Graph theory

ordered pair : $(a,b) \neq (b,a)$
unordered pair : $(a,b) = (b,a)$

Graph : (Def)

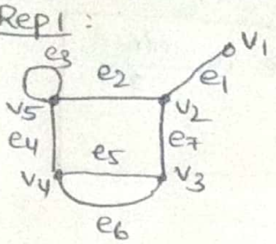
$G=(V,E)$ $V=\{v_1, v_2 \dots v_n\} \rightarrow$ points

$E=\{E_1, E_2 \dots E_n\} \rightarrow$ lines -

Each Edge is associated with unorder pair of vertices (v_i, v_j)

Representation of Graph :

Rep1 :



Rep2 :

$G\{V,E\}$

$V=\{v_1 \text{ to } v_6\}$

$E=\{E_1 \text{ to } E_6\}$

$e_1 \rightarrow (v_1,v_2) / (v_2,v_1)$ $e_4 \rightarrow [v_5,v_4]$

$e_2 \rightarrow (v_2,v_5)$ $e_5 \rightarrow [v_4,v_3]$

$e_3 \rightarrow (v_5,v_5)$ $e_6 \rightarrow [v_4,v_3]$

Rep3

$G=(V,E,\delta)$

$V=\{v_1 \text{ to } v_6\}$

$E=\{E_1 \text{ to } E_6\}$

δ = Transition fn

Terminology :

① End vertices :

Each Edge is associated with unorder vertices called End vertices.

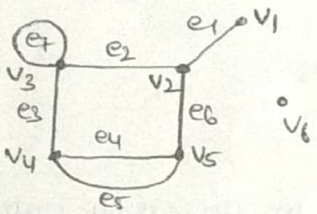
② loop / self loop :

If End vertices are same then that Edge is called loop

③ parallel Edges :

Two Edges share Common End vertices then It is called parallel Edges.

Ex :- ①



$e_4, e_5 \rightarrow$ parallel edges.

$e_7 \rightarrow$ loop

$d(v_1) = 1 \Rightarrow$ pendant vertex

$d(v_2) = 3$

$d(v_3) = 4$

$d(v_4) = 3$

$d(v_5) = 3$

$d(v_6) = 0 \Rightarrow$ Isolated vertex.

④ Incidence :

Meeting point of vertices & edges.

⑤ Degree / valency :

Num of Edges Incident on a vertex

⑥ Null graph

Set of Isolated vertices

⑦ Adjacent

→ vertices (v_1, v_2)

→ Edges (e_1, e_2, e_6)

Note :

Type	Loop	// edges
Simple graph	✗	✗
Multigraph	✗	✓
Pseudo graph	✓	✓

Th1 : sum of degrees of all vertices is Equal to the twice the num of Edges.

$$\boxed{\sum_{i=1}^{n} d(v_i) = 2 * |e|}$$

Th2 : Num of odd vertices in a graph will always be Even

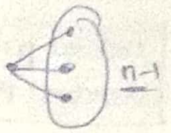
odd vertices ↓ degree odd.

$\text{Some odd vertices} + \underbrace{\text{Some Even vertices}}_{\text{Even}} = \underbrace{2 * e}_{\text{Even}}$

odd —sum↓→ ~~odd~~ ✗

Even ↓sum → Even ✓

Th3: In a simple graph of n vertices then the max possible degree of a vertex is n-1



Th4: In a simple graph of n vertices the max num of possible edges is $\frac{n(n-1)}{2}$

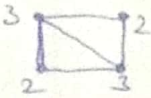

(complete graph)

> **Note**:
> - If a degree of every vertex is n-1, then num of edges will have is $\frac{n(n-1)}{2}$
> - If we have $\frac{n(n-1)}{2}$ edges then the degree of each vertex is n-1

Th5: $\delta(G) \le \frac{2e}{n} \le \Delta(G)$

$\delta(G)$ → min degree; $\Delta(G)$ → max degree

Case 1:



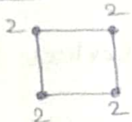
$\delta(G) = 2$

$\frac{2e}{n} = 2.5$

$\Delta(G) = 3$

$\delta(G) < \frac{2e}{n} < \Delta(G)$

Case 2: For regular graph



$\delta(G) = 2$

$\frac{2e}{n} = 2$

$\Delta(G) = 2$

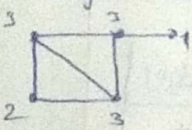
$\delta(G) = \frac{2e}{n} = \Delta(G)$.

> **Note**:
> $\delta(G) \le \frac{2e}{n} \le \Delta(G)$
> atleast — atmost

Degree sequence:

Writing degrees of all vertices either in increasing order or in decreasing order is called degree sequence of a graph.

Ex:-



⇒ 3 3 3 2 1

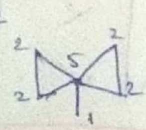
1 2 3 3 3

Ex:- How many edges in {5 2 2 2 2 1}

M1: 5+2+2+2+2+1 = 2e

e = 7

M2:

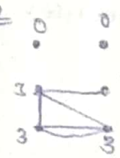


Ex:- How many edges in {3 3 3 1 0 0}

M1: 3+3+3+1+0+0 = 2e

e=5

M2:



Simple graph not possible
Multi graph possible

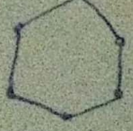
> **Note**:
> - Theorem 1 & 2 - Applicable to all type of graphs
> - Theorem 3 & 4 - Applicable to only simple graphs.

Graphical sequence:

If graph is possible for a degree sequence then that sequence is called Graphical sequence

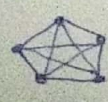
Ex:-

- 2 2 2 2 2 2



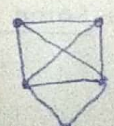
(e=6)
Graphical

- 4 3 3 2 1 — odd vertices is odd ∴ Graph not possible
- 5 4 3 2 1 — Max $\Delta(G) = 4$. Graph not possible
- 4 4 4 4 4 — Complete graph



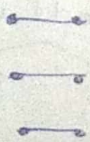
$e = \frac{n(n-1)}{2} = 10$

- 3 3 3 3 2



e=7

- 1 1 1 1 1 1



(e=3)

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If degree sequence is very large then It is difficult to Construct graph

↓ sol

Havel Hakimi

Havel Hakimi theorem :

Procedure : $\cancel{6}$ 3 3 3 3 2 2 1

2 2 2 3 2 2 1 → ordering

→ $\cancel{3}$ 2 2 2 2 2 2 1

1 1 1 2 2 1 → ordering

→ $\cancel{2}$ 2 1 1 1 1

1 0 1 1 1 → ordering

→ $\cancel{1}$ 1 1 1 0

0 1 1 0 → ordering

1 1 0 0 ⟹

Stop until If we are ready to Construct graph

Ex:2 : $\cancel{6}$ 6 6 6 4 3 3 0

$\cancel{5}$ 5 5 3 2 2 0 → ordered

$\cancel{4}$ 4 2 1 1 0 → ordered

$\cancel{3}$ 1 0 0 0 → ordered

0 -1 -1 0

Not possible Graphical Sequence.

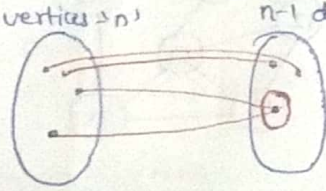
Th6 : In a simple graph atleast two vertices will have same degree $(n \geq 2)$

Proof : Consider a graph having n-vertices

Case1 : $\{0, 1, \ldots n-2\}$

Num of degrees = n − 1

vertices = n , n−1 degrees

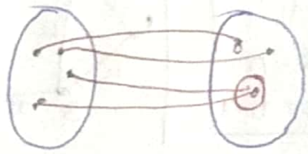


Pigeon hole principle

case 2 : $\{1, 2, \ldots n-1\}$

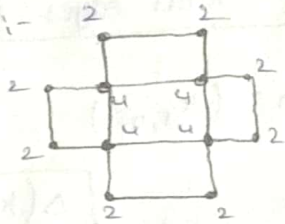
Num of degrees = n − 1

n-vertices , n−1 degrees



Pigeon hole principle.

Ex:-

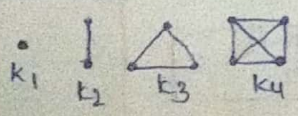


Points to remember :

- Num of odd vertices should be Even
- Max degree should be $\leq n-1$
- If all degrees are distinct then graph is not possible (Th6)
- If sequence is large use Havel Hakemi theorem

Types of Graphs

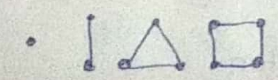
① Complete graph(K_n) $(n \geq 1)$



- $e = \frac{n(n-1)}{2}$
- degree at Each vertex $= n-1$

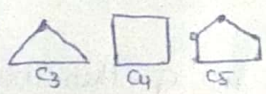
② Regular graph $(n \geq 1)$

- $\boxed{\delta(G) = \frac{2e}{n} = \Delta(G)}$



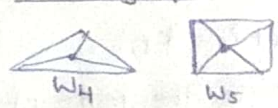
Every K_n is a regular graph but every regular is not K_n

③ cyclic graph $C(n)$ $(n \geq 3)$



- $e = n$
- degree at Each vertex $= 2$

④ wheel graph $W(n)$ $(n \geq 4)$



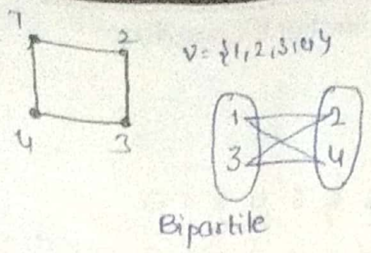
- $e = 2(n-1)$
- degree at Each vertex $= 3$

degree at center $= n-1$

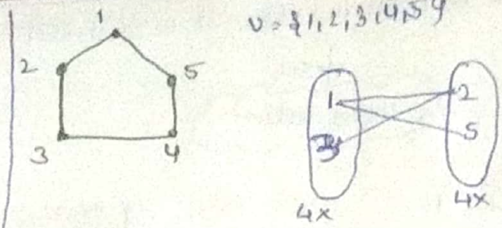
⑤ **Bipartite graph :**

Graph G having vertices V, then V Can be divided into two sets V_1 and V_2 such that Each Edge is from one set to another set but not into Same set.

Ex:-



Bipartile



Not Bipartile.

Note: Even Edges in Cycle → Even len cycle
odd Edges in cycle → odd len cycle

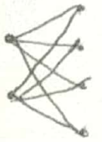
Th 7: Bipartile graph doesn't Contain odd length Cycle.

Note: In any graph if odd len cycle is present then It can't be converted to bipartile graph

⑥ **Complete Bipartile graph**: $K_{m,n}$

$|V_1| = m;\ |V_2| = n$

Ex:- $K_{2,4}$

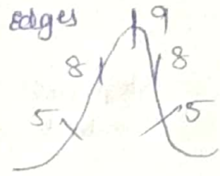


Note:
- $\Delta(K_{m,n}) = \max(m,n)$
- $\delta(K_{m,n}) = \min(m,n)$
- Total vertices $= m+n$
- Total edges $= mn$

* If n is total vertices in Complete bipartile graph then what will be max num of Edges

Ex:- $n = 6$

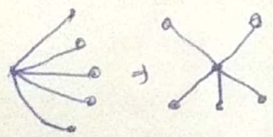
$K_{1,5}$ ⑤ $K_{2,4}$ ⑧ $K_{3,3}$ ⑨ $K_{4,2}$ ⑧ $K_{5,1}$ ⑤



If n is total vertices of Complete bipartile graph then max num of Edges $\left\lfloor \frac{n^2}{4} \right\rfloor$

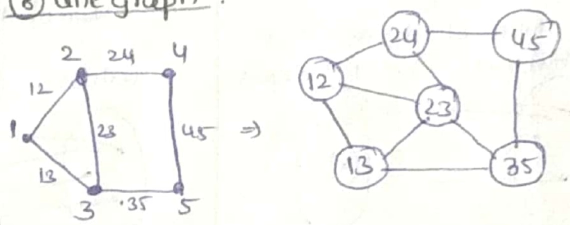
⑦ **Star graph**: $(K_{1,n-1})$

Ex: $K_{1,5}$

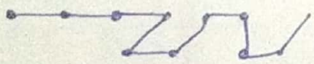


- $\Delta(K_{1,n-1}) = n-1$
- $\delta(K_{1,n-1}) = 1$
- Total vertices $= n$
- Total Edges $= n-1$

⑧ **Line graph**:



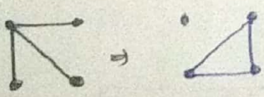
⑨ **path graph**:



⑩ **Compliment graph** $(\bar{G})$

$\boxed{\bar{G} = K_n - G}$

Ex:-



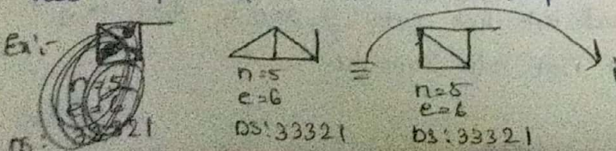
K_n	3	3	3	3
$G \Rightarrow$	3	1	1	1
$\bar{G} \Rightarrow$	0	2	2	2

Note:
- $G + \bar{G} = K_n$
- e is the num of Edges in G then num of Edges in $\bar{G}$ is $\boxed{\frac{n(n-1)}{2} - e}$
- $d_1, d_2 \cdots d_k$ is the degree sequence of G then $(n-1)-d_1, (n-1)-d_2 \cdots (n-1)-d_k$ is the degree sequence of $\bar{G}$

⑪ **Isomorphic graphs**:

Two Graphs G_1 and G_2 are isomorphic to Each other if they have Same incidence property
- → Same num of vertices
- → Same num of edges
- → Same degree seq

Ex:-



Note:
- Inspite of having same num of degree sequence, edges, vertices, two graphs will not gurantee Isomorphism

Ex:-

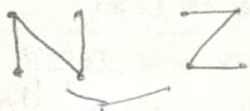
$n=6$, $e=5$, DS: 3 2 2 1 1 1 $\neq$ $n=6$, $e=5$, DS: 3 2 2 1 1 1

Not Isomorphic.

(12) Self Complimentary graph:

If any graph is isomorphic to its own compliment then the graph is called Self complimentary graph.

ex:-



Self Complimentary.

Note:
$$G+G' = K_n$$
$$e+e = \frac{n(n-1)}{2}$$
$$e = \frac{n(n-1)}{4}$$
$\therefore$ Num of Edges in Self Comp graph $= \frac{n(n-1)}{4}$

*) $n=4$

$e = \frac{4*3}{4} = 3$ (self Comp possible)

$n=5$

$e = \frac{5*4}{4} = 5$ (self Comp ✓)

$n=6$

$e = \frac{6\times5}{4} = (\,).(\,)$ (self Comp ×)

$\xrightarrow{\text{Generalize}}$

$\frac{n}{4}$ or $\frac{n-1}{4} \in Z$

$\downarrow$ $n \equiv 0 \bmod 4$ $\quad$ $\downarrow$ $n \equiv 1 \bmod 4$

$\therefore$ $\boxed{n \equiv (0 \text{ or } 1) \bmod 4}$

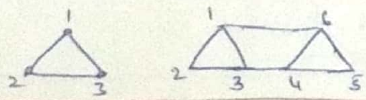
To find a graph is self Comp or not.

Graph functions:

(1) subgraph ($\subset$)

g is called sub graph of G if all the vertices and Edges of $g \in G$

Ex:-



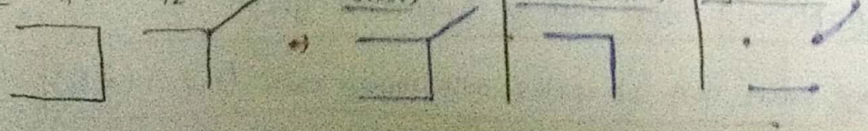
Note
- Single vertex is always sub graph of G
- Single Edge is always subgraph of G
- $x \subset g \subset G$ $\Rightarrow$ subgraph of a subgraph of a graph is subgraph of graph.

(2) Operations.

Union ($\cup$)	Intersection ($\cap$)	Sym diff ($\oplus$)
Let $G_3(V_3, E_3)$	Let $G_3(V_3, E_3)$	Let $G_3(V_3, E_3)$
$G_3 = G_1 \cup G_2$ then	$G_3 = G_1 \cap G_2$ then	$G_3 = G_1 \oplus G_2$ then
$V_3 = V_1 \cup V_2$	$V_3 = V_1 \cap V_2$	$V_1 = V_1 \cup V_2$
$E_3 = E_1 \cup E_2$	$E_3 = E_1 \cap E_2$	$E_3 = (E_1 \cup E_2) - (E_1 \cap E_2)$

Note: $G \cup G = G$
$G \cap G = G$
$G \oplus G$ = Null graph

Ex:- G_1 $\quad$ G_2 $\quad$ union $\quad$ Intersection $\quad$ Sym diff

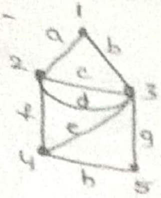


Connectivity:

Walk: Alternating sequences of vertices & edges.
Trail: Alternating sequences of vertices & Edges
Path: Alternating sequences of vertices & Edges

Note:	Repeated vertices	Repeated Edges
Walk	✓	✓
Trail	✓	✗
Path	✗	✗

Ex:-



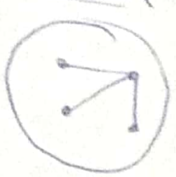
Walk: 1 b $\overbrace{3\ \underbrace{c\ 2\ c}_{RE}\ 3}^{RV}$ d 2 f 4

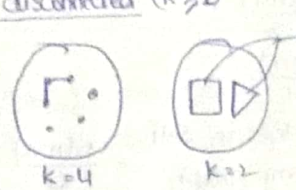
Trail: 1 b $\underbrace{3\ c\ 2\ d\ 3}_{RV}$ e 4

Path: 1 b 3 c 2 f 4

*) Graph

Connected (k=1) | disconnected (k≥2)



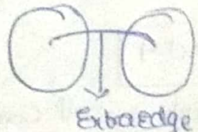


Note:	
Graph	Range of Edges
Connected (k=1)	$n-1 \le e \le \frac{n(n-1)}{2}$ (K_n) (Tree)
disconnected (k≥2)	$n-k \le e \le \frac{(n-k+1)(n-k)}{2}$

*) Tree:
- Graph having n vertices and n-1 Edges is called a tree
- minimally connected
- doesn't contain cycle

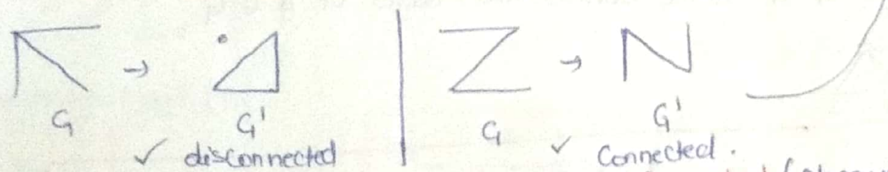
Ex:-

Note:
– If a graph having more than $\frac{(n-1)(n-2)}{2}$ then it will be necessarily connected

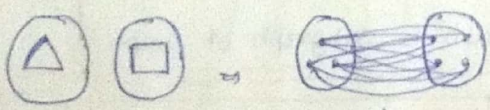


Th 8: If G is connected then $\bar{G}$ will be disconnected (may | may not)

Ex:

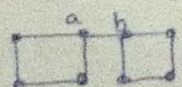


Th 9: If G is disconnected then $\bar{G}$ will be connected (Always true)



Cut edge: (bridge)
Removal of single edge from a graph will make graph as disconnected graph.

Ex:-

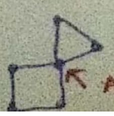


(ab) is Cut Edge

Note:
- If Cut edge will exist then it will not belong to cycle.

Cut vertex: (Articulation point | Cut point)
Removal of single vertex from a graph will make graph as disconnected graph

Ex:-



Note:
If Cut edge Exists then Cut vertex will always exist ($n \ge 3$) ($\ne K_2$)

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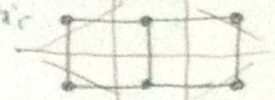
Cut Edge Set:

Removal of set of Edges from a graph will make graph as disconnected graph

Edge Connectivity $[\lambda(G)]$

min num of Edge removal of a graph will make graph as disconnected graph.

Ex:



$\lambda(G) = 2$

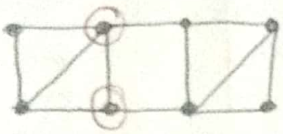
Cut vertex set:

Removal of set of verticies of a graph will make graph as disconnected graph

Vertex Connectivity: $[k(G)]$

Min num of vertices removal of a graph will make graph as disconnected graph

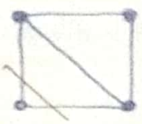
Ex:-



$k(G) = 2$

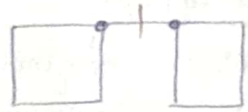
Th 10: $\lambda(G) \le \delta(G)$

Case(i):



$\delta(G) = 2$
$\lambda(G) = 2$

Case(ii):

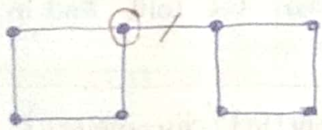


$\delta(G) = 2$
$\lambda(G) = 1$

$\therefore \boxed{\lambda(G) \le \delta(G)}$

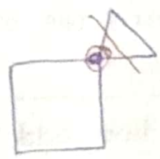
Th 11: $k(G) \le \lambda(G)$

Case (i):



$k(G) = 1$
$\lambda(G) = 1$

Case(ii):



$k(G) = 1$
$\lambda(G) = 2$

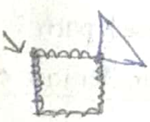
$\therefore \boxed{k(G) \le \lambda(G)}$

Note:

$$\boxed{k(G) \le \lambda(G) \le \delta(G) \le \frac{2e}{n} \le \Delta(G) \le n-1}$$

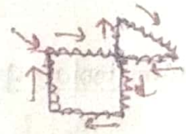
Euler graph:

Trail: RV ✓ RE ×



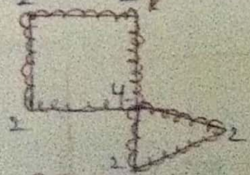
closed trail: If starting and Ending vertices are same then It is Closed trail

Euler circuit/cycle: If any closed trail Covers all the Edges Exactly once then that trail is called Euler circuit/cycle

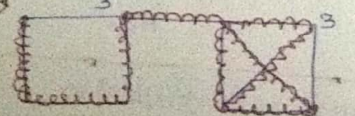


$\boxed{\text{Closed trail + Cover all Edges once}}$

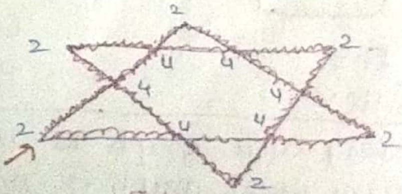
Euler graph: Any graph having Euler ckt then that graph is called Euler graph.



Euler graph

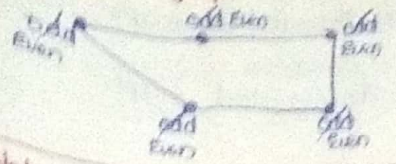


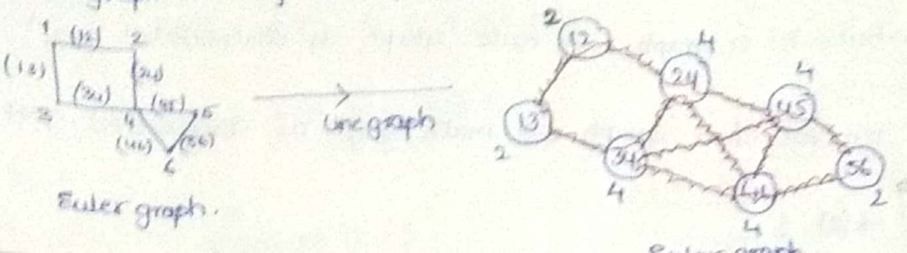
Not Euler graph



Euler graph.

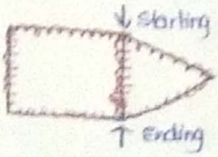
Th12 : Euler graph will only be possible if degree of all the vertices are Even.



> **Note** :
> The line graph of Every Euler graph will always be Euler graph.
>
> 
>

Euler line / Euler path :
Euler line is an open trail that covers all the Edges in the graph

Ex:-

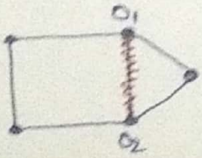


> **Note** : Cover all Edges
> Closed trail → Euler ckt
> open trail → Euler line / Euler path

Th13 : Euler line / Euler path will only be possible if a graph having Exactly two odd vertices.

> **Note** :
> In Euler line if we are starting from one odd vertex then we will End in another odd vertex

Th14 : If in a graph having Exactly two odd vertices (connected / disconnected) then there is always a path between them



Hamiltonian graph :

Path : Alternating sequence of vertices & Edges (RV×, RE×)

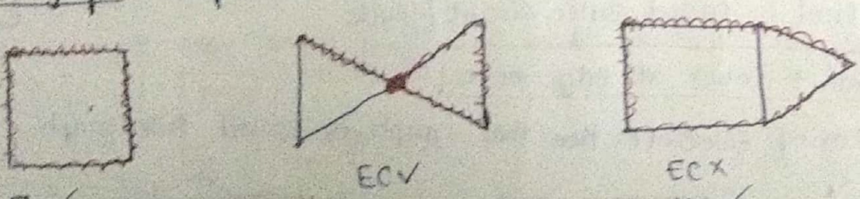
Closed path : In path if starting and ending vertex are same is called as Closed path.

Hamiltonian circuit / cycle : Closed path that covers every vertex once.

> Closed path + Cover all vertices

Hamiltonian graph : Graph that contains Hamiltonian cycle.

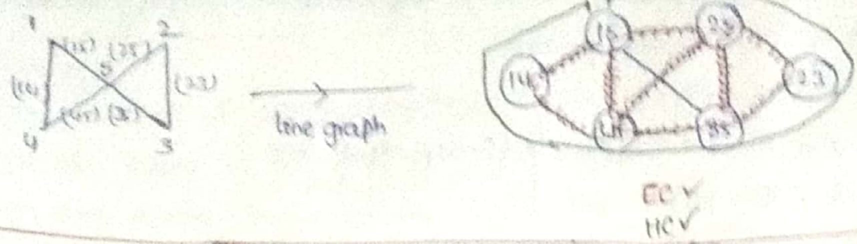
Ex:



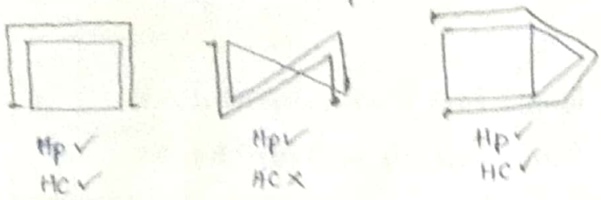
EC✓ HC✓ | EC✓ HC✗ | EC✗ HC✓

> **Note** : Every cyclic graph is both Euler and Hamiltonian
> Every wheel graph is Hamiltonian but not Euler
> Every Complete graph is Hamiltonian and for $K \geq 4$ Every K_n is not Euler

Note:
The line graph of Every Euler graph will always be Hamiltonian as well as Euler graph.



Hamiltonian path: open path that covers all the vertices of a graph exactly once is called Hamiltonian path.

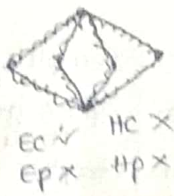


Note:
Every Hamiltonian ckt contains Hamiltonian path but every Hamiltonian path doesn't contain Hamiltonian ckt

Summary:

- Cover
  - All vertices
    - open path → Hamiltonian path
    - Closed path → Hamiltonian ckt ⇑ Hamiltonian graph
  - All Edges
    - open path → Euler line/path
    - Closed path → Euler ckt ⇑ Euler graph

Note:
For symmetric graphs HC and Hp doesn't Exist



Coloring:

properly Coloring: painting all the vertices with colors such that adjacent shouldn't have same color.

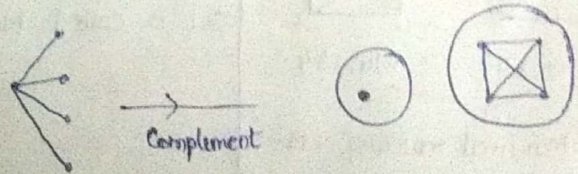
chromatic num ($\chi(G)$)
Min num of colors req'd to paint all the vertices in a graph is called chromatic number of a graph.

Points to remember

Graph	Chromatic num
Isolated	1
Tree	2
Cyclic (C_n)	2 (Even), 3 (odd)
Wheel (W_n)	3 (odd), 4 (Even)
Complete bipartite	2
Star	2
Complete (K_n)	n

Note:
Every 2 chromatic graph can be convertible to bipartite graph

Note:
$K_{1,n-1} = K_{n-1} + \text{Isolated vertex}$



W_4

$\chi(W_4) = 4$

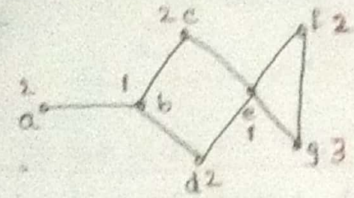
W_5

$\chi(W_5) = 3$

Partitioning problem:

If you paint all the vertices of the graph with different colors and if we put the same color vertices in one set then it is partitioning problem

Ex:



$1 \Rightarrow A = \{b, e\}$
$2 \Rightarrow B = \{a, c, d, f\}$
$3 \Rightarrow C = \{g\}$
} Independent sets.

3 partitions

Independent sets:

- Set of non-adjacent vertices
- Maximal Independent set:

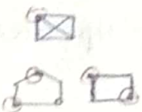
If we take $\{a\}$ we can add $\{c\}$ to form $\{a, c\}$ → Independent set
$\{a, c\}$ we can add $\{d\}$ to form $\{a, c, d\}$ → Independent set
$\{a, c, d\}$ we can add $\{f\}$ to form $\{a, c, d, f\}$ → Independent set
$\{a, c, d, f\}$ If we add $\{g\}$ to form $\{a, c, d, f, g\}$ → not Independent set

∴ Maximal Independent set (MIS) = $\{a, c, d, f\}$

Note:
A graph can contain many maximal independent set like $\{a, c, d, f\}$, $\{b, e\}$, $\{b, f\}$, $\{b, g\}$

largest MIS
⇓
Independence num i.e 4

Note:
Independence num for complete graph = 1



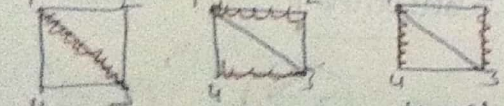
Independence num for cyclic graph $(n \geq 4)$ = 2
Independence num for Complete bipartite graph $K_{m,n}$ = $\max(m, n)$
Independence num for stargraph $K_{1,n-1}$ = $n-1$
Independence num for wheel graph = $\lceil \frac{n}{2} \rceil - 1$

→ $W_4 \Rightarrow 1$
→ $W_5 \Rightarrow 2$
→ $W_6 \Rightarrow 2$
→ $W_7 \Rightarrow 3$
→ $W_8 \Rightarrow 3$
→ $W_9 \Rightarrow 5$

Matching:

Nonadjacent
- vertices → Independence set
- Edges → matching

Matching set:
set of non adjacent Edges.



MS: $\{13\}$ $\{12, 34\}$ $\{14, 23\}$
→ maximal matching sets

Matching number = 2

Maximal matching set:
It is a matching set in which we cannot add any new Element

Note: A graph may contain many maximal matching set

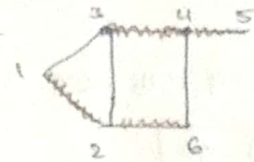
Matching number:
The number of Edges in the largest maximal matching set is called Matching number

Note: $C_3 + W_4 = K_7$
- Matching number for Complete graph $(K_n) = \lfloor \frac{n}{2} \rfloor$

Covering:
- It is the set of edges such that each vertex is incident on atleast one edge
- All the edges in the graph is by default a covering set
- We can remove some of the edges still it would be the covering set
- But at one point we cannot remove edges from it then that set is called

minimal covering set

Ex:-



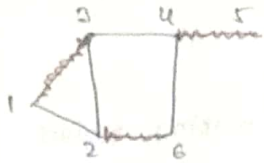
CS: { ~~13~~, 12, 23, 34, 46, 45 }

CS: { ~~13~~, 12, 34, 45, 26 }

{ 12 34 45 26 }

↓

Minimal Covering set



CS: { 13, ~~12~~, 34, 45, 26 }

{ 13, ~~34~~, 45, 26 }

{ 13, 45, 26 }

↓

Minimal Covering set.

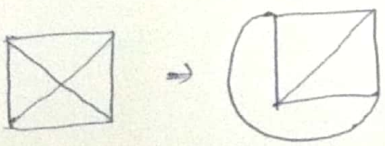
Covering num = 3.

Covering number:

Number of edges present in the smallest minimal covering set is called covering number of a graph

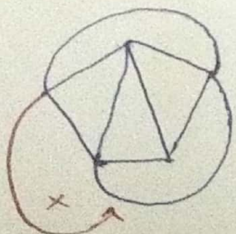
Planarity:
Planar graph: If a graph can be drawn without intersection of its edges then the graph is called planar graph.

Ex:-



K_4

Th 15: K_5 is non-planar

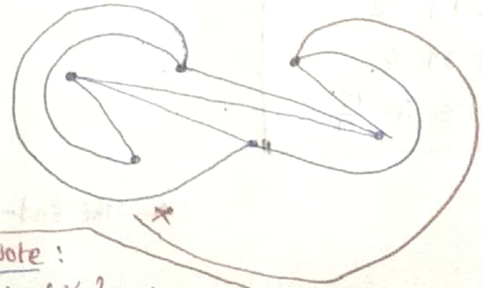


Note:

$n = 5V$
$e = 9e$ } planar

$+1e$ → Non-planar

Th 16: $K_{3,3}$ is non-planar



Note:

$n = 6V$
$e = 8e$ } → planar

$+1e$ → Non-planar

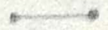
Note:
- Both are regular graphs
- Removal of single edge from the graph will make as planar graph.
- First graph having min num of vertices
- Second graph having min num of edges.

Eulers Formula: (Applicable only to planar graphs)
- when ever we draw a graph on a plane it provides region based on closed edges

region → Bounded (R_1, R_2), Unbounded (R_3)

Th 17: In a graph, n is the total num of vertices, e is num of Edges and f is regions / faces then $\boxed{n-e+f=2}$

Case 1:



$2-1+f=2$

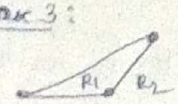
$f=1$

Case 2:

$3-2+f=2$

$f=1$

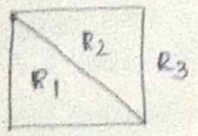
Case 3:



$3-3+f=2$

$f=2$

Conditions to check planarity:



$d(R_i)$ = num of edges present while making region

$d(R_1)=3$; $d(R_2)=3$; $d(R_3)=4$

$\Downarrow$

$d(R_1)\geq 3 \quad d(R_2)\geq 3 \quad d(R_3)\geq 3$

$d(R_1)+d(R_2)+d(R_3)\geq 3\cdot 3$

$\Sigma d(R_i)\geq 3f$

$\boxed{2e\geq 3f} \Rightarrow \boxed{e\leq 3n-6}$

$\Sigma d(R_i) = 3+3+4 = 10 = 2e$

$\boxed{\Sigma d(R_i)=2e}$

we know that

$n-e+f=2 \rightarrow f=2-n+e$

$\delta(G)\leq \frac{2e}{n}$

$\delta(G)\leq \frac{2(3n-6)}{n}$

$\boxed{\delta(G)\leq 6-\frac{12}{n}}$

Note:

For planar graph

(I) $\Sigma d(R_i)=2e$

(II) $2e<3f$

(III) $n-e+f=2$

(IV) $e\leq 3n-6$

(V) $\delta(G)\leq 6-\frac{12}{n}$

×— The End —×