

Digital Signal Processing

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Abstract—This manual provides a simple introduction to digital signal processing.

1 SOFTWARE INSTALLATION

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
-sciPy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 DIGITAL FILTER

2.1 Download the sound file from

```
wget https://github.com/samar2605/EE3900/
blob/master/Assignment%201/codes/
Sound_Noise.wav
```

2.2 You will find a spectrogram at <https://academo.org/demos/spectrum-analyzer>. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?

Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the

synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.

Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frequency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
input_signal)
#output_signal = signal.lfilter(b, a,
input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2.

What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 DIFFERENCE EQUATION

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \quad (3.1)$$

Sketch $x(n)$.

3.2 Let

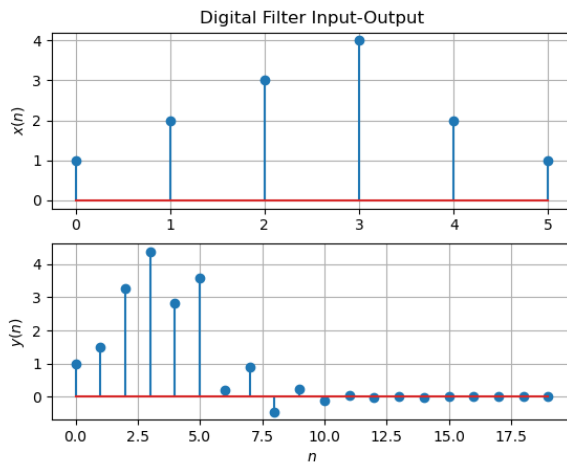
$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$

$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch $y(n)$.

Solution: The following code yields Fig. 3.2.

```
wget https://raw.githubusercontent.com/
samar2605/EE3900/master/filter/
codes/A1_3.py
```



Solution:

$$Z(x(n)) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.7)$$

$$= x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \quad (4.8)$$

$$x(4)z^{-4} + x(5)z^{-5} \\ = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \quad (4.9)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.10)$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.11)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.12)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.13)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.14)$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.15)$$

Solution: It is easy to show that

$$\delta(n) \stackrel{Z}{=} 1 \quad (4.16)$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (4.17)$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.18)$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \stackrel{Z}{=} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.19)$$

Solution:

$$Z(a^n u(n)) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \quad (4.20)$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \quad (4.21)$$

$$= \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1 \quad (4.22)$$

$$= \frac{1}{1 - az^{-1}}, \quad |a| < |z| \quad (4.23)$$

using the formula for the sum of an infinite geometric progression.

4.6 Let

$$H(e^{j\omega}) = H(z = e^{j\omega}). \quad (4.24)$$

Plot $|H(e^{j\omega})|$. Comment. $H(e^{j\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $x(n)$.

Solution: The graph is symmetric and periodic it is attaining high of value 4 and minimum between (0 - 0.5). It is bounded between (0, 4) and periodic with period (2π) because in the below equation $\cos(\omega)$ is periodic function having period 2π

$$H(e^{j\omega}) = \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \quad (4.25)$$

$$\Rightarrow |H(e^{j\omega})| = \frac{|1 + e^{-2j\omega}|}{|1 + \frac{e^{-j\omega}}{2}|} \quad (4.26)$$

$$= \frac{|1 + e^{2j\omega}|}{|e^{2j\omega} + \frac{e^{j\omega}}{2}|} \quad (4.27)$$

$$= \frac{|1 + \cos 2\omega + j \sin 2\omega|}{|e^{j\omega} + \frac{1}{2}|} \quad (4.28)$$

$$= \frac{|4 \cos^2(\omega) + 4j \sin(\omega) \cos(\omega)|}{|2e^{j\omega} + 1|} \quad (4.29)$$

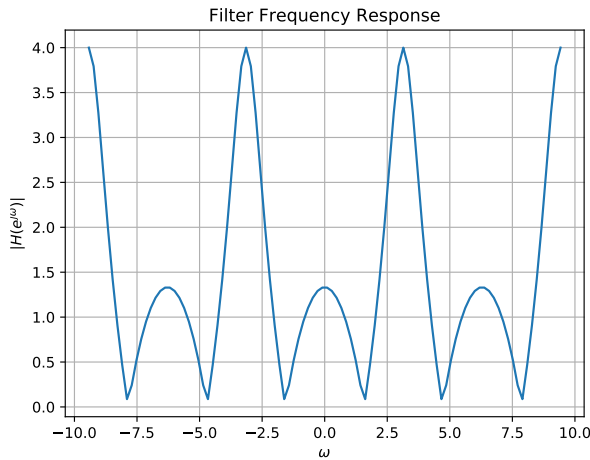
$$= \frac{|4 \cos(\omega)| |\cos(\omega) + j \sin(\omega)|}{|2 \cos(\omega) + 1 + 2j \sin(\omega)|} \quad (4.30)$$

$$\therefore |H(e^{j\omega})| = \frac{|4 \cos(\omega)|}{\sqrt{5 + 4 \cos(\omega)}} \quad (4.31)$$

The following code plots Fig. 4.6.

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filter/codes/dtft.
```

py

Fig. 4.6: $|H(e^{j\omega})|$

4.7 Express $x(n)$ in terms of $H(e^{j\omega})$.

Solution:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \quad (4.32)$$

and

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad (4.33)$$

Now,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad (4.34)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} e^{j\omega n} d\omega \quad (4.35)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega \quad (4.36)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} \cos \omega(n-k) d\omega \quad (4.37)$$

$$d\omega + \int_{-\pi}^{\pi} \sin \omega(n-k) d\omega$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} \cos \omega(n-k) d\omega \quad (4.38)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} h(k) \left. \frac{\sin \omega(n-k)}{n-k} \right|_{-\pi}^{\pi} \quad (4.39)$$

$$= \frac{1}{2\pi} \sum_{k \neq n} h(k) \frac{\sin \pi(n-k)}{n-k} + \sum_{k=n} h(n) \frac{\sin \pi(n-k)}{n-k} \quad (4.40)$$

$$= \frac{0 + 2\pi h(n)}{2\pi} \quad (4.41)$$

$$= h(n) \quad (4.42)$$

5 IMPULSE RESPONSE

5.1 Using long division, find

$$h(n), \quad n < 5 \quad (5.1)$$

for $H(z)$ in (4.12).

Solution: We substitute $x := z^{-1}$, and perform the long division.

$$\begin{array}{r} 2x - 4 \\ \frac{1}{2}x + 1 \overline{) x^2 + 1} \\ \underline{-x^2 - 2x} \\ -2x + 1 \\ \underline{2x + 4} \\ 5 \end{array}$$

$$\Rightarrow (1 + z^{-2}) = \left(\frac{1}{2}z^{-1} + 1\right)(2z^{-1} - 4) + 5 \quad (5.2)$$

$$\Rightarrow \frac{(1 + z^{-2})}{\frac{1}{2}z^{-1} + 1} = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \quad (5.3)$$

$$\Rightarrow H(z) = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \quad (5.4)$$

Now, consider $\frac{5}{\frac{1}{2}z^{-1} + 1}$

The denominator $\frac{1}{2}z^{-1} + 1$ can be expressed as sum of an infinite geometric progression, which as its first term equal to 1 and common ratio $\frac{-1}{2}z^{-1}$

Therefore, we can write $\frac{5}{\frac{1}{2}z^{-1} + 1}$ as $5\left(1 + \left(\frac{-1}{2}z^{-1}\right) + \left(\frac{-1}{2}z^{-1}\right)^2 + \left(\frac{-1}{2}z^{-1}\right)^3 + \left(\frac{-1}{2}z^{-1}\right)^4 + \dots\right)$

Therefore, $H(z)$ can be given by,

$$H(z) = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \quad (5.5)$$

$$(5.6)$$

$$= 2z^{-1} - 4 + 5 + \frac{-5}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} \quad (5.7)$$

$$\Rightarrow H(z) = 1z^0 + \frac{-1}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} \quad (5.8)$$

Comparing the above expression to (4.1) we get $h(n)$ for $n < 5$ as,

$$h(0) = 1 \quad (5.9)$$

$$h(1) = \frac{-1}{2} \quad (5.10)$$

$$h(2) = \frac{5}{4} \quad (5.11)$$

$$h(3) = \frac{-5}{8} \quad (5.12)$$

$$h(4) = \frac{5}{16} \quad (5.13)$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \stackrel{Z}{\Leftrightarrow} H(z) \quad (5.14)$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.12),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (5.15)$$

$$\Rightarrow h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.16)$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.

Solution: Yes, it is bounded between and convergent. We can clearly see in the plot it is not tending to infinite and remain finite. The following code plots Fig. 5.3.

```
wget https://raw.githubusercontent.com/
samar2605/EE3900/master/filter/codes/
A1_5_3.py
```

we know that

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.17)$$

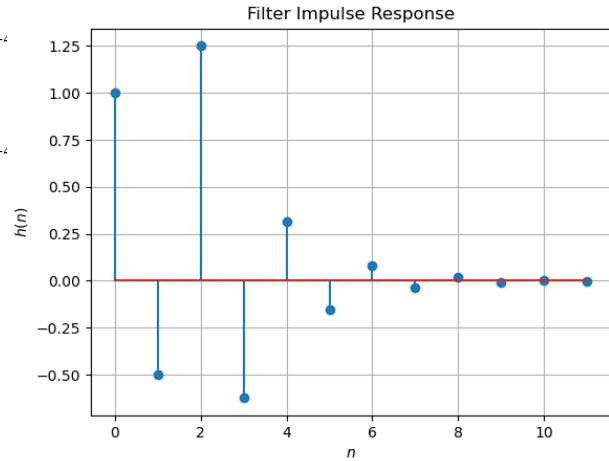


Fig. 5.3: $h(n)$ as the inverse of $H(z)$

Implies we can write that

$$h(n) = \begin{cases} 0 & , n < 0 \\ \left(-\frac{1}{2}\right)^n & , 0 \leq n < 2 \\ 5\left(-\frac{1}{2}\right)^n & , n \geq 2 \end{cases} \quad (5.18)$$

A sequence is said to be bounded when

$$|x_n| \leq M, \forall n \in \mathcal{N} \quad (5.19)$$

Now consider (5.18),

For $n < 0$,

$$|h(n)| \leq 0 \quad (5.20)$$

For $0 \leq n < 2$,

$$|h(n)| = \left(\frac{1}{2}\right)^n \quad (5.21)$$

$$\Rightarrow |h(n)| \leq 1 \quad (5.22)$$

For $n \geq 2$,

$$|h(n)| = 5\left(\frac{1}{2}\right)^n \quad (5.23)$$

$$\Rightarrow |h(n)| \leq 5 \quad (5.24)$$

From above we can say that,

$$M = \max\{0, 1, 5\} \quad (5.25)$$

$$= 5 \quad (5.26)$$

Therefore since M exists and is a real value, we can say that $h(n)$ is bounded.

5.4 Convergent? Justify using the ratio test.

Solution: We see that $h(n)$ is bounded. For

large n , we see that

$$h(n) = \left(-\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^{n-2} \quad (5.27)$$

$$= \left(-\frac{1}{2}\right)^n (4 + 1) = 5 \left(-\frac{1}{2}\right)^n \quad (5.28)$$

$$\Rightarrow \left| \frac{h(n+1)}{h(n)} \right| = \frac{1}{2} \quad (5.29)$$

and therefore, $\lim_{n \rightarrow \infty} \left| \frac{h(n+1)}{h(n)} \right| = \frac{1}{2} < 1$. Hence, we see that $h(n)$ converges.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.30)$$

Is the system defined by (3.2) stable for the impulse response in (5.14)?

Solution: By using $h(n)$ from 5.3

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.31)$$

$$= \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.32)$$

$$= \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.33)$$

$$= \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} \quad (5.34)$$

$$= \frac{2}{3} + \frac{2}{3} < \infty \quad (5.36)$$

5.6 Verify the above result using a python code.

Solution:

```
wget https://raw.githubusercontent.com/
samar2605/EE3900/master/filter/codes/
A1_5_6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.37)$$

This is the definition of $h(n)$.

Solution: The following code plots Fig. 5.7.

```
wget https://raw.githubusercontent.com/
samar2605/EE3900/master/filter/
codes/A1_5_7.py
```

Note that this is the same as Fig. 5.3.

$$= h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2) \quad (5.38)$$

$$= H(z) + \frac{1}{2}z^{-1}H(z) = 1 + z^{-2} \quad (5.39)$$

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (5.40)$$

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.41)$$

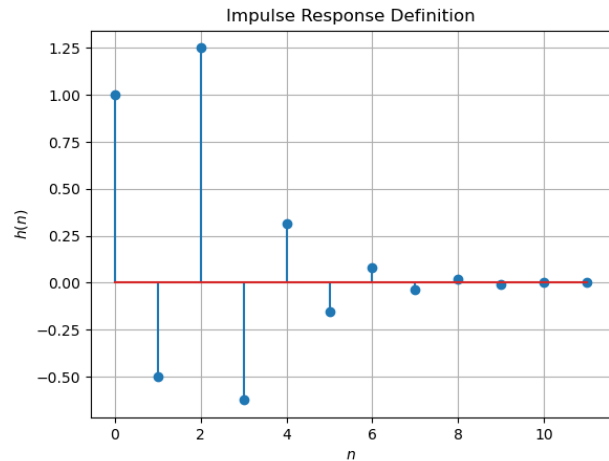


Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.42)$$

Comment. The operation in (5.42) is known as *convolution*.

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. ??.

```
wget https://raw.githubusercontent.com/
samar2605/EE3900/master/filter/codes/
A1_5_8.py
```

5.9 Express the above convolution using a Teoplitz matrix.

Solution:

We know that from, (5.42),

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.43)$$

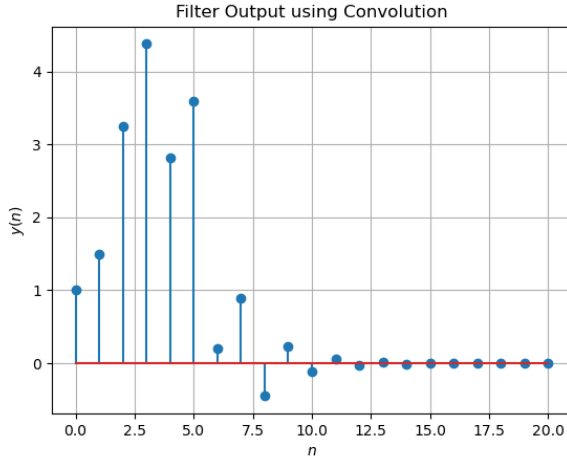


Fig. 5.8: $y(n)$ from the definition of convolution

This can also be written as a matrix-vector multiplication given by the expression,

$$y = T(h) * x \quad (5.44)$$

In the equation (5.44), $T(h)$ is a Teoplitz matrix.

The equation (5.44) can be expanded as,

$$y = x \otimes h \quad (5.45)$$

$$y = \begin{pmatrix} h_1 & 0 & \cdot & \cdot & \cdot & 0 \\ h_2 & h_1 & \cdot & \cdot & \cdot & 0 \\ h_3 & h_2 & h_1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdot & \cdot & 0 \\ h_n & h_{n-1} & h_{n-2} & \cdot & \cdot & h_1 \\ 0 & h_n & h_{n-1} & h_{n-2} & \cdot & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & h_{n-1} \\ 0 & \cdot & \cdot & \cdot & 0 & h_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \quad (5.46)$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.47)$$

Solution: From (5.42), we substitute $k := n-k$

to get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.48)$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \quad (5.49)$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.50)$$

6 DFT AND FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (6.1)$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.2)$$

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \quad (6.3)$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filter/codes/yndft.py
```

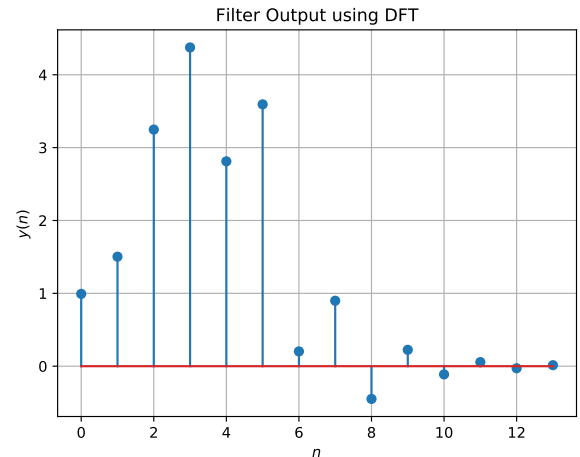


Fig. 6.3: $y(n)$ from the DFT

- 6.4 Repeat the previous exercise by computing $X(k)$, $H(k)$ and $y(n)$ through FFT and IFFT.
- 6.5 Wherever possible, express all the above equations as matrix equations.
- 6.6 Verify the above equations by generating the DFT matrix in python.

7 EXERCISES

Answer the following questions by looking at the python code in Problem 2.3.

- 7.1 The command

```
output_signal = signal.lfilter(b, a,
                               input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^M a(m) y(n-m) = \sum_{k=0}^N b(k) x(n-k) \quad (7.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

- 7.2 Repeat all the exercises in the previous sections for the above a and b .
- 7.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHz.
- 7.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.
- 7.5 Modifying the code with different input parameters and to get the best possible output.