Formulas:

Sets

P(A|B) = $\frac{P(B|A)P(A)}{P(B)}$

P(A|B) = $\frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|\bar{A})P(\bar{A})}$

P(A|B) = $\frac{P(A\cap B)}{P(B)}$

Additive Law:

I. $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

II. If A, B are disjoint: $P(A\cup B) = P(A) + P(B)$

III. If A,B are indep. : $P(A\cup B) = P(A) + P(B) - P(A)P(B)$

Independent Events

Let A and B be two indep. events iff

I. $P(A|B) = P(A)$ or

II. $P(B|A) = P(B)$ or

III. $P(A\cap B) = P(A)\times P(B)$

IV. $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)\times P(B)}{P(B)} = P(A)$

Properties of Conditional Probability

I. $P(A\cap B) = P(A|B)P(B)$

$P(A\cap B) = P(B|A)P(A)$

II. $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$

Combinatorics

$P^n_k = \frac{n!}{(n-k)!}$ Order matters

$C^n_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ Order doesn't matter

---

Distributions $(\mu = E(Y),\ \sigma^2 = V(Y)\ or\ Var(y))$

Binomial: Binom(n, p)

A binomial experiment possesses the following properties:

1. The experiment consists of a fixed number, n, of identical trials.
2. Each trial results in one of two outcomes: success, S, or failure, F.
3. The probability of success on a single trial is equal to some value p and remains the same from trial to trial. The probability of a failure is equal to q = (1 − p).
4. The trials are independent.
5. The random variable of interest is Y , the number of successes observed during the n trials.

n= sample size, p = p(success)

p(y) = $\binom{n}{y}p^y(1-p)^{n-y}$

E(Y) = np, V(Y) = np(1-p)

Geometric: Geom(p) p = p(success)

The geometric probability distribution is often used to model distributions of lengths of waiting times. Used to find the number of trials until the first success.

p(y) = $p(1-p)^{y-1}$

E(Y) = $\frac{1}{p}$, V(Y) = $\frac{1-p}{p^2}$

Neg Binomial: NBinom(r, p)

The number of the trial on which the rth success occurs.

r = r-th success, p = p(success)

p(y) = $\binom{y-1}{r-1}p^r(1-p)^{y-r}$

E(Y) = $\frac{r}{p}$, V(Y) = $\frac{r(1-p)}{p^2}$

Hypergeometric: Hyp(N, n, r)

1. A sample of size n is randomly selected without replacement from a population of N items.
2. In the population, k items can be classified as successes, and N - k items can be classified as failures.

N = Total number, n = sample of larger set, r = total # success in N

p(y) = $\frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$

E(Y) = $\frac{nr}{N}$, V(Y) = $n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$

Hypergeometric Continued…

P(next 5 blue|first 5 red) =

Poisson: Poisson(λ) λ = average rate

The Poisson probability distribution often provides a good model for the probability distribution of the number Y of rare events that occur in space, time, volume, or any other dimension, where λ is the average value of Y.

p(y) = $e^{-\lambda}\frac{\lambda^y}{y!}$

E(Y) = λ, V(Y) = λ

$E(Y) = \sum_y yp(y)$,

$V(Y) = \sigma^2 = E[(Y-\mu)^2] = E[Y] - \mu^2$

Expected Value and Variance Rules

If E(Y) = 2, V(Y) = 3

E(5Y+3) = 5E(Y) + E(3) = 5 * 2 + 3 = 13

V(3Y+1) = $3^2 V(Y+1)$ = 9V(Y+1) = 9V(Y) = 27

V(Y) = $E(Y^2) - (E(Y))^2$,

$E(Y^2) = V(Y) + (E(Y))^2$

Sample Questions:

Permutations

Four types of insects are randomly selected out of 50 given types. How many sample points are associated with the experiment?

P^{50}_4

Cardinality

1a)We record the birthday for each of 20 randomly selected people. We assume that there are only 365 possible distinct birthdays. Find the cardinality of the sample space.

|S| = 365 x 365 x 365… = 365^{20}

1b) We assume that each of the possible sets of birthdays are equiprobable what is the probability each person in the 20 has a different birthday?

A = {1st ≠ 2nd ≠ 3rd ….≠20th} → P(A) = $\frac{|A|}{365^{20}}$

|A| = 365 x 365 x 363 x … x 346

Bayes Rule:

1) Three companies produce colored balls. 20% of balls from company A are red, 30% of balls from company B are red, and 70% of balls from company C are red. You buy a ball from a store (which buys its balls with equal probability from A, B, or C) and observe that it is not red. What is the conditional probability that the store bought the ball from company C

N the event that the ball is not red, then by Bayes' Rule,

$$P(C|N) = \frac{P(C)P(N|C)}{P(A)P(N|A)+P(B)P(N|B)+P(C)P(N|C)}$$
$$= \frac{\frac{1}{3}\times 30\%}{\frac{1}{3}\times 80\% + \frac{1}{3}\times 70\% + \frac{1}{3}\times 30\%} = \frac{10\%}{60\%} = \frac{1}{6}.$$

2) Assume an HIV test is 99% correct for people with HIV and 99% for people without HIV. Assume in a country 0.3% of the population have HIV. If a randomly selected person tests positive, what is the probability that he has HIV?

A = {Person has HIV}, B = {Person tests positive}

P(A|B) = $\frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|\bar{A})P(\bar{A})}$

= $\frac{0.99\times 0.003}{0.99\times 0.003+0.99(0.997)}$

Binomial:

1) An oil exploration firm has raised enough capital to carry out 10 explorations, while the probability of success of each one is 10%. The explorations are independent, find the probability that at least 3 out of the 10 will be successful.

Y = # of successful explorations, Y~Binom(10, 0.1)

P(Y ≥ 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)

2) A particular sale involves 4 items randomly selected from a large lot that is known to contain 10% defectives. Let Y be the number of defectives among the four sold. The purchaser of the items will return the defectives for repair, and the repair cost is given by C = $3Y^2 + Y + 2$. Find E(C)

Y = #(Defectives in 4 sold)

Y ~ Binom(4, 0.1), $\mu = np = 0.4$, $\sigma^2 = np(1-p) = 0.9$

E(C) = $E(3Y^2 + Y + 2) = 3E(Y^2) + E(Y) + E(2)$

= $3(\sigma^2 + \mu^2) + 0.4 + 2$ *= 3(0.9+0.16)+0.4+2*

= 0.75+0.4+2 = 3.15✓

3) A company produces colored balls, and colors 20% blue, 50% red, and 30% white. Twenty balls are purchased at random. What is the probability that at least 15 of them are red

Y = # of red balls, Y ~ Binom(20, 0.5)

Negative Binomial

The employees of a firm that manufactures insulation are being tested for indications of asbestos in their lungs. The firm is requested to send 3 employees who have positive indications of asbestos on to a medical center for further testing. If 30% have positive indications of asbestos in their lungs, find the probability that 11 must be tested in order to find 3 positives.

Y = # of trials to get 3rd success

Y ~ NegBinom(3, 0.3), find P(Y = 11)

A high school basketball player has a 70% free throw success percentage. During the season, what is the probability that they make their third free throw on their 5th shot?

Y = # of trials to get 5th shot, Y ~ NegBinom(5, 0.7)

Geometric

A company produces colored balls, 20% blue, 50% red, and 30% white. A person buys balls from the company until the ball that they buy is red. What is the probability that he buys exactly 3 balls.

Y = # of balls, Y ~ Geom(0.5), Find P(Y = 3)

Let there be n people in the universe. THe evil Thanos thinks that n is too large and the universe suffers as a result. Therefore he decides to transport people into a parallel universe. With each snap, there is a 0.5 probability that he will lose his power. Assume that, if he does not lose his power, he will snap again and again, until he eventually loses his power. What is the expected number of snaps Thanos will make?

Y = # of snaps, Y ~ Geom(0.5). Find E(Y).

E(Y) = $\frac{1}{p} = \frac{1}{0.5} = 2\ snaps$

Hypergeometric

1) 100 balls are purchased, 20 blue, 50 red, and 30 white. 10 balls are drawn at random from these 100 balls. What is the probability that three out of these 10 are red?

Y = # of red balls, Y ~Hyp(100, 50, 10)

P(Y = 3) = $\frac{\binom{50}{3}\binom{50}{7}}{\binom{100}{10}}$

What is the probability that the first five draws are red and the next five draws are blue?

=P(first five are red)P(the next five are blue | first five are red)

P(first 5 red) = $\frac{\binom{50}{5}\binom{100-50}{5-5}}{\binom{100}{5}}$

Other Notes & Theorems

$$\frac{\binom{20}{5}\binom{95-20}{5-5}}{\binom{95}{5}}$$

Altogether = $\frac{\binom{50}{5}\binom{20}{5}}{\binom{100}{5}\binom{95}{5}}$

2) A student prepares for an exam by studying a list on 10 problems. She can solve 6 of them. For the exam, the instructor selects 5 problems at random from the 10 on the list given to the students. What is the probability that the student can solve all 5 problems on the exam?
Y = # of problems they know in the 5 selected
Y ~ Hyp(10, 5, 6)

3) A deck of cards contains 20 cards: 6 red cards and 14 black cards. 5 cards are drawn randomly without replacement. What is the probability that exactly 4 red cards are drawn?
Y = # of red cards drawn
Y ~ Hyp(20, 5, 6)

4) A small voting district has 101 female voters and 95 male voters. A random sample of 10 voters is drawn. What is the probability exactly 7 of the voters will be female?
Y = # voters that are female
Y ~ Hyp(196, 10, 101)

Poisson
1) The average number of patients admitted per day to the emergency room of a hospital is 10, while the corresponding random variable of the number of patients is assumed to be Poisson.
  a. If, on any given day, there are only 12 beds available for new patients, what is the probability that the hospital will not have enough beds to accommodate its newly admitted patients?
Y = # patients / day, Y ~ Poisson(10). → P(Y ≥ 13)
  b. What is the probability that more than 50 patients are admitted in a period of 4 days?
Z = # patients / 4 days, Z~Poisson(40), P(Z ≥ 51)

2) A certain typing agency employs two typists. The average number of errors per article is 1.2 when typed by the first typist and 3 when typed by the second. If your article is equally likely to be typed by either typist, find the probability that it will have no errors.
Y = # of errors in the article
$E_1(Errors/article) = 1.2 = \lambda_1$
$E_2(Errors/article) = 3 = \lambda_2$
Y_1 *~ Poisson(1.2),* Y_2 *~ Poisson(3)*
P(Y = 0) = $\frac{1}{2}(P(Y_1 = 0) + P(Y_2 = 0))$

3) The eruptions of a certain volcano occur according to a Poisson process with an average of 3 eruptions every year. A scientist observes the volcano for a period of 10 years.
  a. Let the random variable X be the total number of eruptions observed by the scientist. Compute E(X) and Var(X).
The average of X is 3 x 10 = 30, therefore,
X ~ Poisson(30) and E(X) = 30, V(X) = 30
  b. Find the probability that the scientist observes 2 or more eruptions over this period of 10 years.
P(X ≥ 2) = 1 - P(X = 0) - P(X = 1)

Partition
There are 9 people in a party, including Hannah and Sarah. We arrange the 9 people into 3 groups (A, B, C), where each group has 3 people. How many arrangements are there for Hannah and Sarah to be in the same group?
If Hanna and Sarah are in the same group the remaining 7 will be placed into 3 groups of 1, 3, and 3
Total arrangements = 3
$\frac{7!}{3!3!1!}$ *= 140*
(multiplied by 3 since they could be in A, B, C.)

Other Sample Problems
1) Consider the experiment, called the **birthday problem**, where our task is to determine the probability that in a group of k people there are at least two people who have the same birthday (assume no leap years). Show the following fact: it takes k = 23 people to have at least two people with the same birthday with probability 50% or more.
A = {at least 2 people have the same birthday}
P(A) = 1 -
$\frac{365!}{365^{23}(365-23)!} = 1 - \frac{365!}{365^{23}342!}$ *= 1 -*
0.04927 = 0.5073 > 0.5✓

2) Let p(y) = c(1+y^2), where y = 0, 1, 2, 3, and p(y) is equal to zero otherwise. Determine c so that p(y) is a discrete probability distribution function.
C(1) + c(1+1) + c(1+4) + c(1+9) = 1
18c = 1, c = $\frac{1}{18}$
3) If E(X) = -1 and V(X) = 0.5, compute E($(X-4)^2$)
= $E(X^2 - 8X + 16) = E(X^2) - 8E(X) + 16 = E(X^2) + 24$
$V(X) = E(X^2) - (E(X))^2$
0.5 = $E(X^2)$ *- 1*
1.5 = $E(X^2)$*, so thus,* $E((X-4)^2) = 1.5 + 24 = 25.5$

Pigeonhole Principle:
For natural numbers k and m, if n = km+1 objects are distributed among m sets, then the pigeonhole principle asserts that at least one of the sets will contain at least k+1 objects. For arbitrary n and m, k+1 = [(n-1)/m]+1 = [n/m]

Partition Theorem
The number of ways of partitioning n distinct objects into k distinct groups containing $n_1, \ldots, n_k$ distinct objects respectively where each object appears in exactly one group

$$N = \frac{n!}{n_1!n_2!n_3!\ldots}$$

r-Permutations
An ordered arrangement of r distinct objects is called a “permutation” the number of ways of ordering an distinct objects taken r at a time is denoted by P_r^n

Permutations
Number of ways to order n objects: n!

Probability Axioms
Let S be a sample space associated with an experiment. To every event A, we assign a #, P(A), called probability the following axioms hold
  I. P(A) ≥ 0, P(Ø) = 0
  II. P(S) = 1
  III. Let A_1, A_2, A_3 be a sequence of events which are pairwise disjoint, that is: i, j , i ≠ *j,* $A_i \cap A_j \neq Ø$ *then,*

$$P(A_1 \cup A_2 \cup \ldots) = P(\bigcup_{i=2}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$
$$= (A_1) + P(A_2)\ldots$$