Probability & Statistics :

Random Experiment : unpredictable outcome of the experiment

Ex:- Tossing a unbiased coin
Rolling a six faced die
Drawing a card from pack of 52

Sample space (S) : The collection of all possible outcomes of the random experiment

Event (E) : The outcomes of the experiment is known as event $\boxed{E \subseteq S}$

Definition of probability :

The prob of an event is defined as the ratio between the favourable cases to the event and the total num of ways in the experiment

Note ; outcomes of an experiment are mutually exclusive → (disjoint) ~~(independent)~~
Equally likely → (Equal prob of occurence)
Exhaustive. → (max num of possibilities of a experiment)

$\therefore \boxed{P(E) = \frac{m}{n} ; m \le n}$

Note :
$P(E) = 0$ → uncertain event
$0 < P(E) < 1$ → chanced event
$P(E) = 1$ → certain event

Note :
- $\boxed{P(S) = 1}$
- $\boxed{P(\bar{A}) = 1 - P(A)}$
- $\boxed{0 \le P(E) \le 1}$
- $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$; iff E_i's are disjoint (or) ME

$P(E_1 \cup E_2 \cup \dots \cup E_n) = P(E_1) + P(E_2) + \dots + P(E_n)$

$\therefore \boxed{P(E_1) + P(E_2) + \dots P(E_n) = 1}$

Note: Tossing a coin 2 times

HH	HT	TH	TT

2 coins tossed single time

H/T T/H

Types of Events

Dependent : occurence of an event is associating with other events within the trail

Mutually Exclusive (ME) : occurence of an event prevents to happen other event within the trail

A B

$A \cap B = \phi$
$P(A \cap B) = 0$

Independent Event : occurence of an event doesn't depends upon the occurence of some event in a different trail

A B

$\boxed{P(A \cap B) = P(A) \cdot P(B)}$

Note : ME events never be independent and independent events never be ME

Addition theorem : If A & B are two events then $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

Dependent :
$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B/A)$
$P(A \cup B) = P(A) + P(B) - P(B) \cdot P(A/B)$

Mutually Exclusive
$P(A \cup B) = P(A) + P(B)$
$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Independent Event :
$\boxed{P(A \cup B) = P(A) + P(B) - P(A)P(B)}$

Product rule : If A & B are two events then $P(A \cap B)$ can be defined as

Dependent
$P(A \cap B) = P(A) \cdot P(B/A)$
$P(A \cap B) = P(B) \cdot P(A/B)$
unknown event
known event
Conditional probability

Mutually Exclusive
$\boxed{P(A \cap B) = 0}$

Independent Event :
$\boxed{P(A \cap B) = P(A) \cdot P(B)}$

If A & B are two dependent Events

- $\boxed{P(A\cap B') = P(A) - P(A\cap B)}$ → only A
- $\boxed{P(A'\cap B) = P(B) - P(A\cap B)}$ → only B
- $\boxed{P(A'\cap B') = P(\overline{A\cup B}) = 1 - P(A\cup B)}$ → none of AB, neither A nor B
- $P(A\Delta B) = P(A\cap B') + P(A'\cap B)$ → Exactly once, only once
  $= P(A) + P(B) - 2P(A\cap B)$
  $= P(A\cup B) - P(A\cap B)$
- $P(A\cap B)$ → both
- $P(A\cup B)$ → Either A or B, atleast one
- $\underbrace{P(A'\cap B')}_{0} + \underbrace{P(A\Delta B)}_{1}$ → Atmost one

$= 1 - \cancel{P(A\cup B)} + \cancel{P(A\cup B)} - P(A\cap B)$

$= 1 - P(A\cap B)$

$\therefore \boxed{P(A'\cap B') + P(A\Delta B) = 1 - P(A\cap B)}$

- $P(A'/B) = \dfrac{P(A'\cap B)}{P(B)} = 1 - \dfrac{P(A\cap B)}{P(B)} = 1 - P(A/B)$

$P(A'/B) = 1 - P(A/B)$

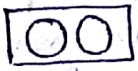
$P(A/B) + P(A'/B) = 1$

- $P(A/B') = \dfrac{P(A\cap B')}{P(B')} = \dfrac{P(A) - P(A\cap B)}{1 - P(B)}$, $P(B) \neq 1$
- $P(A'/B') = \dfrac{P(A'\cap B')}{P(B')} = \dfrac{1 - P(A\cup B)}{1 - P(B)}$; $P(B) \neq 1$

$\boxed{P(A/B') + P(A'/B') = 1}$

Note: Sum of the reverse probabilities of all the events $(E_1, E_2 \ldots E_n)$ are Equal to 1

If A and B are ME and Exhaustive Events
↓
$P(A) + P(B) = 1$


ME

- $P(A\cap B') = P(A)$
- $P(A'\cap B) = P(B)$
- $P(A'\cap B') = 1 - P(A\cup B) = 1 - 1 = 0$
- $P(A\Delta B) = P(A\cap B') + P(A'\cap B) = P(A) + P(B) = 1$

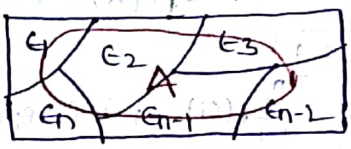
$\boxed{P(A\Delta B) = 1}$

If A and B are Independent Events:

- $P(A\cap B') = P(A) - P(A\cap B) = P(A)\cdot P(B')$
- $P(A'\cap B) = P(B) - P(A\cap B) = P(A')P(B)$
- $P(A'\cap B') = P(A')P(B')$
- $P(A\Delta B) = P(A'\cap B) + P(A\cap B')$

$\boxed{P(A\Delta B) = P(A')P(B) + P(A)P(B')}$

Baye's theorem: If $E_1, E_2 \ldots E_n$ are ME Events $(P(E_i) \neq 0)$ such that A is an Arbitary event which is a subset of "$\bigcup_{i=1}^{n} E_i$" then $P(A)$ is



$A = (E_1\cap A) \cup (E_2\cap A) \cup \ldots \cup (E_n\cap A)$

$P(A) = P(E_1\cap A) + P(E_2\cap A) + \ldots + P(E_n\cap A)$

$P(A) = \sum_{i=1}^{n} P(E_i\cap A) = \sum_{i=1}^{n} P(E_i)\cdot P(A/E_i)$

Now, Inverse prob.

$$P(E_i/A) = \frac{P(E_i\cap A)}{P(A)} = \frac{P(E_i)\cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\cdot P(A/E_i)}$$

- Baye's theorem is applied to known Events
- Min 2 ME Events should be there to apply Baye's theorem

From

Ex:- Rolling a dice

Prime	
O	E

$P(O) = \frac{3}{6}$; $P(E) = \frac{3}{6}$; $P(\text{Prime}) = \{2,3,5\}$

$P(O/\text{Prime}) = \frac{2}{3}$ $P(E/\text{Prime}) = \frac{1}{3}$

$P(O/\text{Prime}) + P(E/\text{Prime}) = 1$

Standard problems:

1 Coin → 2^1
2 coins → 2^2
⋮
n coins → 2^n

1 die → 6^1
2 dice → 6^2
⋮
n dice → 6^n

AND	product	$\cap$
OR	sum	$\cup$

When to apply Add^n theorem:
- Either / or
- atleast one

When to apply product rule
- AND
- simultaneously
- Alternatively
- one after other

Face cards

	K_{13}	Q_{12}	J_{11}	
red H	1	1	1	3
D	1	1	1	3
C	1	1	1	3
S	1	1	1	3
Black	4	4	4	12 Face Cards

Atleast	min	$\geq$	min to max
Atmost	max	$\leq$	max to min

Cards (52)
H (13) D (13) C (13) S (13)

Shortcut for coins:

Atleast one Tail & Atmost 2 Heads in 4 Coins

$$\begin{Bmatrix} {}^4C_2^H = 6 = {}^4C_2^T \\ {}^4C_1^H = 4 = {}^4C_3^T \\ {}^4C_0^H = 1 = {}^4C_4^T \end{Bmatrix}$$

Imp

$\boxed{11} \to$ fav cases

Total $\to 2^4 \to 16$

Prob $= \frac{11}{16}$

Atleast 2 Head & Atleast 2 Tail in 6 Coins

$$\begin{Bmatrix} {}^6C_2^H = 15 = {}^6C_4^T \\ {}^6C_3^H = 20 = {}^6C_3^T \\ {}^6C_4^H = 15 = {}^6C_2^T \end{Bmatrix}$$

Imp

$\boxed{50}$ fav

Total $2^6 = 64$

Prob $= \frac{50}{64} = \frac{25}{32}$

Note:

- ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$
- ${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots + {}^nC_n = 2^{n-1}$
- ${}^nC_1 + {}^nC_3 + {}^nC_5 + \dots + {}^nC_n = 2^{n-1}$

Random Variables and Expectation:

Random variable: Connecting the out comes of the random experiment with real values is known as Random variable and its data is known as univariate data.

- It is represented Mathematically as

$$\boxed{X : S \Longrightarrow \mathbb{R}}$$

X → RV, S → Sample space, $\mathbb{R}$ → Real num

Note: Random variables are always real

Types of Random variables:

- **Discrete RV** (Integers)
  - Probability Mass function $P(x)$
  - Discrete distribution
    - Binomial
    - Poisson
- **Continuos RV** (Fractions)
  - Probability density fn $F(x)$
  - Continuos distribution
    - Uniform
    - Exponential
    - Normal

Note:

- pmf defines the probabilities at Integers
- pdf defines the probabilities b/w the points as a area since it is piece wise Continuos fn
- * Pmf : Σ (Summation) $\to P(X=x_0)$
- * Pdf : $\int$ (Integration) $\to P(a<x<b)$

Expectation (Mean) $E(x)$

$$\boxed{\begin{aligned} \bar{X} = E(x) &= \sum_{x=0}^{n} x \cdot p(x) \quad ; \text{ If } x \text{ is a Discrete RV} \\ &= \int_{-\infty}^{\infty} x f(x)\,dx \quad ; \text{ If } x \text{ is a Continuos RV} \end{aligned}}$$

Note:

$$\boxed{\sum_{x=0}^{n} p(x) = 1}$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

Variance $V(x)(\sigma^2)$

- $\sqrt{\text{variance}}$ = standard deviation
- $V(x) = E(x-E(x))^2 = E(x^2) - (E(x))^2$
- $$\boxed{\begin{aligned} \sigma^2 = V(x) &= \sum_{x=0}^{n} x^2 p(x) - \left(\sum_{x=0}^{n} x\, p(x)\right)^2 \quad ; \text{ If } x \text{ is a discrete RV} \\ &= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2 ; \text{ If } x \text{ is a Continuos RV} \end{aligned}}$$

Properties of Expectation :

If 'x' is a RV and 'a' is a Constant

- $E(ax) = aE(x)$
- $E(X \pm Y) = E(x) \pm E(Y)$
- $E(a) = a$
- $E(ax+b) = aE(x)+b$
- $E(E(E\cdots E(x))\cdots) = E(x)$
- If x and y are Independent RV then
  $E(x \cdot y) = E(x) \cdot E(y)$

Properties of variance :

If x is a RV and a is a constant

- $V(ax) = a^2 V(x)$
- $V(x \pm y) = V(x) + V(y)$
- $V(a) = 0$
- $V(ax+b) = a^2 V(x)$
- $V(ax-by) = a^2 V(x) + b^2 V(y)$
- $V\left(\frac{x}{a} - \frac{y}{b}\right) = \frac{1}{a^2} V(x) + \frac{1}{b^2} V(y)$

Note : variance is always non-negative ie $V(x) \geq 0$

$E(x^2) - (E(x))^2 \geq 0 \Rightarrow E(x^2) \geq (E(x))^2$

$E(x^2) = (E(x))^2$; x is a const

$E(x^2) > (E(x))^2$; x is a var

Note : The Expectation and variance for sum on the number of dice are.

$E(x) = \frac{7}{2} n \qquad V(x) = \frac{35}{12} n$

Note : If Every RV has same probability then

$E(x) = \frac{n+1}{2} \qquad V(x) = \frac{n^2-1}{12}$

- sum of Even num (0 Included) - $n^2 - n$
- sum of Even natural num - $n^2 + n$
- sum of odd nums - n^2

Probability distributions :

① Binomial distribution :

If x is said to be a binomial RV, It allows values from 0 to n with parameters n(num of repetitions), P(success prob) then

$$\text{Pmf} \rightarrow B(x, n, p) = p(x) = \begin{cases} {}^{n}C_{x}\, p^{x} q^{n-x} & ; \; 0 \leq x \leq n, \; p+q=1 \Rightarrow q = 1-p \\ 0 & ; \; \text{otherwise} \end{cases}$$

Mean $= E(x) = np$

Variance $(V(x)) = npq$

When to apply?

- If observations are Independent (drawn by the 'with replacement technique')
- probability of success is Constant
- Mean should be greater than variance.

② Poisson distribution :

If x is said to be a poisson RV defined in the interval 0 to ∞ with the parameter 'λ' ($\lambda > 0$) then

$$\text{Pmf} : P(x, \lambda > 0) = p(x) = \begin{cases} \frac{e^{-\lambda} \lambda^{x}}{x!} & ; \; \lambda > 0, \; 0 \leq x \leq \infty \\ 0 & ; \; \text{otherwise} \end{cases}$$

Mean $= E(x) = \lambda$

Variance $= V(x) = \lambda$

When to Apply ?

Keywords

- Arrival rate
- defect item
- Time dependent
- rare occurrences

- observations are infinitely large
- prob of success is very small
- $np = \lambda \Rightarrow \boxed{p = \frac{\lambda}{n}}$

Note : sum of Independent poisson RV is also a poisson RV

$X \rightarrow \lambda_1 \qquad Y \rightarrow \lambda_2$

$$P(x+y) = \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^{x+y}}{(x+y)!}$$

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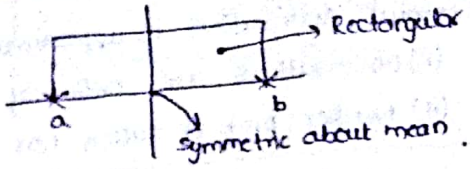
③ Uniform distribution :

If x is a uniform RV defined in (a,b) and $a<b$ then

$$\boxed{\text{Pdf} \rightarrow U(a,b) = f(x) = \begin{cases} \frac{1}{b-a} & ; a<x<b \\ 0 & ; \text{otherwise} \end{cases}}$$

$$\boxed{\begin{aligned} \text{Mean} &= E(x) = \frac{a+b}{2} \\ \text{Variance} &= V(x) = \frac{(b-a)^2}{12} \end{aligned}}$$

Note :
- Every interval has its own prob density function
- The shape of uniform RV curve is rectangular and it is symmetric about mean



④ Normal distribution :

If x is said to be Normal RV defined in $(-\infty,\infty)$ with mean (μ) and variance (σ^2) then

$$\boxed{\text{Pdf} \rightarrow N(x,\mu,\sigma^2) = f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} & ; \begin{matrix} -\infty<x<\infty \\ -\infty<\mu<\infty \\ 0<\sigma<\infty \end{matrix} \\ 0 & ; \text{otherwise} \end{cases}}$$

→ ∴ Probability can't be determined directly
↓ sol
Standard Normal RV

Standard Normal RV :

If 'z' is a RV with mean '0' and variance '1' then

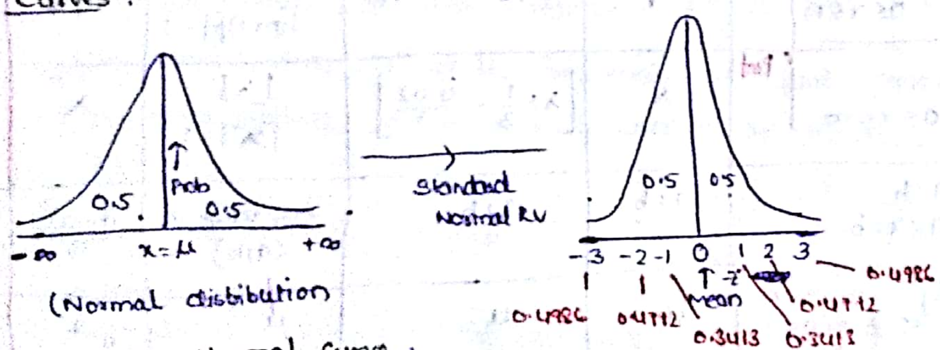
$$\boxed{\text{Pdf} \rightarrow N(z,0,1) = f(z) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} & ; -3 \le z \le 3 \\ 0 & ; \text{otherwise} \end{cases}}$$

→ Just remember

Mathematically

$$\boxed{z = \frac{x-E(x)}{\sqrt{V(x)}} \quad \& \quad -3 \le z \le 3}$$

Curves :



Solving problems :

If x is Given

① find z

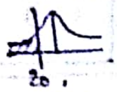
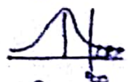
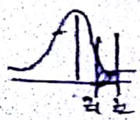
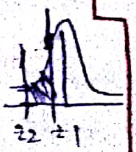
$z = \frac{x-E(x)}{\sqrt{V(x)}}$ & z is present in $-3 \le z \le 3$

② find $P(z)$ using Areas under Normal curve

Note :

$$\boxed{\begin{aligned} &\mu-\sigma < x < \mu+\sigma \\ &P(\mu-\sigma < x < \mu+\sigma) = 0.66 \end{aligned}}$$

Areas under Normal curve :

- $P(z \le z_0)$ → $0.5 + \text{Area}(z_0(+ve))$  z_0
  → $0.5 - \text{Area}(z_0(-ve))$  z_0
- $P(z \ge z_0)$ → $0.5 - \text{Area}(z_0(+ve))$  z_0
  → $0.5 + \text{Area}(z_0(-ve))$ 
- $P(z_1 < z < z_2)$ → $A_1 + A_2$  z_1 z_2 → $z_1(-ve)$, $z_2(+ve)$
  → $A_2 - A_1$ $\left[\begin{matrix} z_1 \& z_2 \rightarrow +ve \\ z_1 \& z_2 \rightarrow -ve \end{matrix}\right]$  z_1 z_2  z_2 z_1

Note :

- Normal Random variable is symmetric about mean
- The shape of curve in Normal distribution is bell shaped

Imp - Normal probabilities at a particular points are zero.

- Sum and differences between the Normal RV is also a Normal RV
- Binomial distribution is approximated to Normal if
  - (i) observations are infinitely large
  - (ii) Neither prob of success nor prob of failure are very small

Mean = μ

$= \int_{-3}^{3} z f(z)\, dz$

variance = σ^2

$= \int_{-3}^{3} z^2 f(z) - \mu^2$

⑤ Exponential distribution :

If x is a Exponential RV defined in 0 to ∞ with the parameter θ $(\theta>0)$ then

Pdf $\rightarrow E(x,\theta) = f(x) = \begin{cases} \theta e^{-\theta x} & ; \theta>0,\ 0\leq x<\infty \\ 0 & ; \text{otherwise} \end{cases}$

Keywords

- Service rate
- Reliability Component

Note :

Mean = $E(x) = \frac{1}{\theta}$

Variance $= V(x) = \frac{1}{\theta^2} \Rightarrow \frac{E(x)}{\theta}$

Note :

$E(x) = V(x)$ Iff $\theta = 1$

$E(x) < V(x)$ Iff $0<\theta<1$

$E(x) > V(x)$ Iff $\theta > 1$

Points to remember : Imp

Distribution	function	Mean	Median	Mode	variance
Binomial	$^nC_x p^x q^{n-x}$; $p+q=1$, $0\leq x\leq n$ (Pmf)	np	$\lfloor np \rfloor$ (or) $\lceil np \rceil$	$\lfloor (n+1)p \rfloor$ $\lceil (n+1)p \rceil - 1$	npq
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}$; $\lambda>0$, $0\leq x\leq\infty$ (Pmf)	λ	$\left[\lambda+\frac{1}{3}-\frac{0.02}{\lambda}\right]$	$\lfloor\lambda\rfloor$ $\lceil\lambda\rceil - 1$	λ
Uniform	$\frac{1}{b-a}$; $a<b$, $a<x<b$	$\frac{a+b}{2}$	$\frac{a+b}{2}$	Any value in (a,b)	$\frac{(b-a)^2}{12}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$; $-\infty<x<\infty$, $-\infty<\mu<\infty$, $0<\sigma<\infty$	μ	μ	μ	σ^2
Exponential	$\theta e^{-\theta x}$; $\theta>0$, $0\leq x<\infty$	$\frac{1}{\theta}$	$\frac{\ln(2)}{\theta}$	0	$\frac{1}{\theta^2}$

- Statistics Continued in printout

x — The end — x

Statistics

Definition: It is defined as Collection of data, Analysis of data & Interpretation of data

Data → Grouped, Ungrouped

- Grouped
  - class intervals
  - frequency
- Ungrouped
  - frequency

Data → closed, open $(\overset{-0.5}{l} - \overset{+0.5}{u})$ (Converted to)

- closed
  - If class Intervals are in a Continuous form without any discontinuity
- open
  - If class Intervals are in discontinuous form without any Continuity

Note:

	Ungrouped	Grouped
Mean (Best measure)	$\frac{\Sigma x_i}{n}$	$\frac{\Sigma f_i x_i}{\Sigma f_i}$
Median	Arrange Data in Asc/Desc; n: odd – middle, Even – Average	$l + \frac{\left(\frac{N}{2} - (\lt cf)\right) * cd}{f}$
Mode	Most frequently repeated observation	$l + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) cd$

For grouped data
Mode = 3Median − 2 Mean

$N = \Sigma f_i$, n = num of observations

$\lt cf$ → Cumulative freq of above Class

cd → class difference

$\Delta_1 = f_{high} - f_{above}$

$\Delta_2 = f_{high} - f_{below}$

l = lower bound of ideal class.

Note:

- If the first class itself is ideal then $(\lt cf = f)$ and Median = l
- In Mode finding, If max frequencies are repeated first, last and in between, Select in between as ideal class
- In Mode finding, If max frequencies are repeated inbetween then select randomly (bimodal)
- In Mode finding, If all the frequencies are Equal, Mode is undefined
- In Mode finding, If the max frequencies are repeated first and last then select randomly (Bimodal)

Measures of central tendencies

- Mean
- Median
- Mode

→ Mean is the best measure

Measures of Dispersion: (Identify deviation within the data)

- Range
- Deviation
  - Mean
  - Standard
  - Quartile
- Coefficient of variation

Range:

- Max − Min
- Greatest value − least value

standard deviation: (σ)

- $\sqrt{\text{variance}}$ = SD
- $SD = \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{n}}$ (Ungrouped)
- $= \sqrt{\frac{1}{N} \Sigma f_i (x_i - \bar{x})^2}$ (Grouped)

- $6QD = 5MD = 4SD$

$QD = \frac{2}{3}\sigma$ $MD = \frac{4}{5}\sigma$

Coefficient of variance: (cv)

$CV = \frac{SD}{Mean} * 100$

Note:

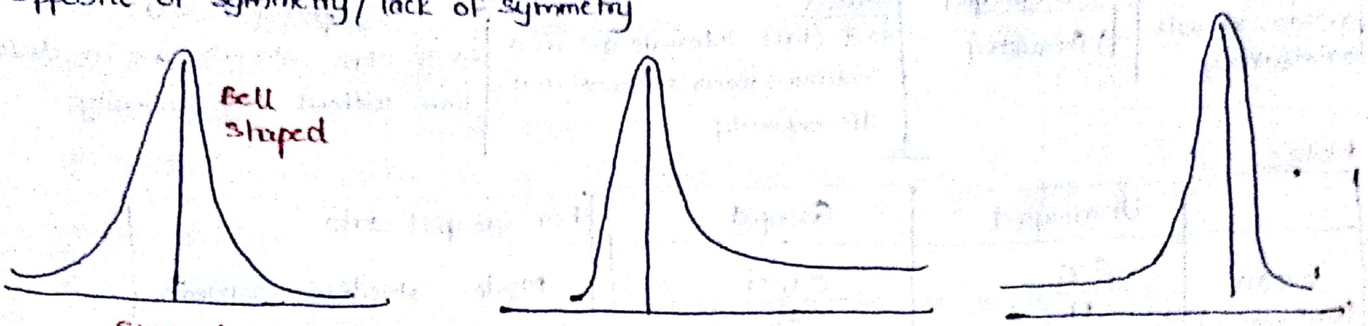
- lesser variance is more Consistent
- $V(x) \geq 0$
  - $V(x) = 0$ → constant
  - $V(x) > 0$ → variable
- Sum of differences from mean is always zero
  $\Sigma(x_i - \bar{x}) = 0$
- If variances are Equal for different groups then greater mean is more Consistent

Note :

- lesser SD → lesser CV → Data is more Consistent

Skewness :

- Opposite of symmetry / lack of symmetry



Symmetry | +vely Skewed | -vely Skewed.

$\boxed{\text{Mean} = \text{Median} = \text{Mode}}$. | $\boxed{\text{Mean} > \text{Median} > \text{Mode}}$ | $\boxed{\text{Mean} < \text{Median} < \text{Mode}}$

Ex:- Normal distribution
Uniform distribution

Ex:- Exponential distribution with skewness +2

Note :

skewness for poisson distribution, binomial distribution depends upon parameters

$\lambda^{-1/2}$ $\dfrac{1-2p}{\sqrt{npq}}$

>0 → +vely Skewed
$=0$ → Symmetric
<0 → -vely Skewed.

Coefficient of Skewness : (SK)

$$SK = \frac{\text{Mean} - \text{Mode}}{\sigma} \quad (\text{or}) \quad \frac{3(\text{Mean} - \text{Median})}{\sigma}$$

min = -1, max = +1

$\therefore \boxed{-3 \le SK \le 3}$ → reason for limit of '3' in Normal distribution

X— The End —X

- Standard Error of mean = $\dfrac{\sigma}{\sqrt{n}}$
- Num of divisor of $10^n = (n+1)^2$
- The area (in %) under standard normal distribution curve of var Z within limits from -3 to $+3$, is 99.73%
- In Exponential distribution
  $y = \min(x_1, x_2) = \sum_{i=1}^{n} \theta_i$