

# 第八周

- 教学计划

- 贝叶斯网络（上次课）
- 项目进度报告：0106，0202，0211
- 马可夫模型（这次课）

- 任务

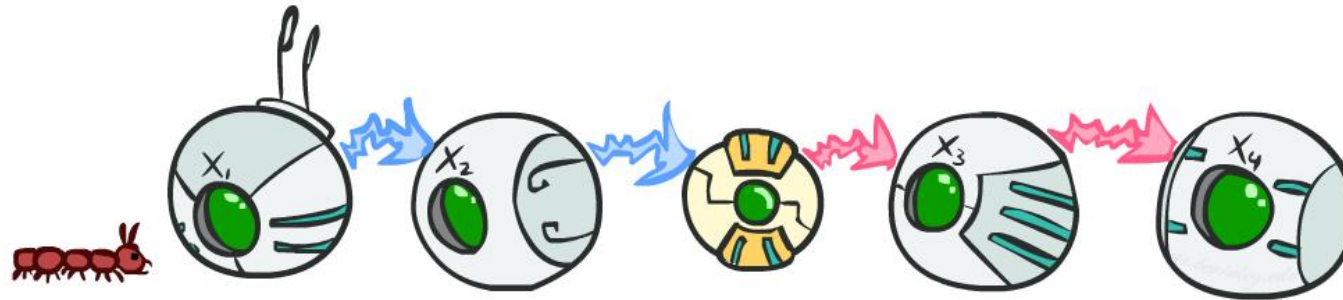
- 家作4：家作3的后两题+家作4
- 项目4：贝叶斯网络，马可夫模型（14道题）
- 有三周的时间，5/3提交

# Probability Recap

- Conditional probability  $P(x|y) = \frac{P(x, y)}{P(y)}$
- Product rule  $P(x, y) = P(x|y)P(y)$
- Chain rule 
$$\begin{aligned} P(X_1, X_2, \dots, X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots \\ &= \prod_{i=1}^n P(X_i|X_1, \dots, X_{i-1}) \end{aligned}$$
- X, Y independent if and only if:  $\forall x, y : P(x, y) = P(x)P(y)$
- X and Y are conditionally independent given Z if and only if:  $X \perp\!\!\!\perp Y | Z$   
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

# CS 188: Artificial Intelligence

## Markov Models



Instructors: Dan Klein and Pieter Abbeel --- University of California, Berkeley

[These slides were created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at <http://ai.berkeley.edu>.]

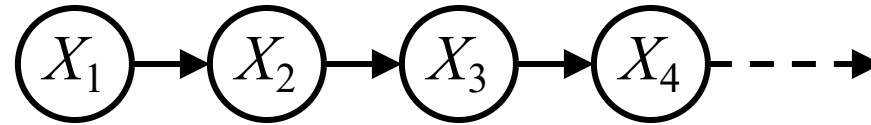
# Reasoning over Time or Space

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- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time (or space) into our models

# Markov Models

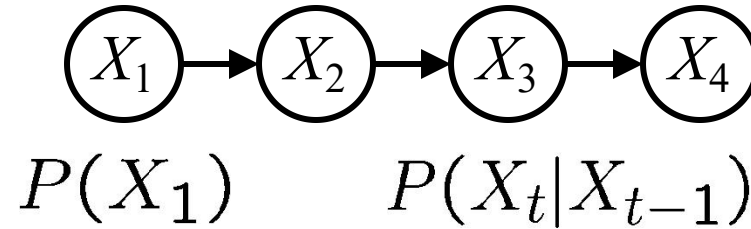
- Value of  $X$  at a given time is called the **state**



$$P(X_1) \quad P(X_t|X_{t-1})$$

- Parameters: called **transition probabilities** or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

# Joint Distribution of a Markov Model



- Joint distribution:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

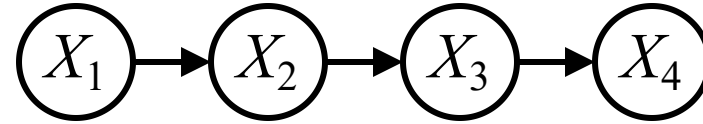
- More generally:

$$\begin{aligned} P(X_1, X_2, \dots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2) \dots P(X_T|X_{T-1}) \\ &= P(X_1) \prod_{t=2}^T P(X_t|X_{t-1}) \end{aligned}$$

- Questions to be resolved:

- Does this indeed define a joint distribution?
- Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

# Chain Rule and Markov Models



- From the chain rule, every joint distribution over  $X_1, X_2, X_3, X_4$  can be written as:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)P(X_4|X_1, X_2, X_3)$$

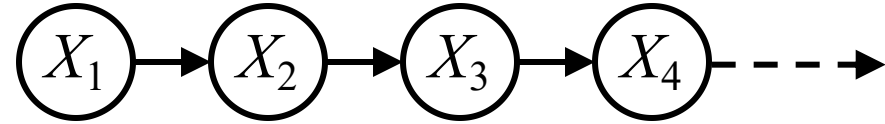
- Assuming that

$$X_3 \perp\!\!\!\perp X_1 \mid X_2 \quad \text{and} \quad X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$$

results in the expression posited on the previous slide:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

# Chain Rule and Markov Models



- From the chain rule, every joint distribution over  $X_1, X_2, \dots, X_T$  can be written as:

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_1, X_2, \dots, X_{t-1})$$

- Assuming that for all  $t$ :

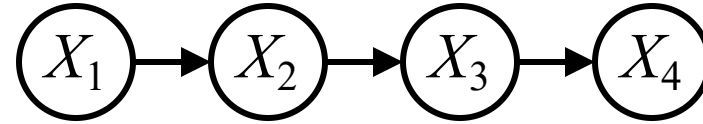
$$X_t \perp\!\!\!\perp X_1, \dots, X_{t-2} \mid X_{t-1}$$

gives us the expression posited on the earlier slide:

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_{t-1})$$



# Implied Conditional Independencies



- We assumed:  $X_3 \perp\!\!\!\perp X_1 \mid X_2$  and  $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$

- Do we also have  $X_1 \perp\!\!\!\perp X_3, X_4 \mid X_2$  ?

- Yes!

- Proof:

$$\begin{aligned} P(X_1 \mid X_2, X_3, X_4) &= \frac{P(X_1, X_2, X_3, X_4)}{P(X_2, X_3, X_4)} \\ &= \frac{P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_2)P(X_4 \mid X_3)}{\sum_{x_1} P(x_1)P(X_2 \mid x_1)P(X_3 \mid X_2)P(X_4 \mid X_3)} \\ &= \frac{P(X_1, X_2)}{P(X_2)} \\ &= P(X_1 \mid X_2) \end{aligned}$$

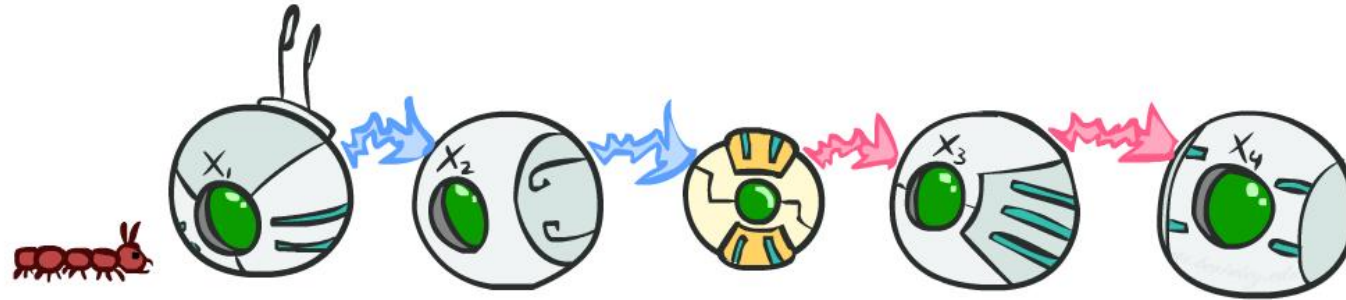
# Markov Models Recap

- Explicit assumption for all  $t$ :  $X_t \perp\!\!\!\perp X_1, \dots, X_{t-2} \mid X_{t-1}$
- Consequence, joint distribution can be written as:

$$\begin{aligned} P(X_1, X_2, \dots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2) \dots P(X_T|X_{T-1}) \\ &= P(X_1) \prod_{t=2}^T P(X_t|X_{t-1}) \end{aligned}$$

- Implied conditional independencies: (try to prove this!)
  - Past variables independent of future variables given the present  
i.e., if  $t_1 < t_2 < t_3$  or  $t_1 > t_2 > t_3$  then:  $X_{t_1} \perp\!\!\!\perp X_{t_3} \mid X_{t_2}$
- Additional explicit assumption:  $P(X_t \mid X_{t-1})$  is the same for all  $t$

# Conditional Independence

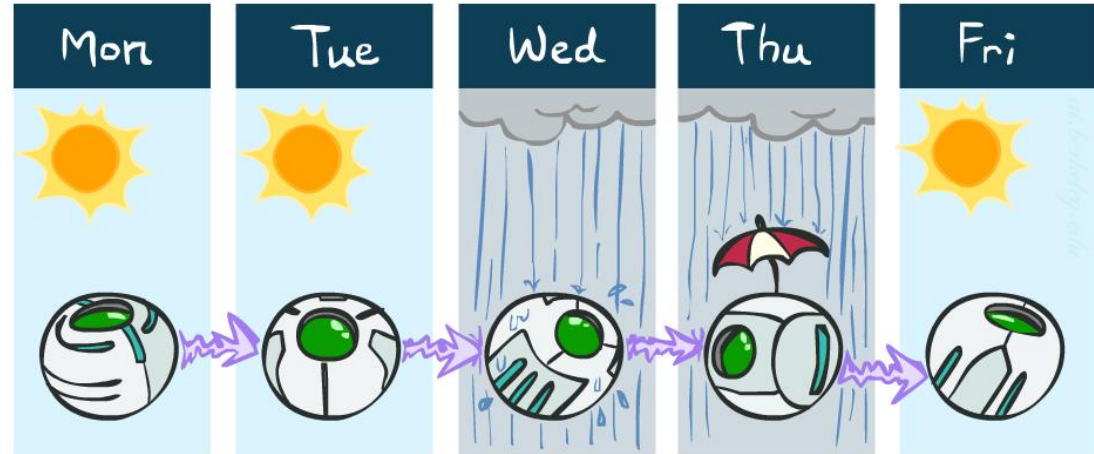


- Basic conditional independence:
  - Past and future independent of the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property
- Note that the chain is just a (growable) BN
  - We can always use generic BN reasoning on it if we truncate the chain at a fixed length

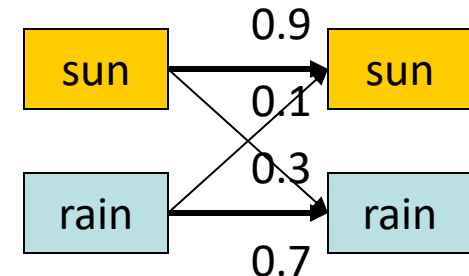
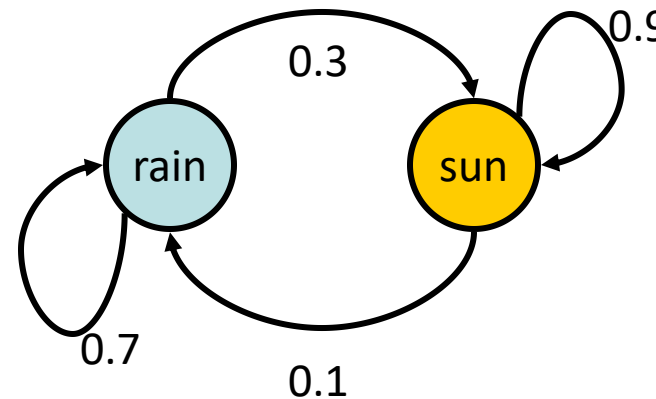
# Example Markov Chain: Weather

- States:  $X = \{\text{rain}, \text{sun}\}$
- Initial distribution: 1.0 sun
- CPT  $P(X_t \mid X_{t-1})$ :

$X_{t-1}$	$X_t$	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

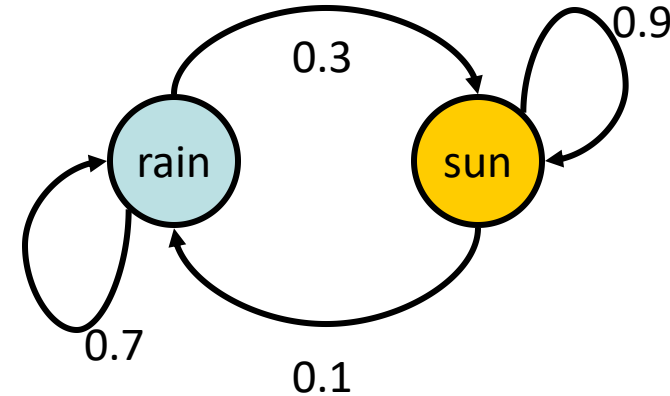


Two new ways of representing the same CPT



# Example Markov Chain: Weather

- Initial distribution: 1.0 sun



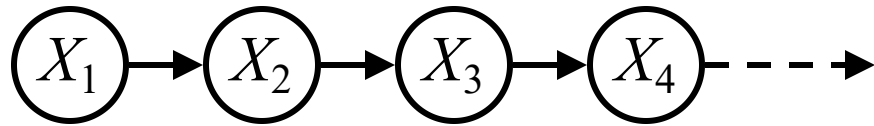
- What is the probability distribution after one step?

$$P(X_2 = \text{sun}) = P(X_2 = \text{sun} | X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun} | X_1 = \text{rain})P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

# Mini-Forward Algorithm

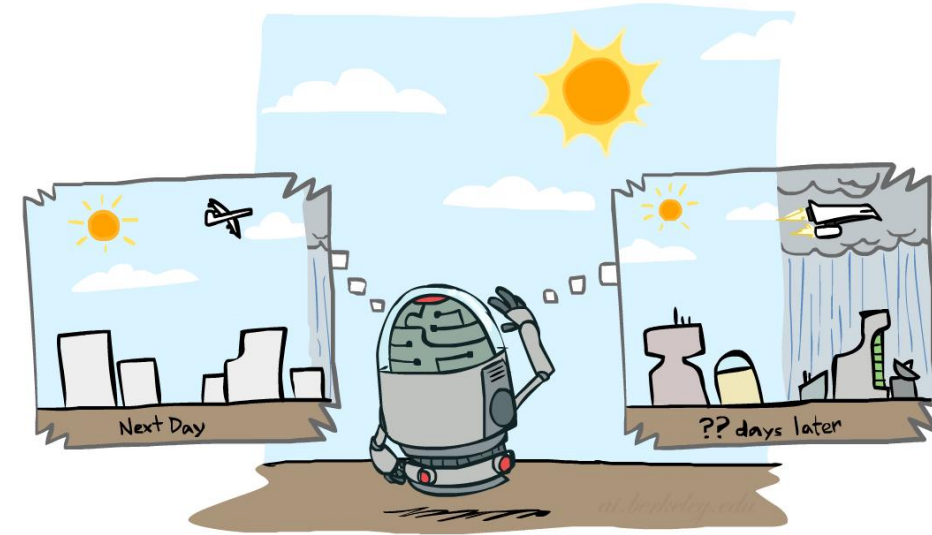
- Question: What's  $P(X)$  on some day  $t$ ?



$$P(x_1) = \text{known}$$

$$\begin{aligned} P(x_t) &= \sum_{x_{t-1}} P(x_{t-1}, x_t) \\ &= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1}) \end{aligned}$$

*Forward simulation*



# Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\begin{array}{ccccccc}
 \left\langle \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.9 \\ 0.1 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.84 \\ 0.16 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.804 \\ 0.196 \end{array} \right\rangle & \longrightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & & P(X_\infty)
 \end{array}$$

- From initial observation of rain

$$\begin{array}{ccccccc}
 \left\langle \begin{array}{c} 0.0 \\ 1.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.3 \\ 0.7 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.48 \\ 0.52 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.588 \\ 0.412 \end{array} \right\rangle & \longrightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & & P(X_\infty)
 \end{array}$$

- From yet another initial distribution  $P(X_1)$ :

$$\begin{array}{ccc}
 \left\langle \begin{array}{c} p \\ 1 - p \end{array} \right\rangle & \dots & \longrightarrow \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & & P(X_\infty)
 \end{array}$$

# Video of Demo Ghostbusters Basic Dynamics

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# Video of Demo Ghostbusters Circular Dynamics

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# Video of Demo Ghostbusters Whirlpool Dynamics

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# Stationary Distributions

- For most chains:

- Influence of the initial distribution gets less and less over time.
- The distribution we end up in is independent of the initial distribution

- Stationary distribution:

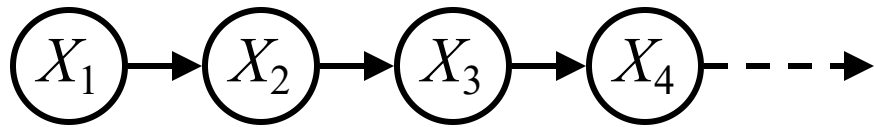
- The distribution we end up with is called the **stationary distribution**  $P_\infty$  of the chain
- It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$



# Example: Stationary Distributions

- Question: What's  $P(X)$  at time  $t = \text{infinity}$ ?



$$P_{\infty}(\text{sun}) = P(\text{sun}|\text{sun})P_{\infty}(\text{sun}) + P(\text{sun}|\text{rain})P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = P(\text{rain}|\text{sun})P_{\infty}(\text{sun}) + P(\text{rain}|\text{rain})P_{\infty}(\text{rain})$$

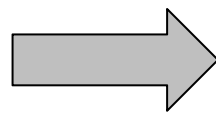
$$P_{\infty}(\text{sun}) = 0.9P_{\infty}(\text{sun}) + 0.3P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = 0.1P_{\infty}(\text{sun}) + 0.7P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{sun}) = 3P_{\infty}(\text{rain})$$

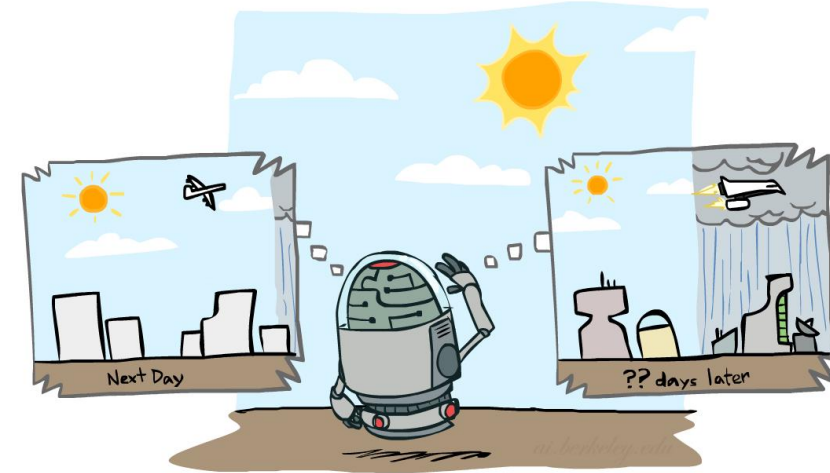
$$P_{\infty}(\text{rain}) = 1/3P_{\infty}(\text{sun})$$

Also:  $P_{\infty}(\text{sun}) + P_{\infty}(\text{rain}) = 1$



$$P_{\infty}(\text{sun}) = 3/4$$

$$P_{\infty}(\text{rain}) = 1/4$$

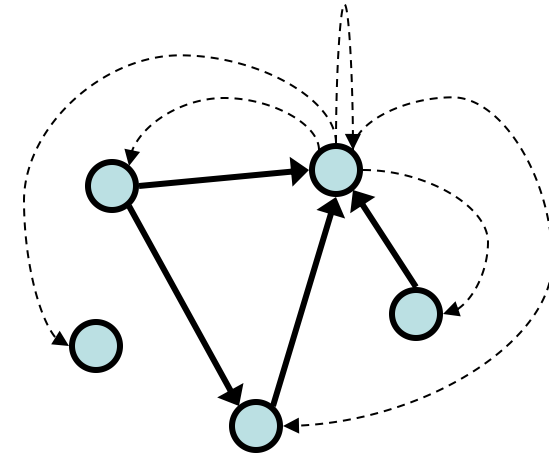


$X_{t-1}$	$X_t$	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

# Application of Stationary Distribution: Web Link Analysis

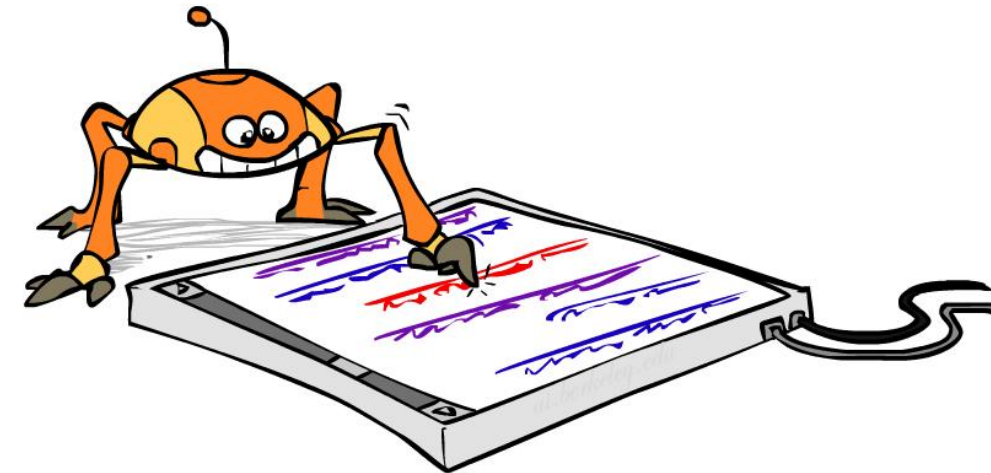
- PageRank over a web graph

- Each web page is a state
- Initial distribution: uniform over pages
- Transitions:
  - With prob.  $c$ , uniform jump to a random page (dotted lines, not all shown)
  - With prob.  $1-c$ , follow a random outlink (solid lines)



- Stationary distribution

- Will spend more time on highly reachable pages
- E.g. many ways to get to the Acrobat Reader download page
- Somewhat robust to link spam
- Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



# Application of Stationary Distributions: Gibbs Sampling\*

- Each joint instantiation over all hidden and query variables is a state:  $\{X_1, \dots, X_n\} = H \cup Q$

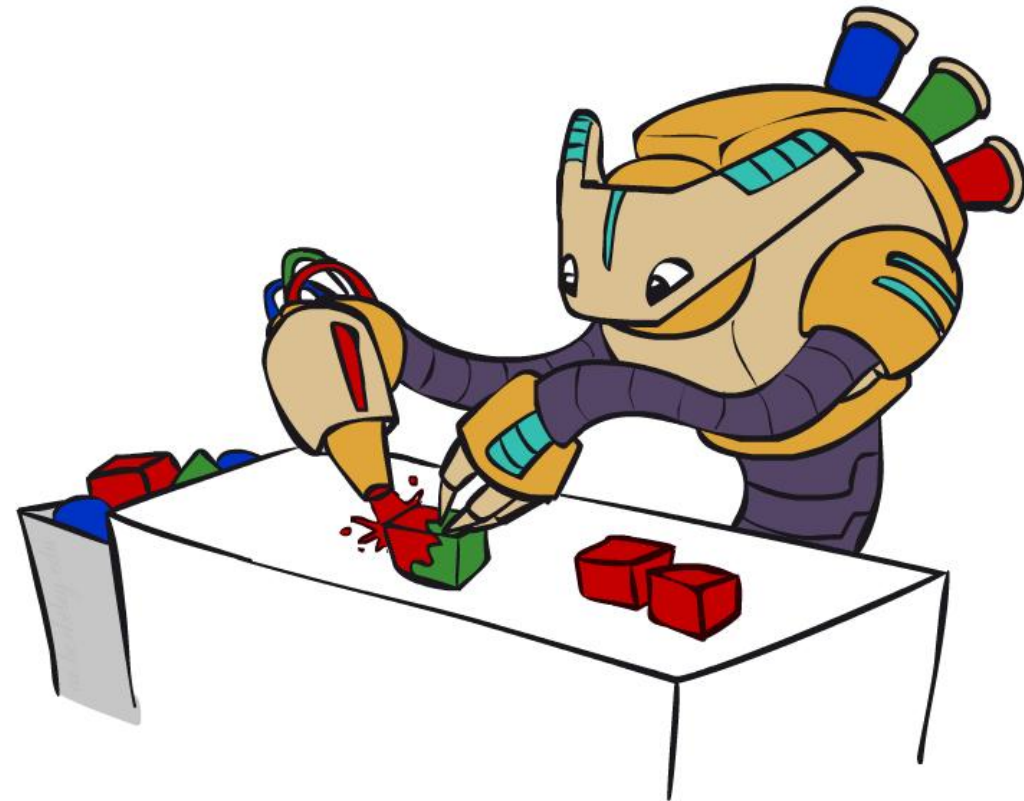
- Transitions:

- With probability  $1/n$  resample variable  $X_j$  according to

$$P(X_j \mid x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, e_1, \dots, e_m)$$

- Stationary distribution:

- Conditional distribution  $P(X_1, X_2, \dots, X_n \mid e_1, \dots, e_m)$
- Means that when running Gibbs sampling long enough we get a sample from the desired distribution
- Requires some proof to show this is true!



# Next Time: Hidden Markov Models!

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