare Φ with $\mathcal{W}$, giving us iteration trees $\mathcal{T}, \mathcal{U}$ on $\Phi, \mathcal{M}$ with last models $\mathcal{H}, \mathcal{N}$, respectively. As $\kappa \subseteq \mathrm{Def}^{\mathcal{W}}$, Φ is stable and $\mathcal{W}$ is stable and universal, the thick trick 8.6 implies we get that $\mathcal{H}$ lies above $\mathcal{S}$ in $\mathcal{T}$, $\mathcal{W}$ wins the comparison and letting $i : \mathcal{S} \to \mathcal{H}$ be the iteration map, $\mathrm{crit}\, i \geqslant \kappa$. The situation thus looks as follows:

$$\begin{tikzcd} \mathcal{H} \arrow[r, phantom, "\trianglelefteq"] & \mathcal{N} \\ \mathcal{S} \arrow[u, "i"'] \arrow[u, phantom, "\mathcal{T}", shift left=3] & \mathcal{W} \arrow[u, dash, "\mathcal{U}"] \end{tikzcd}$$

Since $\mathcal{S}$ agrees with $\mathcal{W}$ below λ, we get that $\mathcal{P}^{\mathcal{S}}(\kappa) = \mathcal{P}^{\mathcal{W}}(\kappa) = \mathcal{P}^{\mathcal{H}}(\kappa)$. We then have two cases, whether or not $\mathcal{S}$ is a weasel. We will only treat the case where $\mathcal{S}$ is not a weasel in some detail. In the case where it's a weasel one argues that $S(\mathcal{S})$ is $\mathcal{S}$-thick and the $\mathcal{W}$-to-$\mathcal{N}$ branch of $\mathcal{U}$ doesn't drop so that $\mathcal{H} = \mathcal{N}$, and then proceeds to a technical argument involving the hull- and definability property to conclude that $\mathcal{P}^{\mathcal{W}}(\kappa)$ has cardinality κ in $\mathcal{W}$, the desired contradiction.

If $\mathcal{S}$ isn't a weasel then it is κ-sound and projects across κ (Jensen & Steel, 2013, Claim 5.19). But since $\mathrm{crit}\, i \geqslant \kappa$, we can't have that $\mathcal{T}$ is non-trivial as then there would be a drop along the $\mathcal{S}$-to-$\mathcal{H}$ branch. Thus $\mathcal{S} = \mathcal{H}$ and $i = \mathrm{id}$. But then since $\mathrm{On}^{\mathcal{S}} < \mathrm{On}^{\mathcal{W}} = \kappa^+$, we get that $\mathcal{S} \lhd \mathcal{W}$ and thus $\mathcal{S} \in \mathcal{W}$. But because of κ-soundness, $\mathcal{S}$ can be coded as a subset of κ, so this contradicts that $\mathcal{P}^{\mathcal{W}}(\kappa) \subseteq \mathcal{S}$.

This finishes the (sketch of the) proof of the weak covering theorem. ■

8.4 Definition of K

We'll now prove that $\tilde{K}(\mathrm{cof}\, \kappa, \kappa^+) | \mathrm{cof}\, \kappa$ satisfies a certain inductive definition for κ a singular strong limit cardinal, which will be independent on the specific choice of κ.

Definition 8.10. A premouse $\mathcal{M}$ is 0-**strong** if it's stable, and given any weasel $\mathcal{W}$ such that $S(\mathcal{W})$ is $\mathcal{W}$-thick and Σ an iteration strategy for $\mathcal{W}$, there is an iteration tree $\mathcal{T}$ on $\mathcal{W}$ by Σ with last model $\mathcal{P}$ such that there is a fully elementary embedding $\pi : \mathcal{M} \to \mathcal{Q}$ with $\mathcal{Q} \trianglelefteq \mathcal{P}$. $\circ$

Maybe also satisfies ZF$^-$? Maybe not.

The last condition in the definition of 0-strong is that we can "synthetically" compare $\mathcal{M}$ to every weasel $\mathcal{W}$ satisfying that $S(\mathcal{W})$ is $\mathcal{W}$-thick, even though $\mathcal{M}$ might not be iterable.

Definition 8.11. Let $\mathcal{M}$ be a premouse and κ an $\mathcal{M}$-cardinal. Then $\mathcal{M}$ is $\kappa^{+\mathcal{M}}$-**strong** if whenever $\Phi := (\langle \mathcal{N}, \mathcal{M} \rangle, \langle \kappa \rangle)$ is a phalanx with $\mathcal{N}$ being μ-strong for every $\mathcal{M}$-cardinal $\mu < \kappa^{+\mathcal{M}}$, then Φ is $(\Omega + 1)$-iterable. $\circ$

Definition 8.12. A premouse $\mathcal{M}$ **satisfies the local inductive definition of** K if for every $\mathcal{M}$-cardinal κ of $\mathcal{M}$,

$$\kappa^{+\mathcal{M}} := \sup\{\kappa^{+\mathcal{N}} \mid \mathcal{N} \text{ is } \kappa\text{-strong and properly 1-small}\}$$

and for every $\alpha < \kappa^{+\mathcal{M}}$, $\mathcal{P} = \mathcal{M}|\alpha$ iff $\mathcal{P} = \mathcal{N}|\alpha$ for some κ-strong properly 1-small $\mathcal{N}$. $\circ$

Lemma 8.13. *Let κ be a singular strong limit cardinal, $\tau :=$ cof κ and $\Omega := \kappa^+$. Then $\tilde{K}(\operatorname{cof}\kappa, \kappa^+)|\operatorname{cof}\kappa$ satisfies the local inductive definition of K.*

Proof. By Weak Covering 8.7 we get that $\kappa^{+\mathcal{M}} = \kappa^+ = \Omega$ for some mouse $\mathcal{M}$. But $\mathcal{W}_0 := \mathcal{M}|\Omega$ is then a collapsing weasel with κ as largest cardinal. If $\mathcal{W}_0$ is stable then set $\mathcal{W}_1 := \mathcal{W}_0$ and otherwise set $\mathcal{W}_1 := \operatorname{Ult}(\mathcal{W}_0, U)|\Omega$, where U is the order zero measure on $\eta^{\mathcal{W}_0} = \operatorname{cof}^{\mathcal{W}_0}\kappa \geqslant \tau$, which is then a stable collapsing weasel still having κ as its largest cardinal by Proposition 7.3. This implies that Ω is $\mathcal{W}_1$-thick, so that there is some stable collapsing weasel $\mathcal{W}$ with

$$(\operatorname{Def}^{\mathcal{W}_1}, \in) \cong (\operatorname{Def}^{\mathcal{W}}, \in)$$

and such that $\tau \subseteq \operatorname{Def}^{\mathcal{W}}$ by the construction given before the Thick Trick 8.6, so that $\mathcal{W}|\tau = \tilde{K}(\tau, \Omega)|\tau$. Note that $\mathcal{W}$ still have κ as its largest cardinal. Indeed, since $\mathcal{W}$ and $\mathcal{R}$ are both collapsing, letting $\sigma : \mathcal{R} \to \mathcal{W}$ be the collapse we have that $\sigma(\kappa) = \sigma(\gamma^{\mathcal{R}}) = \gamma^{\mathcal{W}}$, so that $\gamma^{\mathcal{W}} \leqslant \kappa$. But since κ is a V-cardinal it's also a $\mathcal{W}$-cardinal and we thus have that $\gamma^{\mathcal{W}} = \kappa$.

We thus have to show that $\mathcal{W}|\tau$ satisfies the local inductive definition of K. The argument in the proof of (Steel, 1996, Theorem 6.11) can be used directly here, if we just show the following claim.

Claim 8.13.1. Let $\mu \leqslant \tau$ be a cardinal of $\mathcal{W}$ and assume $\Phi := (\langle \mathcal{W}, \mathcal{M} \rangle, \langle \mu \rangle)$ is an iterable phalanx with $|\mathcal{M}| < \Omega$. Then there's an iteration tree $\mathcal{U}$ on $\mathcal{W}$ with last model $\mathcal{Q}$ and in which all extenders have length $\geqslant \mu$, an initial segment $\mathcal{P} \trianglelefteq \mathcal{Q}$ and a fully elementary $\pi : \mathcal{M} \to \mathcal{P}$ such that $\pi \upharpoonright \mu = \mathrm{id}$.

Should this be a k-embedding for $\mathcal{M}$ being k-sound?

Proof of claim. We first show that both Φ and $\mathcal{W}$ are stable. $\mathcal{W}$ is stable by construction, and since $|\mathcal{M}| < \Omega$ it's stable as well, so we just need to show that if $(\eta^{\mathcal{W}})^{+\mathcal{W}} \leqslant \mu$ then $\eta^{\mathcal{W}}$ isn't a measurable cardinal of neither $\mathcal{W}$ nor $\mathcal{M}$. But since $\eta^{\mathcal{W}} \geqslant \tau$, using that κ is the largest cardinal in $\mathcal{W}$, this is vacously true.

Now compare Φ with $\mathcal{W}$, giving us iteration trees $\mathcal{T}, \mathcal{U}$ on $\Phi, \mathcal{W}$ with last models $\mathcal{P}, \mathcal{Q}$, respectively. As $\mu \subseteq \mathrm{Def}^{\mathcal{W}}$, Φ is stable and $\mathcal{W}$ is stable and universal, the thick trick 8.6 implies that $\mathcal{P}$ lies above $\mathcal{M}$ in $\mathcal{T}$, $\mathcal{W}$ wins the comparison and letting $i : \mathcal{M} \to \mathcal{P}$ be the iteration map, $\operatorname{crit} i \geqslant \mu$. Thus taking $\pi := i$ works. $\dashv$

This thus finishes the proof of the local inductive definition. ■

Corollary 8.14. *For singular strong limit cardinals μ and ν with $\operatorname{cof}\mu \leqslant \operatorname{cof}\nu$, it holds that $\tilde{K}(\operatorname{cof}\mu, \mu^+)|\operatorname{cof}\mu = \tilde{K}(\operatorname{cof}\nu, \nu^+)|\operatorname{cof}\mu$.*

Proof. This is directly by Lemma 8.13, since the local inductive definition of K is independent of μ. ■

Definition 8.15. **The core model** K is the unique proper class premouse such that given any singular strong limit cardinal μ, $K|\operatorname{cof}\mu = \tilde{K}(\operatorname{cof}\mu, \mu^+)|\operatorname{cof}\mu$. ∘

To see that K does in fact satisfy all the properties mentioned in Theorem 8.1, we refer to chapters 5 and 6 of (Steel, 1996). Even though a measurable cardinal is assumed throughout, the proofs of these will still go through in our framework.

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