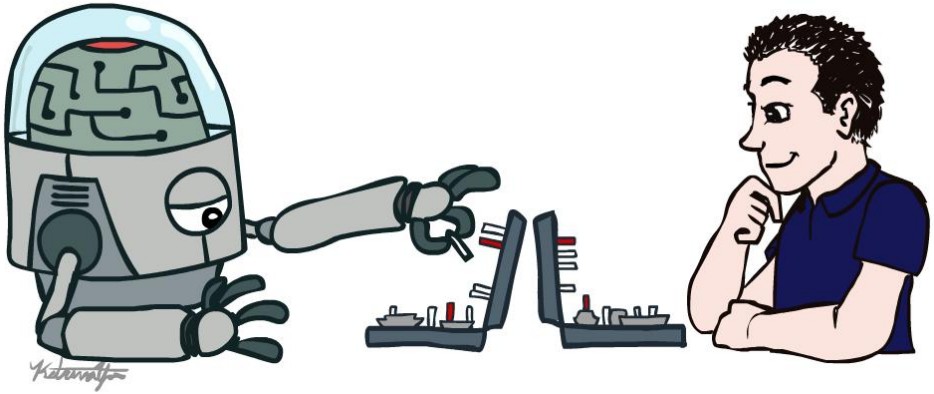
第十三周

- 教学计划
  - 期末回顾
- 任务
  - 家作5/项目5：机器学习，5/31日提交
  - 项目5口令：**machinelearning**

人工智能导论

Final Review



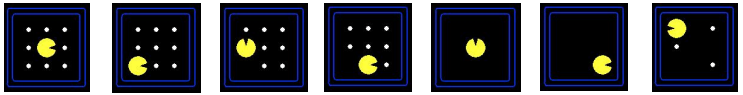
Agenda

- 期末回顾
  - Search： BFS/DFS/UCS/A*/CPS/Minimax/Expectimax
  - MDP: value iteration, policy iteration
  - RL: temporal difference learning, approximate Q learning
  - Bayes net: Bayes rule, conditional independence, variable elimination
  - Markov models: hidden markov model, particle filtering
  - Machine learning: naive Bayes, perceptor, kernels, K-mean
  - Neural network: gradient ascent, back propagation

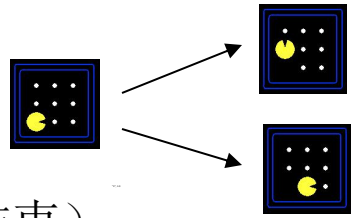
SEARCH

Search Problems

- A search problem consists of:
  - A state space（状态空间）



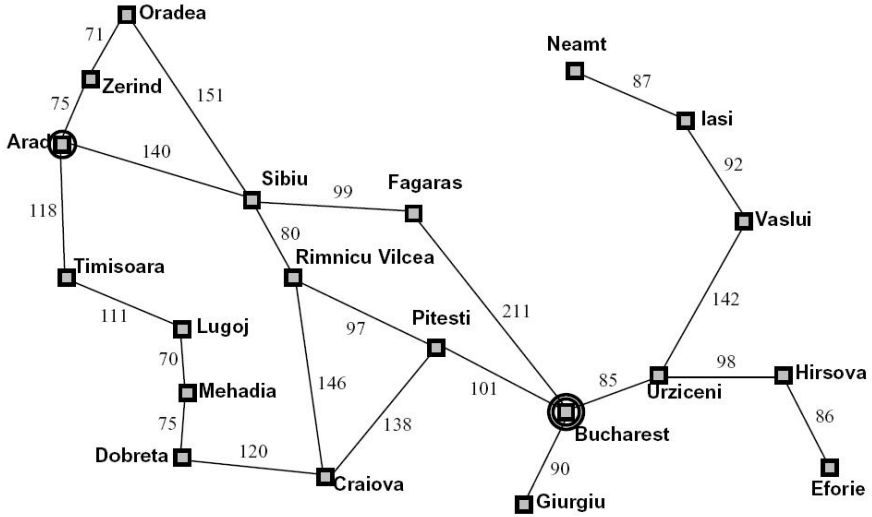
  - A successor function（后续函数）
    (with actions, costs)



  - A start state and a goal test（初始、结束）

- A solution is a sequence of actions (a plan) which transforms the start state to a goal state

Example: Traveling in Romania



- State space:
  - Cities
- Successor function:
  - Roads: Go to adjacent city with cost = distance
- Start state:
  - Arad
- Goal test:
  - Is state == Bucharest?
- Solution?

State Space Sizes?

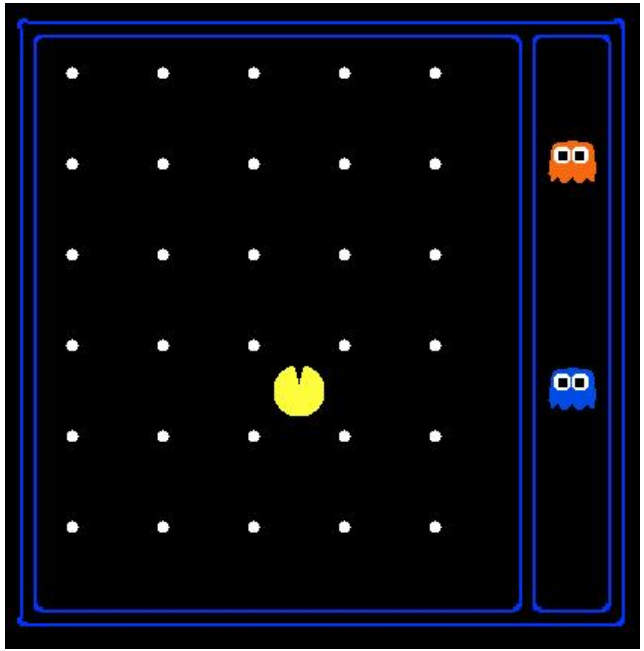
- World state:
  - Agent positions: 120
  - Food count: 30
  - Ghost positions: 12
  - Agent facing: NSEW

- How many
  - World states?

    $120x(2^{30})x(12^2)x4$
  - States for pathing?

    120
  - States for eat-all-dots?

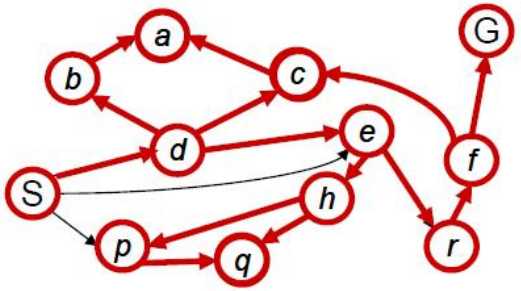
    $120x(2^{30})$

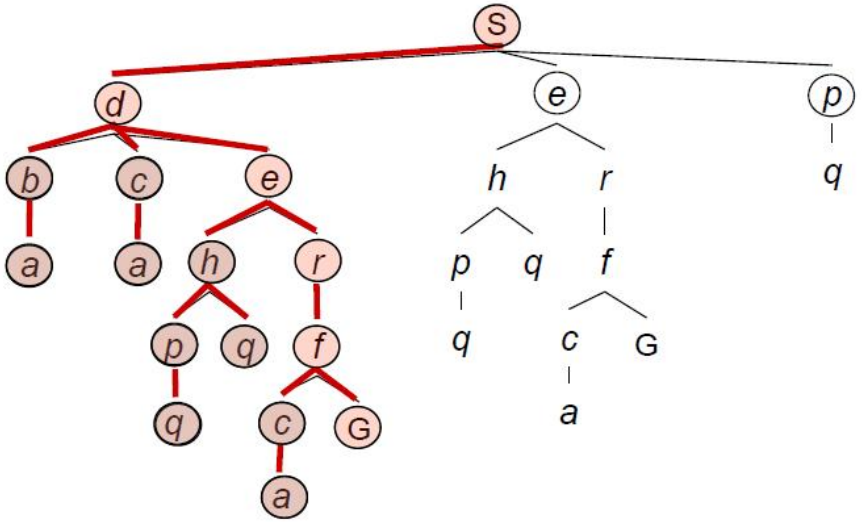


Depth-First Search

Strategy: expand a deepest node first

Implementation: Fringe is a LIFO stack

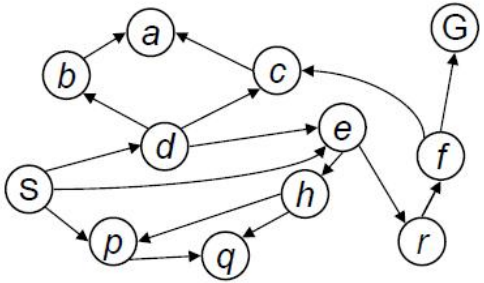


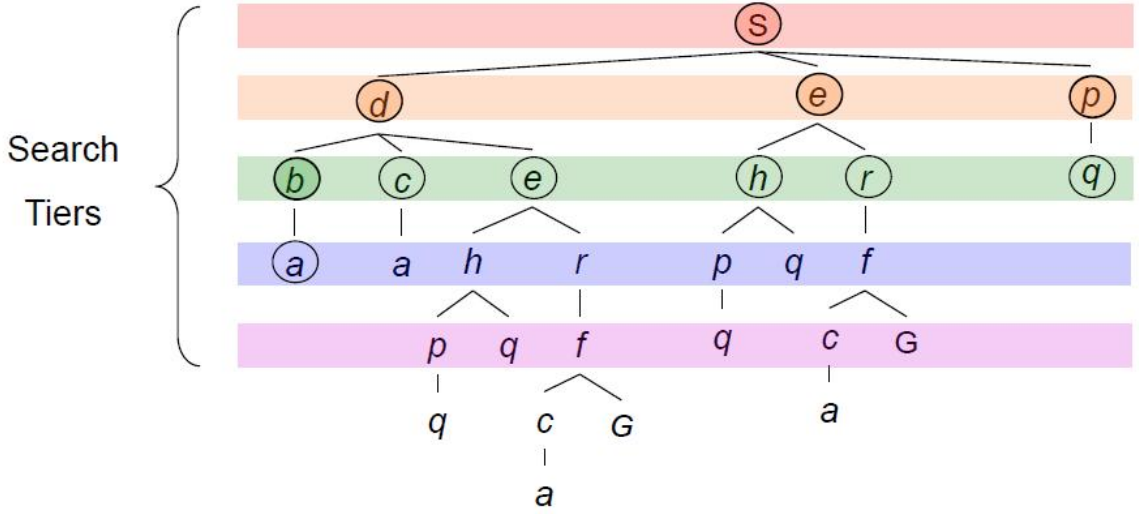


Breadth-First Search

Strategy: expand a shallowest node first

Implementation: Fringe is a FIFO queue

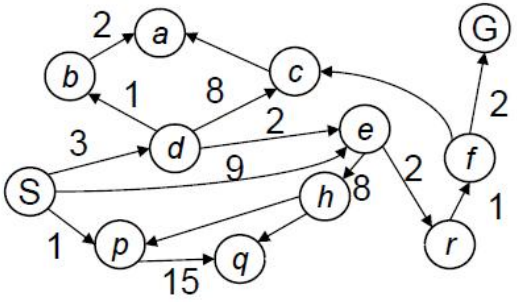


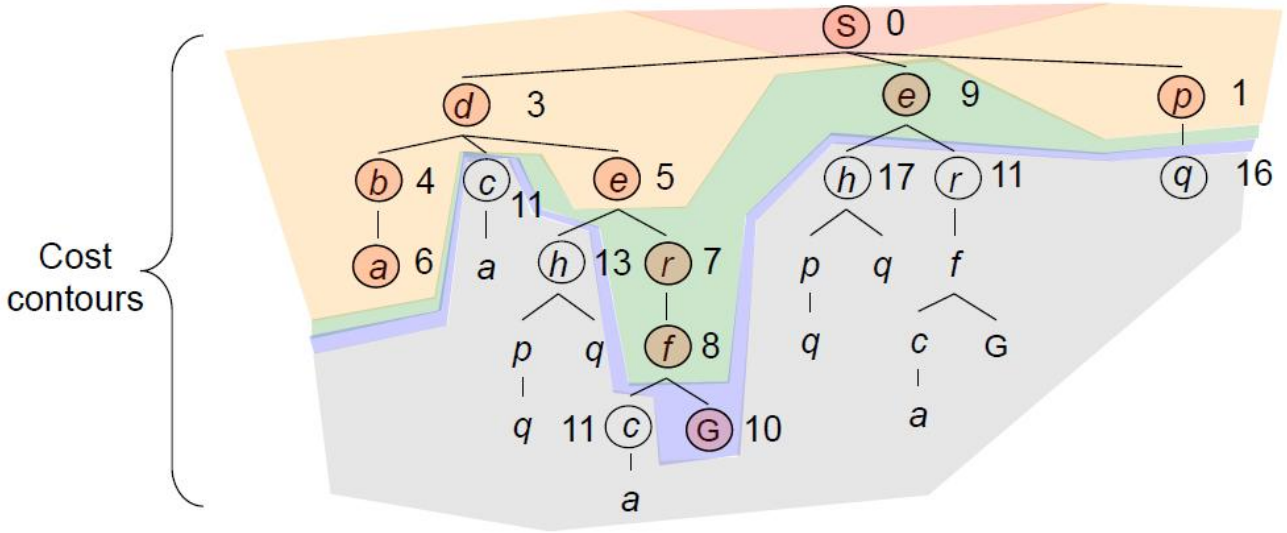


Uniform Cost Search

Strategy: expand a cheapest node first:

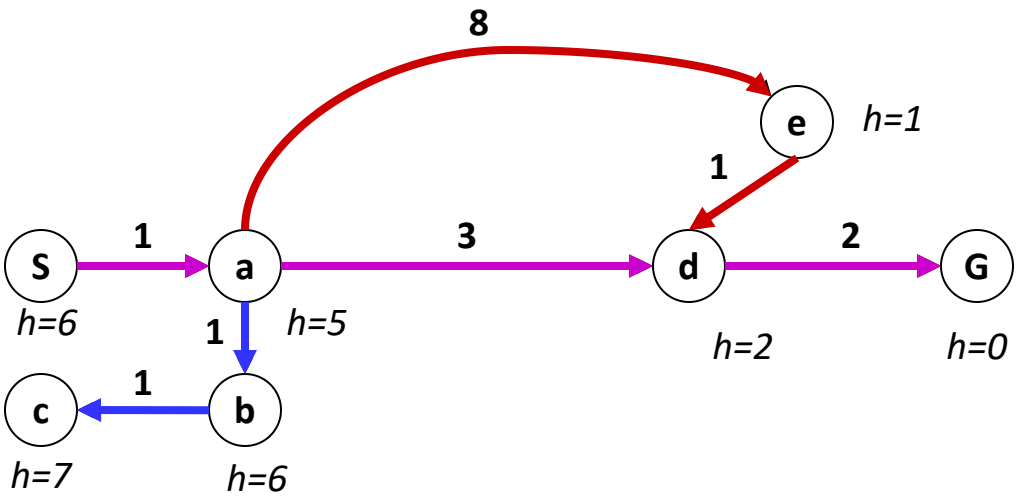
Fringe is a priority queue (priority: cumulative cost)

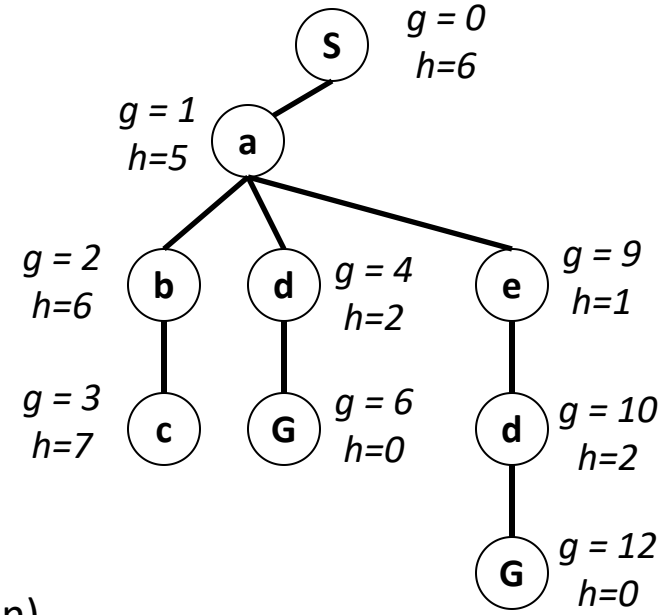




Combining UCS and Greedy

- Uniform-cost orders by path cost, or *backward cost* g(n)
- Greedy orders by goal proximity, or *forward cost* h(n)





- A* Search orders by the sum: f(n) = g(n) + h(n)

Example: Teg Grenager

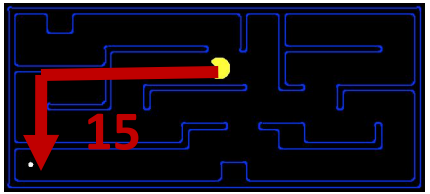
Admissible Heuristics

- A heuristic h is *admissible* (optimistic) if:

$$0 \leq h(n) \leq h^*(n)$$

where $h^*(n)$ is the true cost to a nearest goal

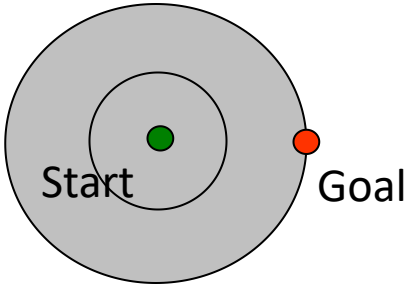
- Examples:



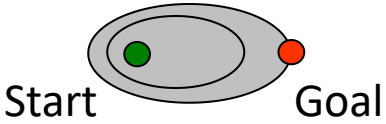
- Coming up with admissible heuristics is most of what's involved in using A* in practice.

UCS vs A* Contours

- Uniform-cost expands equally in all "directions"

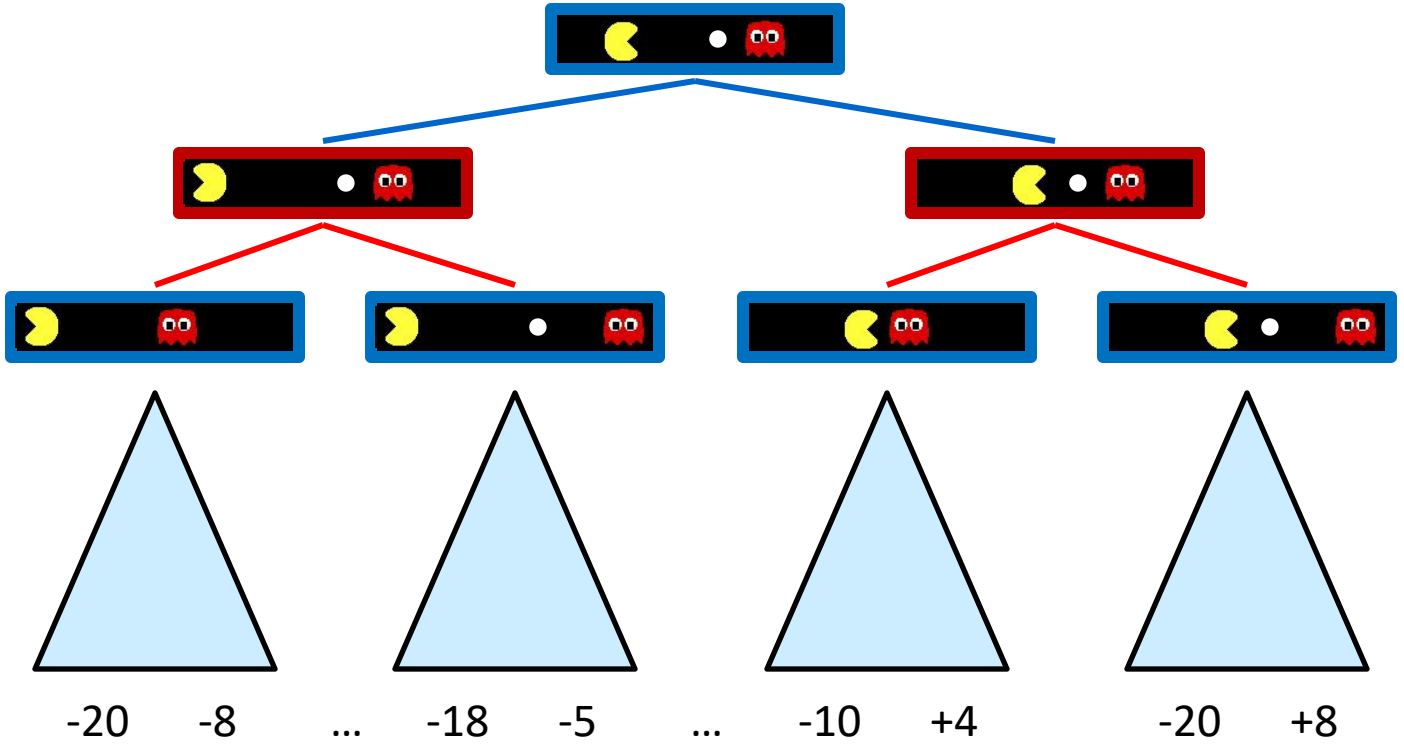


- A* expands mainly toward the goal, but does hedge its bets to ensure optimality

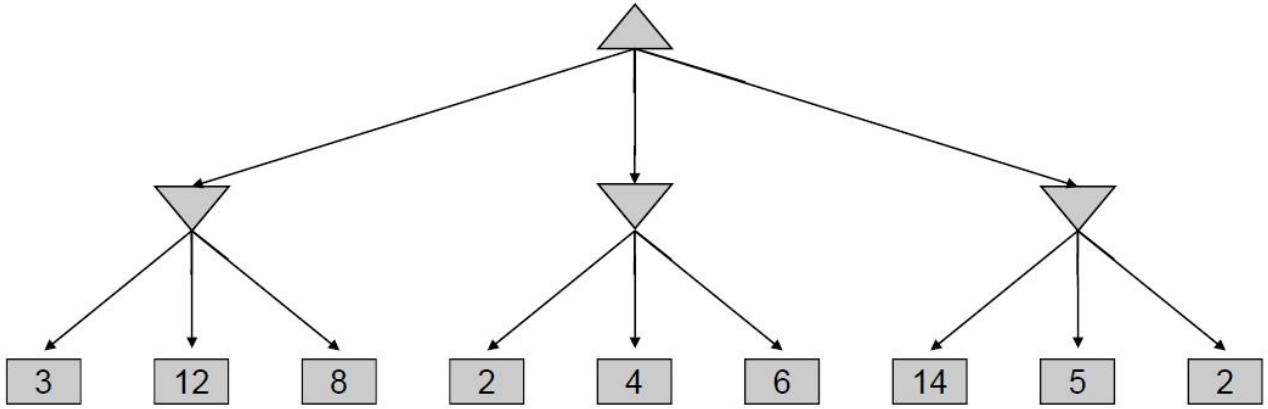


[Demo: contours UCS / greedy / A* empty (L3D1)]
[Demo: contours A* pacman small maze (L3D5)]

Adversarial Game Trees



Minimax Example



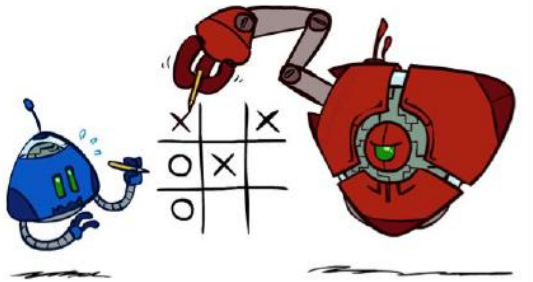
Minimax Implementation (Dispatch)

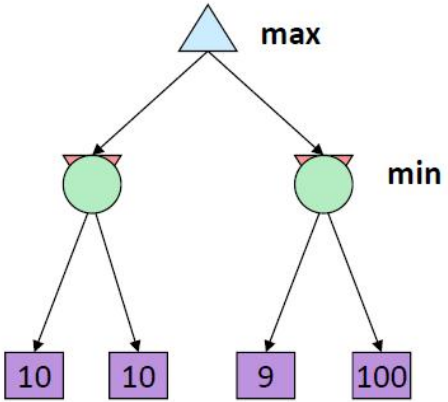
```
def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is MIN: return min-value(state)
```

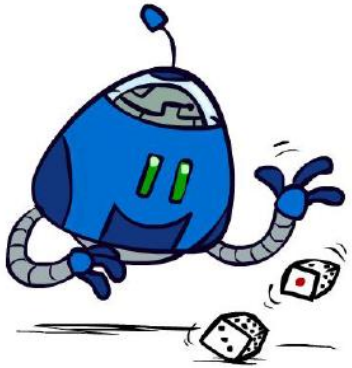
```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def min-value(state):
    initialize v = +∞
    for each successor of state:
        v = min(v,
            value(successor))
    return v
```

Worst-Case vs. Average Case







Idea: Uncertain outcomes controlled by chance, not an adversary!

Expectimax Pseudocode

```
def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)
```

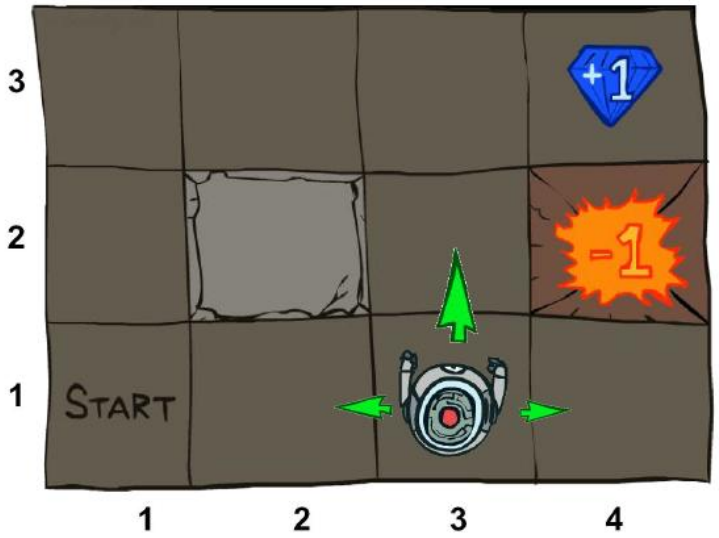
```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

MARKOV DECISION PROCESSES

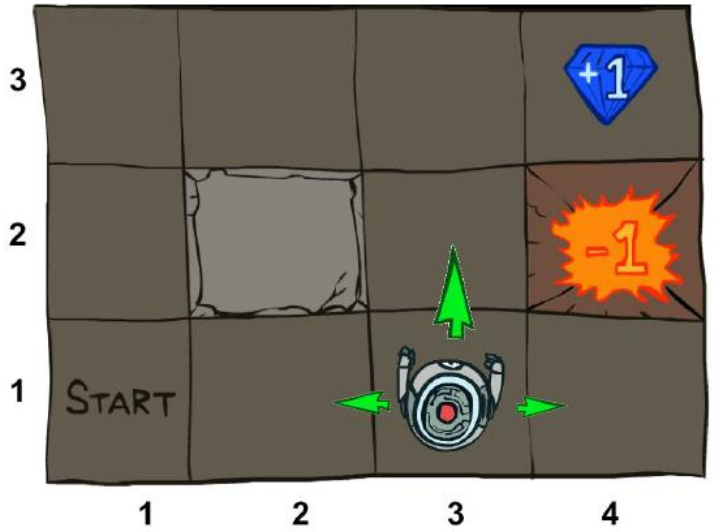
Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards



Markov Decision Processes

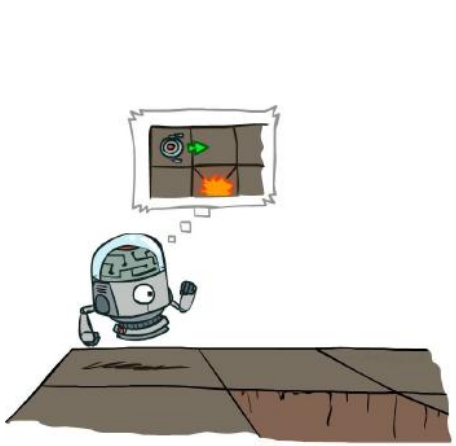
- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function $T(s, a, s')$
    - Probability that a from s leads to s', i.e., $P(s' | s, a)$
    - Also called the model or the dynamics
  - A reward function $R(s, a, s')$
    - Sometimes just $R(s)$ or $R(s')$
  - A start state
  - Maybe a terminal state

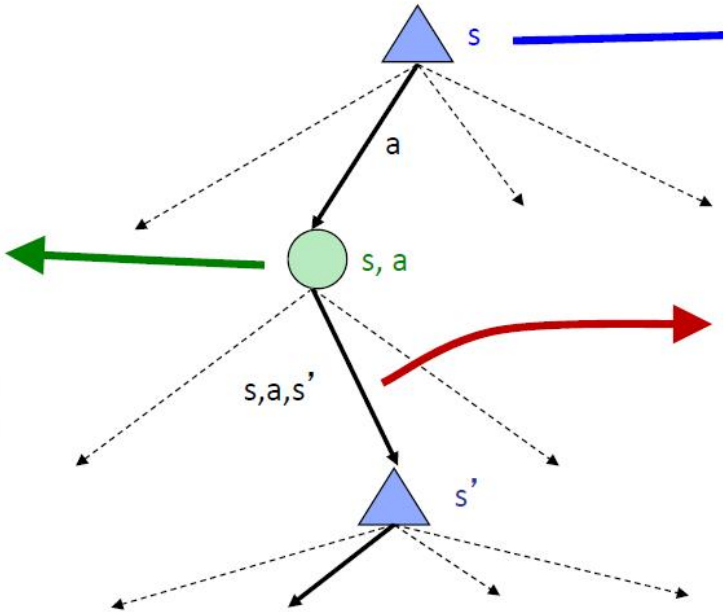


- MDPs are non-deterministic search problems
  - One way to solve them is with expectimax search
  - We'll have a new tool soon

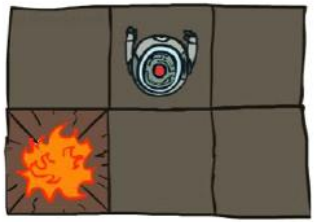
MDP search trees

- Each MDP state projects an expectimax-like search tree





s is a *state*



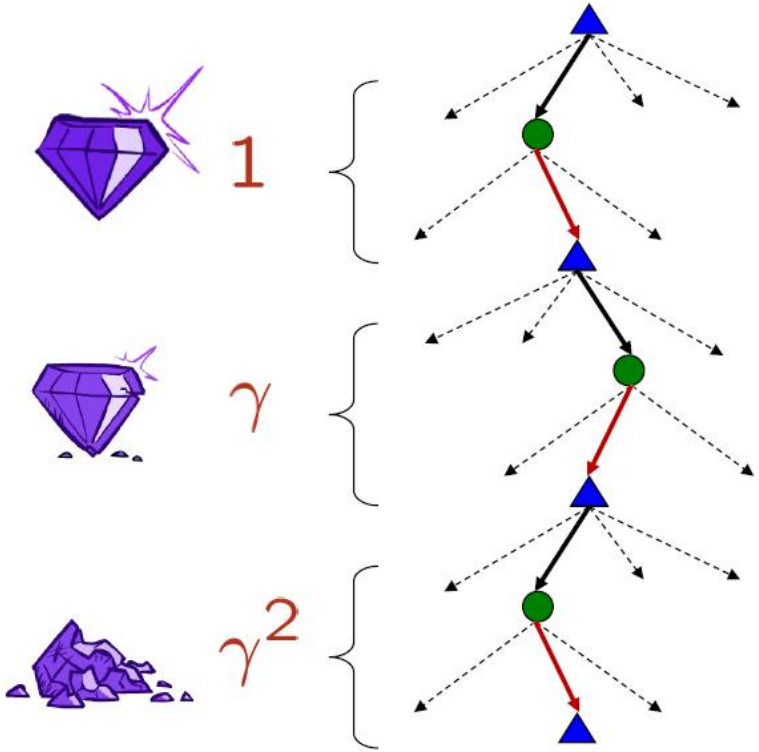
(s,a,s') called a *transition*

T(s,a,s') = P(s'|s,a)

R(s,a,s')

Discounting

- How to discount?
  - Each time we descend a level, we multiply in the discount once
- Why discount?
  - Sooner rewards probably do have higher utility than later rewards
  - Also helps our algorithms converge
- Example: discount of 0.5
  - U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3
  - U([1,2,3]) < U([3,2,1])



Optimal Quantities

- The value (utility) of a state s:

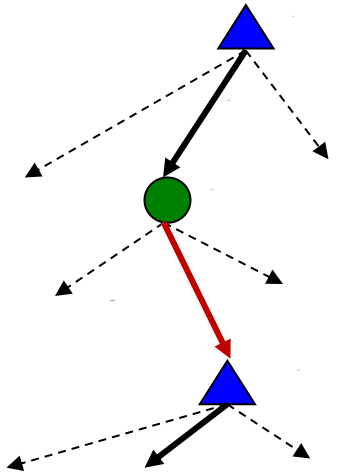
  $V^*(s)$ = expected utility starting in s and acting optimally

- The value (utility) of a q-state (s,a):

  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally

- The optimal policy:

  $\pi^*(s)$ = optimal action from state s



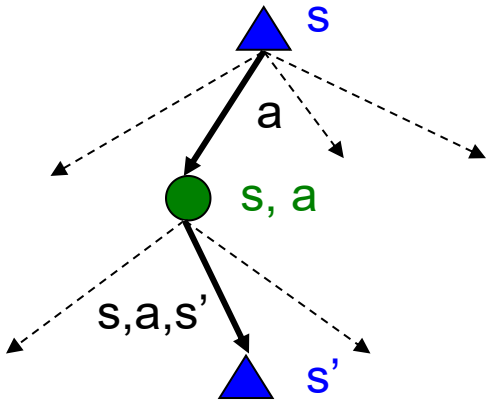
s is a *state*

(s, a) is a *q-state*

(s,a,s') is a *transition*

Values of States

- Fundamental operation: compute the (expectimax) value of a state
  - Expected utility under optimal action
  - Average sum of (discounted) rewards
  - This is just what expectimax computed!



- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s,a)$$

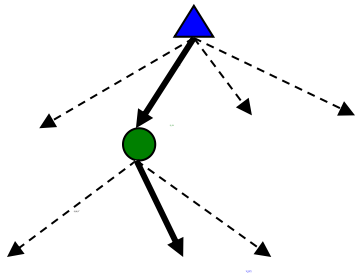
$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

Value Iteration

- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one play of expectimax from each state:

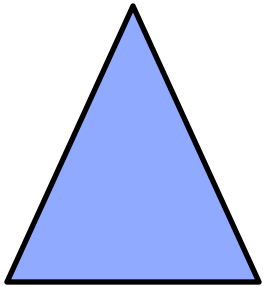
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$



- Repeat until convergence

- Complexity of each iteration: $O(S^2A)$

- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

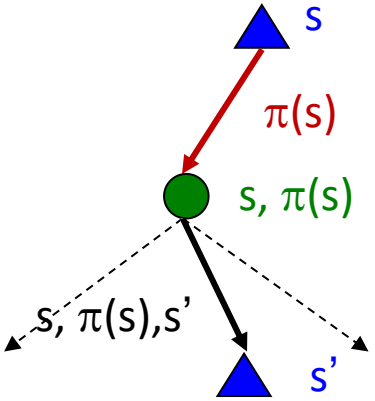


Policy Evaluation

- How do we calculate the V's for a fixed policy π?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$



- Efficiency: $O(S^2)$ per iteration
- Idea 2: Without the maxes, the Bellman equations are just a linear system
  - Solve with Matlab (or your favorite linear system solver)

Policy Iteration

- Evaluation: For fixed current policy π, find values with policy evaluation:
  - Iterate until values converge:

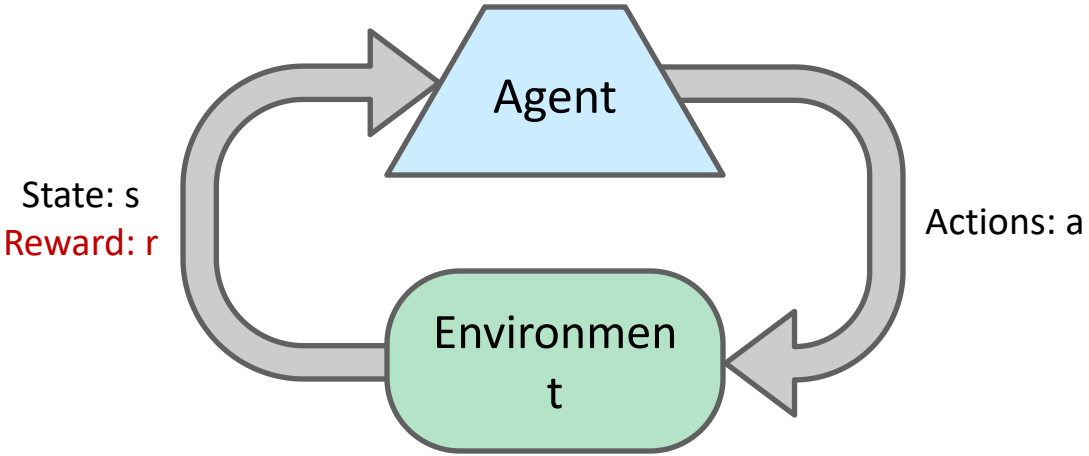
$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s') \right]$$

- Improvement: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s') \right]$$

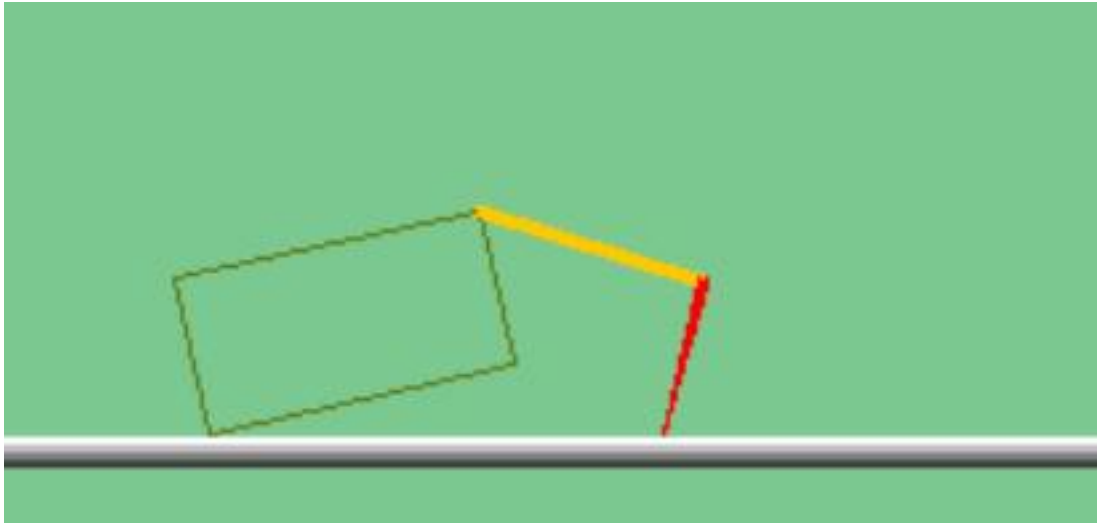
REINFORCEMENT LEARNING

Reinforcement Learning



- Basic idea:
  - Receive feedback in the form of rewards
  - Agent's utility is defined by the reward function
  - Must (learn to) act so as to maximize expected rewards
  - All learning is based on observed samples of outcomes!

The Crawler!



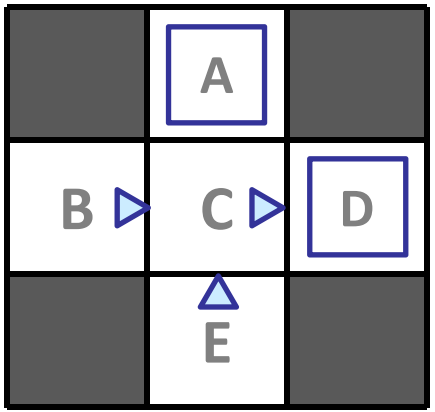
Reinforcement Learning

- Still assume a Markov decision process (MDP):
  - A set of states $s \in S$
  - A set of actions (per state) A
  - A model T(s,a,s')
  - A reward function R(s,a,s')
- Still looking for a policy $\pi(s)$

- New twist: don't know T or R
  - I.e. we don't know which states are good or what the actions do
  - Must actually try actions and states out to learn

Example: Model-Based Learning

Input Policy π



Observed Episodes (Training)

Episode 1

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 2

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 3

E, north, C, -1
C, east, D, -1
D, exit, x, +10

Episode 4

E, north, C, -1
C, east, A, -1
A, exit, x, -10

Learned Model

$\hat{T}(s, a, s')$

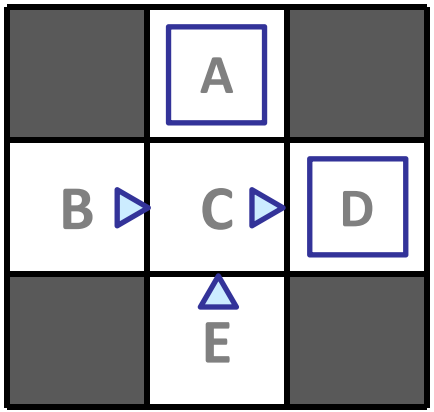
T(B, east, C) = 1.00
T(C, east, D) = 0.75
T(C, east, A) = 0.25
...

$\hat{R}(s, a, s')$

R(B, east, C) = -1
R(C, east, D) = -1
R(D, exit, x) = +10
...

Example: Direct Evaluation

Input Policy π



Observed Episodes (Training)

Episode 1

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 2

B, east, C, -1
C, east, D, -1
D, exit, x, +10

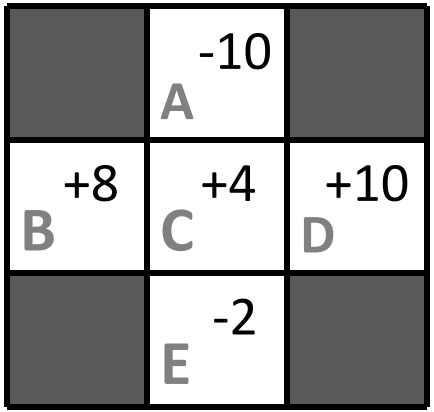
Episode 3

E, north, C, -1
C, east, D, -1
D, exit, x, +10

Episode 4

E, north, C, -1
C, east, A, -1
A, exit, x, -10

Output Values



Sample-Based Policy Evaluation?

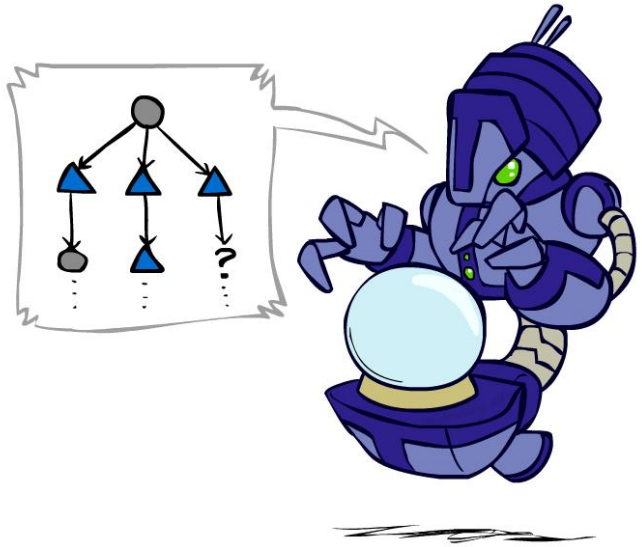
- We want to improve our estimate of V by computing these averages:

$$V^{\pi}_{k+1}(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}_{k}(s')]$$

- Idea: Take samples of outcomes s' (by doing the action!) and average

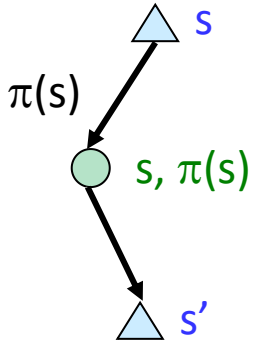
$$sample_1 = R(s, \pi(s), s'_1) + \gamma V^{\pi}_{k}(s'_1)$$
$$sample_2 = R(s, \pi(s), s'_2) + \gamma V^{\pi}_{k}(s'_2)$$
$$\ldots$$
$$sample_n = R(s, \pi(s), s'_n) + \gamma V^{\pi}_{k}(s'_n)$$

$$V^{\pi}_{k+1}(s) \leftarrow \frac{1}{n} \sum_{i} sample_i$$



Temporal Difference Learning

- Big idea: learn from every experience!
  - Update V(s) each time we experience a transition (s, a, s', r)
  - Likely outcomes s' will contribute updates more often



- Temporal difference learning of values
  - Policy still fixed, still doing evaluation!
  - Move values toward value of whatever successor occurs: running average

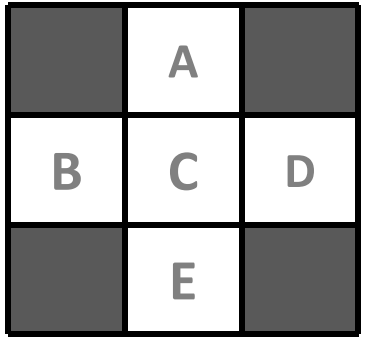
Sample of V(s): $sample = R(s, \pi(s), s') + \gamma V^{\pi}(s')$

Update to V(s): $V^{\pi}(s) \leftarrow (1-\alpha)V^{\pi}(s) + (\alpha)sample$

Same update: $V^{\pi}(s) \leftarrow V^{\pi}(s) + \alpha(sample - V^{\pi}(s))$

Example: Temporal Difference Learning

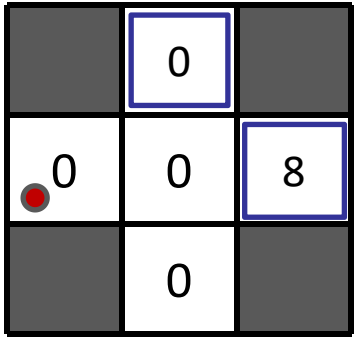
States

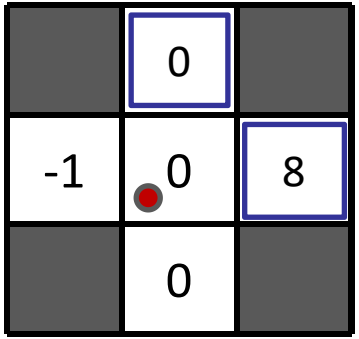


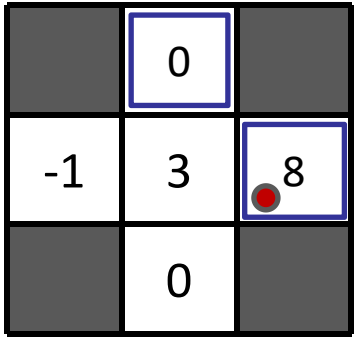
Observed Transitions

B, east, C, -2

C, east, D, -2







$$V^{\pi}(s) \leftarrow (1-\alpha)V^{\pi}(s) + \alpha\left[R(s,\pi(s),s') + \gamma V^{\pi}(s')\right]$$

Q-Learning

- Q-Learning: sample-based Q-value iteration

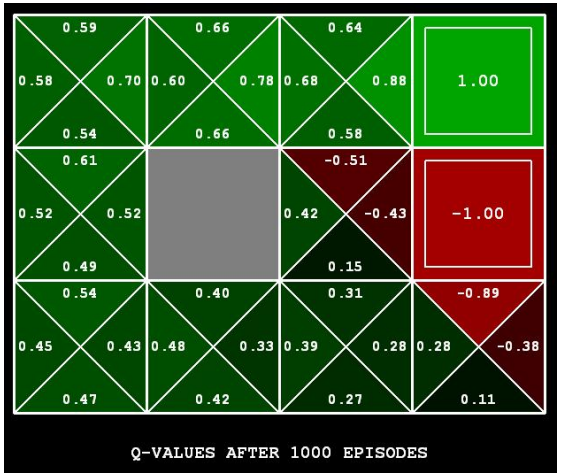
$$Q_{k+1}(s,a) \leftarrow \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \max_{a'} Q_k(s',a') \right]$$

- Learn Q(s,a) values as you go
  - Receive a sample (s,a,s',r)
  - Consider your old estimate: $Q(s,a)$
  - Consider your new sample estimate:

$$sample = R(s,a,s') + \gamma \max_{a'} Q(s',a')$$

  - Incorporate the new estimate into a running average:

$$Q(s,a) \leftarrow (1-\alpha)Q(s,a) + (\alpha)\,[sample]$$



Approximate Q-Learning

$$Q(s,a) = w_1 f_1(s,a) + w_2 f_2(s,a) + \ldots + w_n f_n(s,a)$$

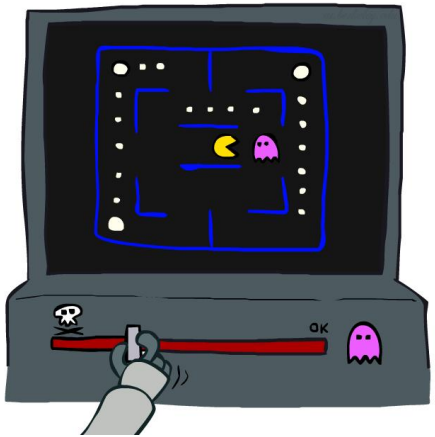
- Q-learning with linear Q-functions:

$$\text{transition} = (s, a, r, s')$$

$$\text{difference} = \left[r + \gamma \max_{a'} Q(s', a') \right] - Q(s,a)$$

$$Q(s,a) \leftarrow Q(s,a) + \alpha \, [\text{difference}]$$

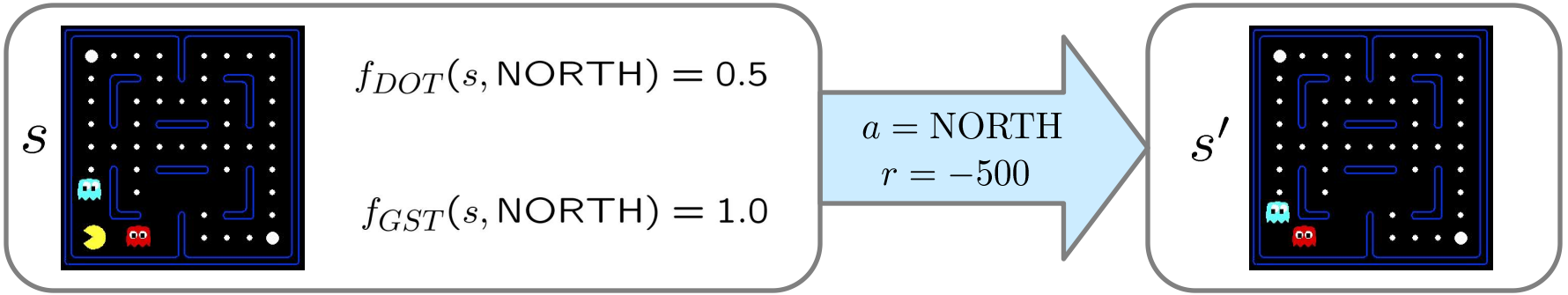
$$w_i \leftarrow w_i + \alpha \, [\text{difference}] \, f_i(s,a)$$



- Intuitive interpretation:
  - Adjust weights of active features
  - E.g., if something unexpectedly bad happens, blame the features that were on: disprefer all states with that state's features
- Formal justification: online least squares

Example: Q-Pacman

$$Q(s,a) = 4.0 f_{DOT}(s,a) - 1.0 f_{GST}(s,a)$$



$Q(s, \text{NORTH}) = +1$

$Q(s', \cdot) = 0$

$$r + \gamma \max_{a'} Q(s', a') = -500 + 0$$

difference $= -501$

$$w_{DOT} \leftarrow 4.0 + \alpha\,[-501]\,0.5$$
$$w_{GST} \leftarrow -1.0 + \alpha\,[-501]\,1.0$$

$$Q(s,a) = 3.0 f_{DOT}(s,a) - 3.0 f_{GST}(s,a)$$

[Demo: approximate Q-learning pacman

BAYES NET

Probability

- Conditional probability $P(x|y) = \frac{P(x,y)}{P(y)}$
- Product rule $P(x,y) = P(x|y)P(y)$
- Chain rule $$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots \\ &= \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1}) \end{aligned}$$
- X, Y independent if and only if: $\forall x, y : P(x,y) = P(x)P(y)$
- X and Y are conditionally independent given Z if and only if: $X \perp\!\!\!\perp Y|Z$

$$\forall x, y, z : P(x,y|z) = P(x|z)P(y|z)$$

Bayes' Rule

- Two ways to factor a joint distribution over two variables:

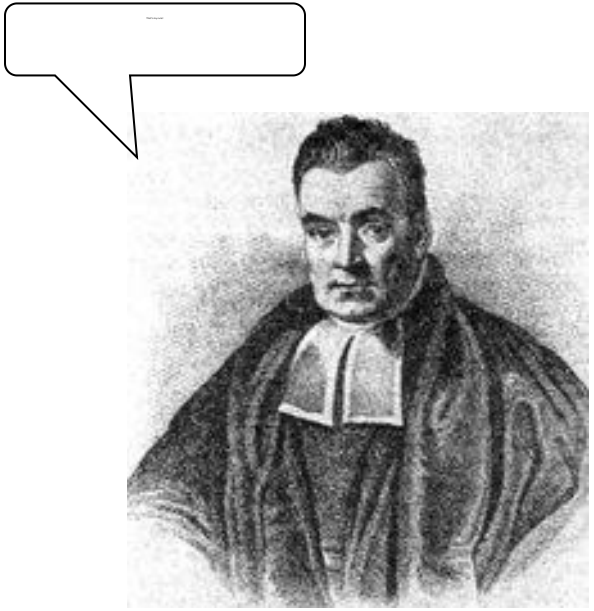
$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

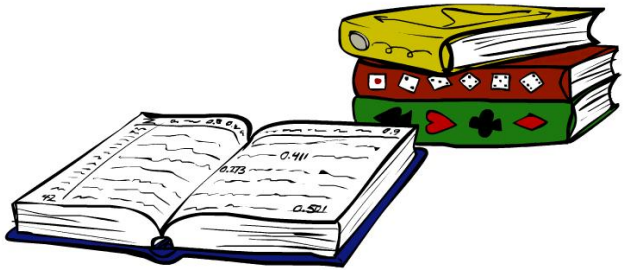
- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we'll see later (e.g. ASR, MT)

- In the running for most important AI equation!

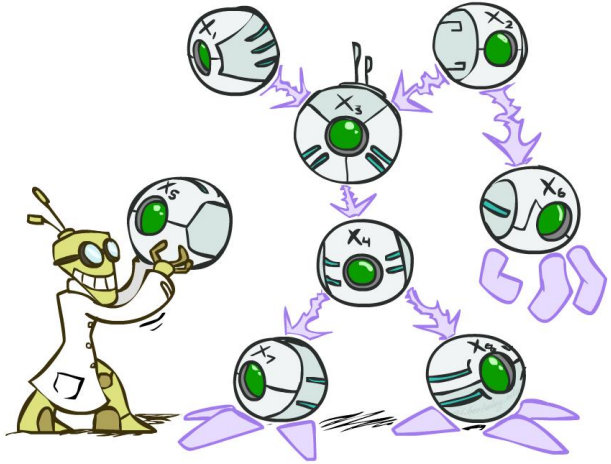


Bayes' Nets: Big Picture

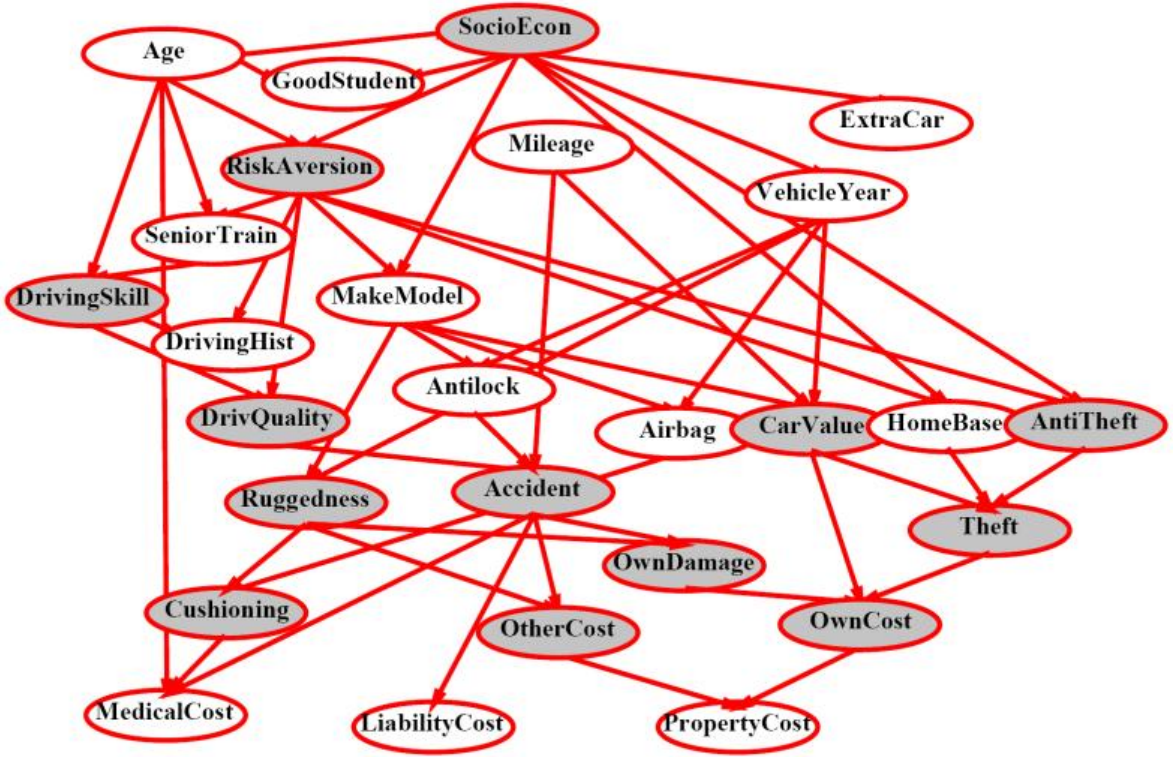
- Two problems with using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time



- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - More properly called graphical models
  - We describe how variables locally interact
  - Local interactions chain together to give global, indirect interactions
  - For about 10 min, we'll be vague about how these interactions are specified



Example Bayes' Net: Insurance

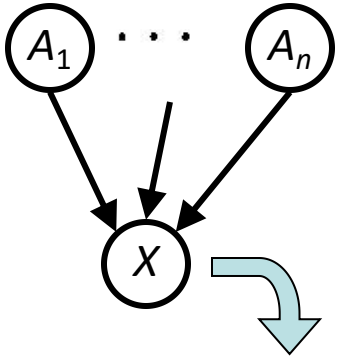


Bayes' Net Semantics

- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \ldots a_n)$$

  - CPT: conditional probability table
  - Description of a noisy "causal" process



$$P(X|A_1 \ldots A_n)$$

A Bayes net = Topology (graph) + Local Conditional Probabilities

Causal Chains

- This configuration is a "causal chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)}$$

$$= P(z|y)$$

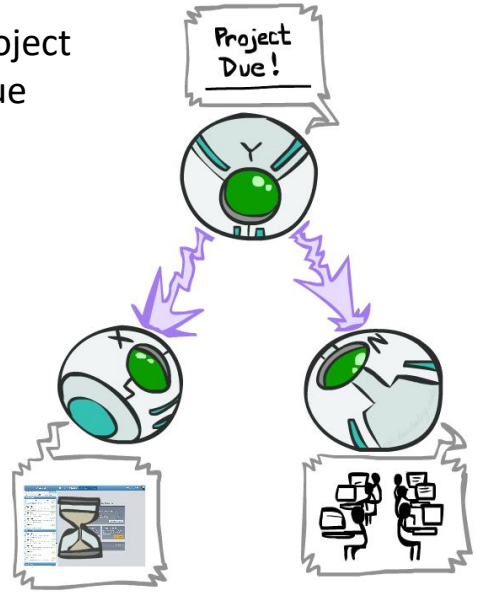
Yes!

- Evidence along the chain "blocks" the influence

Common Cause

- This configuration is a "common cause"

Y: Project due



X: Forums busy

Z: Lab full

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X and Z independent given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)}$$

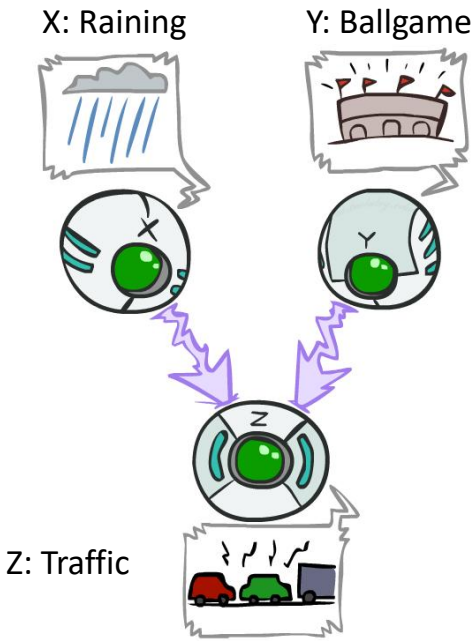
$$= P(z|y)$$

Yes!

- Observing the cause blocks influence between effects.

Common Effect

- Last configuration: two causes of one effect (v-structures)



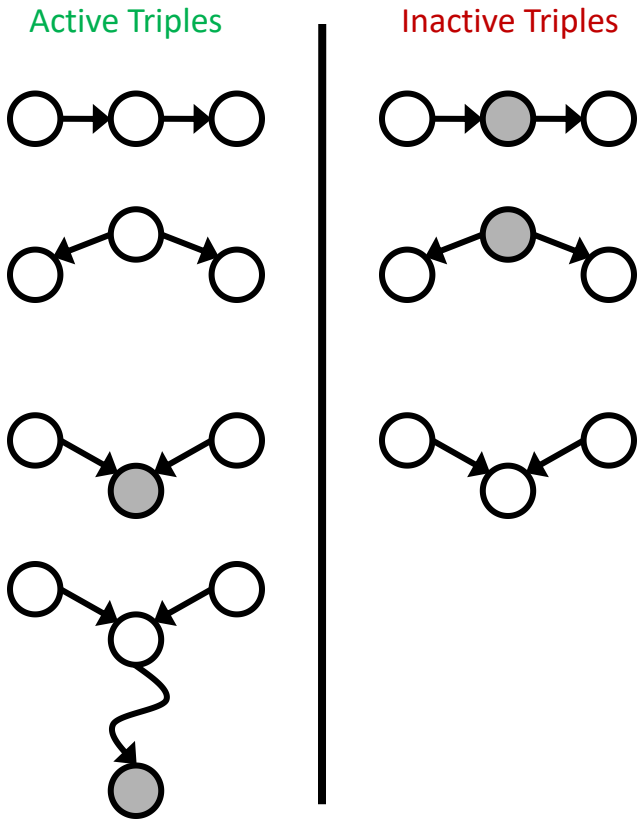
- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the other cases
  - Observing an effect activates influence between possible causes.

Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y "d-separated" by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = independence!

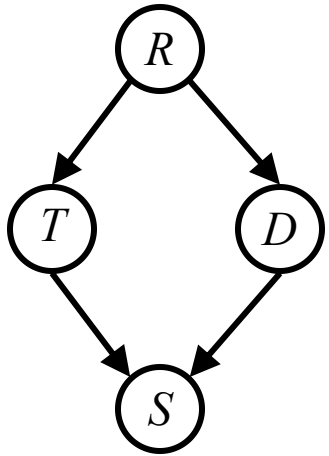
- A path is active if each triple is active:
  - Causal chain A → B → C where B is unobserved (either direction)
  - Common cause A ← B → C where B is unobserved
  - Common effect (aka v-structure)
    A → B ← C where B *or one of its descendents* is observed

- All it takes to block a path is a single inactive segment



Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad
- Questions:



$$T \perp\!\!\!\perp D$$

$$T \perp\!\!\!\perp D | R$$ *Yes*

$$T \perp\!\!\!\perp D | R, S$$

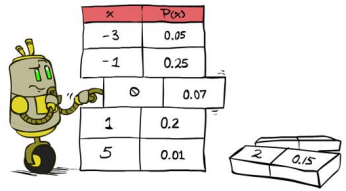
General Variable Elimination

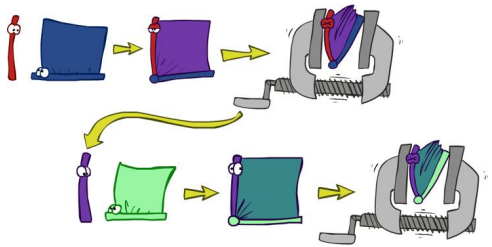
- Query: $P(Q|E_1 = e_1, \ldots E_k = e_k)$

- Start with initial factors:
  - Local CPTs (but instantiated by evidence)

- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H

- Join all remaining factors and normalize

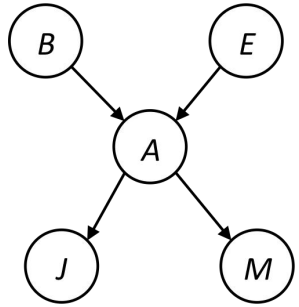




$\times \frac{1}{Z}$

Example

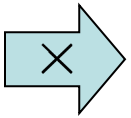
$$P(B|j,m) \propto P(B,j,m)$$



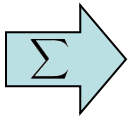
$P(B)$	$P(E)$	$P(A\|B,E)$	$P(j\|A)$	$P(m\|A)$

Choose A

$P(A|B,E)$
$P(j|A)$
$P(m|A)$



$P(j,m,A|B,E)$

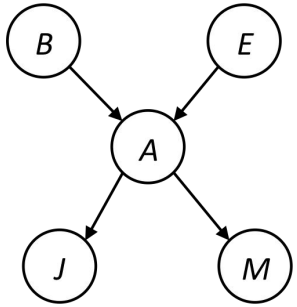


$P(j,m|B,E)$

$P(B)$	$P(E)$	$P(j,m\|B,E)$

Example

$P(B)$ $P(E)$ $P(j, m|B, E)$



Choose E

$P(E)$

$P(j, m|B, E)$ $\times$ $P(j, m, E|B)$ $\sum$ $P(j, m|B)$

$P(B)$ $P(j, m|B)$

Finish with B

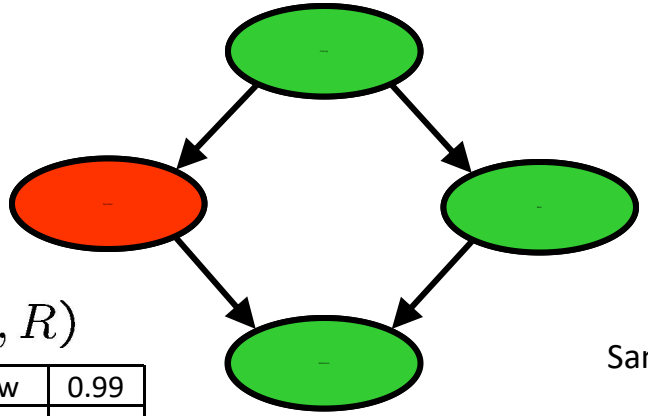
$P(B)$

$P(j, m|B)$ $\times$ $P(j, m, B)$ $-$ $P(B|j, m)$

Prior Sampling

$P(C)$

+c	0.5
-c	0.5



$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5

$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

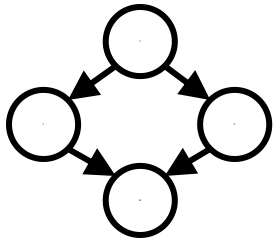
+c, -s, +r, +w

-c, +s, -r, +w

...

Rejection Sampling

- Let's say we want P(C)
  - No point keeping all samples around
  - Just tally counts of C as we go
- Let's say we want P(C| +s)
  - Same thing: tally C outcomes, but ignore (reject) samples which don't have S=+s
  - This is called rejection sampling
  - It is also consistent for conditional probabilities (i.e., correct in the limit)



+c, -s, +r, +w
+c, +s, +r, +w
-c, +s, +r, -w
+c, -s, +r, +w
-c, -s, -r, +w

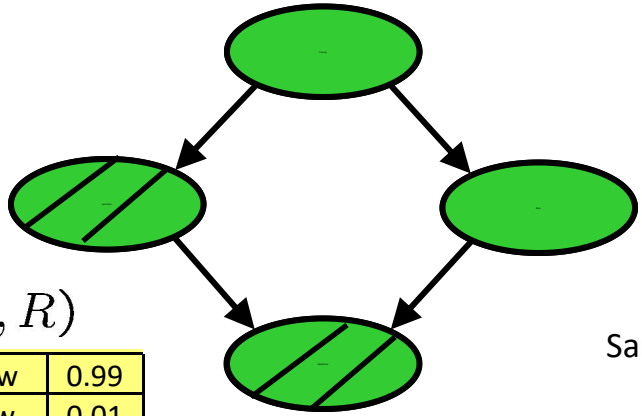
Likelihood Weighting

$P(C)$

+c	0.5
-c	0.5

$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5



$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

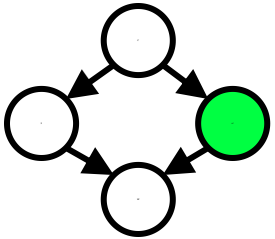
+c, +s, +r, +w

…

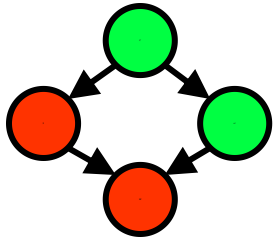
$$w = 1.0 \times 0.1 \times 0.99$$

Gibbs Sampling Example: P(S | +r)

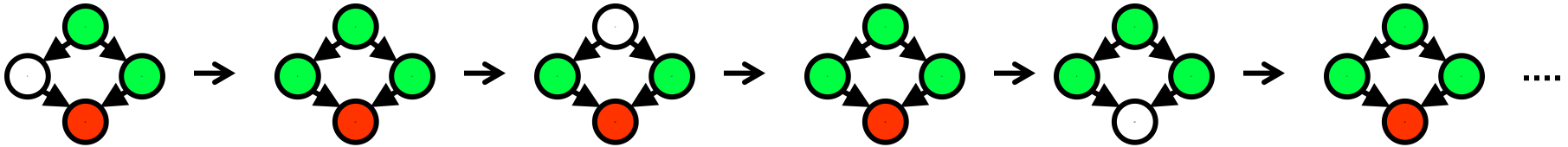
- Step 1: Fix evidence
  - R = +r



- Step 2: Initialize other variables
  - Randomly



- Steps 3: Repeat
  - Choose a non-evidence variable X
  - Resample X from P(X | all other variables)

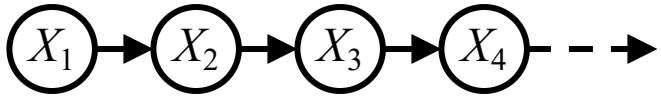


Sample from $P(S|+c,-w,+r)$ Sample from $P(C|+s,-w,+r)$ Sample from $P(W|+s,+c,+r)$

MARKOV MODELS

Markov Models

- Value of X at a given time is called the state

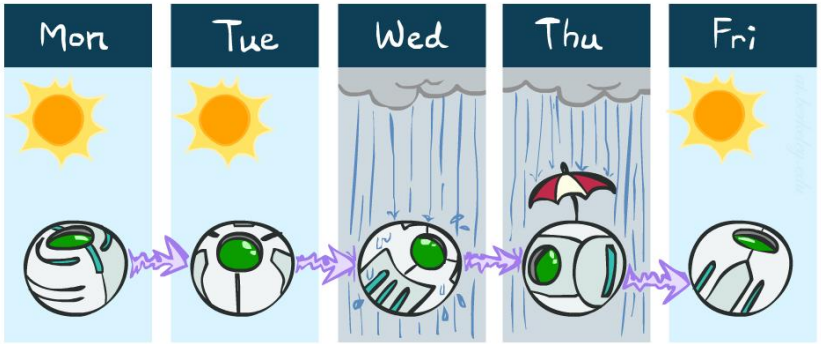


$P(X_1)$ $P(X_t|X_{t-1})$

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

Example Markov Chain: Weather

- States: X = {rain, sun}

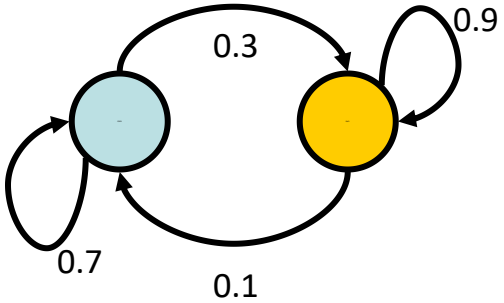


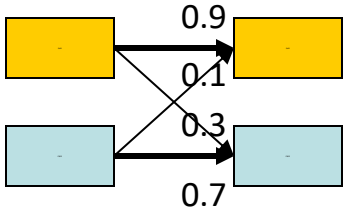
- Initial distribution: 1.0 sun

- CPT $P(X_t \mid X_{t-1})$:

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

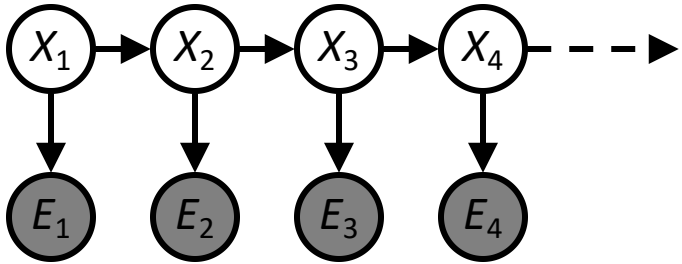
Two new ways of representing the same CPT



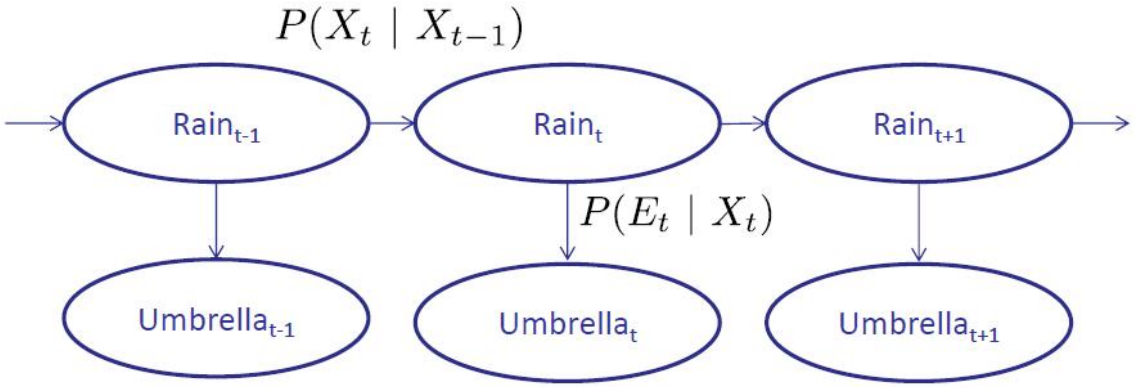


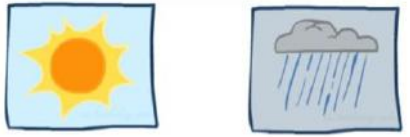
Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe outputs (effects) at each time step



Example: Weather HMM





- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t \mid X_{t-1})$
  - Emissions: $P(E_t \mid X_t)$

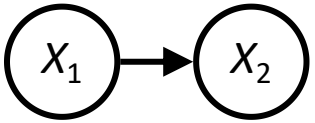
R_t	R_{t+1}	$P(R_{t+1} \mid R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t \mid R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Passage of Time

- Assume we have current belief P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$



- Then, after one time step passes:

$$\begin{aligned} P(X_{t+1}|e_{1:t}) &= \sum_{x_t} P(X_{t+1}, x_t|e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t}) P(x_t|e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1}|x_t) P(x_t|e_{1:t}) \end{aligned}$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t) B(x_t)$$

- Basic idea: beliefs get "pushed" through the transitions
  - With the "B" notation, we have to be careful about what time step t the belief is about, and what evidence it includes

Observation

- Assume we have current belief P(X | previous evidence):

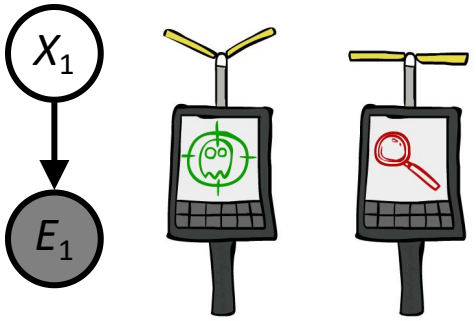
$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

- Then, after evidence comes in:

$$\begin{aligned} P(X_{t+1}|e_{1:t+1}) &= P(X_{t+1}, e_{t+1}|e_{1:t})/P(e_{t+1}|e_{1:t}) \\ &\propto_{X_{t+1}} P(X_{t+1}, e_{t+1}|e_{1:t}) \\ &= P(e_{t+1}|e_{1:t}, X_{t+1})P(X_{t+1}|e_{1:t}) \\ &= P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t}) \end{aligned}$$

- Or, compactly:

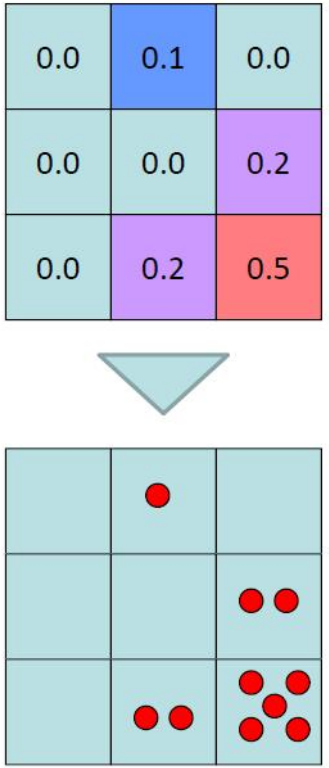
$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$



- Basic idea: beliefs "reweighted" by likelihood of evidence
- Unlike passage of time, we have to renormalize

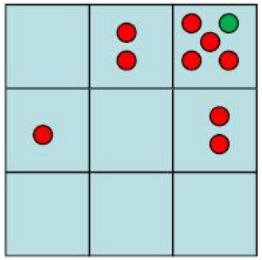
Particle Filtering

- Filtering: approximate solution
- Sometimes |X| is too big to use exact inference
  - |X| may be too big to even store B(X)
  - E.g. X is continuous
- Solution: approximate inference
  - Track samples of X, not all values
  - Samples are called particles
  - Time per step is linear in the number of samples
  - But: number needed may be large
  - In memory: list of particles, not states
- This is how robot localization works in practice
- Particle is just new name for sample



Particles

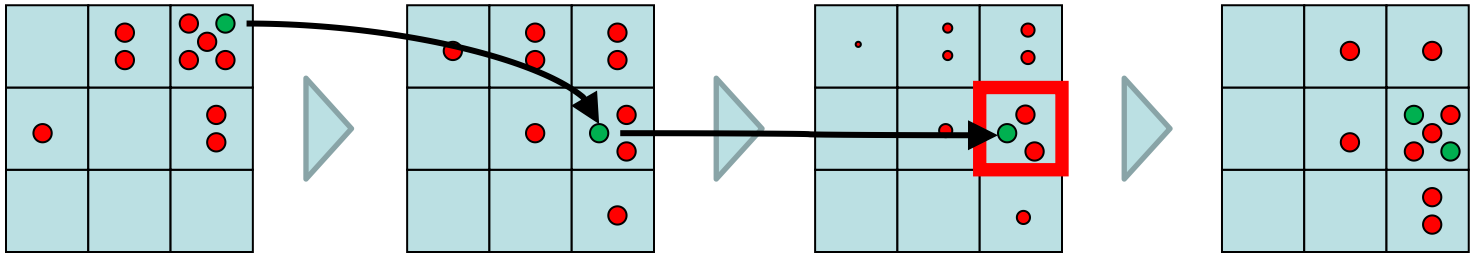
- Our representation of P(X) is now a list of N particles (samples)
  - Generally, N << |X|
  - Storing map from X to counts would defeat the point
- P(x) approximated by number of particles with value x
  - So, many x may have P(x) = 0!
  - More particles, more accuracy
- For now, all particles have a weight of 1



Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)

Particle Filtering

- Particles: track samples of states rather than an explicit distribution



Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)

Particles:
(3,2)
(2,3)
(3,2)
(3,1)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(2,2)

Particles:
(3,2) w=.9
(2,3) w=.2
(3,2) w=.9
(3,1) w=.4
(3,3) w=.4
(3,2) w=.9
(1,3) w=.1
(2,3) w=.2
(3,2) w=.9
(2,2) w=.4

(New) Particles:
(3,2)
(2,2)
(3,2)
(2,3)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(3,2)

[Demos: ghostbusters particle filtering (L15D3,4,5)]

MACHINE LEARNING

Example: Spam Filter

- Input: an email
- Output: spam/ham

- Setup:
  - Get a large collection of example emails, each labeled "spam" or "ham"
  - Note: someone has to hand label all this data!
  - Want to learn to predict labels of new, future emails

- Features: The attributes used to make the ham / spam decision
  - Words: FREE!
  - Text Patterns: $dd, CAPS
  - Non-text: SenderInContacts
  - …

Dear Sir.

First, I must solicit your confidence in this transaction, this is by virture of its nature as being utterly confidencial and top secret. …

TO BE REMOVED FROM FUTURE MAILINGS, SIMPLY REPLY TO THIS MESSAGE AND PUT "REMOVE" IN THE SUBJECT.

99 MILLION EMAIL ADDRESSES FOR ONLY $99

Ok, Iknow this is blatantly OT but I'm beginning to go insane. Had an old Dell Dimension XPS sitting in the corner and decided to put it to use, I know it was working pre being stuck in the corner, but when I plugged it in, hit the power nothing happened.

Example: Digit Recognition

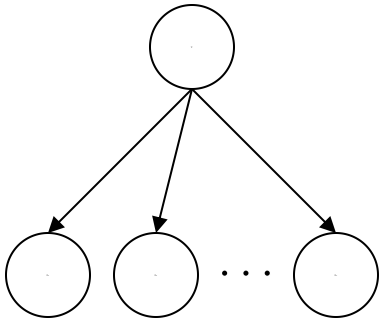
- Input: images / pixel grids
- Output: a digit 0-9

- Setup:
  - Get a large collection of example images, each labeled with a digit
  - Note: someone has to hand label all this data!
  - Want to learn to predict labels of new, future digit images

- Features: The attributes used to make the digit decision
  - Pixels: (6,8)=ON
  - Shape Patterns: NumComponents, AspectRatio, NumLoops
  - …

Naïve Bayes for Digits

- Naïve Bayes: Assume all features are independent effects of the label



- Simple digit recognition version:
  - One feature (variable) F_{ij} for each grid position <i,j>
  - Feature values are on / off, based on whether intensity is more or less than 0.5 in underlying image
  - Each input maps to a feature vector, e.g.

$$\rightarrow \langle F_{0,0} = 0 \;\; F_{0,1} = 0 \;\; F_{0,2} = 1 \;\; F_{0,3} = 1 \;\; F_{0,4} = 0 \;\; \ldots F_{15,15} = 0 \rangle$$

  - Here: lots of features, each is binary valued

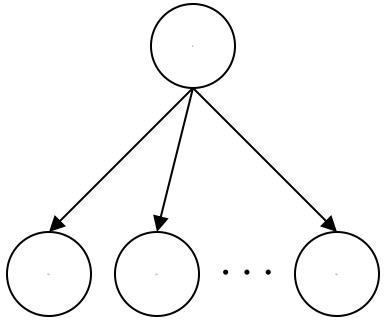
- Naïve Bayes model: $P(Y|F_{0,0} \ldots F_{15,15}) \propto P(Y) \prod_{i,j} P(F_{i,j}|Y)$

- What do we need to learn?

General Naïve Bayes

- A general Naive Bayes model:

$$P(\mathsf{Y}, \mathsf{F}_1 \ldots \mathsf{F}_n) = P(\mathsf{Y}) \prod_i P(\mathsf{F}_i|\mathsf{Y})$$



- We only have to specify how each feature depends on the class
- Total number of parameters is *linear* in n
- Model is very simplistic, but often works anyway

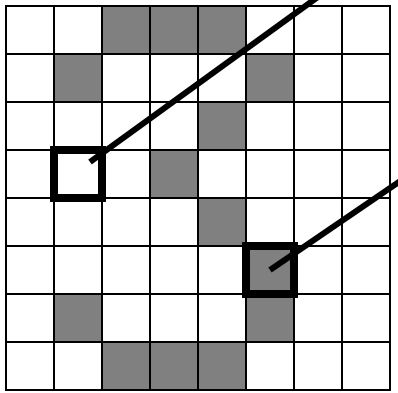
General Naïve Bayes

- What do we need in order to use Naïve Bayes?
  - Inference method (we just saw this part)
    - Start with a bunch of probabilities: P(Y) and the $P(F_i|Y)$ tables
    - Use standard inference to compute $P(Y|F_1 \ldots F_n)$
    - Nothing new here
  - Estimates of local conditional probability tables
    - P(Y), the prior over labels
    - $P(F_i|Y)$ for each feature (evidence variable)
    - These probabilities are collectively called the *parameters* of the model and denoted by θ
    - Up until now, we assumed these appeared by magic, but…
    - …they typically come from training data counts: we'll look at this soon

Example: Conditional Probabilities

$P(Y)$

1	0.1
2	0.1
3	0.1
4	0.1
5	0.1
6	0.1
7	0.1
8	0.1
9	0.1
0	0.1



$P(F_{3,1} = on|Y)$

1	0.01
2	0.05
3	0.05
4	0.30
5	0.80
6	0.90
7	0.05
8	0.60
9	0.50
0	0.80

$P(F_{5,5} = on|Y)$

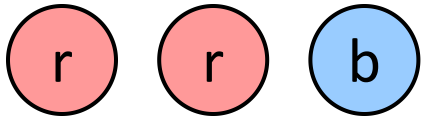
1	0.05
2	0.01
3	0.90
4	0.80
5	0.90
6	0.90
7	0.25
8	0.85
9	0.60
0	0.80

Laplace Smoothing

- Laplace's estimate:
  - Pretend you saw every outcome once more than you actually did

$$P_{LAP}(x) = \frac{c(x)+1}{\sum_x [c(x)+1]}$$

$$= \frac{c(x)+1}{N+|X|}$$



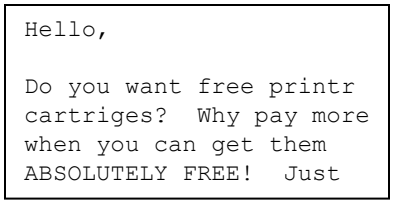
$$P_{ML}(X) =$$

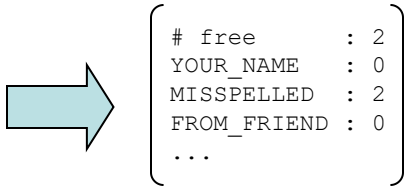
$$P_{LAP}(X) =$$

  - Can derive this estimate with *Dirichlet priors* (see cs281a)

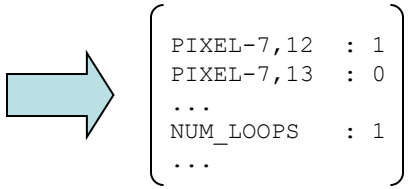
Feature Vectors

x $f(x)$ y





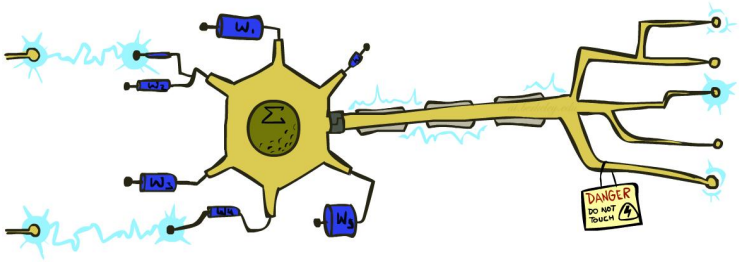
SPAM
or
+



"2"

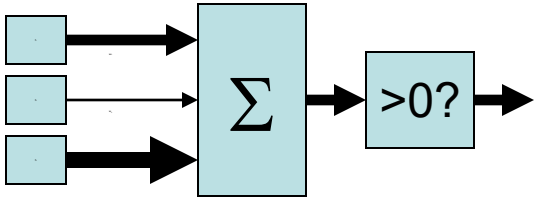
Linear Classifiers

- Inputs are feature values
- Each feature has a weight
- Sum is the activation



$$\text{activation}_w(x) = \sum_i w_i \cdot f_i(x) = w \cdot f(x)$$

- If the activation is:
  - Positive, output +1
  - Negative, output -1



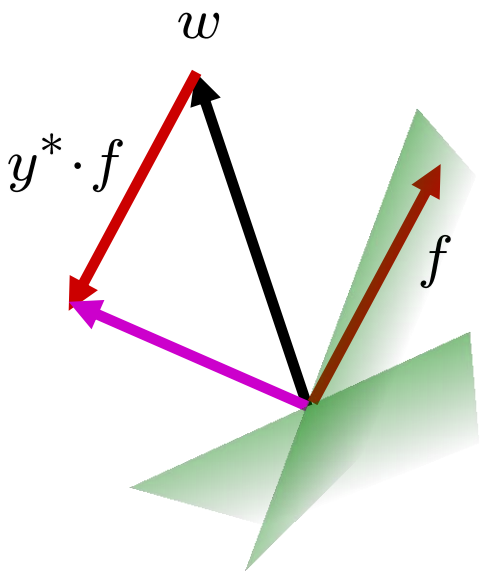
Learning: Binary Perceptron

- Start with weights = 0
- For each training instance:
  - Classify with current weights

$$y = \begin{cases} +1 & \text{if } w \cdot f(x) \geq 0 \\ -1 & \text{if } w \cdot f(x) < 0 \end{cases}$$

  - If correct (i.e., y=y*), no change!
  - If wrong: adjust the weight vector by adding or subtracting the feature vector. Subtract if y* is -1.

$$w = w + y^* \cdot f$$



Perceptron Weights

- What is the final value of a weight w_y of a perceptron?
  - Can it be any real vector?
  - No! It's built by adding up inputs.

$$w_y = \mathbf{0} + f(x_1) - f(x_5) + \dots$$

$$w_y = \sum_i \alpha_{i,y} f(x_i)$$

- Can reconstruct weight vectors (the primal representation) from update counts (the dual representation)

$$\alpha_y = \langle \alpha_{1,y} \;\; \alpha_{2,y} \;\; \dots \;\; \alpha_{n,y} \rangle$$

Dual Perceptron

- How to classify a new example x?

$$\begin{aligned}\text{score}(y,x) &= w_y \cdot f(x) \\ &= \left(\sum_i \alpha_{i,y} f(x_i)\right) \cdot f(x) \\ &= \sum_i \alpha_{i,y} \left(f(x_i) \cdot f(x)\right) \\ &= \sum_i \alpha_{i,y} K(x_i, x)\end{aligned}$$

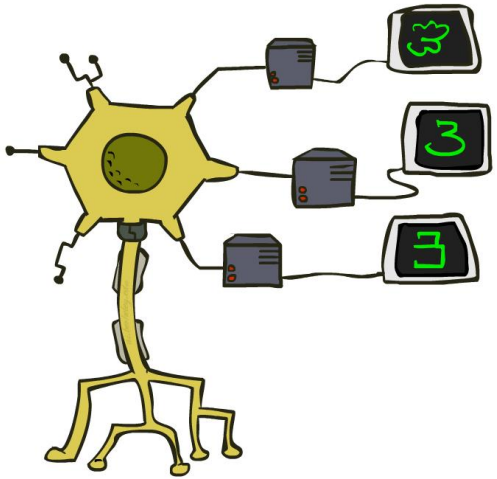
- If someone tells us the value of K for each pair of examples, never need to build the weight vectors (or the feature vectors)!

Kernelized Perceptron

- If we had a black box (kernel) K that told us the dot product of two examples x and x':
  - Could work entirely with the dual representation
  - No need to ever take dot products ("kernel trick")

$$\text{score}(y, x) = w_y \cdot f(x)$$

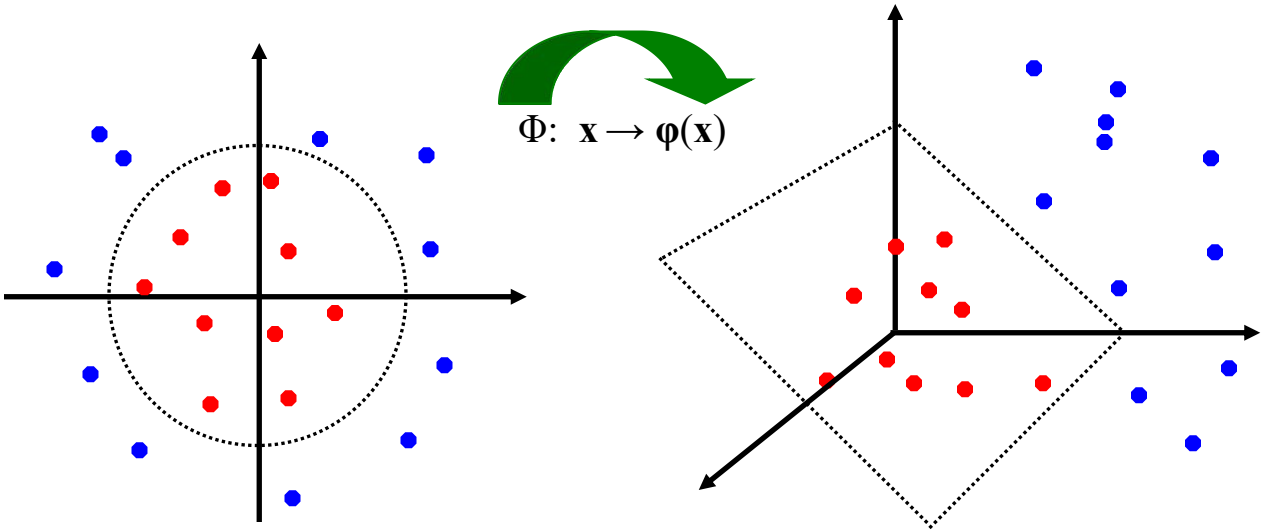
$$= \sum_i \alpha_{i,y} \, K(x_i, x)$$



- Like nearest neighbor – work with black-box similarities
- Downside: slow if many examples get nonzero alpha

Non-Linear Separators

- General idea: the original feature space can always be mapped to some higher-dimensional feature space where the training set is separable:



Some Kernels

- Kernels implicitly map original vectors to higher dimensional spaces, take the dot product there, and hand the result back

- Linear kernel: $K(x, x') = x' \cdot x' = \sum_i x_i \, x'_i$

- Quadratic kernel: $K(x, x') = (x \cdot x' + 1)^2$

$$= \sum_{i,j} x_i x_j \, x'_i x'_j + 2 \sum_i x_i \, x'_i + 1$$

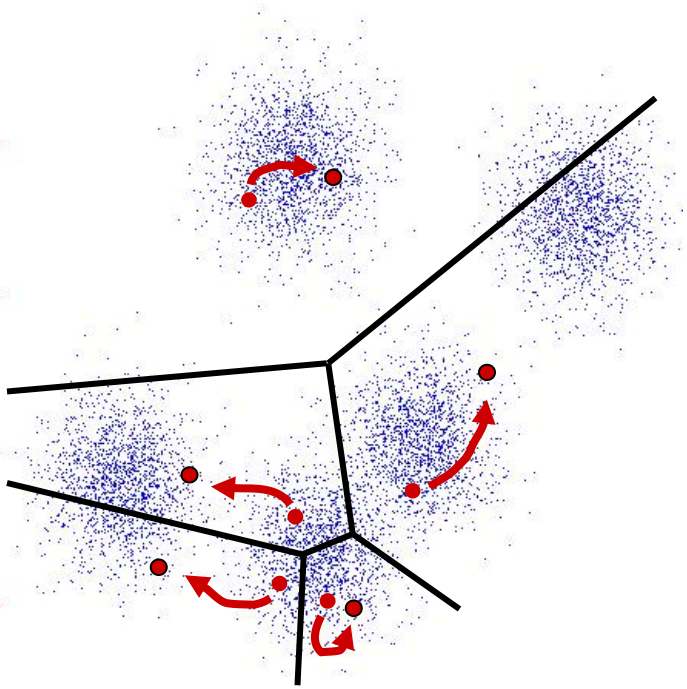
- RBF: infinite dimensional representation

$$K(x, x') = \exp(-||x - x'||^2)$$

- Discrete kernels: e.g. string kernels

K-Means

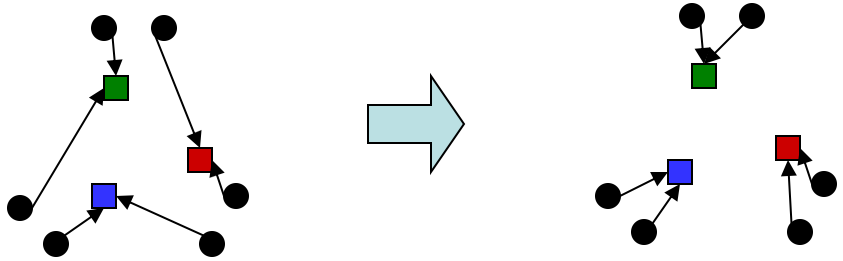
- An iterative clustering algorithm
  - Pick K random points as cluster centers (means)
  - Alternate:
    - Assign data instances to closest mean
    - Assign each mean to the average of its assigned points
  - Stop when no points' assignments change



Phase I: Update Assignments

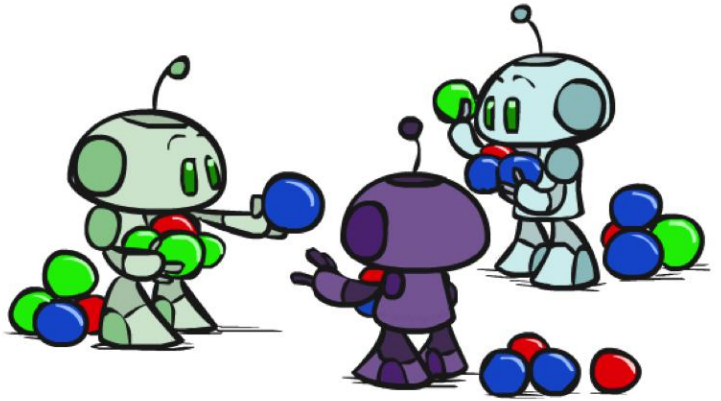
- For each point, re-assign to closest mean:

$$a_i = \underset{k}{\text{argmin}}\, \text{dist}(x_i, c_k)$$



- Can only decrease total distance phi!

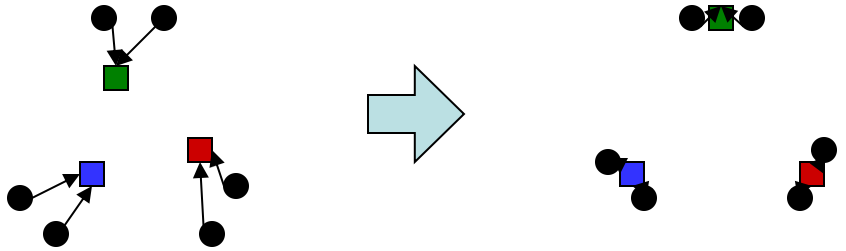
$$\phi(\{x_i\}, \{a_i\}, \{c_k\}) = \sum_i \text{dist}(x_i, c_{a_i})$$



Phase II: Update Means

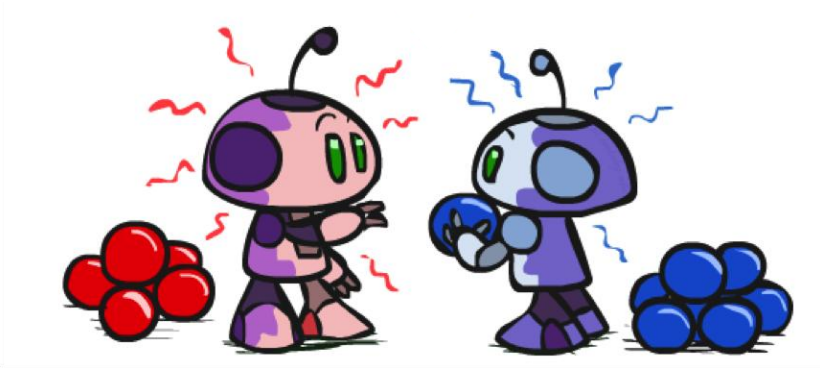
- Move each mean to the average of its assigned points:

$$c_k = \frac{1}{|\{i : a_i = k\}|} \sum_{i:a_i=k} x_i$$



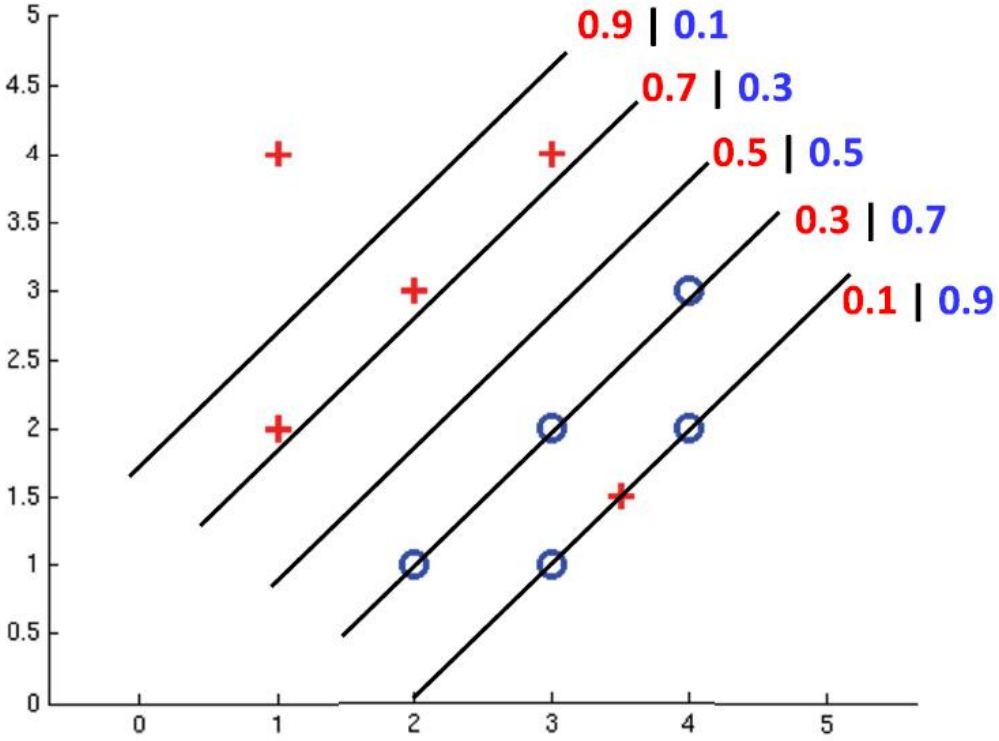
- Also can only decrease total distance… (Why?)

- Fun fact: the point y with minimum squared Euclidean distance to a set of points {x} is their mean



NEURAL NETWORKS

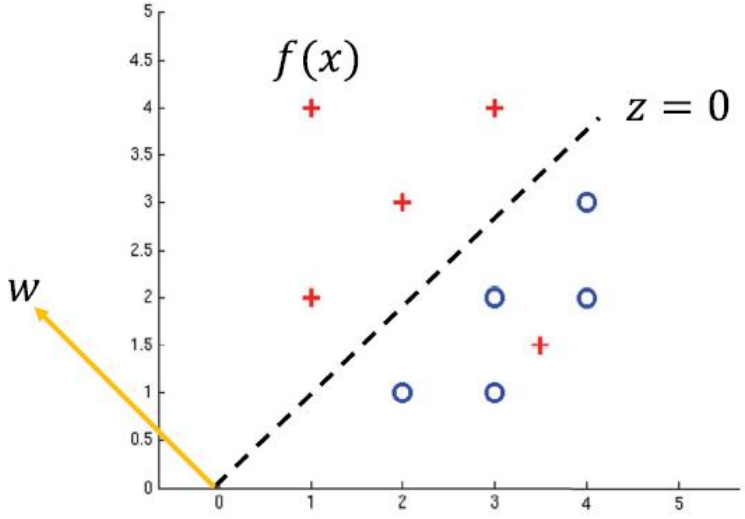
Probabilistic decision



How to get probabilistic decision?

- Perceptron scoring: $z = w \cdot f(x)$
- If $z = w \cdot f(x)$ very positive → want probability going to 1
- If $z = w \cdot f(x)$ very negative → want probability going to 0

- Example:



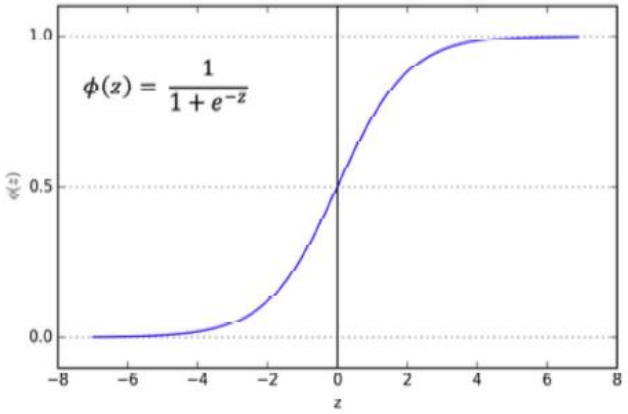
Sigmoid function

- Perceptron scoring: $z = w \cdot f(x)$
- If $z = w \cdot f(x)$ very positive → want probability going to 1
- If $z = w \cdot f(x)$ very negative → want probability going to 0

- Sigmoid function

$$\phi(z) = \frac{1}{1 + e^{-z}}$$

$$= \frac{e^{z}}{e^{z} + 1}$$



Logistic regression

- Perceptron scoring: $z = w \cdot f(x)$
- If $z = w \cdot f(x)$ very positive → want probability going to 1
- If $z = w \cdot f(x)$ very negative → want probability going to 0

- Sigmoid function

$$\phi(z) = \frac{1}{1 + e^{-z}}$$

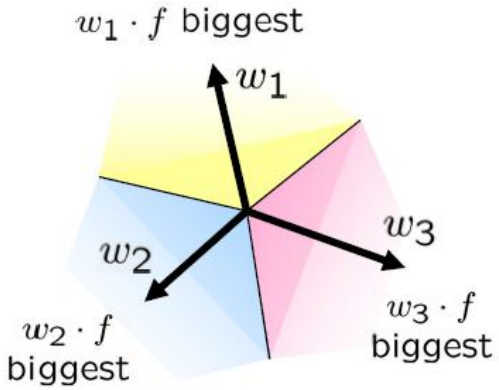
$$P(y^{(i)} = +1 | x^{(i)}; w) = \frac{1}{1 + e^{-w \cdot f(x^{(i)})}}$$

$$P(y^{(i)} = -1 | x^{(i)}; w) = 1 - \frac{1}{1 + e^{-w \cdot f(x^{(i)})}}$$

= Logistic Regression

Multiclass logistic regression

- Recall Perceptron:
  - A weight vector for each class: w_y
  - Score (activation) of a class y: $z = w_y \cdot f(x)$
  - Prediction highest score wins $y = \arg\max_y \ w_y \cdot f(x)$



- How to make the scores into probabilities?

$$z_1, z_2, z_3 \rightarrow \frac{e^{z_1}}{e^{z_1}+e^{z_2}+e^{z_3}}, \frac{e^{z_2}}{e^{z_1}+e^{z_2}+e^{z_3}}, \frac{e^{z_3}}{e^{z_1}+e^{z_2}+e^{z_3}}$$

original activations

softmax activations

- In general: $\text{softmax}(z_1, \ldots, z_n)_i = \frac{e^{z_i}}{\sum_j e^{z_j}}$

Want to find best w

- Given data pairs $x^{(i)}, y^{(i)}$ maximize log-likelihood:

$$\hat{w} = \underset{w}{\operatorname{argmax}} \sum_i \log P(y^{(i)}|x^{(i)}; w)$$

with:

$$P(y^{(i)}|x^{(i)}; w) = \frac{e^{w_{y^{(i)}} \cdot f(x^{(i)})}}{\sum_y e^{w_y \cdot f(x^{(i)})}}$$

Hill climbing

$$\hat{w} = \underset{w}{\operatorname{argmax}} \sum_i \log P(y^{(i)} | x^{(i)}; w)$$

In general, cannot always take derivative and set to 0

Use numerical optimization!

Gradient ascent

Perform update in uphill direction for each coordinate

The steeper the slope (i.e. the higher the derivative) the bigger the step for that coordinate

E.g., consider: $g(w_1, w_2)$

Updates:

$$w_1 \leftarrow w_1 + \alpha * \frac{\partial g}{\partial w_1}(w_1, w_2)$$

$$w_2 \leftarrow w_2 + \alpha * \frac{\partial g}{\partial w_2}(w_1, w_2)$$

- Updates in vector notation:

$$w \leftarrow w + \alpha * \nabla_w g(w)$$

with: $\nabla_w g(w) = \begin{bmatrix} \frac{\partial g}{\partial w_1}(w) \\ \frac{\partial g}{\partial w_2}(w) \end{bmatrix}$ **= gradient**

Gradient ascent

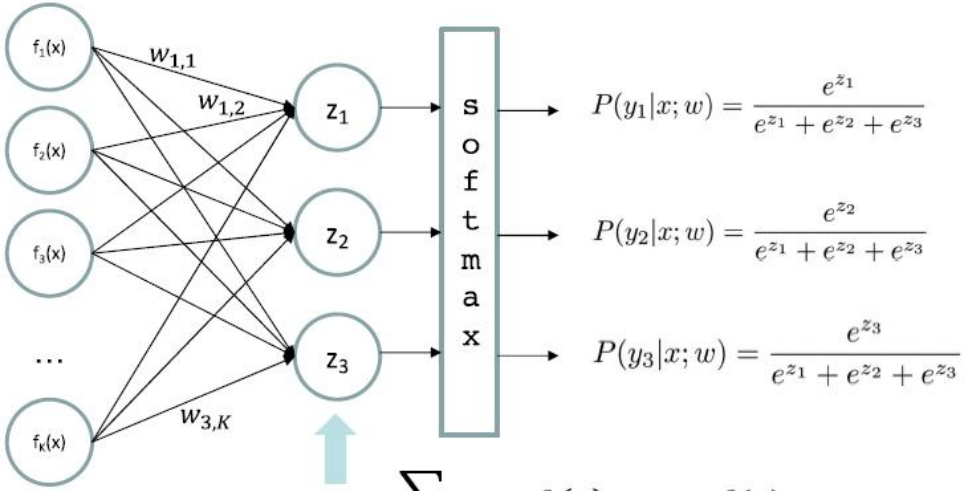
```
init w
for iter = 1, 2, …
```

$$w \leftarrow w + \alpha * \nabla g(w)$$

- α: learning rate --- tweaking parameter that needs to be chosen carefully
- How? Try multiple choices
  - Crude rule of thumb: update changes w about 0.1 – 1 %

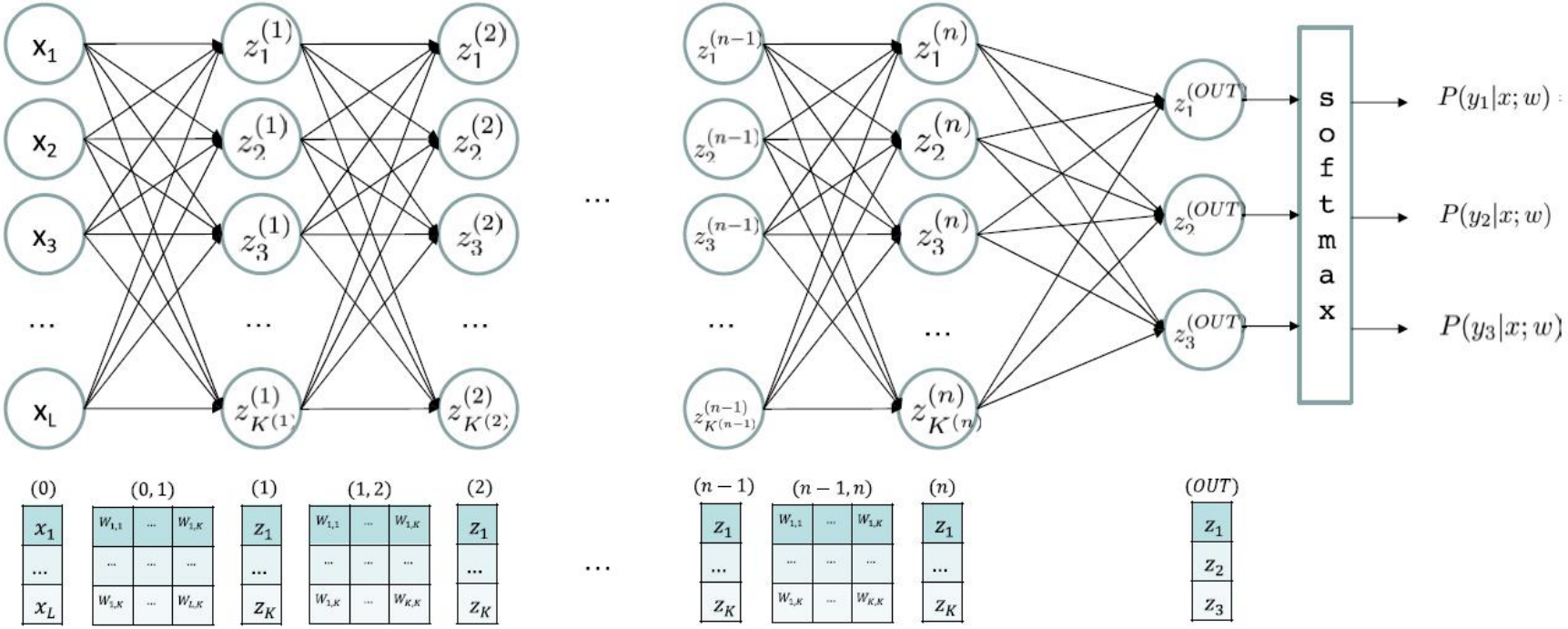
Multiclass logistic regression

= special case of neural network



$$z_i = \sum_j w_{i,j} \cdot f_i(x) = w_i \cdot f(x)$$

Deep neural network: also learn features

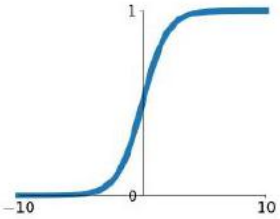


More compactly as matrix multiplication: $z^{(k)} = g(W^{(k-1,k)} z^{(k-1)})$

Common activation functions

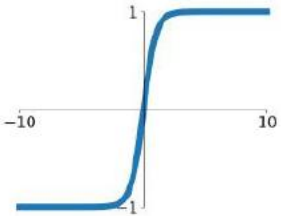
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



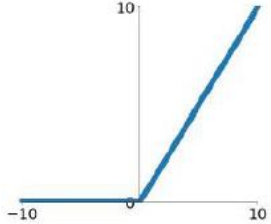
tanh

$$\tanh(x)$$



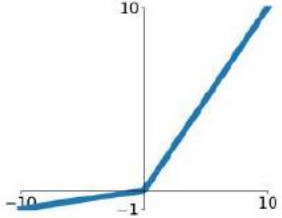
ReLU

$$\max(0, x)$$



Leaky ReLU

$$\max(0.1x, x)$$

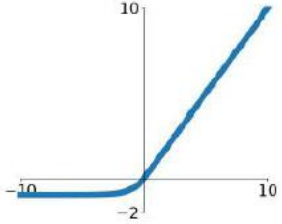


Maxout

$$\max(w_1^T x + b_1, w_2^T x + b_2)$$

ELU

$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$



Universal function approximation theorem

Hornik theorem 1: Whenever the activation function is *bounded and nonconstant*, then, for any finite measure μ, standard multilayer feedforward networks can approximate any function in $L^p(\mu)$ (the space of all functions on R^k such that $\int_{R^k} |f(x)|^p d\mu(x) < \infty$) arbitrarily well, provided that sufficiently many hidden units are available.

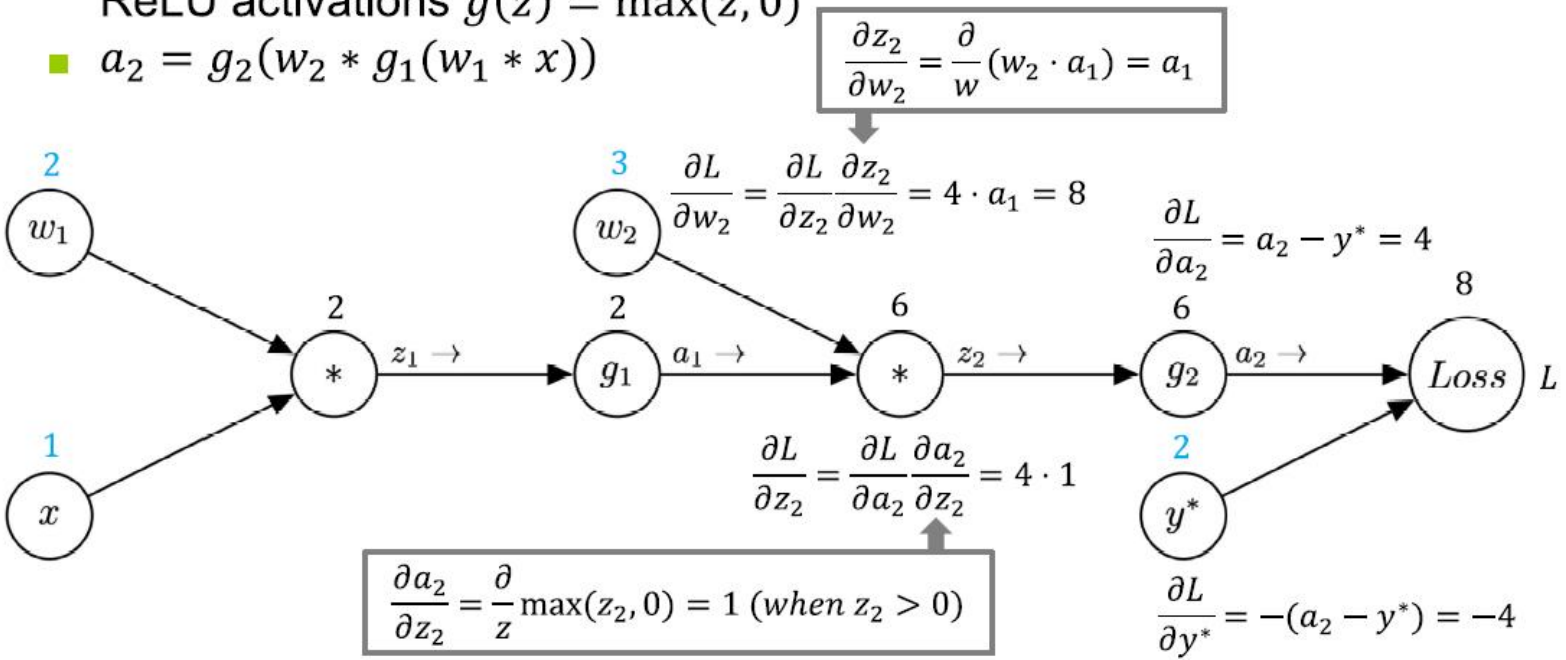
Hornik theorem 2: Whenever the activation function is *continuous, bounded and nonconstant*, then, for arbitrary compact subsets $X \subseteq R^k$, standard multilayer feedforward networks can approximate any continuous function on X arbitrarily well with respect to uniform distance, provided that sufficiently many hidden units are available.

In words: Given any continuous function f(x), if a 2-layer neural network has enough hidden units, then there is a choice of weights that allow it to closely approximate f(x).

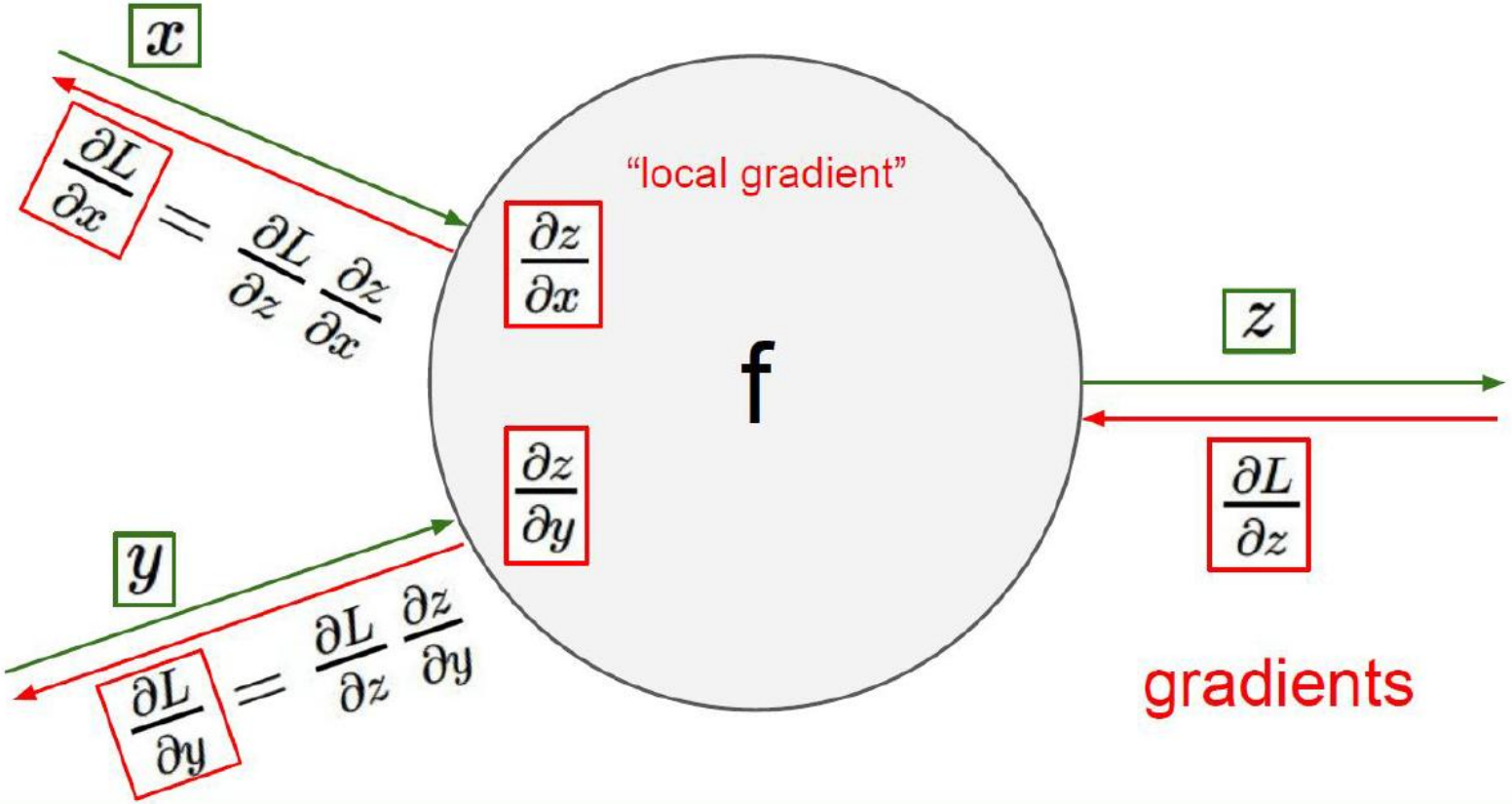
Cybenko (1989) "Approximations by superpositions of sigmoidal functions"
Hornik (1991) "Approximation Capabilities of Multilayer Feedforward Networks"
Leshno and Schocken (1991) "Multilayer Feedforward Networks with Non-Polynomial Activation Functions Can Approximate Any Function"

Differentiation

- Build a *computation graph* and use chain rule: $f(x) = g(h(x)) \quad f'(x) = g'(h(x))h'(x)$
- Example: neural network with quadratic loss $L(a_2, y^*) = \frac{1}{2}(a_2 - y^*)^2$ and ReLU activations $g(z) = \max(z, 0)$
- $a_2 = g_2(w_2 * g_1(w_1 * x))$



Gradient flow

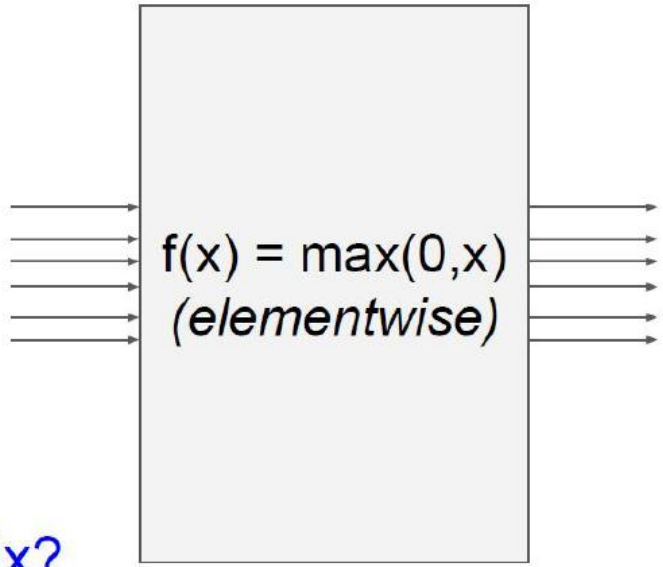


Vectors

Vectorized operations

$$\frac{\partial L}{\partial x} = \boxed{\frac{\partial f}{\partial x}} \frac{\partial L}{\partial f}$$

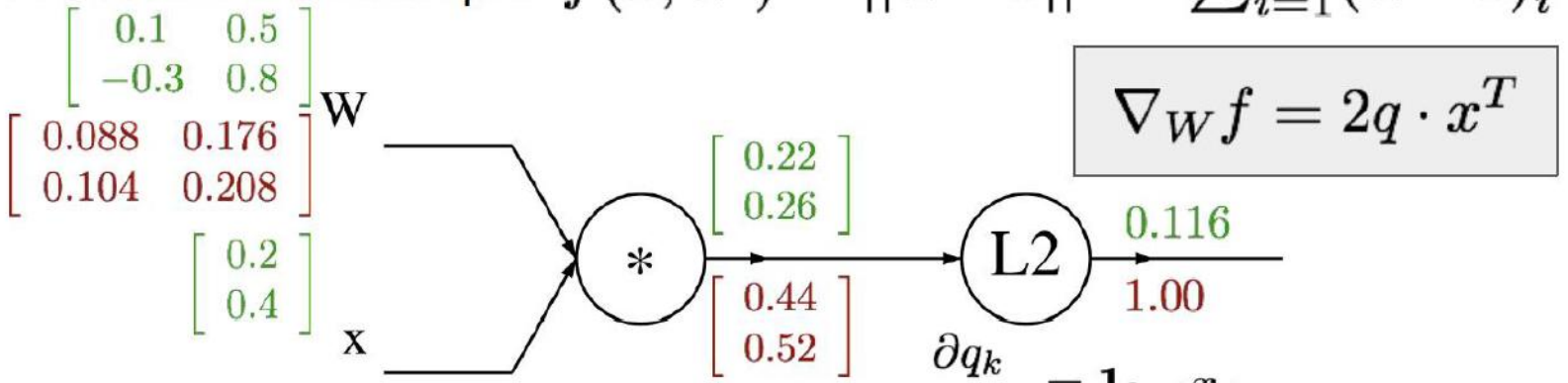
Jacobian matrix



Q: what is the size of the Jacobian matrix? [4096 x 4096!]

Vector example

A vectorized example: $f(x, W) = ||W \cdot x||^2 = \sum_{i=1}^{n} (W \cdot x)_i^2$



$$\nabla_W f = 2q \cdot x^T$$

$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \cdots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \cdots + W_{n,n}x_n \end{pmatrix}$$

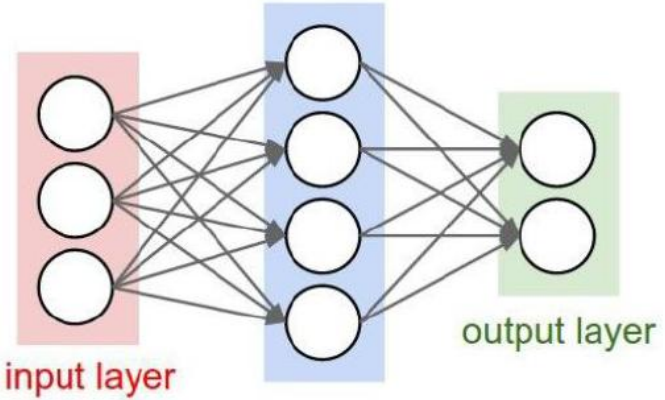
$$f(q) = ||q||^2 = q_1^2 + \cdots + q_n^2$$

$$\frac{\partial q_k}{\partial W_{i,j}} = \mathbf{1}_{k=i} x_j$$

$$\begin{aligned} \frac{\partial f}{\partial W_{i,j}} &= \sum_k \frac{\partial f}{\partial q_k} \frac{\partial q_k}{\partial W_{i,j}} \\ &= \sum_k^k (2q_k)(\mathbf{1}_{k=i} x_j) \\ &= 2q_i x_j \end{aligned}$$

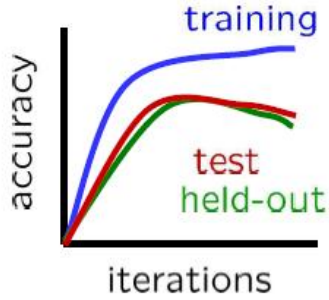
2 layer neural network

```python
import numpy as np
from numpy.random import randn

N, D_in, H, D_out = 64, 1000, 100, 10
x, y = randn(N, D_in), randn(N, D_out)
w1, w2 = randn(D_in, H), randn(H, D_out)

for t in range(2000):
  h = 1 / (1 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
  print(t, loss)

  grad_y_pred = 2.0 * (y_pred - y)
  grad_w2 = h.T.dot(grad_y_pred)
  grad_h = grad_y_pred.dot(w2.T)
  grad_w1 = x.T.dot(grad_h * h * (1 - h))

  w1 -= 1e-4 * grad_w1
  w2 -= 1e-4 * grad_w2
```



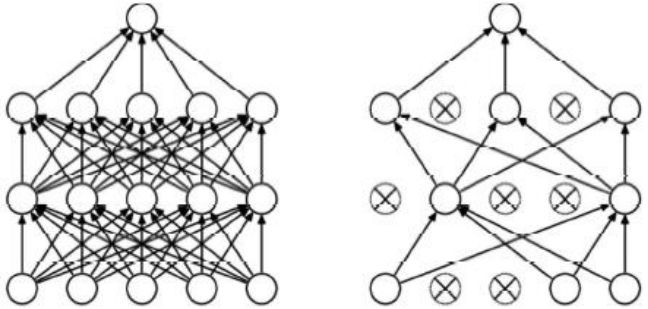
Prevent overfitting in NN

Early stopping:



Weight regularization: $\max_w \sum_i \log P(y^{(i)}|x^{(i)}; w) - \frac{\lambda}{2}\sum_j w_j^2$

Dropout:



(a) Standard Neural Net

(b) After applying dropout.