Neyman Pearson Detector for Variance Detection

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Outline

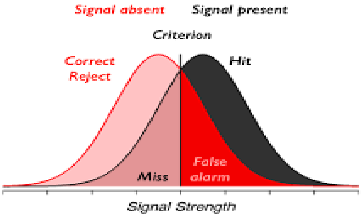
False Alarm and Missed detection?

False Alarm

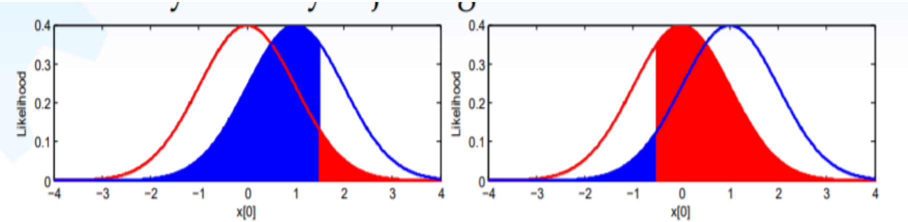
- Also called Type-I Error. H_0 is true but we decide H_1.
- Raising an alarm that signal is present even it is not there, hence **False Alarm**.

Missed Detection

- Also called Type-II Error. H_1 is true but we decide H_0.
- Missed signal detection even if signal is present, hence **Missed Detection**.



Trade Off between False Alarm and Missed Detection



Left One

- Large threshold; small probability of false match (red), but a lot of misses (blue).Less P_{FA} and less P_D(1-P_{MD}).

Right One

- Small threshold; only a few missed detections (blue), but a huge number of false matches. (red).More P_{FA} and more P_D(1-P_{MD}).

Ideally: Less P_{FA} and more P_D!!

Neyman-Pearson Theorem

- Since P_{FA} and P_D depend on each other, we would like to maximize P_D subject to given maximum allowed P_{FA}.
- Neyman Pearson theorem provides a region with **maximum $\boldsymbol{P_D}$** for a given $\boldsymbol{P_{FA}}$

Theorem

$$L(\underline{x}) = \frac{p(\underline{x}; H_1)}{p(\underline{x}; H_0)} > \gamma,$$

where the threshold γ is found by

$$\boldsymbol{P_{FA}} = \int_{R_1 : L(\underline{x}) > \gamma} p(\underline{x}; H_0) d\underline{x} = \alpha \text{ then } \boldsymbol{P_D} = \int_{R_1 : L(\underline{x}) > \gamma} p(\underline{x}; H_1) d\underline{x}$$

Proof..

We can use Lagrangian multipliers to maximize P_D for a given P_{FA}. Forming the Lagrangain

Optimization Problem

$$\begin{aligned} F &= \mathrm{P}_D + \lambda(\mathrm{P}_{FA} - \alpha) \\ &= \int_{R_1} p(\underline{x}; H_1) d\underline{x} + \lambda\left(\int_{R_1} p(\underline{x}; H_0) d\underline{x} - \alpha\right) \\ &= \int_{R_1} (p(\underline{x}; H_1) + \lambda p(\underline{x}; H_0)) d\underline{x} - \lambda\alpha \end{aligned} \tag{1}$$

To maximize equation we should include x in R_1 if that integrand is positive for that value of x or if $p(\underline{x}; H_1) + \lambda p(\underline{x}; H_0) > 0$.

$$L(\underline{x}) = \frac{p(\underline{x}; H_1)}{p(\underline{x}; H_0)} > -\lambda => L(\underline{x}) = \frac{p(\underline{x}; H_1)}{p(\underline{x}; H_0)} > \gamma \text{ with } \gamma > 0$$

Neyman Pearson Detector for detection of variance

Real Case Scenario: Problem Statement

There are two air crafts present in space and there is a base station present on earth. There is a condition that only one of the two air crafts will be in the region specified by the base station. If **aircraft** 1 is present, it sends a sample from normally distributed distribution with mean 0 and variance σ_0^2 =1 and when **aircraft** 2 is present, it sends a sample from same distribution but with variance σ_1^2 =4. It is observed that near base station at $t = 0s$, $t = 2s$ and $t = 4s$, we received three samples 0.5 1 and -3. Test which air crafts are possibly present at $t = 0s$, $t = 2s$ and $t = 4s$.Assume Probability of False Alarm = 0.4.

Problem solving approach

- H_0 : $\mathrm{x}[0] \sim \mathcal{N}(0, \sigma_0^2)$. Null Hypothesis
- H_1 : $\mathrm{x}[0] \sim \mathcal{N}(0, \sigma_1^2)$. Alternative Hypothesis

By using Neyman Pearson approach,

$$L(\underline{x}) = \frac{p(\underline{x}; H_1)}{p(\underline{x}; H_0)} > \gamma,$$

$$p(x[0]; H_0) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(\frac{-x[0]^2}{2\sigma_0^2}\right) \quad p(x[0]; H_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-x[0]^2}{2\sigma_1^2}\right)$$

Solving, we finally get $x[0]^2 > \left(2 \log_e \gamma + \log \frac{\sigma_1^2}{\sigma_0^2}\right) / \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) = \gamma^1$

So we decide R_1 when $|x[0]| > \sqrt{\gamma^1}$ where γ^1 can be calculated from given $P_{FA} = 0.4$. $\boldsymbol{P_{FA}} = \int_{R_1 : |x[0]| > \sqrt{\gamma^1}} p(x[0]; H_0) dx[0] = \alpha = 0.4$.

$$2Q\left(\frac{\sqrt{\gamma^1}}{\sigma_0}\right) = \alpha \implies \gamma^1 = \sigma_0^2 \left(Q^{-1}(\alpha/2)\right)^2$$

Probability of detection

P_D

- Probability of detection $P_D = \int_{R_1 : |x[0]| > \sqrt{\gamma^1}} p(x[0]; H_1) dx[0] = 2Q(\frac{\sqrt{\gamma^1}}{\sigma_1})$

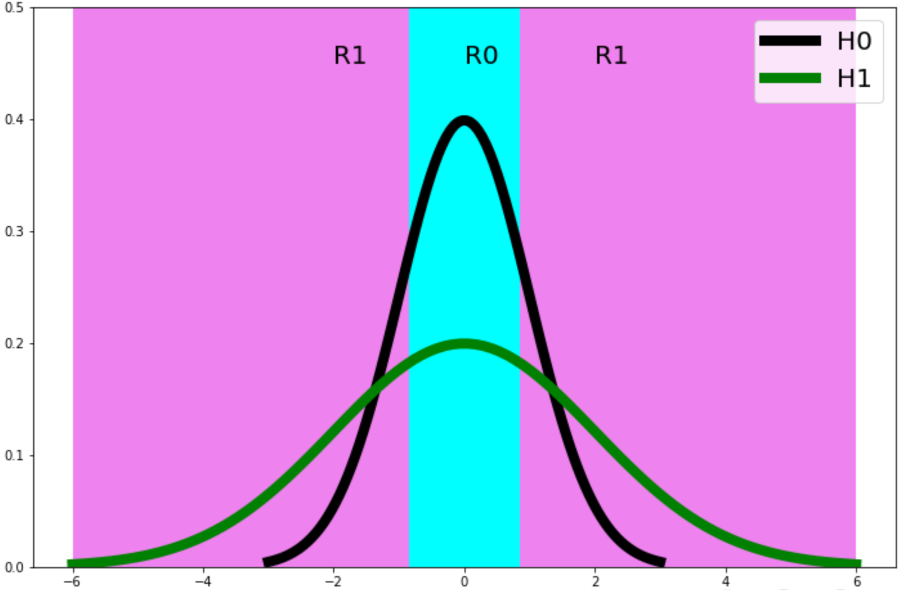
Given Parameter substitution

- Calculated $\gamma^1 = 0.7083263008007937$
- Calculated $P_D = 0.8333513498891151$

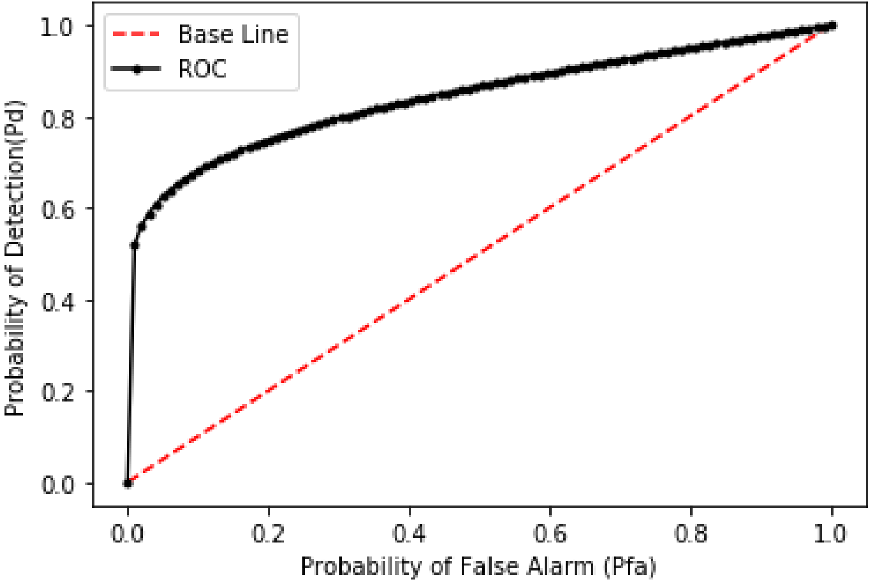
Testing given Sample Points

- We accept Hypothesis H_1 when $|x[0]| > \sqrt{\gamma^1}$ else accept Hypothesis H_0
- So sample 0.5 belongs to H_0. (Accepting Null Hypothesis).
- So sample 1 belongs to H_1. (Rejecting Null Hypothesis).
- So sample -3 belongs to H_1. (Rejecting Null Hypothesis).

Looking Graphically



Receiver Operating Characteristiccs (ROC)



Neyman Pearson Detector for detection of variance

Real Case Scenario 2: Problem Statement

There are two air crafts present in space and there is a base station present on earth. There is a condition that only one of the two air crafts will be in the region specified by the base station. If **aircraft 1** is present, it sends N= 1000 samples from normally distributed distribution with mean 0 and variance σ_0^2 =1 and when **aircraft 2** is present, it sends 1000 samples from same distribution but with variance σ_1^2 =4. It is observed that at time *tsec* , 1000 samples are detected by base station. Test which air craft is present at given time. Assume Probability of False Alarm is 0.3.

Problem solving approach

- H_0 : x[n] $\sim \mathcal{N}(0, \sigma_0^2)$. Null Hypothesis
- H_1 : x[n] $\sim \mathcal{N}(0, \sigma_1^2)$. Alternative Hypothesis

By using Neyman Pearson approach,

$$L(\underline{x}) = \frac{p(\underline{x}; H_1)}{p(\underline{x}; H_0)} > \gamma,$$

$$p(\underline{x}; H_0) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(\frac{-x[n]^2}{2\sigma_0^2}\right) \quad p(\underline{x}; H_1) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-x[n]^2}{2\sigma_1^2}\right)$$

Solving, we finally get $T(\underline{x}) = \sum_{n=0}^{N-1} \frac{x[n]^2}{\sigma_0^2} > \gamma^1 \Rightarrow$ We decide H_1

Contd...

- H_0 : $T(\underline{x}) \sim \chi^2(N)$.
- H_1 : $T(\underline{x}) \sim \Gamma(k = N/2, \theta = 2 * \frac{\sigma_1^2}{\sigma_0^2})$ where k is the shape parameter and θ is the scale parameter.

- $\boldsymbol{P_{FA}} = \int_{R_1 : T(\underline{x}) > \gamma^1} p(T(\underline{x}); H_0) d\underline{x} = \alpha = 0.3$. So γ^1 is ${\chi^2}_{N,\alpha}$
- $\boldsymbol{P_D} = \int_{R_1 : T(\underline{x}) > \gamma^1} p(T(\underline{x}); H_1) d\underline{x}$

Testing Hypothesis

- Reject H_0 when : $T(\underline{x}) > {\chi^2}_{N,\alpha}$
- So we compute $T(\underline{x})$ from the samples from aircraft and compare with ${\chi^2}_{N,\alpha}$ to test hypothesis.

Numerical Calculations

Solved Numericals

- Calculated ${\chi^2}_{N,\alpha}$ is 1022.9598734718893
- Calculated P_D is 1.0 for a given P_{FA}=0.3

Testing Given Samples

- For testing purpose I generated 1000 random samples from normal distribution of mean 0 and standard Deviation 2.
- Calculated test statistic value is 3814.087438239313.
- So we reject Null Hypothesis and Accept Alternative hypothesis H_1.

Looking Graphically

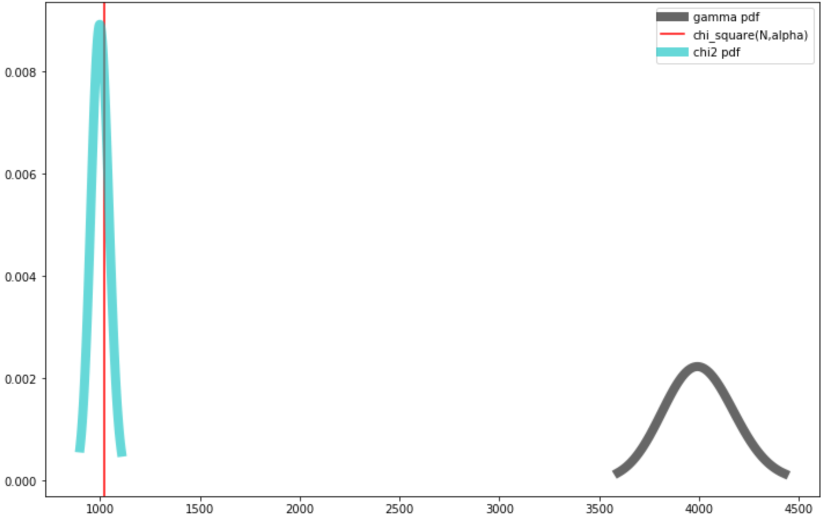


Figure: Regions H1 and H0; H0 is chi2 H1 is gamma

Receiver Operating Characteristics

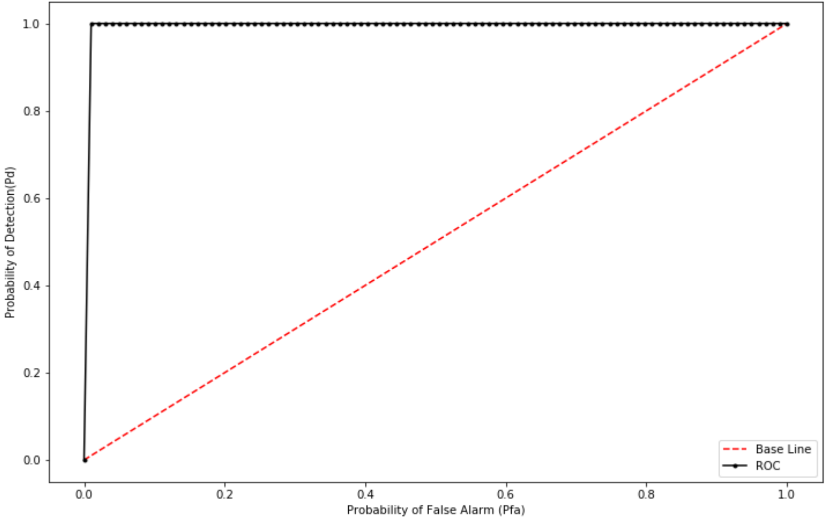


Figure: ROC curve

Other applications of Neyman Pearson Detector

Applications

- Useful in electronics engineering, namely in the design and use of radar systems, digital communication systems, and in signal processing systems
- Economics of land value (the land parcel with the largest utility, whose price is at most his budget).
- Applied to the construction of analysis-specific likelihood-ratios.

References

Steven M. Kay *Fundamentals of statistical signal processing Volume-II Detection theory*

Wikipedia *Neyman–Pearson lemma*

Thank you