Watermarking Cryptographic Functionalities from Standard Lattice Assumptions

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Joint work with David J. Wu

Digital Watermarking

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- Content is (mostly) viewable

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- Difficult to remove (without destroying the image)

Watermarking Programs

[NSS99, BGIRSVY01, HMW07, YF11, Nis13, CHNVW16, BLW17]

Embed a “mark” within a program

Watermarking Programs

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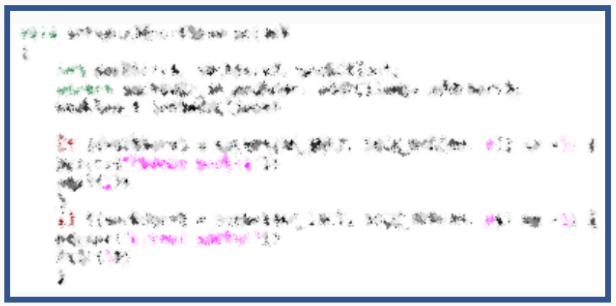
```
void serveur1(portServ ports)
{
    int sockServ1, sockServ2, sockClient;
    struct sockaddr_in monAddr, addrClient, addrServ2;
    socklen_t lenAddrClient;

    if ((sockServ1 = socket(AF_INET, SOCK_STREAM, 0)) == -1) {
    perror("Erreur socket");
    exit(1);
    }
    if ((sockServ2 = socket(AF_INET, SOCK_STREAM, 0)) == -1) {
    perror("Erreur socket");
    exit(1);
    }
```

CRYPTO



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If mark is removed, then program is destroyed

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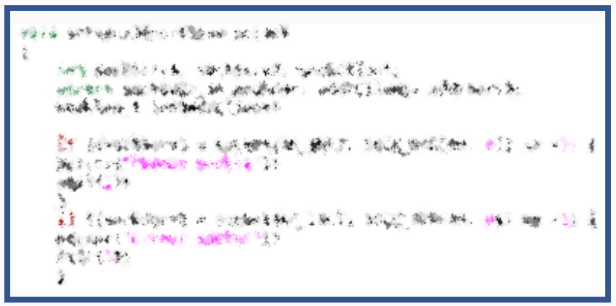

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Two main algorithms:

- Mark(wsk, C) $\rightarrow C'$: Takes circuit C and outputs marked circuit C'
- Verify(wsk, C') $\rightarrow \{0,1\}$: Tests whether a circuit C' is marked or not

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$$C(x) = C'(x)$$

for all inputs x.

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Too Strong: Perfect functionality preserving is impossible for watermarking assuming indistinguishability obfuscation [BGIRSVY12].

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on all but a *negligible fraction* of inputs x.

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Unremovability: Given a marked program C, no efficient adversary can construct a circuit C' where

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Unforgeability: Given a set of marked programs $C_1, \ldots, C_\ell$, no efficient adversary can construct a circuit C' where

- for all i: $C_i(x) \neq C'(x)$ on a noticeable fraction of inputs x
- $\mathsf{Verify}(\mathsf{wsk}, C') = 1$

Can we watermark any programs?

Watermarking only achievable for functions that are not learnable.

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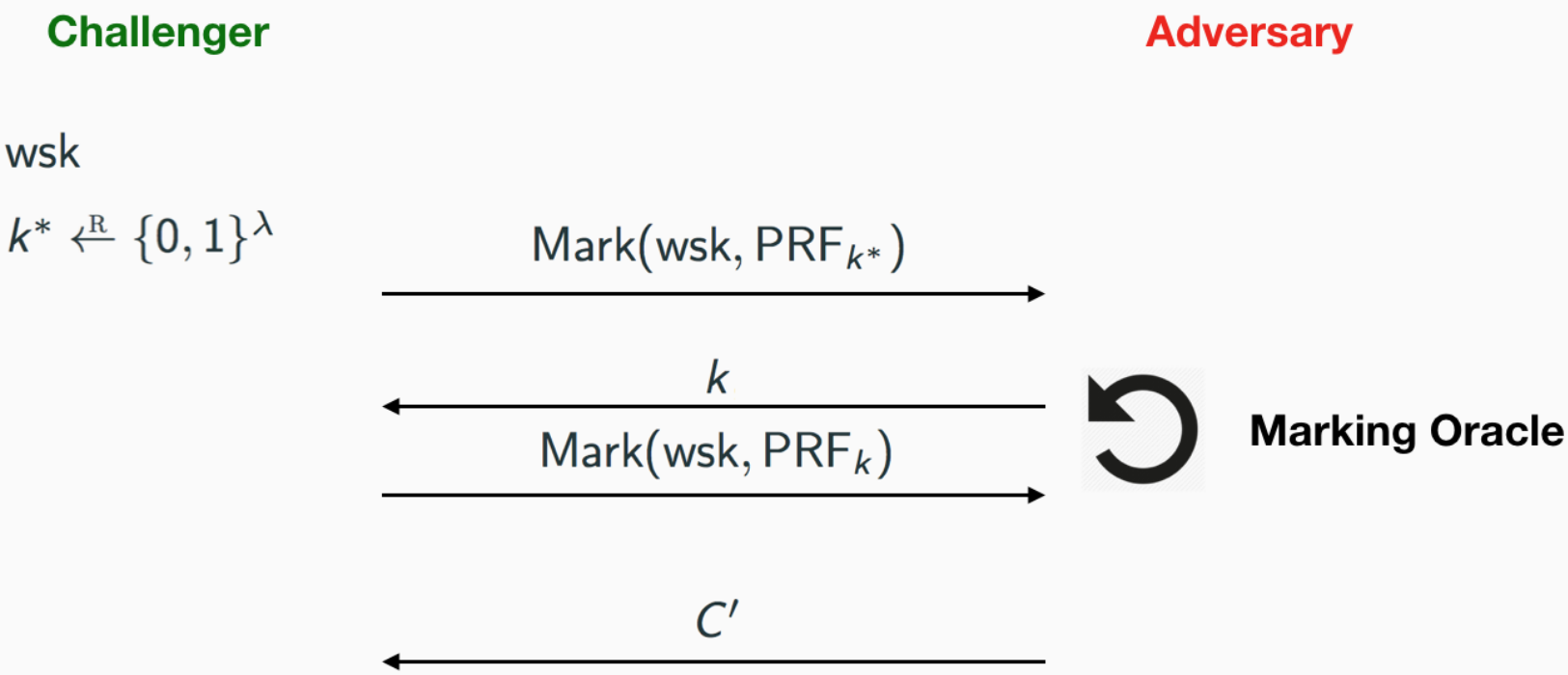
This work: Focus on watermarking **PRFs** [CHNVW16, BLW17].

Enables watermarking of symmetric primitives built from PRFs (e.g., encryption, MACs, etc.)

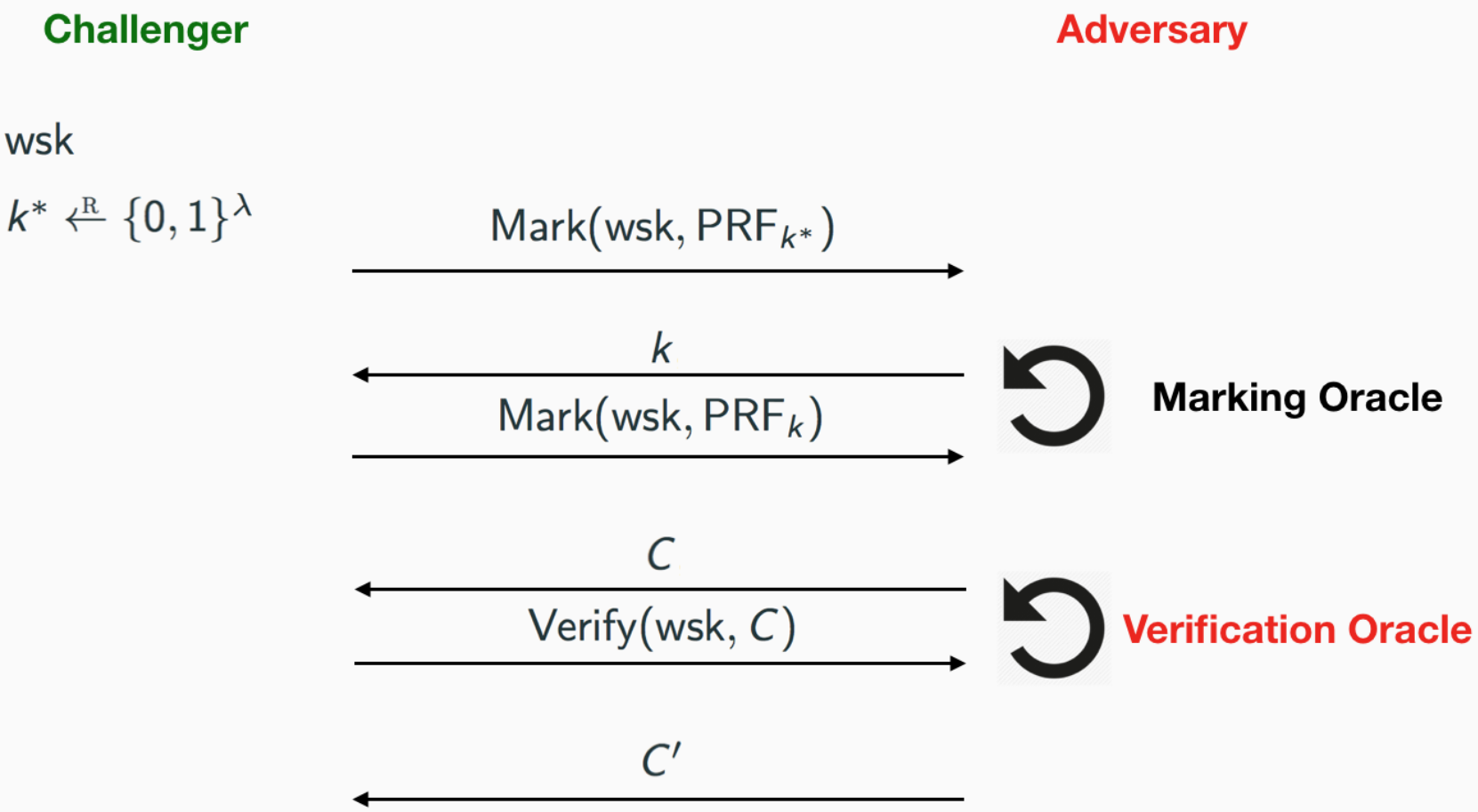
Security Definition (Weaker)



Adversary **wins** if

- $\mathsf{PRF}_{k^*}(x) = C'(x)$ on all but negligible fraction of inputs x
- $\mathsf{Verify}(\mathsf{wsk}, C') = 0$

Security Definition (Stronger)



Adversary **wins** if

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- $\mathsf{Verify}(\mathsf{wsk}, C') = 0$

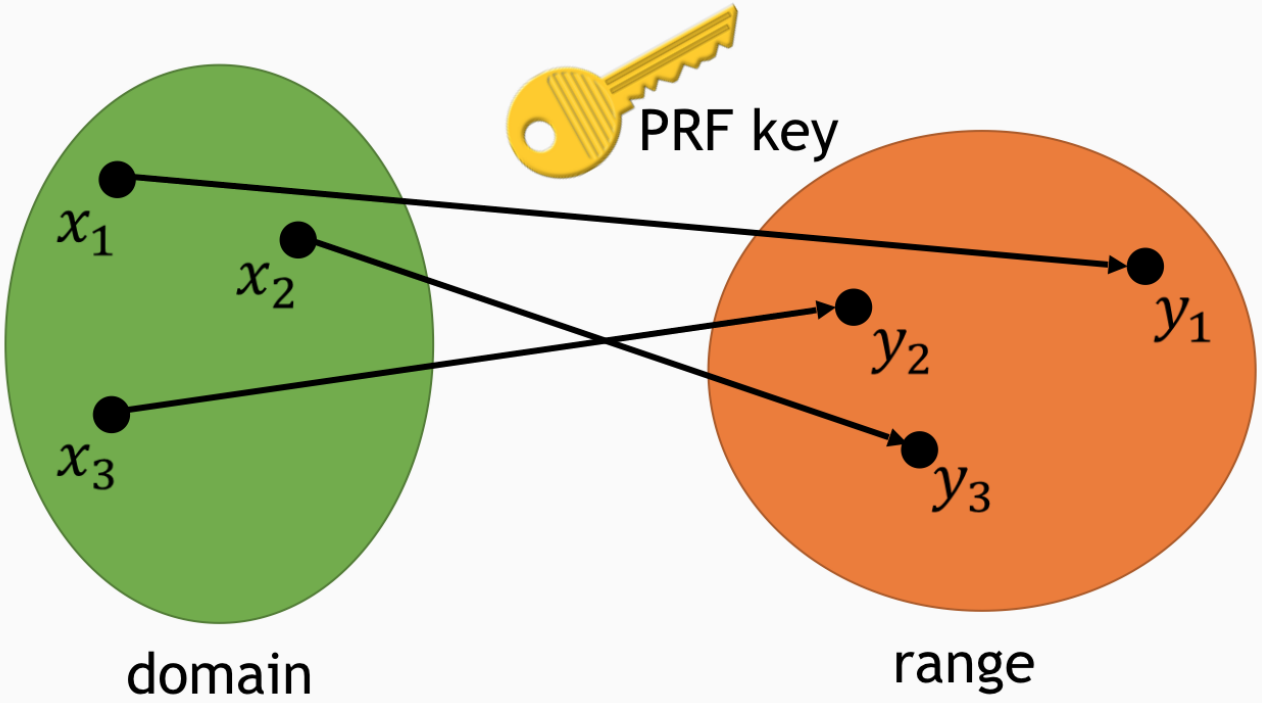
Main Results

Construct watermarking scheme for PRFs satisfying
(weaker) unremovability security definition (and unforgeability)
from LWE

Previous works:

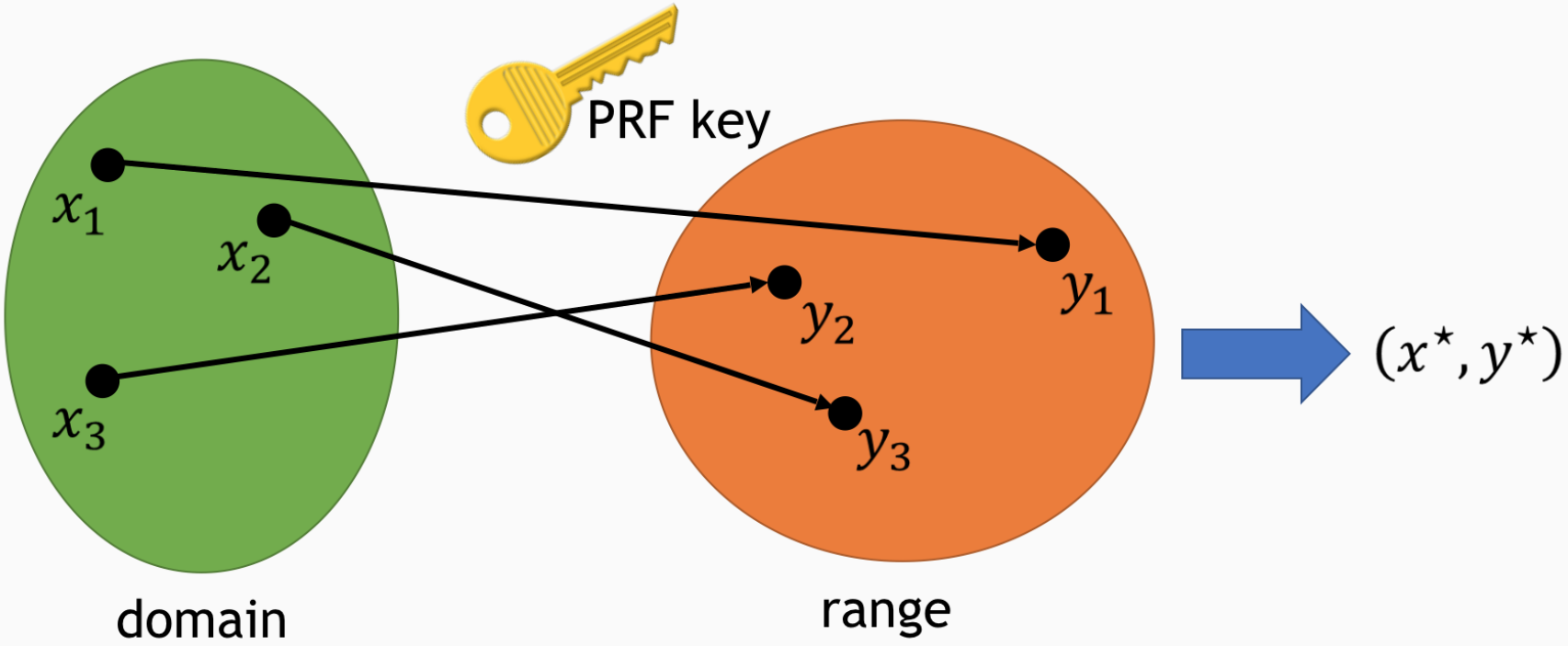
- [CHNVW16]: *(publicly verifiable)* unremovability from iO
- [BLW17]: *(weaker)* unremovability from iO

Blueprint for Watermarking PRFs [CHNVW16, BLW17]



Step 1: Evaluate PRF on test points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ (part of the watermarking secret key)

Blueprint for Watermarking PRFs [CHNVW16, BLW17]



Step 2: Derive a pair $(\mathbf{x}^\star, \mathbf{y}^\star)$ from $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$

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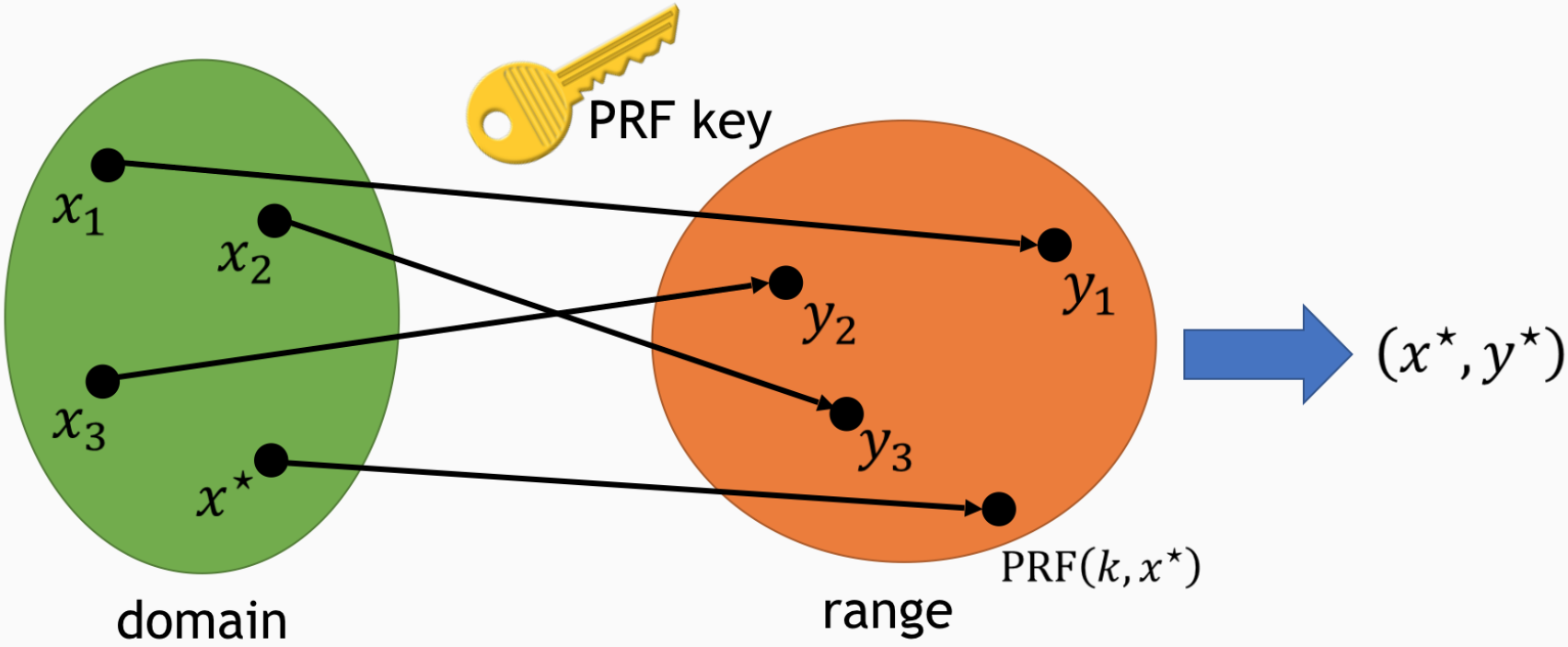

Step 3: "Marked key" is a circuit that implements the PRF at all points, except at $\mathbf{x}^\star$, the output is changed to $\mathbf{y}^\star$

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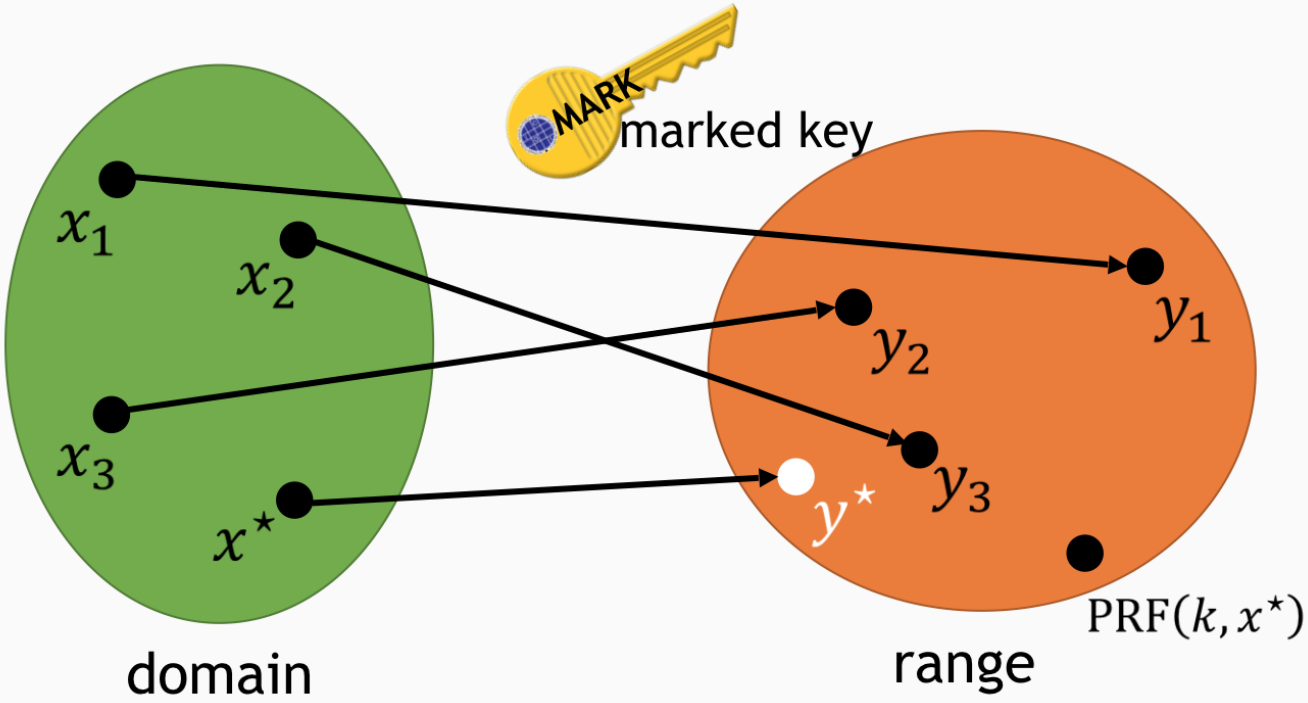

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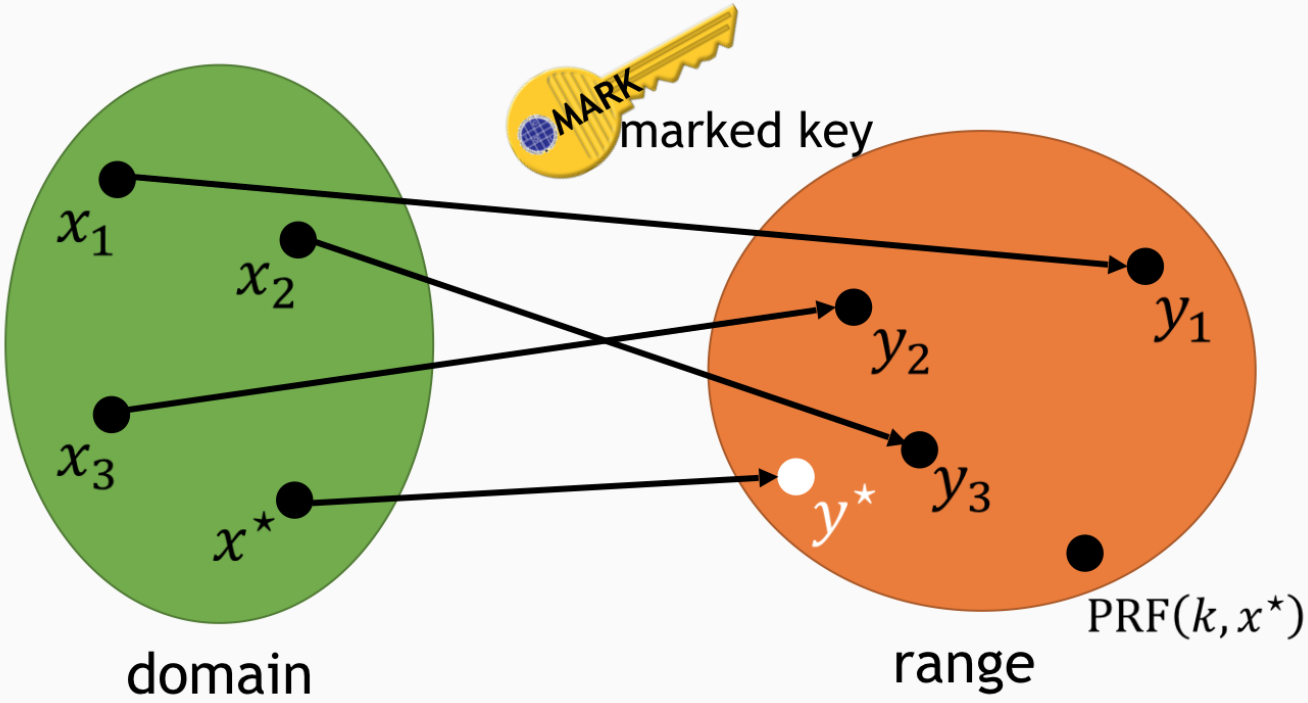

Verification: Evaluate function at x_1, x_2, x_3, derive $(\mathbf{x}^\star, \mathbf{y}^\star)$ and check if the value at $\mathbf{x}^\star$ matches $\mathbf{y}^\star$.

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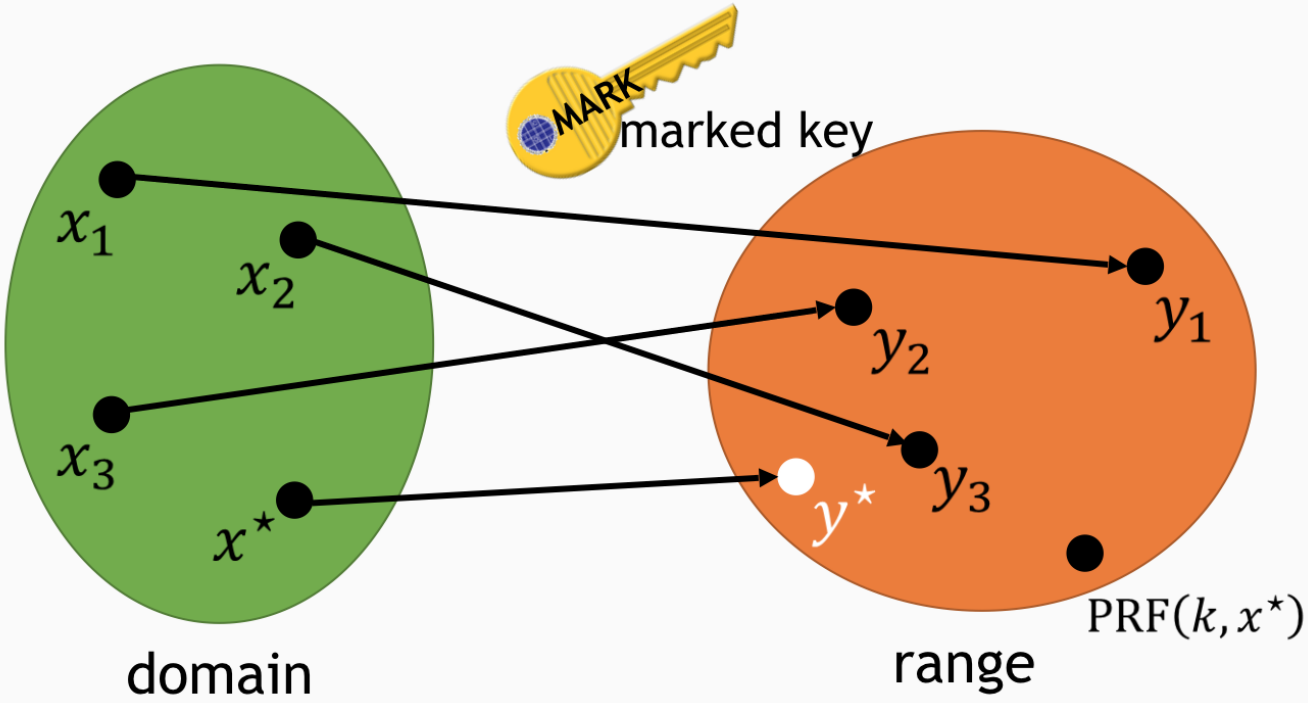


Functionality-preserving: Function differs at a single point

Unremovable: As long as adversary cannot tell that $(\mathbf{x}^\star, \mathbf{y}^\star)$ is "special"

Blueprint for Watermarking PRFs [CHNVW16, BLW17]

Obfuscated program:

$P_{(x^\star, y^\star)}(x)$:

- if $x = x^\star$, output $y^\star$
- else, output $\mathsf{PRF}(k, x)$

Prior solutions: Use obfuscation to hide $(\mathbf{x}^\star, \mathbf{y}^\star)$

Obfuscated program has PRF embedded inside and outputs $\mathsf{PRF}(k, \mathbf{x})$ on all inputs $\mathbf{x} \neq \mathbf{x}^\star$ and $\mathbf{y}^\star$ when $\mathbf{x} = \mathbf{x}^\star$

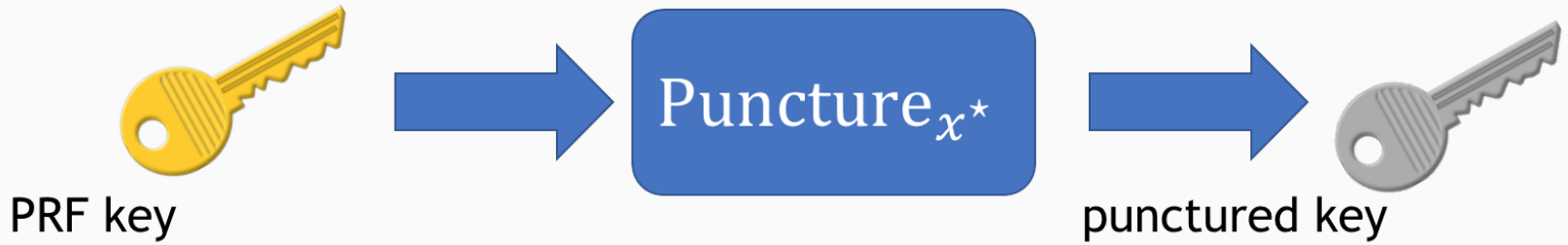
Question: How can we implement the blueprint just from **LWE**?

Private Puncturable PRFs [BLW17, BKM17, CC17, BTVW17]

Watermarked PRF implements PRF at all but a single point

Structurally very similar to a puncturable PRF [BW13, BGI13, KPTZ13]

Puncturable PRF:



Punctured key can be used to evaluate PRF on all points $\mathbf{x} \neq \mathbf{x}^\star$

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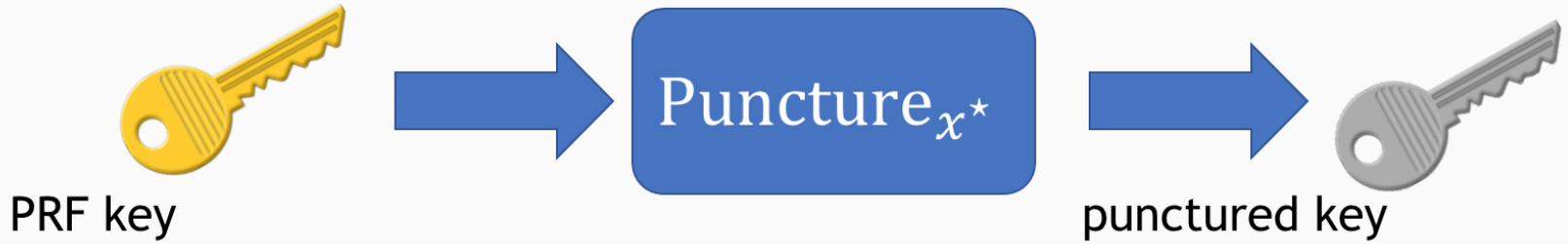

Recall general approach for watermarking:

1. Derive $(\mathbf{x}^\star, \mathbf{y}^\star)$ from input/output behavior of PRF
2. Give out a key that agrees with PRF everywhere, except has value $\mathbf{y}^\star$ at $\mathbf{x} = \mathbf{x}^\star$

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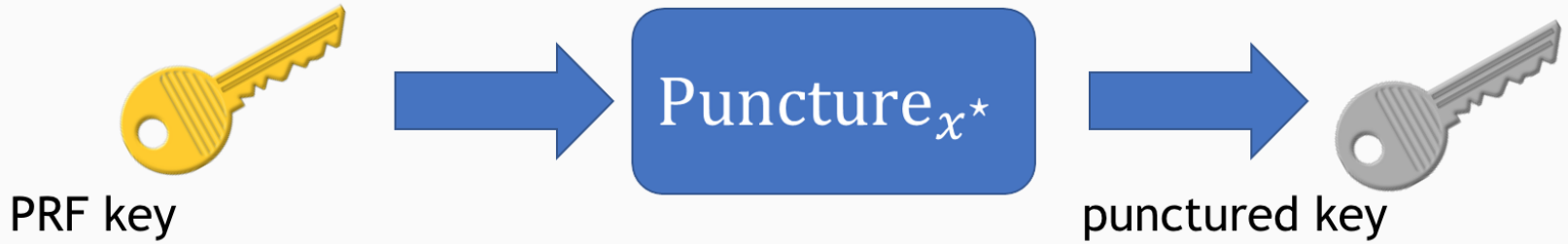

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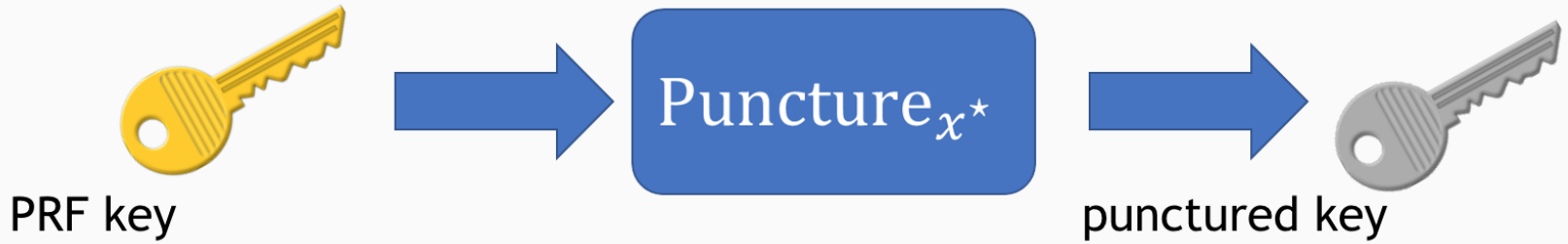

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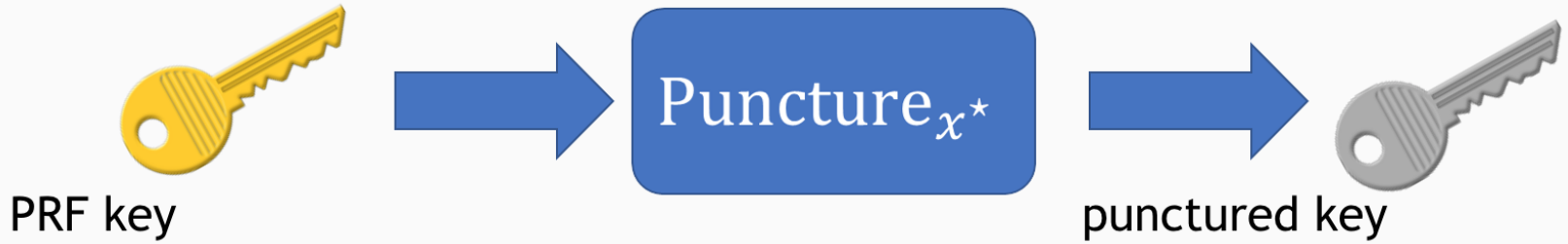

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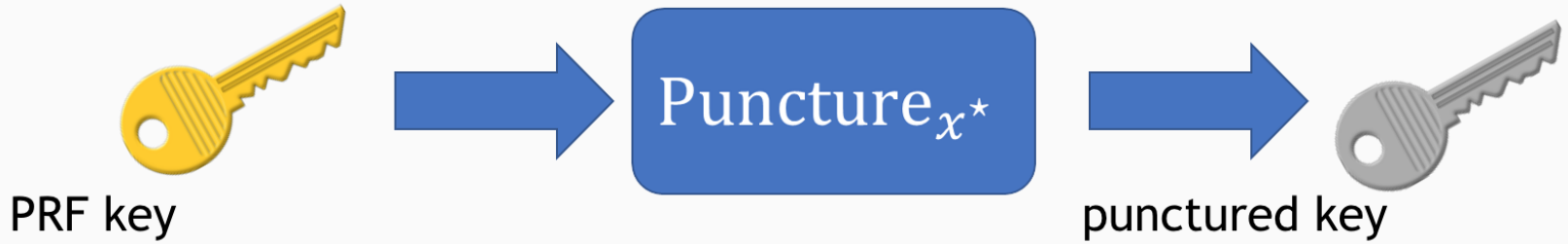

Problem 1: Punctured key does not necessarily hide $\mathbf{x}^\star$, which allows adversary to remove watermark

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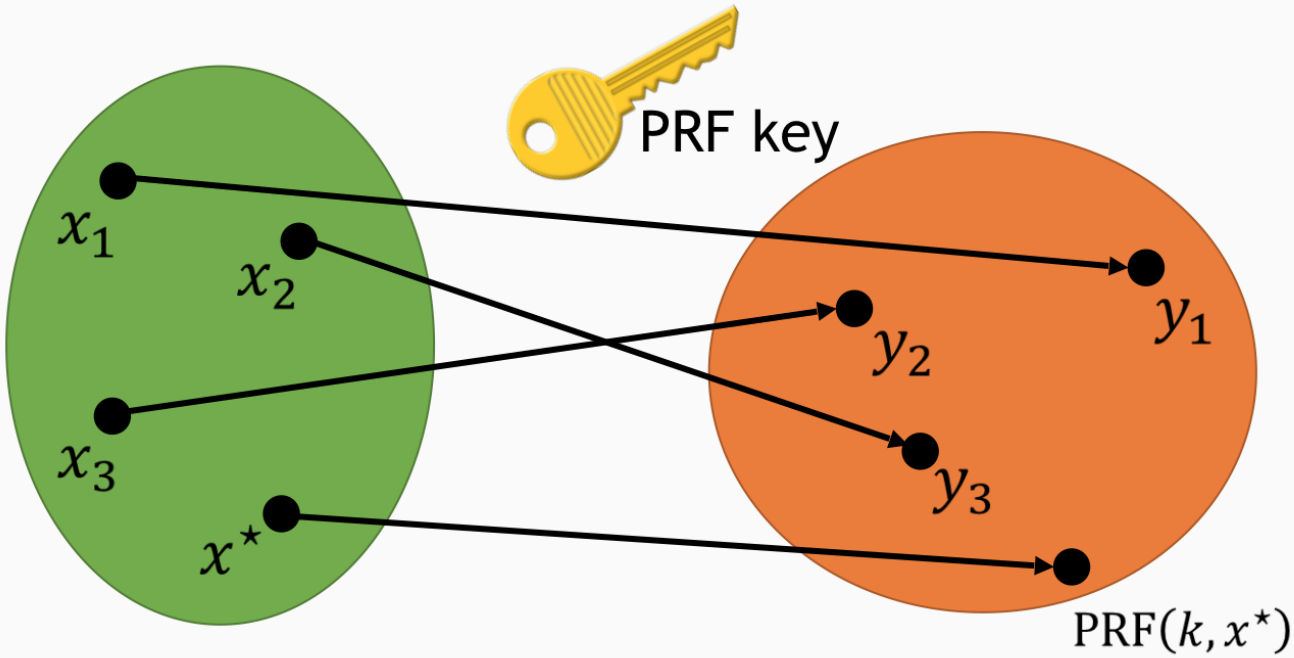
Problem 2: Punctured keys typically do not provide flexibility in programming value at punctured point

- Relax programmability requirement

Private Translucent PRFs



Output of any punctured key on a punctured point lies in a sparse, hidden subspace

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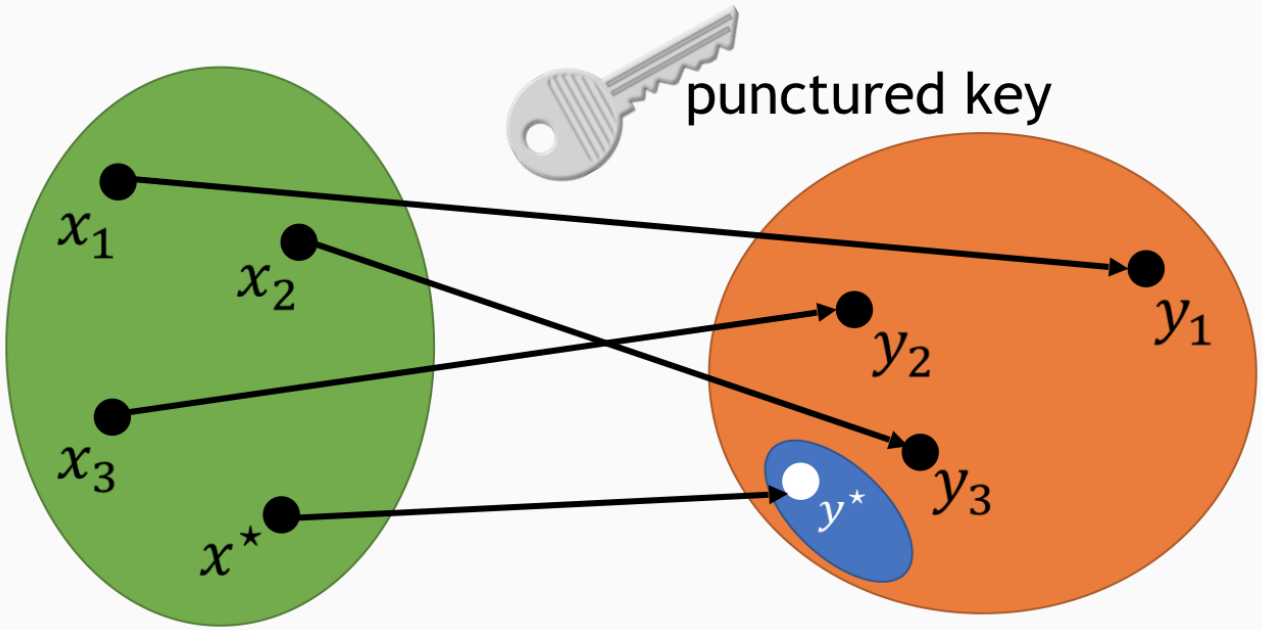

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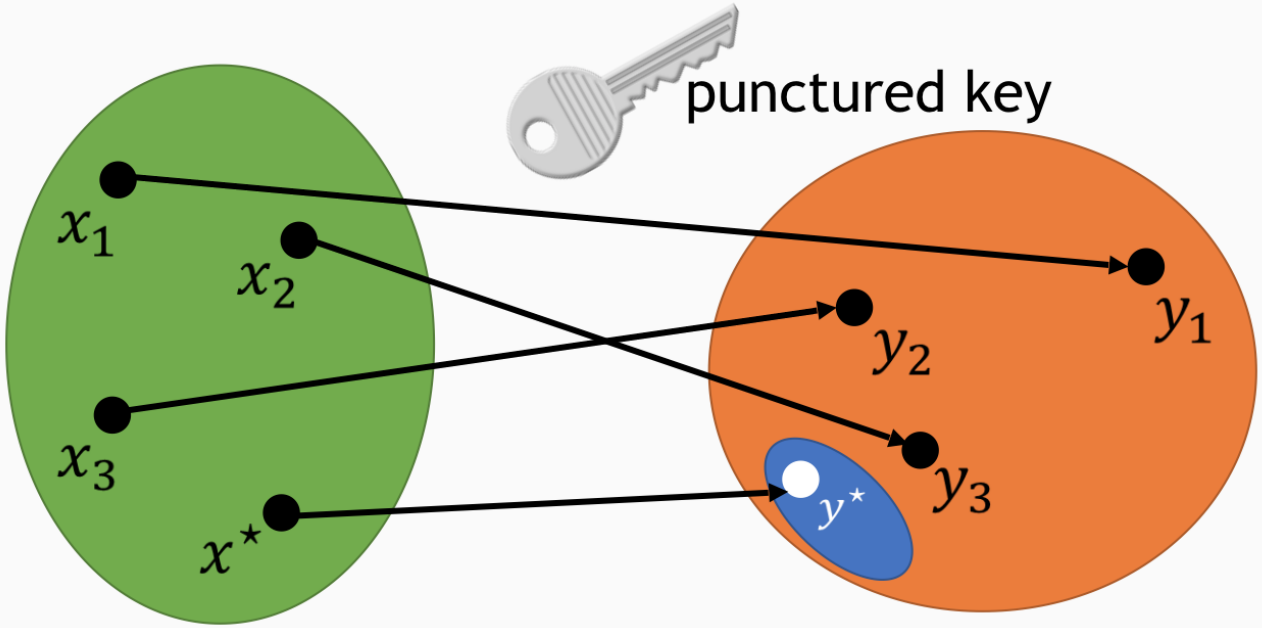

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Secret testing key can be used to test for memership in the hidden subspace

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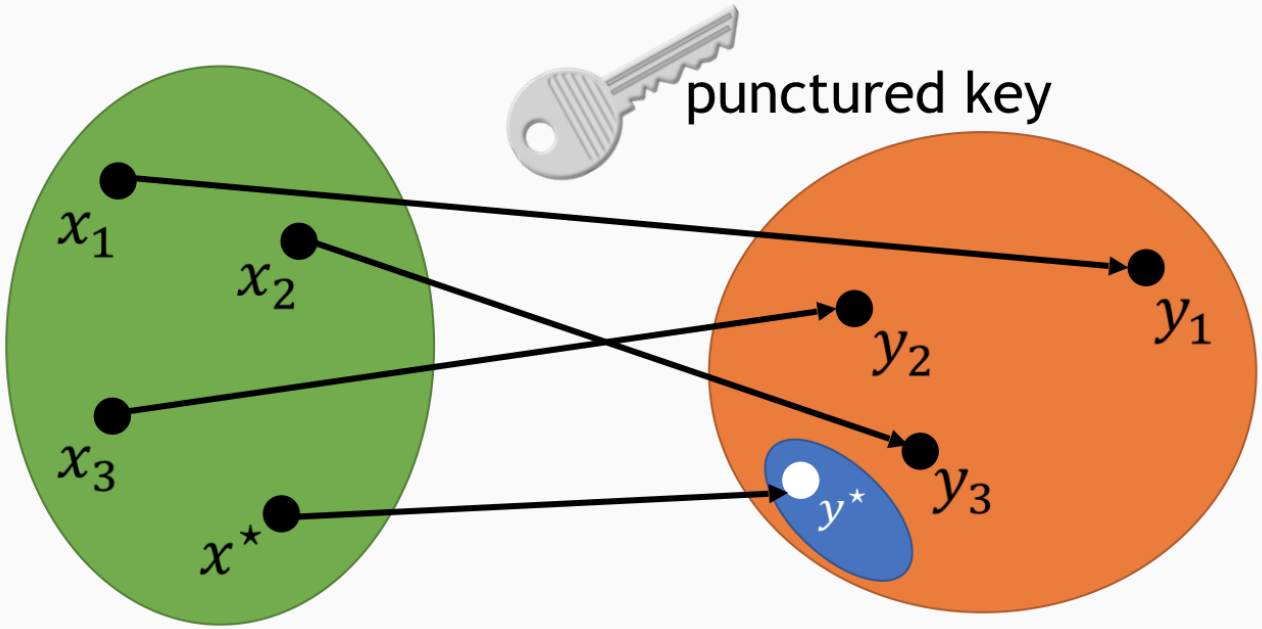
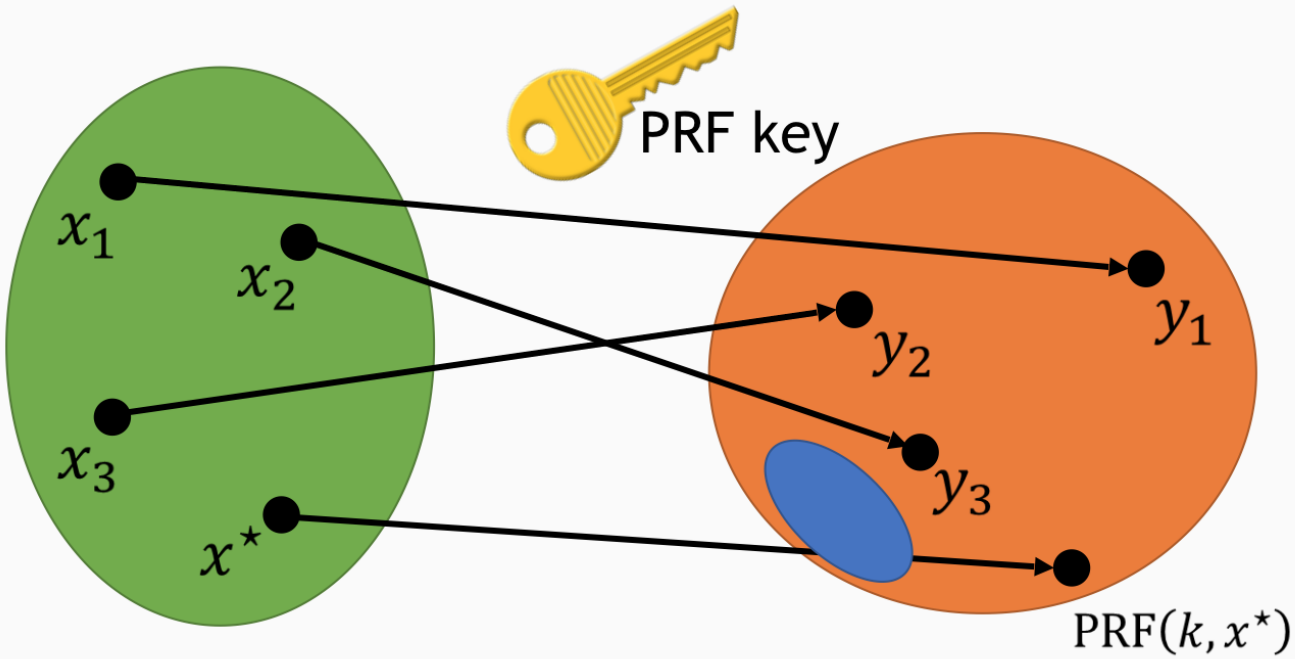


Sets satisfying such properties are called translucent [CDNO97]

- Values in special set looks indistinguishable from a random value (without secret testing key)
- Indistinguishable even through it is easy to sample values from the set

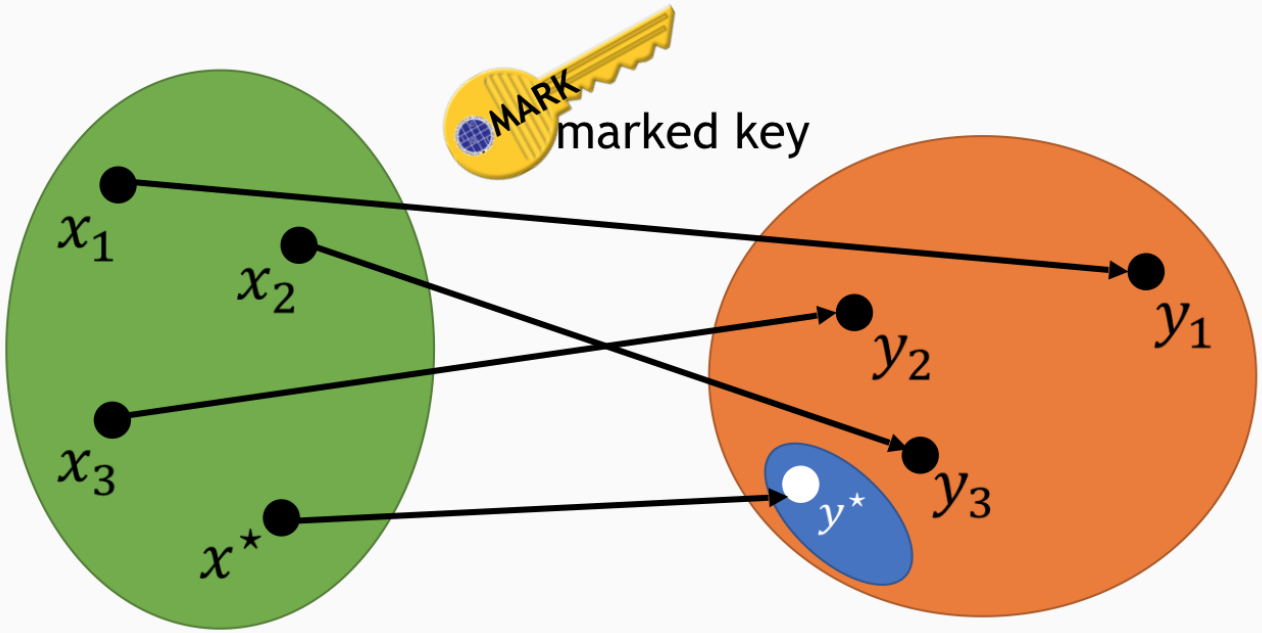
Private Translucent PRFs



Define watermarking secret key (wsk):

- Test points $\mathbf{x}_1, \ldots, \mathbf{x}_d$
- Testing key for private translucent PRF

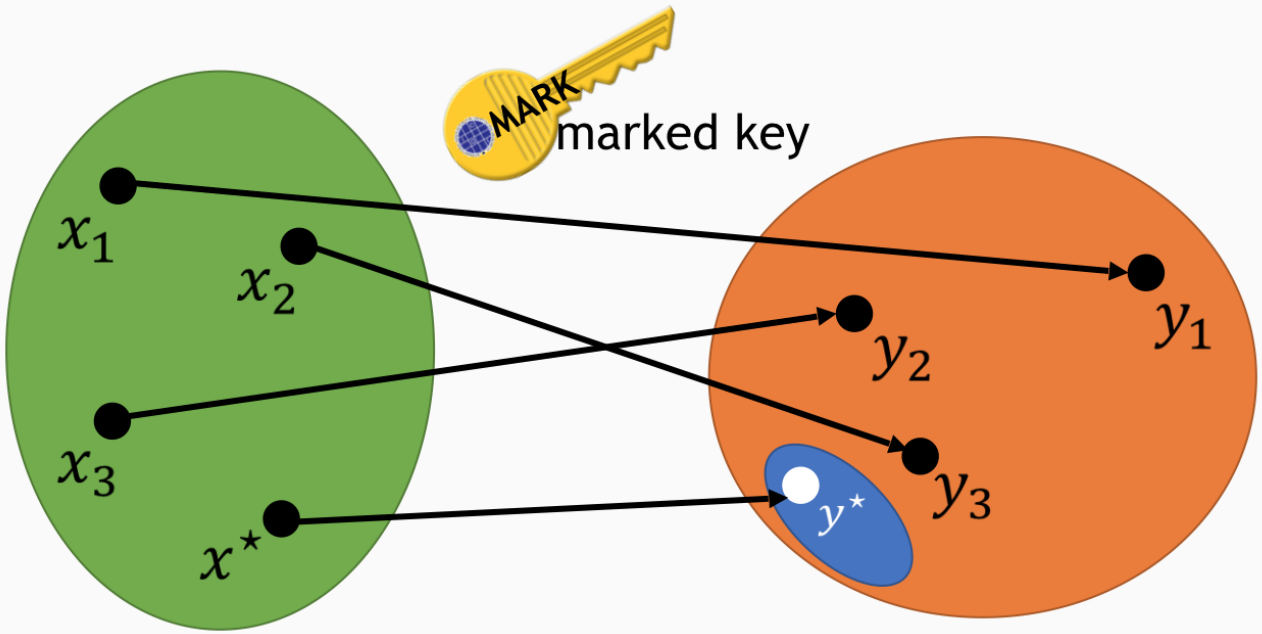
Private Translucent PRFs



To mark a PRF key k: derive special point $\mathbf{x}^\star$ and puncture k at $\mathbf{x}^\star$

Watermarked program is a program C' that evaluates punctured key

Private Translucent PRFs



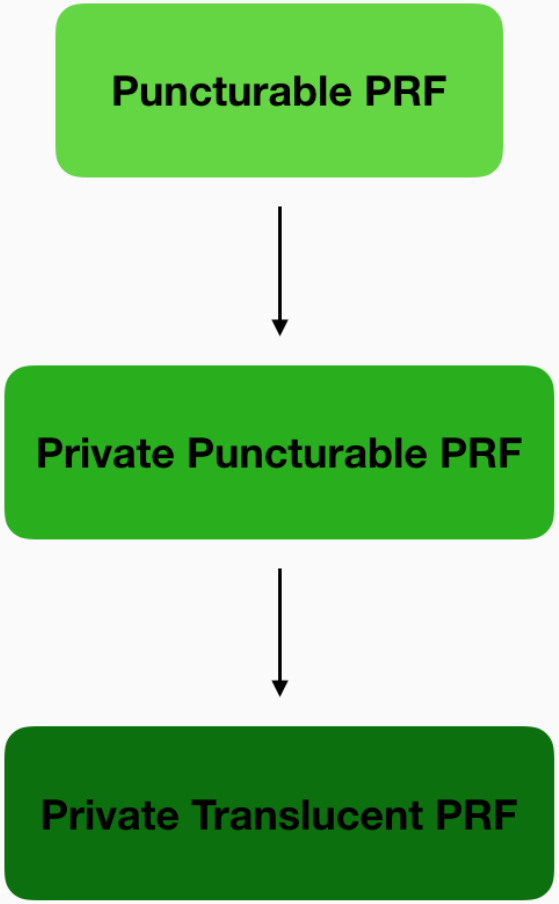
To mark a PRF key k: derive special point $\mathbf{x}^\star$ and puncture k at $\mathbf{x}^\star$

To test a program C':

1. Derive test point $\mathbf{x}^\star$
2. Check whether $C'(\mathbf{x}^\star)$ is in the translucent set

Constructing Private Translucent PRFs from Lattices

Road Map


Puncturable PRF
Private Puncturable PRF
Private Translucent PRF

Learning with Errors [Reg05]

Parameters: n, m, q, χ

Problem: Distinguish the following distributions

$$(\mathbf{A}, \mathbf{s}^T\mathbf{A} + \mathbf{e}^T) \approx (\mathbf{A}, \mathbf{u}^T).$$

for $\mathbf{A} \xleftarrow{R} \mathbb{Z}_q^{n \times m}$, $\mathbf{s} \xleftarrow{R} \mathbb{Z}_q^n$, $\mathbf{e} \xleftarrow{R} \chi^m$, $\mathbf{u} \xleftarrow{R} \mathbb{Z}_q^m$

Learning with Rounding [BPR12]

Parameters: n, m, q, p

Problem: Distinguish the following distributions

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Lattice PRFs [BPR12,...]

LWE says that

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Randomized PRF(?): For any $\mathbf{x} \in \mathcal{X}$, choose $\mathbf{A}_\mathbf{x} \xleftarrow{R} \mathbb{Z}_q^{n \times m}$ and define

$$\mathrm{PRF}(k, \mathbf{x}) = \lfloor \mathbf{s}^T \mathbf{A}_\mathbf{x} \rceil_p$$

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Approach: Fix matrix $\mathbf{A}_1, \ldots, \mathbf{A}_\ell \in \mathbb{Z}_q^{n \times m}$ and set key $k = \mathbf{s} \in \mathbb{Z}_q^n$.

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Question: What is a suitable way to derive $\mathbf{A}_\mathbf{x}$?

Gadget matrix [MP12]

Gadget matrix $\mathbf{G} \in \mathbb{Z}_q^{n \times m}$

There is an efficiently computable function $\mathbf{G}^{-1}(\cdot)$ such that:

- $\mathbf{G}^{-1} : \mathbb{Z}_q^{n \times m} \to \{0,1\}^{m \times m}$
- for all $\mathbf{C} \in \mathbb{Z}_q^{n \times m}$:

$$\mathbf{G} \cdot \mathbf{G}^{-1}(\mathbf{C}) = \mathbf{C}$$

Implementation:

- $\mathbf{G}^{-1}$ is the *bit decomposition* function
- $\mathbf{G}$ consists of *powers-of-2*

Matrix Embeddings [BGGHNSVV14]

Encode $x \in \{0,1\}$ with respect to matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$

$$\mathbf{s}^T(\mathbf{A} + x \cdot \mathbf{G}) + \mathbf{e}^T$$

Matrix Embeddings [BGGHNSVV14]

Encode $x \in \{0,1\}$ with respect to matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$

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Useful: Matrices $\mathbf{A}_{(+)}$, $\mathbf{A}_{(\times)}$ are independent of x_i, x_j.

Matrix Embeddings [BGGHNSVV14]

Compute any circuit $f : \{0,1\}^\ell \to \{0,1\}$:

$$\mathbf{s}^T(\mathbf{A}_1 + x_1 \cdot \mathbf{G}) + \mathbf{e}_1$$

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Useful property:
Matrix $\mathbf{A}_f$ does not depend on encoded values $x_1, \ldots, x_\ell$

$\Rightarrow$ Can derive $\mathbf{A}_1, \ldots, \mathbf{A}_\ell \to \mathbf{A}_f$ without knowing $x_1, \ldots, x_\ell$.

Puncturable PRF [BV14]

Idea: Define PRF with respect to $\mathsf{eq}_{\mathbf{x}}(\mathbf{c}) = \begin{cases} 1 & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$:

$$\mathsf{PRF}(k, \mathbf{x}) = \left\lfloor \mathbf{s}^T \mathbf{A}_{\mathsf{eq}_{\mathbf{x}}} \right\rceil_p .$$

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When $\mathbf{x} \neq \mathbf{c}$, then $\mathsf{eq}_{\mathbf{x}}(\mathbf{c}) = 0$:

$$\left\lfloor \mathbf{s}^T \mathbf{A}_{\mathsf{eq}_{\mathbf{x}}} + \text{noise} \right\rceil_p = \left\lfloor \mathbf{s}^T \mathbf{A}_{\mathsf{eq}_{\mathbf{x}}} \right\rceil_p$$

When $\mathbf{x} = \mathbf{c}$,

$$\left\lfloor \mathbf{s}^T (\mathbf{A}_{\mathsf{eq}_{\mathbf{x}}} + \mathbf{G}) + \text{noise} \right\rceil_p \neq \left\lfloor \mathbf{s}^T \mathbf{A}_{\mathsf{eq}_{\mathbf{x}}} \right\rceil_p$$

Private Constrained PRFs [BKM17, BTVW17]

Hide punctured point?

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Hide punctured point? Recall...

Addition: To compute $c_i + c_j$:

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Punctured key must include $\mathbf{c}$ in order to compute!

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Punctured key must include $\mathbf{c}$ in order to compute!

Idea: Use FHE to hide encoded values [GVW15]

Fully Homomorphic Encryption

For message $x_1, \ldots, x_\ell$ and circuit $f : \{0,1\}^\ell \to \{0,1\}$, compute only on ciphertext

$$\left(\ \mathsf{Enc}(x_1), \ldots, \mathsf{Enc}(x_\ell) \ , \ f \ \right) \Longrightarrow \mathsf{Enc}(f(\mathbf{x}))$$

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For LWE-based constructions, $\mathsf{Enc}(f(x)) \in \mathbb{Z}_q^m$, $\mathsf{sk} \in \mathbb{Z}_q^m$, and decryption is an inner product

$$\left\langle \ \mathsf{Enc}(f(\mathbf{x})) \ , \ \mathsf{sk} \ \right\rangle = \frac{q}{2} \cdot f(\mathbf{x}) + \text{noise}.$$

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$$\Big\langle \ \mathsf{Enc}(f(\mathbf{x})) \ , \ \mathsf{sk} \ \Big\rangle = f(\mathbf{x}).$$

(cheating for simplicity)

Private Puncturable PRF [BKM17, BTVW17]

First encrypt $\mathsf{ct} \leftarrow \mathsf{Enc}(\mathbf{c})$.

Then define punctured key $\mathsf{sk}_{\mathbf{c}}$:

$$\mathbf{s}^T(\mathbf{A}_1 + \mathsf{ct}_1 \cdot \mathbf{G}) + \mathbf{e}_1$$

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We have: $\tilde{\mathsf{ct}} = \mathsf{Enc}(\mathsf{eq}_{\mathbf{x}}(\mathbf{c}))$

How do we **extract** this value?

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\end{array}$$

$$\begin{array}{c}
\mathbf{s}^T(\mathbf{B}_1 + \mathsf{sk}_1 \cdot \mathbf{G}) + \mathbf{e}'_1 \\
\vdots \\
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Give out ct, but not $\mathsf{sk}_1, \ldots, \mathsf{sk}_m$!

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$$\xrightarrow{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x}}}} \mathbf{s}^T(\mathbf{A}^*_{\mathbf{x}} + \langle \mathsf{sk}, \tilde{\mathsf{ct}} \rangle \cdot \mathbf{G}) + \text{noise}$$

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Define PRF with respect to $\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x}}}$: $\mathsf{PRF}(k, \mathbf{x}) = \left\lfloor \mathbf{s}^T \mathbf{A}_{\mathsf{IP}\circ\mathsf{Eval}_{\mathsf{eq}_{\mathbf{x}}}} \right\rceil_p$

Programming

Encode punctured point $\mathbf{c}$ (under FHE)

Define PRF with respect to equality circuit:

$$\mathsf{eq}_{\mathbf{x}}(\mathbf{c}) = \begin{cases} 1 & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$$

Constrained key evaluation at $\mathbf{c}$:

$$\mathbf{s}^T(\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x}}}} + \mathsf{eq}_{\mathbf{x}}(\mathbf{c}) \cdot \mathbf{G}) + \text{noise}$$

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Programming

Encode punctured point $(\mathbf{c}, w)$ (under FHE) for some $w \in \mathbb{Z}_q$

Define PRF with respect to equality circuit:

$$\mathsf{eq}_{\mathbf{x}}(\mathbf{c}) = \begin{cases} 1 & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$$

Constrained key evaluation at $\mathbf{c}$:

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Programming

Encode punctured point $(\mathbf{c}, w)$ (under FHE) for $w \in \mathbb{Z}_q$

Define PRF with respect to equality circuit:

$$\mathsf{eq}_{\mathbf{x}}(\mathbf{c}, w) = \begin{cases} w & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$$

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$$\mathbf{s}^T(\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x}}}} + w \cdot \mathbf{G}) + \text{noise}$$

Programming

Encode punctured point $(\mathbf{c}, w)$ (under FHE) for $w \in \mathbb{Z}_q$

Define PRF with respect to equality circuit:

$$\mathsf{eq}_{\mathbf{x}}(\mathbf{c}, w) = \begin{cases} w & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$$

Constrained key evaluation at $\mathbf{c}$:

$$\mathbf{s}^T(\underbrace{\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x}}}} + w \cdot \mathbf{G}}) + \text{noise}$$

There is some programmability here!

We can choose w at key generation.

Programming

Encode punctured point $(\mathbf{c}, w)$ (under FHE) for $w_i \in \mathbb{Z}_q$

Define PRF with respect to equality circuit:

$$\mathsf{eq}_{\mathbf{x}}(\mathbf{c}, w) = \begin{cases} w & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$$

Constrained key evaluation at $\mathbf{c}$:

$$\mathbf{s}^T(\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x}}}} + w \cdot \mathbf{G}) + \text{noise}$$

Programming

Encode punctured point $(\mathbf{c}, w_1, \ldots, w_N)$ (under FHE) for $w_i \in \mathbb{Z}_q$

Define PRF with respect to equality circuit:

$$\mathsf{eq}_{\mathbf{x}}(\mathbf{c}, w) = \begin{cases} w & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$$

Constrained key evaluation at $\mathbf{c}$:

$$\mathbf{s}^T(\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x}}}} + w \cdot \mathbf{G}) + \text{noise}$$

Programming

Encode punctured point $(\mathbf{c}, w_1, \ldots, w_N)$ (under FHE) for $w_i \in \mathbb{Z}_q$

Define PRF with respect to equality circuit:

$$\mathsf{eq}_{\mathbf{x},i}(\mathbf{c}, w_i) = \begin{cases} w_i & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$$

Constrained key evaluation at $\mathbf{c}$:

$$\mathbf{s}^T(\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_\mathbf{x}}} + w \cdot \mathbf{G}) + \text{noise}$$

Programming

Encode punctured point $(\mathbf{c}, w_1, \ldots, w_N)$ (under FHE) for $w_i \in \mathbb{Z}_q$

Define PRF with respect to equality circuit:

$$\mathsf{eq}_{\mathbf{x},i}(\mathbf{c}, w_i) = \begin{cases} w_i & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$$

Constrained key evaluation at $\mathbf{c}$:

$$\mathbf{s}^T(\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x},1}}} + w_1 \cdot \mathbf{G}_1) + \text{noise}$$

$$\vdots$$

$$\mathbf{s}^T(\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x},N}}} + w_N \cdot \mathbf{G}_N) + \text{noise}$$

Programming

Encode punctured point $(\mathbf{c}, w_1, \ldots, w_N)$ (under FHE) for $w_i \in \mathbb{Z}_q$

Define PRF with respect to equality circuit:

$$\mathsf{eq}_{\mathbf{x},i}(\mathbf{c}, w_i) = \begin{cases} w_i & \mathbf{x} = \mathbf{c} \\ 0 & \text{otherwise} \end{cases}$$

Constrained key evaluation at $\mathbf{c}$:

$$\mathbf{s}^T(\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x},1}}} + w_1 \cdot \mathbf{G}_1) + \text{noise}$$

$$\vdots$$

$$\mathbf{s}^T(\mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x},N}}} + w_N \cdot \mathbf{G}_N) + \text{noise}$$

Choose $w_1, \ldots, w_N$ such that

$$\mathbf{D} = \sum_{i \in [N]} \mathbf{A}_{\mathsf{IP} \circ \mathsf{Eval}_{\mathsf{eq}_{\mathbf{x}_i}}} + \sum_{i \in [N]} w_i \cdot \mathbf{G}_i$$

Programming

Constrained key evaluation at $\mathbf{c}$:

$$\left\lfloor \mathbf{s}^T \mathbf{D} \right\rceil_p$$

Watermark key includes short vector $\mathbf{w}$ such that

$$\mathbf{D} \cdot \mathbf{w} = \mathbf{0}$$

To Conclude...

Open Problems:

- Achieve stronger security guarantees?
- Watermark other primitives?

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- Achieve stronger security guarantees?
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Thanks!