End Semester examination

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(4) Differential entropy in nats

b) $f(x) = \frac{1}{\beta-\alpha} \qquad \alpha \le x \le \beta.$

Sol:- Differential entropy $= -\int f \log f \, dx$

$$= -\int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} \log\left(\frac{1}{\beta-\alpha}\right) dx$$

$$= \frac{1}{\cancel{\beta-\alpha}} \log(\beta-\alpha) \, (\cancel{\beta-\alpha})$$

$$= \log(\beta-\alpha)$$

c) $f(x) = \frac{x}{b^2} e^{-\frac{x^2}{2b^2}}$

The given distribution is Rayleigh distribution

$h(X) = -\int f \log f \, dx$

$\log f \Rightarrow \log \frac{x}{b^2} - \frac{x^2}{2b^2}$

$$h(X) = -\underbrace{\int_0^{\infty} \frac{x}{b^2} e^{-\frac{x^2}{2b^2}} \log \frac{x}{b^2}}_{I_1} + \underbrace{\int_0^{\infty} \frac{x}{b^2} e^{-\frac{x^2}{2b^2}} \frac{x^2}{2b^2} dx}_{I_2}$$

Integral I (I_1):

$$I_1 = -\int_0^{\infty} x e^{-\frac{x^2}{2b^2}} \left[\log(x) - 2\log b\right] dx.$$

$$= -\int_0^{\infty} x e^{-\frac{x^2}{2b^2}} \log x \, dx + \int_0^{\infty} 2x e^{-\frac{x^2}{2b^2}} \log b \, dx$$

(let $x^2 = t \longrightarrow 2\log x = \log t$
$2x\,dx = dt \qquad \log x = \frac{\log t}{2}$) $\Rightarrow$ Substitution

$$= -\int e^{-\frac{t^2}{2b^2}} \frac{\log t}{4} dt + \int_0^{\infty} e^{-\frac{t^2}{2b^2}} \log b \, dt$$

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$$\left(\text{Let, } \quad \frac{t}{2b^2} = z \longrightarrow \log t = \log(2b^2) + \log z \right.$$

$$t = 2b^2 z$$

$$\left. dt = 2b^2 dz \right)$$

$$I_1 = -\frac{1}{4}\int_0^\infty e^{-z}(\ln z + \ln 2b^2)\, 2b^2\, dz + 2b^2\log b$$

$$= -\frac{b^2}{2}\underbrace{\int_0^\infty e^{-z}\ln z} - \frac{b^2}{2}\int_0^\infty \ln 2b^2 e^{-z} dz + 2b^2 \log b.$$

$$= \frac{-b^2}{2}\gamma - \frac{b^2}{2}\ln 2b^2 + 2b^2 \log b \quad -(1)$$

Integral 2 (I_2):

$$I_2 = \int_0^\infty \frac{x}{b^2} e^{-\frac{x^2}{2b^2}} \frac{x^2}{2b^2}\, dx$$

$$\left(\text{let } \quad x^2 = t \quad 2x\,dx = dt \right)$$

$$I_2 = \int_0^\infty \frac{1}{4b^4} e^{\frac{-t}{2b^2}}\, t\, dt$$

(By using the concept of Byparts)

$$I_2 = \frac{1}{4b^4}\left[t \int e^{\frac{-t}{2b^2}} dt \Big|_0^\infty - \int_0^\infty 1 \cdot \int e^{\frac{-t}{2b^2}} dt \; dt \right]$$

$$= \frac{1}{4b^4}\left[-2b^2 t e^{\frac{-t}{2b^2}} \Big|_0^\infty - \int_0^\infty -2b^2 e^{\frac{-t}{2b^2}} dt \right]$$

$$= \frac{1}{4b^4}\left[0 - (2b^2)^2 [0-1] \right] = \frac{4b^4}{4b^4}$$

By (1),(2) $= 1 \quad -(2)$

$$h(x) = -\frac{I_1}{b^2} + I_2$$

$$= \frac{\gamma}{2} - \frac{\ln 2b^2}{2} + 2\log b + 1$$

$$= 1 + \frac{\gamma}{2} + \log \frac{b^2}{\sqrt{2b^2}}$$

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$$h(x) = 1 + \frac{\nu}{2} + \log\left(\frac{b}{\sqrt{2}}\right)$$

$$\boxed{\therefore \quad h(x) = 1 + \frac{\gamma}{2} + \log\left(\frac{b}{\sqrt{2}}\right)}$$

a) $$f(x) = \frac{c}{\alpha} x^{c-1} e^{-\frac{x^c}{\alpha}} \quad ; \quad x, c, \alpha > 0$$

sol:- $$\log f(x) = \log\frac{c}{\alpha} + (c-1)\log x - \frac{x^c}{\alpha}$$

$$h(x) = -\underbrace{\int_0^\infty \frac{c}{\alpha} x^{c-1} e^{-\frac{x^c}{\alpha}} \log\frac{c}{\alpha}\, dx}_{I_1} - \underbrace{\int_0^\infty \frac{c}{\alpha} x^{c-1} e^{-\frac{x^c}{\alpha}} (c-1)\log x\, dx}_{I}$$

$$+ \underbrace{\int_0^\infty \frac{c}{\alpha} x^{c-1} e^{-\frac{x^c}{\alpha}} \frac{x^c}{\alpha}\, dx}_{I_3}$$

$$I_1 = \int_0^\infty \frac{c}{\alpha} x^{c-1} e^{-\frac{x^c}{\alpha}} \log\left(\frac{c}{\alpha}\right) dx$$

let $x^c = t$ $\quad$ ~~$c\log x = \log t$~~

$c x^{c-1} dx = dt$

$$= \int_0^\infty \frac{e^{-\frac{t}{\alpha}}}{\alpha} \log\left(\frac{c}{\alpha}\right) dt$$

$$= \frac{\log\left(\frac{c}{\alpha}\right)}{\alpha} \int_0^\infty \frac{e^{-t/\alpha}}{1}\, dt$$

$$I_1 \Rightarrow \frac{\log\frac{c}{\alpha}}{\alpha}\left[\frac{e^{-t/\alpha}}{-1/\alpha}\right|_0^\infty$$

$$= \frac{\log\frac{c}{\alpha}}{\left(\frac{-1}{\cancel{\alpha}}\right)\cancel{\alpha}}\left[0 - 1\right]$$

$$= \log\left(\frac{c}{\alpha}\right)$$

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I_2 :-

$$I_2 = \int_0^\infty \frac{c}{\alpha} x^{c-1} e^{-\frac{x^c}{\alpha}} (c-1) \log x \, dx$$

let $x^c = t \implies c \log x = \log t$.

$c x^{c-1} dx = dt$

$$= \int_0^\infty \frac{e^{-\frac{t}{\alpha}}}{c\alpha} \ln t \, dt$$

$$= \int_0^\infty \frac{1}{c\alpha} e^{-\frac{t}{\alpha}} \ln t \, dt$$

let $\frac{t}{\alpha} = z \qquad \ln t = \ln(z\alpha)$

$dt = dz \, \alpha$.

$$I_2 = \int_0^\infty \frac{1}{c\cancel{\alpha}} e^{-z} (\ln z + \ln \alpha) \cancel{\alpha} dz$$

$$= \int_0^\infty \frac{1}{c} e^{-z} \ln \alpha \, dz + \int_0^\infty \frac{1}{c} e^{-z} \ln z \, dz$$

$$= -\frac{\ln \alpha}{c} [0-1] + \frac{1}{c} \int_0^\infty e^{-z} \ln z \, dz$$

$$= \frac{\ln \alpha}{c} + \frac{1}{c} \int_0^\infty e^{-z} \ln z \, dz$$

$$\int e^{-z} \ln z \, dz = \ln z \int e^{-z} dz - \int \frac{1}{z} \int e^{-z} dz \, dz$$

$$= -e^{-z} \ln z - \int -\frac{1}{z} e^{-z} dz$$

$$-\int \frac{e^{-z} dz}{z} = E_1(z)$$

$$I_2 = \frac{\ln \alpha}{c} + \frac{1}{c} \left[-e^{-z} \ln z - E_1(z) \right]_0^\infty$$

$$= \frac{\ln \alpha}{c} + \frac{\gamma}{c}$$

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let $I_3 = \int_0^\infty c\frac{x^{c-1}}{\alpha} e^{-\frac{x^c}{\alpha}} \frac{x^c}{\alpha} dx$

$x^c = t$

$cx^{c-1} dx = dt$

$I_3 = \int_0^\infty \frac{t e^{-t/\alpha}}{\alpha^2} dt$

$\int_0^\infty t e^{-\frac{t}{\alpha}} dt = t \int e^{-t/\alpha} dt \Big|_0^\infty - \int_0^\infty 1 \cdot \int e^{-t/\alpha} dt \cdot dt$

$= \underbrace{-t\alpha\, e^{-t/\alpha}\Big|_0^\infty}_{0} + \int_0^\infty \alpha e^{-t/\alpha} dt$

$= -\alpha^2 e^{-t/\alpha}\Big|_0^\infty$

$= -\alpha^2 [0-1]$

$= \alpha^2$

$I_3 = \frac{\int_0^\infty t \cdot e^{-t/\alpha} dt}{\alpha^2} = \frac{\alpha^2}{\alpha^2} = 1$

Finally,

$h(x) = -I_1 - I_2 + I_3$

$= -\log\left(\frac{c}{\alpha}\right) - \frac{\ln \alpha}{c} + \frac{\gamma}{c} + 1$

$= 1 + \log\left(\frac{\alpha}{c}\right) - \frac{\log \alpha}{c} + \frac{\gamma}{c}$

Rearranging terms,

$$\boxed{h(x) = \frac{(c-1)\gamma}{c} + \log\left(\alpha^{1/c}/c\right) + 1}$$

4) d) $f(x) = \dfrac{a k^a}{x^{a+1}}$

Generalized Pareto

$$f_X(x) = \frac{\alpha\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}}{\theta\gamma\left(1+\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{\alpha+1}} \ ; x>\mu$$

let $x_1\ x_2 \ldots x_n$ be random sample of pdf

$$Y_{1:n} \leq Y_{2:n} \leq \ldots Y_{n-1:n} \leq Y_{n:n}$$

$$g_{i:n}(y) = n\binom{n-1}{i-1}\left[F_{i:n}(y)\right]^{i-1}\left[1-F_{i:n}(y)\right]^{n-i} f_X(y)$$

Entropy

$$H_{i:n}(Y) = -\ln\left[\frac{n\alpha}{\theta\gamma}\binom{n-1}{i-1}\right] + \left(1-\frac{1}{\gamma}\right)E\left[\ln\left(\frac{Y-\mu}{\theta}\right)\right]$$

$$+ \left[\alpha(n-i+1)+1\right]E\left[\ln\left(1+\left(\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)\right] +$$

$$(1-i)\,E\left[\ln\left(1-\left(1+\left(\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}\right)\right]$$

We need to calculate,

$$E\left[\left(\frac{Y-\mu}{\theta}\right)^{r}\right] = \int_{\mu}^{\infty}\left(\frac{y-\mu}{\theta}\right)^{r} g_{i:n}(y)\,dy$$

$$= n\alpha\binom{n-1}{i-1}\sum_{l=0}^{i-1}(-1)^{l}\binom{i-1}{l}\frac{\Gamma(r\gamma+1)\,\Gamma(-r\gamma+\alpha(n-i+1+l))}{\Gamma(1+\alpha(n-i+1+l))}$$

Solving,

$$H_{1:n}(Y) = -\ln\left(\frac{n\alpha}{\theta\gamma}\right) + (\gamma-1)\left[\psi(1)-\psi(n\alpha)\right] = H_X(n\alpha)$$

By setting $\alpha = 1 \Rightarrow$

$$F_X(x) = 1-\left(1+\left(\frac{x-\mu}{\theta}\right)^{1/\gamma}\right)^{-1} \quad x>\mu$$

$$f_X(x) = \frac{1}{\gamma\theta}\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}\left(1-\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{2}$$

$x>\mu$

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substituting

$i=1$ and $i=n$

$$H_{1:1}(Y) = -\ln\left(\frac{n}{\gamma\theta}\right) + (\alpha-1)\left[\psi(1)-\psi(n)\right]$$

$$= H_X(\alpha=n)$$

Entropy of order statistics,

$$H_{n:n} = -\ln\left(\frac{n\alpha}{\theta}\right) + \left[\frac{\psi(1)-\psi(\alpha(1+\alpha 1)}{1+\alpha}\right]$$

$$+\left(\frac{n-1}{n}\right) + \left(\frac{\alpha+1}{\alpha}\right)\left[\psi(n+1)-\psi(1)\right]$$

$$H_{1:n} = \ln\left(\frac{\theta}{n\alpha}\right) + \frac{n\alpha+1}{n\alpha} = H_X(n\alpha)$$

let $\gamma=1$ $\mu=\theta$ for Pareto Type [I]

$n=1$

$$\boxed{H_X(\alpha) \Rightarrow \ln\left(\frac{\theta}{\alpha}\right) + \frac{\alpha+1}{\alpha}}$$

(our approach:- We solved first generalized Pareto and substituted for given pareto (I))

For given question $\theta=k$
$\alpha=a$

substituting,

$$H_X(a) = \ln\left(\frac{k}{a}\right) + \frac{a+1}{a}$$

$$\boxed{H_X(a) = \ln\left(\frac{k}{a}\right) + 1 + \frac{1}{a}}$$

$$\boxed{H_X(a) = \log\left(\frac{k}{a}\right) + 1 + \frac{1}{a}}$$ ("answer")

"Some results of integration taken from"
(IEEE Papers: "PARETO-TYPES AND ITS ORDER STATISTICS DISTRIBUTIONS" By G.H. Yari and G.R.M. Borzadaran.)

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(2) let $\{X_i\}$ be stationary stochastic process with entropy rate $H(\mathcal{X})$. Relation b/w $H(\mathcal{X})$ and $H(X_i)$

Sol: We know entropy rate is defined as,

$$H(\mathcal{X}) = H(X_1 / X_0, X_{-1} \ldots)$$

we also know that, conditioning reduces entropy,

hence.

$$H(X_1 / X_0, X_{-1} \ldots) \leq H(X_1)$$

Lets see, when equality holds,

If X_1 is independent of past $X_0, X_{-1} \ldots$ i.e if and only if X_i is an iid process.

↳ Then equality holds.

(3) Let $Y_i = (X_{i-1}, X_i)$ X_i is a stationary process.

$Z_i = (X_{3i-1}, X_{3i})$ Relation b/w entropy rates

$W_i = X_{3i}$ $H(\mathcal{Y})$, $H(\mathcal{X})$, $H(\mathcal{Z})$ and $H(\mathcal{W})$

Sol:- $H(Y_1, Y_2 \ldots Y_n) = H(X_0, X_1 \ldots X_{n-1}, X_n)$

(Y_i past) $H(Y_1 \ldots Y_n) = H(X_1 \ldots X_{n+1})$ (By stationarity)

Dividing by n and taking limit, we get equality.

$$\therefore H(\mathcal{X}) = H(\mathcal{Y}) \quad \text{—(1)}.$$

$$H(Z_1, Z_2 \ldots Z_n) = H(X_2, X_3, \quad X_5, X_6 \quad X_8, X_9 \ldots X_{3n-1}, X_{3n})$$

$$H[Z_1 / Z_0 \ldots] = H(X_2, X_3 / (X_{-1}, X_0)\ (X_{-4}, X_{-3}))$$

$$= H(X_2 / X_{-1}, X_0, X_{-4}, X_{-3} \ldots) + H(X_3 / X_{-1}, X_0 \ldots X_2)$$

(1) (2)

(1) $\Rightarrow H(X_2 / X_{-1}, X_0, X_{-4}, X_{-3}) \le H(X_2 / X_1, X_0, X_{-1} - - -) = H(\mathcal{X})$

(1) $\Rightarrow H(X_3 / X_2, X_{-1} - - -) \le H(X_3 / X_2, X_1, X_0, X_{-1} - - -) = H(\mathcal{X})$.

$H(W_1 W_2 - - W_n) = H(X_3, X_6 - - X_{3n})$ (W_i part)

$H(W_1 / W_0 - -) = H(X_3 / X_0 . X_{-3} - - -)$

$$\underbrace{H(X_3 / X_0, X_{-3} - -)}_{} \le \lim_{n\to\infty} \underbrace{H(X_3 / X_2, X_1, X_0 \; \cancel{X_{-1}} - -)}_{}$$

$H(W) \le H(X)$.

$\therefore H(\mathcal{X}) > H(W) \longrightarrow$ (3).

(Z_i part)

Following Z_i and Y_i we can notice that

$$Z_i = Y_{3i}$$
$$\downarrow \qquad \downarrow$$
$$(X_{3i-1}, X_{3i}) \quad (X_{3i-1}, X_{3i})$$

$H(Z_1 / Z_0, Z_{-1} - -) = H(Y_3 / Y_0, Y_{-3} - - -)$

$H(Y_3 / Y_0, Y_{-3} - -) \le \lim_{n\to\infty} H(Y_3 / Y_2, Y_1, Y_{-1} - - -)$

From (1) $\therefore H(Z) \le H(Y) = H(\mathcal{X})$

$\therefore H(Z) < H(Y) = H(\mathcal{X})$. ——— (2).

$\Rightarrow$ Answers:
(1) $H(\mathcal{X}) = H(Y) \longrightarrow$ (1)
(2) $H(\mathcal{X}) > H(Z) \longrightarrow$ (2)
(3) $H(\mathcal{X}) > H(W) \longrightarrow$ (3)

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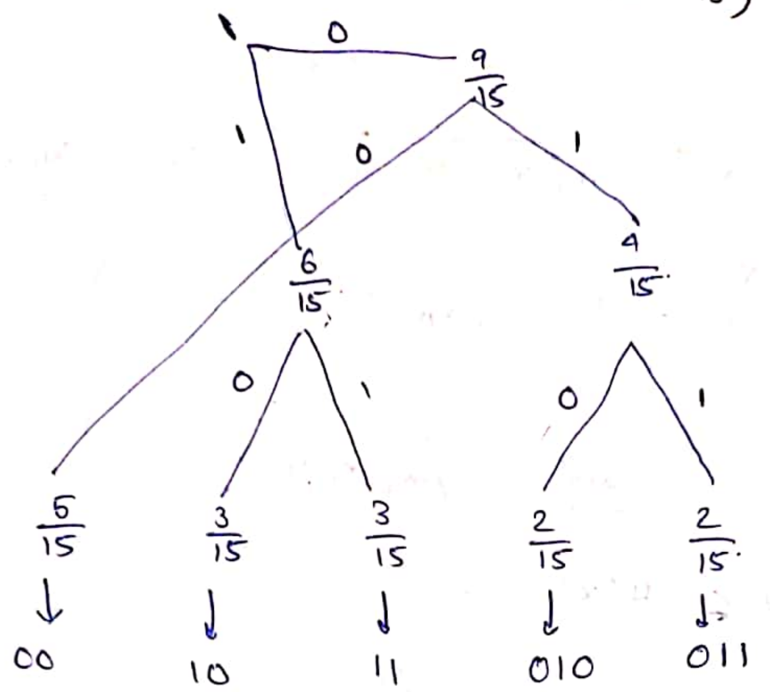
6) a) Coding scheme for,

$$\left(\frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{2}{15}, \frac{2}{15}\right)$$



Code word	Probabilities
0 0	$1/3$
1 0	$1/5$
1 1	$1/5$
0 1 0	$2/15$
0 1 1	$2/15$

(b) $x_1, x_2 \ldots x_n$ be iid acc^n to $p(x)$

Find $\lim_{n \to \infty} \left[p(x_1 \ldots x_n) \right]^{1/n}$

Sol:- As $x_1, x_2 \ldots x_n$ are iid,

$\log p(x_i)$ are also iid's.

$$\lim_{n \to \infty} \left(p(x_1 \ldots x_n)\right)^{1/n} = \lim_{n \to \infty} 2^{\log p(x_1 \ldots x_n)^{1/n}}$$

$$= \lim_{n \to \infty} 2^{\frac{1}{n} \log p(x_1 \ldots x_n)}$$

As $p(x_1 \ldots x_n) = p(x_1)\, p(x_2) \ldots p(x_n)$

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$$= \lim_{n\to\infty} \left(2^{\frac{1}{n}\log\left(\prod_i P(x_i)\right)}\right)$$

$$= \lim_{n\to\infty} 2^{\frac{1}{n}\sum_{i=1}^{n}\log P(x_i)}$$

From law of large numbers

$\log P(x_i)$ are also iids.

$$\frac{\sum_{i=1}^{n} \log P(x_i)}{n} \longrightarrow \mathbb{E}(\log P(x))$$

$$= 2^{\mathbb{E}[\log P(x)]}$$

$$= 2^{-H(x)}$$

$$\therefore \lim_{n\to\infty} \left(P(x_1 - x_n)\right)^{1/n} = 2^{-H(x)}$$

1. $f : X \to [0,1]$

$x \in X$, $f_A(x)$ degree of inclusion of x in set A.

(a) Design entropy to quantify uncertainty associated with inclusion of x in A. Here $\boxed{\mu_A(x) \Rightarrow f_A(x)}$

Sol:- Let P be a probability measure over R^n.

$$P(A) = \int_A dP$$

(or) equivalently

contin, discrete

$$P(A) = \int_{R^n} \mu_A(x)\, dP = \sum_{x\in R^n} \mu_A(x)$$

$$= E(\mu_A) \quad \text{—(1)}$$

$$\mu_A : R^n \to [0,1]$$

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Probability of fuzzy set Event A is defined by the Lebesgue-Stieltjes integral given in eq ①

Some properties of fuzzy set

$$A \subset B \Rightarrow \mu_A(x) \le \mu_B(x) \quad \forall x$$

$$A \cup B \quad \mu_{A\cup B}(x) = \mathrm{Max}[\mu_A(x), \mu_B(x)] \quad \forall x$$

Turning to notion of entropy, we note that its usual defn in information theory is as follows:

let x be a random variable takes $x_1 - - x_n$, $P_1 - - P_n$

Entropy given by $H(x) = -\sum_{i=1}^{n} P_i \log P_i$

Here for fuzzy sets, we quantify the entropy as a weighted shannon entropy.

For a random variable z an event occurs with probability P_1 here uncertainty defined as $\mu_A(x_i)$ → Event

Event x.

$$\to -\underbrace{\mu_A(x)}_{\text{weighted}} P \log P \Rightarrow -\mu_A(x) P(A) \log P(A)$$

For a binary $\begin{cases} \to P \to x_1 \\ \to 1-P \to x_2 \end{cases}$ events

$$H(A) = -\mu_A(x_1) P \log P - \mu_A(x_2)(1-P)\log(1-P)$$

$$= -\mu_A(x_1) P \log P - \mu_A(x_2)(1-P)\log(1-P).$$

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(b) let $X = \{x_1 - - x_n\}$ with probabilities
$\{p_1 - p_n\}$
$H(X)$. relation with (a).

Sol: By using the defn of above entropy as a weighted shannon entropy, For all events $1 - - n$

This definition, suggets that entropy of a fuzzy ~~set~~ subset A, of finite set $\{x_1 - x_n\}$ with respect to a probability distribution $P = \{P_1 - - P_n\}$ be defined as follows

$$H^P(A) = - \sum_{i=1}^{n} \mu_A(x_i) P_i \log P_i$$

$(\mu_A \to f_A)$

Entropy of fuzzy set A w.r.t probabilistic distribution P.

If x and y are indepent R.V's with P, Q (probabilit $(P_1 - P_n)$ $(q_1 - q_m)$)

$$H(x, y) = H(x) + H(y)$$
$$H^{PQ}(A, B) = P(A)\, H^P(A) + P(B)\, H^Q(B)$$

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5) Consider a channel $(X^n, H(y^n/x^n), Y^n)$ DMC

$\hookrightarrow$ Entropy transition matrix.

Channel capacity, Max. probab of error, avg. probab of error and rate.

Sol: $\Rightarrow$ We know Rate $= \frac{1}{n}\log M$. For a (n,M) code.

By property of chain rule,

$$H(X^n, Y^n) = H(X^n) + H(Y^n/X^n) \quad -①$$
$$= H(Y^n) + H(X^n/Y^n) \quad -①a$$

By property of DMC,

$$P(y^n/x^n) = \prod_{i=1}^{n} P(y_i/x_i)$$

Applying E and log on B.S.

$$-E\log P(y^n/x^n) = \sum_{i=1}^{n} -E[\log P(y_i/x_i)]$$

$$\boxed{H(Y^n/X^n) = \sum_{i=1}^{n} H(Y_i/X_i)} \quad -②$$

From the achievability of channel coding theorm, for every i/p distribution $P(x^n)$ $I(x^n;y^n)$ is achievable as rate for large n^{th} extension such that

① Rate R arbitarily close to $I(X^n;Y^n)$

② λ_{max} (Max error probability) ~~close~~ arbitaly small.

Without loss of generality,

Choose $P(x^n)$ to be one that achieves channel capacity i.e $I(X^n;Y^n) = C$

$$I(X^n;Y^n) = H(X^n) - H(X^n/Y^n) \quad -③$$
$$= H(Y^n) - H(Y^n/X^n)$$

From ②,

$$H(Y^n/X^n) = \sum_{i=1}^{n} H(Y_i/X_i)$$

$\hookrightarrow$ This can be obtained from entropy transition matrix

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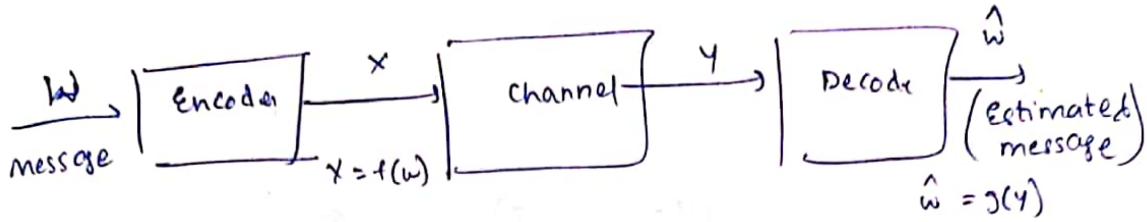
we know $H(X^n)$ distribution at transmitter, $H(Y^n)$ at receiver

From ①,

$H(Y^n)$ can be known. [$\because$ from $H(X^n/Y^n)$.]

$\therefore$ $C = I(X^n; Y^n)$ can be calculated. — ④



λ_w : Error probability given message w sent,

$\lambda_w = Pr\{\hat{w} \neq w / W = w\}$

$\lambda_w = Pr\{ g(Y^n) \neq w / X^n = f(x^n(w))\}$

$\lambda_{max} = \max\limits_{w} \lambda_w$

Avg. probability of error

$Pr\{Err\} = P_e = \frac{1}{M} \sum\limits_{w} \lambda_w$.

$Pr\{Err\} = \sum\limits_{w=1}^{M} Pr\{Err / W = w\} \, Pr\{W = w\}$

$= Pr\{Err / W = 1\} \sum\limits_{w=1}^{M} Pr\{W = w\}$

$= Pr\{Err / W = 1\}$

$\boxed{\therefore \; P_e = \lambda_1 = \lambda_2 \; -- \; = \lambda_{max}}$

E_w : Event $= \{ (\tilde{x}(w), Y) \in T^n_{[X,Y]\delta} \}$

Jointly typical.

$Pr\{Err\}_{W=1} \leq Pr\{E_1^c / W = 1\} + \sum\limits_{w=2}^{M} Pr\{E_w / W = 1\}$ — ④a

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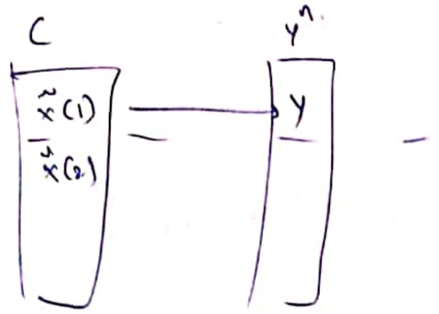
By strong JAEP,

$$\Pr\{G^c / w=1\} = \Pr\{\underbrace{(\tilde{x}(1), y) \notin T^n_{[x,y]}}_{\text{Not jointly typical}} / w=1\} < \nu$$



$$\Pr\{Err / w=1\} \leq 2^{-n(I(x;y)-\tau)} \quad - (5)$$

From eq-(4) getting $I(x;y)$ we can calculate

and $\frac{1}{n}\log M < I(x;y) - \frac{\varepsilon}{4} \Rightarrow M < 2^{n(I(x;y) - \frac{\varepsilon}{4})}$ (6)

$$\boxed{R < I(x;y) - \frac{\varepsilon}{4}} \Rightarrow \text{[Rate can be calculated from } I(x;y) \text{ calculated]}$$

Substituting (6a) in (5)

$$\Pr\{Err\} < \nu + \underbrace{2^{n(I(x;y) - \frac{\varepsilon}{4})}}_{M \text{ terms}} \underbrace{2^{-n(I(x;y)-\tau)}}_{\text{Each one. } \Pr\{E\omega/\omega\}}$$

$$Pe \leq \nu + 2^{-n(\frac{\varepsilon}{4} - \tau)}$$

$\boxed{Pe < \frac{\varepsilon}{2}} \Rightarrow$ small. $< \frac{\varepsilon}{3}$ $< \frac{\varepsilon}{6}$

$\left(\tau \to 0 \quad \delta \to 0 \quad \frac{\varepsilon}{4} - \tau > 0\right)$

Here (5), calculates Pe,

By our assumption $\lambda_1, \lambda_2 - \lambda_w = \lambda_{max} = Pe$

Pe can also be calculated by Other calculation,

$$\boxed{Pe \geq 1 - \frac{C}{\frac{1}{n}\log M}} \quad - (7)$$

we know C, M from above eqn

Hence from eq - ① to ⑦,

Capacity, P_e, λ_{max}, R can be calculated from the entropy transition matrix

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