

BLM2041 Signals and Systems

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BLM2041 Signals and Systems

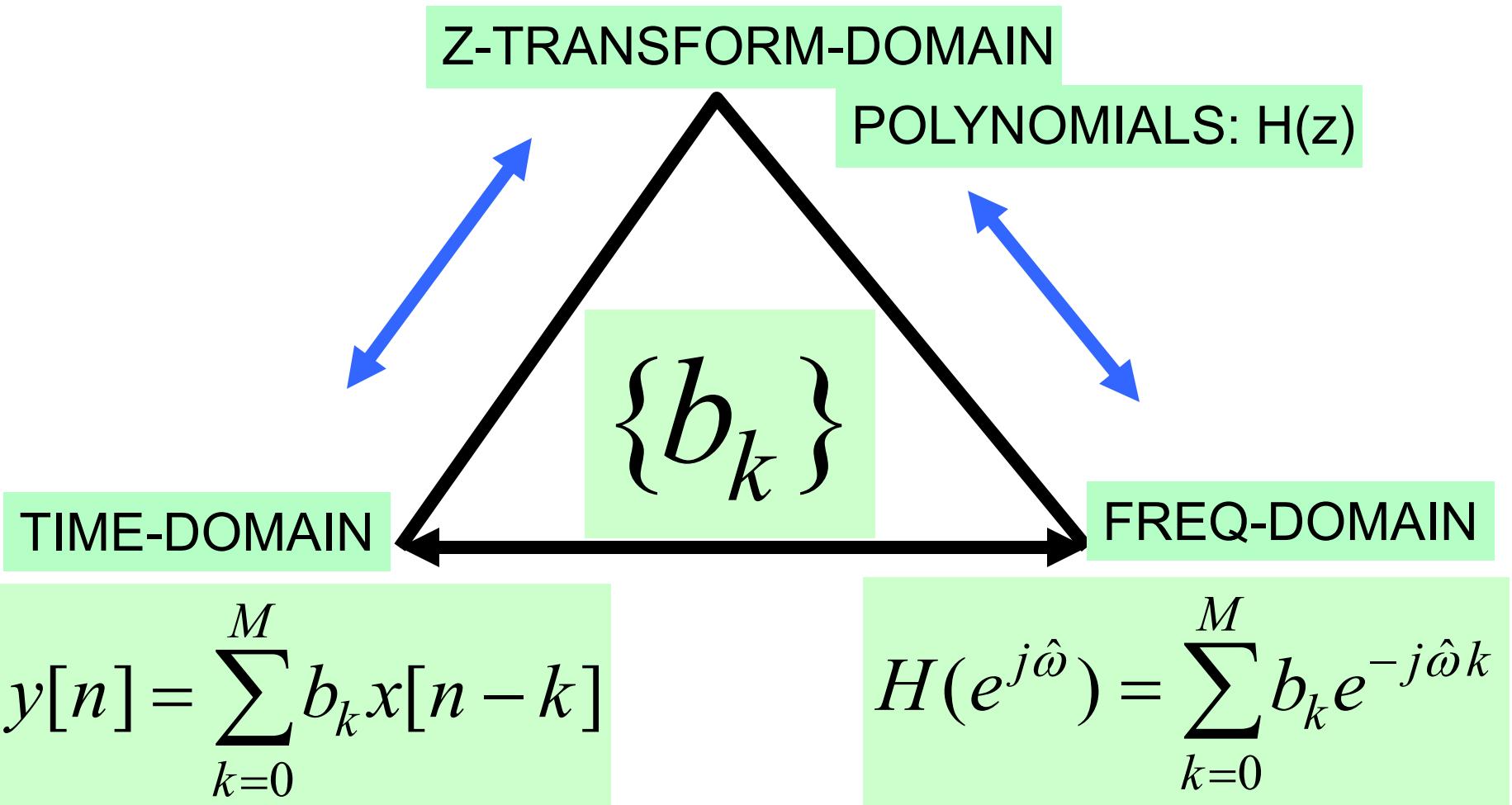
Z Transforms: Introduction

LECTURE OBJECTIVES

- INTRODUCE the Z-TRANSFORM
 - Give Mathematical Definition
 - Show how the $H(z)$ POLYNOMIAL simplifies analysis
 - CONVOLUTION is SIMPLIFIED !
- Z-Transform can be applied to
 - FIR Filter: $h[n] \rightarrow H(z)$
 - Signals: $x[n] \rightarrow X(z)$

$$H(z) = \sum_n h[n]z^{-n}$$

THREE DOMAINS



Three main reasons for Z-Transform

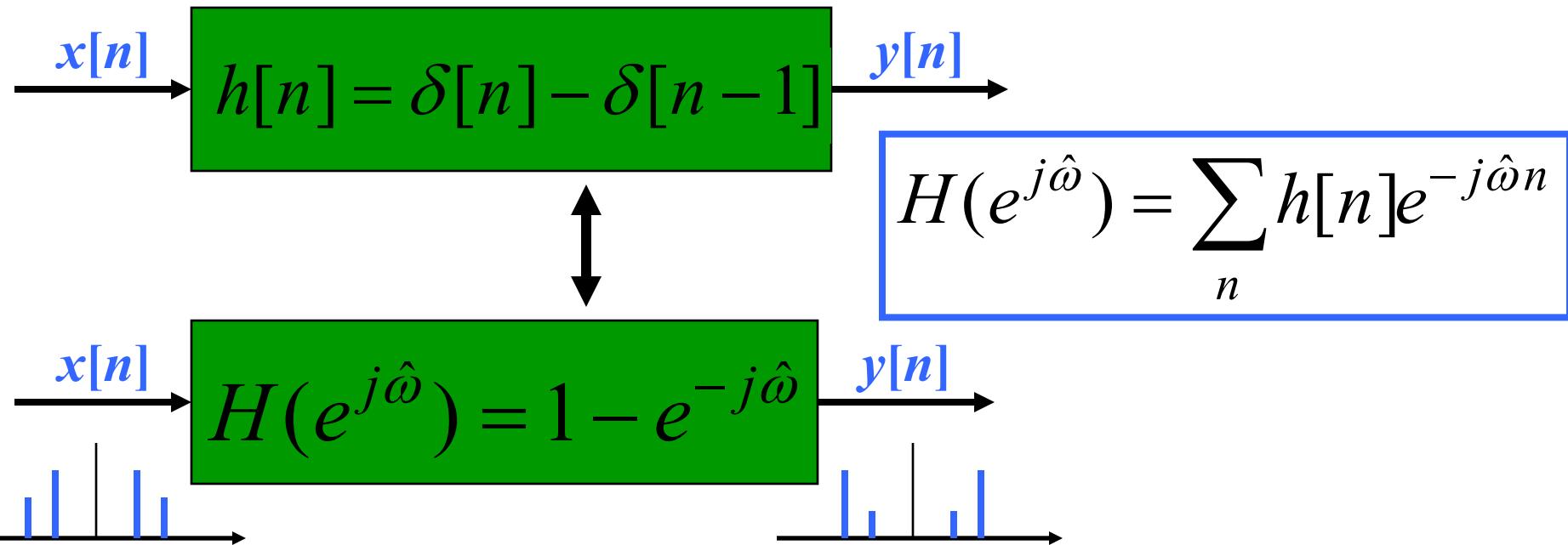
- Offers compact and convenient notation for describing digital signals and systems
- Widely used by DSP designers, and in the DSP literature
- Pole-zero description of a processor is a great help in visualizing its stability and frequency response characteristic

TRANSFORM CONCEPT

- Move to a new domain where
 - OPERATIONS are EASIER & FAMILIAR
 - Use POLYNOMIALS
- TRANSFORM both ways
 - $x[n] \rightarrow X(z)$ (into the z domain)
 - $X(z) \rightarrow x[n]$ (back to the time domain)

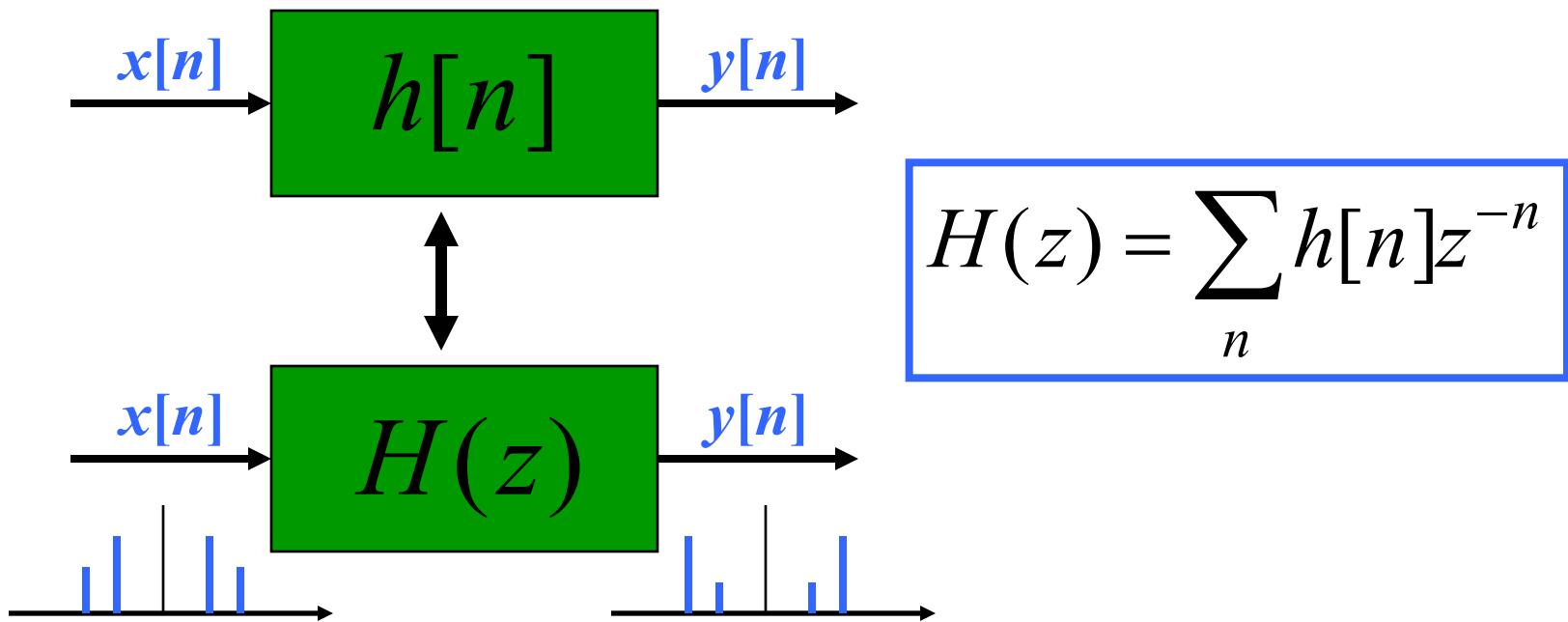
“TRANSFORM” EXAMPLE

- Equivalent Representations



Z-TRANSFORM IDEA

- POLYNOMIAL REPRESENTATION



Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM:

$$H(z) = \sum_n h[n]z^{-n}$$

- EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

APPLIES to
Any SIGNAL

$$H(z) = 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4}$$

$$= 2 - 3z^{-2} + 2z^{-4}$$

$$= 2 - 3(z^{-1})^2 + 2(z^{-1})^4$$

POLYNOMIAL in z^{-1}

Z-Transform EXAMPLE

- ANY SIGNAL has a z-Transform:

$$X(z) = \sum_n x[n]z^{-n}$$

Example 7.1

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ?$$

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Z-Transform EXAMPLE

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

EXONENT GIVES
TIME LOCATION

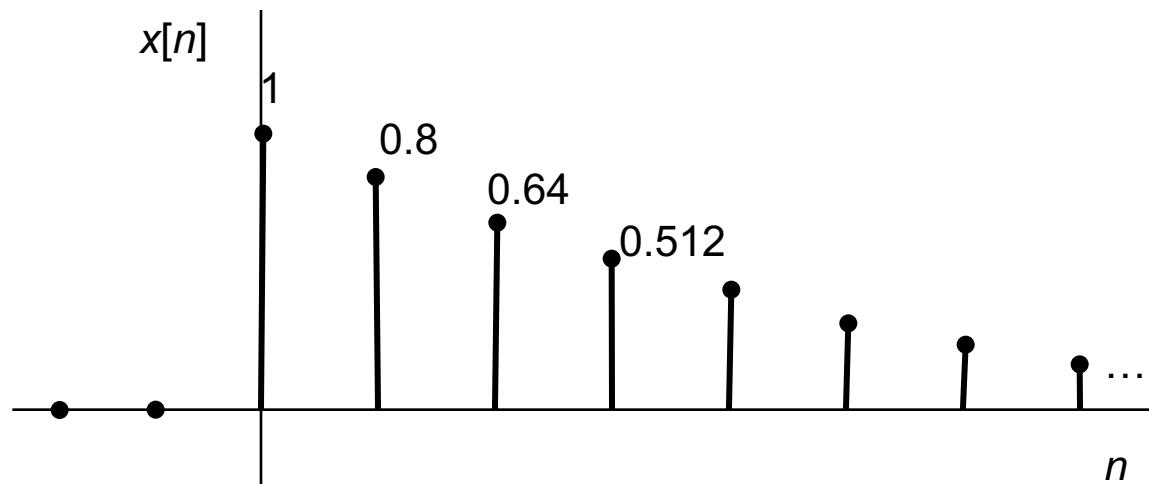
$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

$$x[n] = ?$$

$$x[n] = \delta[n] - 2\delta[n - 1] + 3\delta[n - 3] - \delta[n - 5]$$

Example

- Find the Z-Transform of the exponentially decaying signal shown in the following figure, expressing it as compact as possible.



Example

- The Z-Transform of the signal:

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n]z^{-n} \\ &= 1 + 0.8z^{-1} + 0.64z^{-2} + 0.512z^{-3} + \dots \\ &= 1 + (0.8z^{-1}) + (0.64z^{-1})^2 + (0.512z^{-1})^3 + \dots \\ &= \frac{1}{1 - 0.8z^{-1}} = \frac{z}{z - 0.8} \end{aligned}$$

Example

- Find and sketch, the signal corresponding to the Z-Transform:

$$X(z) = \frac{1}{z + 1.2}$$

Example

- Recasting $X(z)$ as a power series in z^{-1} , we obtain:

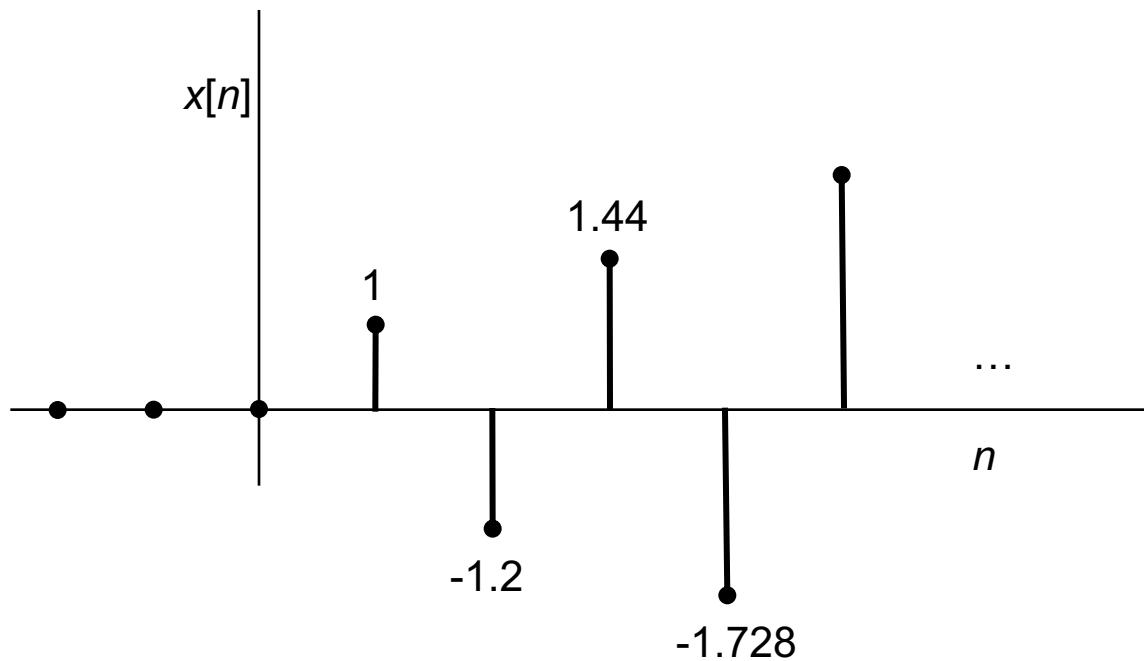
$$\begin{aligned} X(z) &= \frac{1}{(z+1.2)} = \frac{z^{-1}}{(1+1.2z^{-1})} = z^{-1}(1+1.2z^{-1})^{-1} \\ &= z^{-1}\{1 + (-1.2z^{-1}) + (-1.2z^{-1})^2 + (-1.2z^{-1})^3 + \dots\} \\ &= z^{-1} - 1.2z^{-2} + 1.44z^{-3} - 1.728z^{-4} + \dots \end{aligned}$$

- Successive values of $x[n]$, starting at $n=0$, are therefore:

$$0, 1, -1.2, 1.44, -1.728, \dots$$

Example

- $x[n]$ is shown in the following figure:



Z-Transform of FIR Filter

- CALLED the **SYSTEM FUNCTION**
 - $h[n]$ is same as $\{b_k\}$

**SYSTEM
FUNCTION**

$$H(z) = \sum_{k=0}^M b_k z^{-k} = \sum_{k=0}^M h[k] z^{-k}$$

$$y[n] = \sum_{k=0}^M b_k x[n-k] = \sum_{k=0}^M h[k] x[n-k]$$

FIR DIFFERENCE EQUATION

CONVOLUTION

Z-Transform of FIR Filter

- Get $H(z)$ DIRECTLY from the $\{b_k\}$
- Example 7.3 in the book:

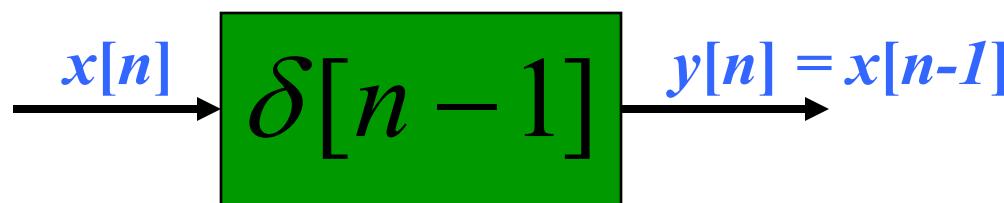
$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

$$\{b_k\} = \{6, -5, 1\}$$

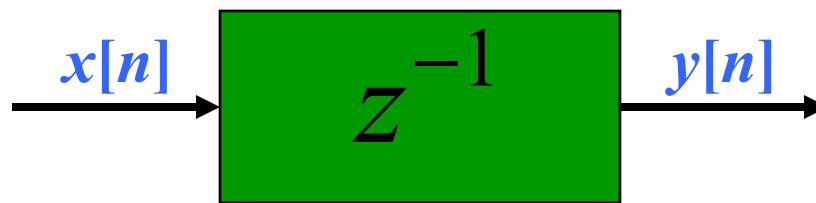
$$H(z) = \sum b_k z^{-k} = 6 - 5z^{-1} + z^{-2}$$

Ex. DELAY SYSTEM

- UNIT DELAY: find $h[n]$ and $H(z)$



$$H(z) = \sum \delta[n - 1] z^{-n} = z^{-1}$$



DELAY EXAMPLE

- UNIT DELAY: find $y[n]$ via polynomials
 - $x[n] = \{3, 1, 4, 1, 5, 9, 0, 0, 0, \dots\}$

$$Y(z) = z^{-1} X(z)$$

$$Y(z) = z^{-1}(3 + z^{-1} + 4z^{-2} + z^{-3} + 5z^{-4} + 9z^{-5})$$

$$Y(z) = 0z^0 + 3z^{-1} + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5} + 9z^{-6}$$

n	$n < 0$	0	1	2	3	4	5	6	$n > 6$
$y[n]$	0	0	3	1	4	1	5	9	0

DELAY PROPERTY

A delay of one sample multiplies the z-transform by z^{-1} .

$$x[n - 1] \iff z^{-1}X(z)$$

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \iff z^{-n_0}X(z)$$

GENERAL I/O PROBLEM

- Input is $x[n]$, find $y[n]$ (for FIR, $h[n]$)
- How to combine $X(z)$ and $H(z)$?

Example 7.5

$$x[n] = \delta[n - 1] - \delta[n - 2] + \delta[n - 3] - \delta[n - 4]$$

and $h[n] = \delta[n] + 2\delta[n - 1] + 3\delta[n - 2] + 4\delta[n - 3]$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

and $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

FIR Filter = CONVOLUTION

$x[n], X(z)$

0 +1 -1 +1 -1

$h[n], H(z)$

1 2 3 4

0	+1	-1	+1	-1
0	+2	-2	+2	-2
0	+3	-3	+3	-3
0	+4	-4	+4	-4

$y[n], Y(z)$

0 +1 +1 +2 +2 -3 +1 -4

$$y[n] = \sum_{k=0}^M b_k x[n-k] = \sum_{k=0}^M h[k] x[n-k]$$

CONVOLUTION

CONVOLUTION PROPERTY

- PROOF:

$$y[n] = x[n] * h[n] = \sum_{k=0}^M h[k]x[n - k]$$

$$Y(z) = \sum_{k=0}^M h[k] (z^{-k} X(z))$$

MULTIPLY
Z-TRANSFORMS

$$= \left(\sum_{k=0}^M h[k] z^{-k} \right) X(z) = H(z)X(z).$$



CONVOLUTION EXAMPLE

- MULTIPLY the z-TRANSFORMS:

Example 7.5

$$x[n] = \delta[n - 1] - \delta[n - 2] + \delta[n - 3] - \delta[n - 4]$$

and $h[n] = \delta[n] + 2\delta[n - 1] + 3\delta[n - 2] + 4\delta[n - 3]$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

and $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

MULTIPLY $H(z)X(z)$

CONVOLUTION EXAMPLE

- Finite-Length input $x[n]$
- FIR Filter ($L=4$)

$$Y(z) = H(z)X(z)$$

MULTIPLY
Z-TRANSFORMS

$$\begin{aligned} &= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4}) \\ &= z^{-1} + (-1 + 2)z^{-2} + (1 - 2 + 3)z^{-3} + (-1 + 2 - 3 + 4)z^{-4} \\ &\quad + (-2 + 3 - 4)z^{-5} + (-3 + 4)z^{-6} + (-4)z^{-7} \\ &= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7} \end{aligned}$$

$y[n] = ?$

CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
 - NO, LTI SYSTEMS can be rearranged !!!
 - Remember: $h_1[n] * h_2[n]$
 - How to combine $H_1(z)$ and $H_2(z)$?

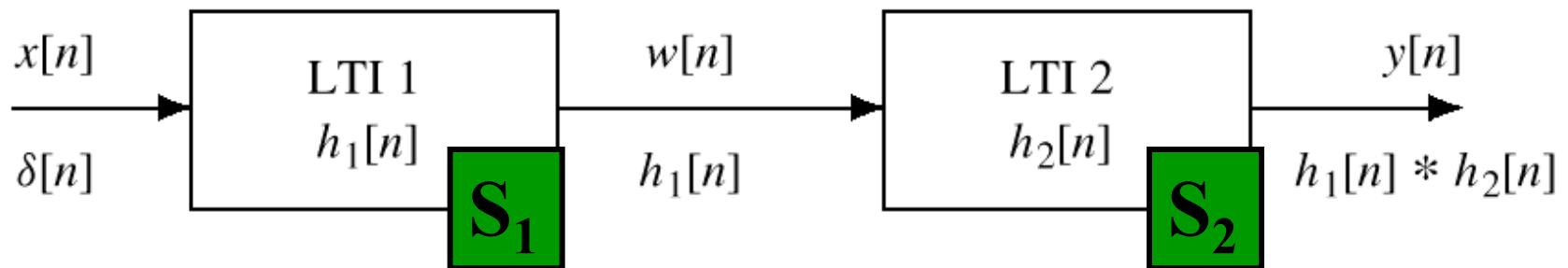
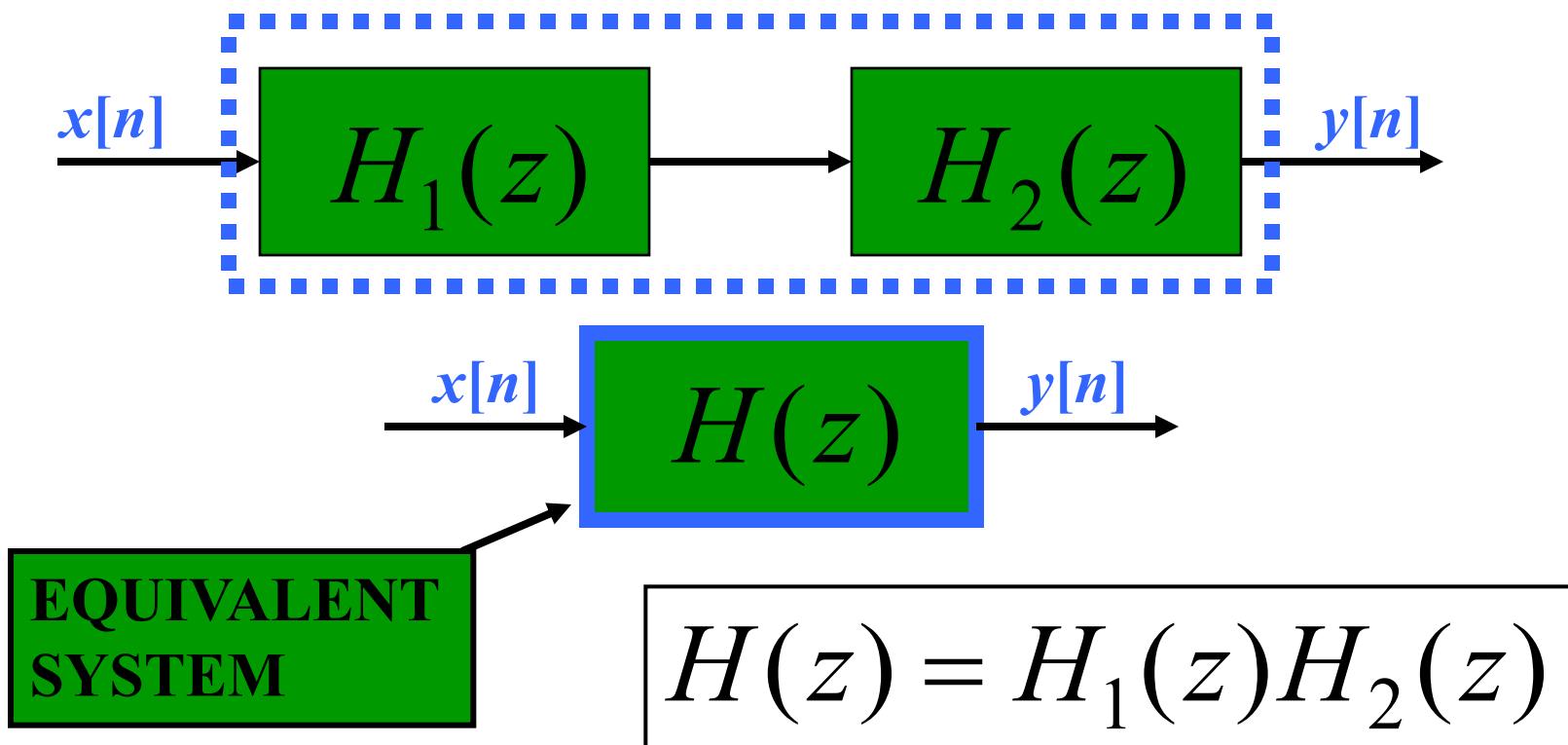


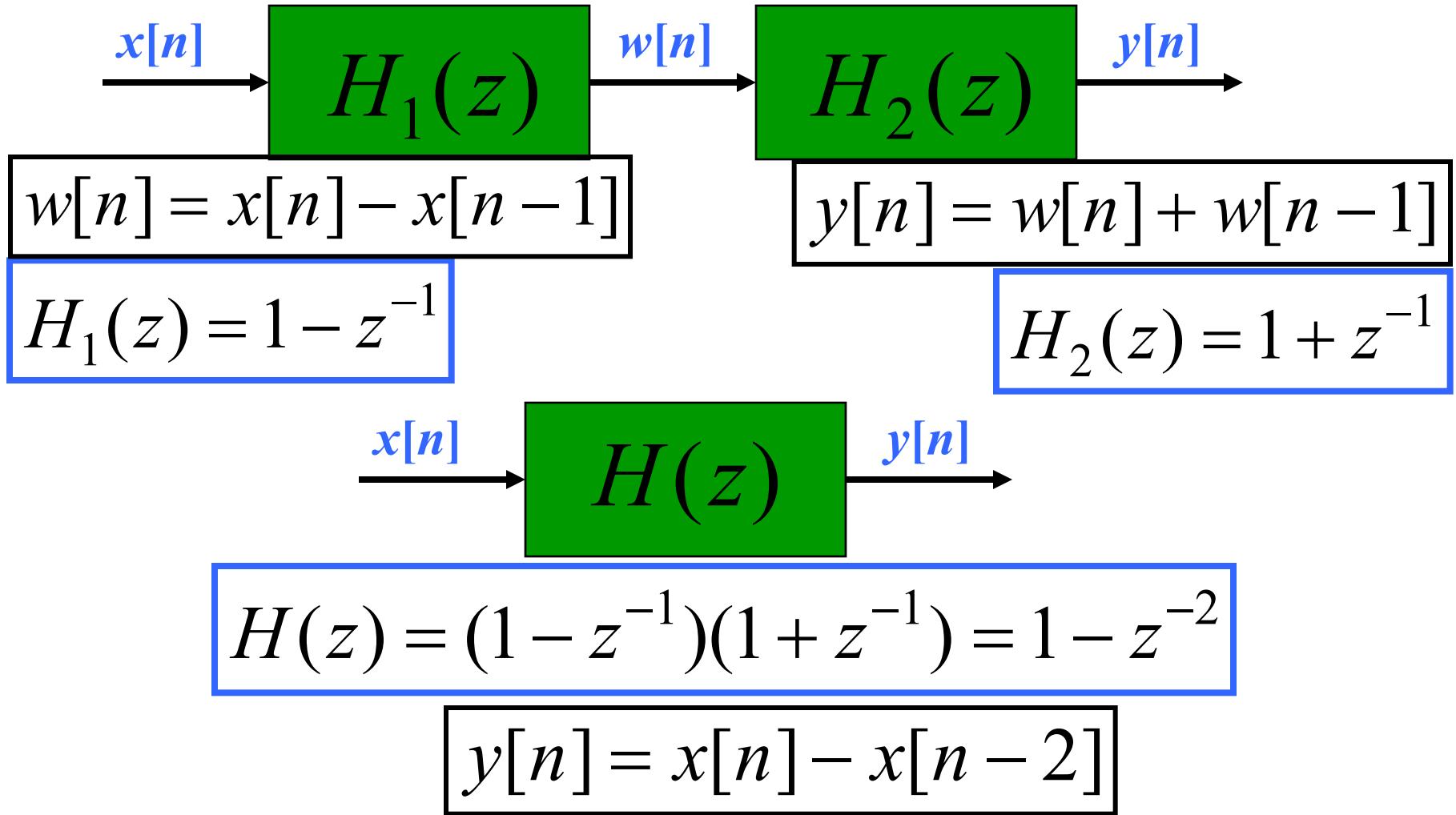
Figure 5.19 A Cascade of Two LTI Systems.

CASCADE EQUIVALENT

- Multiply the System Functions



CASCADE EXAMPLE



Zeros of $H(z)$ and the Frequency Domain

LECTURE OBJECTIVES

- ZEROS and POLES
- Relate $H(z)$ to FREQUENCY RESPONSE

$$H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}}$$

- THREE DOMAINS:
 - Show Relationship for FIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

DESIGN PROBLEM

- Example:
 - Design a Lowpass FIR filter
 - Find b_k
 - Reject completely 0.7π , 0.8π , and 0.9π
 - This is NULLING
 - Estimate the filter length needed to accomplish this task.
 - How many b_k ?
- Z POLYNOMIALS provide the TOOLS

Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM:
$$H(z) = \sum_n h[n]z^{-n}$$
- EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

APPLIES to
Any SIGNAL

$$H(z) = 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4}$$

$$= 2 - 3z^{-2} + 2z^{-4}$$

$$= 2 - 3(z^{-1})^2 + 2(z^{-1})^4$$

POLYNOMIAL in z^{-1}

CONVOLUTION PROPERTY

- Convolution in the **n**-domain
 - SAME AS
- Multiplication in the **z**-domain



$$y[n] = h[n] * x[n] \leftrightarrow Y(z) = H(z)X(z)$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=0}^M h[k]x[n-k] \end{aligned}$$

FIR Filter

MULTIPLY
z-TRANSFORMS



CONVOLUTION EXAMPLE



$$x[n] = \delta[n - 1] + 2\delta[n - 2]$$

$$h[n] = \delta[n] - \delta[n - 1]$$

$$y[n] = x[n] * h[n]$$

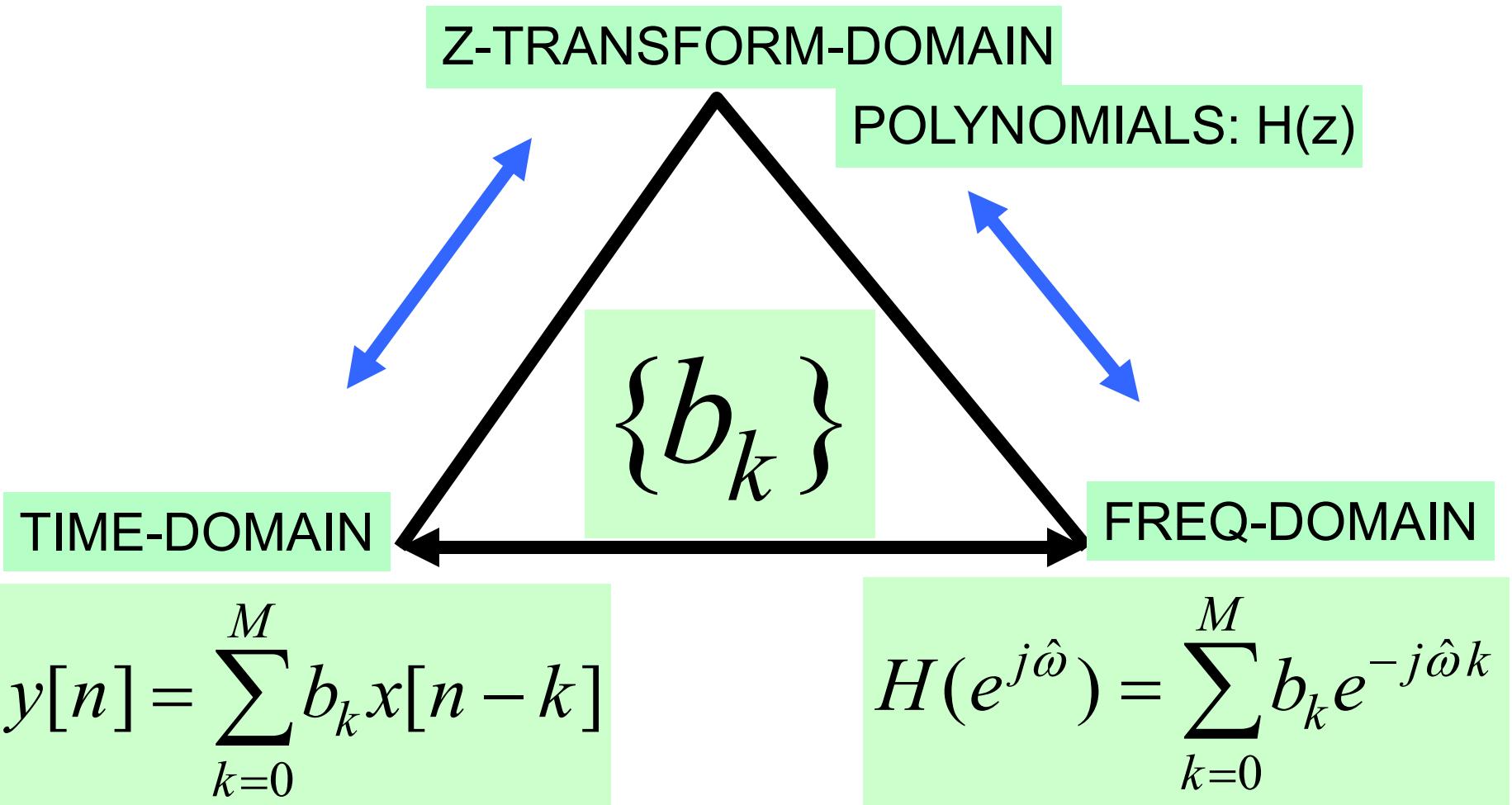
$$X(z) = z^{-1} + 2z^{-2}$$

$$H(z) = 1 - z^{-1}$$

$$Y(z) = (z^{-1} + 2z^{-2})(1 - z^{-1}) = z^{-1} + z^{-2} - 2z^{-3}$$

$$y[n] = \delta[n - 1] + \delta[n - 2] - 2\delta[n - 3]$$

THREE DOMAINS



FREQUENCY RESPONSE ?

- Same Form:

$\hat{\omega}$ - Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^M b_k e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^M b_k (e^{j\hat{\omega}})^{-k}$$

$$z = e^{j\hat{\omega}}$$

z - Domain

$$H(z) = \sum_{k=0}^M b_k z^{-k}$$

SAME COEFFICIENTS

ANOTHER ANALYSIS TOOL

- z-Transform POLYNOMIALS are EASY !
 - ROOTS, FACTORS, etc.
- ZEROS and POLES: where is $H(z) = 0$?
- The z-domain is COMPLEX
 - $H(z)$ is a COMPLEX-VALUED function of a COMPLEX VARIABLE z .

ZEROS of $H(z)$

- Find z , where $H(z)=0$

$$H(z) = 1 - \frac{1}{2} z^{-1}$$

$$1 - \frac{1}{2} z^{-1} = 0 ?$$

$$z - \frac{1}{2} = 0$$

Zero at : $z = \frac{1}{2}$

ZEROS of $H(z)$

- Find z , where $H(z)=0$
 - Interesting when z is ON the unit circle.

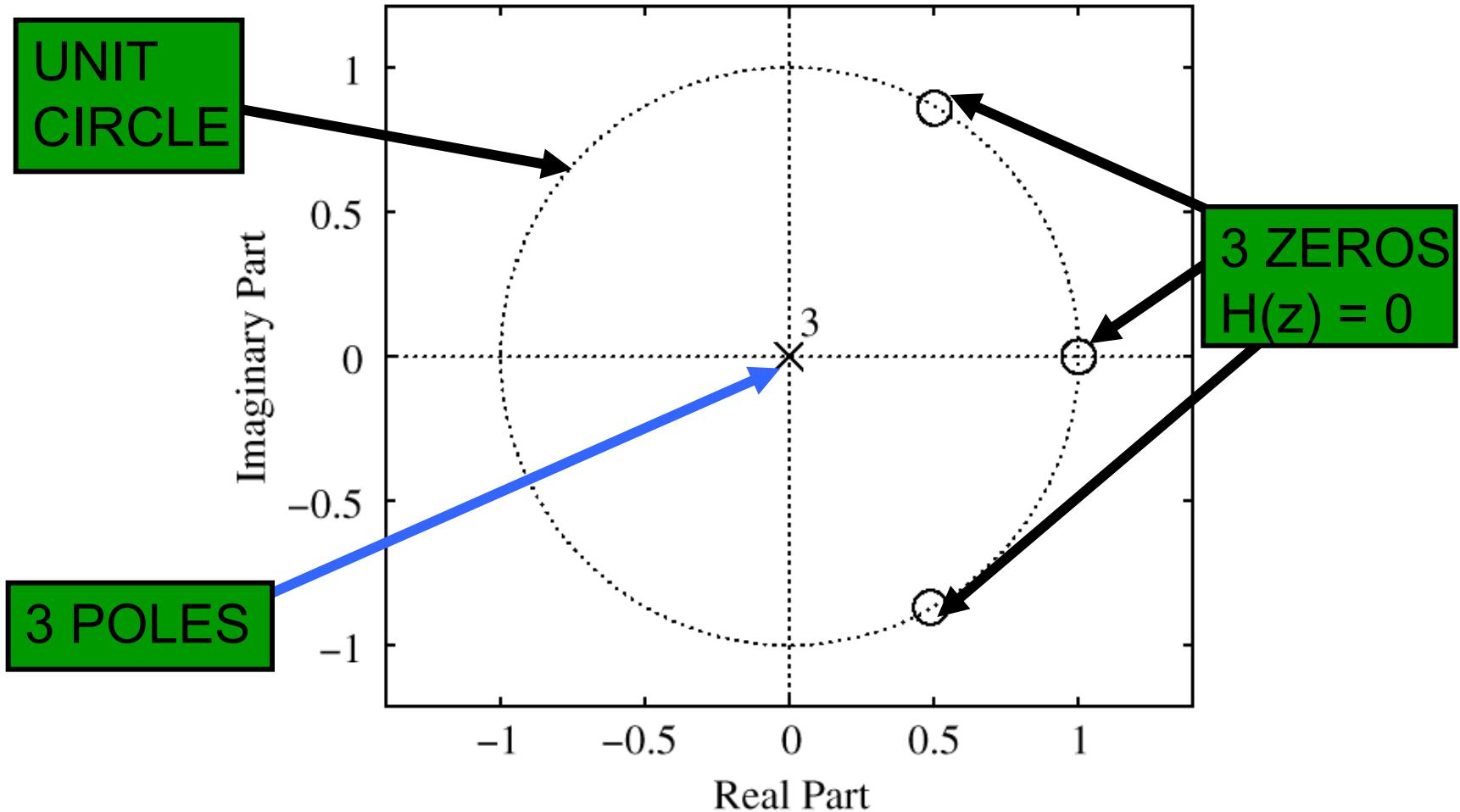
$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})$$

$$\text{Roots: } z = 1, \frac{1}{2} \pm j \frac{\sqrt{3}}{2}$$

$$e^{\pm j\pi/3}$$

PLOT ZEROS in z-DOMAIN



POLES of H(z)

- Find z , where $H(z) \rightarrow \infty$
 - Not very interesting for the FIR case

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = \frac{z^3 - 2z^2 + 2z - 1}{z^3}$$

Three Poles at : $z = 0$

FREQ. RESPONSE from ZEROS

$$H(e^{j\hat{\omega}}) = H(z) \Big|_{z=e^{j\hat{\omega}}}$$

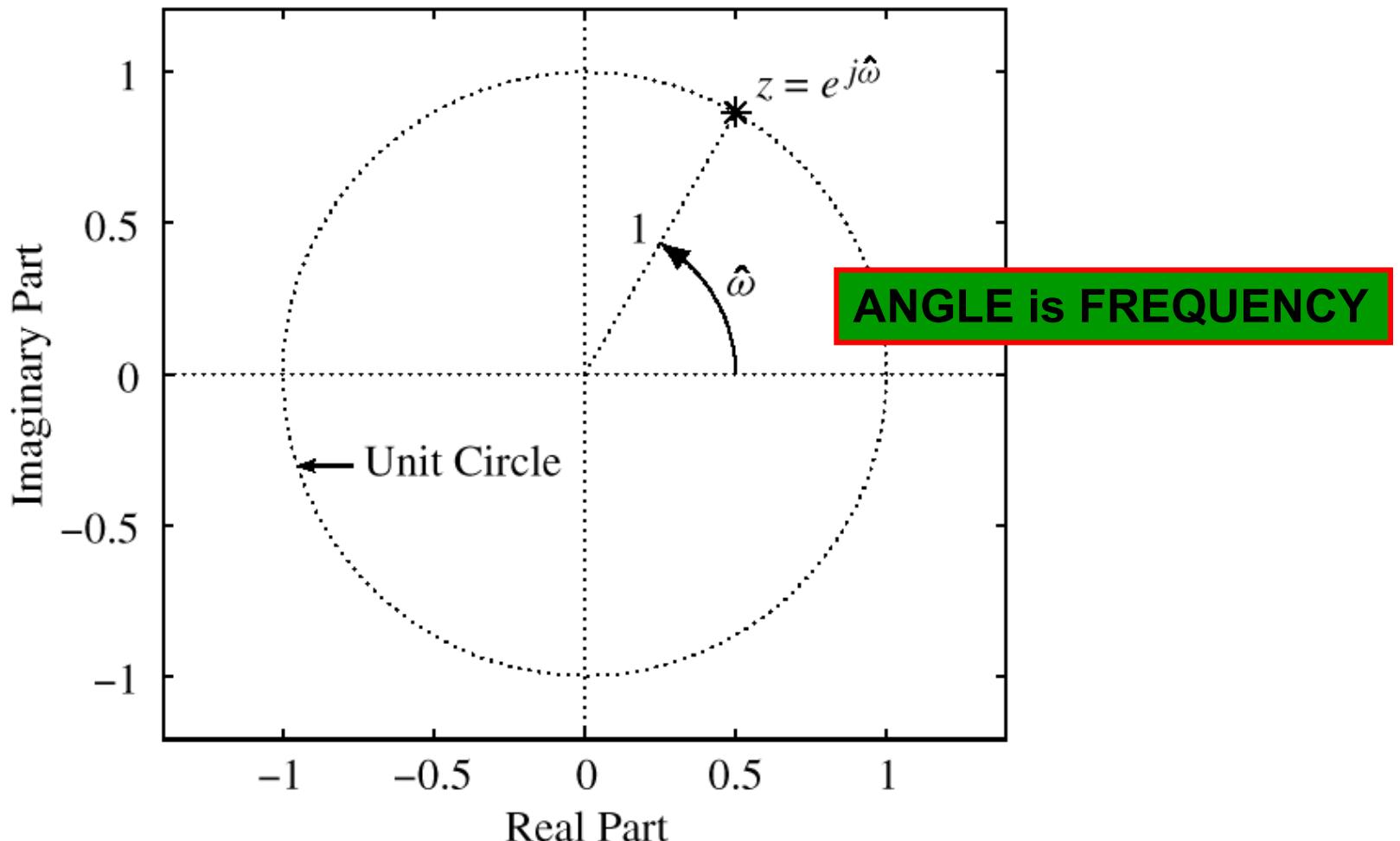
- Relate $H(z)$ to FREQUENCY RESPONSE
- EVALUATE $H(z)$ on the **UNIT CIRCLE**
 - ANGLE is same as FREQUENCY

$z = e^{j\hat{\omega}}$ (as $\hat{\omega}$ varies)

defines a CIRCLE, radius = 1

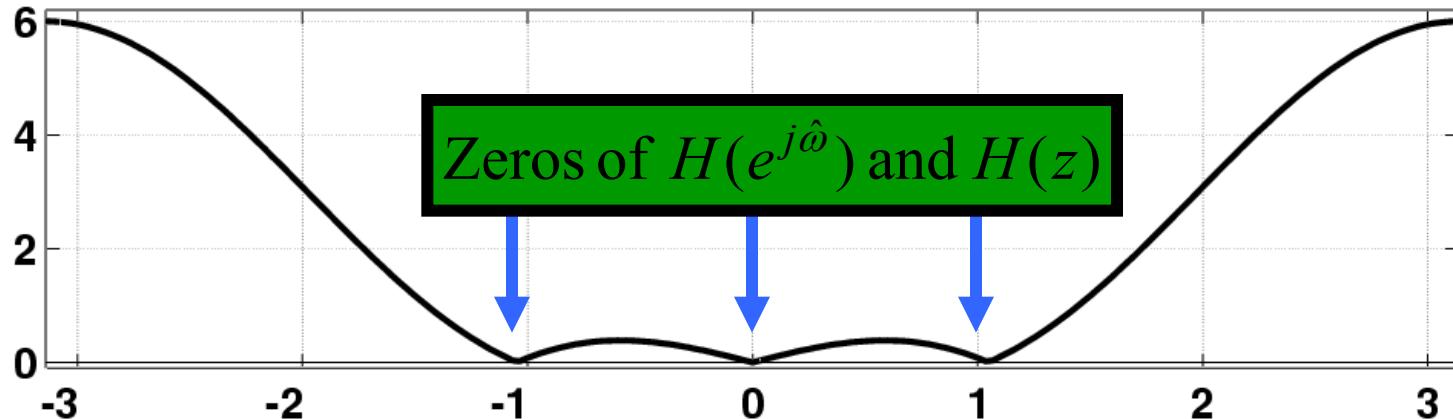
$$H(e^{j\hat{\omega}}) = H(z) \Big|_{z=e^{j\hat{\omega}}}$$

The Complex z -Plane

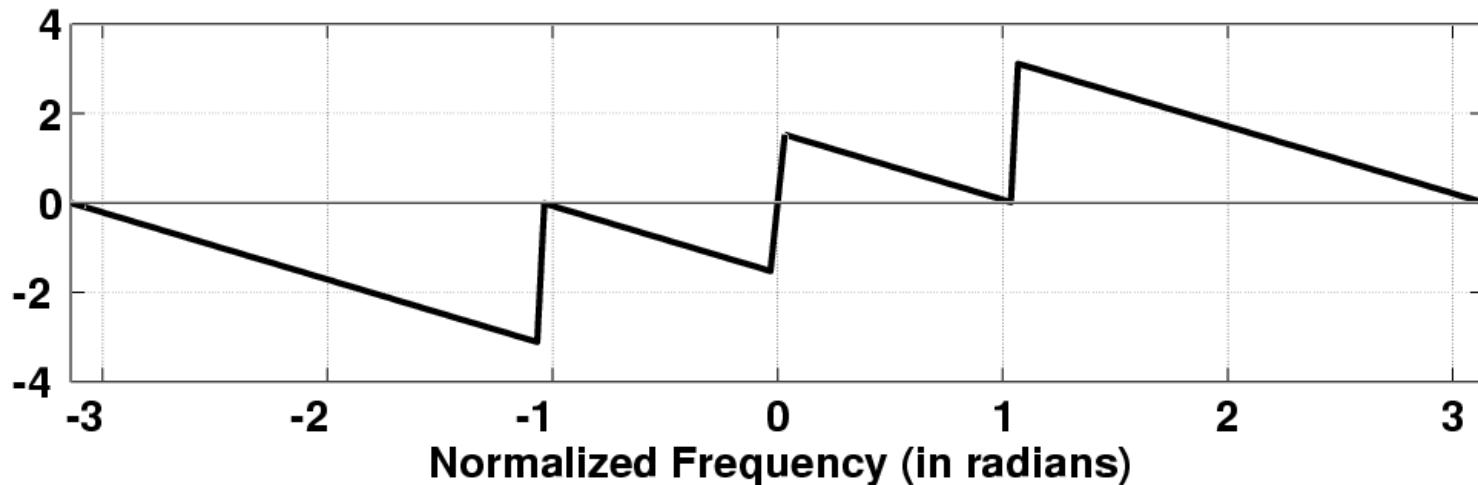


FIR Frequency Response

Magnitude of Frequency Response for $h[n] = 1, -2, 2, -1$



Phase Angle of Frequency Response for $h[n] = 1, -2, 2, -1$

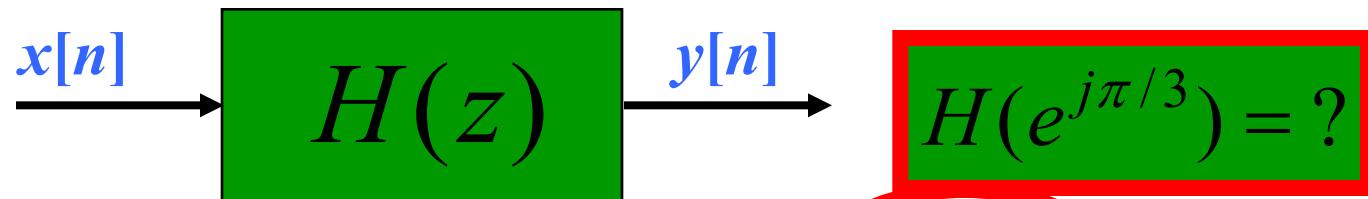


NULLING PROPERTY of $H(z)$

- When $H(z)=0$ on the unit circle.
 - Find inputs $x[n]$ that give zero output

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

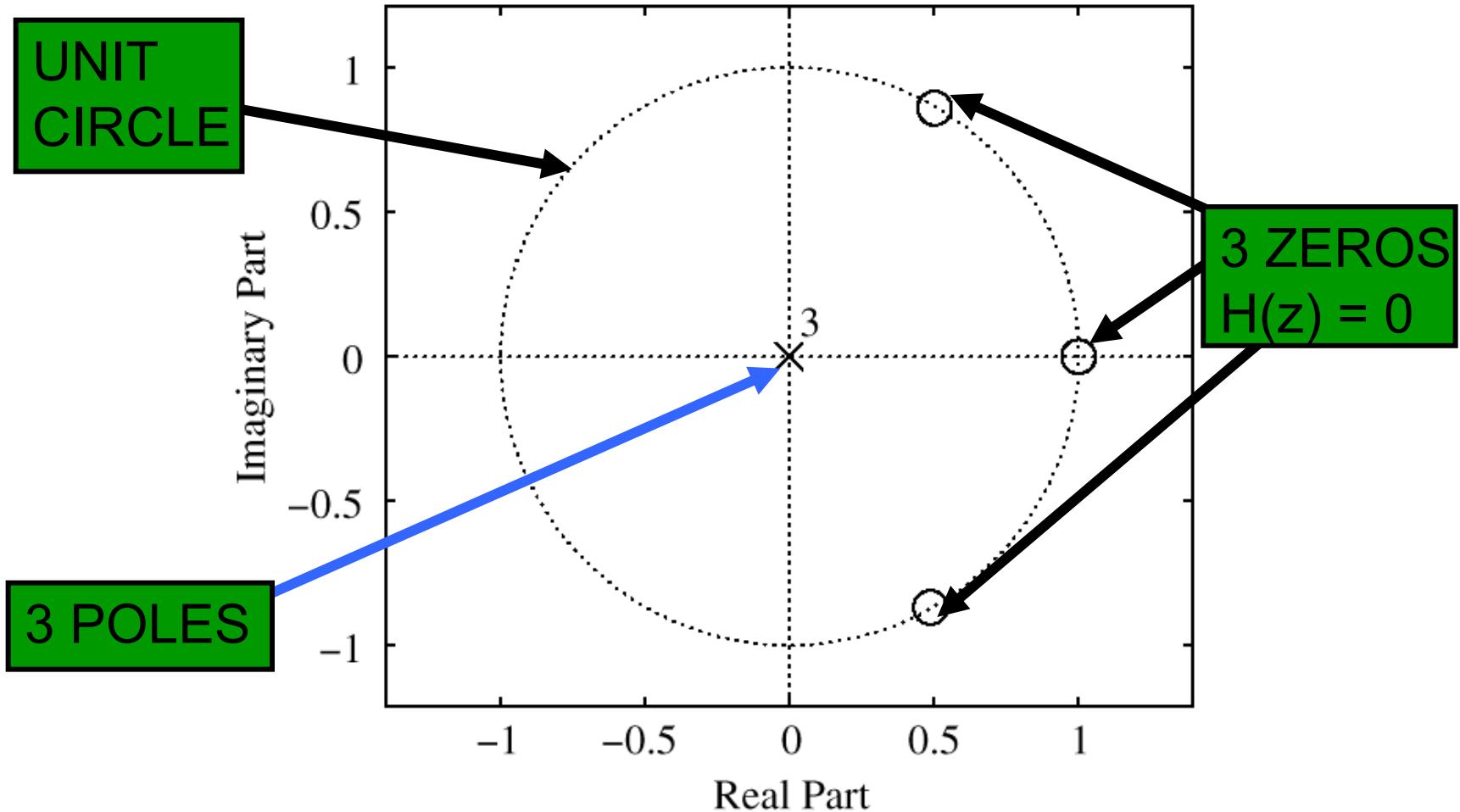
$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$



$$x[n] = e^{j(\pi/3)n}$$

$$y[n] = H(e^{j(\pi/3)}) \cdot e^{j(\pi/3)n}$$

PLOT ZEROS in z-DOMAIN



NULLING PROPERTY of $H(z)$

- Evaluate $H(z)$ at the input “frequency”

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

$$y[n] = H(e^{j\pi/3}) \cdot e^{j(\pi/3)n}$$

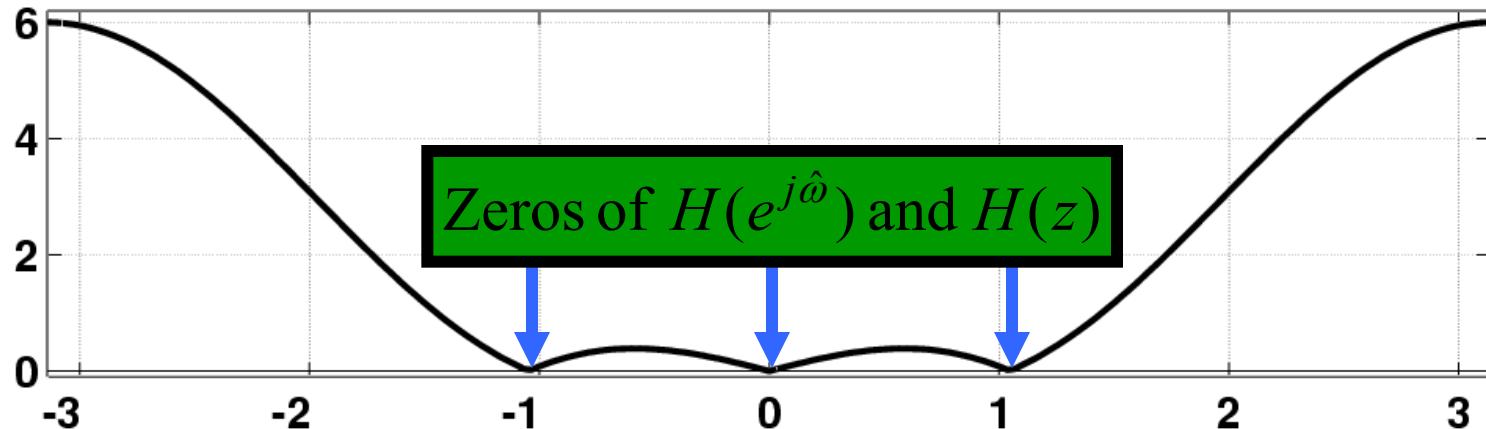
$$y[n] = (1 - 2e^{-j\pi/3} + 2e^{-j2\pi/3} - e^{-j3\pi/3}) \cdot e^{j(\pi/3)n}$$

$$(1 - 2\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + 2\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) - (-1))$$

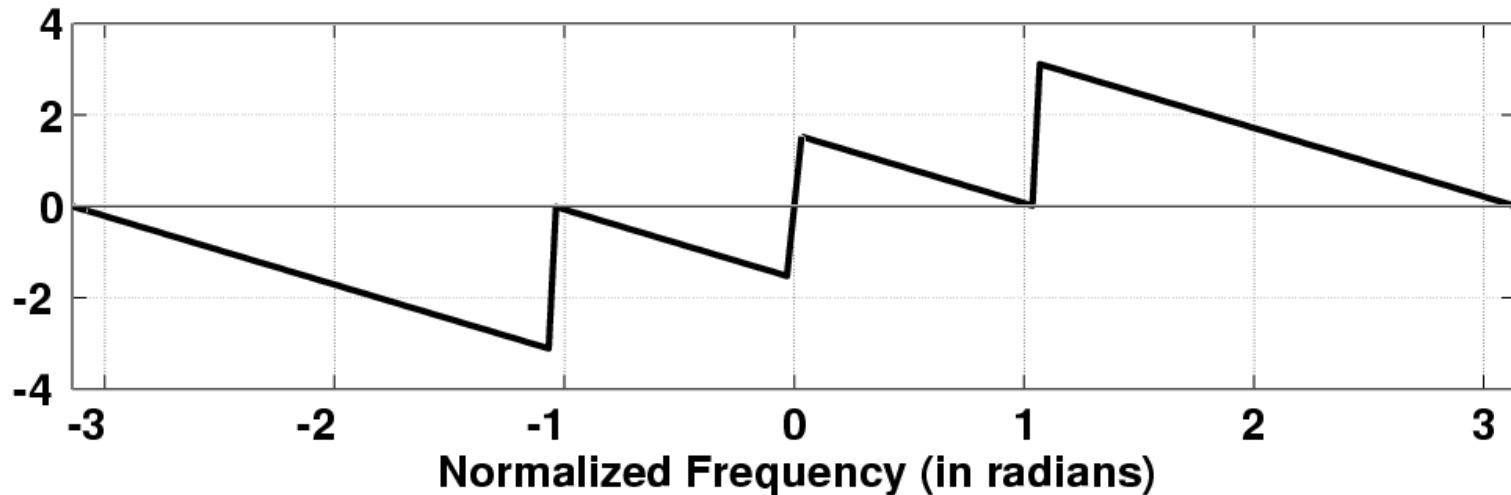
$$y[n] = (1 - 1 + j\sqrt{3} - 1 - j\sqrt{3} + 1) \cdot e^{j(\pi/3)n} = 0$$

FIR Frequency Response

Magnitude of Frequency Response for $h[n] = 1, -2, 2, -1$



Phase Angle of Frequency Response for $h[n] = 1, -2, 2, -1$



DESIGN PROBLEM

- Example:
 - Design a Lowpass FIR filter
 - Find b_k
 - Reject completely 0.7π , 0.8π , and 0.9π
 - Estimate the filter length needed to accomplish this task.
 - How many b_k
- Z POLYNOMIALS provide the TOOLS

IIR Filters: Feedback and $H(z)$

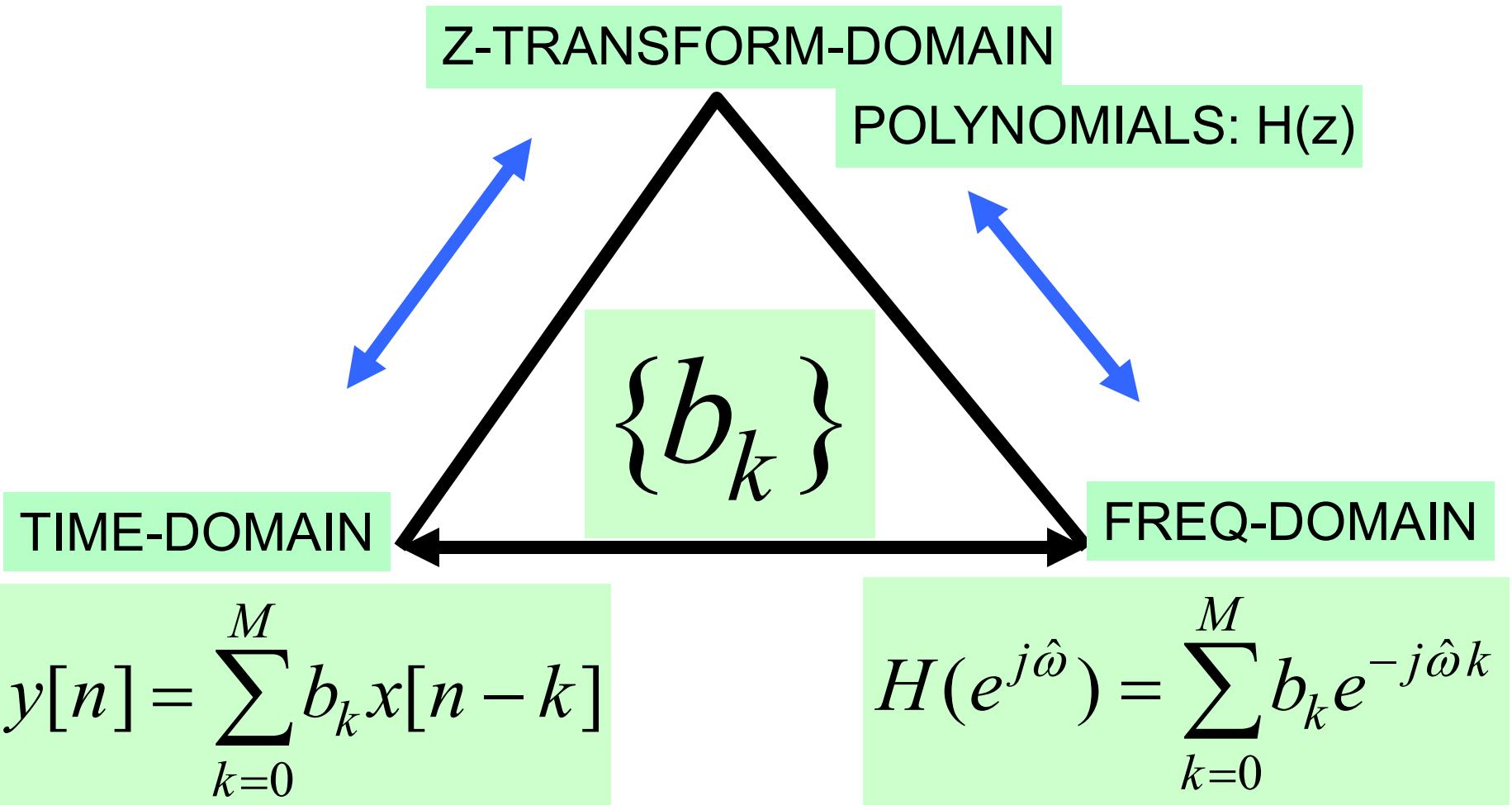
LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
 - Define **IIR** DIGITAL Filters
 - Have **FEEDBACK**: use PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^N a_\ell y[n - \ell] + \sum_{k=0}^M b_k x[n - k]$$

- Show how to compute the output $y[n]$
 - FIRST-ORDER CASE ($N=1$)
 - Z-transform: Impulse Response $h[n] \leftrightarrow H(z)$

THREE DOMAINS



Quick Review: Delay by n_d

$$y[n] = x[n - n_d]$$

IMPULSE RESPONSE

$$h[n] = \delta[n - n_d]$$

SYSTEM FUNCTION

$$H(z) = z^{-n_d}$$

FREQUENCY RESPONSE

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_d}$$

LOGICAL THREAD

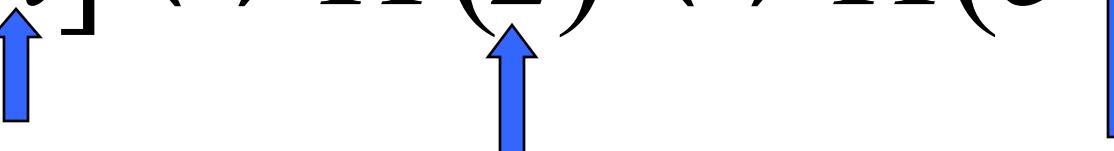
- FIND the IMPULSE RESPONSE, $h[n]$

- INFINITELY LONG
 - IIR Filters

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$$

- EXPLOIT THREE DOMAINS:

- Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$


ONE FEEDBACK TERM

- ADD **PREVIOUS** OUTPUTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

PREVIOUS
FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
 - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

FEEDBACK COEFFICIENT

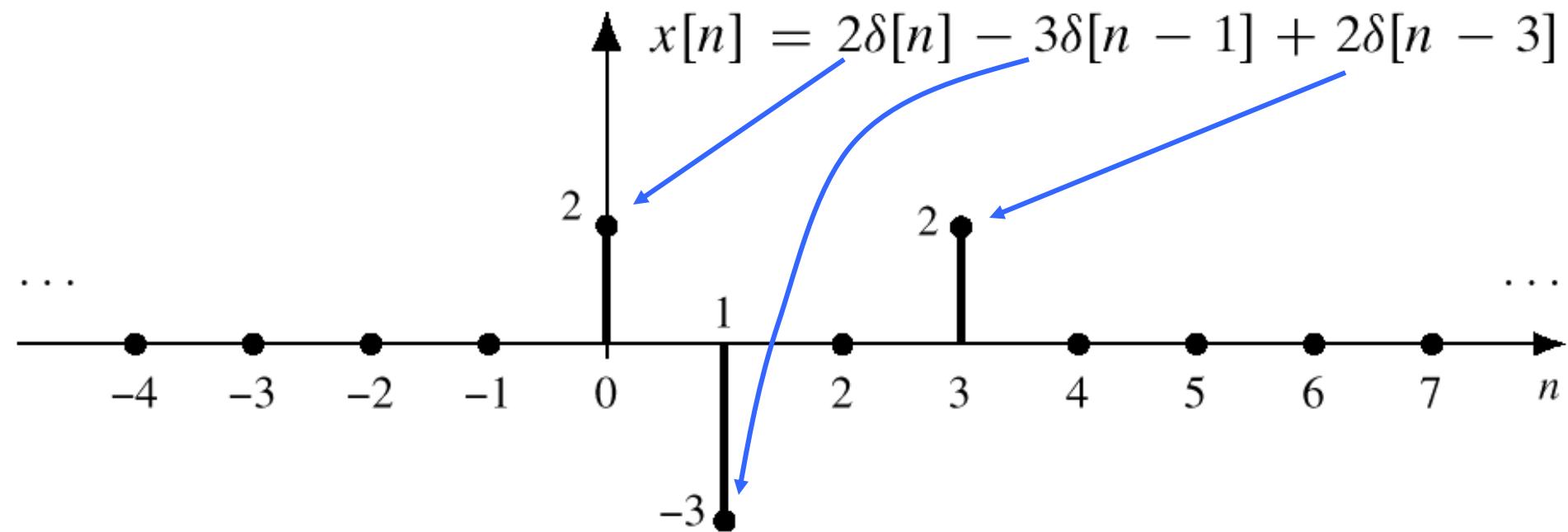
SIGN CHANGE

- MATLAB

– `yy = filter([3, -2], [1, -0.8], xx)`

COMPUTE OUTPUT

$$y[n] = 0.8y[n - 1] + 5x[n]$$



COMPUTE $y[n]$

- FEEDBACK DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 5x[n]$$

- NEED $y[-1]$ to get started


$$y[0] = 0.8y[-1] + 5x[0]$$

AT REST CONDITION

- $y[n] = 0$, for $n < 0$
- BECAUSE $x[n] = 0$, for $n < 0$

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0 , i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied*.
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE $y[0]$

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- SAME with MORE FEEDBACK TERMS

$$y[n] = a_1y[n-1] + a_2y[n-2] + \sum_{k=0}^2 b_k x[n-k]$$

COMPUTE MORE $y[n]$

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

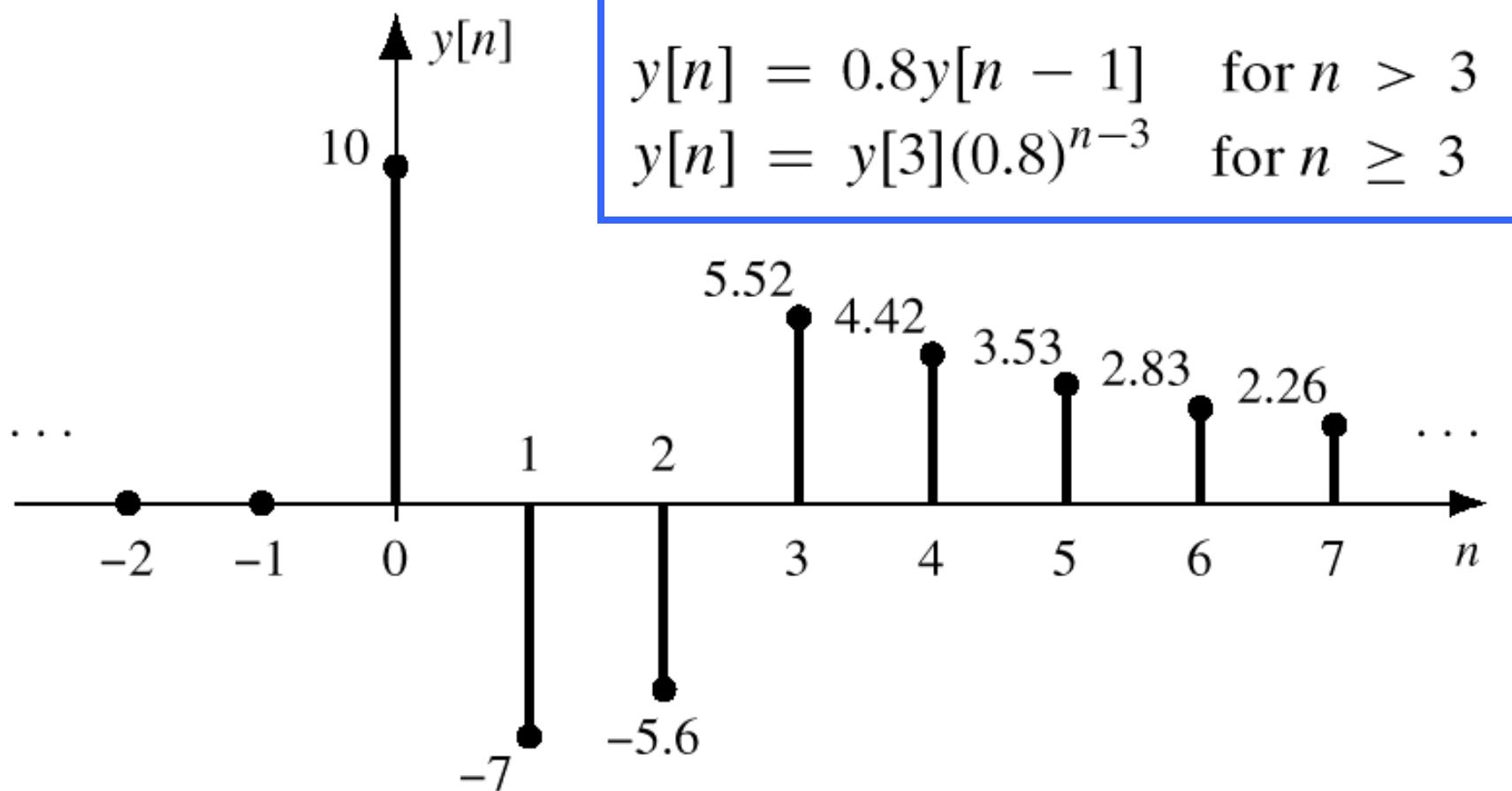
$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLLOT $y[n]$



IMPULSE RESPONSE

$$h[n] = a_1 h[n-1] + b_0 \delta[n]$$

n	$n < 0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0(a_1)^n u[n]$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

IMPULSE RESPONSE

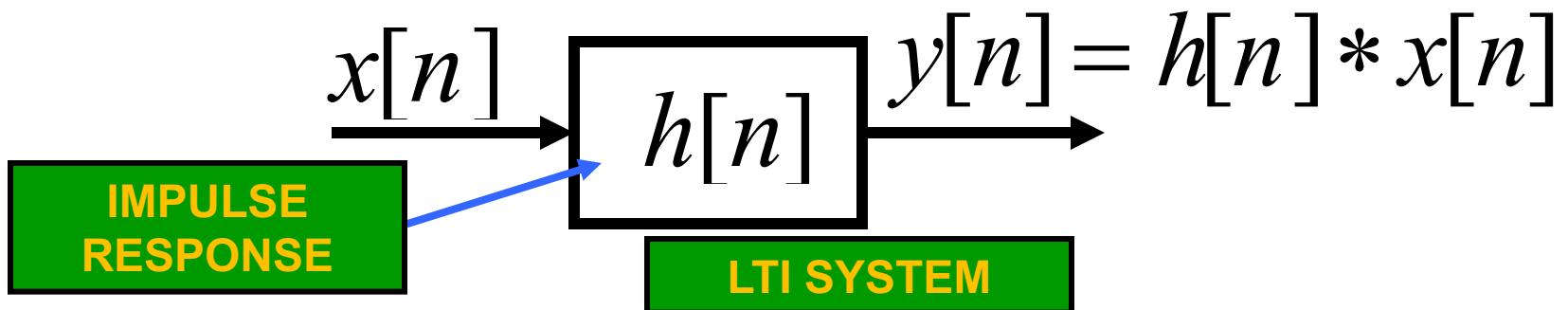
- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n - 1] + 3x[n]$$

- Find $h[n]$

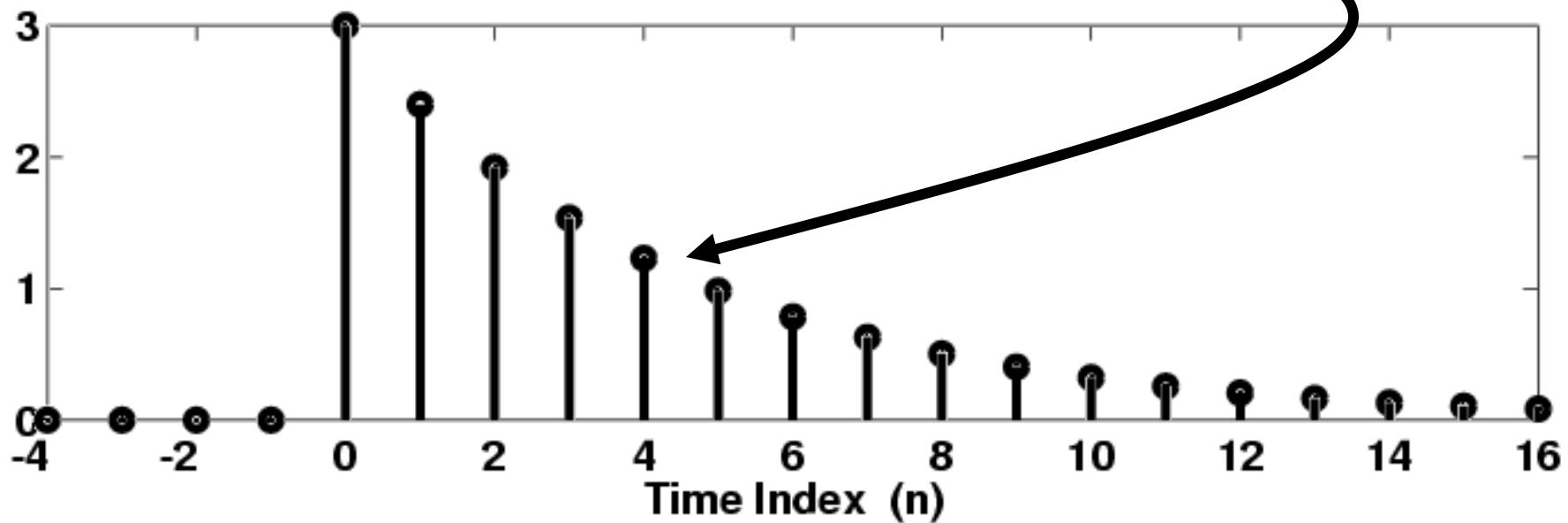
$$h[n] = 3(0.8)^n u[n]$$

- CONVOLUTION in TIME-DOMAIN



PLOT IMPULSE RESPONSE

$$h[n] = b_0(a_1)^n u[n] = 3(0.8)^n u[n]$$



Infinite-Length Signal: $h[n]$

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

APPLIES to
Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z) = \sum_{n=-\infty}^{\infty} b_0(a_1)^n u[n] z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n}$$

Derivation of $H(z)$

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$\begin{aligned} H(z) &= b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n \\ &= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1| \end{aligned}$$

$$H(z) = \text{z-Transform}\{ h[n] \}$$

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$H(z) = \text{z-Transform}\{ h[n] \}$$

- ANOTHER FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

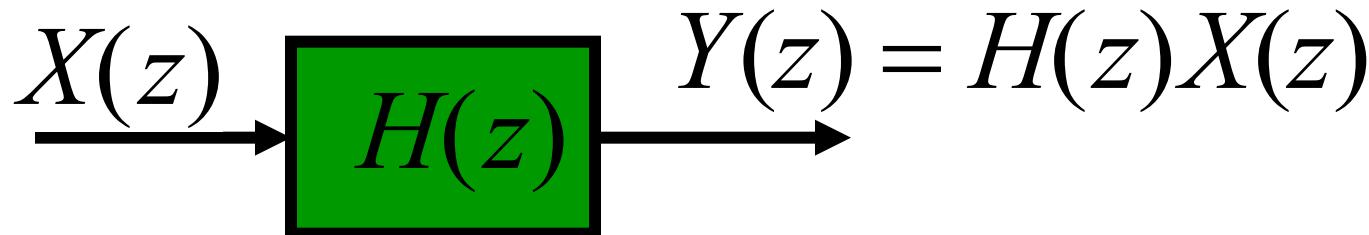
$$h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]$$

z^{-1} is a shift

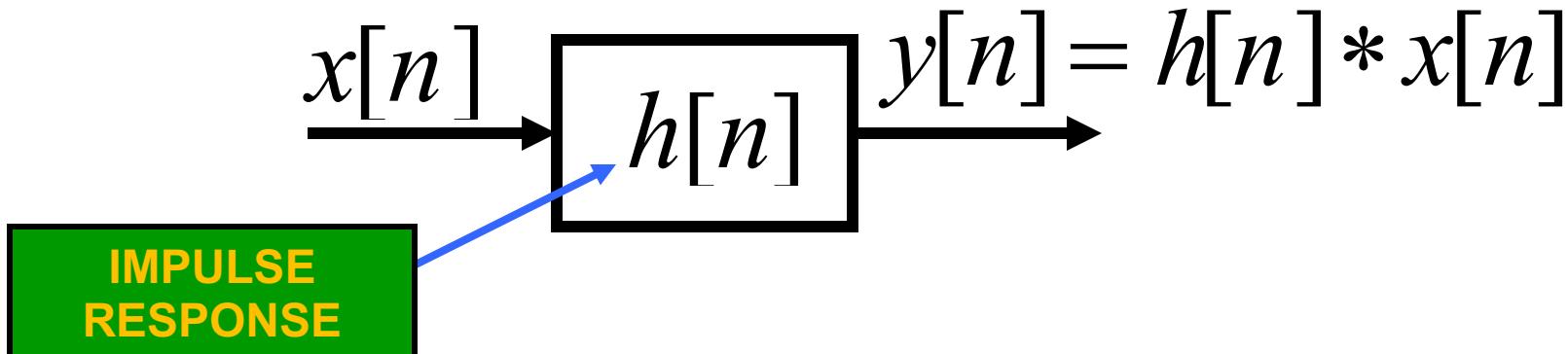
$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS



- CONVOLUTION in TIME-DOMAIN



STEP RESPONSE: $x[n]=u[n]$

$$y[n] = a_1 y[n - 1] + b_0 x[n]$$

n	$x[n]$	$y[n]$
$n < 0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
\vdots	1	\vdots

DERIVE STEP RESPONSE

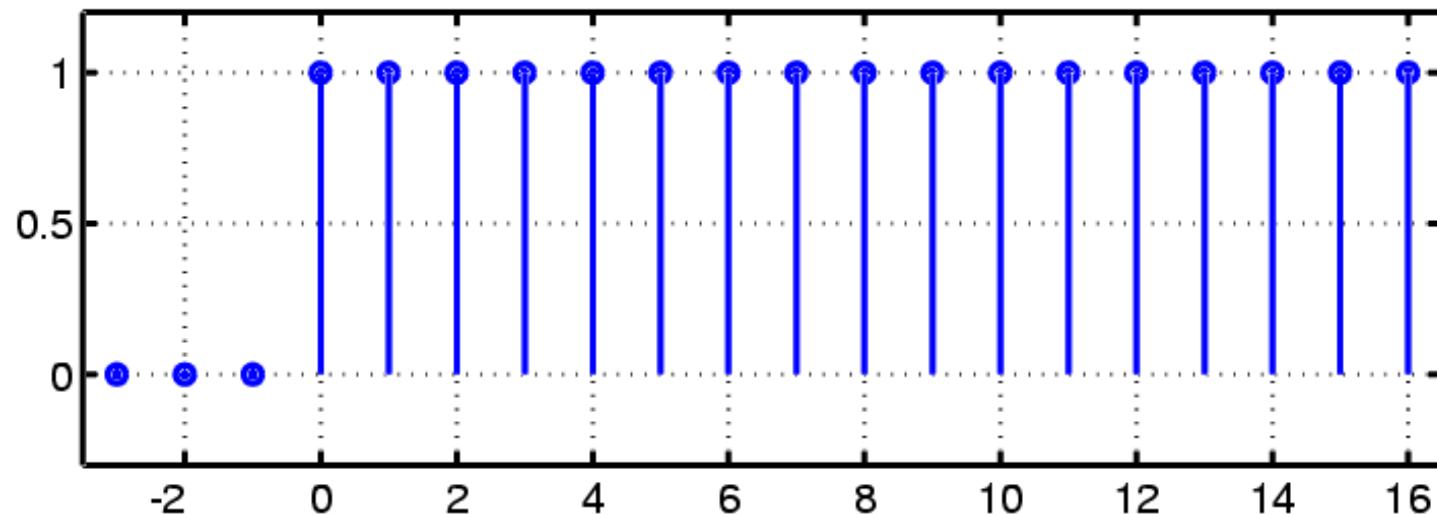
$$y[n] = b_0(1 + a_1 + a_1^2 + \dots + a_1^n) = b_0 \sum_{k=0}^n a_1^k$$

$$\sum_{k=0}^L r^k = \begin{cases} \frac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \quad \text{for } n \geq 0, \quad \text{if } a_1 \neq 1$$

PLOT STEP RESPONSE

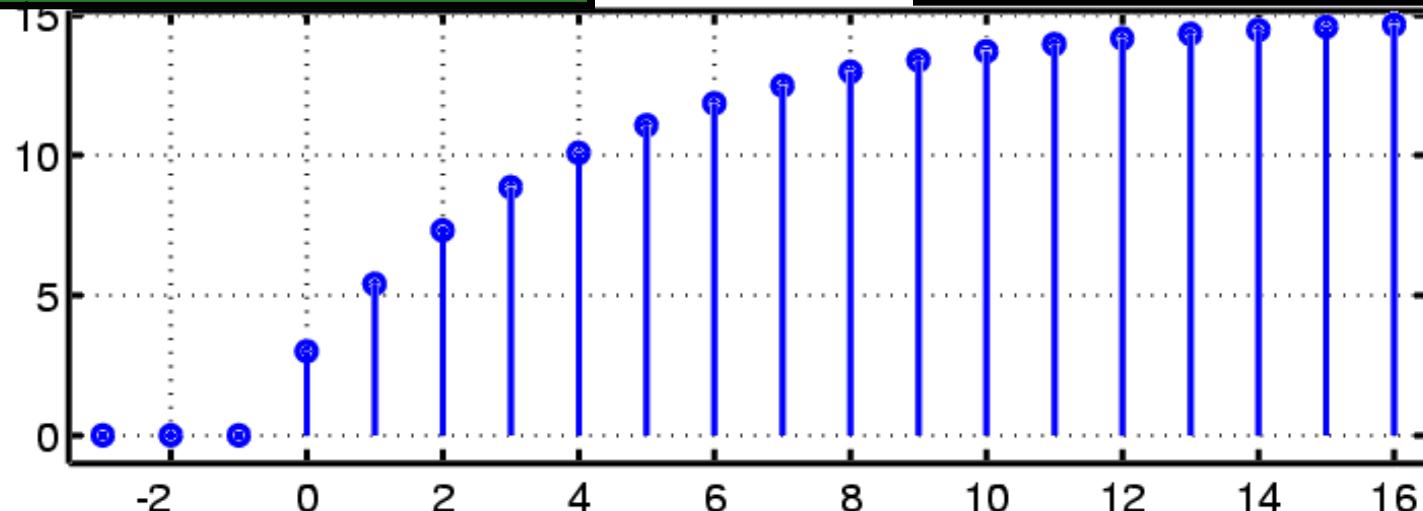
Step Input



$$y[n] = 0.8y[n-1] + 3u[n]$$

Step Response

$$y[n] = 15(1 - 0.8^{n+1})u[n]$$



IIR Filters: $H(z)$ and Frequency Response

LECTURE OBJECTIVES

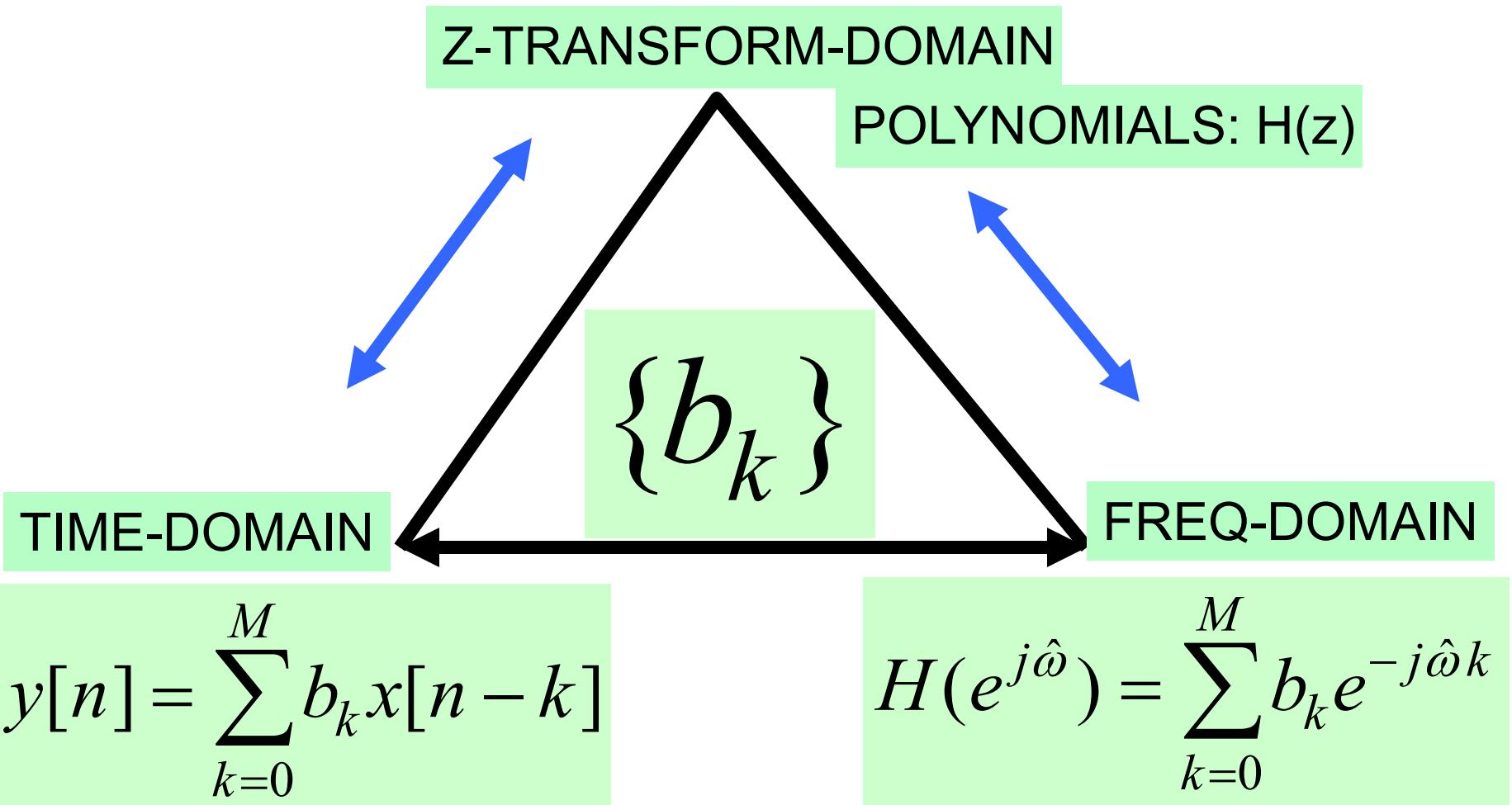
- SYSTEM FUNCTION: $H(z)$
- $H(z)$ has POLES and ZEROS
- FREQUENCY RESPONSE of IIR
 - Get $H(z)$ first

$$H(e^{j\hat{\omega}}) = H(z) \Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

THREE DOMAINS



$$H(z) = \text{z-Transform}\{ h[n] \}$$

- FIRST-ORDER IIR FILTER:

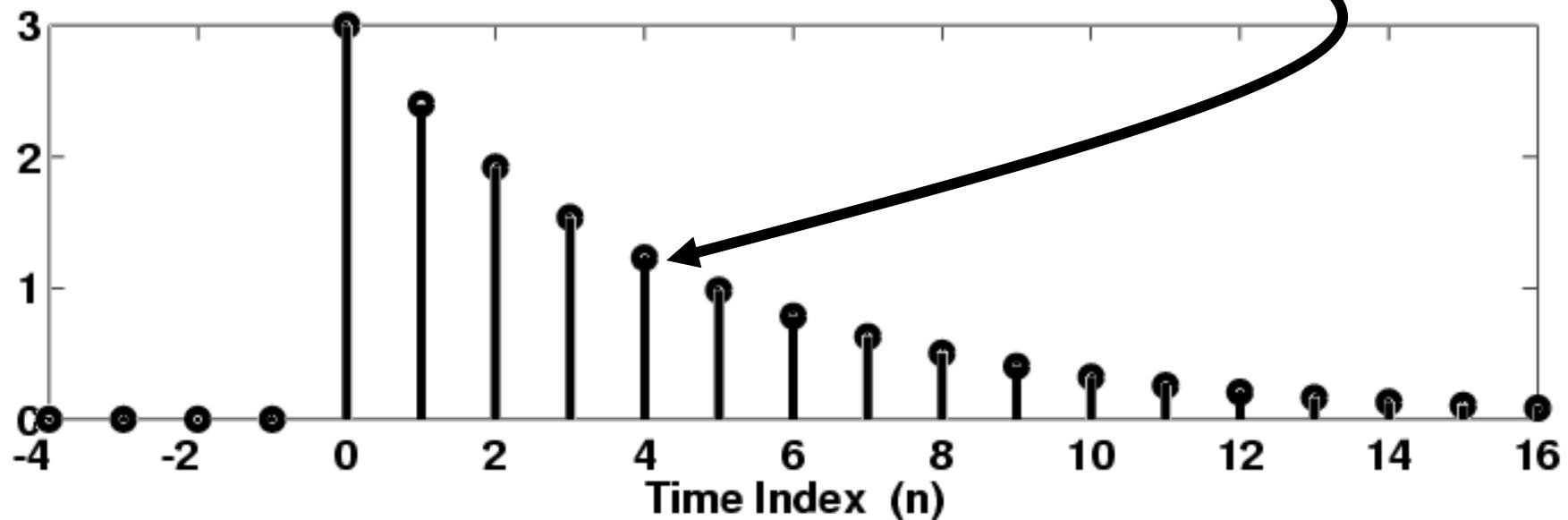
$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

Typical IMPULSE Response

$$h[n] = b_0(a_1)^n u[n] = 3(0.8)^n u[n]$$



First-Order Transform Pair

$$h[n] = ba^n u[n] \leftrightarrow H(z) = \frac{b}{1 - az^{-1}}$$

- GEOMETRIC SEQUENCE:

$$\begin{aligned} H(z) &= b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n \\ &= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1| \end{aligned}$$

DELAY PROPERTY of $X(z)$

- DELAY in TIME \leftrightarrow Multiply $X(z)$ by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1}X(z)$$

Proof:

$$\sum_{n=-\infty}^{\infty} x[n-1]z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+1)}$$
$$= z^{-1} \sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell} = z^{-1}X(z)$$

Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION $H(z)$
 - Use DELAY PROPERTY

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z -transform by z^{-n_0}

$$x[n - n_0] \iff z^{-n_0} X(z)$$

SYSTEM FUNCTION of IIR

- NOTE the FILTER COEFFICIENTS

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

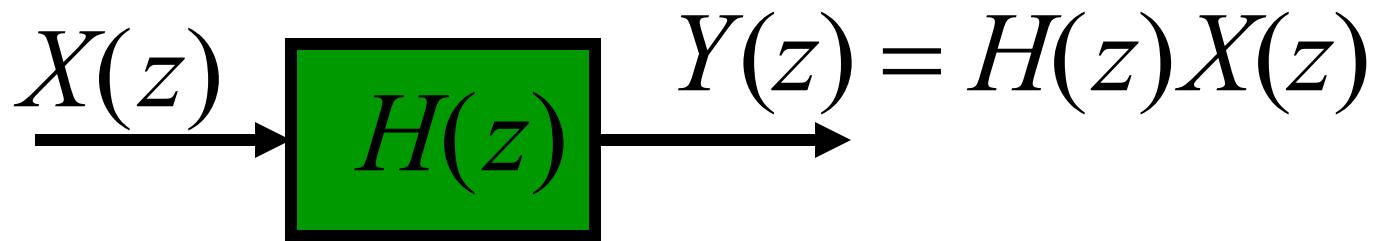
- READ the FILTER COEFFS:

$$H(z)$$

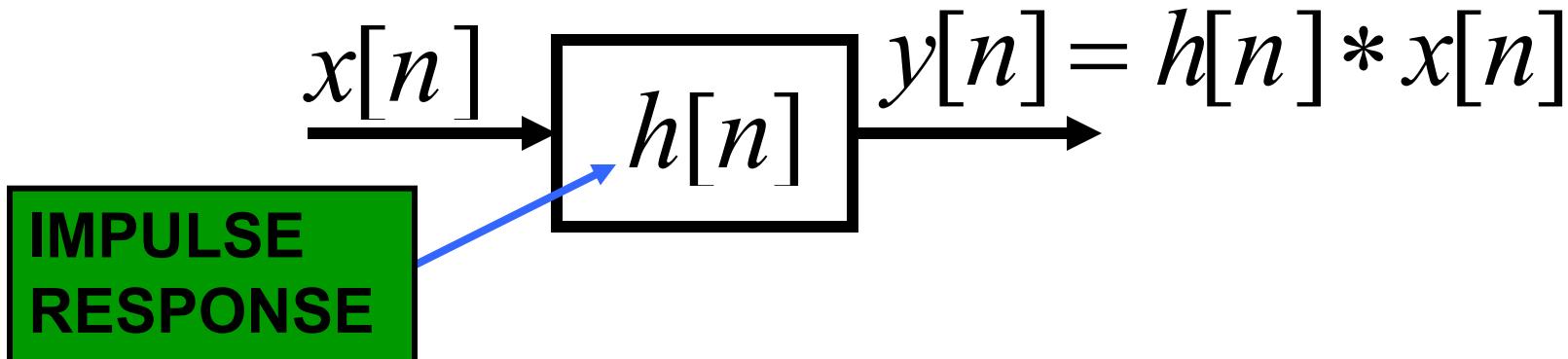
$$Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8z^{-1}} \right) X(z)$$

CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS



- CONVOLUTION in TIME-DOMAIN



POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

ZERO:
 $H(z)=0$

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

POLE: $H(z) \rightarrow \infty$

EXAMPLE: Poles & Zeros

- VALUE of $H(z)$ at POLES is INFINITE

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

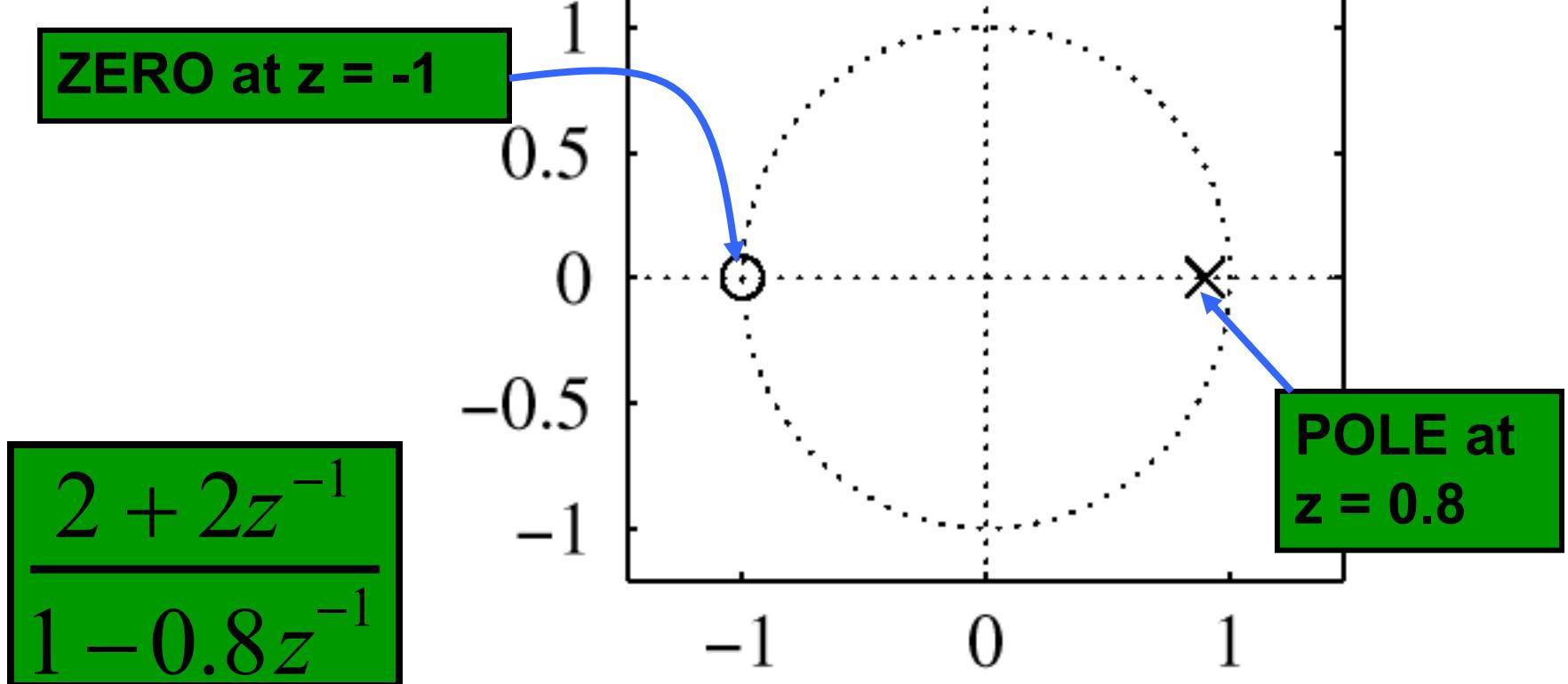
$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

$$H(z) = \frac{2 + 2\left(\frac{4}{5}\right)^{-1}}{1 - 0.8\left(\frac{4}{5}\right)^{-1}} = \frac{\frac{9}{2}}{0} \rightarrow \infty$$

ZERO at $z= -1$

POLE at $z=0.8$

POLE-ZERO PLOT



FREQUENCY RESPONSE

- SYSTEM FUNCTION: $H(z)$
- $H(z)$ has DENOMINATOR
- FREQUENCY RESPONSE of IIR
 - We have $H(z)$

$$H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$



FREQ. RESPONSE FORMULA

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$$

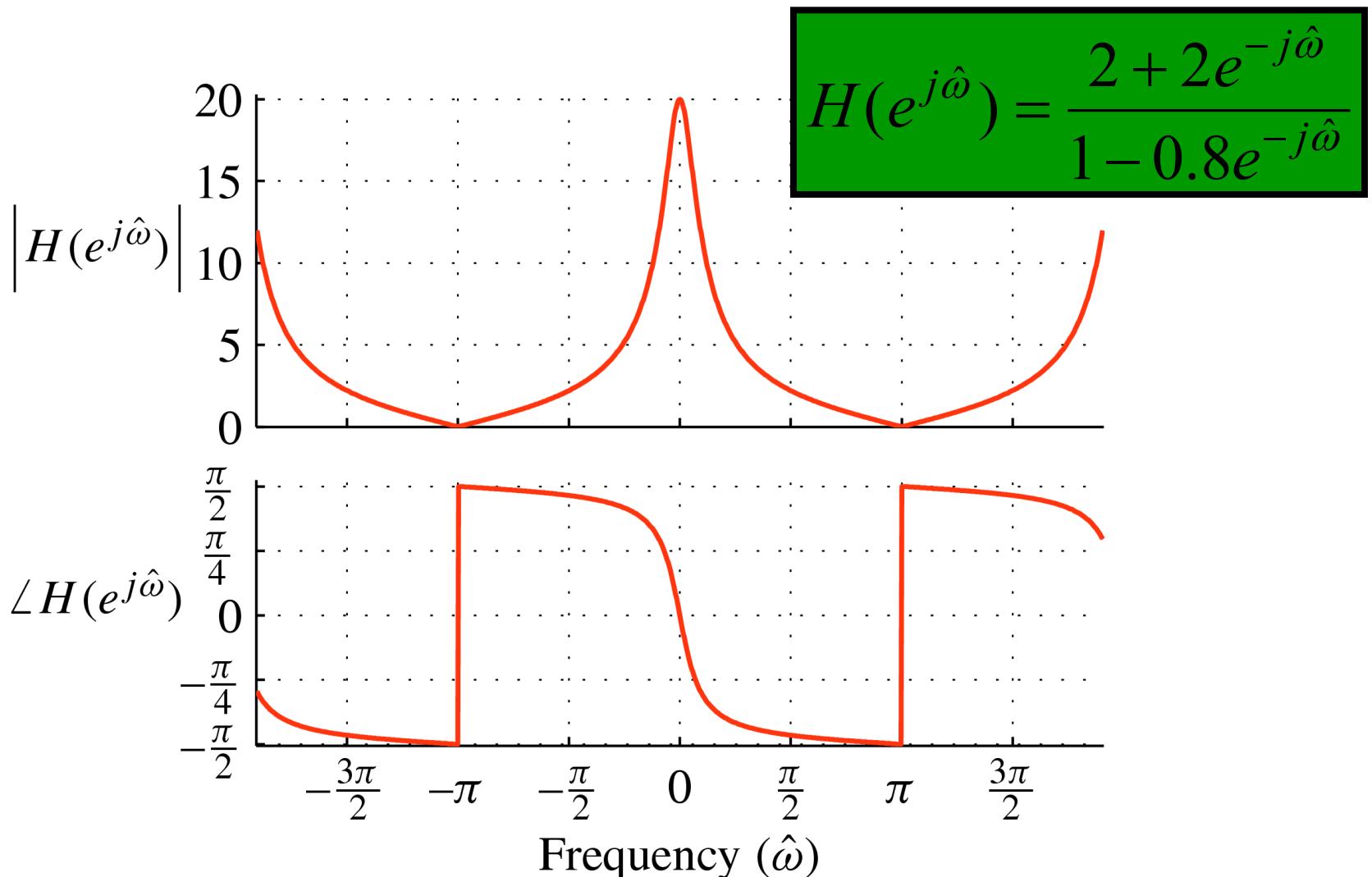
$$\left|H(e^{j\hat{\omega}})\right|^2 = \left| \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}} \right|^2 = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}} \cdot \frac{2 + 2e^{j\hat{\omega}}}{1 - 0.8e^{j\hat{\omega}}}$$

$$\frac{4 + 4 + 4e^{-j\hat{\omega}} + 4e^{j\hat{\omega}}}{1 + 0.64 - 0.8e^{-j\hat{\omega}} - 0.8e^{j\hat{\omega}}} =$$

$$\frac{8 + 8 \cos \hat{\omega}}{1.64 - 1.6 \cos \hat{\omega}}$$

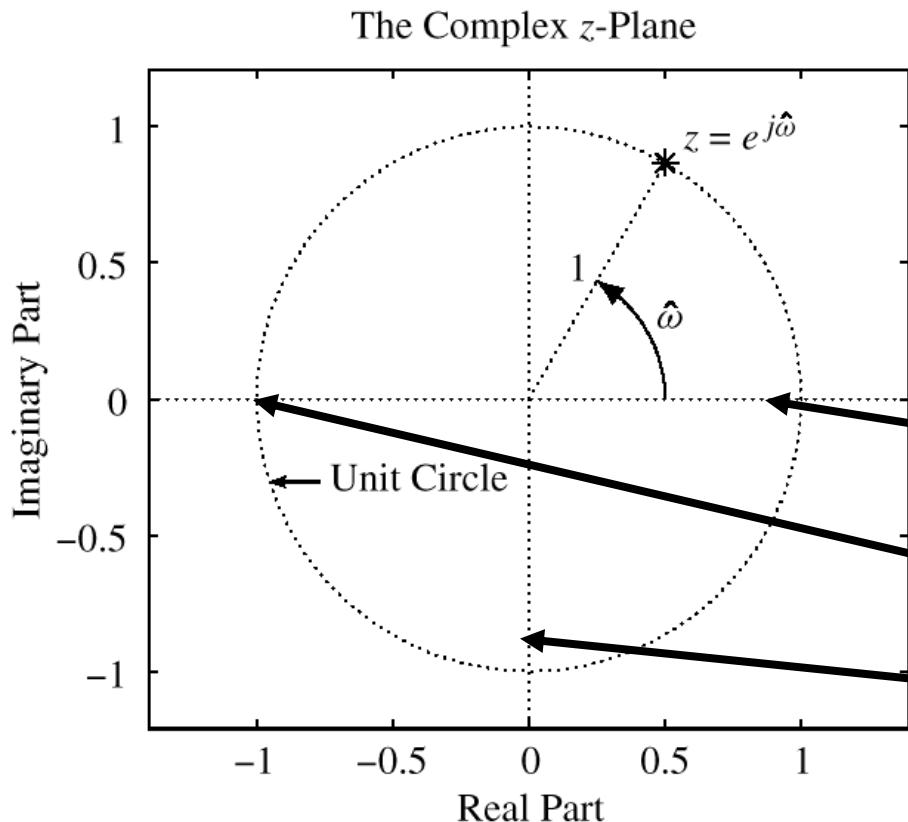
$$@ \hat{\omega} = 0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{8 + 8}{0.04} = 400, \quad @ \hat{\omega} = \pi ?$$

Frequency Response Plot



UNIT CIRCLE

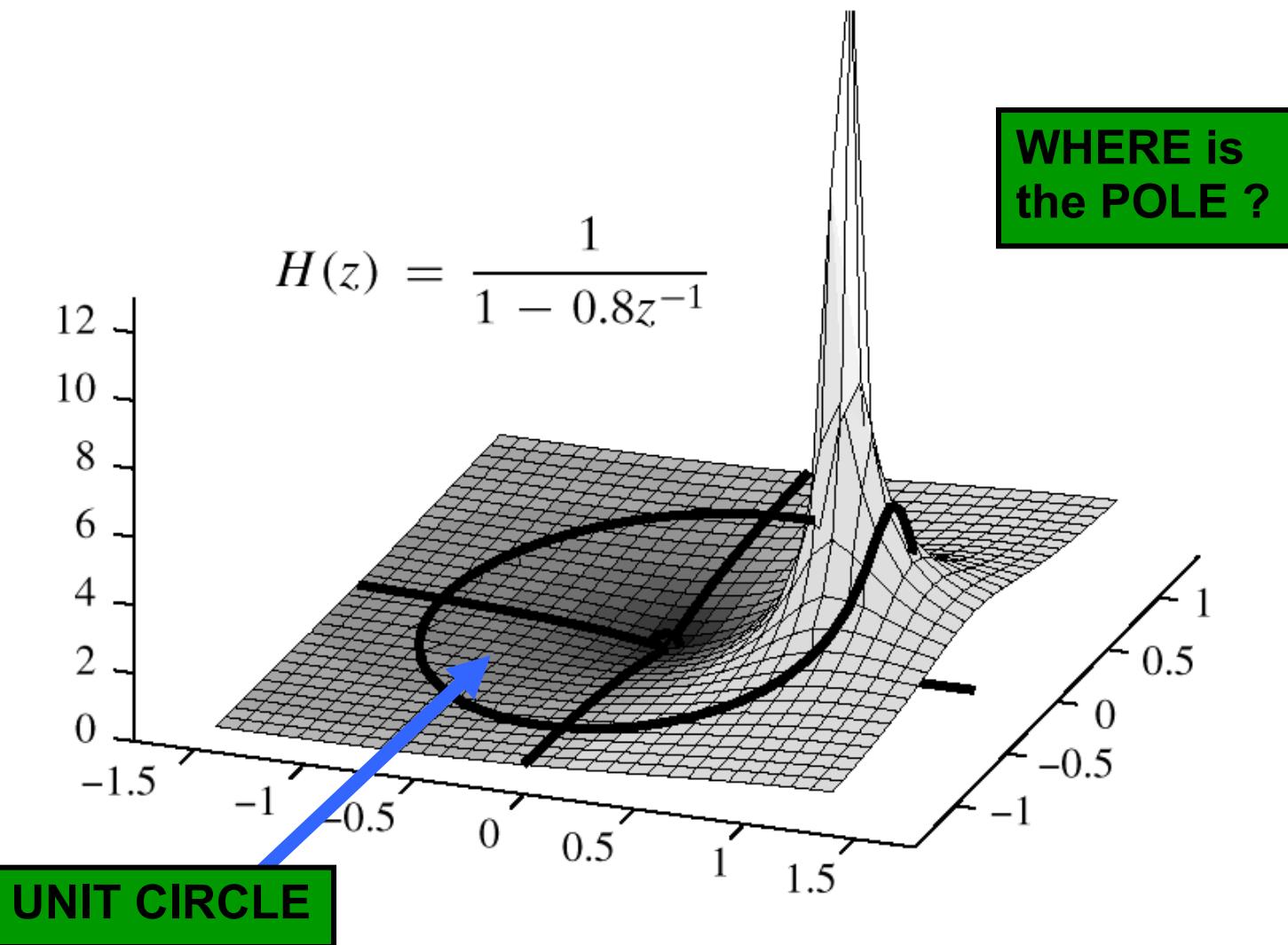
- MAPPING BETWEEN z and $\hat{\omega}$



$$Z = e^{j\hat{\omega}}$$

$$\begin{array}{lll} z = 1 & \leftrightarrow & \hat{\omega} = 0 \\ z = -1 & \leftrightarrow & \hat{\omega} = \pm\pi \\ z = \pm j & \leftrightarrow & \hat{\omega} = \pm\frac{1}{2}\pi \end{array}$$

3-D VIEWPOINT: EVALUATE H(z) EVERYWHERE



SINUSOIDAL RESPONSE

- $x[n] = \text{SINUSOID} \Rightarrow y[n] \text{ is SINUSOID}$
- Get MAGNITUDE & PHASE from $H(z)$

if $x[n] = e^{j\hat{\omega}n}$

then $y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$

where $H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}}$



POP QUIZ

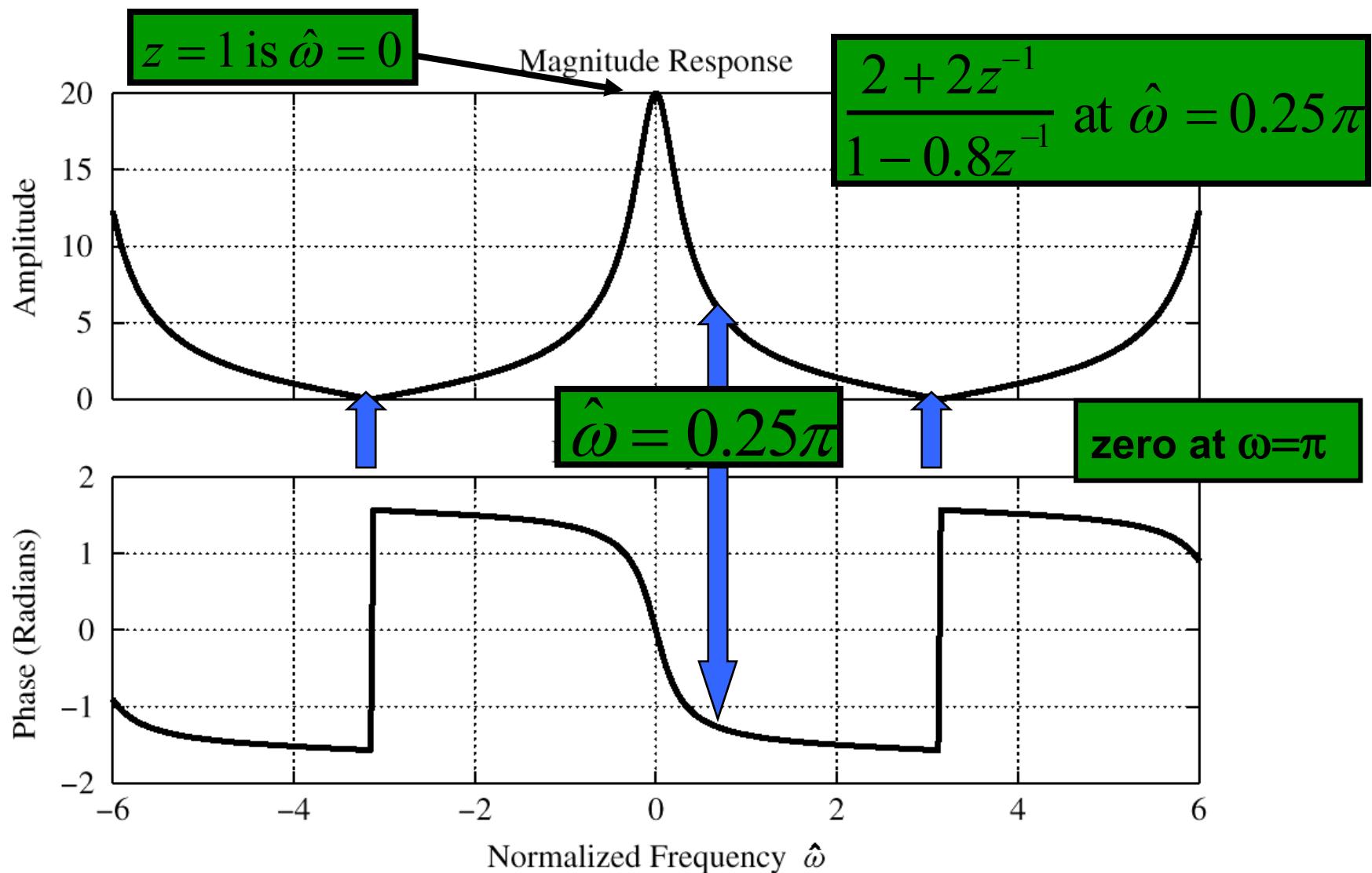
- Given:

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

- Find the Impulse Response, $h[n]$
- Find the output, $y[n]$
 - When

$$x[n] = \cos(0.25\pi n)$$

Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

- Given:

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

- Find output, $y[n]$, when

$$x[n] = \cos(0.25\pi n)$$

- Evaluate at

$$z = e^{j0.25\pi}$$

$$H(z) = \frac{2 + 2(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2})}{1 - 0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182 \cos(0.25\pi n - 0.417\pi)$$

CASCADE EQUIVALENT

- Multiply the System Functions

