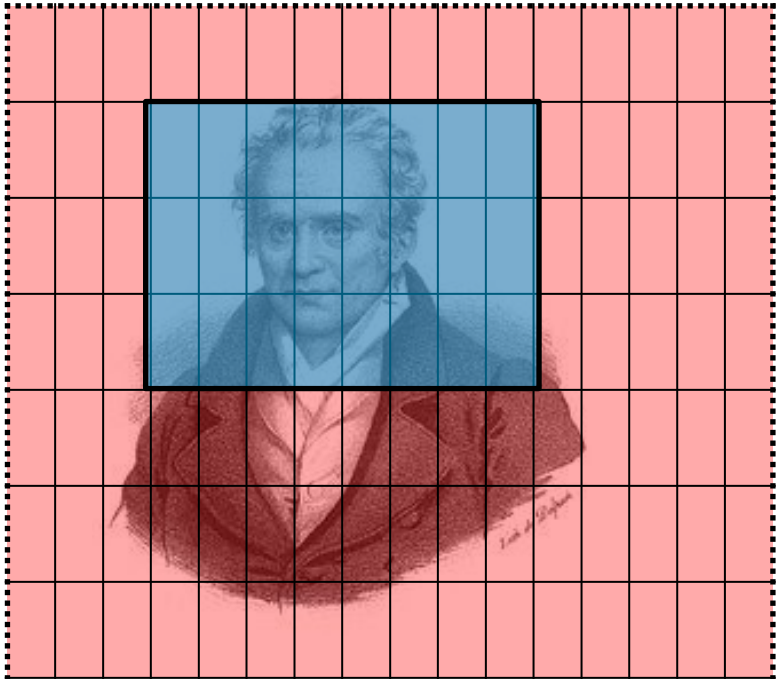
Data structures for totally monotone matrices

Submatrix maximum queries in (inverse) Monge matrices

Input: n x n (inverse) Monge matrix M represented implicitly (constant time oracle access)

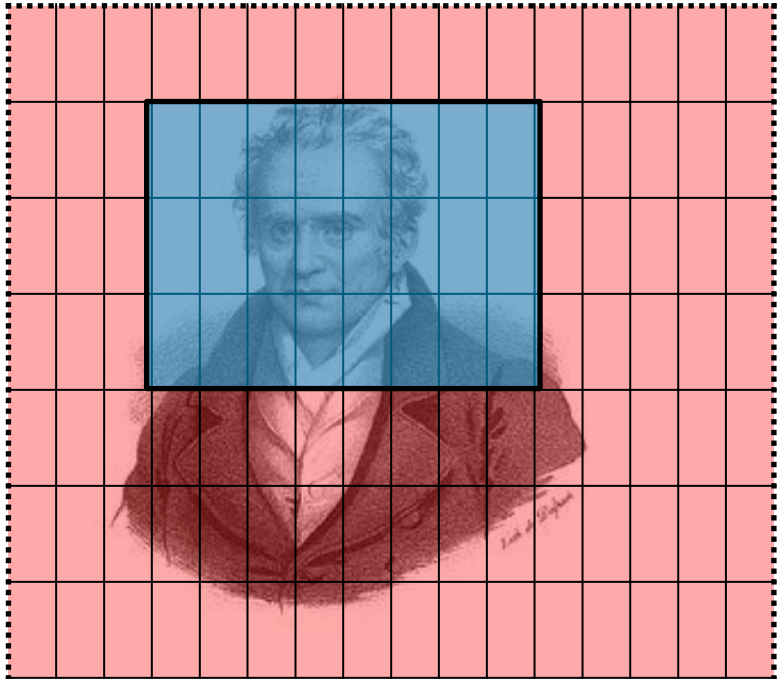
Output: data structure answering maximum queries in any rectangular submatrix of M



Submatrix maximum queries

Main Theorem:
A data structure with the following performance:

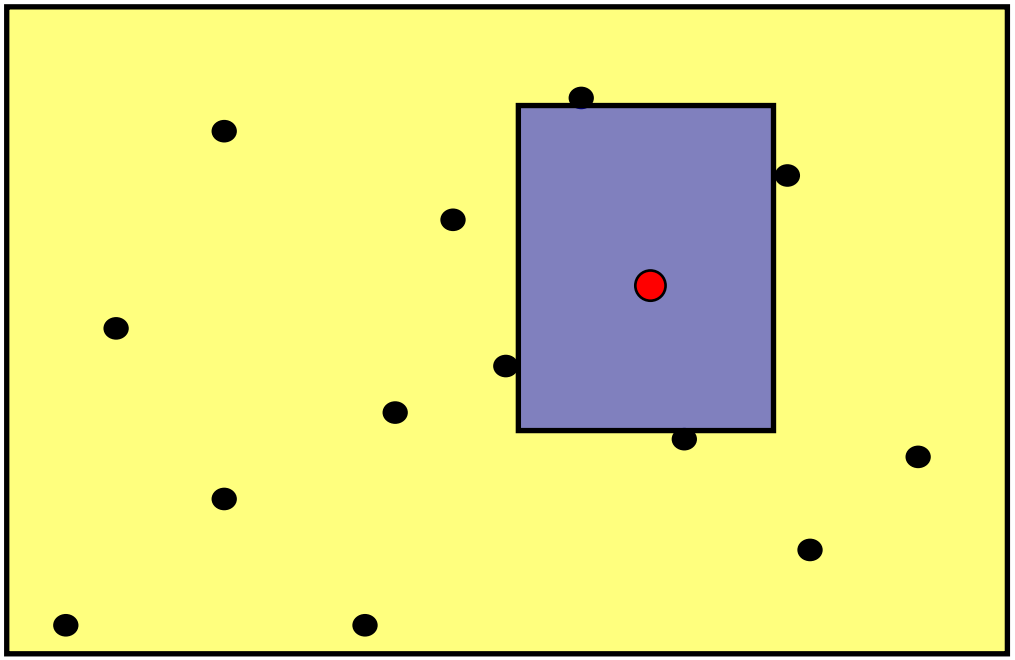
preprocessing time	$O(n \log^2 n)$
space	$O(n \log n)$
query time	$O(\log^2 n)$



Extends to rectangular matrices and certain partial matrices

Application first…

Largest Empty Rectangle

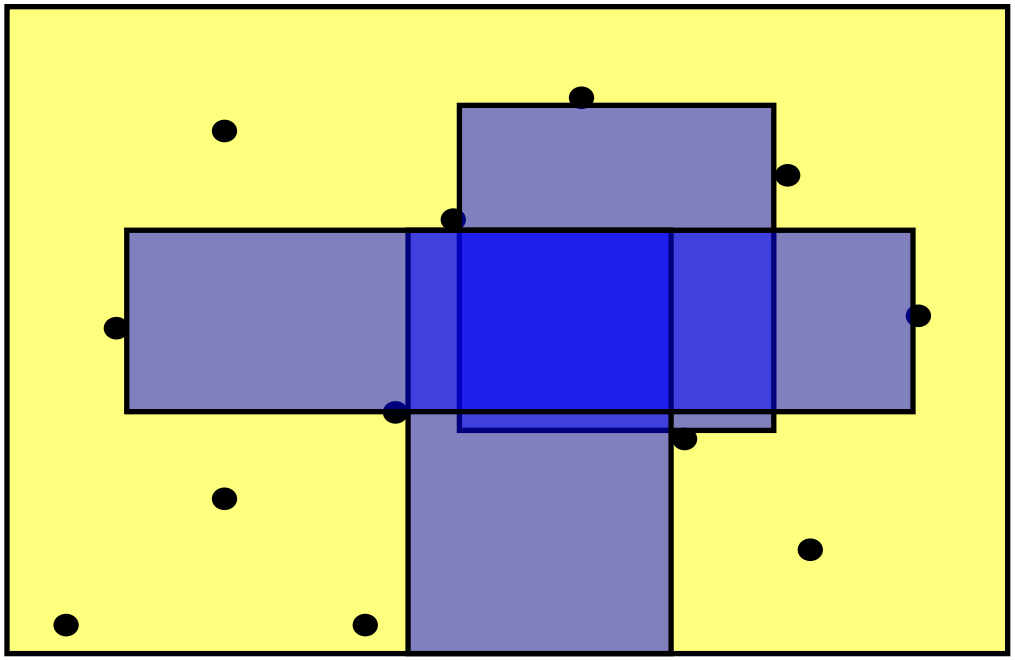


For a set P of n points in the plane build a data structure such that:

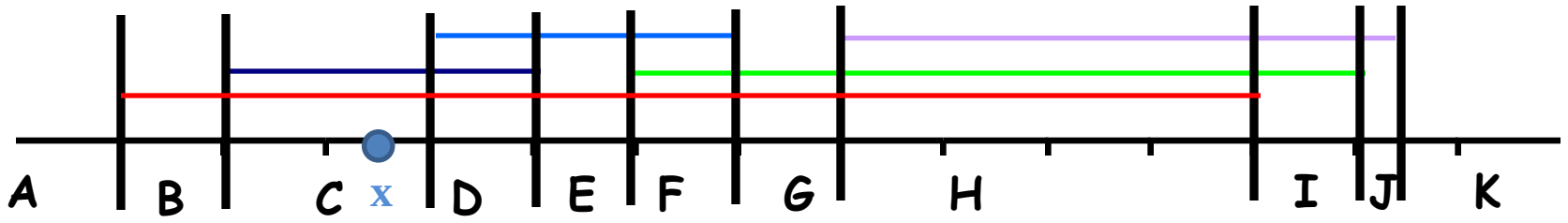
Given a query q, efficiently finds the largest empty (of pts of P) rectangle containing q

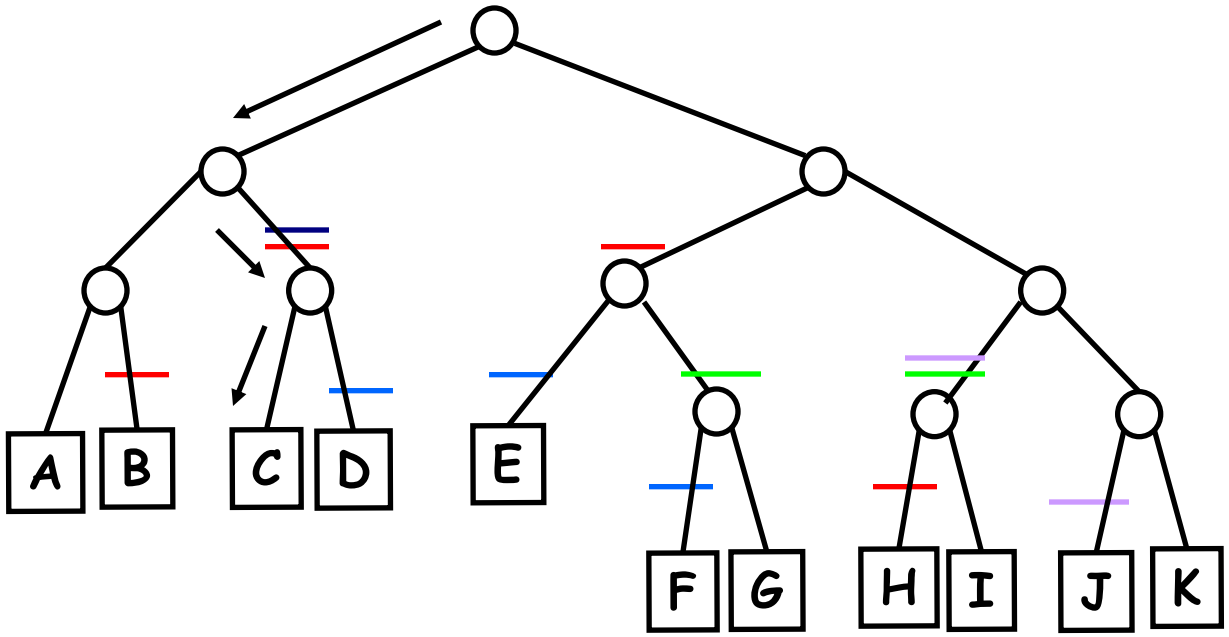
Naïve Approach

Store all N maximal empty rectangles in a two-dimensional segment tree.

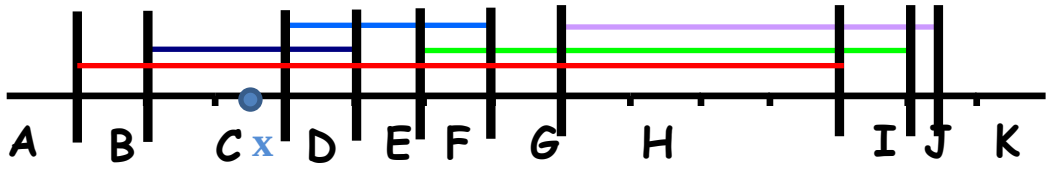


(detour) A 1D Segment tree

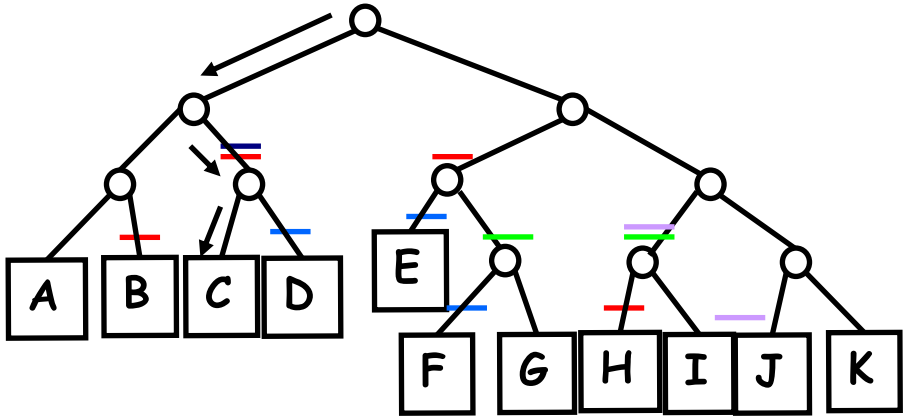




A 2D Segment tree

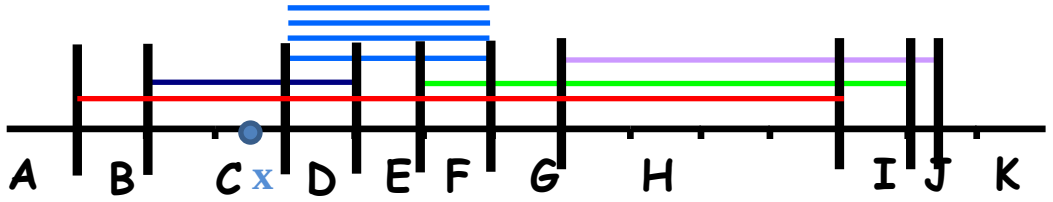


The size of the primary tree is prop. to the number of different x-coordinates



Since rectangles share x-coordinates there could be many rectangles mapped even to deep nodes.

A 2D Segment tree



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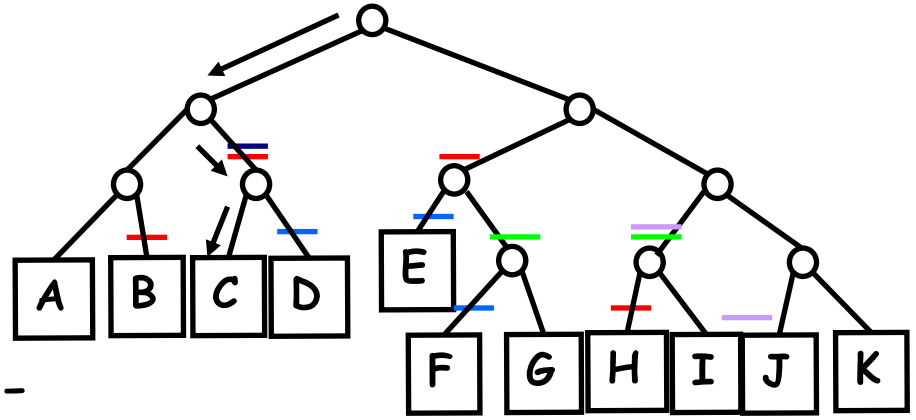
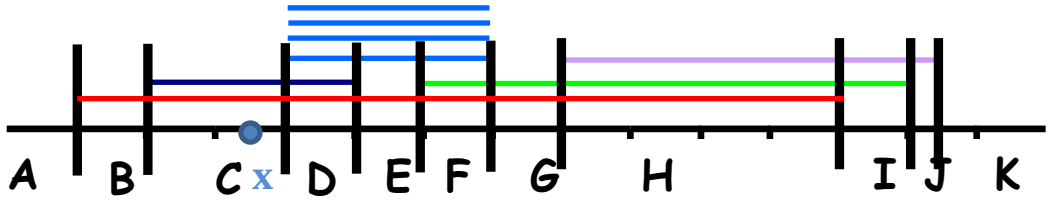


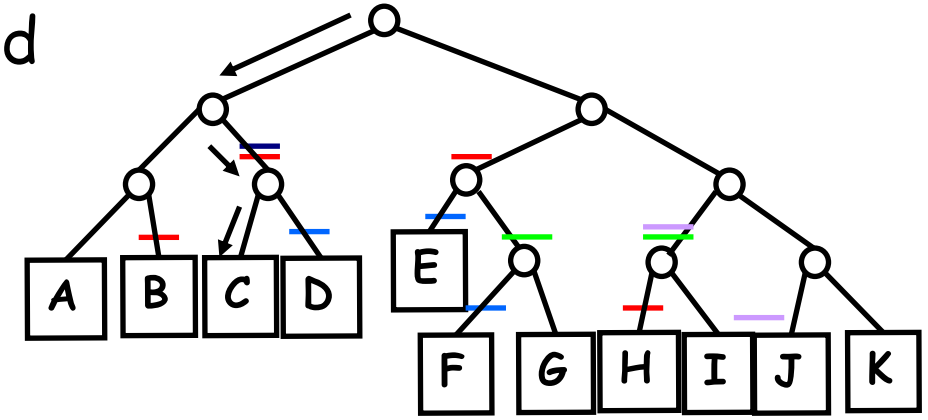
Since rectangles share x-coordinates there could be many rectangles mapped even to deep nodes.

In particular many rectangles can share their x-projections

A 2D Segment tree



Each rectangle is mapped to O(log(n)) primary nodes



So if we have N rectangles the total size of the secondary trees is O(Nlog(n))

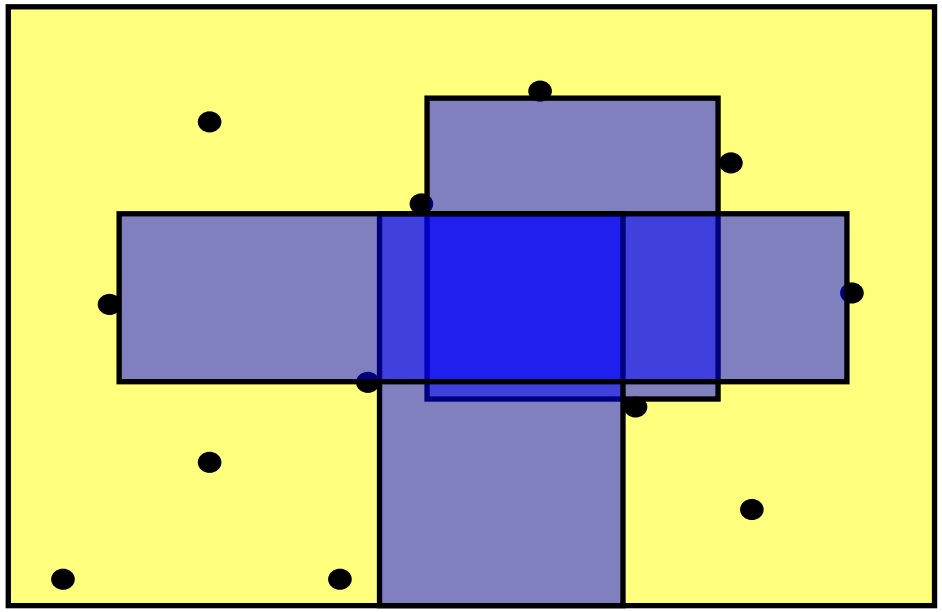
Naïve approach

Store all N maximal empty rectangles in a

two-dimensional segment tree.

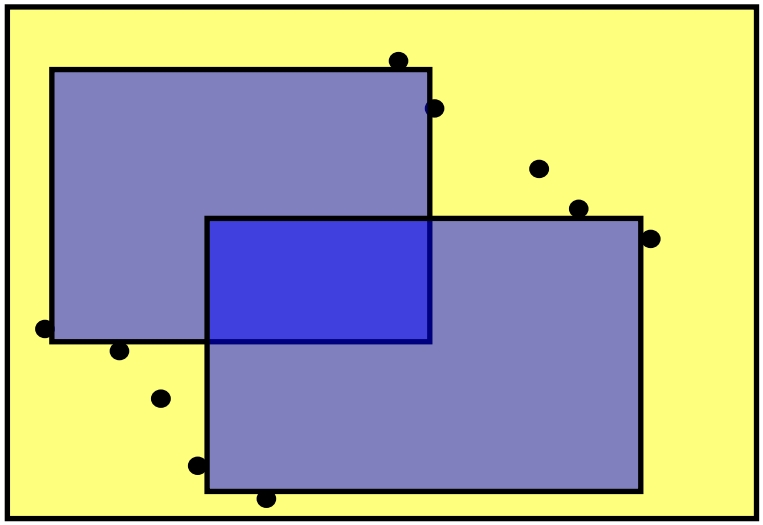
Space required is $O(N \log N)$, preprocessing time $O(N \log^2 N)$ and query time is $O(\log^2 N)$.

How many such rectangles could there be ?



Naïve approach

Caveat: There may be $N = \Theta(n^2)$ maximal empty rectangles

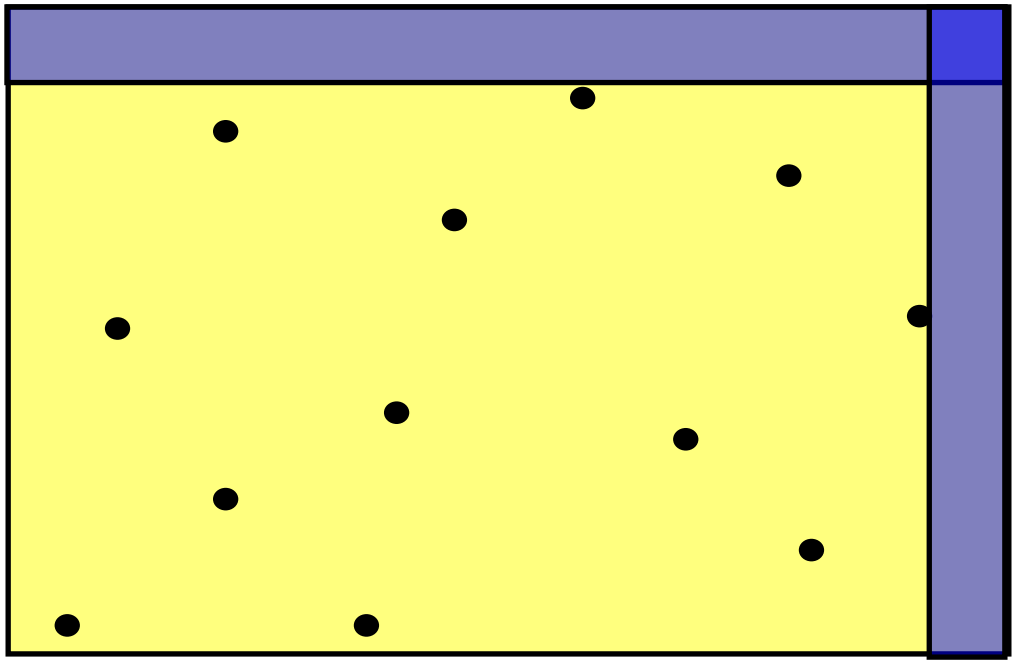


There could be a Quadratic number of rectangles supported by two points in each of the 1st and 3rd quadrants (symmetrically, 2nd and 4th quadrants)

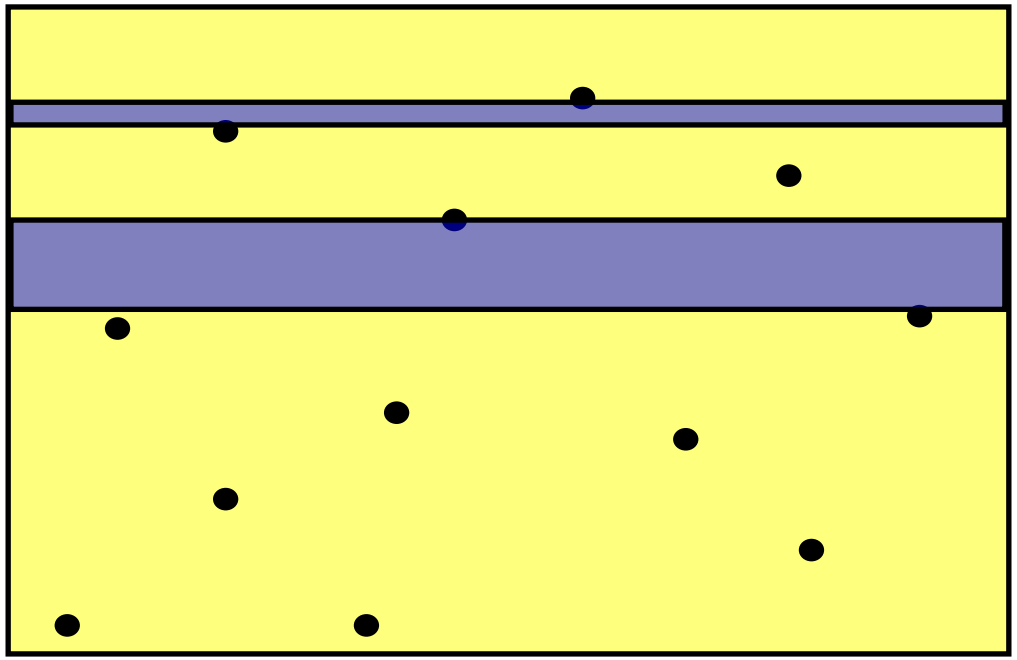
Different kinds of rectangles

- We will classify them, starting with rectangles that have an edge on the boundary

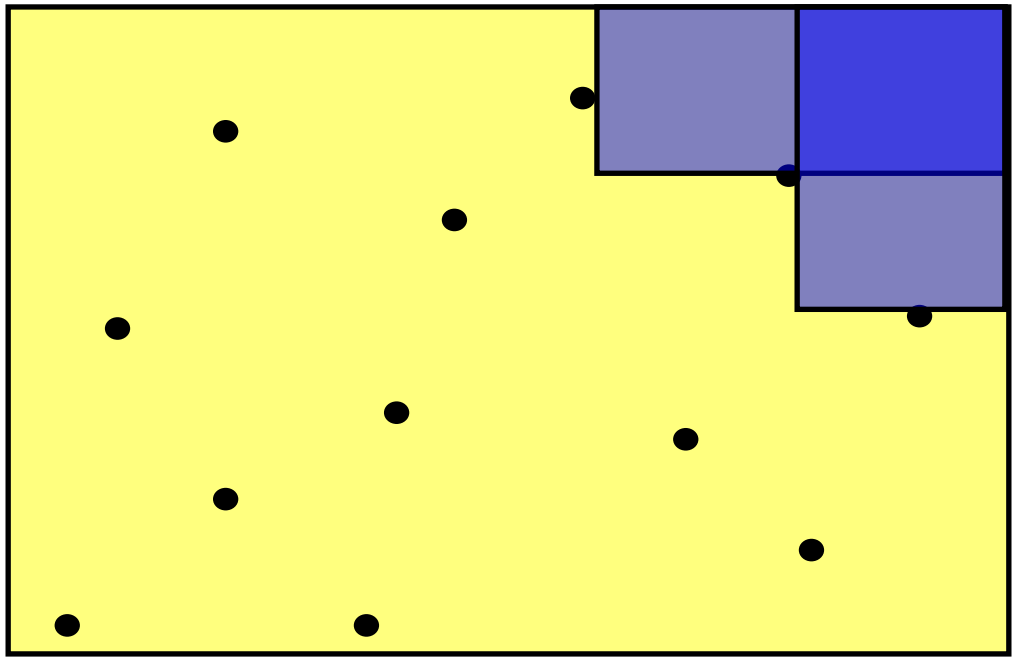
3 Edges on the boundary



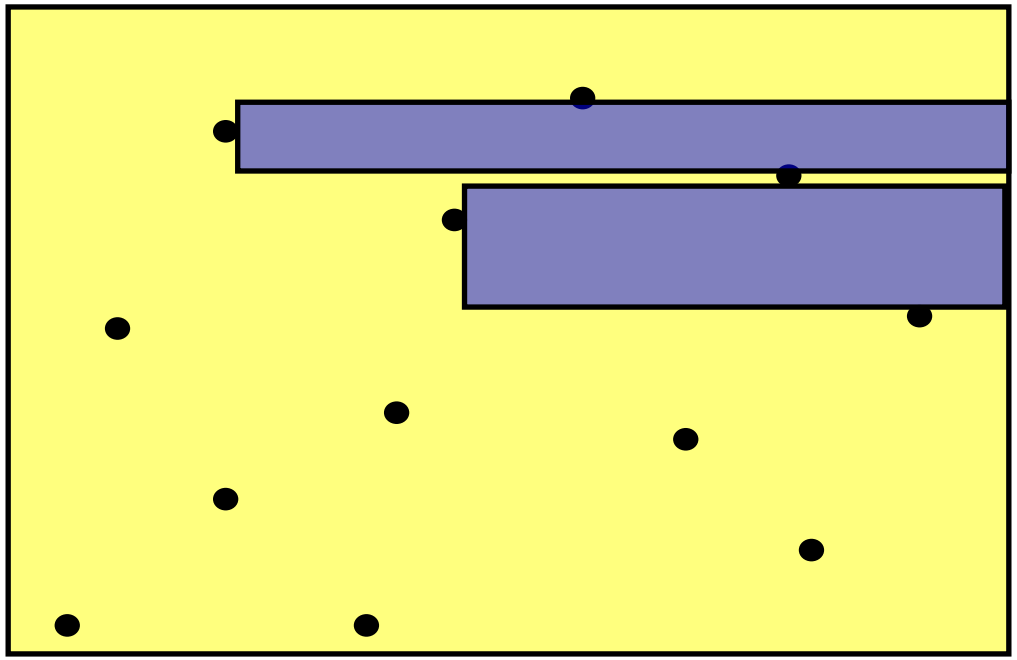
2 edges on the boundary (type 1)



2 Edges on the boundary (type 2)



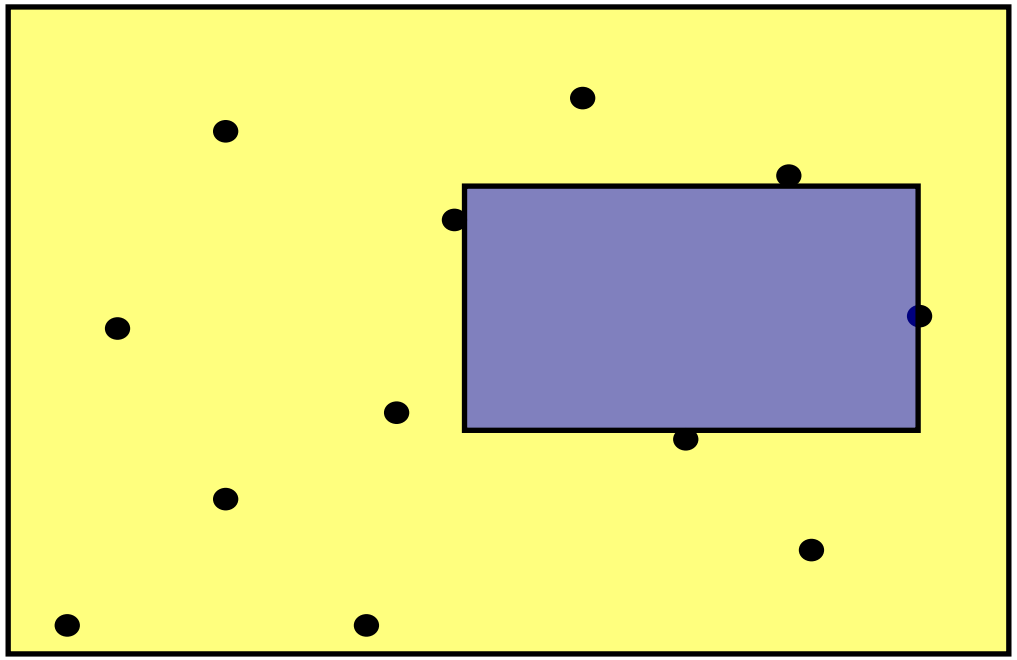
1 Edge on the boundary



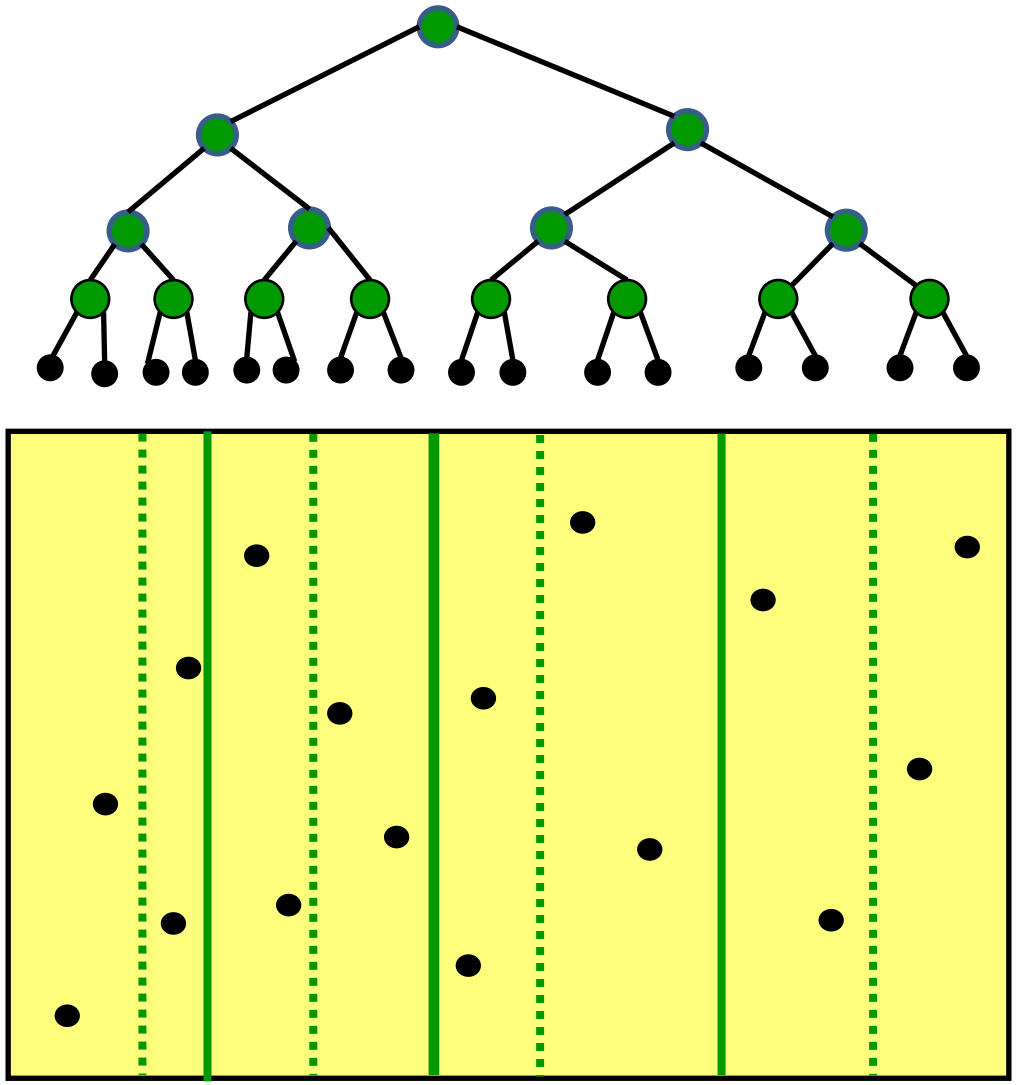
Rectangle with edges on the boundary (summary)

- $O(n)$ rectangles, compute then in $O(n\log(n))$ time and put them in the segment tree in $O(n\log^2(n))$

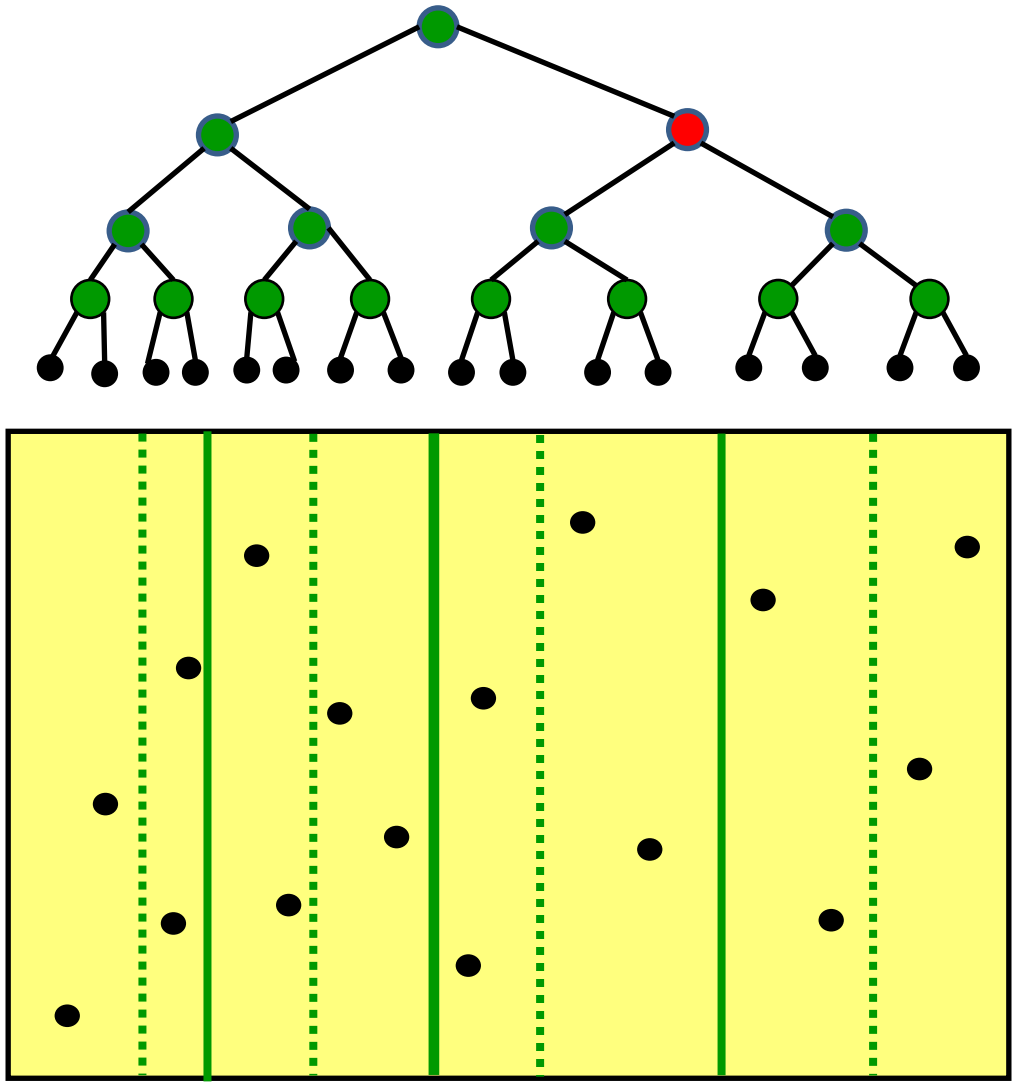
Rectangle without edges on the boundary



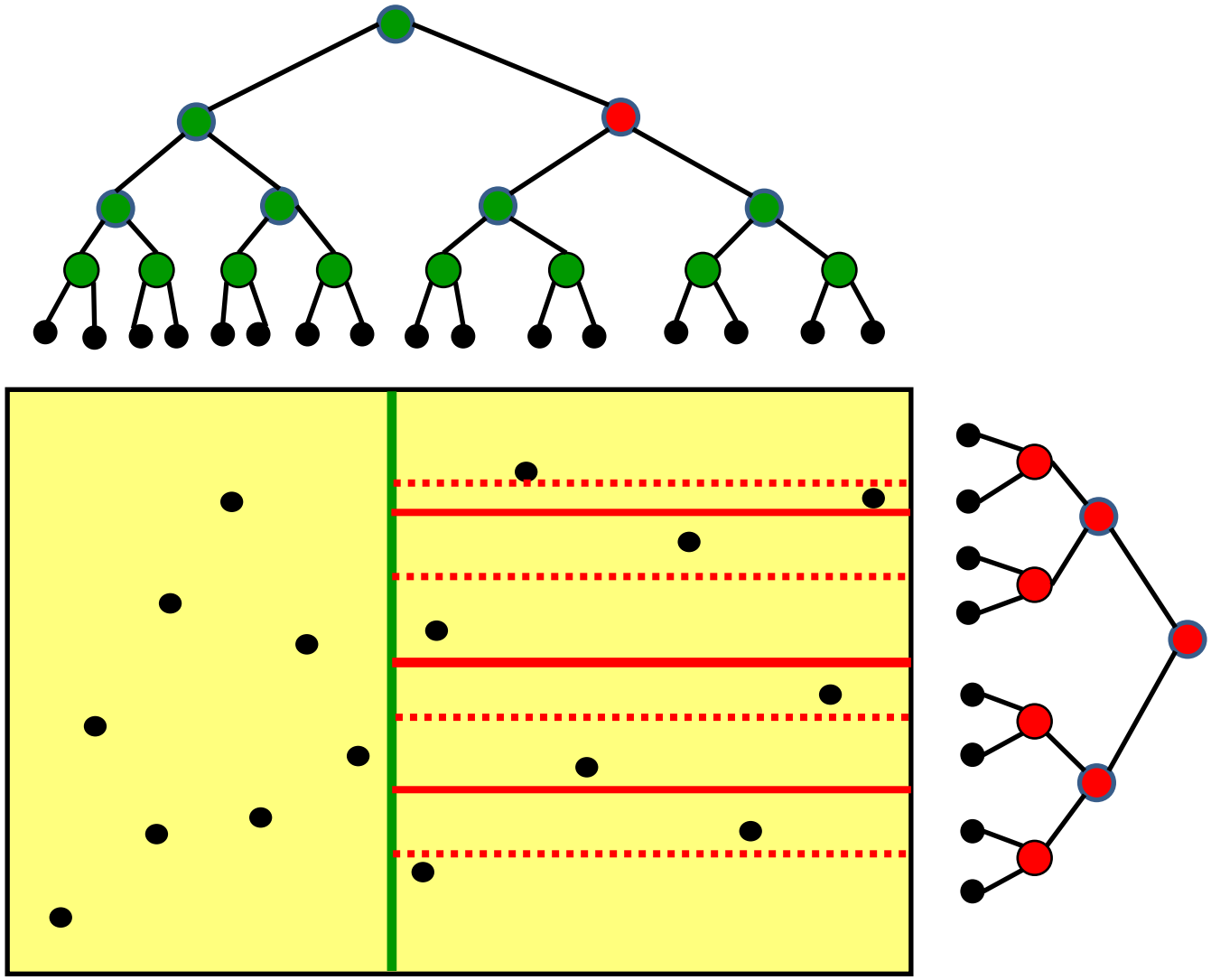
2D Range tree



2D Range tree

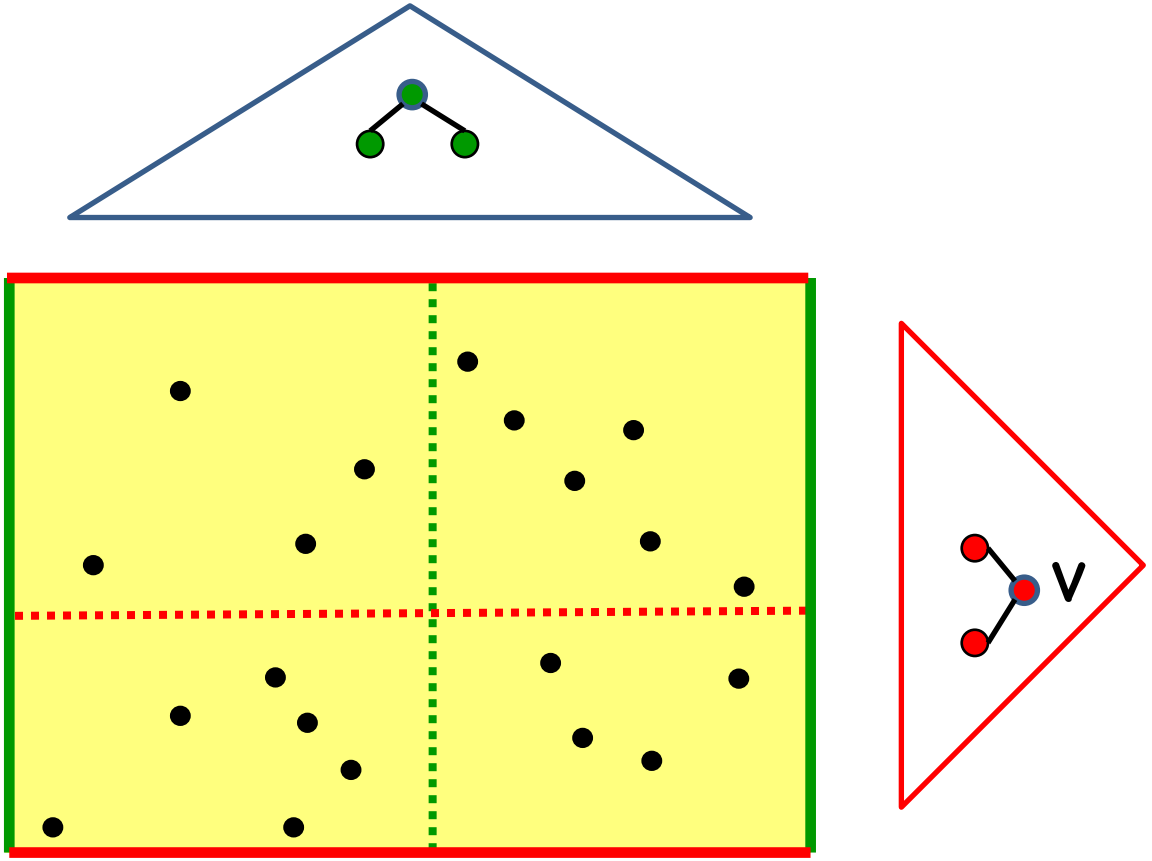


2D Range tree



An internal secondary node in the range tree

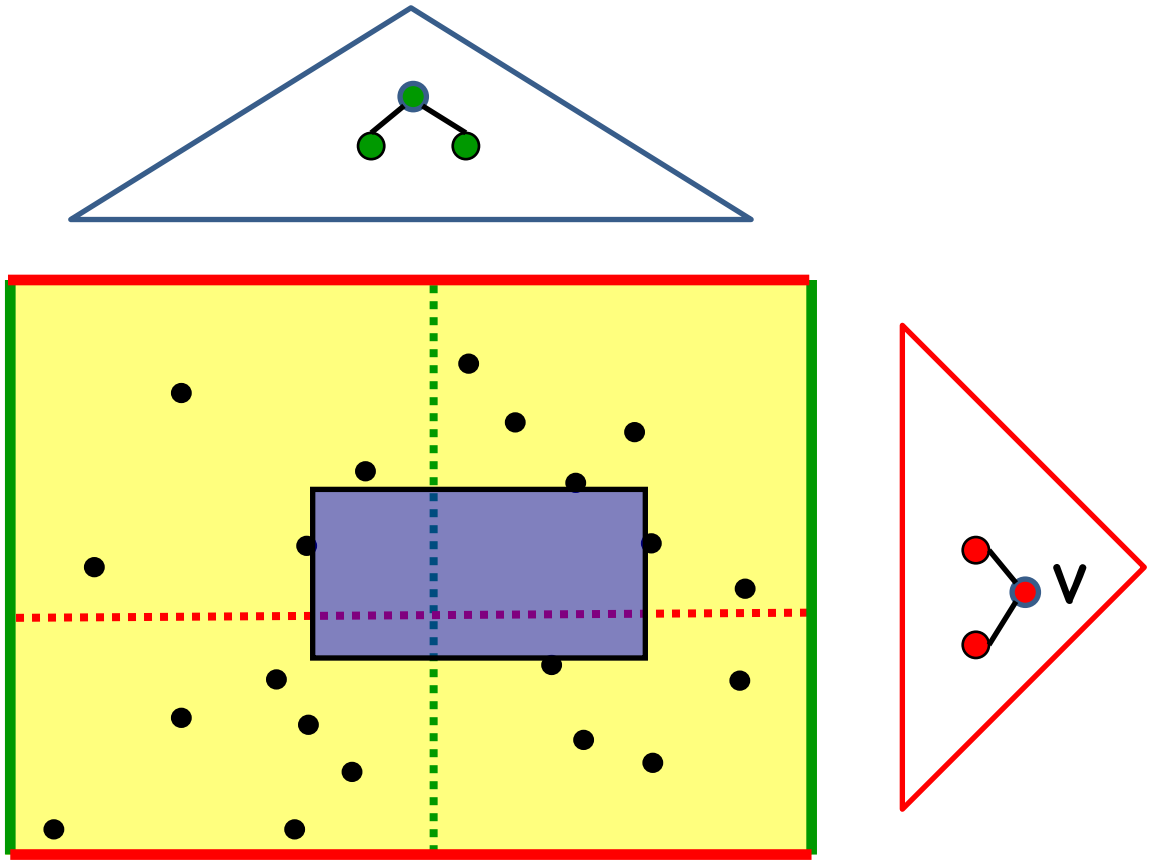
Corresponds to some rectangle with two dividers



A maximal empty rectangle

Contains the "origin" in some internal node v

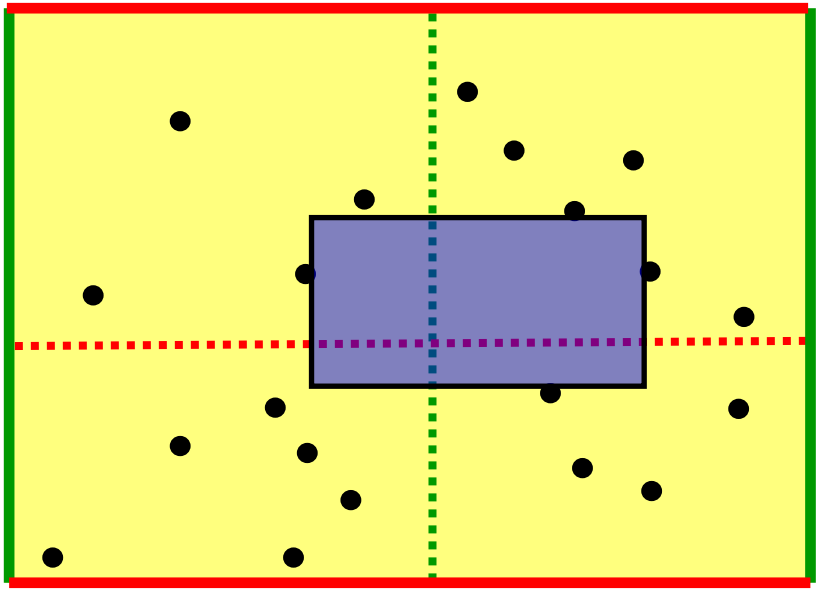
Each node is responsible for the maximal empty rectangles containing its "origin"

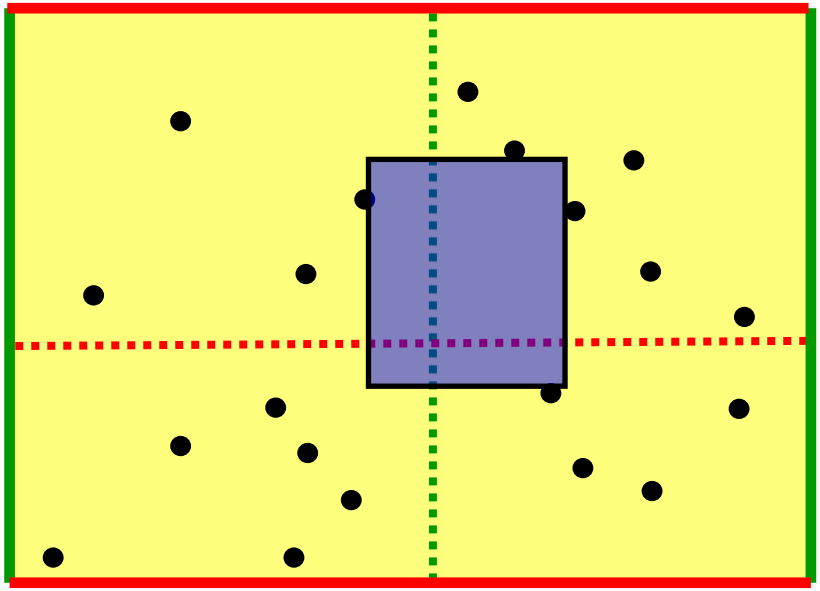


Classify maximal emty rectangles containing the origin

We are now focusing on one secondary node v of the range tree, n_v is the size of the subtree of v

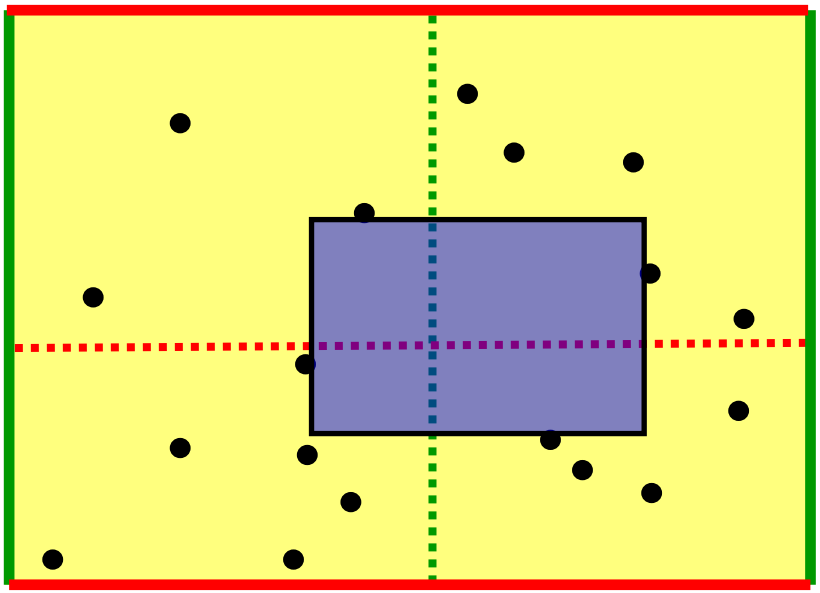
3 points on one side of a divider

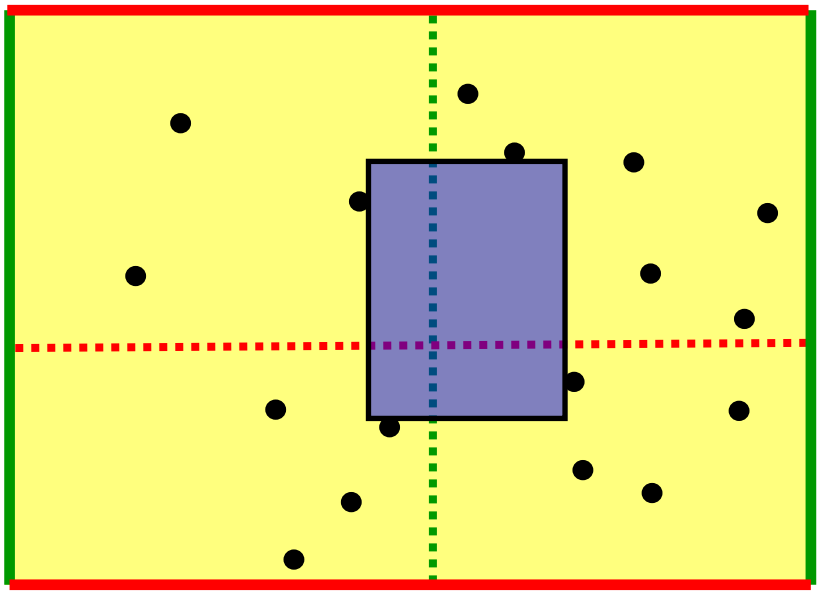




$O(n_v)$ of these, can compute then in $O(n_v \log n_v)$ time

1 point in each quadrant



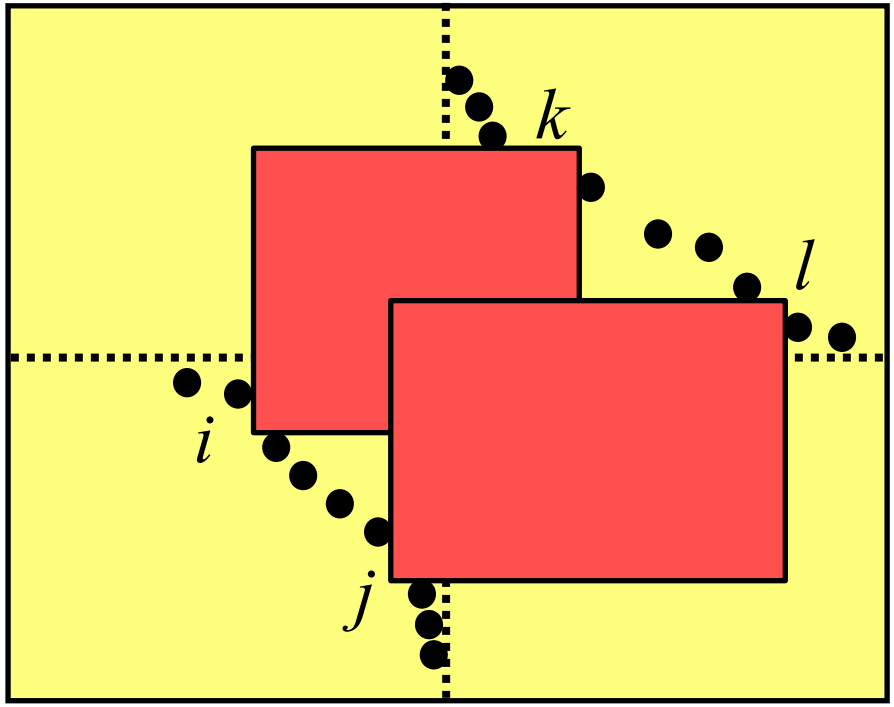


Linearly many $O(n_v)$, can compute then in $O(n_v \log n_v)$ time

Summary so far

- Summing up over the nodes of the range tree we have $O(n\log^2(n))$ rectangles that we compute in $O(n\log^3(n))$ time
- We insert them into the segment tree in $O(n\log^4(n))$ time
- The segment tree takes $O(n\log^3 n)$ space and can answer a query in $O(\log^2 n)$ time

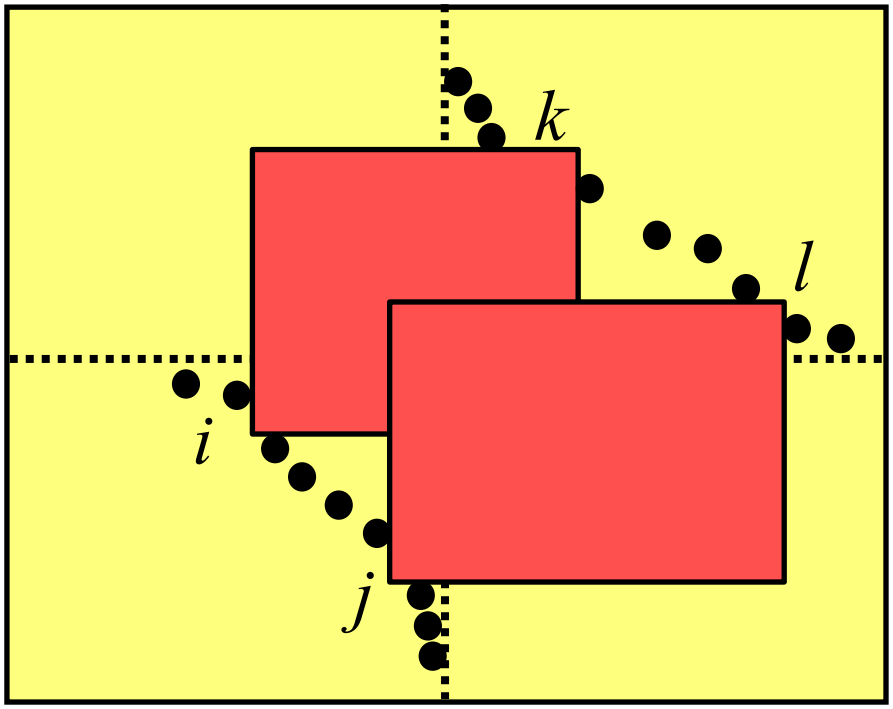
2 Points in each quadrant

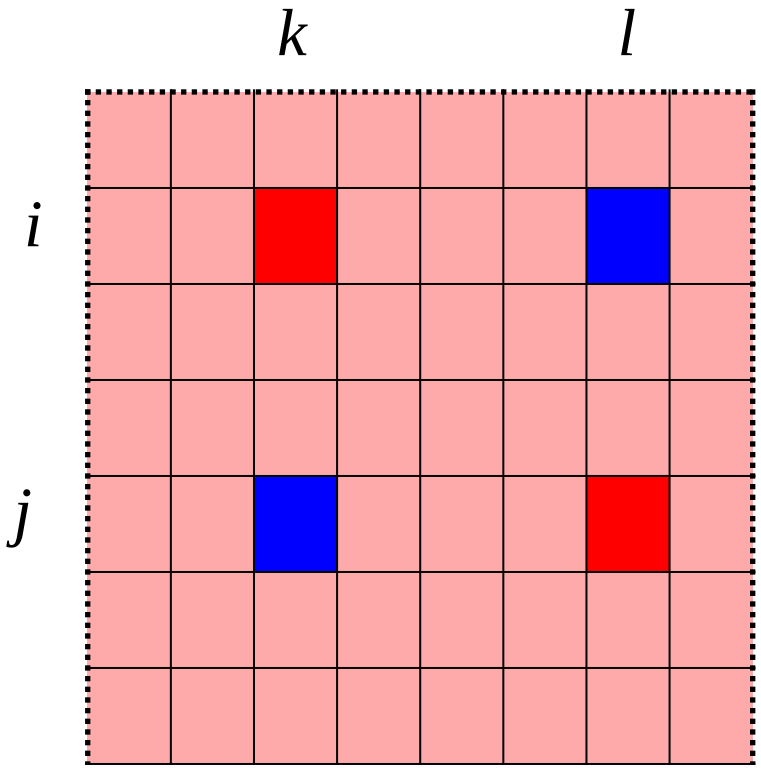


We can compute the chains in $O(n_v \log(n_v))$ time, but there are too many rectangles to put in the segment tree

The (inverse) Monge property

Define a matrix **M** such that M_{ik} is the area of maximal rectangle supported by i^{th} pair in 3^{rd} quadrant and k^{th} pair in 1^{st} quadrant





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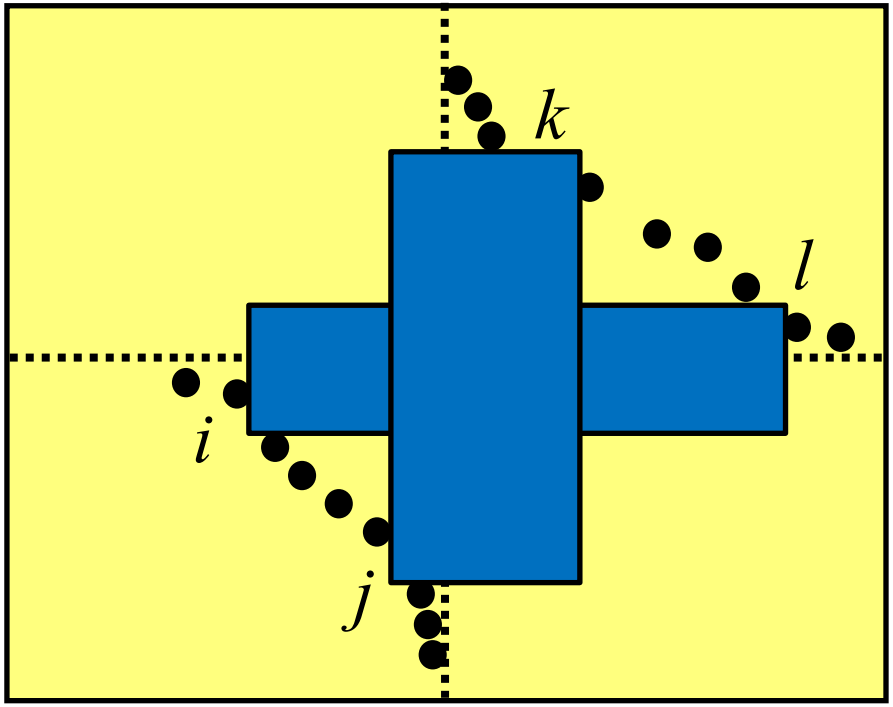
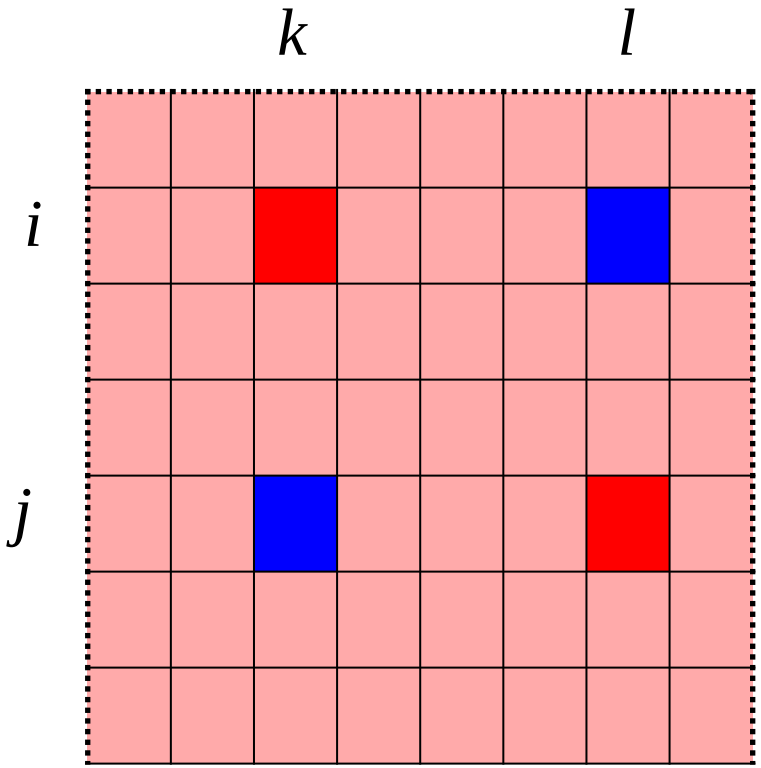
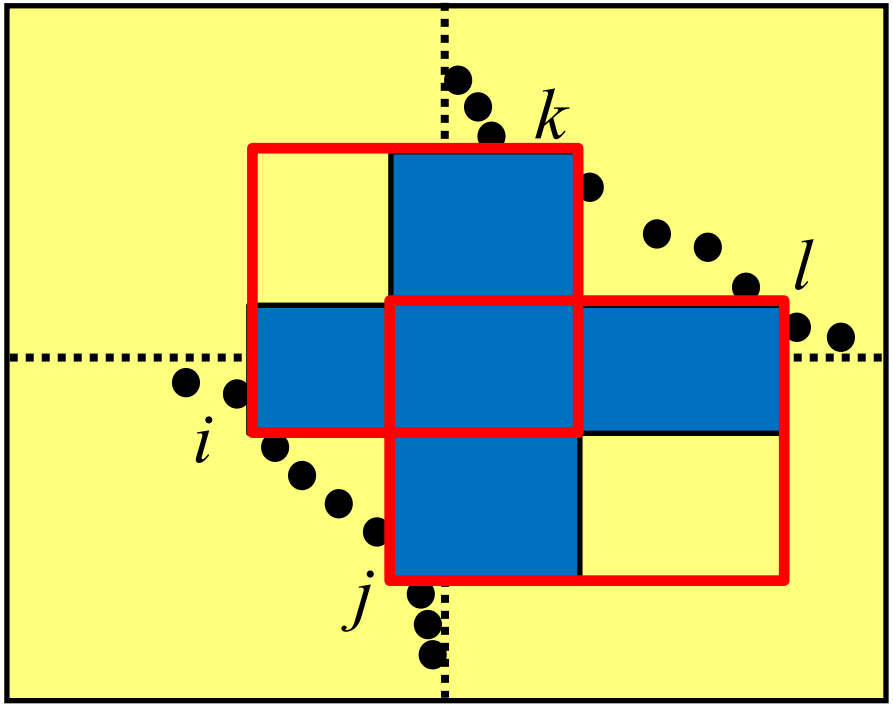
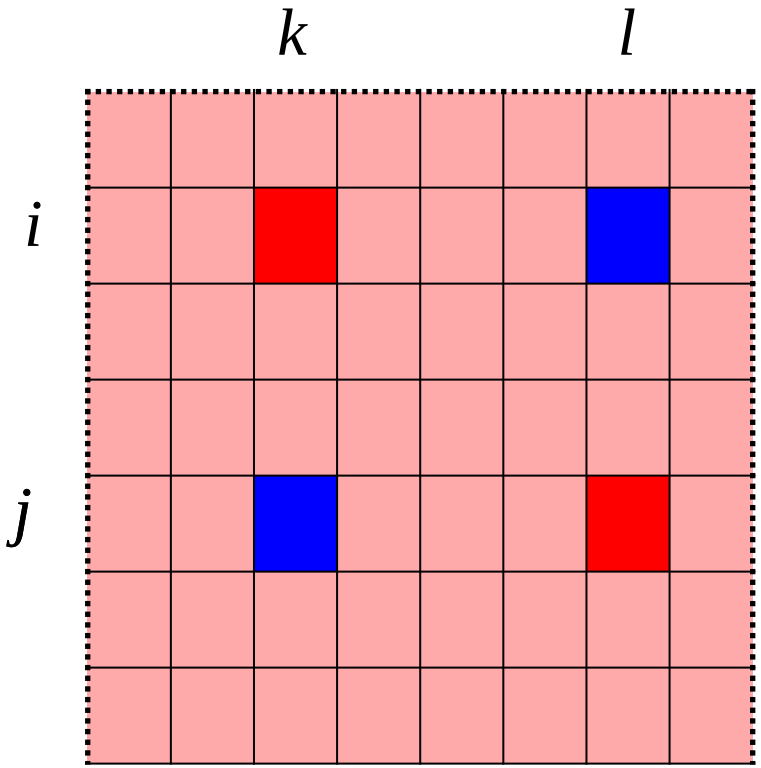




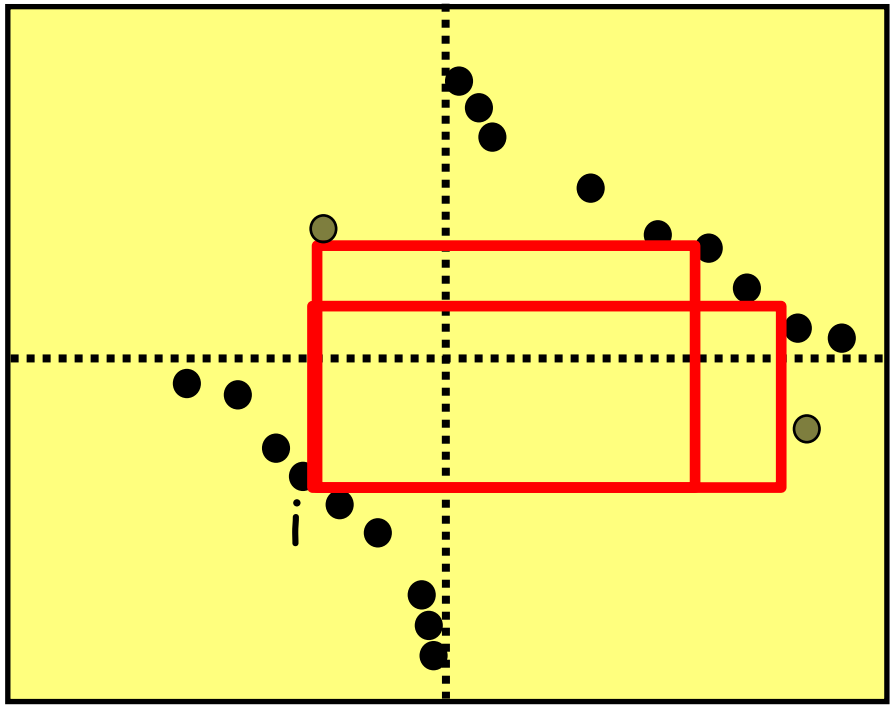
The (inverse) Monge property

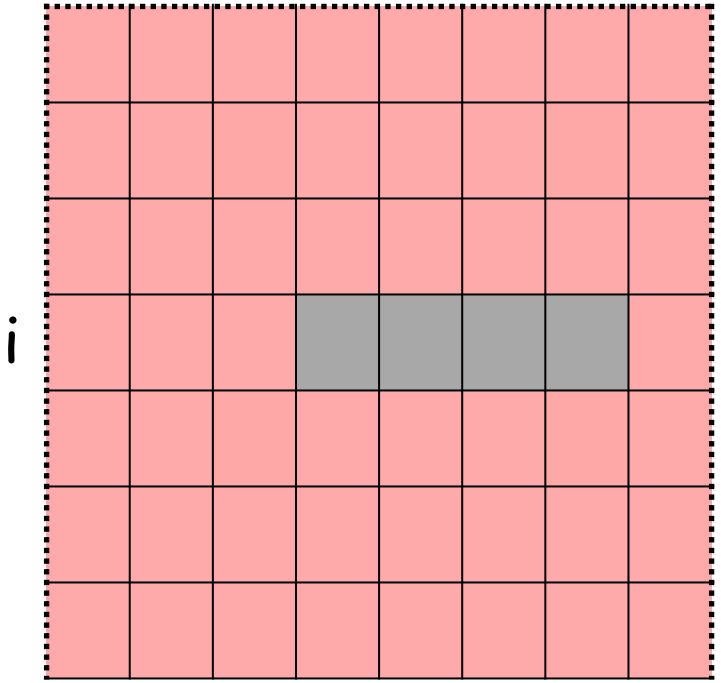
For every $i<j$, $k<l$, $M_{ik} + M_{jl} \geq M_{il} + M_{jk}$



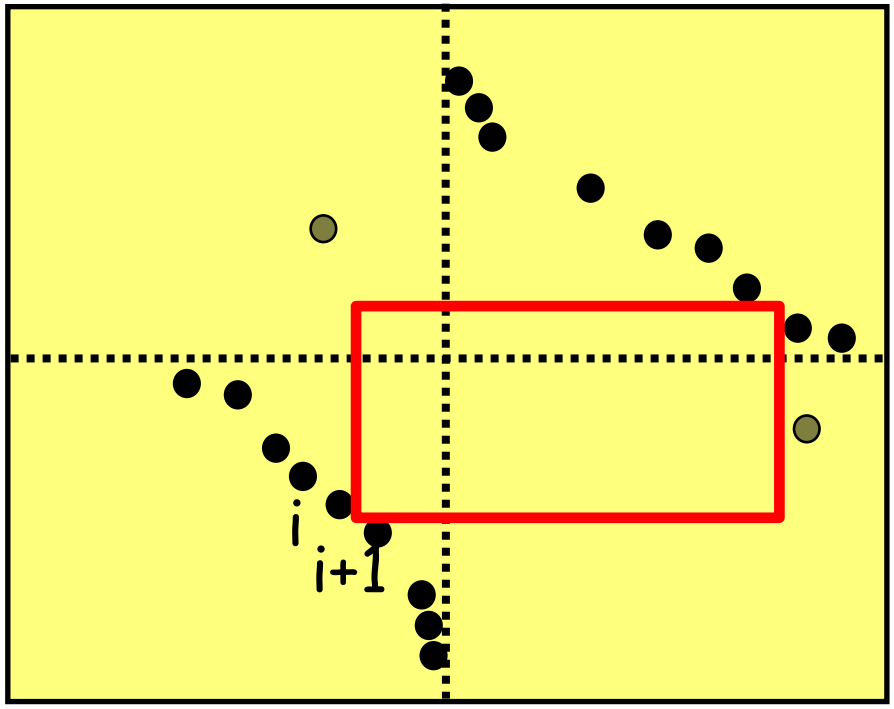


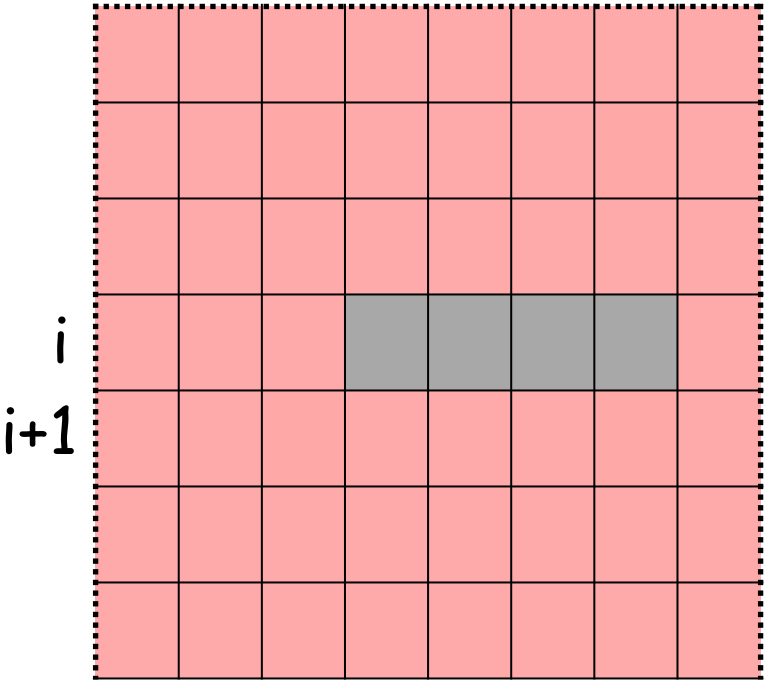
The matrix is not fully defined



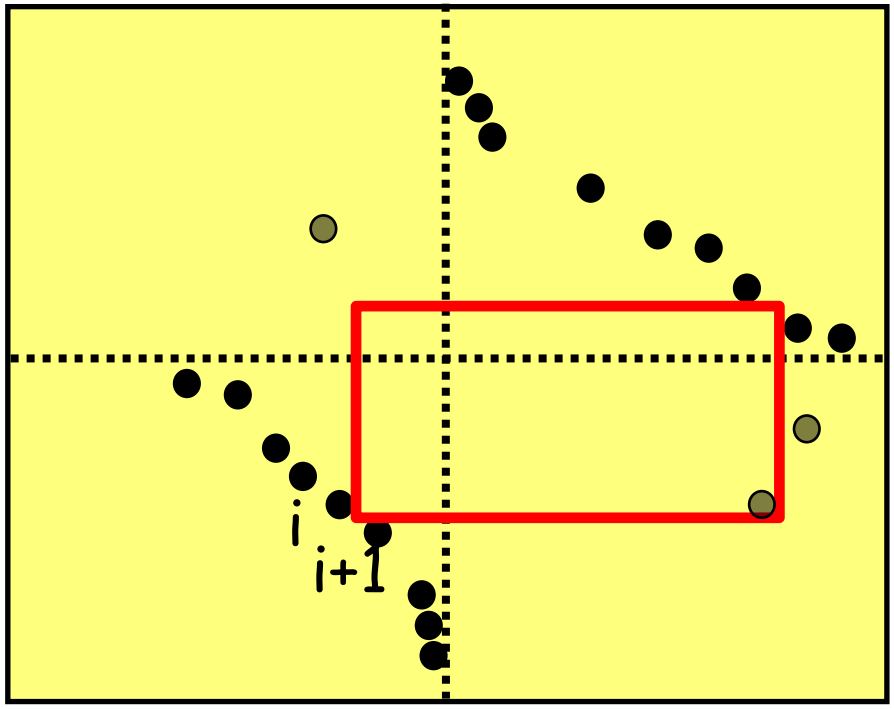


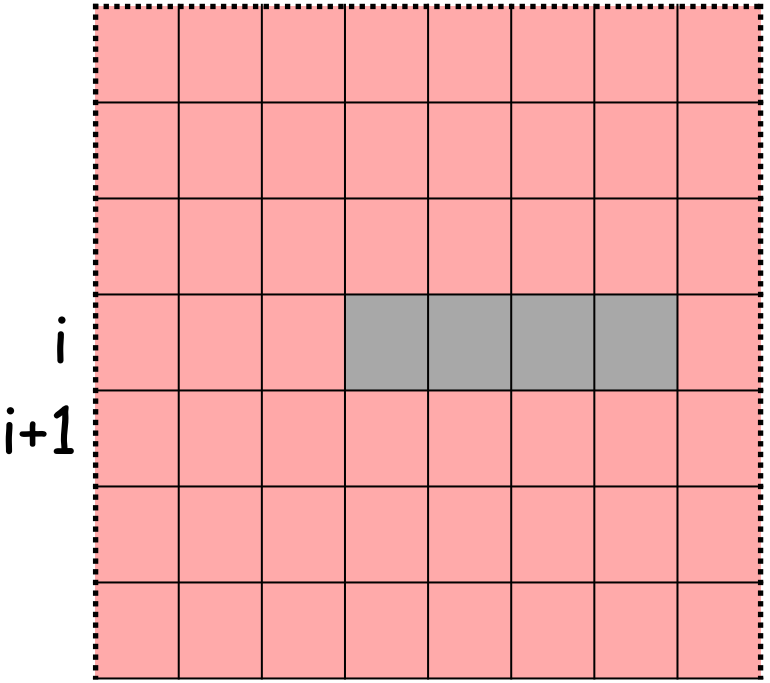
The relation between the defined parts of consecutive rows





The relation between the defined parts of consecutive rows





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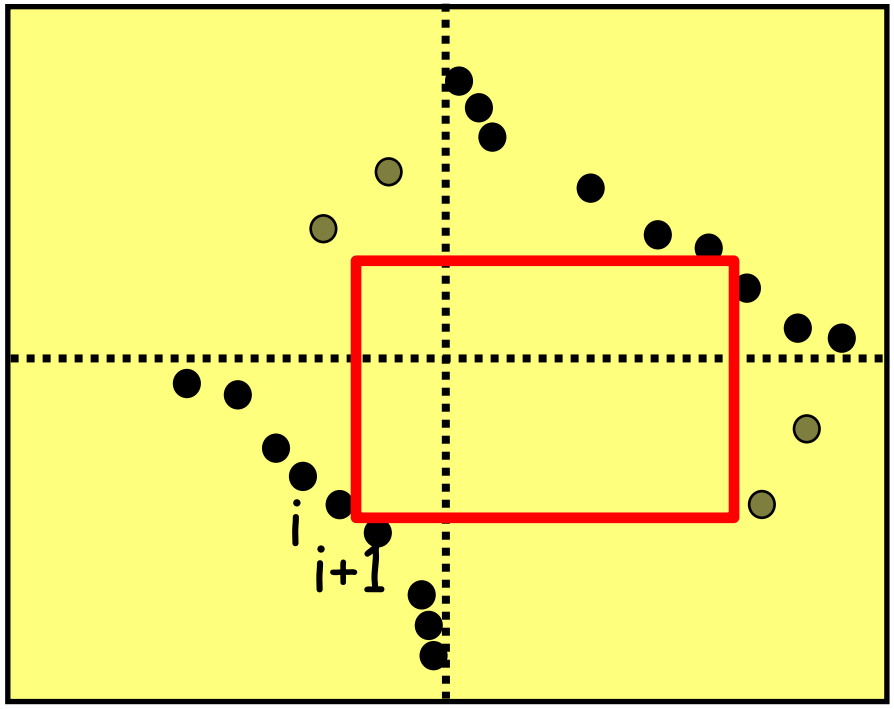
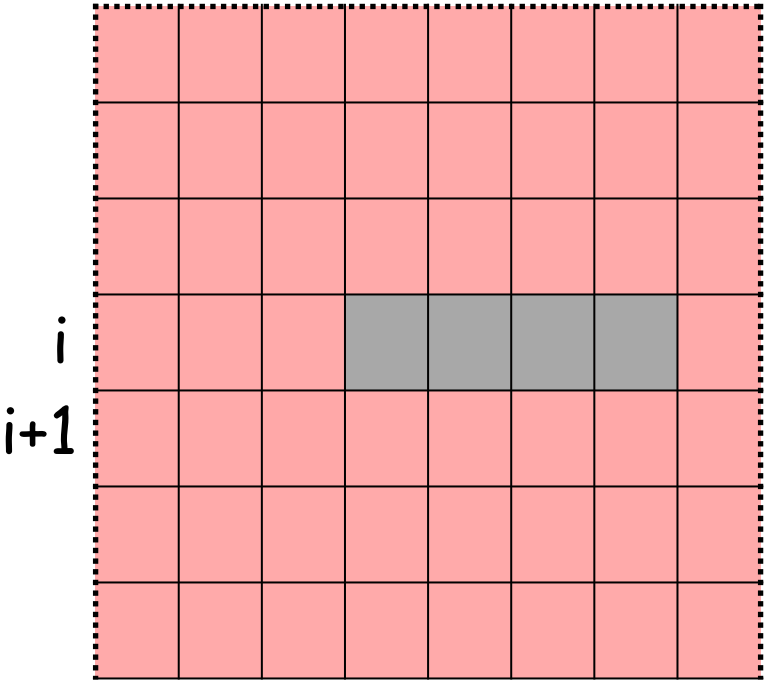
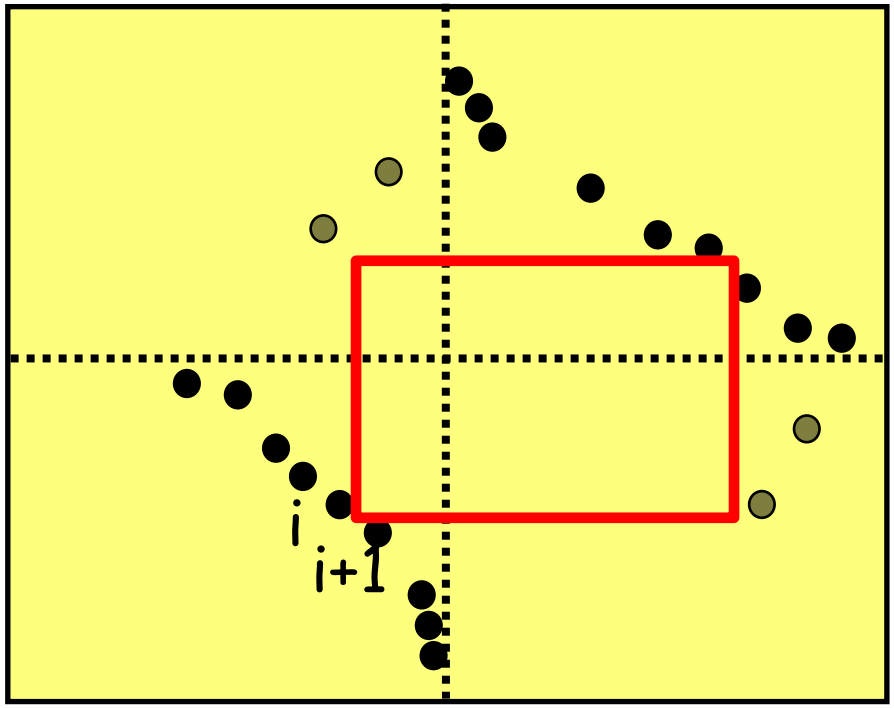
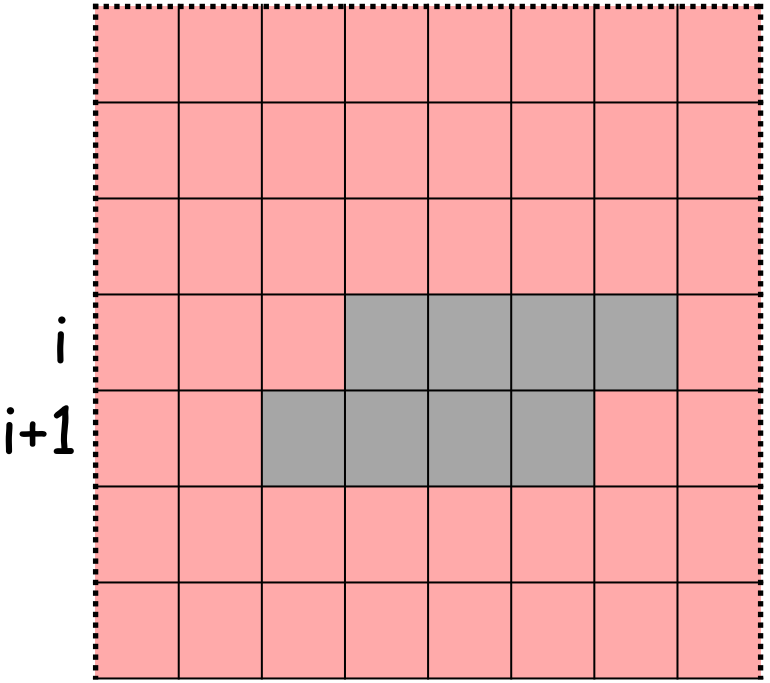


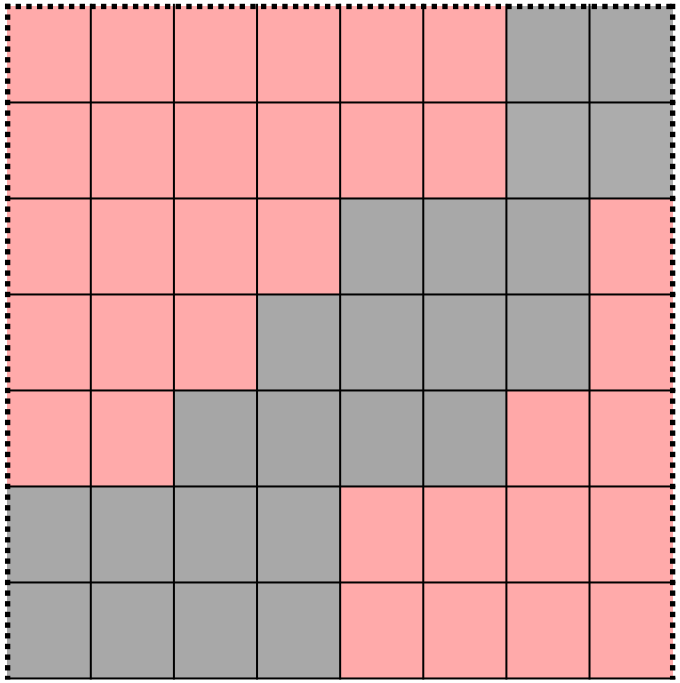


The matrix is "staircase"





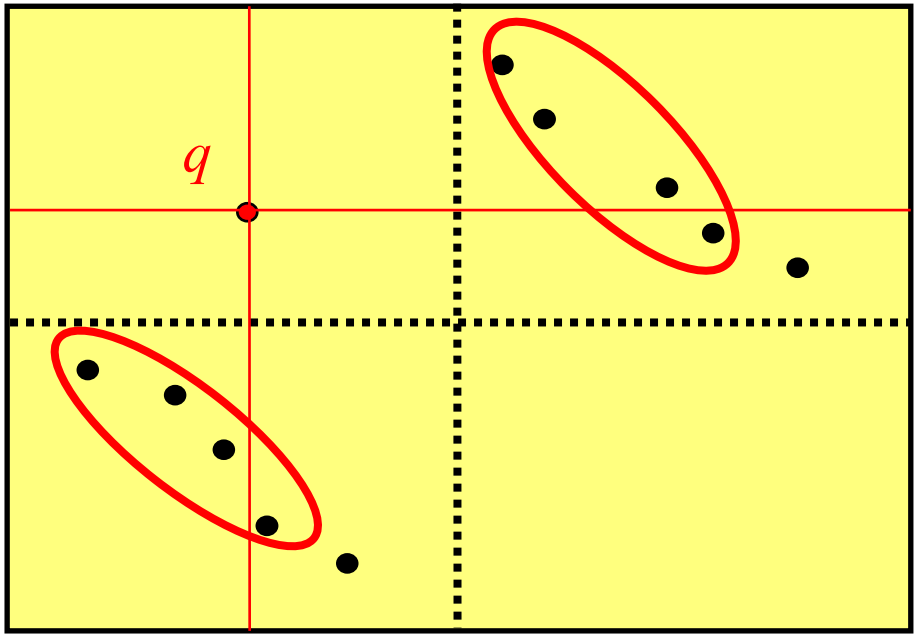
The matrix is "staircase"

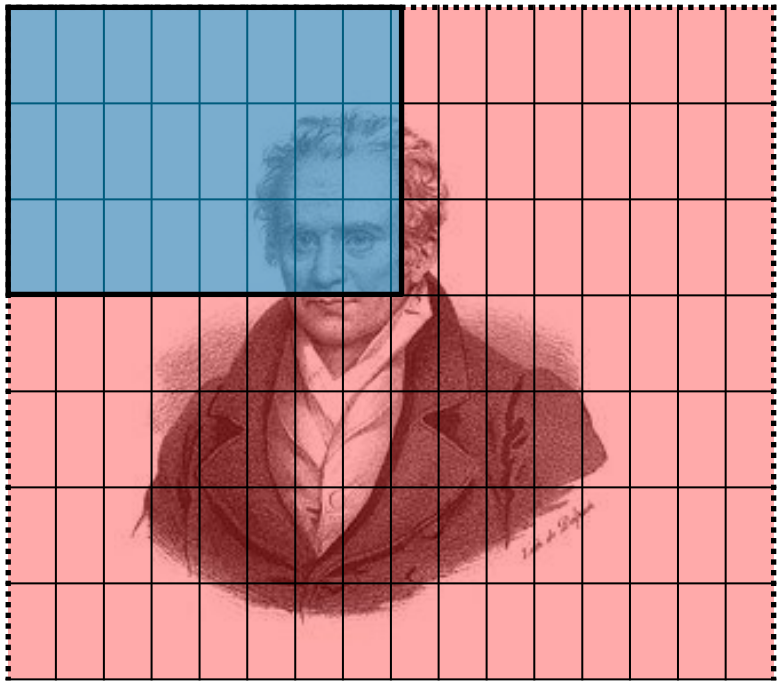


Query

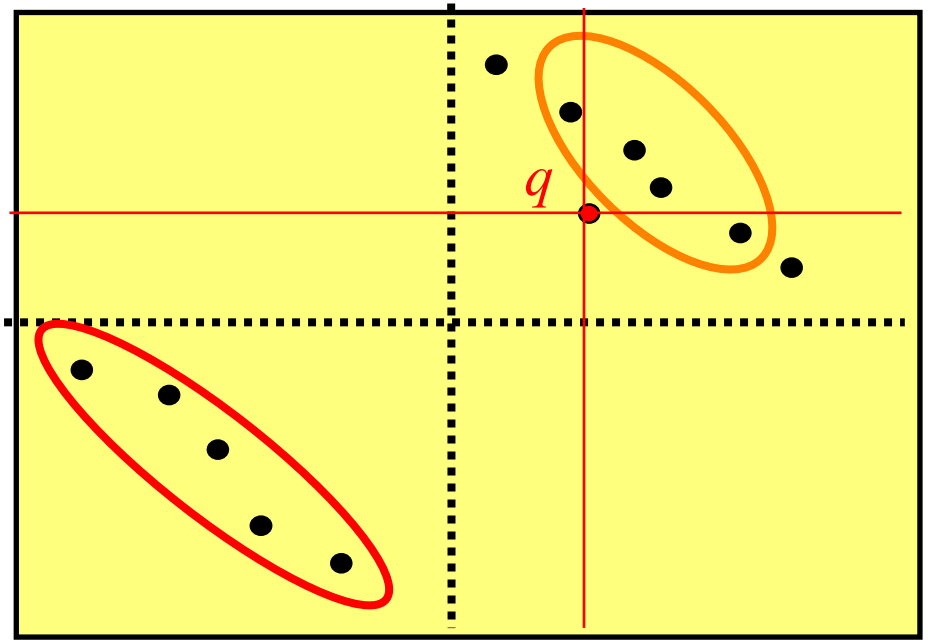
A query point q defines ranges of relevant pairs in the 1st and 3rd quadrants

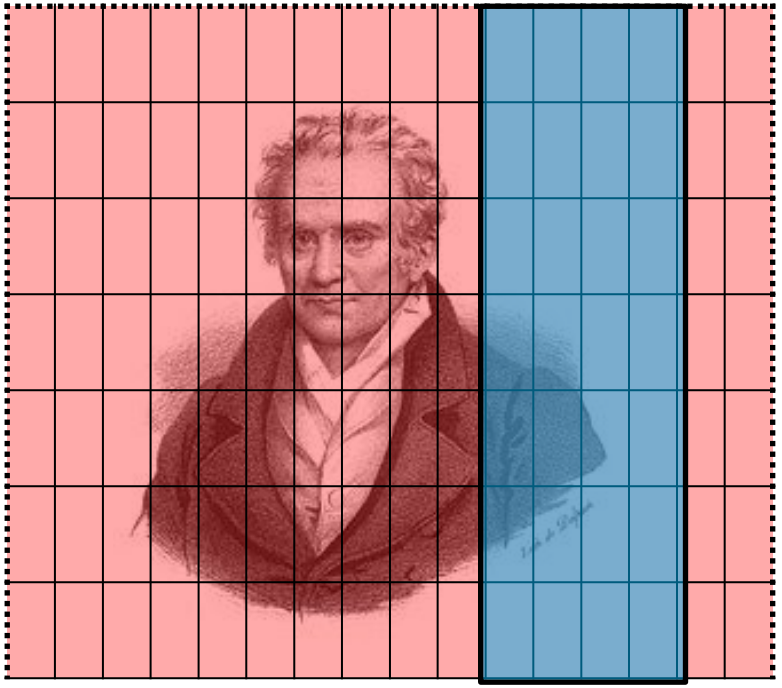
Use the Monge submtarix maximum data structure to find the maximum-area rectangle containing q in the relevant range





Query





Analysis

It takes $O(n_v a(n_v)\log(n_v))$ space $O(n_v a(n_v)\log^2 n_v)$ preprocessing time per secondary node v

Summing over all secondary nodes of the range tree we get that the total space is $O(na(n)\log^3(n))$ and the preprocessing time is $O(na(n)\log^4(n))$

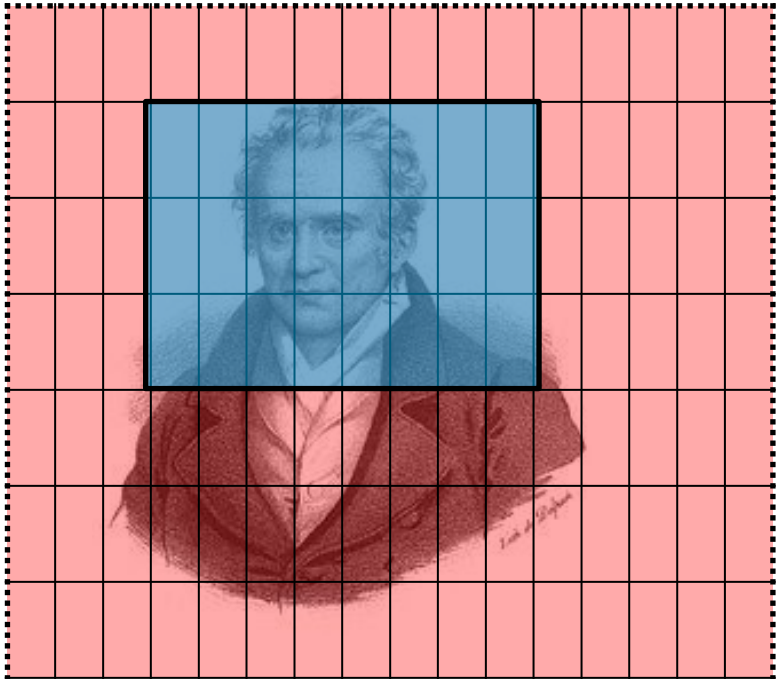
Query time is $O(\log^2 n_v)$ at a secondary node v and $O(\log^4 n)$ overall

Thats it as far as the empty rectangle application is concerned

Back to our data structure

Input: n x n Monge matrix M represented implicitly (constant time oracle access)

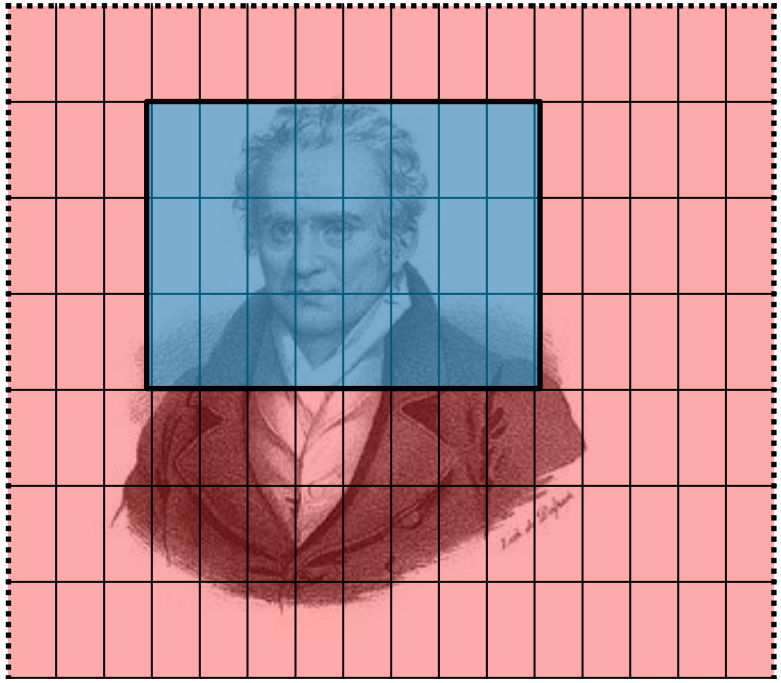
Output: Data structure answering maximum queries in any rectangular submatrix of M



Submatrix maximum queries in Monge matrices

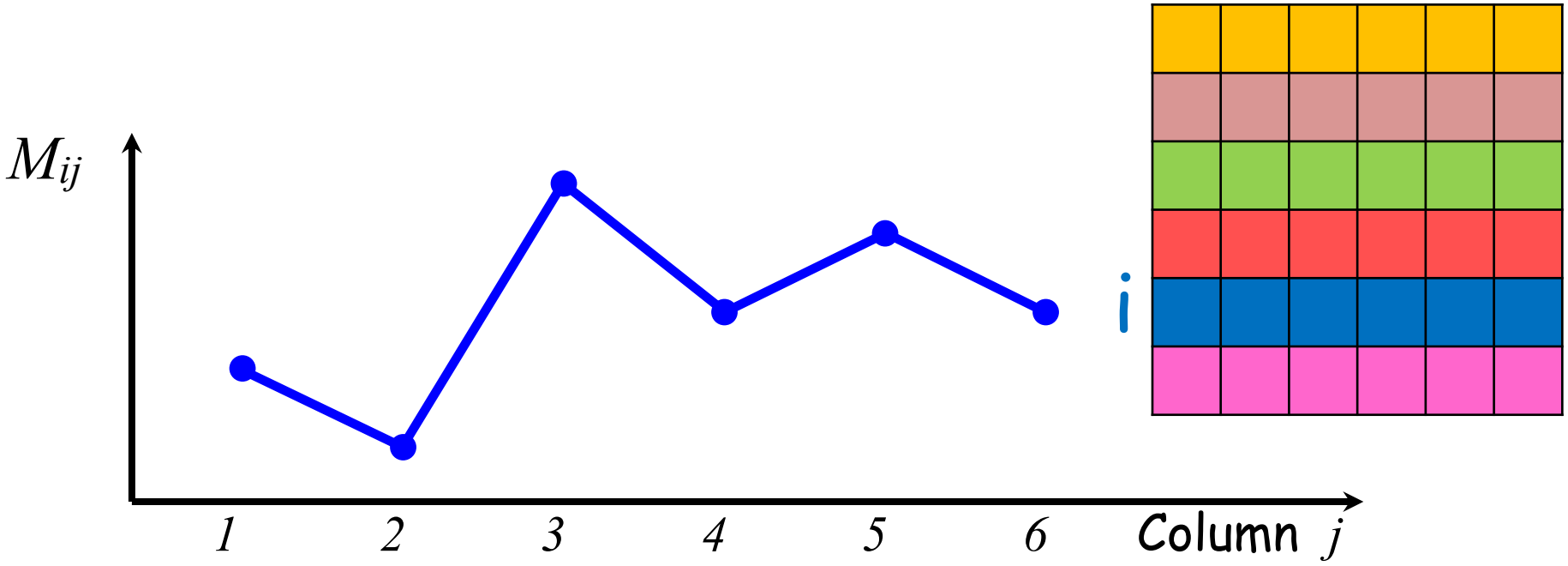
Main Theorem:
A data structure with the following performance:

preprocessing time	$O(n \log^2 n)$
space	$O(n \log n)$
query time	$O(\log^2 n)$



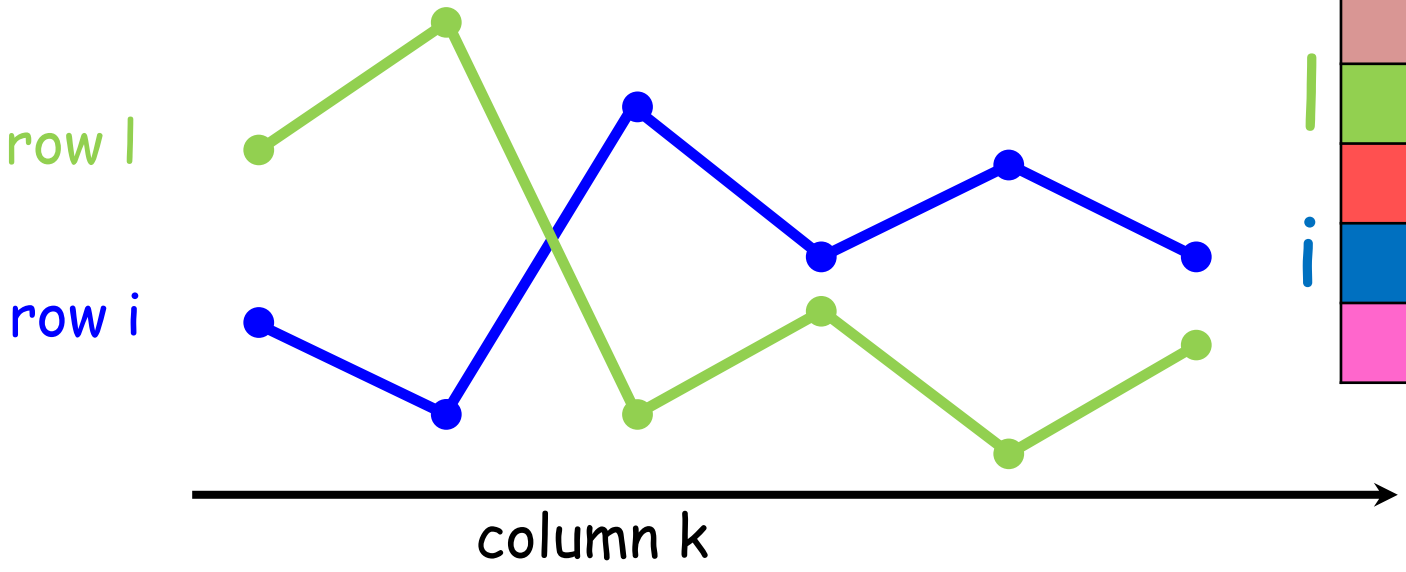
Totally monotone matrices and pseudo-lines

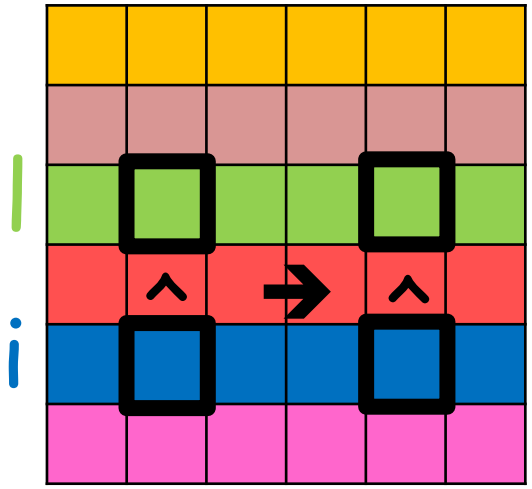
Think of row i of M as a discrete function mapping column j to M_{ij}. Extend the domain to the interval [1,n] by linear interpolation



Totally monotone matrices and pseudo-lines

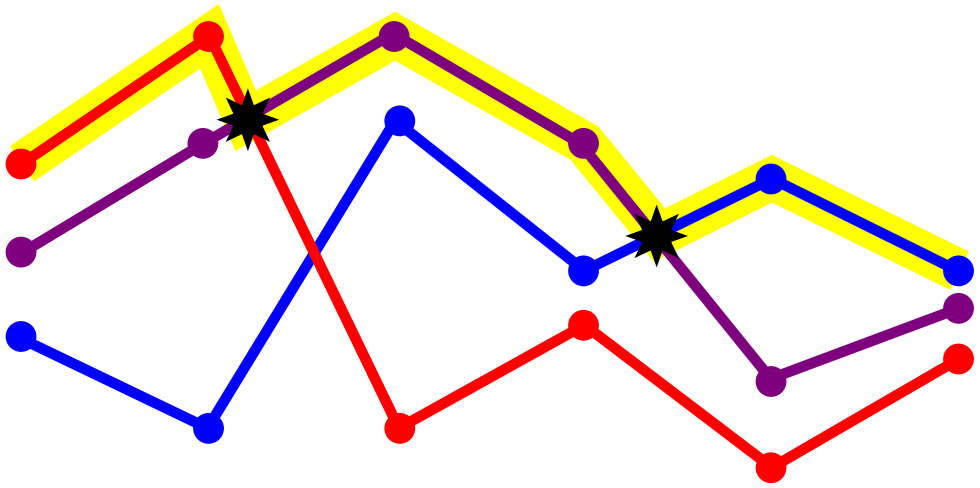
These are pseudo-lines:
Every pair of curves intersect in at most one point



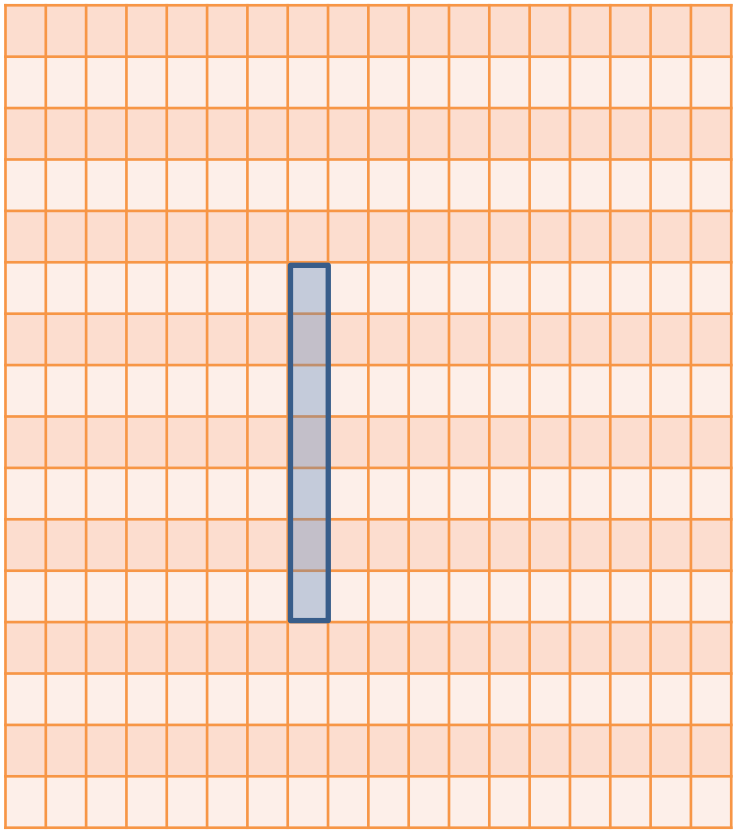


Upper envelopes of pseudo-lines

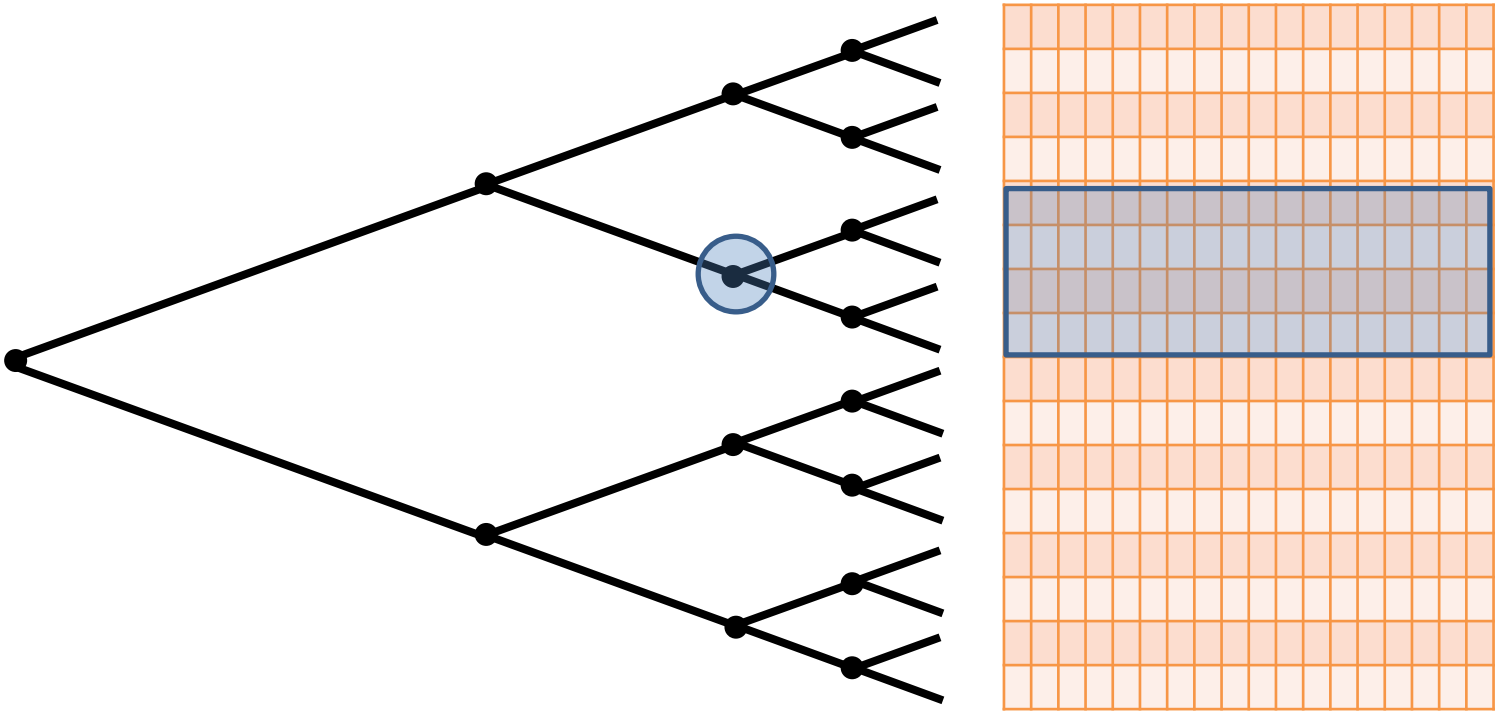
- Column maxima are on the upper envelope of the pseudo-lines
- For m rows there are O(m) breakpoints
- Represent an upper envelope by a search tree over the breakpoints



Column maxima in a range of rows

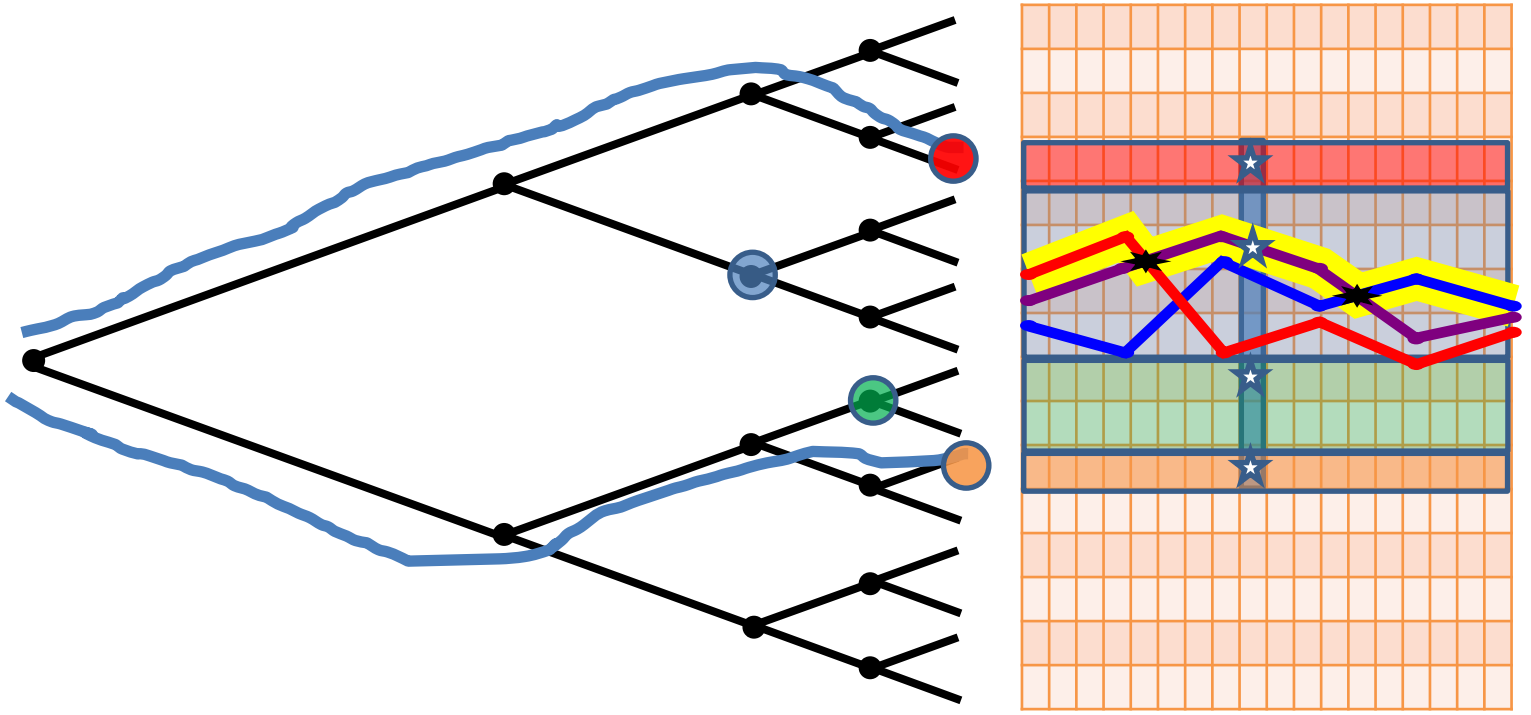


Column maxima in a range of rows



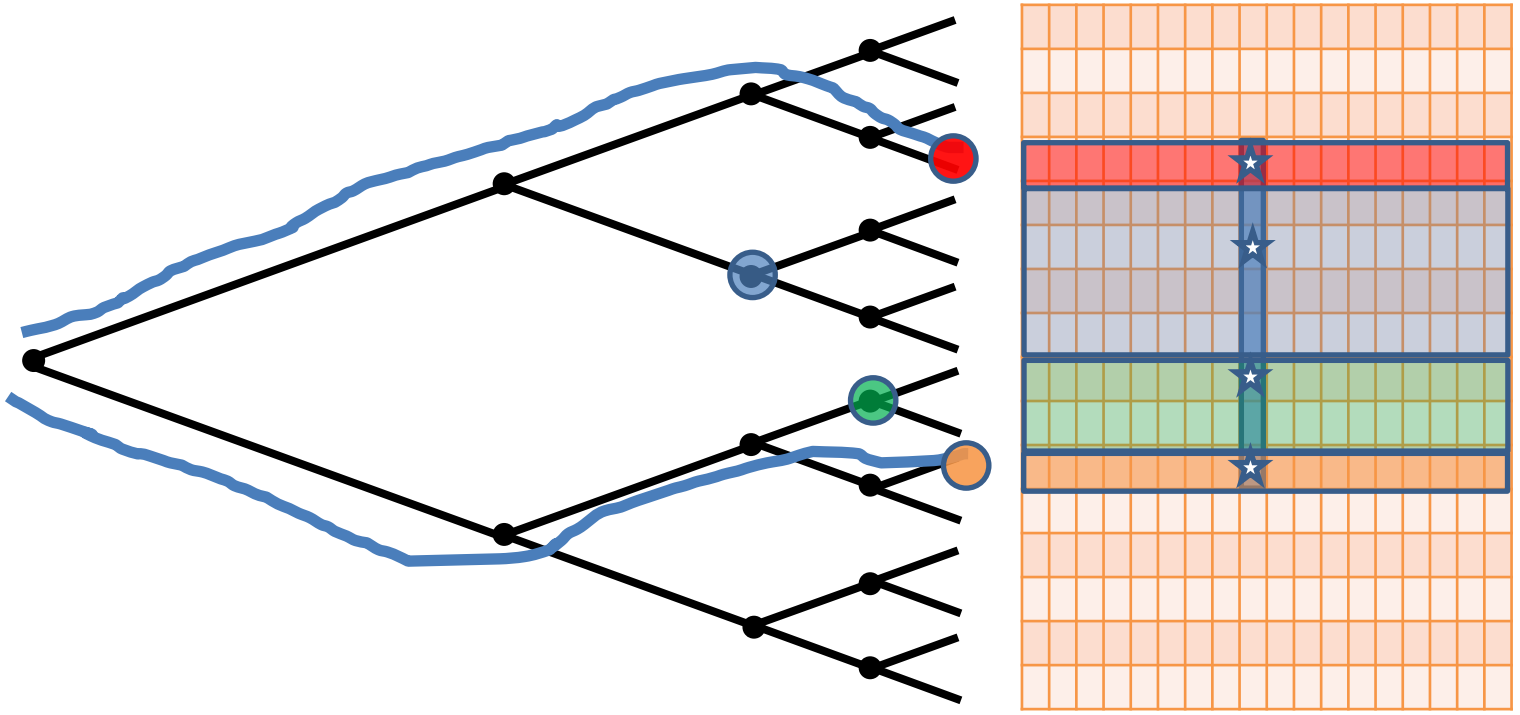
- Balanced search tree over the rows
- Each node stores the upper envelope of its rows
- Takes $O(n \log n)$ time in a bottom-up computation

Column maxima in a range of rows: a query



- A query is "covered" by O(log n) canonical nodes
- Search the envelope of each canonical node for the interval containing the column
- Return max of entries of the appropriate rows

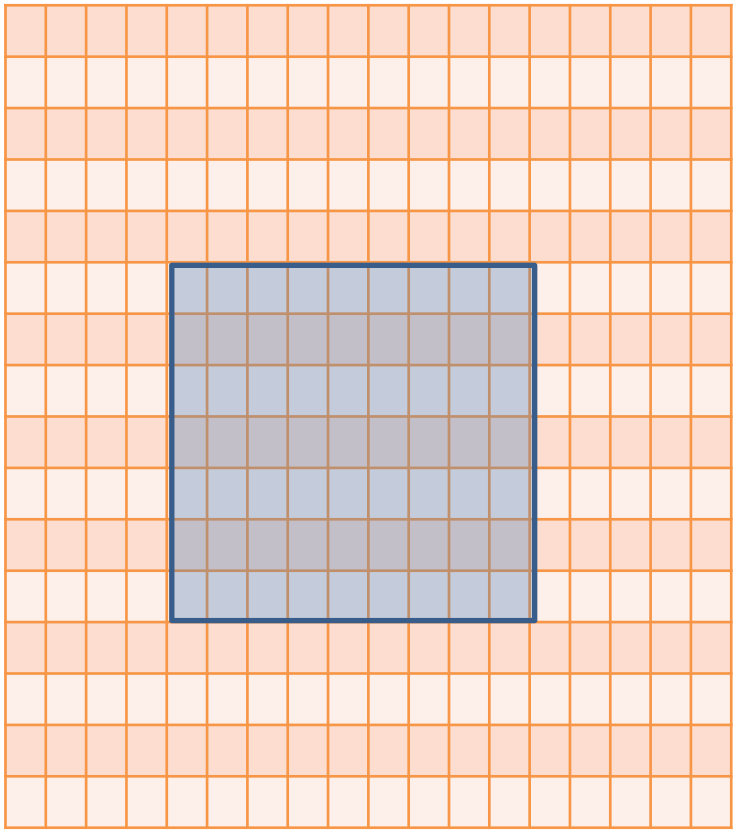
Column maxima in a range of rows: a query



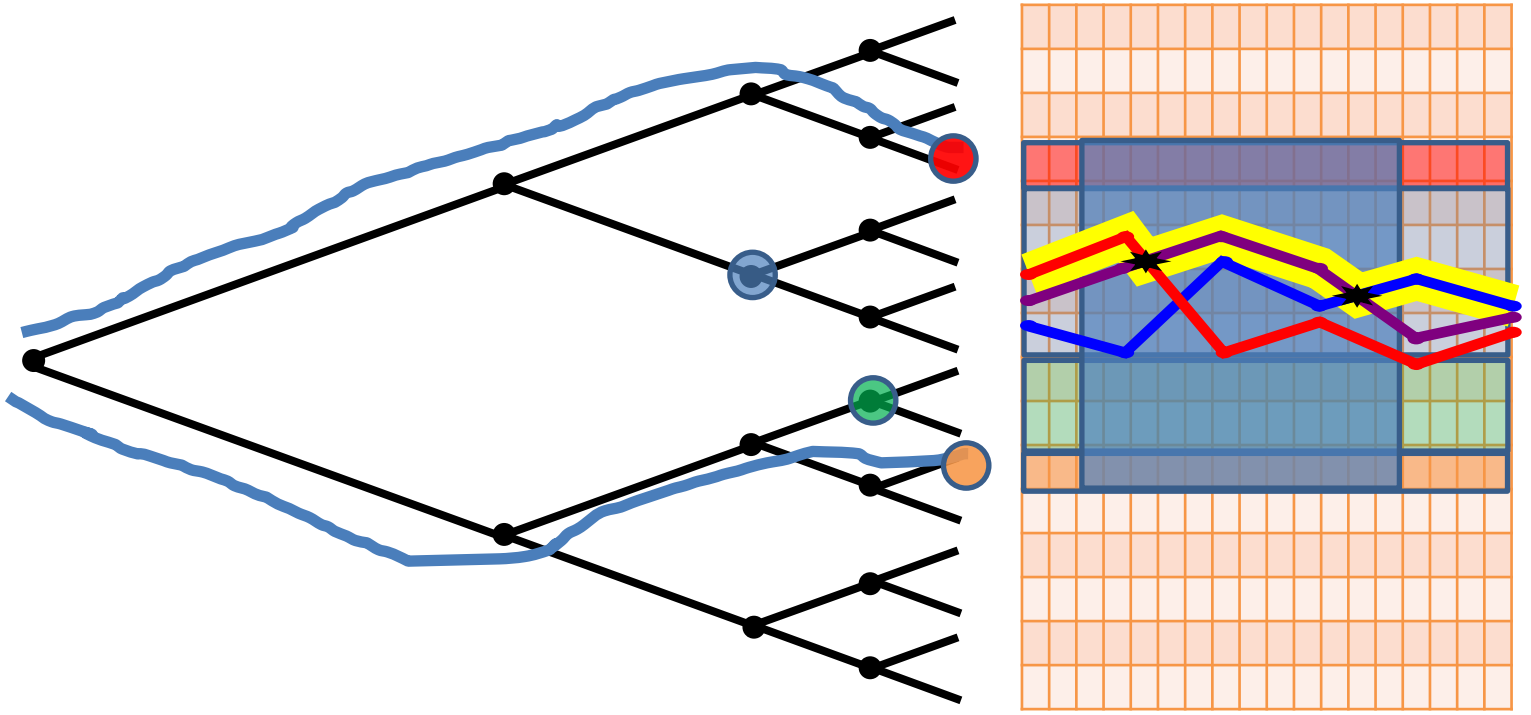
A query can be answered in $O(\log^2 n)$ time by querying each of the canonical nodes in $O(\log n)$ time.

Using fractional cascading reduces query time to $O(\log n)$

Maximum in a general submatrix



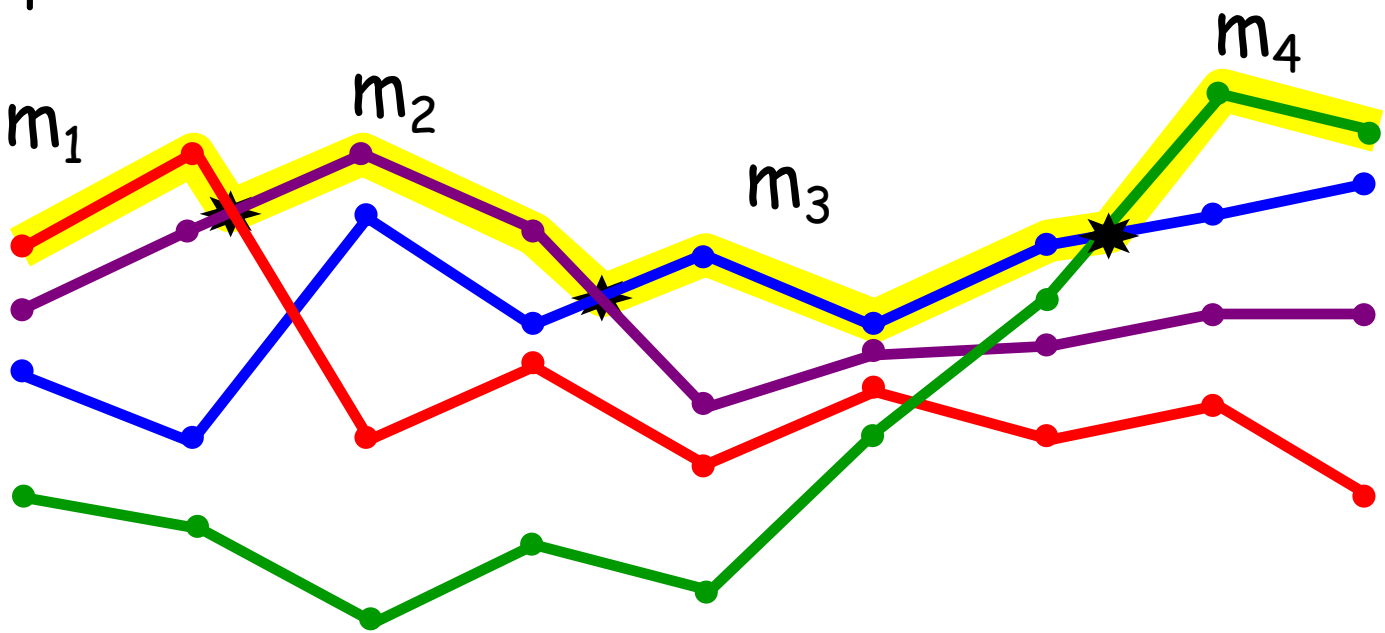
Maximum in a general submatrix



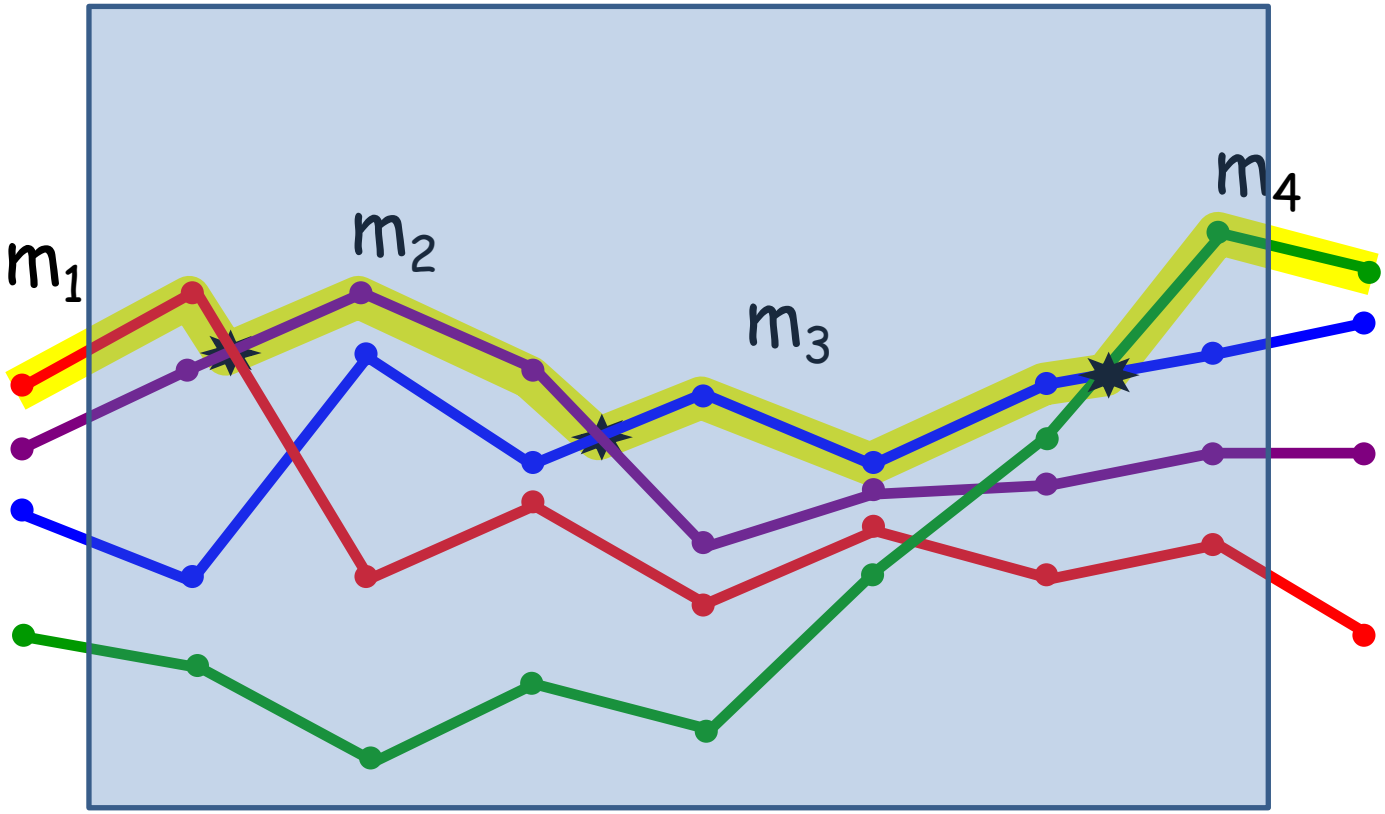
From each envelope we need the maximum in a sequence of complete intervals and a prefix+suffix of two intervals

Store additional information on the upper envelope

- Compute the maxima in each interval of the envelope
- This is a maxima in a subinterval of a row
- Can compute using a stucture as before over the columns
- Store subtree maxima in the search tree of the breakpoints

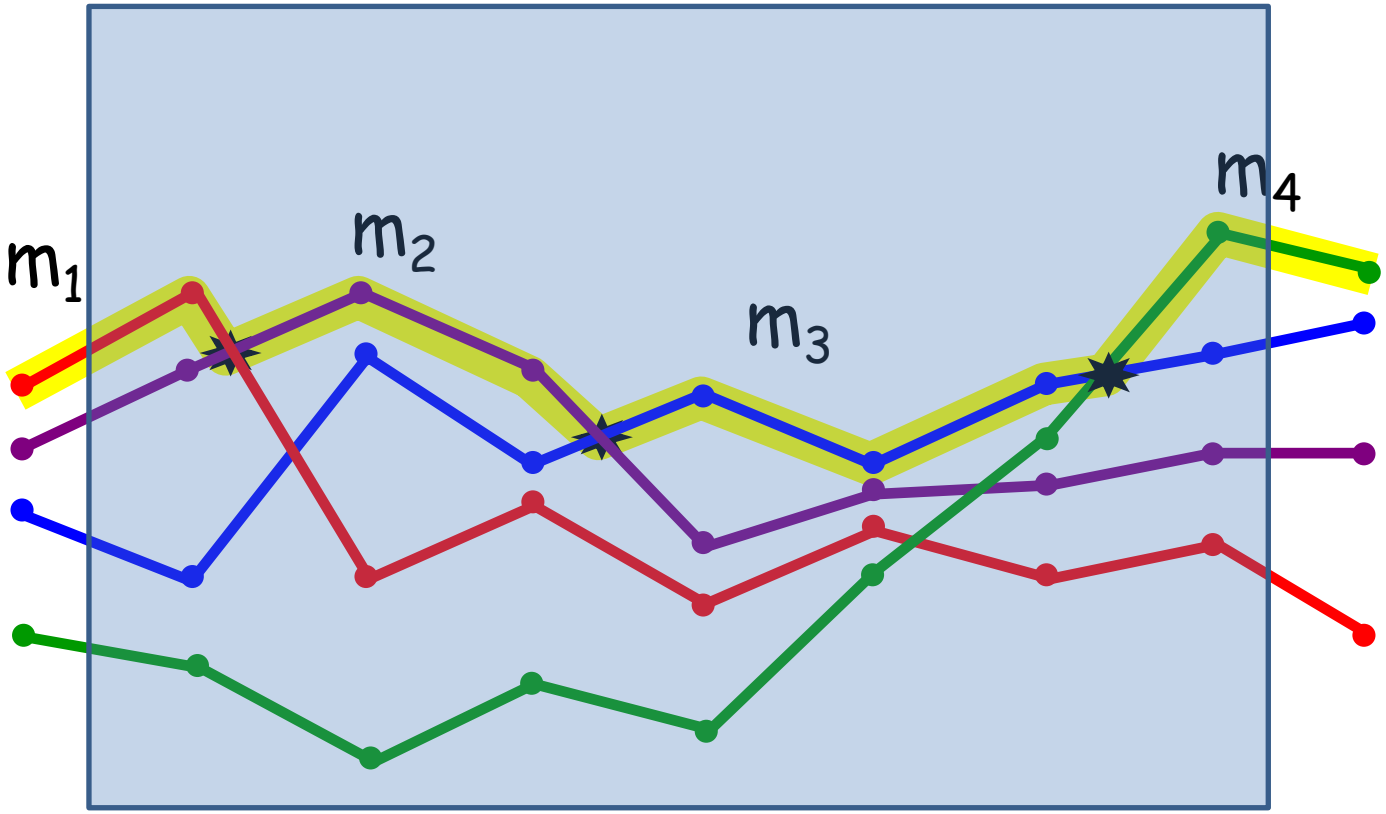


Submatrix query



- Find the maxima in the complete intervals of the envelope contained in the query

Submatrix query



- Find the maxima in the suffix and the prefix by queries to the structure for maxima in subintervals of rows

Submatrix queries: summary

preprocessing time $O(n \log n)$ (the total number of different intervals on all envelopes is $O(n)$)

space $O(n \log n)$

query time $O(\log^2 n)$

Submatrix queries: Extensions

- extension to certain Monge partial matrices using pseudo-segments instead of pseudo-lines
- preprocessing time $O(n\alpha(n)\log^2 n)$
  space $O(n\alpha(n)\log n)$
  query time $O(\log^2 n)$