Formulations with and without the $\frac{1}{L}$ factor are equivalent (Carcione 2001 and Moczo and Kristek 2005 say that a $\frac{1}{L}$ factor is missing, but in fact the two ways of writing it are equivalent). Here is how to obtain the formulation with $\frac{1}{L}$ from the formulation without.

with infinite values at the low- and high-frequency limits. A combination of several relaxation mechanisms can model any quality factor function versus frequency where the fitting parameters are the relaxation times.

The theory, developed in Carcione et al. (1988c), circumvents the convolutional relation between the stress and strain tensors by the introduction of the memory variables. In the 3-D case, the resulting wave equation is solved for the displacement field, one memory variable for each dissipation mechanism related to the dilatational wave, and five memory variables for each mechanism related to the shear wave. In the 2-D case, two memory variables are used for each shear relaxation mechanism. The problem is solved in the time domain by a new time integration method based on an optimum polynomial interpolation of the evolution operator (Tal-Ezer et al., 1990). This method is especially designed to solve wave propagation in linear viscoelastic media and greatly improves the spectral technique used in Carcione et al. (1988c). The spatial derivative terms are computed by means of the Fourier pseudospectral method. Similar approaches based on finite difference in time (a Taylor expansion of the evolution operator) and space were given in Day and Minster (1984), and in Emmerich and Korn (1987) for the viscoacoustic wave equation.

In the earth, there are cases where the impedance contrast is very weak but the contrast in attenuation is significant, i.e., if one of the materials is very unconsolidated or has fluid-filled pores. To simulate this situation, I present an example of waves impinging on a plane interface separating an elastic material of a viscoelastic medium with similar elastic moduli and density but different quality factors (Q interface). A second example displays a common shot time section in a medium that includes highly anelastic lens-shaped bodies. Then, I compute the seismic response to a single shot of a complex structure containing a gas cap in an anticlinal fold, a typical trap in exploration geophysics. Finally, I consider examples of wave simulation in 3-D homogeneous and inhomogeneous structures. The algorithm is tested against the analytical solution, which is based on a 3-D viscoelastic Green's function derived from the correspondence principle.

EQUATION OF MOTION

The time-domain equation of motion of an $n - D$ viscoelastic medium is formed with the following equations (Carcione, 1987; Carcione et al., 1988c):

1) The linearized equations of momentum conservation:

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i, \qquad i = 1, \ldots, n,$$

where $\mathbf{x} = (x_1, x_2, x_3) \equiv (x, y, z)$ is the position vector, $\sigma_{ij}(\mathbf{x}, t)$ are the stress components, $u_i(\mathbf{x}, t)$ are the displacements, $\rho(\mathbf{x})$ denotes the density, and $f_i(\mathbf{x}, t)$ are the body forces, t being the time variable. Repeated indices imply summation and a dot above a variable indicates time differentiation.

2) The stress-strain relations:

$$\sigma_{xx} = (\lambda_u + 2\mu_u)\frac{\partial u_x}{\partial x} + \lambda_u\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \left(\lambda_r + \frac{2}{n}\mu_r\right)\sum_{\ell=1}^{L_1} e_{1\ell} + 2\mu_r \sum_{\ell=1}^{L_2} e_{11\ell}, \tag{2a}$$

$$\sigma_{yy} = (\lambda_u + 2\mu_u)\frac{\partial u_y}{\partial y} + \lambda_u\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + \left(\lambda_r + \frac{2}{n}\mu_r\right)\sum_{\ell=1}^{L_1} e_{1\ell} + 2\mu_r \sum_{\ell=1}^{L_2} e_{22\ell}, \tag{2b}$$

$$\sigma_{zz} = (\lambda_u + 2\mu_u)\frac{\partial u_z}{\partial z} + \lambda_u\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \left(\lambda_r + \frac{2\mu_r}{n}\right)\sum_{\ell=1}^{L_1} e_{1\ell} - \frac{2}{n}\mu_r \sum_{\ell=1}^{L_2} (e_{11\ell} + e_{22\ell}), \tag{2c}$$

$$\sigma_{xy} = \mu_u\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \mu_r \sum_{\ell=1}^{L_2} e_{12\ell}, \tag{2d}$$

$$\sigma_{xz} = \mu_u\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \mu_r \sum_{\ell=1}^{L_2} e_{13\ell}, \tag{2e}$$

$$\sigma_{yz} = \mu_u\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) + \mu_r \sum_{\ell=1}^{L_2} e_{23\ell}, \tag{2f}$$

where

$$\lambda_u = \underbrace{\left(\lambda_r + \frac{2}{n}\mu_r\right)}_{\text{this is the bulk modulus Kappa}} M_{u1} - \frac{2}{n}\mu_r M_{u2}, \tag{3a}$$

Thus M_{u1} corresponds to Q_K, (not to Q_P)

and

$$\mu_u = \mu_r M_{u2}, \tag{3b}$$

$K = \lambda + \frac{2}{n}\mu$ → $K = \lambda + \mu$ in 2D, $K = \lambda + \frac{2}{3}\mu$ in 3D

are the unrelaxed or high-frequency Lame constants, with λ_r and μ_r the relaxed or low-frequency Lame constants. M_{uv}, $v = 1, 2$ are relaxation functions evaluated at $t = 0$, with $v = 1$, the dilatational mode, and $v = 2$, the shear mode. For the general standard linear solid rheology, they are given by

$$M_{uv} = 1 - \frac{1}{L}\sum_{\ell=1}^{L_v}\left(1 - \frac{\tau_{\epsilon\ell}^{(v)}}{\tau_{\sigma\ell}^{(v)}}\right) = +\frac{1}{L}\sum_{\ell=1}^{L}\frac{\tau_{\epsilon\ell}}{\tau_{\sigma\ell}}, \qquad v = 1, 2, \tag{4}$$

Carcione 2007 eq. 2.199

with $\tau_{\sigma\ell}^{(v)}$ and $\tau_{\epsilon\ell}^{(v)}$ material relaxation times. The quantities $e_{1\ell}(\mathbf{x}, t)$ are memory variables related to the L_1 mechanisms which describe the anelastic characteristics of the dilatational wave, and $e_{11\ell}(\mathbf{x}, t)$, $e_{22\ell}(\mathbf{x}, t)$, $e_{12\ell}(\mathbf{x}, t)$, $e_{13\ell}(\mathbf{x}, t)$, and $e_{23\ell}(\mathbf{x}, t)$ are memory variables related to the L_2 mechanisms of the quasi-shear wave.

Remark from Dimitri Komatitsch : equation (2c) for sigma_zz is not correct, in 2D it should be:
sigma_zz = lambdaplus2mu_unrelaxed*duz_dz + lambda_unrelaxed*dux_dx + (lambda_relaxed + mu_relaxed) * sum_of_e1 - 2 * mu_relaxed * sum_of_e11

Carcione, Seismic modeling in viscoelastic media, Geophysics vol. 58, p. 110-120 (1993)

In the 2D case, equation (2c) of Carcione 1993 should be:

$$\sigma_{zz} = (\lambda_u + 2\mu_u)\frac{\partial u_z}{\partial z} + \lambda_u \frac{\partial u_x}{\partial x} + (\lambda_r + \mu_r)\sum e_1 - 2\mu_r \sum e_{11} \tag{1}$$

i.e., n in equation (2c) is not at the right place.

Fixed in this scan

3) The memory variable first-order equations in time:

$$\dot{e}_{1\ell} = \Theta\phi_{1\ell} - \frac{e_{1\ell}}{\tau_{\sigma\ell}^{(1)}}, \qquad \ell = 1, \ldots, L_1, \tag{5a}$$

$$\dot{e}_{11\ell} = \left(\frac{\partial u_x}{\partial x} - \frac{\Theta}{n}\right)\phi_{2\ell} - \frac{e_{11\ell}}{\tau_{\sigma\ell}^{(2)}}, \qquad \ell = 1, \ldots, L_2, \tag{5b}$$

$$\dot{e}_{22\ell} = \left(\frac{\partial u_y}{\partial y} - \frac{\Theta}{n}\right)\phi_{2\ell} - \frac{e_{22\ell}}{\tau_{\sigma\ell}^{(2)}}, \qquad \ell = 1, \ldots, L_2, \tag{5c}$$

$$\dot{e}_{12\ell} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\phi_{2\ell} - \frac{e_{12\ell}}{\tau_{\sigma\ell}^{(2)}}, \qquad \ell = 1, \ldots, L_2, \tag{5d}$$

$$\dot{e}_{13\ell} = \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\phi_{2\ell} - \frac{e_{13\ell}}{\tau_{\sigma\ell}^{(2)}}, \qquad \ell = 1, \ldots, L_2, \tag{5e}$$

$$\dot{e}_{23\ell} = \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)\phi_{2\ell} - \frac{e_{23\ell}}{\tau_{\sigma\ell}^{(2)}}, \qquad \ell = 1, \ldots, L_2, \tag{5f}$$

where

$$\Theta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{6}$$

is the dilatation field, and

$$\phi_{\nu\ell} = \frac{1}{\tau_{\sigma\ell}^{(\nu)}}\left(1 - \frac{\tau_{\varepsilon\ell}^{(\nu)}}{\tau_{\sigma\ell}^{(\nu)}}\right), \qquad \nu = 1, 2, \tag{7}$$

$\times \dfrac{1}{\frac{1}{L}\sum_{\ell=1}^{L} \frac{\tau_{\varepsilon\ell}}{\tau_{\sigma\ell}}}$

eq 2.330 of 3rd edition, Same as eq 2.293 of second edition. Also eqs. 3.192 and 3.193

are the response function components evaluated at $t = 0$. The low-frequency or elastic limit ins obtained when $\tau_{\varepsilon\ell}^{(\nu)} \rightarrow \tau_{\sigma\ell}^{(\nu)}$, $\forall \ell$; thus, $M_{uv} \rightarrow 1$ and $\phi_{\nu\ell} \rightarrow 0$, and the memory variables vanish. On the other hand, at the high-frequency limit the system also behaves elastically, corresponding to the instantaneous response. As can be seen from the stress-strain equations, the mean stress depends only on the parameters and memory variables with index $\nu = 1$ which involve dilatational dissipation mechanisms. Similarly, the deviatoric stress components depend on the parameters and memory variables with index $\nu = 2$, involving shear mechanisms. The 3-D case is obtained with $n = 3$, the 2-D case with $n = 2$ and, say, $\partial/\partial y\,[\cdot] = 0$, $\sigma_{yy} = \sigma_{xy} = \sigma_{zy} = 0$, and $e_{22\ell} = e_{12\ell} = e_{23\ell} = 0$. The elastic case is obtained by taking $\tau_{\varepsilon\ell} = \tau_{\sigma\ell}$, $\forall \ell$ (low-frequency limit), or by zeroing the memory variables and taking the unrelaxed Lame constants as the elastic Lame constants (high-frequency limit). Viscoacoustic wave propagation is simply obtained by setting $\mu_r = 0$; the resulting equation can be written in terms of the dilatation 0, or in terms of the pressure $p = -\sigma_{xx} = -\sigma_{yy} = -\sigma_{zz}$. The system of equations (1), (2a, f) and (5a, f) is solved for the displacement field and memory variables by using a new spectral algorithm as a time marching scheme (Tal-Ezer et al., 1990). To balance time integration and spatial accuracies, the spatial derivatives are computed by means of the Fourier pseudospectral method.

2-D WAVE PROPAGATION

Q interface

This example considers wave propagation across an interface separating media with different quality factors but similar elastic moduli. The left half-space is elastic and the right half-space is viscoelastic (see Figure 4). The viscoelastic medium has almost constant quality factors in the seismic exploration band, as can be seen in Figure 1 where the bulk (Q_k), compressional (Q_P), and shear (Q_S) quality factors are plotted. Relaxation times are $\tau_{\varepsilon 1}^{(1)} = 0.0334$ s, $\tau_{\sigma 1}^{(1)} = 0.0303$ s, $\tau_{\varepsilon 1}^{(2)} = 0.0352$ s, $\tau_{\sigma 1}^{(2)} = 0.0287$ s, $\tau_{\varepsilon 2}^{(1)} = 0.0028$ s, $\tau_{\sigma 2}^{(1)} = 0.0025$ s, $\tau_{\varepsilon 2}^{(2)} = 0.0029$ s, and $\tau_{\sigma 2}^{(2)} = 0.0024$ s. The group and phase velocities are displayed in Figure 2a and 2b for P- and S-waves, respectively; they indicate strong wave dispersion. Expressions for the quality factors and wave velocities in viscoelastic media can be found in Carcione et al. (1988c).

The compressional-and shear-wave velocities of the elastic medium are chosen in such a way as to minimize the normal *PP* and *SS* reflection coefficients at the central frequency of the source (25 Hz) whose spectra is plotted in Figure 1 with a dotted line. As stated by the correspondence principle (Bland, 1960), the reflection and transmission coefficients for an interface in attenuating media may be obtained from their analogues in elastic media by merely substituting the elastic velocities for the complex anelastic velocities. Assuming constant density in the whole space, the normal incidence reflection coefficient simply becomes $R(x) = (V - x)/(V + x)$, where V is the P-wave (S-wave)

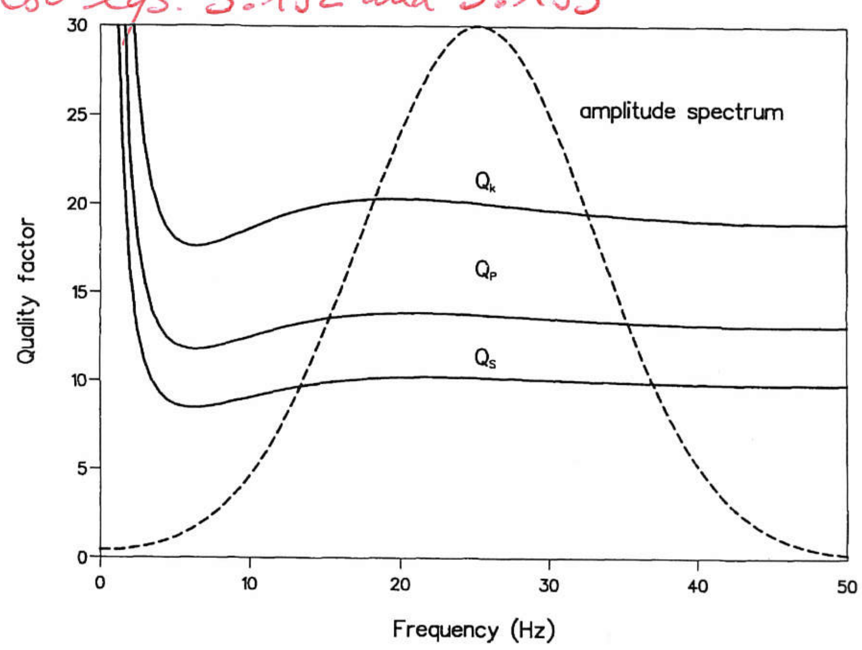

FIG. 1. P-wave, S-wave, and bulk quality factors for the viscoelastic medium of the Q interface. The dashed line represents the amplitude spectrum of the source.

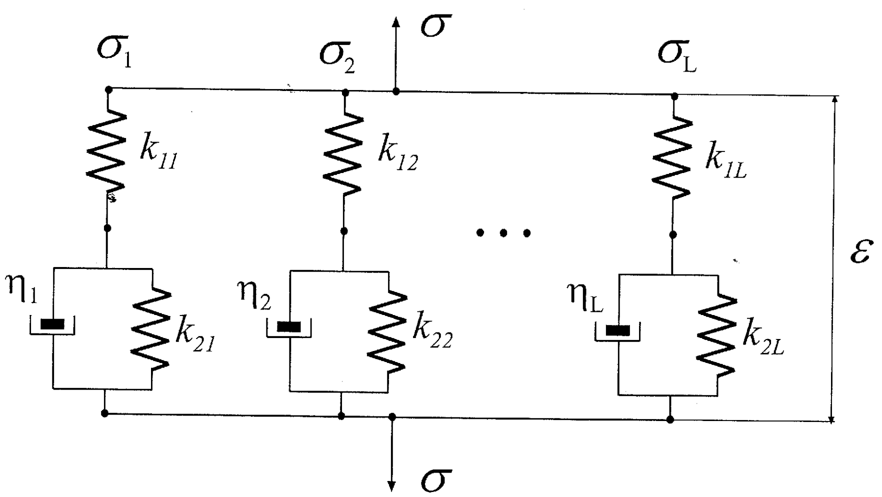

Figure 2.11: Mechanical model for a generalized Zener material.

The total stress acting on the system is $\sigma = \sum_{l=1}^{L} \sigma_l$. Therefore, the stress-strain relation in the frequency domain is

$$\sigma = \sum_{l=1}^{L} M_l \epsilon = \sum_{l=1}^{L} M_{Rl} \left(\frac{1 + i\omega\tau_{\epsilon l}}{1 + i\omega\tau_{\sigma l}} \right) \epsilon. \tag{2.165}$$

We can choose $M_{Rl} = M_R/L$, and the complex modulus can be expressed as

$$M(\omega) = \sum_{l=1}^{L} M_l(\omega), \quad M_l(\omega) = \frac{M_R}{L} \left(\frac{1 + i\omega\tau_{\epsilon l}}{1 + i\omega\tau_{\sigma l}} \right), \tag{2.166}$$

thereby reducing the number of independent constants to $2L + 1$.

The relaxation function is easily obtained from the time-domain constitutive equation

$$\sigma = \sum_{l=1}^{L} \sigma_l = \sum_{l=1}^{L} \psi_l * \partial_t \epsilon \equiv \psi * \partial_t \epsilon, \tag{2.167}$$

where ψ_l has the form (2.157), and

$$\psi(t) = M_R \left[1 - \frac{1}{L} \sum_{l=1}^{L} \left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}} \right) \exp(-t/\tau_{\sigma l}) \right] H(t). \tag{2.168}$$

The unrelaxed modulus is obtained for $t = 0$,

$$M_U = M_R \left[1 - \frac{1}{L} \sum_{l=1}^{L} \left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}} \right) \right] = \frac{M_R}{L} \sum_{l=1}^{L} \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}. \tag{2.169}$$

The relaxation function obtained by Liu, Anderson and Kanamori (1976) lacks the factor $1/L$.

Notes from Emmanuel Chaljub, CNRS Grenoble, France, November 2015

1 Stress relaxation functions

The stress-strain relation in a viscoelastic solid writes:

$$\sigma(t) \;=\; \Psi(t) \star \dot{\varepsilon}(t) \;=\; M(t) \star \varepsilon(t) \tag{1}$$

where Ψ is the stress relaxation function (i.e. the stress response to a unit step in strain), and M is the viscoelastic modulus, $M(t) = \dot{\Psi}(t)$.

The strain response for a generalized Zener body (see e.g. Moczo and Kristek 2005) is:

$$\Psi(t) \;=\; \left\{ \sum_{l=1}^{n} \frac{1}{n} M_R \left[1 - \left(1 - \frac{\tau_l^{\varepsilon}}{\tau_l^{\sigma}} \right) \exp\left(-\frac{t}{\tau_l^{\sigma}} \right) \right] \right\} H(t) \tag{2}$$

where M_R is the relaxed elastic modulus, τ^{σ} and τ^{ε} are the stress and strain relaxation times, respectively, and H is the Heaviside step function. The $\frac{1}{n}$ factor is important at this stage to properly interpret the relaxation function as the sum of individual Zener mechanisms. The equation is simplified as:

$$\Psi(t) \;=\; M_R \left[1 - \frac{1}{n} \sum_{l=1}^{n} \left(1 - \frac{\tau_l^{\varepsilon}}{\tau_l^{\sigma}} \right) \exp\left(-\frac{t}{\tau_l^{\sigma}} \right) \right] H(t) \tag{3}$$

In Komatitsch and Tromp (2002), the stress relaxation function is

$$\Psi(t) \;=\; M_R \left[1 - \sum_{l=1}^{n} \left(1 - \frac{\tau_l^{\varepsilon}}{\tau_l^{\sigma}} \right) \exp\left(-\frac{t}{\tau_l^{\sigma}} \right) \right] H(t) \tag{4}$$

At this stage, we define the coefficients:

$$\theta_l \;=\; \frac{1}{\tau_l^{\sigma}} \tag{5}$$

$$\kappa_l \;=\; \frac{\tau_l^{\varepsilon}}{\tau_l^{\sigma}} - 1 \tag{6}$$

$$\kappa_l^{\star} \;=\; n\,\kappa_l \tag{7}$$

The generalized Zener body relaxation function (3) writes:

$$\Psi(t) \;=\; M_R \left[1 + \frac{1}{n} \sum_{l=1}^{n} \kappa_l \exp\left(-\theta_l\, t \right) \right] H(t) \tag{8}$$

The relaxation function used in Komatitsch and Tromp (2002) writes:

$$\Psi(t) \;=\; M_R \left[1 + \sum_{l=1}^{n} \kappa_l \exp\left(-\theta_l\, t \right) \right] H(t) \tag{9}$$

or

$$\Psi(t) \;=\; M_R \left[1 + \frac{1}{n} \sum_{l=1}^{n} \kappa_l^{\star} \exp\left(-\theta_l\, t \right) \right] H(t) \tag{10}$$

Note that the two relaxation functions (with or without the $1/n$ factor) only differ because of the definition of the κ coefficients.

2 Elastic moduli

Next we have to define the elastic moduli in the frequency domain. We apply the time derivative in the frequency domain: $\hat{M} = \imath\omega\,\hat{\Psi}$ where $\hat{}$ denotes Fourier transform, and we use the known transform:

$$\widehat{\exp(-at)H(t)} = \frac{1}{a + i\omega} \tag{11}$$

to get the elastic modulus of the generalized Zener body:

$$M(\omega) \;=\; M_R\left(1 + \frac{\imath\,\omega}{n}\sum_{l=1}^{n}\frac{\kappa_l}{\theta_l + \imath\,\omega}\right) \tag{12}$$

The real (M_1) and imaginary (M_2) parts of the modulus are defined by:

$$M(\omega) \;=\; M_1(\omega) \;+\; \imath\,M_2(\omega) \tag{13}$$

$$\frac{M_1(\omega)}{M_R} \;=\; 1 + \frac{1}{n}\sum_{l=1}^{n}\frac{\omega^2\,\kappa_l}{\theta_l^2 + \omega^2} \tag{14}$$

$$\frac{M_2(\omega)}{M_R} \;=\; \frac{1}{n}\sum_{l=1}^{n}\frac{\omega\,\kappa_l\,\theta_l}{\theta_l^2 + \omega^2} \tag{15}$$

To get the complex elastic modulus for the relaxation function used in Komatitsch and Tromp (2002), we just have to replace κ by $\kappa^\star$ (or equivalently to remove the $\frac{1}{n}$ term) in eqs 12, 14 and 15:

$$M(\omega) \;=\; M_R\left(1 + \imath\,\omega\sum_{l=1}^{n}\frac{\kappa_l}{\theta_l + \imath\,\omega}\right) \tag{16}$$

$$\frac{M_1(\omega)}{M_R} \;=\; 1 + \sum_{l=1}^{n}\frac{\omega^2\,\kappa_l}{\theta_l^2 + \omega^2} \tag{17}$$

$$\frac{M_2(\omega)}{M_R} \;=\; \sum_{l=1}^{n}\frac{\omega\,\kappa_l\,\theta_l}{\theta_l^2 + \omega^2} \tag{18}$$

3 Solving for the coefficients

The κ and θ coefficients are then computed so that the ratio of real to imaginary parts of the elastic modulus fits a target Quality factor Q at discrete frequencies $\omega_j, j = 1...J$:

$$M_1(\omega_j) \;=\; Q(\omega_j)\,M_2(\omega_j) \tag{19}$$

For the generalized Zener body, this writes:

$$\frac{1}{n}\sum_{l=1}^{n}\left(\frac{\omega_j\,\theta_l\,Q(\omega_j) - \omega_j^2}{\theta_l^2 + \omega_j^2}\right)\kappa_l \;=\; 1 \tag{20}$$

For the relaxation function used in Komatitsch and Tromp (2002) and implemented in `specfem`, the system to solve is:

$$\sum_{l=1}^{n} \left(\frac{\omega_j \, \theta_l \, Q(\omega_j) - \omega_j^2}{\theta_l^2 + \omega_j^2} \right) \kappa_l \;=\; 1 \tag{21}$$

Here, we should note that the non-linear solver `Solvopt` does not solve the system (20) for θ_l and κ_l but actually solves the system (21) for θ_l and $\kappa_l^{\texttt{Solvopt}} = \frac{1}{n}\kappa_l$, which are stored in the arrays `point` and `weight` in the code, respectively. The easiest way to include `Solvopt` in `specfem` is thus to use directly the $\kappa_l^{\texttt{Solvopt}}$ values.

4 Memory variables

We write again Hooke's law (1) with the stress relaxation function of a generalized Zener body:

$$\sigma(t) \;=\; M(t) \star \varepsilon(t) \;=\; \dot{\Psi}(t) \star \varepsilon(t) \tag{22}$$

$$\Psi(t) \;=\; M_R \left[1 + \frac{1}{n} \sum_{l=1}^{n} \kappa_l \exp\left(-\,\theta_l \, t\right) \right] H(t) \tag{23}$$

The elastic modulus is the time derivative of the relaxation function, which writes

$$M(t) \;=\; M_R \left[1 + \frac{1}{n} \sum_{l=1}^{n} \kappa_l \right] \delta(t) - \frac{1}{n} M_R \sum_{l=1}^{n} \theta_l \kappa_l \exp\left(-\,\theta_l \, t\right) \; H(t) \tag{24}$$

or

$$M(t) \;=\; M_U \, \delta(t) - \frac{1}{n} M_R \sum_{l=1}^{n} \theta_l \kappa_l \exp\left(-\,\theta_l \, t\right) \; H(t) \tag{25}$$

Next, we introduce the auxiliary variables

$$\Phi_l(t) \;=\; -\theta_l \, \kappa_l \; \exp\left(-\,\theta_l \, t\right) \tag{26}$$

and the memory variables

$$R_l(t) \;=\; \left[\Phi_l(t) \, H(t)\right] \star \varepsilon(t) \tag{27}$$

Hooke's law can be rewritten as:

$$\sigma(t) \;=\; M_U \, \varepsilon(t) + \frac{1}{n} M_R \sum_{l=1}^{n} R_l(t) \tag{28}$$

and each memory variable statisfies the first order differential equation in time

$$\partial_t \, R_l(t) \;+\; \theta_l \, R_l(t) \;=\; -\kappa_l \, \theta_l \, \varepsilon(t) \tag{29}$$

5 Dispersion

If practice, the moduli are given at a reference frequency ω_{ref}, so that the relaxed and unrelaxed values are not known. Once the κ and θ coefficients are computed, it is enough to scale the real values of the moduli using Eq. 14 or 17, following that we use or not the $\frac{1}{n}$ factor.

For the generalized Zener body rheology (i.e. with the $\frac{1}{n}$ factor), the unrelaxed and relaxed values are

$$M_U = M_{ref}\left(\frac{1+\frac{1}{n}\sum_{l=1}^{n}\kappa_l}{1+\frac{1}{n}\sum_{l=1}^{n}\frac{\omega_{ref}^2\kappa_l}{\theta_l^2+\omega_{ref}^2}}\right) \tag{30}$$

$$M_R = M_{ref}\left(\frac{1}{1+\frac{1}{n}\sum_{l=1}^{n}\frac{\omega_{ref}^2\kappa_l}{\theta_l^2+\omega_{ref}^2}}\right) \tag{31}$$

$$\text{or } M_R = M_U - M_{ref}\left(\frac{\frac{1}{n}\sum_{l=1}^{n}\kappa_l}{1+\frac{1}{n}\sum_{l=1}^{n}\frac{\omega_{ref}^2\kappa_l}{\theta_l^2+\omega_{ref}^2}}\right) \tag{32}$$

$$\text{or } M_R = M_U\left(\frac{1}{1+\frac{1}{n}\sum_{l=1}^{n}\kappa_l}\right) \tag{33}$$

Note that using the definition of θ_l (Eq. 5), Eq. (30) can be rewritten:

$$M_U = M_{ref}\left(\frac{1+\frac{1}{n}\sum_{l=1}^{n}\kappa_l}{1+\frac{1}{n}\sum_{l=1}^{n}\frac{\kappa_l}{1+\frac{1}{\left(\omega_{ref}*\tau_l^{\sigma}\right)^2}}}\right) \tag{34}$$

Note: Viscoelastic Media

C.-H. Zhang
(Dated: October 7, 2015)

I. VISCOELASTIC ISOTROPIC CONTINUUM

The stress-strain relation in a viscoelastic medium can be defined as

$$\sigma_{ij}(t) = \psi_{ijkl} * \dot{\varepsilon}_{kl}(t) = \dot{\psi}_{ijkl} * \varepsilon_{kl}(t), \tag{1}$$

where ψ_{ijkl} is the relaxation function, and symbol * is used for the convolution. Defining

$$M_{ijkl}(t) = \dot{\psi}_{ijkl} \tag{2}$$

as the modulus function, I have

$$\sigma_{ij}(t) = M_{ijkl} * \varepsilon_{kl}(t). \tag{3}$$

The strain-stress relation

$$\varepsilon_{ij}(t) = \chi(t) * \dot{\sigma}(t) = \dot{\chi}(t) * \sigma(t) = J(t) * \sigma(t), \tag{4}$$

where χ or $J(t)$ is referred to as the creep function. The relation between J and M is that

$$M(t) * J(t) = \delta(t) \tag{5}$$

in time space, and

$$M(\omega)J(\omega) = 1 \tag{6}$$

in frequency space.

For an isotropic system, the stress-strain relation can be written as

$$\sigma_{ij}(t) = \delta_{ij} M_\kappa(t) * \varepsilon_{kk}(t) + 2M_\mu(t) * \left[\varepsilon_{ij}(t) - \frac{1}{3}\delta_{ij}\varepsilon_{kk}(t)\right] \tag{7}$$

where $M_\kappa(t)$ and $M_\mu(t)$ are relaxation function for compression and shear waves, respectively. Applying the Fourier transformation, we have

$$\sigma_{ij}(\omega) = \delta_{ij} M_\kappa(\omega)\varepsilon_{kk}(\omega) + 2M_\mu(\omega)\left[\varepsilon_{ij}(\omega) - \frac{1}{3}\delta_{ij}\varepsilon_{kk}(\omega)\right]. \tag{8}$$

Given the viscoelastic modulus, the quality factor $Q(\omega)$ is

$$Q_\nu(\omega) = \frac{\mathrm{Re}M_\nu(\omega)}{\mathrm{Im}M_\nu(\omega)}, \text{with } \nu = \kappa, \mu. \tag{9}$$

Now we use a rheological model called generalized standard linear solid (GSLS) to incorporate the anelasticity, which is composed of L Maxwell bodies (spring $\delta M_{\nu,l}$ and dashpot $\eta_{\nu,l}$ in series) connected in parallel with a spring M_ν, as shown schematically in Fig. (1). For each Maxwell body l, a given stress σ_l produces a deformation $\varepsilon_{l,1}$ on the spring, and a deformation $\varepsilon_{l,2}$ on the dashpot. The stress-strain relation in the spring is given (Carcione's 2001 book, page 61)

$$\sigma_l = \delta M_{\nu,l}\varepsilon_{l,1} \tag{10}$$

and the stress-strain relation in the dashpot is

$$\sigma_l = \eta_l \partial_t \varepsilon_{l,2}. \tag{11}$$

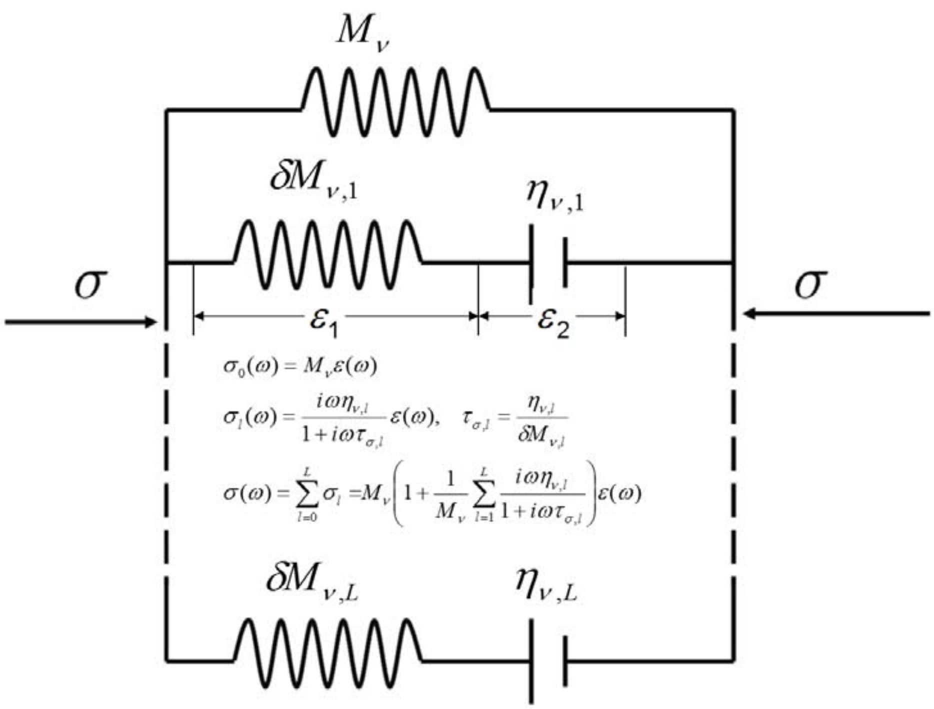

FIG. 1: Schematic diagram of generalized standard linear solid (GSLS) composed of L parallel so-called Maxwell bodies in parallel with a spring with a spring constant M_ν.

The total deformation of the Maxwell body is

$$\varepsilon_l = \varepsilon_{l,1} + \varepsilon_{l,2}. \tag{12}$$

Using the above three equations, one obtains the stress-strain relation of the Maxwwell element as

$$\frac{\partial_t \sigma}{\delta M_{\nu,l}} + \frac{\sigma_l}{\eta} = \partial_t \varepsilon_l. \tag{13}$$

Transforming into Fourier space, one has

$$\sigma_l(\omega) = \frac{i\omega\eta_l}{1 + i\omega\frac{\eta_l}{\delta M_{\nu,l}}}\varepsilon_l(\omega) = M_l(\omega)\varepsilon_l(\omega) \tag{14}$$

where

$$M_l(\omega) = \frac{i\omega\eta_l}{1 + i\omega\frac{\eta_l}{\delta M_{\nu,l}}} = \frac{i\omega\eta_l}{1 + i\omega\tau_{\sigma,l}^{(\nu)}}, \tau_{\sigma,l}^{(\nu)} = \frac{\eta_l}{\delta M_{\nu,l}}. \tag{15}$$

Now the total strain on the whole GSLS is

$$\varepsilon = \varepsilon_1 = \cdots = \varepsilon_l \tag{16}$$

and stress on the GSLS is

$$\sigma(\omega) = \sum_{l=0}^{L}\sigma_l = M_\nu\left[1 + \frac{1}{M_\nu}\sum_{l=1}^{L} M_l(\omega)\right]\varepsilon(\omega) = M_\nu\left[1 + \frac{1}{M_\nu}\sum_{l=1}^{L}\frac{i\omega\eta_l}{1 + i\omega\tau_{\sigma,l}^{(\nu)}}\right]\varepsilon(\omega). \tag{17}$$

This equation is the same one as (7) in Moczo and Kristek (Geophysical Research Letters, 32, L01306, 2005), including the definition of $\tau_{\sigma,l}$. Now I introduce

$$\tau_{\epsilon,l}^{(\nu)} = \eta_l\left(\frac{1}{M_\nu} + \frac{1}{\delta M_{\nu,l}}\right), \tag{18}$$

This is where the result obtained here is different from that in oczo and Kristek (Geophysical Research Letters, 32, L01306, 2005). Using (18), one has the complex modulus M in this GSLS as

$$\begin{aligned}M_\nu(\omega) &= M_\nu\left[1-L+\sum_{l=1}^{L}\frac{1+i\omega\tau_{\epsilon,l}^{(\nu)}}{1+i\omega\tau_{\sigma,l}^{(\nu)}}\right] = M_\nu\left[1-\sum_{l=1}^{L}\left(1-\frac{\tau_{\epsilon,l}^{(\nu)}}{\tau_{\sigma,l}^{(\nu)}}\right)+\sum_{l=1}^{L}\left(1-\frac{\tau_{\epsilon,l}^{(\nu)}}{\tau_{\sigma,l}^{(\nu)}}\right)\frac{1}{1+i\omega\tau_{\sigma,l}^{(\nu)}}\right],\\ &= M_\nu\left[M_{\nu,1}(\omega)+iM_{\nu,2}(\omega)\right],\end{aligned} \tag{19}$$

where

$$\begin{aligned}M_{\nu,1}(\omega) &= 1+\sum_{l=1}^{L}\frac{\omega^2\tau_{\sigma,l}^{(\nu)}\left(\tau_{\epsilon,l}^{(\nu)}-\tau_{\sigma,l}^{(\nu)}\right)}{1+\omega^2\left(\tau_{\sigma,l}^{(\nu)}\right)^2},\\ M_{\nu,2}(\omega) &= \sum_{l=1}^{L}\frac{\omega\left(\tau_{\epsilon,l}^{(\nu)}-\tau_{\sigma,l}^{(\nu)}\right)}{1+\omega^2\left(\tau_{\sigma,l}^{(\nu)}\right)^2}.\end{aligned} \tag{20}$$

According to (6), I obtain

$$\begin{aligned}J_\nu(\omega) = [M_\nu(\omega)]^{-1} &= \frac{1}{M_\nu}\left[1-L+\sum_{l=1}^{L}\frac{1+i\omega\tau_{\sigma,l}^{(\nu)}}{1+i\omega\tau_{\epsilon,l}^{(\nu)}}\right]\\ &= \frac{1}{M_\nu}\left[1-\sum_{l=1}^{L}\left(1-\frac{\tau_{\sigma,l}^{(\nu)}}{\tau_{\epsilon,l}^{(\nu)}}\right)+\sum_{l=1}^{L}\left(1-\frac{\tau_{\sigma,l}^{(\nu)}}{\tau_{\epsilon,l}^{(\nu)}}\right)\frac{1}{1+i\omega\tau_{\epsilon,l}^{(\nu)}}\right]\\ &= \frac{1}{M_\nu}\left[A_\nu-iB_\nu\right],\end{aligned} \tag{21}$$

where

$$\begin{aligned}A_\nu(\omega) &= 1-\sum_{l=1}^{L}\frac{\omega^2\tau_{\epsilon,l}^{(\nu)}\left(\tau_{\epsilon,l}^{(\nu)}-\tau_{\sigma,l}^{(\nu)}\right)}{1+\omega^2\left(\tau_{\epsilon,l}^{(\nu)}\right)^2},\\ B_\nu(\omega) &= \sum_{l=1}^{L}\frac{\omega\left(\tau_{\epsilon,l}^{(\nu)}-\tau_{\sigma,l}^{(\nu)}\right)}{1+\omega^2\left(\tau_{\epsilon,l}^{(\nu)}\right)^2}.\end{aligned} \tag{22}$$

Doing inverse Fourier transformation of (19), (the Fourier transformation of 1 is delta(t), and $1/(1+i\omega\tau)$ is $e^{-t/\tau}$), we obtain the relaxation function

$$M_\nu(t) = M_\nu\left[1-\sum_{l=1}^{L}\left(1-\frac{\tau_{\epsilon,l}^{(\nu)}}{\tau_{\sigma,l}^{(\nu)}}\right)\right]\delta(t)+M_\nu\sum_{l=1}^{L}\left(1-\frac{\tau_{\epsilon,l}^{(\nu)}}{\tau_{\sigma,l}^{(\nu)}}\right)e^{-t/\tau_{\sigma,l}^{(\nu)}}H(t),\quad \nu=\kappa,\mu, \tag{23}$$

and the creep function

$$J_\nu(t) = \frac{1}{M_\nu}\left[1-\sum_{l=1}^{L}\left(1-\frac{\tau_{\sigma,l}^{(\nu)}}{\tau_{\epsilon,l}^{(\nu)}}\right)\right]\delta(t)+\frac{1}{M_\nu}\sum_{l=1}^{L}\left(1-\frac{\tau_{\sigma,l}^{(\nu)}}{\tau_{\epsilon,l}^{(\nu)}}\right)e^{-t/\tau_{\epsilon,l}^{(\nu)}}H(t),\quad \nu=\kappa,\mu. \tag{24}$$

Therefore, an instantaneous elastic response of the viscoelastic material is given by the so-called unrelaxed modulus $M_{\nu,U}$, a long-term equilibrium response is given by the relaxed modulus $M_{\nu,R}$,

$$M_{\nu,R} = \lim_{t\to\infty}\psi_\nu(t) = \lim_{\omega\to 0}M_\nu(\omega) = M_\nu, \tag{25}$$

$$M_{\nu,U} = \lim_{t\to 0}\psi_\nu(t) = \lim_{\omega\to\infty}M_\nu(\omega) = M_\nu\left[1-\sum_{l=1}^{L}\left(1-\frac{\tau_{\epsilon,l}^{(\nu)}}{\tau_{\sigma,l}^{(\nu)}}\right)\right] = M_{\nu,R}+\sum_{l=1}^{L}\delta M_{\nu,l}. \tag{26}$$

The frequency dependent Q factor is then

$$Q_\nu(\omega) = \frac{1+\displaystyle\sum_{l=1}^{L}\frac{\omega^2\tau_{\sigma,l}^{(\nu)}(\tau_{\epsilon,l}^{(\nu)}-\tau_{\sigma,l}^{(\nu)})}{1+\omega^2(\tau_{\sigma,l}^{(\nu)})^2}}{\displaystyle\sum_{l=1}^{L}\frac{\omega(\tau_{\epsilon,l}^{(\nu)}-\tau_{\sigma,l}^{(\nu)})}{1+\omega^2(\tau_{\sigma,l}^{(\nu)})^2}}. \tag{27}$$

The isotropic stress-strain relation under this GSLS can be written as

$$\begin{aligned}
\sigma_{ij}(t) =& \delta_{ij}M_\kappa\left[1-\sum_{l=1}^{L}\left(1-\frac{\tau_{\epsilon,l}^{(\kappa)}}{\tau_{\sigma,l}^{(\kappa)}}\right)\right]\varepsilon_{kk}(t)+2M_\mu\left[1-\sum_{l=1}^{L}\left(1-\frac{\tau_{\epsilon,l}^{(\mu)}}{\tau_{\sigma,l}^{(\mu)}}\right)\right]\left[\varepsilon_{ij}(t)-\frac{1}{3}\delta_{ij}\varepsilon_{kk}(t)\right] \\
&+\delta_{ij}\sum_{l=1}^{L}R_l^{(\kappa)}(t)+\sum_{l=1}^{L}R_{ij,l}^{(\mu)}(t) \\
&=\delta_{ij}M_{\kappa,U}\varepsilon_{kk}(t)+2M_{\mu,U}\left[\varepsilon_{ij}(t)-\frac{1}{3}\delta_{ij}\varepsilon_{kk}(t)\right]+\delta_{ij}\sum_{l=1}^{L}R_l^{(\kappa)}(t)+\sum_{l=1}^{L}R_{ij,l}^{(\mu)}(t)
\end{aligned} \tag{28}$$

where the memory variables are defined as

$$\begin{aligned}
R_l^{(\kappa)}(t) &= M_{\kappa,R}\left[\left(1-\frac{\tau_{\epsilon l}^{(\kappa)}}{\tau_{\sigma l}^{(\kappa)}}\right)e^{-t/\tau_{\sigma l}^{(\kappa)}}H(t)\right]*\varepsilon_{kk}(t) \\
&= M_{\kappa,R}\left(1-\frac{\tau_{\epsilon l}^{(\kappa)}}{\tau_{\sigma l}^{(\kappa)}}\right)\int_{-\infty}^{t}d\tau\varepsilon_{kk}(\tau)e^{-(t-\tau)/\tau_{\sigma l}^{(\kappa)}},
\end{aligned} \tag{29}$$

$$\begin{aligned}
R_{ij,l}^{(\mu)}(t) &= 2M_{\mu,R}\left[\left(1-\frac{\tau_{\epsilon l}^{(\mu)}}{\tau_{\sigma l}^{(\mu)}}\right)e^{-t/\tau_{\sigma l}^{(\mu)}}H(t)\right]*\left[\varepsilon_{ij}(t)-\frac{1}{3}\delta_{ij}\varepsilon_{kk}(t)\right] \\
&= 2M_{\mu,R}\left(1-\frac{\tau_{\epsilon l}^{(\mu)}}{\tau_{\sigma l}^{(\mu)}}\right)\int_{-\infty}^{t}d\tau\left[\varepsilon_{ij}(\tau)-\frac{1}{3}\delta_{ij}\varepsilon_{kk}(\tau)\right]e^{-(t-\tau)/\tau_{\sigma l}^{(\mu)}}.
\end{aligned} \tag{30}$$

The equations of motion of the memory variables are

$$\tau_{\sigma,l}^{(\kappa)}\frac{dR_l^{(\kappa)}(t)}{dt}+R_l^{(\kappa)}(t) = M_{\kappa,R}\left(1-\frac{\tau_{\epsilon,l}^{(\kappa)}}{\tau_{\sigma,l}^{(\kappa)}}\right)\varepsilon_{kk}(t), \tag{31}$$

$$\tau_{\sigma,l}^{(\mu)}\frac{dR_{ij,l}^{(\mu)}(t)}{dt}+R_{ij,l}^{(\mu)}(t) = 2M_{\mu,R}\left(1-\frac{\tau_{\epsilon,l}^{(\mu)}}{\tau_{\sigma,l}^{(\mu)}}\right)\left[\varepsilon_{ij}(t)-\frac{1}{3}\delta_{ij}\varepsilon_{kk}(t)\right]. \tag{32}$$

One can easily see that $R_{11}^{(\mu)}+R_{22}^{(\mu)}+R_{33}^{(\mu)}=0$. Therefore, for each SLS, we only have 6 memory variables: $R_l^{(\kappa)}(t)$, $R_{11,l}^{(\mu)}(t)$, $R_{22,l}^{(\mu)}(t)$, $R_{23,l}^{(\mu)}(t)$, $R_{13,l}^{(\mu)}(t)$ and $R_{12,l}^{(\mu)}(t)$.

From a physical point of view, (28) shows that an attenuated medium can be regarded as a physical lossy medium which depends on the medium stiffness constant.

Given the complex modulus (21) or (19), one can calculate the complex wavenumber as

$$k_\nu^2(\omega) = \left[\frac{\omega}{V_\nu}-i\alpha(\omega)\right]^2 = \frac{\omega^2}{V_{\nu,R}^2}\left[A_\nu(\omega)-iB_\nu(\omega)\right], \tag{33}$$

where $V_{\nu,R}=\sqrt{M_{\nu,R}/\rho}$ is the phase velocity associated with the relaxed elastic modulus $M_{\nu,R}$, and ρ is the density. From (33), one obtains

$$V_\nu^2(\omega) = \frac{2V_{\nu,R}^2}{B_\nu^2(\omega)}\Omega_\nu(\omega), \tag{34}$$

$$\alpha_\nu^2(\omega) = \frac{\omega^2}{2V_{\nu,R}^2}\Omega_\nu(\omega), \tag{35}$$

where

$$\Omega_\nu(\omega) = \sqrt{A_\nu^2(\omega) + B_\nu^2(\omega)} - A_\nu(\omega). \tag{36}$$

In practice, we are given a phase velocity V_P or V_S measured or estimated at a certain reference frequency ω_r. we have

$$M_{\nu,R} = \rho V_\nu^2(\omega_r)\frac{B_\nu^2(\omega_r)}{2\Omega_\nu(\omega_r)}. \tag{37}$$

On the other hand, the modulus at given frequency ω can be approximated determined by

$$M_\nu(\omega) \approx M_\nu(\omega_r)\left[1 + \frac{2}{\pi Q_\nu}\ln\frac{\omega}{\omega_r}\right]. \tag{38}$$

Therefore, at the central of the absorption band ω_c, we have

$$M_\nu(\omega_c) \approx M_\nu(\omega_r)\left[1 + \frac{2}{\pi Q_\nu}\ln\frac{\omega_c}{\omega_r}\right]. \tag{39}$$

Because of

$$\dot{\sigma}_{ij}(t) = \dot{M}_{ijkl}(t) * \varepsilon_{kl}(t) = M_{ijkl}(t) * \dot{\varepsilon}_{kl}(t), \tag{40}$$

one can see everything is still valid for the relation between $\dot{\sigma}_{ij}(t)$ and $\dot{\varepsilon}_{kl}(t)$.

In this relaxation model, we need determine $\tau_{\epsilon l}^{(\nu)}$ and $\tau_{\sigma l}^{(\nu)}$ according to the target Q_ν value using Eq. (27) for each SLS and each mode $\nu = \kappa$ andμ, and on each grid points which will need a lot memory.

First, we assume that $\tau_{\sigma,l}^{(\mu)} = \tau_{\sigma,l}^{(\kappa)}$ because it is mainly determined by the maximum and minimum frequencies the mesh can resolve. Supposing that the minimum frequency is f_{min}, we obtain the maximum frequency f_{max} for L SLS as

$$f_{max} = f_{min} \times 10^{\theta(L)}. \tag{41}$$

For example, for $L = 4$, the optimal $\theta(4) = 2.25$. We than set $\tau_{\sigma,l}$ to be L-equally spaced in $\log 10$ frequency. The central frequency of the absorption band is given as $\omega_c = 2\pi\sqrt{f_{min}f_{max}}$.

For a constant Q-spectrum within a limited frequency range described by the GSLSs, the optimal values of $\tau_{\epsilon l}$ is determined by a least-squared inversion of the following function

$$J_\nu\left\{\tau_{\epsilon l}\right\} = \int_{\omega_1}^{\omega_2}\left[Q_\nu\left\{\omega, \tau_{\epsilon l}^{(\nu)}\right\} - \tilde{Q}_\nu\right]^2 d\omega, \tag{42}$$

where $Q_\nu\left\{\omega, \tau_{\epsilon,l}^{(\nu)}\right\}$ is given by (27), and $\tilde{Q}_\nu$ is the desired constant Q. This optimization has to be performed for the desired Q's for both P- and S-waves which yields $\tau_{\epsilon,l}$-values for P- and S-waves on each grid points.

After obtaining $\tau_{\sigma,l}$ and $\tau_{\epsilon,l}$, one can calculate the modulus (37) at the reference frequency, and (39) at the central frequency.

A. Wave equation in the presence of attenuation

Here I summary the wave equation in which the attenuation is incorporated for isotropic media. Here $f_{ext,i}$ is the i-th component of the external source force vector.

- Displacement-stress form. Let's u_i be the i-th component of the displacement vector. The wave equation is then

$$\begin{aligned}
\rho\ddot{u}_i &= \sigma_{ij,j} + f_{ext,i} = f_{int,i} + f_{ext,i}, \\
\sigma_{ij}(t) &= \delta_{ij}M_{\kappa,U}\varepsilon_{kk}(t) + 2M_{\mu,U}\left[\varepsilon_{ij}(t) - \frac{1}{3}\delta_{ij}\varepsilon_{kk}(t)\right] + \delta_{ij}\sum_{l=1}^{L}R_l^{(\kappa)}(t) + \sum_{l=1}^{L}R_{ij,l}^{(\mu)}(t),
\end{aligned}$$