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Why the missing 1/L in the relaxation function does not cause problem in our viscoelastic simulation? Because the error pointed out by Peter Moczo is not an error.

1 Boltzmann principle

For a linear isotropic viscoelastic material, the stress-strain relation is given by Boltzmann principle. In scalar notation, we have:

$$\sigma(t)=\int_{-\infty}^{t}\psi(t-t')\dot{\varepsilon}(t')dt' \tag{1}$$

where $\sigma(t)$ is stress, $\dot{\varepsilon}(t)$ time derivative of strain, and $\psi(t)$ stress relaxation function defined as a stress response to Heaviside unit step function in strain. The stress at a given time *t* is determined by the entire history of the strain until *t*.

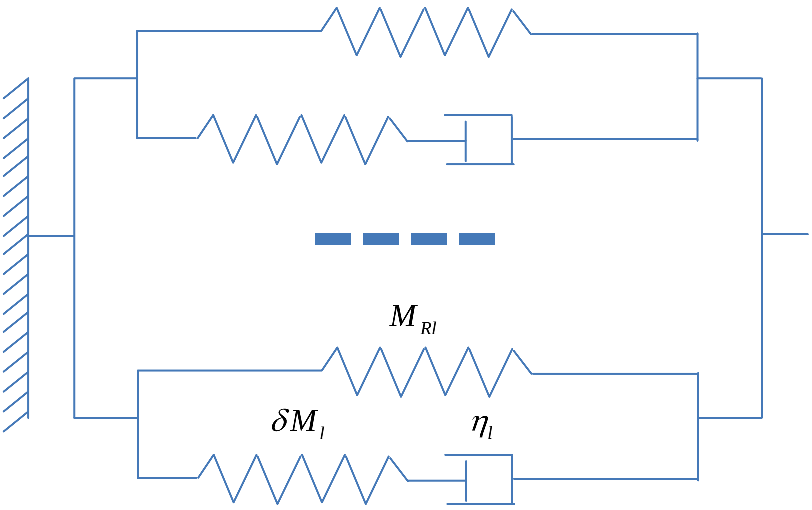
Using symbol $*$ for the convolution and applying the convolution's property, we have:

$$\sigma(t)=\psi(t)*\dot{\varepsilon}(t)=\dot{\psi}(t)*\varepsilon(t) \tag{2}$$

In following, we set $\dot{\psi}(t)=M(t)$.

2 Representation of $M(t)$ by a set of standard linear solid

In seismology, we typically encounter viscoelastic material with attenuation is observed to be relatively constant over a broad frequency range. For this kind of material, we could approximate the $M(t)$ by a set of standard linear solid:



For viscoelastic material described by *N* standard linear solids, we have:

$$M(\omega)=\sum_{l=1}^{N}M_{Rl}\frac{1+\mathbf{i}\tau_{\varepsilon l}\omega}{1+\mathbf{i}\tau_{\sigma l}\omega} \tag{3}$$

Where $\tau_{\varepsilon l}=\frac{\eta_l}{\delta M_l}\frac{M_{Ul}}{M_{Rl}}$, $\tau_{\sigma l}=\frac{\eta_l}{\delta M_l}$, $\frac{\tau_{\varepsilon l}}{\tau_{\sigma l}}=\frac{M_{Ul}}{M_{Rl}}$. M_R is the total relaxed modulu.

Correspondingly, for $M_{Rl}=\frac{M_R}{N}$ we have:

$$\psi(t)=M_R\left[1-\frac{1}{N}\sum_{l=1}^{N}\left(1-\frac{\tau_{\varepsilon l}}{\tau_{\sigma l}}\right)e^{-t/\tau_{\sigma l}}\right]H(t) \tag{4}$$

3 Error pointed out by Peter Moczo

In his 2005 paper : "On the rheological models used for time-domain methods of seismic wave propagation", Moczo pointed out that the $\frac{1}{N}$ is missing in Liu's paper: "Velocity dispersion due to anelasticity: Implications for seismology and mantle composition". That Liu use (4) wrongly as:

$$\psi(t) = M_R\left[1-\sum_{l=1}^{N}\left(1-\frac{\tau_{\varepsilon l}}{\tau_{\sigma l}}\right)e^{-t/\tau_{\sigma l}}\right]H(t) \tag{5}$$

Taking into account that the SPECFEM2D and SPECFEM3D are based on (5), that means the two codes would give wrong results in case for viscoelastic simulation.

4 Error pointed out by Peter Moczo is not an error.

First by setting $M_{Rl} = \frac{M_R}{N}$, (3) can be rewritten as:

$$\begin{aligned} M(\omega) &= \sum_{l=1}^{N} M_{Rl}\frac{1+\mathbf{i}\tau_{\varepsilon l}\omega}{1+\mathbf{i}\tau_{\sigma l}\omega} = \frac{M_R}{N}\sum_{l=1}^{N}\frac{1+\mathbf{i}\tau_{\varepsilon l}\omega}{1+\mathbf{i}\tau_{\sigma l}\omega} = \frac{M_R}{N}\sum_{l=1}^{N}\left(1+\frac{\mathbf{i}\omega(\tau_{\varepsilon l}-\tau_{\sigma l})}{1+\mathbf{i}\tau_{\sigma l}\omega}\right) \\ &= \frac{M_R}{N}\sum_{l=1}^{N}\left(1+\frac{\mathbf{i}\omega(\tau_{\varepsilon l}-\tau_{\sigma l})(1-\mathbf{i}\tau_{\sigma l}\omega)}{1+\omega^2\tau_{\sigma l}^2}\right) = M_R + \frac{M_R}{N}\sum_{l=1}^{N}\left(\frac{\mathbf{i}\omega(\tau_{\varepsilon l}-\tau_{\sigma l})(1-\mathbf{i}\tau_{\sigma l}\omega)}{1+\omega^2\tau_{\sigma l}^2}\right) \\ &= M_R\left[1+\sum_{l=1}^{N}\frac{\omega^2\tau_{\sigma l}^2}{1+\omega^2\tau_{\sigma l}^2}\left(\frac{1}{N}\frac{(\tau_{\varepsilon l}-\tau_{\sigma l})}{\tau_{\sigma l}}\right)+\sum_{l=1}^{N}\frac{\mathbf{i}\omega\tau_{\sigma l}}{1+\omega^2\tau_{\sigma l}^2}\left(\frac{1}{N}\frac{(\tau_{\varepsilon l}-\tau_{\sigma l})}{\tau_{\sigma l}}\right)\right] \end{aligned} \tag{4}$$

Introducing $\frac{\tau'_{\varepsilon l}-\tau_{\sigma l}}{\tau_{\sigma l}} = \frac{1}{N}\frac{(\tau_{\varepsilon l}-\tau_{\sigma l})}{\tau_{\sigma l}}$ that is:

$$\tau'_{\varepsilon l} = \frac{1}{N}(\tau_{\varepsilon l}-\tau_{\sigma l})+\tau_{\sigma l} \tag{5}$$

then the (4) reduces to

$$M(\omega) = M_R\left[1+\sum_{l=1}^{N}\frac{\omega\tau_{\sigma l}(\tau'_{\varepsilon l}-\tau_{\sigma l})}{1+\omega^2\tau_{\sigma l}^2}+\sum_{l=1}^{N}\frac{\mathbf{i}\omega\tau_{\sigma l}(\tau'_{\varepsilon l}-\tau_{\sigma l})}{1+\omega^2\tau_{\sigma l}^2}\right] \tag{6}$$

(6) corresponds exactly as

$$\psi(t) = M_R\left[1-\sum_{l=1}^{N}\left(1-\frac{\tau'_{\varepsilon l}}{\tau_{\sigma l}}\right)e^{-t/\tau_{\sigma l}}\right]H(t) \tag{7}$$

Thus if in frequency domain we use (6) to obtain the value of M_R, $\tau'_{\varepsilon l}$ and $\tau_{\sigma l}$, no error exists. In our codes SPECFEM3D and SPECFEM2D, we use (6) together with (7). Thus, no modification has to be made in our code.

It worth note here, $\tau'_{\varepsilon l}$ do not share the same physical meaning as $\tau_{\varepsilon l}$.

Appendix A: For constant τ method, relationship between M_R, $\tau_{\varepsilon l}$ and $\tau_{\sigma l}$ determined by (3) and that by (6)

Considering using one standard linear solid, for both (3) and (6), we have

$$M(\omega) = M_R\left(1-\frac{\omega^2\tau_{\sigma 1}(\tau_{\sigma 1}-\tau_{\varepsilon 1})}{1+\omega^2\tau_{\sigma 1}^2}+\mathbf{i}\frac{\omega(\tau_{\varepsilon 1}-\tau_{\sigma 1})}{1+\omega^2\tau_{\sigma 1}^2}\right) \tag{A-1}$$

Thus

$$Q^{-1} = \frac{\mathrm{Im}\left[M(\omega)\right]}{\mathrm{Re}\left[M(\omega)\right]} = \frac{\dfrac{\omega\left(\tau_{\varepsilon 1} - \tau_{\sigma 1}\right)}{1+\omega^2\tau_{\sigma 1}^2}}{1 - \dfrac{\omega^2\tau_{\sigma 1}\left(\tau_{\sigma 1} - \tau_{\varepsilon 1}\right)}{1+\omega^2\tau_{\sigma 1}^2}} = \frac{\omega\left(\tau_{\varepsilon 1} - \tau_{\sigma 1}\right)}{1+\omega^2\tau_{\sigma 1}\tau_{\varepsilon 1}} \tag{A-2}$$

Introducing $\tau = \dfrac{\tau_{\varepsilon l} - \tau_{\sigma l}}{\tau_{\sigma l}}$, we can rewritten (A-2) as:

$$Q^{-1} = \frac{\omega\tau\tau_{\sigma 1}}{1+\omega^2\tau_{\sigma 1}^2\left(1+\tau\right)} \tag{A-3}$$

In paper: “Modeling of a constant Q: Methodology and algorithm for an efficient and optimally inexpensive viscoelastic technique”, Blanch, Robertsson and Symes observed the L2 approximation of constant Q near the interesting band $\left[\omega_0 - \Delta\omega, \omega_0 + \Delta\omega\right]$ with (A-3) gives $\tau << 1$ for bigger Q (Q > 20).Moreover, by setting a constant τ, the change of $\tau_{\sigma 1}$ is enough to give good approximation if ω_0 changes. Thus they prosper to use constant τ in approximation of constant Q in wide frequency band.

Thus, in general case, for $\tau << 1$, with (3) we have:

$$Q^{-1} = \frac{\dfrac{1}{N}\displaystyle\sum_{l=1}^{N}\frac{\omega\tau_{(3)}\tau_{\sigma 1}}{1+\omega^2\tau_{\sigma l}^2}}{1 - \tau_{(3)}\dfrac{1}{N}\displaystyle\sum_{l=1}^{N}\frac{\omega^2\tau_{\sigma l}\tau_{\sigma 1}}{1+\omega^2\tau_{\sigma l}^2}} \approx \frac{1}{N}\sum_{l=1}^{N}\frac{\omega\tau_{(3)}\tau_{\sigma 1}}{1+\omega^2\tau_{\sigma l}^2} \tag{A-4}$$

While (6) we have

$$Q^{-1} = \frac{\displaystyle\sum_{l=1}^{N}\frac{\omega\tau_{(6)}\tau_{\sigma 1}}{1+\omega^2\tau_{\sigma l}^2}}{1 - \tau_{(6)}\displaystyle\sum_{l=1}^{N}\frac{\omega^2\tau_{\sigma l}\tau_{\sigma 1}}{1+\omega^2\tau_{\sigma l}^2}} \approx \sum_{l=1}^{N}\frac{\omega\tau_{(6)}\tau_{\sigma 1}}{1+\omega^2\tau_{\sigma l}^2} \tag{A-5}$$

If we use same set of $\tau_{\sigma 1}$. Different value of τ will be obtained by (3) compared that obtained by (6) denoted as $\tau_{(6)}$, which gives that

$$\tau_{(3)} / N = \tau_{(6)} \tag{A-6}$$

That is

$$\tau'_{\varepsilon l} = \left(\tau_{\varepsilon l} - \tau_{\sigma l}\right)\frac{1}{N} + \tau_{\sigma l} \tag{A-6}$$