

EE3025 Assignment-1

M SAI MEHAR - EE18BTECH11029

Download all python codes from

<https://github.com/saimehar31/EE3025/tree/main/Assignment1/codes>

and latex-tikz codes from

<https://github.com/saimehar31/EE3025/blob/main/Assignment1/ee18btech11029.tex>

1 PROBLEM

1.1. Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \quad (1.1.1)$$

$$h(n) = \left(-\frac{1}{2} \right)^n u(n) + \left(-\frac{1}{2} \right)^{n-2} u(n-2) \quad (1.1.2)$$

1.2. Compute $X(k)$, $H(k)$ and $y(n)$ using FFT and IFFT methods

2 SOLUTION

2.1. Input signal $x(n)$ is given as

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \quad (2.1.1)$$

2.2. Impulse Response of the System is

$$h(n) = \left(-\frac{1}{2} \right)^n u(n) + \left(-\frac{1}{2} \right)^{n-2} u(n-2) \quad (2.2.1)$$

2.3. FFT of a Input Signal $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (2.3.1)$$

2.4. FFT of a Impulse Response $h(n)$ is

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (2.4.1)$$

2.5. FFT of output Signal $y(n)$ can be calculated by

$$Y(k) = X(k)H(k) \quad (2.5.1)$$

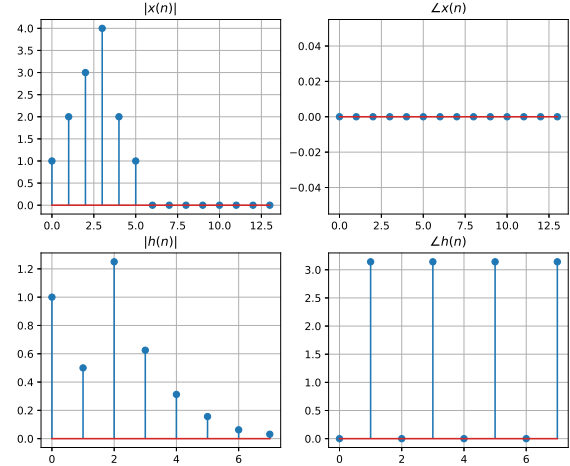


Fig. 2.1: Input signal $x(n)$ and Impulse response $h(n)$

2.6. $y(n)$ can be calculated by doing IFFT for $Y(k)$

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (2.6.1)$$

2.7. Plotting FFT of output signal

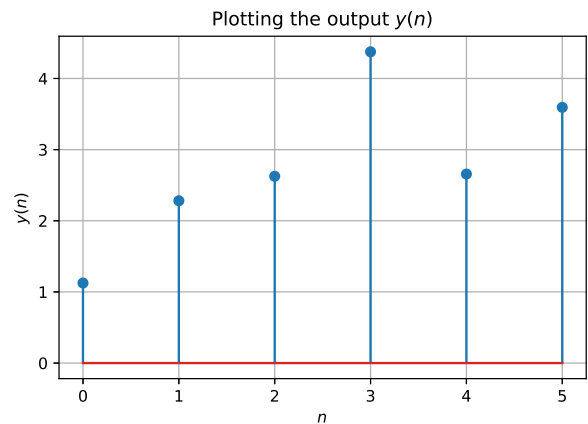


Fig. 2.7: Output signal $y(n)$

3 PROBLEM

3.1. Wherever possible, express all the above equations as matrix equations.

4 SOLUTION

4.1. FFT of signal $X(n)$

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (4.1.1)$$

4.2. Let $W_N^{nk} = e^{-j2\pi kn/N}$ then this can be expressed in terms of matrices as:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & W_N^4 & W_N^5 \\ 1 & W_N^2 & W_N^4 & W_N^6 & W_N^8 & W_N^{10} \\ 1 & W_N^3 & W_N^6 & W_N^9 & W_N^{12} & W_N^{15} \\ 1 & W_N^4 & W_N^8 & W_N^{12} & W_N^{16} & W_N^{20} \\ 1 & W_N^5 & W_N^{10} & W_N^{15} & W_N^{20} & W_N^{25} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \end{bmatrix} \quad (4.2.1)$$

4.3. Given that $x(n) = \{1, 2, 3, 4, 2, 1\}$ and As, $N = 6$ then above equation on multiplying matrices becomes

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 1+2+3+4+2+1 \\ 1+(2)e^{-j\pi/3} + \dots + (1)e^{-j5\pi/3} \\ 1+(2)e^{-2j\pi/3} + \dots + (1)e^{-2j5\pi/3} \\ 1+(2)e^{-3j\pi/3} + \dots + (1)e^{-3j5\pi/3} \\ 1+(2)e^{-4j\pi/3} + \dots + (1)e^{-4j5\pi/3} \\ 1+(2)e^{-5j\pi/3} + \dots + (1)e^{-5j5\pi/3} \end{bmatrix} \quad (4.3.1)$$

4.4. On solving we get,

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{bmatrix} \quad (4.4.1)$$

$$\Rightarrow X(0) = 13 + 0j, \quad (4.4.2)$$

$$X(1) = -4 - 1.732j, \quad (4.4.3)$$

$$X(2) = 1 + 0j, \quad (4.4.4)$$

$$X(3) = -1 + 0j, \quad (4.4.5)$$

$$X(4) = 1 + 0j, \quad (4.4.6)$$

$$X(5) = -4 + 1.732j \quad (4.4.7)$$

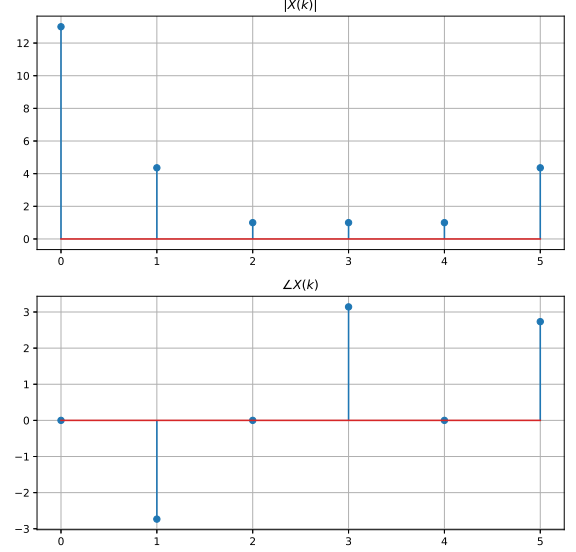


Fig. 4.4: FFT of $x(n)$

4.5. Now to find $H(k)$ we need to know the $h(n)$. For that we need to first find the $Y(z)$ by applying Z-transform on equation i.e.,

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.5.1)$$

$$\Rightarrow Y(z) = \frac{2(z^2 + 1)}{z(2z + 1)}X(z) \quad (4.5.2)$$

Now we can find $H(z)$ using $Y(z)$ i.e.,

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.5.3)$$

$$H(z) = \frac{2(z^2 + 1)}{z(2z + 1)} \quad (4.5.4)$$

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.5.5)$$

By applying inverse Z - transform we can find the value of $h(n)$

$$h(n) = Z^{-1} \left[\frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \right] \quad (4.5.6)$$

$$h(n) = \left[\frac{-1}{2} \right]^n u(n) + \left[\frac{-1}{2} \right]^{n-2} u(n-2) \quad (4.5.7)$$

Assuming that length of $h(n)$ is same as length of $x(n)$ i.e., $N = 6$. Now finding the value of $h(n)$ using matrix method we get

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & W_N^4 & W_N^5 \\ 1 & W_N^2 & W_N^4 & W_N^6 & W_N^8 & W_N^{10} \\ 1 & W_N^3 & W_N^6 & W_N^9 & W_N^{12} & W_N^{15} \\ 1 & W_N^4 & W_N^8 & W_N^{12} & W_N^{16} & W_N^{20} \\ 1 & W_N^5 & W_N^{10} & W_N^{15} & W_N^{20} & W_N^{25} \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \\ h(4) \\ h(5) \end{bmatrix} \quad (4.5.8)$$

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} = \begin{bmatrix} h(0) + h(1) + h(2) + h(3) + h(4) + h(5) \\ h(0) + h(1)e^{-j\pi/3} + \dots + h(5)e^{-j5\pi/3} \\ h(0) + h(1)e^{-2j\pi/3} + \dots + h(5)e^{-2j5\pi/3} \\ h(0) + h(1)e^{-3j\pi/3} + \dots + h(5)e^{-3j5\pi/3} \\ h(0) + h(1)e^{-4j\pi/3} + \dots + h(5)e^{-4j5\pi/3} \\ h(0) + h(1)e^{-5j\pi/3} + \dots + h(5)e^{-5j5\pi/3} \end{bmatrix}$$

4.6. On solving we get

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} = \begin{bmatrix} 1.28125 \\ 0.51625 - 0.5142j \\ -0.07813 + 1.1096j \\ 3.84375 \\ -0.07183 - 1.1096j \\ 0.51625 + 0.5142j \end{bmatrix} \quad (4.6.1)$$

$$\Rightarrow H(0) = 1.28125 + 0j, \quad (4.6.2)$$

$$H(1) = 0.51625 - 0.5141875j, \quad (4.6.3)$$

$$H(2) = -0.078125 + 1.1095625j, \quad (4.6.4)$$

$$H(3) = 3.84375 + 0j, \quad (4.6.5)$$

$$H(4) = -0.071825 - 1.1095625j, \quad (4.6.6)$$

$$H(5) = 0.515625 + 0.5141875j \quad (4.6.7)$$

So, These values which we got are same as that of from the plots.

4.7. Now to find $H(k)$ we need to know the $h(n)$. For that we need to first find the $Y(z)$ by applying Z-transform on equation

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} \times \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} \quad (4.7.1)$$

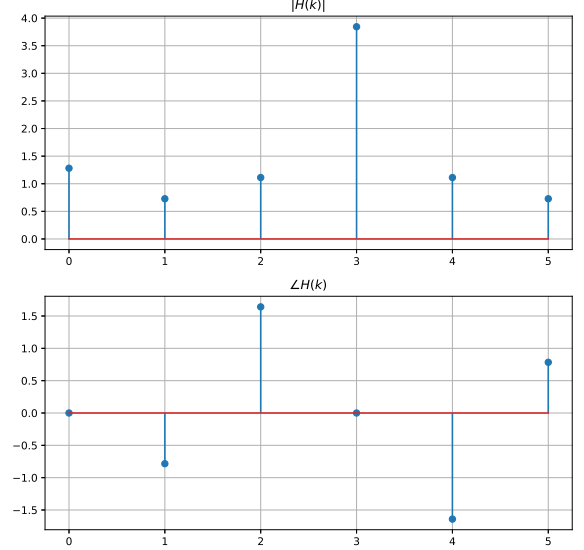


Fig. 4.6: FFT of $h(n)$

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \end{bmatrix} = \begin{bmatrix} 16.65625 \\ -2.95312 + 1.16372j \\ -0.07813 + 1.1096j \\ -3.84375 \\ -0.07813 - 1.1096j \\ -2.95312 - 1.16372j \end{bmatrix} \quad (4.7.2)$$

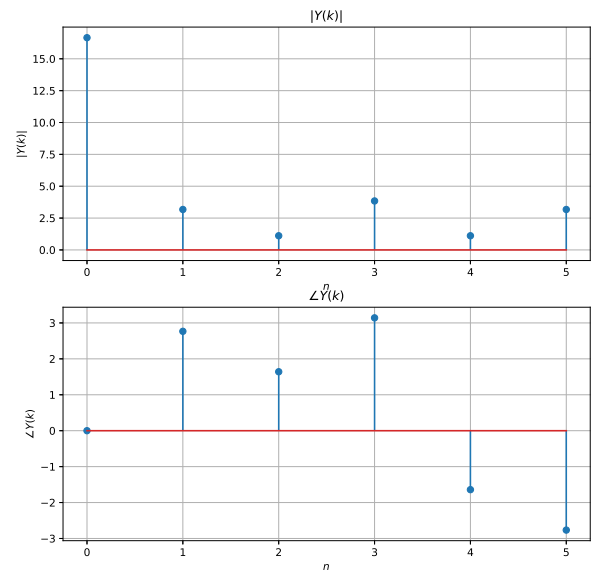


Fig. 4.7: FFT of $y(n)$

- 4.13. If we have $N = 2^M$ where $M \in \mathbb{Z}^+$ then we can recursively breakdown N/2 point DFT Matrix into N/4 point DFT Matrix and so on till we reach 2-point DFT Matrix.
for $N = 8$, we can write,

$$F_8 = \begin{bmatrix} I_4 & D_4 \\ I_4 & -D_4 \end{bmatrix} \begin{bmatrix} F_4 & 0 \\ 0 & F_4 \end{bmatrix} P_8 \quad (4.13.1)$$

$$F_4 = \begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 & 0 \\ 0 & F_2 \end{bmatrix} P_4 \quad (4.13.2)$$

Finally, the 2-point DFT Matrix is the base case

$$F_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} \quad (4.13.3)$$

- 4.14. Visualization of computing 8-Point DFT recursively using 4-point DFT's and 2-point DFT's. Expressing 8-point DFT's in terms of 4-point DFT's.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_e(0) \\ X_e(1) \\ X_e(2) \\ X_e(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_o(0) \\ X_o(1) \\ X_o(2) \\ X_o(3) \end{bmatrix} \quad (4.14.1)$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_e(0) \\ X_e(1) \\ X_e(2) \\ X_e(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_o(0) \\ X_o(1) \\ X_o(2) \\ X_o(3) \end{bmatrix} \quad (4.14.2)$$

Here representing X_e as X_1 and X_o as X_2
4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (4.14.3)$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (4.14.4)$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (4.14.5)$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (4.14.6)$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \quad (4.14.7)$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \quad (4.14.8)$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \quad (4.14.9)$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \quad (4.14.10)$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \quad (4.14.11)$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \quad (4.14.12)$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \quad (4.14.13)$$

X_3 and X_4 both combined would give X_1
 X_5 and X_6 both combined would give X_2

- 4.15. The following C program will compute and print the FFT (N-point where N is of the form 2^n)

<https://github.com/saimehar31/EE3025/tree/master/Assignment1/codes/fft.c>

- 4.16. Using the above properties recursively we have implemented radix-2 Fast-Fourier transform algorithm.

- In a N-point DFT matrix multiplication consists of $2N^2$ multiplication so the overall time complexity is in the order of $O(N^2)$
- In FFT we recursively break down each stage into two N/2-point FFT's resulting in $\log N$ term and we do this for entire N so combining both we get $O(N \log N)$

4.17. Taking an example of 8-point function,

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1, 0, 0 \right\} \quad (4.17.1)$$

We know that,

$$X(k) \triangleq W_N^{nk} x(n), \quad k = 0, 1, \dots, N-1 \quad (4.17.2)$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 \\ W_8^0 W_8^1 W_8^2 W_8^3 W_8^4 W_8^5 W_8^6 W_8^7 \\ W_8^0 W_8^2 W_8^4 W_8^6 W_8^8 W_8^{10} W_8^{12} W_8^{14} \\ W_8^0 W_8^3 W_8^6 W_8^9 W_8^{12} W_8^{15} W_8^{18} W_8^{21} \\ W_8^0 W_8^4 W_8^8 W_8^{12} W_8^{16} W_8^{20} W_8^{24} W_8^{28} \\ W_8^0 W_8^5 W_8^{10} W_8^{15} W_8^{20} W_8^{25} W_8^{30} W_8^{35} \\ W_8^0 W_8^6 W_8^{12} W_8^{18} W_8^{24} W_8^{30} W_8^{36} W_8^{42} \\ W_8^0 W_8^7 W_8^{14} W_8^{21} W_8^{28} W_8^{35} W_8^{42} W_8^{49} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} \quad (4.17.3)$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 \\ W_8^0 W_8^1 W_8^2 W_8^3 W_8^4 W_8^5 W_8^6 W_8^7 \\ W_8^0 W_8^2 W_8^4 W_8^6 W_8^8 W_8^{10} W_8^{12} W_8^{14} \\ W_8^0 W_8^3 W_8^6 W_8^9 W_8^{12} W_8^{15} W_8^{18} W_8^{21} \\ W_8^0 W_8^4 W_8^8 W_8^{12} W_8^{16} W_8^{20} W_8^{24} W_8^{28} \\ W_8^0 W_8^5 W_8^{10} W_8^{15} W_8^{20} W_8^{25} W_8^{30} W_8^{35} \\ W_8^0 W_8^6 W_8^{12} W_8^{18} W_8^{24} W_8^{30} W_8^{36} W_8^{42} \\ W_8^0 W_8^7 W_8^{14} W_8^{21} W_8^{28} W_8^{35} W_8^{42} W_8^{49} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (4.17.4)$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 13 \\ -3.121 - 6.535j \\ 1j \\ 1.121 - 0.535j \\ -1 \\ 1.121 + 0.535j \\ -1j \\ -3.121 + 6.535j \end{bmatrix} \quad (4.17.5)$$

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \\ H(6) \\ H(7) \end{bmatrix} = \begin{bmatrix} W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 W_8^0 \\ W_8^0 W_8^1 W_8^2 W_8^3 W_8^4 W_8^5 W_8^6 W_8^7 \\ W_8^0 W_8^2 W_8^4 W_8^6 W_8^8 W_8^{10} W_8^{12} W_8^{14} \\ W_8^0 W_8^3 W_8^6 W_8^9 W_8^{12} W_8^{15} W_8^{18} W_8^{21} \\ W_8^0 W_8^4 W_8^8 W_8^{12} W_8^{16} W_8^{20} W_8^{24} W_8^{28} \\ W_8^0 W_8^5 W_8^{10} W_8^{15} W_8^{20} W_8^{25} W_8^{30} W_8^{35} \\ W_8^0 W_8^6 W_8^{12} W_8^{18} W_8^{24} W_8^{30} W_8^{36} W_8^{42} \\ W_8^0 W_8^7 W_8^{14} W_8^{21} W_8^{28} W_8^{35} W_8^{42} W_8^{49} \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.65 \\ 0.3125 \\ -0.15625 \\ 0.078125 \\ -0.0390625 \end{bmatrix} \quad (4.17.6)$$

$$\Rightarrow \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \\ H(6) \\ H(7) \end{bmatrix} = \begin{bmatrix} 1.32 \\ 0.858 - 0.514j \\ -0.015 - 0.007j \\ 0.516 + 1.829j \\ 3.96 \\ 0.516 - 1.829j \\ -0.015 + 0.007j \\ 0.858 + 0.514j \end{bmatrix} \quad (4.17.7)$$

So,

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \\ Y(6) \\ Y(7) \end{bmatrix} = \begin{bmatrix} X(0) \cdot H(0) \\ X(1) \cdot H(1) \\ X(2) \cdot H(2) \\ X(3) \cdot H(3) \\ X(4) \cdot H(4) \\ X(5) \cdot H(5) \\ X(6) \cdot H(6) \\ X(7) \cdot H(7) \end{bmatrix} \quad (4.17.8)$$

Solving,

$$\Rightarrow \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \\ Y(6) \\ Y(7) \end{bmatrix} = \begin{bmatrix} 17.16 \\ -6.04 - 4j \\ -0.007 - 0.015j \\ 1.55 + 1.77j \\ -3.96 \\ 1.55 - 1.77j \\ 0.007 + 0.015j \\ -6.04 + 4j \end{bmatrix} \quad (4.17.9)$$

Similary we get

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0.53125 \\ 1.69 \\ 3.09 \\ 4.375 \\ 2.773 \\ 3.593 \\ 0.203 \\ 0.8984 \end{bmatrix} \quad (4.17.10)$$