Threshold Cryptosystems from Threshold Fully Homomorphic Encryption

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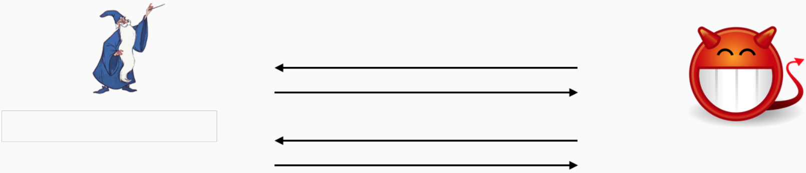
Stanford University

Joint work with Dan Boneh, Rosario Gennaro, Steven Goldfeder, Aayush Jain, Peter M. R. Rasmussen, and Amit Sahai

Key Management



Key Management

 $\xleftarrow{R}$ KeyGen

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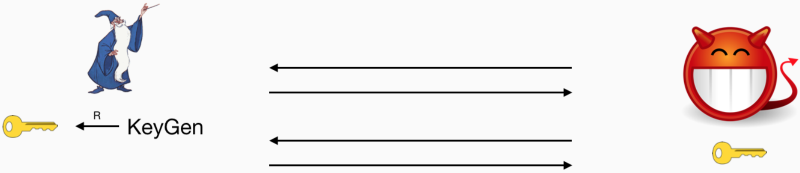

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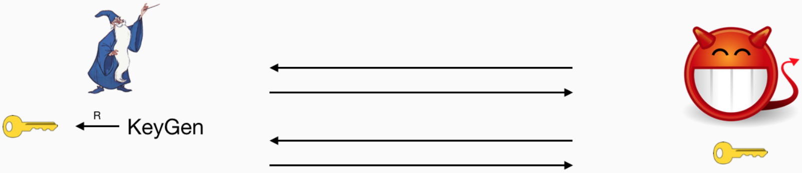
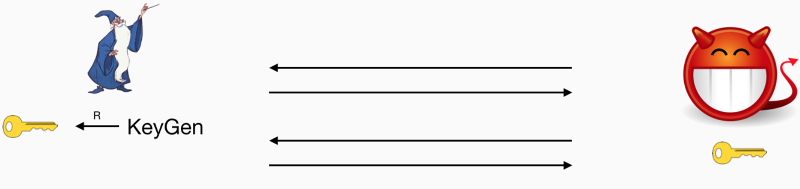


Key management is **hard** in practice!

(difficult to implement to crypto, systems get hacked, human error, ...)



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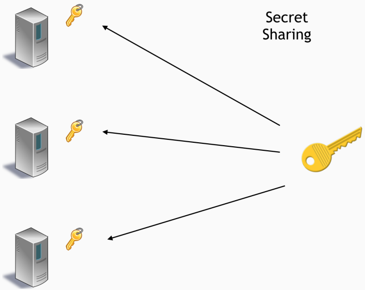
Can we address key management at a more **fundamental** level?

Threshold Cryptography

Can we divide the key into shares and store them separately?



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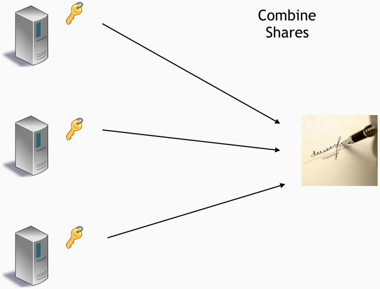

Correctness: Each server can independently compute an evaluation share, which can later be publicly combined to form final evaluation.

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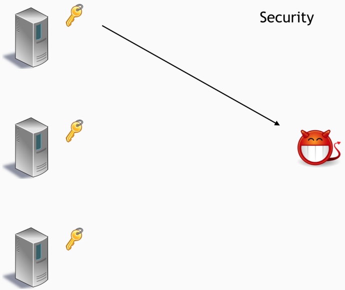

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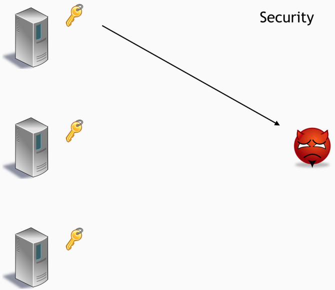

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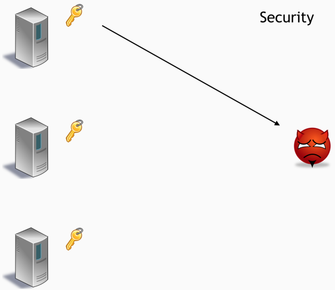

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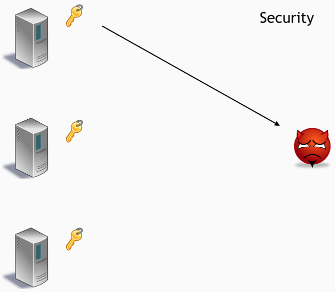
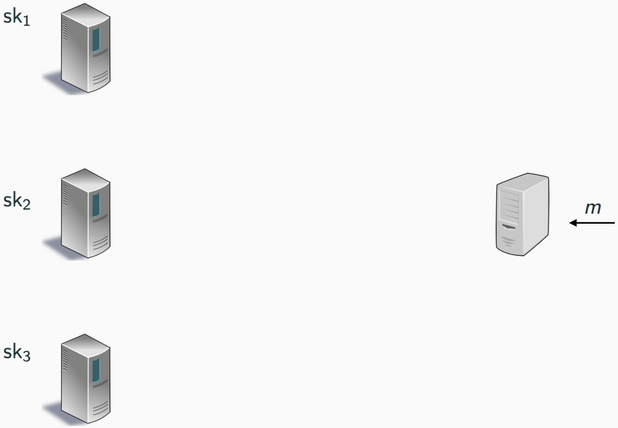


Correctness: Each server can independently compute an evaluation share. Any **t** evaluation shares can later be publicly combined to form final evaluation.

Security: Hard to form final evaluation without **t** evaluation shares.

Threshold Signatures



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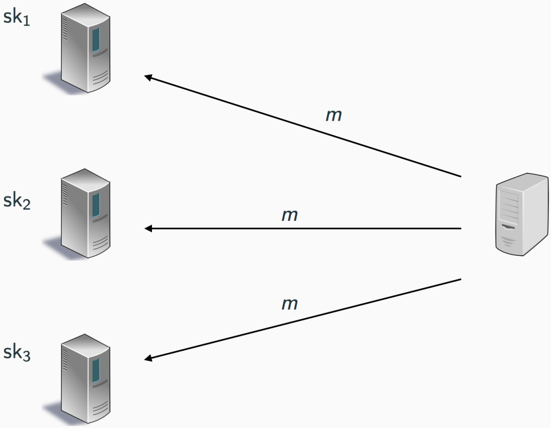

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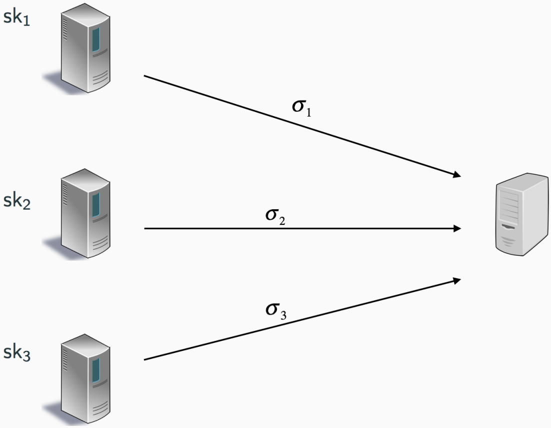

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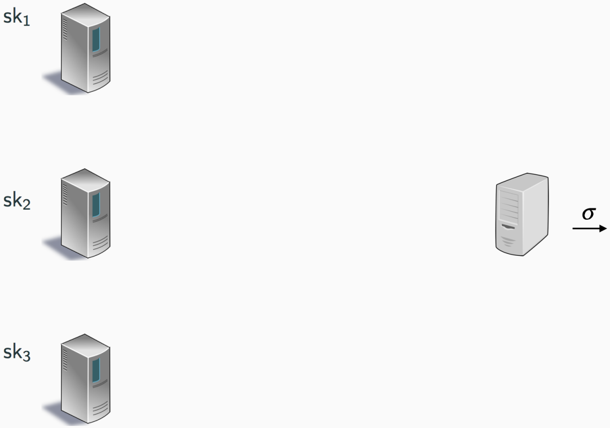

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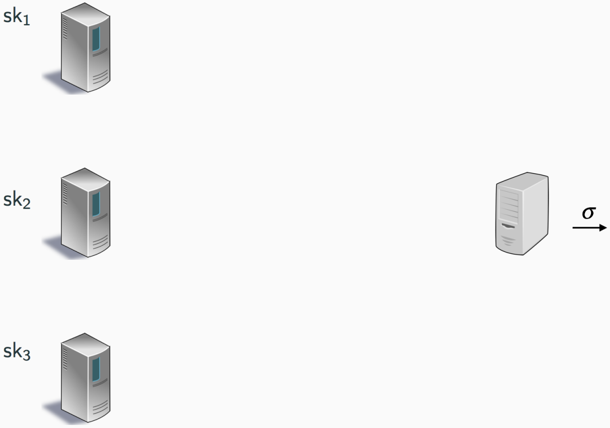
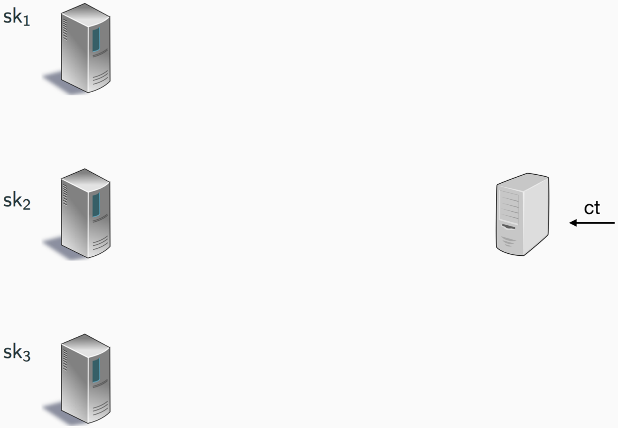


Requirements: Compactness, Correctness, Unforgeability, Anonymity, Robustness, ...

Threshold Public Key Encryption



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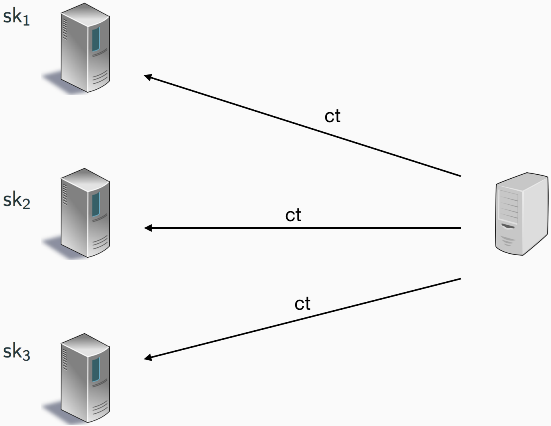

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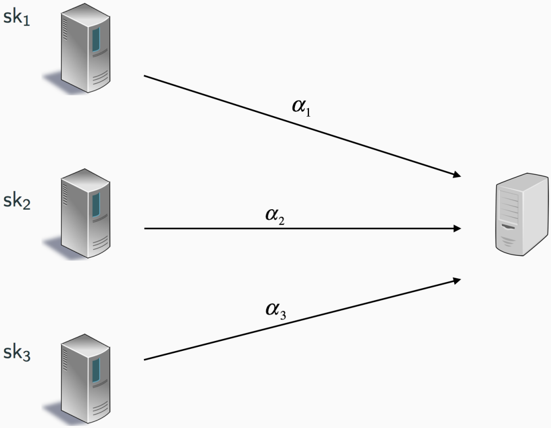

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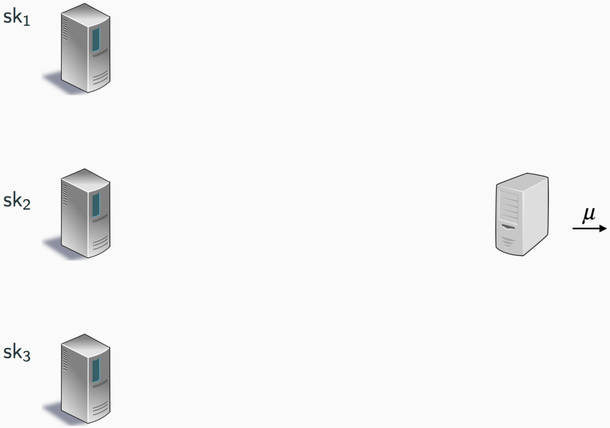

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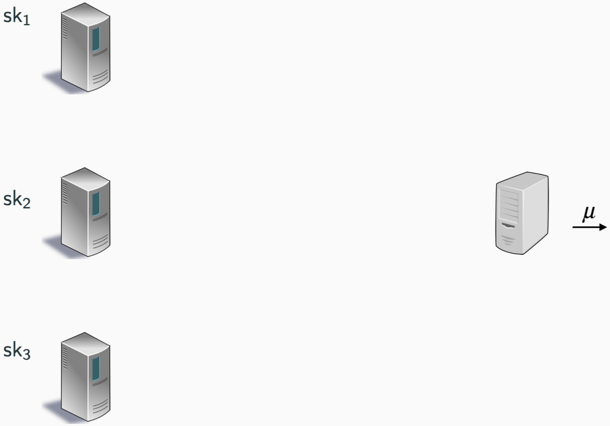

Requirements: Compactness, Correctness, CCA security, Robustness, ...

Works on Threshold Cryptography

- RSA signatures [Fra89, DDFY94, GRJK07, Sho00]
- Schnorr signatures [SS01]
- (EC)DSA signatures [GJKR01, GGN16]
- BLS signatures [BLS04, Bol03]
- Cramer-Shoup encryption [CG99]
- Many more [SG02, DK05, BBH06]

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Not much progress on lattice-based schemes.

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- Define a new primitive Universal Thresholdizer (UT).
- A general framework for constructing threshold cryptosystems using UT.
- New constructions for threshold signatures, threshold PKE, distributed PRFs, functional encryption with distributed key generation, ...

Defining Threshold Fully Homomorphic Encryption (TFHE)

(Standard) Fully Homomorphic Encryption

- $\mathsf{Setup}(1^\lambda) \to (\mathsf{pk}, \mathsf{sk})$
- $\mathsf{Encrypt}(\mathsf{pk}, \mu) \to \mathsf{ct}$
- $\mathsf{Eval}(\mathsf{pk}, C, \mathsf{ct}_1, \ldots, \mathsf{ct}_k) \to \hat{\mathsf{ct}}$
- $\mathsf{Decrypt}(\mathsf{sk}, \hat{\mathsf{ct}}) \to \hat{\mu}$

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Correctness

For any $C : \{0,1\}^k \to \{0,1\}$, $\hat{\mathsf{ct}} \leftarrow \mathsf{Eval}(\mathsf{pk}, C, \mathsf{ct}_1, \ldots, \mathsf{ct}_k)$, $|S| \geq t$,

$$\mathsf{FinDec}(\mathsf{pk}, \{\mathsf{PartDec}(\mathsf{sk}_i, \hat{\mathsf{ct}})\}_{i \in S}) = C(\mu_1, \ldots, \mu_k).$$

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Compactness

$$|\hat{\mathsf{ct}}| \leq \mathsf{poly}(\lambda) \quad |\mathsf{p}_i| \leq \mathsf{poly}(\lambda, N).$$

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Semantic Security

Standard PKE semantic security
(Adversary given $\{\mathsf{sk}_i\}_{i \in S^*}$ for $|S^*| < t$)

Simulation Security (Real World)

Challenger		Adversary
pk $\mathsf{sk}_1, \ldots, \mathsf{sk}_N$	$\xrightarrow{\mathsf{pk}}$	
	$\xleftarrow{S^*}$	$\lvert S^* \rvert = t - 1$
	$\xrightarrow{\{\mathsf{sk}_i\}_{i \in S^*}}$	
	$\xleftarrow{\mu_1, \ldots, \mu_k}$	
	$\xrightarrow{\mathsf{ct}_1, \ldots, \mathsf{ct}_k}$	
$\hat{\mathsf{ct}} \leftarrow \mathsf{Eval}(\mathsf{pk}, C, \mathsf{ct}_1, \ldots, \mathsf{ct}_k)$	$\xleftarrow{(i \in [N], C)}$	
$\mathsf{p}_i \leftarrow \mathsf{PartDec}(\mathsf{sk}_i, \mathsf{ct})$	$\xrightarrow{\mathsf{p}_i}$	

Simulation Security (Ideal World)

Challenger		Adversary
pk	pk $\longrightarrow$	
$\mathsf{sk}_1, \ldots, \mathsf{sk}_N$	S^* $\longleftarrow$	$\lvert S^* \rvert = t - 1$
	$\{\mathsf{sk}_i\}_{i \in S^*}$ $\longrightarrow$	
	$\mu_1, \ldots, \mu_k$ $\longleftarrow$	
	$\mathsf{ct}_1, \ldots, \mathsf{ct}_k$ $\longrightarrow$	
	$(i \in [N], C)$ $\longleftarrow$	
$\mathcal{S}(\{\mathsf{sk}_i\}, \{\mathsf{ct}_i\}, C, C(\mu_1, \ldots, \mu_k))$	p_i $\longrightarrow$	

FHE from Approximate Eigenvector [GSW13]

- Ciphertext $\mathbf{C}$ is a matrix in $\{0,1\}^{m \times m}$
- Secret key $\vec{\mathbf{s}}$ is a vector in $\mathbb{Z}_q^m$

Approximate eigenvector property

$$\mathbf{C} \cdot \vec{\mathbf{s}} = \mu \cdot \vec{\mathbf{s}} + \text{noise}$$

- **Homomorphic addition**: $\mathbf{C}_1 + \mathbf{C}_2$

  $\vec{\mathbf{s}}$ is an eigenvector for $(\mathbf{C}_1 + \mathbf{C}_2)$.
- **Homomorphic multiplicaton**: $\mathbf{C}_1 \cdot \mathbf{C}_2$

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IDEA

Observation: Decryption operation is *linear*.

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Question: Can we use Shamir secret sharing to break up the key $\vec{\mathbf{s}}$?

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For any $|S| = t$, $i \in S$, there exists Lagrange coefficient λ_i such that

$$\vec{\mathbf{s}} = \sum_{i \in S} \lambda_i \cdot \vec{\mathbf{s}}_i.$$

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Define partial decryption

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1. Define a linear secret sharing scheme with low-norm reconstruction coefficients
2. Change the scheme itself using clearing out denominators trick

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Linear secret sharing scheme for $\mathsf{k} \in \mathbb{Z}_q$

- $\mathsf{Share}(\mathsf{k}, \mathbb{A}) \to (\mathsf{s}_1, \ldots, \mathsf{s}_N) \in \mathbb{Z}_q^N$
- $\mathsf{Combine}(\{\mathsf{s}_i\}_S)$:
  - There exists efficiently computable coefficients $c_i \in \mathbb{Z}_q$ such that for any set $S \in \mathbb{A}$,

$$\mathsf{k} = \sum_{i \in S} c_i \cdot \mathsf{s}_i .$$

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Example: Shamir secret sharing scheme

$$\phi_t(S) = 1 \Leftrightarrow |S| \geq t.$$

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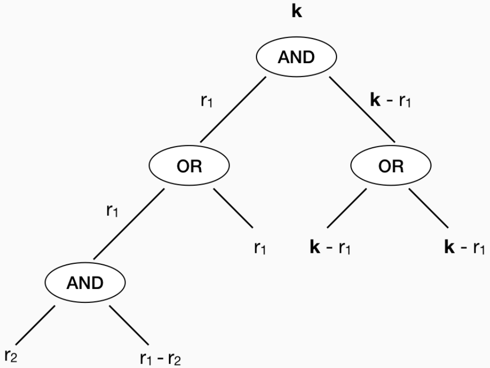
Define $\{0,1\}$-LSSS as a linear secret sharing scheme where the *reconstruction coefficients* are always binary.

Question: How expressive is $\{0,1\}$-LSSS?

Monotone Boolean Formulas



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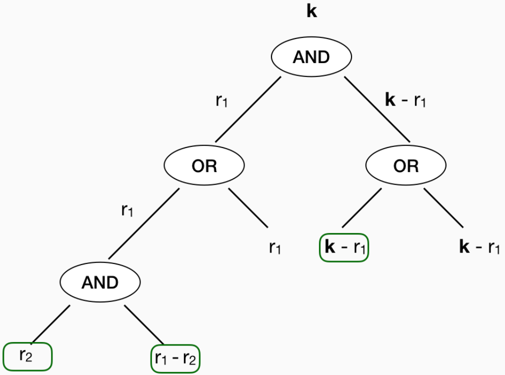

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- [Val84, Gol14] show that threshold function expressible by monotone Boolean formulas.

Recap

Use $\{0,1\}$-LSSS to break up FHE key $\vec{\mathbf{s}}$

$$\vec{\mathbf{s}} \to \vec{\mathbf{s}}_1, \ldots, \vec{\mathbf{s}}_N$$

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$$\mathrm{PartDec}(\mathrm{pk}, \mathbf{C}, \vec{\mathbf{s}}_i) = \mathbf{C} \cdot \vec{\mathbf{s}}_i + \text{noise}.$$

Define final decryption

$$\begin{aligned} \mathrm{FinDec}(\mathrm{pk}, \{\mathbf{C} \cdot \vec{\mathbf{s}}_i\}_S) &= \sum_{i \in S} c_i \cdot (\mathbf{C} \cdot \vec{\mathbf{s}}_i + \text{noise}_i) \\ &= C \cdot \vec{\mathbf{s}} + \underbrace{\text{noise} + \sum_{i \in S} c_i \cdot \text{noise}_i}_{\text{small}} \end{aligned}$$

Recap

Use $\{0,1\}$-LSSS to break up FHE key $\vec{\mathbf{s}}$

$$\vec{\mathbf{s}} \to \vec{\mathbf{s}}_1, \ldots, \vec{\mathbf{s}}_N$$

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Note: Requires careful security analysis!

Clearing out Denominators

Expressing threshold circuit in monotone Boolean formula expensive!

Circuit size $O(N^{5.2}) \Rightarrow$ partial key $|\mathsf{sk}_i| \leq O(N^{4.2})$ on average.

Question: Can we do better?

Idea: Use the technique of clearing out the denominators [Sho00,ABVVW12]

Lemma

For any Lagrange coefficients λ_i,

$$|(N!) \cdot \lambda_i| \leq (N!)^3.$$

Clearing out Denominators

Use Shamir secret sharing to break up FHE key $\vec{\mathbf{s}}$

$$\vec{\mathbf{s}} \to \vec{\mathbf{s}}_1, \ldots, \vec{\mathbf{s}}_N$$

Define partial decryption

$$\mathrm{PartDec}(\mathrm{pk}, \mathbf{C}, \vec{\mathbf{s}}_i) = \mathbf{C} \cdot \vec{\mathbf{s}}_i + (N!)^2 \cdot \mathrm{noise}.$$

Define final decryption

$$\begin{aligned}
\mathrm{FinDec}(\mathrm{pk}, \{\mathbf{C} \cdot \vec{\mathbf{s}}_i\}_S) &= \sum_{i \in S} \lambda_i \cdot (\mathbf{C} \cdot \vec{\mathbf{s}}_i + (N!)^2 \cdot \mathrm{noise}_i) \\
&= C \cdot \vec{\mathbf{s}} + \underbrace{\mathrm{noise} + \sum_{i \in S} \lambda_i \cdot (N!)^2 \cdot \mathrm{noise}_i}_{\textit{relatively}\ \text{small}}
\end{aligned}$$

Universal Thresholdizer

Universal Thresholdizer

- $\mathsf{Setup}(1^\lambda, t, N, x) \to (\mathsf{pp}, \mathsf{s}_1, \ldots, \mathsf{s}_N) \qquad \text{for } x \in \{0,1\}^k$
- $\mathsf{Eval}(\mathsf{pp}, \mathsf{s}_i, C) \to \mathsf{p}_i$
- $\mathsf{Combine}(\mathsf{pp}, \{\mathsf{p}_i\}) \to C(x)$

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  - $\mathsf{Setup}(1^\lambda, t, N) \to (\mathsf{pk}, \mathsf{sk}_1, \ldots, \mathsf{sk}_N)$
  - $\mathsf{Encrypt}(\mathsf{pk}, x) \to \mathsf{ct}$

$$\mathsf{pp} = (\mathsf{pk}, \mathsf{ct}) \qquad \mathsf{s}_i = \mathsf{sk}_i.$$

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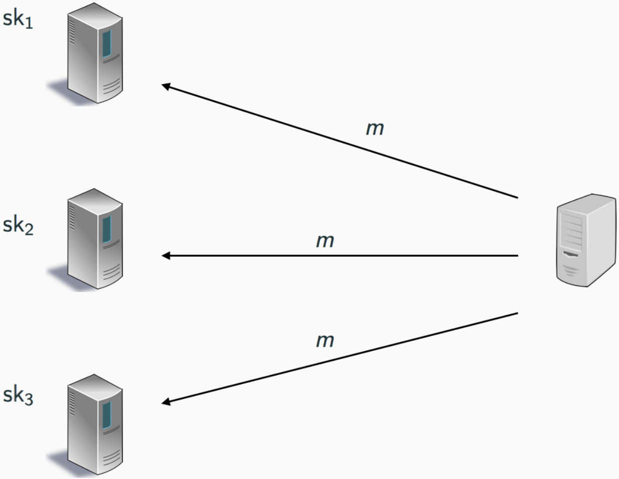
- $\mathsf{Combine}(\mathsf{pp}, \{\mathsf{p}_i\}) \to C(x)$
  - $\mathsf{FinDec}(\mathsf{pk}, \{\mathsf{p}_i\}) \to C(x)$

Threshold Signatures

$\mathsf{pp} = (\mathsf{pk}, \mathsf{Enc}(\mathsf{sk}_{\mathsf{sign}}))$

$C_m(\mathsf{sk}) = \mathsf{Sign}(\mathsf{sk}, m)$



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sk_1

$\mathsf{Enc}(\sigma)$

sk_2

$\mathsf{Enc}(\sigma)$

sk_3

$\mathsf{Enc}(\sigma)$

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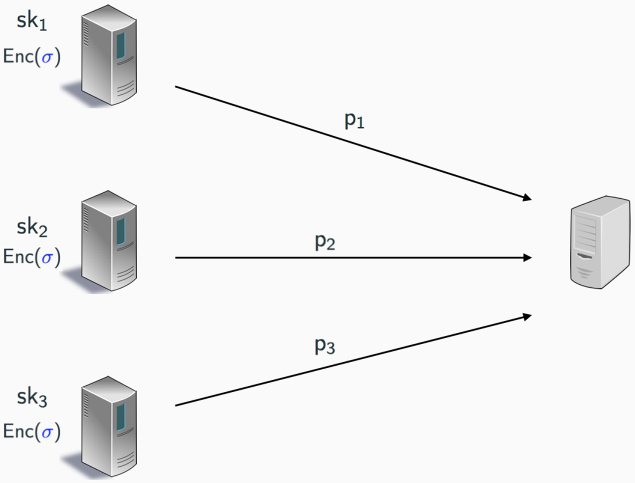

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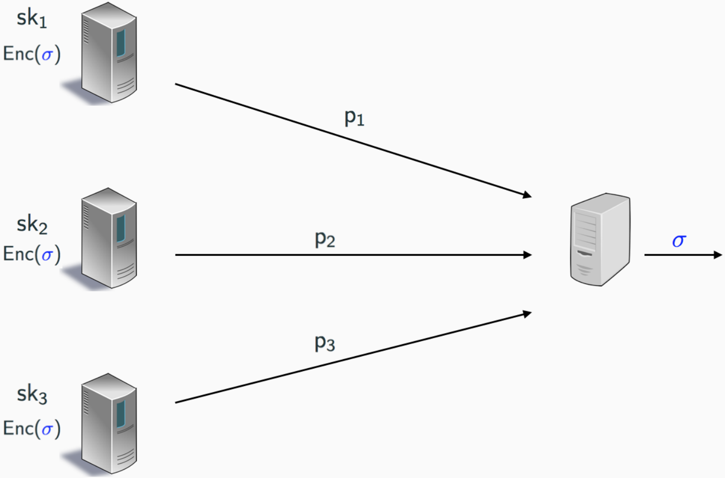
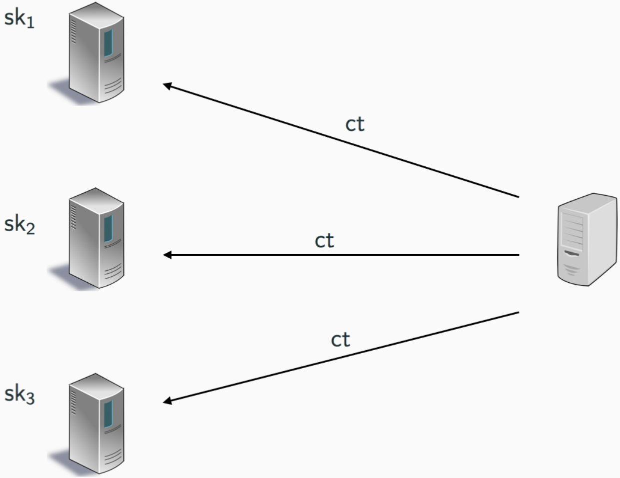


Threshold PKE

$\mathsf{pp} = (\mathsf{pk}, \mathsf{Enc}(\mathsf{sk}_{\mathsf{Dec}}))$



$\mathsf{pp} = (\mathsf{pk}, \mathsf{Enc}(\mathsf{sk}_{\mathsf{Dec}}))$

$C_{\mathsf{ct}}(\mathsf{sk}) = \mathsf{Dec}(\mathsf{sk}, m)$

sk_1

$\mathsf{Enc}(\mu)$

sk_2

$\mathsf{Enc}(\mu)$

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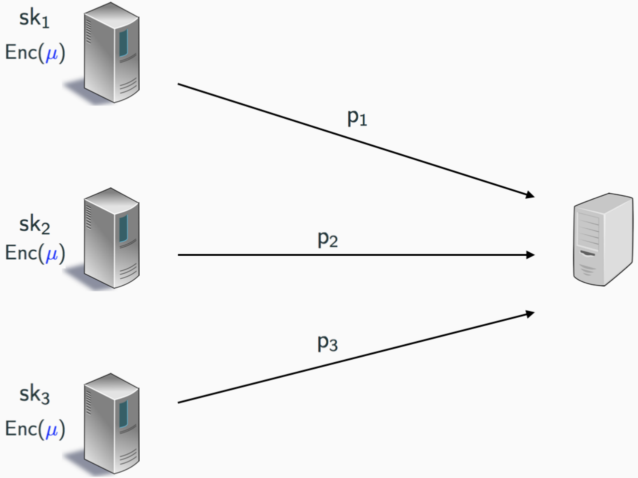

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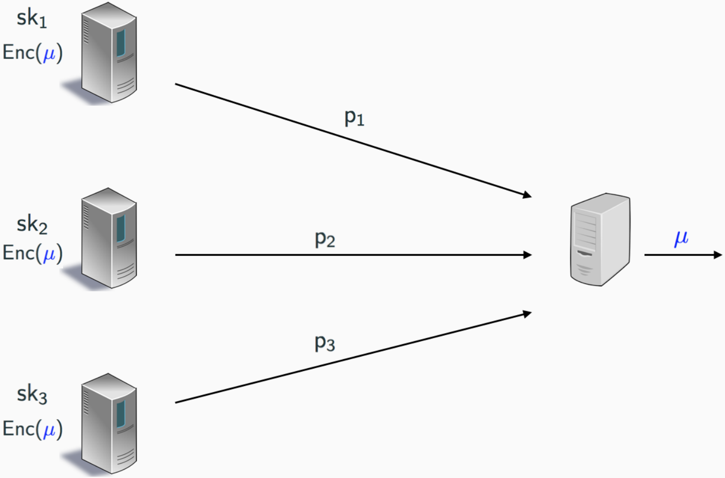

To Conclude...

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- Technical challenged in the analysis
- Decentralizing threshold FHE
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Thanks!