Basics:

Linear Algebra

Trace of a matrix:

The sum of the diagonal elements of a square matrix A is called trace of A

Properties

- $tr(A \pm B) = tr(A) \pm tr(B)$
- $tr(AB) \neq tr(A)\, tr(B)$
- $tr(AB) = tr(BA)$
- $tr(KA) = K\, tr(A)$ where K is scalar.

Diagonal matrix:

If all the non-diagonal elements in a square matrix are zero then the matrix is called diagonal matrix.

Note: If A & B are diagonal matrices.
$A \pm B$, A^2, B^2, $A^n \pm B^n$, A^T, $A^T \pm B^T$, $adj(A), adj(B)$, A^{-1}, B^{-1} are also diagonal matrices.

Upper triangular matrix:

$$\begin{bmatrix} 2 & -3 & 4 \\ 0 & 4 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

Lower triangular matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 8 & 0 \\ 4 & 8 & 9 \end{bmatrix}$$

Triangular matrix: which may be either upper or lower triangular.

Diagonal matrix: Both upper & lower triangular

Properties of transpose.

- $(A^T)^T = A$
- $(A \pm B)^T = A^T \pm B^T$
- $(KA)^T = KA^T$; K is scalar.
- $(AB)^T = B^T A^T$
- $(A^2)^T = (A^T)^2$

Symmetric matrix: $A^T = A$

Skew symmetric matrix: $A^T = -A$ $\Rightarrow$ The diagonal elements for skew symm matrix is equal to zero.

Note: Every square matrix A can be uniquely expressed as a sum of sym & skew sym matrices

$$A = \frac{A - A^T}{2} + \frac{A + A^T}{2}$$

Properties of symmetric matrix.

If A & B are symmetric matrix then

- $A \pm B$, A^2, B^2, $AB + BA$, A^K, B^K, KA – symmetric
- AB, BA need not be symmetric
- $\left.\begin{matrix} A + A^T \\ A \cdot A^T, A^T A \end{matrix}\right\}$ – symmetric

Properties of skew symmetric matrices.

If A & B are skew sym matrix then

$A \pm B$, KA, A^3, A^5, A^7 – skew

$\left.\begin{matrix} A^2, A^4, A^6 \\ A^2 \pm B^2 \end{matrix}\right\}$ – symmetric

AB, BA are not skew symmetric.

$\left.\begin{matrix} AB - BA \\ A - A^T, A^T - A \end{matrix}\right\}$ skew symmetric.

Idempotent matrix: $A^2 = A$

Involutary matrix: $A^2 = I_n$

Nilpotent matrix: $A^m = O_n$ (m → Index of nilpotent)

Orthogonal matrix: $AA^T = A^T A = I$

$A^{-1} = A^T$

Note:

- If $AB = A$ & $BA = B$ then A & B are Idempotent
- If A & B are orthogonal then AB and BA are also orthogonal matrices.

Note:

The number of different $n \times n$ symmetric matrices with each element being 0 or 1 is $2^{\frac{n(n+1)}{2}}$

Note :

Property	Addition	Multiplication
Commutative	✓	✗
Associative	✓	✓
Distributive	–	✓

Additive identity → 0
Additive inverse → −A
Multiplicative identity → I
Multiplicative inverse → A^{-1}

- The product of two matrices can be the null matrix while neither of them is the null matrix

$$\underset{A}{\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} AB$$

- If AB = 0 then BA may or may not be 0

Note :
Sum of odd numbers
$1+3+5+7+\dots+(2n-1) = n^2$
Sum of even numbers
$2+4+6+8+\dots+2n = n^2+n$

Determinants :

- $\text{Det } A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} = a_1(b_2c_3 - b_3c_2) - b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$

$\boxed{\text{sign} = (-1)^{i+j}}$

- Singular matrix : $|A| = 0$

Properties of determinants :

- If A and B are two square matrices of same order then $|AB| = |A||B|$
- $|A^m| = |A|^m \quad (m = 2,3,4 \dots)$
- If $|A| \neq 0$ then $|A^{-1}| = \frac{1}{|A|}$
- $|A| = |A^T|$
- If every element of a row (column) of a determinant of A is zero then $|A| = 0$

Ex:- $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 \quad (\text{or}) \quad \begin{vmatrix} 1 & 0 & 4 \\ 2 & 0 & 5 \\ 3 & 0 & 6 \end{vmatrix} = 0$

- If any two rows (columns) are identical then the value of determinant is zero.

$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0 \quad (\text{or}) \quad \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{vmatrix} = 0.$

- If any two rows (columns) of a determinants are interchanged then sign of det changes.
- $|kA_{n\times n}| = k^n |A_{n\times n}|$
- $\begin{vmatrix} a+\lambda_1 & b+\lambda_2 & c+\lambda_3 \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ d & e & f \\ g & h & i \end{vmatrix}$
- The det of upper Δ^{lar}, lower Δ^{lar}, diagonal, scalar matrix is equal to the product of its diagonals.
- If A is an orthogonal matrix then $|A| = \pm 1$

- If A is an Idempotent matrix then $|A| = 0$ or 1
- $|I_n| = 1$
- The det of skew sym matrix of odd order is zero.

Minor of an Element (M_{ij})

If $A = (a_{ij})$ is a square matrix of order n then the minor of an element a_{ij} in A is the det of a square matrix that remains after deleting i^{th} row and j^{th} col of A.

Ex:- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$; $M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of an Element: $\boxed{A_{ij} = (-1)^{i+j} M_{ij}}$

Note:

If $A = (a_{ij})$ is a square matrix of order 'n' then

- $\sum_{j=1}^{n} a_{ij} M_{ij} = |A|$ (or) $\sum_{i=1}^{n} a_{ij} M_{ij} = |A|$
- $\sum_{j=1}^{n} a_{ij} M_{kj} = 0$ (or) $\sum_{i=1}^{n} a_{ij} M_{ik} = 0$

Adjoint matrix:

If B is Cofactor matrix of A then

$$\boxed{adj(A) = B^T = (\text{Cofactor matrix})^T}$$

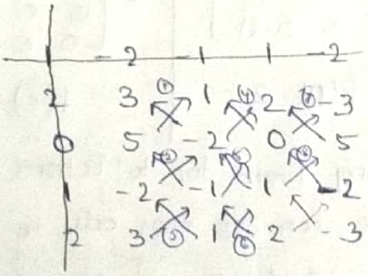
Q) $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$ find $Adj(A)$

$$\begin{matrix} 3 & 1 & 2 & 3 \\ 5 & -2 & 0 & 5 \\ -2 & -1 & 1 & -2 \\ 3 & 1 & 2 & 3 \end{matrix}$$
(or)

Starting from middle row middle Element

$|A| = -2$

Adj A :



$Adj(A) = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & -5 & 1 \end{bmatrix}$

Properties of adjoint matrix:

- $A \cdot adj(A) = adj(A) \cdot A = |A| I_n$
- $Adj(O) = O$; $Adj(I) = I$
- If D is a diagonal matrix then $adj(D)$ is also a diagonal matrix.

$A = \begin{bmatrix} l & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & n \end{bmatrix}$ $adj(A) = \begin{bmatrix} mn & 0 & 0 \\ 0 & ln & 0 \\ 0 & 0 & lm \end{bmatrix}$

- $Adj(A^T) = (adj(A))^T$
- $Adj(AB) = adj(B) \cdot adj(A)$
- $|A| = 0$ then $|adj(A)| = 0$
- $|adj(A)| = |A|^{n-1}$
- $|adj(adj\, A)| = |A|^{(n-1)^2}$
- $adj(adj\, A) = |A|^{n-2} \cdot A$
- If A is symmetric then $adj(A)$ is also symmetric
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverse of a matrix :

If A is non-singular matrix then

$$\boxed{A^{-1} = \frac{1}{|A|} adj(A)}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Properties of Inverse of a square matrix :

- If the inverse of a square matrix A Exists then it is unique.
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- If $AB = AC$ then $B=C$
- If $BA = CA$ then $B=C$
- $(KA)^{-1} = \frac{1}{K}A^{-1}$
- $(I_n)^{-1} = I_n$

If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$

Note :
- Num of mul^n of $A_{m\times n} B_{n\times p}$ = mnp
- Num of Add^n of $A_{m\times n} B_{n\times p}$ = $mp(n-1)$

Rank of a matrix :

- A non negative Integer 'r' is said to be rank of matrix A if

(i) there Exist atleast one non-zero minor of order 'r'

(ii) All minors of order $(r+1)$ if they Exist are zeros.

then $\rho(A) = r$

(or)

- The number of non-zero rows in the Echelon form of a matrix can also be observed as Rank of the matrix.

Echelon form :

$A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & 0 & c_3 & d_3 \end{bmatrix}_{3\times 4}$ $\rho(A) = 3$

$B = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\rho(B) = 2$

$C = \begin{bmatrix} a & b & c & d \\ 0 & 0 & e & f \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\rho(C) = 2$

$D = \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\rho(D) = 1$

- Num of leading zero's is Increasing from top to bottom like row number is Increasing
- All zero rows are present below the non-zero rows only ie at the bottom of the matrix.

Any matrix satisfying the above conditions can be observed in Echelon form.

Properties of Rank of a matrix :-

- $1 \le \rho(A)_{m\times n} \le \min(m,n)$
- $\rho(O)_{m\times n} = 0$
- $\rho(I_n) = n$
- $0 \le \rho(AB) \le \min\{\rho(A), \rho(B)\}$
- Imp If $|A| \neq 0$ then $\rho(A) = n$ (n rows are Independent)
- If $|A| = 0$ then $\rho(A) < n$ (some rows are dependent)

Imp $\boxed{\text{Linearly dependent } |A| = 0}$

- $\rho(A) = \rho(A^T)$
- $\rho(A+B) \le \{\rho(A) + \rho(B)\}$
- $\rho(A-B) \ge \{\rho(A) - \rho(B)\}$

Imp If A has rank 'r' then Interchanging of rows in A doesn't affect the rank $(R_i \leftrightarrow R_j)$ $\rho(A) = r$

Imp If A has rank 'r' and If a row is multiplied by non-zero Element then rank will not be affected. $(R_i \to KR_i$ $(K \neq 0)$ $\Rightarrow \rho(A) = r)$

Imp $R_i \to KR_i + R_j \Rightarrow \rho(A) = r$

- To find rank of a given matrix then we must Exclusively apply Row operations / Column operations but not both.

Imp The rank of a diagonal matrix is Equal to the num of non-zero diagonal Elements.

Note :

- In echelon form of a matrix the rows/Cols which are made zero rows/Cols are dependent and the remaining non-zero rows/Cols are independent rows/Cols.

Imp Num of independent rows | Cols | vector of A = rank of A

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & & \end{bmatrix}$$

row vector

Col vector

System of linear Equations :

Non-Homogenous system

- $\boxed{AX = B}$ → Constant col vector

Coeff matrix, var vector

Note :

(i) $\rho(A) = \rho(A|B) = n$ → only one sol

(ii) $\rho(A) = \rho(A|B) < n$ → Infinite sols

(iii) $\rho(A) \neq \rho(A|B)$ → No sol

Inconsistent sol

Homogenous system

- $\boxed{AX = 0}$

Note :

(i) $\rho(A) = n$ → unique sol (zero sol) ($x_1 = x_2 = \dots = 0$) Trivial

(ii) $\rho(A) < n$ → Infinite sols (zero + non-zero)

Trivial, non-Trivial

*) If $A_{n \times n}$ then

Imp If $|A| \neq 0 \Rightarrow \rho(A) = n \Rightarrow$ unique (zero) sol

If $|A| = 0 \Rightarrow \rho(A) < n \Rightarrow \infty$ (non-zero) sol

Imp If $\rho(A_{n\times n}) = r$ in $AX = 0$ then Num of independent sols = $n - r$

LU Decomposition :

Ex:- $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \cancel{a}\ 2 & 1 & 0 \\ \cancel{b}\ -3 & \cancel{c}\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & \cancel{d}\ -1 & \cancel{e}\ 1 \\ 0 & 0 & \cancel{f}\ 1 \end{bmatrix}$

L, U

In L :-

Now,

$2 = a(1) + 1(0) + 0(0)$

$\therefore a = 2$

$-3 = b(1) + c(0) + 1(0)$

$b = -3$

$-10 = b(2) + c(d)$

$-10 + 6 = cd \Rightarrow cd = 4$

$c = +4$ ($\because d = -1$)

In U :

$3\ d = 2(2) + 1(d) + 0(0)$

$d = -1$

$\cancel{3 = 2(1) + 3e + 3f}$

$\cancel{3e + 3f = 1}$

$3 = 2(1) + e$

$e = 1$

$\cancel{2 = 1(-3) - 10e}$

$2 = -3(1) + 4(1) + f$

$f = 1$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -4 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Eigen values and Eigen vectors :

- $\boxed{\text{These properties are applied only on any square matrix}}$
- To find Eigen values of a given matrix $A_{n\times n}$, First we consider the characteristic Equation $|A-\lambda I| = 0$

The roots of this Equations are called characteristics roots | latent roots | propervalues | Eigen values

ie $(A)(x) = \lambda(x)$

$$(A-\lambda I)(x) = 0 \Rightarrow \boxed{|A-\lambda I| = 0} \text{ (Singular)}$$

$\downarrow$ roots $\Rightarrow$ Eigen values

- To find Eigen vectors of the matrix A then

$$\boxed{(A-\lambda I)x = 0}$$

At diff values of λ and get non-zero sols only and those non-zero sols are called Eigen vectors of A

Properties of Eigen values & Eigen vectors :

- Imp: Sum of Eigen values of A = Trace(A)
  sum of diagonals of A
- Imp: product of Eigen values of A = det A
- If $A_{n\times n}$ has n distinct eigen values then a set of 'n' Independent Eigen vectors of $A_{n\times n}$ Exist
- The Eigen values of upper Δ^{lar} / Lower Δ^{lar} / Diagonal matrix are given by its principal diagonal Elements
- If $A_{n\times n}$ has repeated Eigen values. then atmost a set of n Independent Eigen vectors of $A_{n\times n}$ Exist
- If $A = \begin{bmatrix} -2 & 3 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ $|A| = 24$, $\lambda = -2, 4, 3$

(i) $A+2I \Rightarrow 0, 6, 5$

(ii) $A^3 \Rightarrow -8, 64, 27$

(iii) $5A \Rightarrow -10, 20, 15$

(iv) $A^{-1} \Rightarrow -1/2, 1/4, 1/3$

(v) $Adj(A) \Rightarrow -12, 6, 8$

(vi) $A^T \Rightarrow -2, 4, 3$

$A \cdot A^{-1} = I$
$\lambda \cdot (?) = 1$
$\downarrow$
$\frac{1}{\lambda}$

$A \cdot Adj(A) = |A| \cdot I$
$\lambda \cdot (?) = |A| \cdot 1$
$\downarrow$
$\frac{|A|}{\lambda}$

- The Eigen vectors of any real symmetric matrix are orthogonal to Each other.

ie $\boxed{x_i^T x_j = 0 \quad \forall\, i \neq j}$

Ex:- $A \begin{pmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{pmatrix}$

$\boxed{\lambda = 1,}$

- If λ is Eigen value of orthogonal matrix then $\frac{1}{\lambda}$ is also an Eigen value to the same matrix
- If $\boxed{\begin{matrix} a+\sqrt{b} \\ a+ib \end{matrix}}$ is an Eigen value of A then $\boxed{\begin{matrix} a-\sqrt{b} \\ a-ib \end{matrix}}$ is also an Eigen value of A
- The Eigen values of a skew symmetric matrix is Either zero or purely Imaginary
- '0' is an Eigen value iff $\boxed{|A|=0}$

Eigen vectors :

- Eigen vectors of $\left.\boxed{\begin{matrix} A \text{ and } A^m \\ A \text{ and } A^{-1} \\ A \text{ and } KA \end{matrix}}\right\}$ are same
- Eigen vectors of $\boxed{A \text{ and } A^T}$ are not same
- Eigen vector of Symmetric matrix are orthogonal.

Algebraic mutiplicity

- If an Eigen value λ of a square matrix A of order n is repeated m times then the number 'm' is called algebraic multiplicity of an Eigen value λ

If $\lambda = \overbrace{1, 1, 1, 1}^{4 \text{ times}}, 2$

AM of 1 is 4

Geometric multiplicity :

- If 'P' num of linearly Independent Eigen vectors of matrix A of order n' to an Eigen value 'λ' then P is called Geometric multiplicity of λ ie

$$\boxed{\begin{aligned} P &= n - r \\ P &= (\text{num of var}) - \rho(A - \lambda I) \end{aligned}}$$

Cayley Hamilton theorem :

Every square matrix satisfies its own Characteristic Equation.

Ex:- If $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ is CE then

A/c to CH theorem $\boxed{A^3 - 12A^2 + 36A - 32 = 0}$

Applications :

- To find higher powers of matrix A
- To find Inverse of matrix A

X —— The End —— X