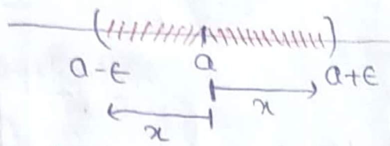
Calculus :

Neighbour hood of a real number :

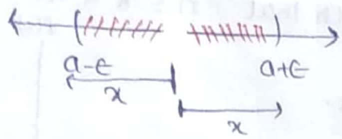
Let $a \in R$ be any real number and let $0 < \epsilon < 1$ be any small +ve real number (R^+) then $|x-a| < \epsilon$ represents all the real numbers in an $(a-\epsilon, a+\epsilon)$ which is called 'ϵ-neighbour hood of a'



Deleted Neighborhood of a real number :

When $0 < |x-a| < \epsilon$

It represents all the real nums in the $(a-\epsilon, a+\epsilon)$ Except at $x=a$ which is called 'deleted Neighborhood of a'.



Limit :

Let $y=f(x)$, be any real function defined for $x \in R$ and let $a \in R$ be a selected real number. Suppose there exist two small +ve real numbers ϵ & δ such that $0<|x-a|<\delta$ (DNBH) $\Rightarrow$ $|f(x)-L| < \epsilon$ (NBH) then we can observe that as x approaches a then $f(x)$ approaches L

ie $\boxed{\lim_{x \to a} f(x) = L}$

$x = a^+/a^-$ but not a

Types of functions :

Explicit function : $y = x^2 - 2x + 2$

Implicit function : $x^3 + y^3 - 3xy = 0$

Composite function : $z = f(x, y)$

Polynomial function : $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

Rational Function : $f(x) = \dfrac{g(x)}{h(x)}$

Algebraic function : $f(x) = x^3\sqrt{x^2+1}$

$f(x) = \sqrt{x^2+4}$

$f(x) = \dfrac{2x^5+4}{\sqrt{x+1}}$

Even Function : $f(-x) = f(x)$

Ex:- $\cos x$, $|x|$, x^2 ...

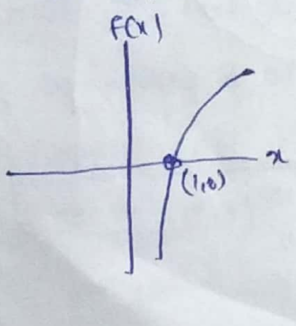
odd Function : $f(-x) = -f(x)$

Ex:- $\sin x$, x, x^3 ...

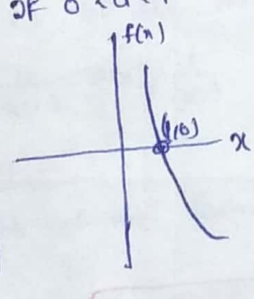
Logarithmic Function :

$f(x) = \log_a x$; $a, x \in R$, $a > 0$ & $a \neq 1$, $x > 0$

If $a > 1$

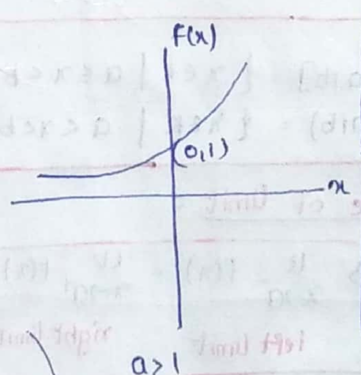


If $0 < a < 1$

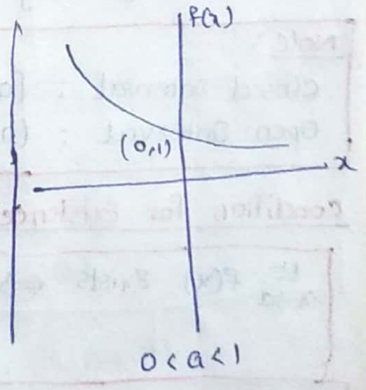


Exponential Function :

$f(x) = a^x = e^{x \log_e a}$; $a > 0, a \neq 1$, $a, x \in R$



$a > 1$



$0 < a < 1$

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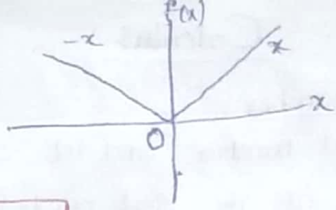
Modulus function :

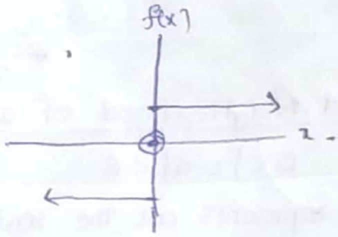
$$f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Note :
$|x|$ is continuos function for every $x \in R$
$|x|$ is not differentiable at $x = 0$
$|x+a|$ is not differentiable at $x = -a$

Signum function :

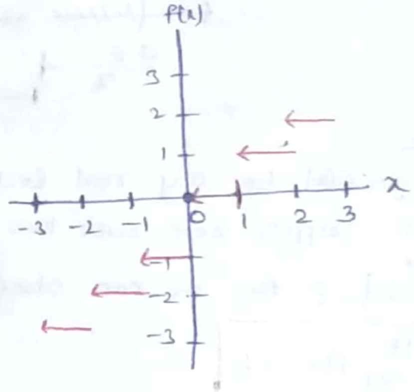
$$f(x) = sgn(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 1 & ; x > 0 \\ -1 & ; x < 0 \\ 0 & ; x = 0 \end{cases}$$



Step function or Greatest Integer function or Bracket function :

- A function of the form $f(x) = [x] = n$ such that $n \le x < n+1$ $\forall n \in Z$ & $x \in R$ is called step function.

$[x] \rightarrow$ greatest integer less than or equal to x.



Ex:- $[7.2] = 7$
$[7] = 7$
$[-1.2] = -2$
$[0] = 0$
$[-1] = -1$

$[x] = -1$; $-1 \le x < 0$
$[x] = 0$; $0 \le x < 1$
$[x] = 1$; $1 \le x < 2$
$[x] = 2$; $2 \le x < 3$

Note : step function is discontinuos at every integer point

Symmetric properties of a curve.

- $f(-x, y) = f(x, y)$ is symmetric about y-axis
  Ex :- $x^2 = 4ay$

- $f(x, -y) = f(x, y)$ is symmetric about x-axis
  Ex:- $y^2 = 4ax$.

- $f(x, y) = f(y, x)$ is symmetric about $y = x$
  Ex: $x^3 + y^3 - 3axy = 0$

Note :
Closed interval : $[a, b] = \{x \in R \mid a \le x \le b\}$
Open interval : $(a, b) = \{x \in R \mid a < x < b\}$

Condition for Existence of limit :

$\lim_{x \to a} f(x)$ Exists $\Longleftrightarrow$ $\underbrace{\lim_{x \to a^-} f(x)}_{\text{left limit}} = \underbrace{\lim_{x \to a^+} f(x)}_{\text{right limit}}$

Note : If $\lim_{x \to a} f(x)$ exist then it is unique

If $f(x)$ is a polynomial function then $\lim_{x \to a} f(x) = f(a)$

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Standard limits :

- $\lim_{x\to a} \frac{x^n - a^n}{x-a} = n \cdot a^{n-1}$, $a \neq 0$ & $n \in Q$
- $\lim_{x\to 0} \frac{e^{mx}-1}{x} = m$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$
- $\lim_{x\to 0} (1+ax)^{\frac{1}{x}} = e^a$
- $\lim_{x\to \infty} \left[1+\frac{a}{x}\right]^x = e^a$
- $\lim_{x\to 0} \frac{\sin x}{x} = 1$; $\lim_{x\to\infty} \frac{\sin x}{x} = 0$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$
- $\lim_{x\to 0} \left[\frac{a^x+b^x}{2}\right]^{\frac{1}{x}} = \sqrt{ab}$
- $\lim_{x\to 0} [\cos x + a \sin bx]^{1/x} = e^{ab}$
- $\lim_{x\to 0} \frac{1-\cos ax}{x^2} = \frac{a^2}{2}$

Proof

- If $k \in N$

$$\lim_{n\to\infty} \frac{1^k + 2^k + 3^k + \dots + n^k}{n^{k+1}} = \frac{1}{k+1}$$

Note: If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$ then.

$$\lim_{x\to a} \underbrace{(f(x))^{g(x)}}_{1^\infty \text{ form}} = e^{\lim_{x\to a} (f(x)-1)\, g(x)}$$

1^∞ form

Indeterminate forms :

$$\boxed{\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^\infty, \infty^0}$$

L- Hospital rule : $\frac{0}{0}$; $\frac{\infty}{\infty}$

- If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ (or) $\frac{\infty}{\infty}$ then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \dots \frac{f^n(a)}{g^n(a)}}$$; where $f^n(a) = g^n(a) \neq 0$

Evaluation of $\infty - \infty$:

- If $\lim_{x\to a} (f(x) - g(x))$ is of the form $\infty - \infty$, It can be transformed to $\frac{0}{0}$ form by writing it as $\lim_{x\to a} \left[\left[\frac{1}{g(x)} - \frac{1}{f(x)}\right] \Big/ \frac{1}{f(x)\cdot g(x)}\right]$ and now Apply L-Hospital rule.

Evaluation of $0\cdot\infty$:

- If $\lim_{x\to a} f(x)\cdot g(x)$ is of the form $0\cdot\infty$, It can be transformed into $\frac{0}{0}$ form by writing it as $\lim_{x\to a} f(x) \Big/ \frac{1}{g(x)}$ and now Apply L-Hospital rule.

Evaluation of 0^0 or ∞^0 :

- If $\lim_{x\to a} (f(x))^{g(x)}$ is of the form 0^0 or ∞^0 then $L = \lim_{x\to a} f(x)^{g(x)}$ Apply log

$$\log_e L = \lim_{x\to a} g(x) \log_e f(x)$$

$$L = e^{\lim_{x\to a} g(x) \cdot \ln f(x)} \Rightarrow 0\cdot\infty \text{ form}$$

$$= e^{\lim_{x\to a} \left[\ln f(x) \Big/ \frac{1}{g(x)}\right]}$$

Apply L-Hospital rule

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Points to remember :

Imp **Evaluation of Indeterminate limits**

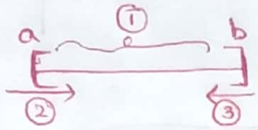
1^∞ form	$\frac{0}{0}, \frac{\infty}{\infty}$ form	$\infty - \infty$ form	$0 \cdot \infty$ form	$0^0, \infty^0$ form
$e^{\lim (f(x)-1)g(x)}$ method	L-Hospital rule	Convert to $\frac{0}{0}$ form $\left[\frac{1}{g(x)} - \frac{1}{f(x)}\right] \Big/ \frac{1}{f(x)\cdot g(x)}$ method. - Apply L-Hospital rule	Convert to $\frac{0}{0}$ form $f(x) \Big/ \frac{1}{g(x)}$ method - Apply L-H rule	- Apply log - $e^{\lim 0\cdot\infty \text{ form}}$ - $e^{\lim \ln f(x) / \frac{1}{g(x)}}$ $\left(\frac{0}{0}\right)$ Apply L-H rule

Continuity at Intervals

- A function f is defined on (a,b) is said to be continuos on (a,b) iff it is continuos at every point of (a,b) ie
  open Interval

$$\boxed{\lim_{x\to d} f(x) = f(d) \quad \forall\ d \in (a,b)}$$

- A function f is defined on $[a,b]$ is said to be continuos on $[a,b]$ iff
  Closed Interval
  (i) f is continuos at (a,b)
  (ii) f is right continuos at a
  $\lim_{x\to a^+} f(x) = f(a)$
  (iii) f is left continuos at b
  $\lim_{x\to b^-} f(x) = f(b)$



Continuity at a point :

Let f be a fn defined at a point 'a' then f is continuos at a

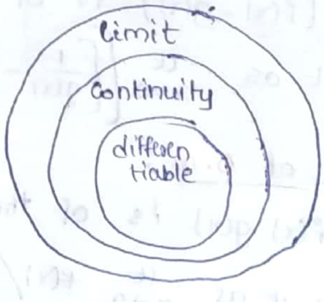
(i) left continuos at a iff $\lim_{x\to a^-} f(x) = f(a)$

(ii) right continuos at a iff $\lim_{x\to a^+} f(x) = f(a)$

Note :

- Any function $f(x)$ is continuos means its graph is continuos without any breakages
  Ex:- e^x, $\sin x$, $\cos x$, constant, $\log x$ $(0,\infty)$, $\sinh x$, $\cosh x$, polynomial Eqns, modulus
- If $f(x)$ and $g(x)$ are continuos then

$$\left.\begin{array}{l} f(x) \pm g(x) \\ f(x)\cdot g(x) \\ \frac{f(x)}{g(x)}\ ;\ g(x) \neq 0 \\ k\cdot f(x) \end{array}\right\} \text{Continuos}$$



- If fn is continuos then limit exists
- If fn has limit then it may or may not be continuos.

Differentiablity :

Derivative of a real function :

Let $y = f(x)$ be any real fn defined for $x \in R$. if the following limits exists finetely then we can say $f(x)$ has derivatives.

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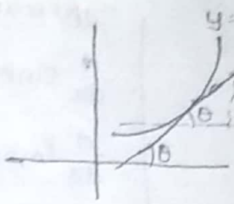
$$\boxed{\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$



In particular

$$[f'(x)]_{x=a} = f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \rightarrow \text{slope } (\tan\theta)$$

Note :

At point $x = a$

Limit $\xrightarrow{\text{may/may not}}$ Continuous $\xrightarrow{\text{may/may not}}$ differentiable

differentiable $\rightarrow$ Continuous ✓ ; Continuous $\rightarrow$ Limit ✓

Note :

- Derivative of a function at $x=a$ Exists If

  Left hand derivative = Right hand derivative

Points

$$f'(a^-) = \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} = f'(a^+)$$

- If the derivative of $f(x)$ Exist at $x=a$ then the function $f(x)$ is said to be differentiable function at $x=a$
- polynomial fns, Exponential fns, sine, Cosine fns are differentiable Every where.
- If $f(x)$ and $g(x)$ are two differentiable fns then

$$\left.\begin{array}{l} f(x) \pm g(x) \\ f(x) \cdot g(x) \\ \dfrac{f(x)}{g(x)} ;\ g(x) \neq 0 \\ k \cdot f(x) \end{array}\right\} \text{differentiable}$$

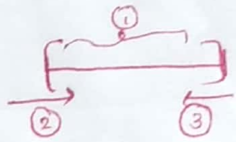
Derivability of a fn in Closed Interval :

A function $f(x)$ is said to be derivable (or) differentiable on closed Interval $[a,b]$ if

(I) $f'(c)$ Exists $\forall c \in (a,b)$

(II) Right hand derivative $(Rf'(a))$ Exists

(III) left hand derivative $(Lf'(a))$ Exists.



Derivability of a function in an open Interval :

A fn $f(x)$ is said to derivable on open Interval (a,b) if

(I) $f'(c)$ Exists $\forall c \in (a,b)$

Standard derivatives :

- $\frac{d}{dx}(uv) = uv' + vu'$
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
- $\frac{dy}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$

Poly, Exp, log

- $\frac{d}{dx} x^n = nx^{n-1}$
- $\frac{d}{dx} (ax+b)^n = n(ax+b)^{n-1} \cdot a$
- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} a^x = a^x \log_e a$
- $\frac{d}{dx} \log_e x = \frac{1}{x}$
- $\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e$

Trignometric

- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$
- $\frac{d}{dx} \sec x = \sec x \tan x$
- $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

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Inverse trignometric

- $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

- $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

- $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

- $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

- $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

- $\frac{d}{dx}(\csc^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

Hyperbolic

- $\frac{d}{dx}\sinh x = \cosh x$

$\frac{d}{dx}\cosh x = \sinh x$

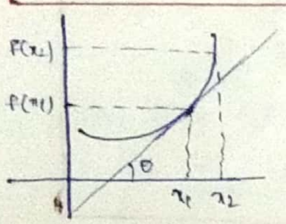
Increasing and decreasing functions :

Increasing function :

$x_2 > x_1$

$f(x_2) > f(x_1)$

$$\boxed{\frac{dy}{dx} = f'(x) = \tan\theta > 0}$$



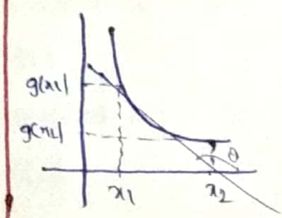
Inc fn.

Decreasing function :

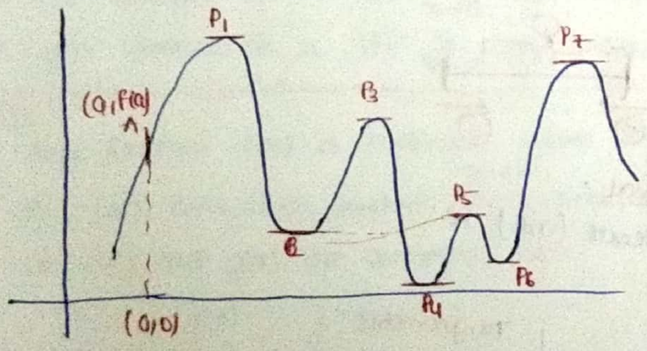
$x_2 > x_1$

$f(x_2) < f(x_1)$

$$\boxed{\frac{dy}{dx} = f'(x) = \tan\theta < 0}$$



Maximum / Minimum value of fn $(y = f(x))$:



P_1 – Global max

P_4 – Global min

$P_3\ P_5\ P_7$ – Local max

$P_2\ P_6$ – Local min

- In the above graph $y = f(x)$ max value occur at $P_1 P_3 P_5 P_7$
  min value occurs at $P_2 P_4 P_6$

- At any point of max/min we can observe that tangents to the curve are // to x-axis

ie $\boxed{\frac{dy}{dx} = f'(x) = 0}$

solving this we get stationary points / critical points

- while passing through any point of maximum, $f'(x)$ changes it's nature from +ve to -ve.

ie $f'(x)$ is a decreasing function at point of max.

$$\boxed{f''(x) < 0}$$

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- While passing through any point of minimum, $f'(x)$ changes its nature from -ve to +ve ie $f'(x)$ is an Increasing function at point of minimum

$$\boxed{f''(x) > 0}$$

- If $\boxed{f''(x) = 0}$ at any stationary point then there is no max/min value of $f(x)$ then we can say the stationary points as $\boxed{\text{point of Inflection}}$
- At any point of max, the shape of the curve looks like Concave down / convex up ($f''(x) < 0$)
- At any point of min, the shape of the curve looks like Concave up / Convex down ($f''(x) > 0$).
- Max value / Min value are also called as Extreme value or optimal value of $f(x)$ and any point of MAX/MIN is also called as point of Extremum/optimum.

Note:
- suppose It is decided to find Extreme value of $f(x)$ in $[a,b]$ then In addition to applying the above procedure we must consider the fn values at $x=a$ and $x=b$ before concluding global max / global min values.

Points to remember

Maxima	Minima	
$y = f(x)$	$y = f(x)$	
$f'(x) = 0$	$f'(x) = 0$	If $f''(x) = 0$
Articulation points.	$\hookrightarrow$ Articulation points	$\hookrightarrow$ Point of Inflection.
$f''(x) < 0$ → dec fn	$f''(x) > 0$ → Inc fn	
⇓	⇓	
Max at that point → Extreme Point	min at particular point → optimum point	
Curve shape: Concave down, Convex up	Shape: Concave up, Convex down.	

Max value or Min value of an Implicit fn $f(x,y)$:

Step 1: Consider $\underbrace{\frac{\partial f}{\partial x} = 0 \; ; \; \frac{\partial f}{\partial y} = 0}$

solving it we get stationary Co-ordinates.

$(x,y) = (a_1,b_1)(a_2,b_2) - - -$

Step 2: calculate

$$r = \frac{\partial^2 f}{\partial x^2} \; ; \; s = \frac{\partial^2 f}{\partial x \partial y} \; ; \; t = \frac{\partial^2 f}{\partial y^2}$$

Step 3: (i) If $rt - s^2 > 0$ and $r < 0$ → max value

(ii) If $rt - s^2 > 0$ and $r > 0$ → Min value.

(iii) If $rt - s^2 < 0$ → No max/min at stationary point → Saddle point

(iv) If $rt - s^2 = 0$ → No Conclusion at that stationary point

Note:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \; ; \; \frac{\partial f}{\partial y} = \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$ → Exists finitely

→ tangents to the surface

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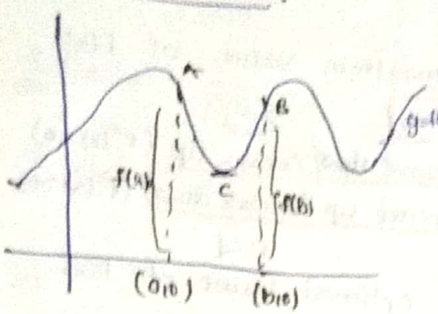
Mean value theorems :

① Rolle's theorem :



(i) $f(x)$ is cont in $[a,b]$

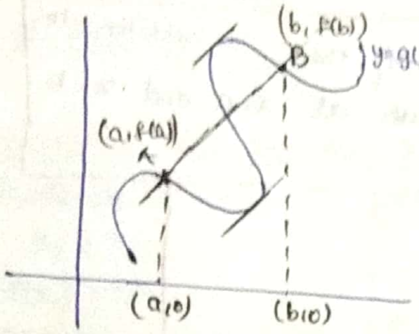
(ii) $f(x)$ is diff in (a,b)

(iii) $f(a) = f(b)$

then there exists atleast one point $c \in (a,b)$ such that

$$f'(c) = 0$$

② lagranges theorem :



(i) $f(x)$ is Continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b)

then there Exists atleast one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

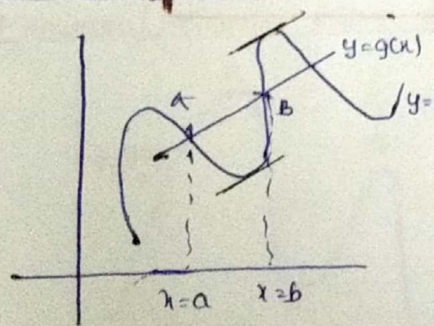
Note :

- On the above stmt if we consider or if we define $f(x) = k$ (const) $\forall x [a,b]$ then $f(a) = k$ and $f(b) = k$ then

legranges theorem reduces to $f'(c) = \frac{k-k}{b-a} = 0$.

$\therefore f'(c) = 0$ (Rolles theorem)

③ Cauchy Mean value theorem :



(i) $f(x)$ and $g(x)$ are Continuos in $[a,b]$

(ii) $f(x)$ and $g(x)$ are differentiable in (a,b)

(iii) $g'(x) \neq 0 \quad \forall x \in (a,b)$

then there Exists atleast one point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Note:

- On the above stmt of we consider $g(x) = x + d \quad \forall x \in [a,b]$ and d is a Constant

then, $\frac{f'(c)}{1} = \frac{f(b) - f(a)}{b-a}$ (legranges theorem)

- On the above stmt of we Consider $f(x) = k$ and $g(x) = x + L \quad \forall x \in [a,b]$ then

$$f'(c) = \frac{k-k}{b-a} = 0 \text{ (Rolle's theorem)}$$

Note :

$$\text{Cauchy} \xrightarrow{f(x)=k,\ g(x)=x+d} \text{Rolles}$$

$$\text{Cauchy} \xrightarrow{g(x)=x+d} \text{legranges} \xrightarrow{f(x)=k} \text{Rolles}$$

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Note :

Taylor Series : $[a, x]$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots \quad \text{at } x = a$$

Maclaurens series $[0, x]$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad \text{at } x = 0$$

Ex:- $f(x) = x^{-2/3}$ $[-1, 1]$

$f(x) = \frac{1}{x^{2/3}}$

Not Cont at $x = 0$, $0 \in [-1, 1]$

$f'(x) = \frac{-2}{3} x^{-5/3}$

Not diff at $x = 0$, $0 \in (-1, 1)$

Not cont, Not diff

$f(x) = x^{2/3}$ $[-1, 1]$

$f(x) = x^{2/3}$

Continuos

$f'(x) = \frac{2}{3} x^{-1/3}$

Not diff at $x = 0$, $0 \in (-1, 1)$

Cont, Not diff

$f(x) = x^{-2/3}$ $[0, 1]$

$f(x) = \frac{1}{x^{2/3}}$

Not cont at $x = 0$, $0 \in [0, 1]$

$f'(x) = -\frac{2}{3} x^{-5/3}$

differentiable -

Not diff at $x = 0$, $0 \notin (0, 1)$

Not Cont, diff

$f(x) = x^{-2/3}$ $(0, 1]$

$f(x) = \frac{1}{x^{2/3}}$

Continuos.

$f'(x) = \frac{-2}{3} x^{-5/3}$

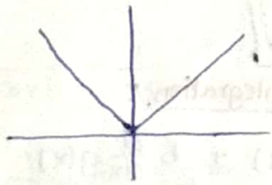
differentiable

Cont, diff

Note :

- $|x|$ is Continuos but not differentiable at $x = 0$

$$f(x) = |x| \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$



$$f'(x) \begin{cases} 1 & ; x \geq 0 \\ -1 & ; x < 0 \end{cases}$$

$$f'(0) = f'(0^+) = f'(0^-)$$

$1 \quad\quad 1 \neq -1$

Not diff

- $|x-a|$ is Continuos at $x = a$ but not differentiable at $x = a$

Integration :

Let $f(x)$ be a function, the Collection of all its primitives is called indefinite Integral of $f(x)$ $(\int f(x)\,dx)$

- Inverse operation of differentiation.

Indefinite Integrals

Poly, e, log

- $\int dx = x + c$
- $\int x^n = \frac{x^{n+1}}{n+1} + c$
- $\int \frac{1}{x} dx = \log_e^x + c$
- $\int e^x dx = e^x + c$
- $\int \log x \, dx = x(\log^x - 1) + c$
- $\int a^x dx = \frac{a^x}{\log a} + c$

Trig

- $\int \sin x = -\cos x + c$
- $\int \cos x = \sin x + c$
- $\int \tan x = \log \sec x + c = -\log \cos x + c$
- $\int \cot x = \log \sin x + c$
- $\int \sec x = \log(\sec x + \tan x) + c$
- $\int \operatorname{cosec} x = \log(\operatorname{cosec} x - \cot x) + c$
- $\int \sec^2 x = \tan x + c$
- $\int \operatorname{cosec}^2 x = -\cot x + c$
- $\int \sec x \tan x = \sec x + c$
- $\int \operatorname{cosec} x \cot x = -\operatorname{cosec} x + c$

Hyp

- $\int \sinh x \, dx = \cosh x + c$
- $\int \cosh x \, dx = \sinh x + c$

Inv

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

$\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + c$

$\int \frac{dx}{1+x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$

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Integrals of some particular fns :

- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
- $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$
- $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c$

- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c$
- $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)+c$
  $= \log\left|x+\sqrt{a^2+x^2}\right|+c$
- $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)+c$
  $= \log\left|x+\sqrt{x^2-a^2}\right|+c$
- $\frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x)+c$

- $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$
- $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+c$
  $= \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right|+c$
- $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)+c$
  $= \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c$

Note :

- $\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c$
- $\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c$
- $\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \operatorname{cosec}^{-1}(x)$; $\int \frac{-1}{a^2+x^2}\,dx = \cot^{-1}(x)+c$
  ; $\int \frac{-1}{\sqrt{a^2-x^2}} = \cos^{-1}\left(\frac{x}{a}\right)+c$.

Note :

$$\boxed{\int uv = u\int v - \int\left(u'\int v\right) \quad ; \quad u \;\; v \;\; u' \to \int v}$$

Comparision b/w differentiation and integration :

(i) $\frac{d}{dx}(a f(x) \pm b g(x)) = a\frac{d}{dx}f(x) \pm b\frac{d}{dx}g(x)$

$\int (af(x) \pm bg(x))\,dx = a\int f(x)\,dx \pm b\int g(x)\,dx$

(ii) All functions are not differentiable, similarly there are some functions which are not integrable

(iii) Integral of a fn is always discussed in an interval but derivative of a function can be discussed in an interval as well as at a point

Methods of Integration :

① Integration by substitution :

Integration can be transformed into another form by changing the independent variable 'x' to t

$\int f(x)\,dx$ can be by substituting $x = g(t)$

ie $I = \int f(x)\,dx$

let $x = g(t)$

$dx = g'(t)\,dt$

$I = \int f(g(t))\,g'(t)\,dt$

Ex:- $\int \sin^3 x \cos^2 x\,dx$

$\int (1-\cos^2 x)\cos^2 x\,(\sin x\,dx)$

$\cos x = t \qquad -\sin x\,dx = dt$

$\int (1-t^2)t^2(-dt)$

$\int (t^4-t^2)\,dt = \frac{t^5}{5}-\frac{t^3}{3}+c$

$\frac{(\cos x)^5}{5}-\frac{(\cos x)^3}{3}+c$

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Integration by parts :

This method is used to Integrate the product of two fns. If $f(x)$ and $g(x)$ be two Integrable functions then

$$\boxed{\int f(x)\cdot g(x)\,dx = f(x)\cdot\int g(x)\,dx - \int f'(x)\cdot\left(\int g(x)\,dx\right)dx}$$

ILATE Rule :

I → Inverse : $\sin^{-1}x, \tan^{-1}x \ldots$

L → Logarithm : $\log x$

A → Algebric : $1, x, x^2+1$

T → Trigrometic : $\sin x, \cos x$ - -

E → Exponential : e^x

Ex:- $\int x\cos x$

$u = x$ | $v = \cos x$

$du = dx$ | $\int v = +\sin x$

$\int uv = u\int v - \int du' \int v$

$= x\sin x - \int dx(\sin x)$

$= x\sin x + \cos x + C$

Ex:- $\int xe^x\,dx$

$u = x \quad v = e^x$

$\int uv = u\int v - \int u'(\int v)$

$= x\int e^x - \int dx \int e^x dx$

$xe^x - \int e^x dx$

$= (x-1)e^x + C$.

Integration by partial fractions :

Ex:- $\int \frac{dx}{(x+1)(x+2)}$

Now, $\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

Comparing $A(x+2) + B(x+1) = 1$

$x(A+B) + 2A+B = 1 \Rightarrow$ $A+B=0$, $2A+B=1$

$A = 1, \quad B = -1$

$\therefore \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

$\int \frac{1}{(x+1)(x+2)}dx = \int \frac{1}{x+1}dx - \int \frac{1}{x+2}dx$

$= \log|x+1| - \log|x+2| + c$

$= \log\left|\frac{x+1}{x+2}\right| + c$.

Definite Integrals :

- Contains limits for Integral

Ex:- $\int_4^9 x\,dx$

Properties of definite Integrals :

- $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $c\in(a,b)$
- $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$
- $\int_0^a f(x)\,dx = \int f(a-x)\,dx$
- Imp $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{b-a}{2}$
- Imp $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$; $f(x)$ is even fn $f(-x)=f(x)$

  $= 0$; $f(x)$ is odd fn $f(-x) = -f(x)$
- $\int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx$; $f(a-x) = f(x)$

  $= 0$; $f(a-x) = -f(x)$

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$\rightarrow \int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(x+a) = f(x)$
ie $f(x)$ is periodic fn with period a

Note	Period
$\sin x$	2π
$\cos x$	2π
$\operatorname{cosec} x$	2π
$\sec x$	2π
$\tan x$	π
$\cot x$	π

$\rightarrow \int_0^a x f(x) = \frac{a}{2}\int_0^a f(x)$ if $f(a-x) = f(x)$

$\rightarrow \int_a^b x f(x) = \frac{b-a}{2}\int_a^b f(x)\,dx$ if $f(a+b-x) = f(x)$

$\rightarrow \int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx.$

$\rightarrow \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx.$

$- \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3} & ; \text{If } n \text{ is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & ; \text{If } n \text{ is Even} \end{cases}$

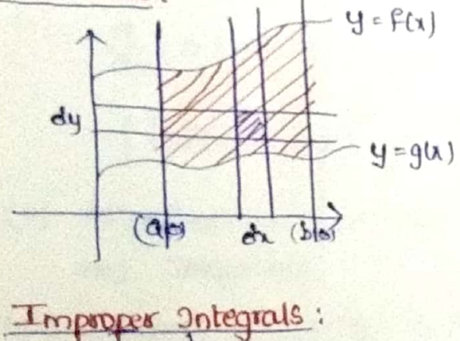
$- \int_0^{\pi/2} \sin^m x \cos^n x\,dx = \left\{\frac{[(m-1)(m-3)(m-5)\cdots(1\text{ or }2)][(n-1)(n-3)(n-5)\cdots(1\text{ or }2)]}{(m+n)(m+n-2)(m+n-4)\cdots\cdots(1\text{ or }2)}\cdot k\right\}$

$k = \begin{cases} \frac{\pi}{2} & \text{If } m \text{ and } n \text{ are Even} \\ 1 & \text{otherwise.} \end{cases}$

For length:

* -2 —— 3 $\Rightarrow$ length $= \int_{-2}^{3} dx = [x]_{-2}^{3} \Rightarrow 3+2 \Rightarrow 5$ units.

For Area:



$\text{Area}(\Delta) = \left[\int_a^b \left[\int_{g(x)}^{f(x)} dy\right] dx\right]$

$\bullet \int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$

Improper Integrals:

Type 1

$\int_a^b f(x)\,dx$

$a = \infty$ / $b = \infty$ or both

Ex:- $\int_0^{\infty} e^x$, $\int_{-\infty}^{0} e^x$, $\int_{-\infty}^{\infty} e^x \ldots$

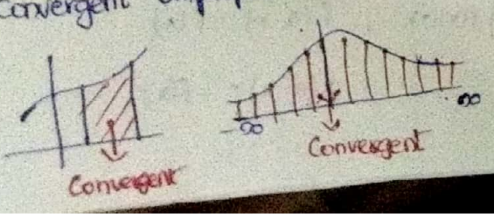
Type 2

$\int_a^b f(x)\,dx$ is said to be Improper Integral if a and b are finite but $f(x)$ is Infinite for some $x \in [a, b]$

Ex:- $\int_0^1 \log(1-x)\,dx$; $\int_{-1}^{1} \frac{1}{x}\,dx$ ---- etc.
At $x=1$ At $x=0$

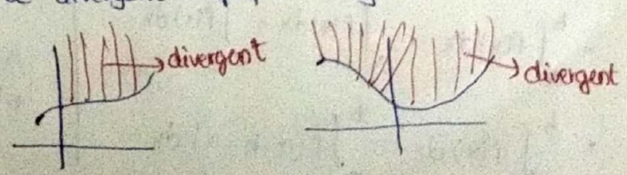
Convergence:

$\rightarrow$ $\int_a^b f(x)\,dx$ is finite then It is said to be Convergent Improper Integral



Divergent:

$\rightarrow$ If $\int_a^b f(x)\,dx$ is Infinite then It is said to be divergent Improper Integral



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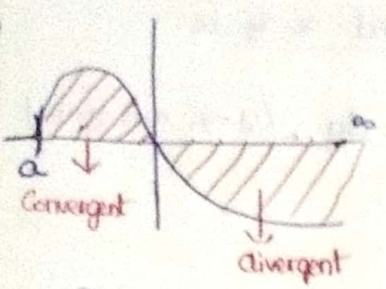
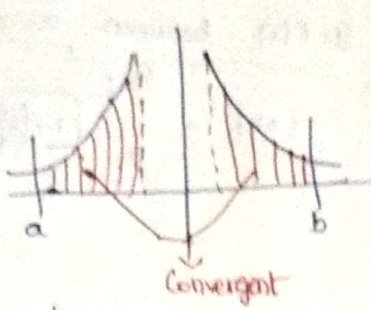
Note:



$\int_a^\infty f(x)\,dx = -\infty$



$\int_a^b f(x)\,dx = $ finite.

*) Convergence / divergence for $\int_a^b f(x)\,dx$ is the value of the Integral.

Note :

Comparision test :

Method 1 :

- If $0 \le f(x) \le g(x)$ $\forall x \in [a,b]$ and $\int_a^b g(x)\,dx$ converges then $\int_a^b f(x)\,dx$ also converges.
- If $0 \le f(x) \le g(x)$ $\forall x \in [a,b]$ and $\int_a^b f(x)\,dx$ diverges then $\int_a^b g(x)\,dx$ also diverges.

Method 2 : [Limit form]

For Type 1 Improper Integral :

- If $f(x)$ and $g(x)$ are two +ve fns such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$ (finite and non-zero) then

$\int_a^\infty f(x)\,dx$ and $\int_b^\infty g(x)\,dx$ converge / diverge together

For Type 2 Improper Integrals :

- If $f(x)$ and $g(x)$ are two +ve fns and.

(i) $f(x) \to \infty$ as $x \to a$ such that $\lim_{x\to a} \frac{f(x)}{g(x)} = l$ /

(ii) $f(x) \to \infty$ as $x \to b$ such that $\lim_{x\to a} \frac{f(x)}{g(x)} = l$ then.

$\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ both converge / diverge together.

Some useful Comparisions :

(i) $\int_a^\infty \frac{dx}{x^p}$ $(a>0)$ Converges to $\frac{1}{(p-1)a^{p-1}}$ $(p>1)$

diverges to ∞ when $(p \le 1)$

(ii) $\int_a^\infty e^{-px}\,dx$ and $\int_{-\infty}^b e^{px}\,dx$ converges for any constant $(p>0)$

diverges to ∞ when $(p \le 0)$

(iii) $\int_a^b \frac{dx}{(b-x)^p}$ converges to $\frac{1}{(1-p)(b-a)^{p-1}}$ $(p<1)$

diverges to ∞ when $(p \ge 1)$

(iv) $\int_a^b \frac{dx}{(x-a)^p}$ Converges to $\frac{1}{(1-p)(b-a)^{p-1}}$ $(p<1)$

diverges to ∞ when $(p \ge 1)$

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Length of an arc of a curve :

- The length of an arc of a curve $y = f(x)$ between $x = x_1$ and $x = x_2$ is.

$$\text{length} = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad (\text{or}) \quad \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \quad (y = y_1 \text{ and } y = y_2)$$

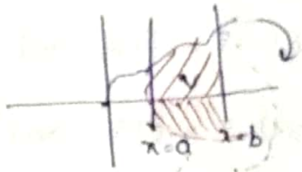
In polar Co-ordinates

$$\text{length} = \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Volume of Solid of revolution :

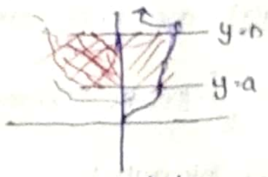
- The volume of the solid generated by revolving the area bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$. about x-axis is

$$V = \int_a^b \pi y^2 \, dx .$$



- The volume of the solid generated by revolving the area bounded by the curve $x = g(y)$, y-axis and the lines $y = a$ and $y = b$ about x-axis is

$$V = \int_a^b \pi x^2 \, dx$$



- In polar Co-ordinates, the volume of solid generated by revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$

(i) about the Initial line $\theta = 0$ is

$$V = \int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta \, d\theta$$

(ii) about the line $\theta = \pi/2$

$$V = \int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta \, d\theta$$

x — The End — x

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