Detection Theory

Detection theory

- A the last topic of the course, we will briefly consider detection theory.
- The methods are based on estimation theory and attempt to answer questions such as
    - Is a signal of specific model present in our time series? E.g., detection of noisy sinusoid; beep or no beep?
    - Is the transmitted pulse present at radar signal at time t?
    - Does the mean level of a signal change at time t?
    - After calculating the mean change in pixel values of subsequent frames in video, is there something moving in the scene?

Detection theory

- The area is closely related to *hypothesis testing*, which is widely used e.g., in medicine: Is the response in patients due to the new drug or due to random fluctuations?
- In our case, the hypotheses could be

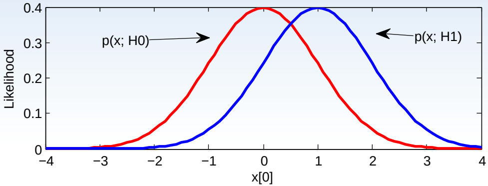
$$\mathcal{H}_1 : x[n] = A\cos(2\pi f_0 n + \phi) + w[n]$$
$$\mathcal{H}_0 : x[n] = w[n]$$

- This example corresponds to detection of noisy sinusoid.
- The hypothesis $\mathcal{H}_1$ corresponds to the case that the sinusoid is present and is called *alternative hypothesis*.
- The hypothesis $\mathcal{H}_0$ corresponds to the case that the measurements consists of noise only and is called *null hypothesis*.

Introductory Example

- *Neyman-Pearson approach* is the classical way of solving detection problems in an optimal manner.
- It relies on so called *Neyman-Pearson theorem*.
- Before stating the theorem, consider a simplistic detection problem, where we observe one sample $x[n]$ from one of two densities: $\mathcal{N}(0, 1)$ or $\mathcal{N}(1, 1)$.
- The task is to choose the correct density in an optimal manner.

Introductory Example



- Our hypotheses are now

$$\mathcal{H}_1 : \mu = 1,$$
$$\mathcal{H}_0 : \mu = 0.$$

- An obvious approach for deciding the density would choose the one, which is higher for a particular $x[0]$.

Introductory Example

- More specifically, study the likelihoods and choose the more likely one.
- The likelihoods are

$$\mathcal{H}_1 : p(x[0] \mid \mu = 1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n]-1)^2}{2}\right).$$

$$\mathcal{H}_0 : p(x[0] \mid \mu = 0) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n])^2}{2}\right).$$

- Now, one should select $\mathcal{H}_1$ if $p(x[0] \mid \mu = 1) > p(x[0] \mid \mu = 0)$.

Introductory Example

- Let's state this in terms of $x[0]$:

$$
\begin{aligned}
& p(x[0] \mid \mu = 1) > p(x[0] \mid \mu = 0) \\
\Leftrightarrow\ & \frac{p(x[0] \mid \mu = 1)}{p(x[0] \mid \mu = 0)} > 1 \\
\Leftrightarrow\ & \frac{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n]-1)^2}{2}\right)}{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n])^2}{2}\right)} > 1 \\
\Leftrightarrow\ & \exp\left(-\frac{(x[n]-1)^2) - x[n]^2}{2}\right) > 1
\end{aligned}
$$

Introductory Example

$$
\begin{aligned}
&\Leftrightarrow (x[n]^2 - (x[n]-1)^2) > 0 \\
&\Leftrightarrow 2x[n] - 1 > 0 \\
&\Leftrightarrow x[n] > \frac{1}{2}.
\end{aligned}
$$

- In other words, choose $\mathcal{H}_1$ if $x[0] > 0.5$ and $\mathcal{H}_0$ if $x[0] < 0.5$.
- Studying the ratio of likelihoods on the second row of the derivation is the key.
- This ratio is called *likelihood ratio*, and comparison to a threshold γ (here $\gamma = 1$) is called *likelihood ratio test* (LRT).

Introductory Example

- Note, that it is also possible to study posterior probability ratios $p(\mathcal{H}_1 \mid \mathbf{x})/p(\mathcal{H}_0 \mid \mathbf{x})$ instead of the above likelihood ratio $p(\mathbf{x} \mid \mathcal{H}_1)/p(\mathbf{x} \mid \mathcal{H}_0)$.
- However, using Bayes rule, this *MAP test* turns out to be

$$\frac{p(\mathbf{x} \mid \mathcal{H}_1)}{p(\mathbf{x} \mid \mathcal{H}_0)} > \frac{p(\mathcal{H}_1)}{p(H_2)},$$

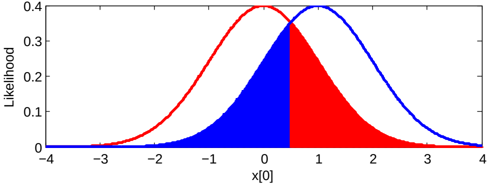
i.e., the only effect of using posterior probability is on the threshold for the LRT.

Error Types

- It might be that the detection problem is not symmetric and some errors are more costly than others.
- For example, when detecting a disease, a missed detection is more costly than a false alarm.
- The tradeoff between misses and false alarms can be adjusted using the threshold of the LRT.

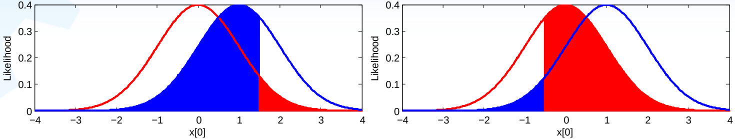
Error Types

- The below figure illustrates the probabilities of the two kinds of errors. The red area on the left corresponds to the probability of choosing $\mathcal{H}_1$ while $\mathcal{H}_0$ would hold (false match). The blue area is the probability of choosing $\mathcal{H}_0$ while $\mathcal{H}_1$ would hold (missed detection).



Error Types

- It can be seen that we can decrease either probability arbitrarily small by adjusting the detection threshold.



- Left: large threshold; small probability of false match (red), but a lot of misses (blue).
- Right: small threshold; only a few missed detections (blue), but a huge number of false matches (red).

Error Types

- Probability of false alarm for the threshold $\gamma = 1.5$ is

$$P_{FA} = P(x[0] > \gamma \mid \mu = 0) = \int_{1.5}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n])^2}{2}\right) dx[n] \approx 0.0668.$$

- Probability of missed detection is

$$P_M = P(x[0] > \gamma \mid \mu = 1) = \int_{-\infty}^{1.5} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n]-1)^2}{2}\right) dx[n] \approx 0.6915.$$

- An equivalent, but more useful term is the complement of P_M: probability of detection:

$$P_D = 1 - P_M = \int_{1.5}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n]-1)^2}{2}\right) dx[n] \approx 0.3085.$$

Neyman-Pearson Theorem

- Since P_{FA} and P_D depend on each other, we would like to maximize P_D subject to given maximum allowed P_{FA}. Luckily the following theorem makes this easy.
- **Neyman-Pearson Theorem:** For a fixed P_{FA}, the likelihood ratio test maximizes P_D with the decision rule

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \gamma,$$

with threshold γ is the value for which

$$\int_{\mathbf{x}:L(\mathbf{x})>\gamma} p(\mathbf{x}; \mathcal{H}_0)\, d\mathbf{x} = P_{FA}.$$

Neyman-Pearson Theorem

- As an example, suppose we want to find the best detector for our introductory example, and we can tolerate 10% false alarms ($P_{FA} = 0.1$).
- According to the theorem, the detection rule is:

$$\text{Select } \mathcal{H}_1 \text{ if } \frac{p(x \mid \mu = 1)}{p(x \mid \mu = 0)} > \gamma$$

The only thing to find out now is the threshold γ such that

$$\int_{\gamma}^{\infty} p(x \mid \mu = 0)\, dx = 0.1.$$

This can be done with Matlab function `icdf`, which solves the inverse cumulative distribution function.

Neyman-Pearson Theorem

- Unfortunately `icdf` solves the γ for which

$$\int_{-\infty}^{\gamma} p(x \mid \mu = 0)\, dx = 0.1 \text{ instead of } \int_{\gamma}^{\infty} p(x \mid \mu = 0)\, dx = 0.1.$$

  Thus, we have to use the function like this:
  `icdf('norm', 1-0.1, 0, 1)`, which gives $\gamma \approx 1.2816$.
- Similarly, we can also calculate the P_D with this threshold:

$$P_D = \int_{1.2816}^{\infty} p(x \mid \mu = 1)\, dx \approx 0.3891.$$

Detector for a known waveform

- The NP approach applies to all cases where likelihoods are available.
- An important special case is that of a known waveform $s[n]$ embedded in WGN sequence $w[n]$:

$$\begin{aligned}\mathcal{H}_1 &: x[n] = s[n] + w[n] \\ \mathcal{H}_0 &: x[n] = w[n].\end{aligned}$$

- An example of a case where the waveform is known could be detection of radar signals, where a pulse $s[n]$ transmitted by us is reflected back after some propagation time.

Detector for a known waveform

- For this case the likelihoods are

$$p(\mathbf{x} \mid \mathcal{H}_1) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x[n]-s[n])^2}{2\sigma^2}\right),$$

$$p(\mathbf{x} \mid \mathcal{H}_0) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x[n])^2}{2\sigma^2}\right).$$

- The likelihood ratio test is easily obtained as

$$\frac{p(\mathbf{x} \mid \mathcal{H}_1)}{p(\mathbf{x} \mid \mathcal{H}_0)} = \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}(x[n]-s[n])^2 - \sum_{n=0}^{N-1}(x[n])^2\right)\right] > \gamma.$$

Detector for a known waveform

- This simplifies by taking the logarithm from both sides:

$$-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}(x[n]-s[n])^2-\sum_{n=0}^{N-1}(x[n])^2\right)>\ln\gamma.$$

- This further simplifies into

$$\frac{1}{\sigma^2}\sum_{n=0}^{N-1}x[n]s[n]-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(s[n])^2>\ln\gamma.$$

Detector for a known waveform

- Since $s[n]$ is a known waveform (= constant), we can simplify the procedure by moving it to the right hand side and combining it with the threshold:

$$\sum_{n=0}^{N-1} x[n]s[n] > \sigma^2 \ln\gamma + \frac{1}{2}\sum_{n=0}^{N-1}(s[n])^2.$$

We can equivalently call the right hand side as our threshold (say γ') to get the final decision rule

$$\sum_{n=0}^{N-1} x[n]s[n] > \gamma'.$$

Examples

- This leads into some rather obvious results.
- The detector for a known DC level in WGN is

$$\sum_{n=0}^{N-1} x[n]A > \gamma \Rightarrow A \sum_{n=0}^{N-1} x[n] > \gamma$$

Equally well we can set a new threshold and call it $\gamma' = \gamma/(AN)$. This way the detection rule becomes: $\bar{x} > \gamma'$. Note that a negative A would invert the inequality.

Examples

- The detector for a sinusoid in WGN is

$$\sum_{n=0}^{N-1} x[n]A\cos(2\pi f_0 n + \phi) > \gamma \Rightarrow A\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n + \phi) > \gamma.$$

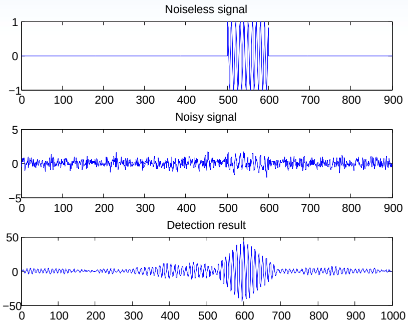
- Again we can divide by A to get

$$\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n + \phi) > \gamma'.$$

- In other words, we check the correlation with the sinusoid. Note that the amplitude A does not affect our statistic, only the threshold which is anyway selected according to the fixed P_{FA} rate.

Examples

- As an example, the below picture shows the detection process with $\sigma = 0.5$.



Detection of random signals

- The problem with the previous approach was that the model was too restrictive; the results depend on how well the phases match.
- The model can be relaxed by considering *random signals*, whose exact form is unknown, but the correlation structure is known. Since the correlation captures the frequency (but not the phase), this is exactly what we want.
- In general, the detection of a random signal can be formulated as follows.

Detection of random signals

- Suppose $\mathbf{s} \sim \mathcal{N}(0, \mathbf{C}_s)$ and $\mathbf{w} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$. Then the detection problem is a hypothesis test

$$\mathcal{H}_0 : \mathbf{x} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$$
$$\mathcal{H}_1 : \mathbf{x} \sim \mathcal{N}(0, \mathbf{C}_s + \sigma^2\mathbf{I})$$

- It can be shown (see Kay-2, p. 145), that the decision rule becomes

$$\text{Decide } \mathcal{H}_1, \text{ if } \mathbf{x}^T\hat{\mathbf{s}} > \gamma,$$

where

$$\hat{\mathbf{s}} = \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x}.$$

Detection of random signals

- The term $\hat{\mathbf{s}}$ is in fact the estimate of the signal; more specifically, the linear Bayesian MMSE estimator, which assumes linearity for the estimator (similar to BLUE).
- A particular special case of a random signal is the Bayesian linear model.
- The Bayesian linear model assumes linearity $\mathbf{x} = \mathbf{H}\theta + \mathbf{w}$ together with a prior for the parameters, such as $\theta \sim \mathcal{N}(0, \sigma^2\mathbf{I})$
- Consider the following detection problem:

$$
\begin{aligned}
\mathcal{H}_0 &: \mathbf{x} = \mathbf{w} \\
\mathcal{H}_1 &: \mathbf{x} = \mathbf{H}\theta + \mathbf{w}
\end{aligned}
$$

Detection of random signals

- Within the earlier random signal framework, this is written as

$$\mathcal{H}_0 : \mathbf{x} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$$
$$\mathcal{H}_1 : \mathbf{x} \sim \mathcal{N}(0, \mathbf{C}_s + \sigma^2 \mathbf{I})$$

  with $\mathbf{C}_s = \mathbf{H}\mathbf{C}_\theta \mathbf{H}^T$.

- The assumption $\mathbf{s} \sim \mathcal{N}(0, \mathbf{H}\mathbf{C}_\theta \mathbf{H}^T)$ states that the exact form of the signal is unknown, and we only know its covariance structure.

Detection of random signals

- This is helpful in the sinusoidal detection problem: we are not interested in the phase (included by the exact formulation $x[n] = A\cos(2\pi f_0 n + \phi)$), but only in the frequency (as described by the covariance matrix $\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T$).
- Thus, the decision rule becomes:

$$\text{Decide } \mathcal{H}_1, \text{ if } \mathbf{x}^T\hat{\mathbf{s}} > \gamma,$$

where

$$\begin{aligned}\hat{\mathbf{s}} &= \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x} \\ &= \mathbf{H}\mathbf{C}_\theta\mathbf{H}^T(\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T + \sigma^2\mathbf{I})^{-1}\mathbf{x}\end{aligned}$$

Detection of random signals

- Luckily the decision rule simplifies quite a lot by noticing that the last part is the MMSE estimate of θ:

$$\begin{aligned}\mathbf{x}^T\hat{\mathbf{s}} &= \mathbf{x}^T\mathbf{C}_\theta\mathbf{H}^T(\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T+\sigma^2\mathbf{I})^{-1}\mathbf{x}\\ &= \mathbf{x}^T\mathbf{H}\hat{\theta}.\end{aligned}$$

- An example of applying the linear model is in Kay: Statistical Signal Processing, vol. 2; Detection Theory, pages 155-158.
- In the example, a Rayleigh fading sinusoid is studied, which has an unknown amplitude A and phase term ϕ. Only the frequency f_0 is assumed to be known.

Detection of random signals

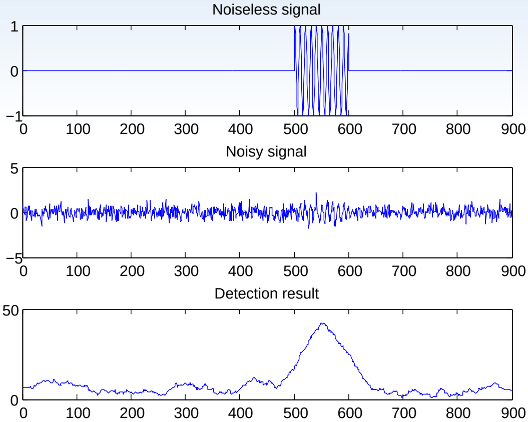
- This can be manipulated into a linear model form with two unknowns corresponding to A and ϕ.
- The final result is the decision rule:

$$\left|\sum_{n=0}^{N-1} x[n] \exp(-2\pi i f_0 n)\right| > \gamma.$$

- As an example, the below picture shows the detection process with $\sigma = 0.5$.
- Note the simplicity of Matlab implementation:

```
h = exp(-2*pi*sqrt(-1)*f0*n);
y = abs(conv(h,x));
```

Detection of random signals



Receiver Operating Characteristics

- A usual way of illustrating the detector performance is the *Receiver Operating Characteristics* curve (ROC curve).
- This describes the relationship between P_{FA} and P_D for all possible values of the threshold γ.
- The functional relationship between P_{FA} and P_D depends on the problem (and the selected detector, although we have proven that LRT is optimal).

Receiver Operating Characteristics

- For example, in the DC level example,

$$P_D(\gamma) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-1)^2}{2}\right) dx$$

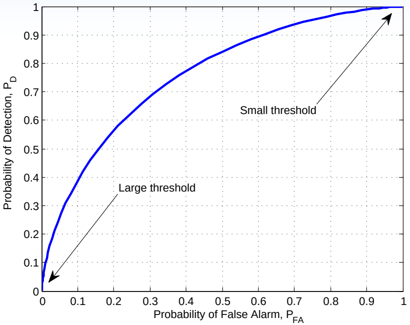
$$P_{FA}(\gamma) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

- It is easy to see the relationship:

$$P_D(\gamma) = \int_{\gamma-1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = P_{FA}(\gamma - 1).$$

Receiver Operating Characteristics

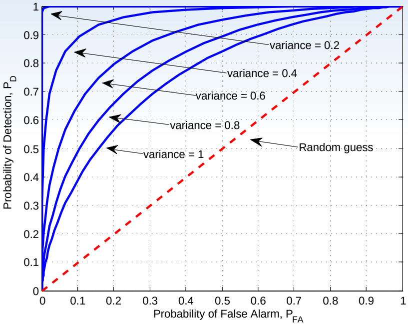
- Plotting the ROC curve for all γ results in the following curve.



Receiver Operating Characteristics

- The higher the ROC curve, the better the performance.
- A random guess has diagonal ROC curve.
- In the DC level case, the performance increases if the noise variance σ^2 decreases. Below are the ROC plots for various values of σ^2.

Receiver Operating Characteristics



- This gives rise to a widely used measure for detector performance: the *Area Under (ROC) Curve*, or AUC criterion.

Composite hypothesis testing

- In the previous examples the parameter values specified the distribution completely; e.g., either $A = 1$ or $A = 0$.
- Such cases are called *simple hypotheses testing*.
- Often we can't specify exactly the parameters for either case, but instead a range of values for each case.
- An example could be our DC model $x[n] = A + w[n]$ with

$$\mathcal{H}_1 : A \neq 0$$
$$\mathcal{H}_0 : A = 0$$

Composite hypothesis testing

- The question can be posed in a probabilistic manner as follows:

  What is the probability of observing $x[n]$ *if* $\mathcal{H}_0$ *would hold?*

- If the probability is small (e.g., all $x[n] \in [0.5, 1.5]$, and let's say the probability of observing $x[n]$ under $\mathcal{H}_0$ is 1 %), then we can conclude that the null hypothesis can be *rejected* with 99% confidence.

An example

- As an example, consider detecting a biased coin in a coin tossing experiment.
- If we get 19 heads out of 20 tosses, it seems rather likely that the coin is biased.
- How to pose the question mathematically?
- Now the hypotheses is

$$\mathcal{H}_1 : \text{coin is biased: } p \neq 0.5$$
$$\mathcal{H}_0 : \text{coin is unbiased: } p = 0.5,$$

where p denotes the probability of a head for our coin.

An example

- Additionally, let's say, we want 99% confidence for the test.
- Thus, we can state the hypothesis test as: "what is the probability of observing at least 19 heads assuming $p = 0.5$?"
- This is given by the binomial distribution

$$\underbrace{\binom{20}{19} 0.5^{19} \cdot 0.5^{1}}_{\text{19 heads}} \underbrace{+}_{\text{or}} \underbrace{0.5^{20}}_{\text{20 heads}} \approx 0.00002.$$

- Since $0.00002 < 1\%$, we can reject the null hypothesis and the coin is biased.

An example

- Actually, the 99% confidence was a bit loose in this case.
- We could have set a 99.98% confidence requirement and still reject the null hypothesis.
- The upper limit for the confidence (here 99.98%) is widely used and called the *p-value*.
- More specifically,

  The p-value is the probability of obtaining a test statistic at least as extreme as the one that was actually observed, assuming that the null hypothesis is true.