PFPL Supplement: Relating Transition and Cost Dynamics for Parallel **PCF**[*]

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1   Introduction

A cost dynamics for a language requires validation: what makes the cost assignments sensible? One form of validation is a Brent-type theorem, which provides a concrete bound on the cost in terms of an abstract machine model. The bounds are stated in terms of one or more parameters—such as the number of processing elements—of the machine model. The proof amounts to the description of a scheduler for the specified platform that achieves those bounds.

Another form of validation is to relate a cost dynamics to a transition dynamics for the same language. The assigned cost should determine the number of transitions required to achieve the value of an expression, if it has one. For a sequential SOS the correspondence validates the work measure of a cost graph, and for a parallel SOS the correspondence validates the span measure of a cost graph.

The purpose of this note is to state and prove these correspondences for the modal formulation of **PCF** enriched with parallelism, as described in Harper (2018).

2   Cost and Transition Dynamics

For convenient reference the cost dynamics of parallel **PCF** is given in Figure 1. The parallel transition dynamics is given in Figure 2. The sequential transition dynamics may be derived from Harper (2019), giving a left-to-right interpretation of the parallel bind construct. When it is necessary to distinguish them, the parallel transition judgment is written $e \underset{\mathsf{par}}{\longmapsto} e'$ and the sequential analogue is written $e \underset{\mathsf{seq}}{\longmapsto} e'$.

Instead of defining a separate sequential transition dynamics for parallel **PCF**, it may be considered to be defined implicitly by the parallel transition dynamics under the interpretation of the lazy product type $\tau_1 \mathbin{\&} \tau_2$ as if it were the eager product type $\tau_1 \, \mathsf{comp} \otimes \tau_2 \, \mathsf{comp}$ of two suspended computations. This interpretation forces a sequential (either left-to-right or right-to-left) evaluation order of the components of a pair, so that the length of the transition sequence corresponds to the work, rather than the span, of a computation.

$$\frac{}{\texttt{ret}(\,v\,) \Downarrow^{\mathbf{1}} v} \tag{1a}$$

$$\frac{e_0 \Downarrow^c v}{\texttt{ifz}\{\tau\}(\,\texttt{z}\,;e_0\,;x\,.\,e_1\,) \Downarrow^{\mathbf{1}\oplus c} v} \tag{1b}$$

$$\frac{[e/x]e_1 \Downarrow^c v}{\texttt{ifz}\{\tau\}(\,\texttt{s}(\,e\,)\,;e_0\,;x\,.\,e_1\,) \Downarrow^{\mathbf{1}\oplus c} v} \tag{1c}$$

$$\frac{[\texttt{fun}\{\tau_2\,;\tau\}(\,f\,.\,x\,.\,e\,),v_2/f,x]e \Downarrow^c v}{\texttt{ap}(\,\texttt{fun}\{\tau_2\,;\tau\}(\,f\,.\,x\,.\,e\,)\,;v_2\,) \Downarrow^{\mathbf{1}\oplus c} v} \tag{1d}$$

$$\frac{e \Downarrow^c v \quad [v/x]e_2 \Downarrow^{c_2} v_2}{\texttt{bnd}(\,\texttt{comp}(\,e\,)\,;x\,.\,e_2\,) \Downarrow^{\mathbf{1}\oplus c \oplus c_2} v_2} \tag{1e}$$

$$\frac{e_1 \Downarrow^{c_1} v_1 \quad e_2 \Downarrow^{c_2} v_2 \quad [v_1 \otimes v_2/x]e \Downarrow^c v}{\texttt{parbnd}(\,e_1 \,\&\, e_2\,;x\,.\,e\,) \Downarrow^{(c_1 \otimes c_2)\oplus \mathbf{1}\oplus c} v} \tag{1f}$$

$$\frac{[v_1,v_2/x_1,x_2]e \Downarrow^c v}{\texttt{split}(\,v_1 \otimes v_2\,;x_1,x_2\,.\,e\,) \Downarrow^{\mathbf{1}\oplus c} v} \tag{1g}$$

Figure 1: Cost Dynamics

3 Validating the Cost Dynamics

Theorem 3.1. *If $e \Downarrow^c v$, then $e \underset{par}{\longmapsto}^n v$, where $n = dp(c)$, and $e \underset{seq}{\longmapsto}^n v$, where $n = wk(c)$.*

Proof. By rule induction on cost dynamics. The validation of the depth is justified by piecing together derivations in the parallel or sequential dynamics, respectively.

Consider rule (1f), so that $\texttt{parbnd}(\,e_1 \,\&\, e_2\,;x\,.\,e_3\,) \Downarrow^c v_3$, where

1. $e_1 \Downarrow^{c_1} v_1$,

2. $e_2 \Downarrow^{c_2} v_2$,

3. $[v_1 \otimes v_2/x]e_3 \Downarrow^{c_3} v_3$, and

4. $c = (c_1 \otimes c_2) \oplus \mathbf{1} \oplus c_3$.

By induction

1. $e_1 \underset{par}{\longmapsto}^{n_1} v_1$ with $n_1 = dp(c_1)$.

2. $e_2 \underset{par}{\longmapsto}^{n_2} v_2$ with $n_2 = dp(c_2)$.

3. $e_3 \underset{par}{\longmapsto}^{n_3} v_3$ with $n_3 = dp(c_3)$.

$$\overline{e \ \mathsf{initial}} \tag{2a}$$

$$\frac{v \ \mathsf{val}}{\mathtt{ret}(\,v\,) \ \mathsf{final}} \tag{2b}$$

$$\overline{\mathtt{ifz\{z\}}(\,e_0\,;\,x\,;\,e_1\,.\longmapsto\,)e_0} \tag{2c}$$

$$\overline{\mathtt{ap}(\,\mathtt{fun}\{\tau_2\,;\,\tau\}(\,f\,.\,x\,.\,e\,)\,;\,v_2\,)\longmapsto[\mathtt{fun}\{\tau_2\,;\,\tau\}(\,f\,.\,x\,.\,e\,),v_2/f,x]e} \tag{2d}$$

$$\frac{e_1 \longmapsto e_1'}{\mathtt{bnd}(\,\mathtt{comp}(\,e_1\,)\,;\,x\,.\,e_2\,)\longmapsto\mathtt{bnd}(\,\mathtt{comp}(\,e_1'\,)\,;\,x\,.\,e_2\,)} \tag{2e}$$

$$\overline{\mathtt{bnd}(\,\mathtt{comp}(\,\mathtt{ret}(\,v_1\,)\,)\,;\,x\,.\,e_2\,)\longmapsto[v_1/x]e_2} \tag{2f}$$

$$\frac{e_1 \longmapsto e_1' \quad e_2 \longmapsto e_2'}{\mathtt{parbnd}(\,e_1 \,\&\, e_2\,;\,x\,.\,e\,)\longmapsto\mathtt{parbnd}(\,e_1' \,\&\, e_2'\,;\,x\,.\,e\,)} \tag{2g}$$

$$\frac{e_1 \longmapsto e_1'}{\mathtt{parbnd}(\,e_1 \,\&\, \mathtt{ret}(\,v_2\,)\,;\,x\,.\,e\,)\longmapsto\mathtt{parbnd}(\,e_1' \,\&\, \mathtt{ret}(\,v_2\,)\,;\,x\,.\,e\,)} \tag{2h}$$

$$\frac{e_2 \longmapsto e_2'}{\mathtt{parbnd}(\,\mathtt{ret}(\,v_1\,) \,\&\, e_2\,;\,x\,.\,e\,)\longmapsto\mathtt{parbnd}(\,\mathtt{ret}(\,v_1\,) \,\&\, e_2'\,;\,x\,.\,e\,)} \tag{2i}$$

$$\overline{\mathtt{parbnd}(\,\mathtt{ret}(\,v_1\,) \,\&\, \mathtt{ret}(\,v_2\,)\,;\,x\,.\,e\,)\longmapsto[v_1 \otimes v_2/x]e} \tag{2j}$$

$$\overline{\mathtt{split}(\,v_1 \otimes v_2\,;\,x_1,x_2\,.\,e\,)\longmapsto[v_1,v_2/x_1,x_2]e} \tag{2k}$$

Figure 2: Parallel Transition Dynamics

Assume that n_1 is no larger than n_2; the other case is handled analogously. Applying the rules of parallel transition,

$$\mathtt{parbnd}(\, e_1 \,\&\, e_2 \,;\, x \,.\, e_3 \,) \xmapsto[\mathit{par}]{n_1} \mathtt{parbnd}(\, \mathtt{ret}(\, v_1 \,) \,\&\, e_2' \,;\, x \,.\, e_3 \,) \text{ (for some } e_2')$$

$$\xmapsto[\mathit{par}]{n_2 - n_1} \mathtt{parbnd}(\, \mathtt{ret}(\, v_1 \,) \,\&\, \mathtt{ret}(\, v_2 \,) \,;\, x \,.\, e_3 \,)$$

$$\xmapsto[\mathit{par}]{1} [v_1 \otimes v_2 / x] e_3$$

$$\xmapsto[\mathit{par}]{n_3} v_3.$$

The length of this sequence is exactly the depth, $dp(c) = \max(n_1, n_2) + 1 + n_3$.

The validation of the work is similar, albeit using the sequential dynamics. In that case the transition sequence has length $n_1 + n_2 + 1 + n_3$, which is $wk(c)$.

$\square$

Lemma 3.2 (Cost Head Expansion). *1. If $e \xmapsto{\ \ }_{\mathit{par}} e'$ and $e' \Downarrow^{c'} v$, then $e \Downarrow^c v$ for some c such that $dp(c) = dp(c') + 1$ and $wk(c) = wk(c') + k$ for some $k > 0$.*

2. If $e \xmapsto[\mathit{seq}]{} e'$ and $e' \Downarrow^{c'} v$, then $e \Downarrow^c v$ for some c such that $wk(c) = wk(c') + 1$.

Proof. Consider the parallel case; the sequential is handled similarly. Proceed by induction on the derivation of the initial transition.

For example, if the initial transition is by rule (2g), then $e = \mathtt{parbnd}(\, e_1 \,\&\, e_2 \,;\, x \,.\, e_3 \,)$, $e' = \mathtt{parbnd}(\, e_1' \,\&\, e_2' \,;\, x \,.\, e_3 \,)$, with $e_1 \xmapsto[\mathit{par}]{} e_1'$ and $e_2 \xmapsto[\mathit{par}]{} e_2'$. By assumption $\mathtt{parbnd}(\, e_1' \,\&\, e_2' \,;\, x \,.\, e_3 \,) \Downarrow^{c'} v$ for some cost graph c'. By inversion $c' = (c_1' \otimes c_2') \oplus \mathbf{1} \oplus c_3$, where

1. $e_1' \Downarrow^{c_1'} v_1$ for some v_1,

2. $e_2' \Downarrow^{c_2'} v_2$ for some v_2, and

3. $[v_1 \otimes v_2 / x] e_3 \Downarrow^{c_3} v$.

By induction

1. $e_1 \Downarrow^{c_1} v_1$ with $dp(c_1) = dp(c_1') + 1$, and

2. $e_2 \Downarrow^{c_2} v_2$ with $dp(c_2) = dp(c_2') + 1$.

Consequently, by rule (1f), $e \Downarrow^c v$, where $c = (c_1 \otimes c_2) \oplus \mathbf{1} \oplus c_3$. Calculate as follows:

$$\begin{aligned}
dp(c) &= \max(dp(c_1), dp(c_2)) + dp(c_3) + 1 \\
&= \max(dp(c_1') + 1, dp(c_2') + 1) + dp(c_3) + 1 \\
&= \max(dp(c_1'), dp(c_2')) + 1 + dp(c_3) + 1 \\
&= dp(c') + 1.
\end{aligned}$$

$\square$

Corollary 3.3. *For any e and v, $e \xmapsto[\mathit{par}]{}{}^* \mathtt{ret}(\, v \,)$ iff $e \xmapsto[\mathit{seq}]{}{}^* \mathtt{ret}(\, v \,)$.*

Proof. It follows from Lemma 3.2 that if $e \longmapsto^* \mathtt{ret}(v)$ sequentially or in parallel, then $e \Downarrow^c v$ for some cost c. The converse is Theorem 3.1. ∎

Theorem 3.4. *If* $e \underset{par}{\longmapsto}^d \mathtt{ret}(v)$ *and* $e \underset{seq}{\longmapsto}^w \mathtt{ret}(v)$, *then there exists c such that $e \Downarrow^c v$ with* $dp(c) = d + 1$ *and* $wk(c) = w + k$ *for some* $k > 0$.

Proof. Either $d = w = 0$ or $d = d'+1$ and $w = w'+1$. In the former case take $c = \mathbf{1}$, whose work and depth are 1, and apply rule (1a). In the latter $e \underset{par}{\longmapsto} e' \underset{par}{\longmapsto}^{d'} \mathtt{ret}(v)$, and $e \underset{seq}{\longmapsto} e' \underset{seq}{\longmapsto}^{w'} \mathtt{ret}(v)$. By induction there exists c' such that $e \Downarrow^{c'} v$ with $dp(c') = d' + 1$ and $wk(c') = w' + k'$, and the result follows by Lemma 3.2. ∎

References

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