Notes on compatibility between Carcione's analytical viscoelastic solution and SPECFEM2D

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The use of terms as unrelaxed or relaxed moduli can be confusing, especially for the construction of quasi-analytical solutions. Carcione used, in his famous paper Carcione et al. (1988), the following formulation for the unrelaxed moduli, in eq. (38) where we added the $(1/L)$ factor that was missing

$$M_{u_\nu} = M_\nu \left(1 - \frac{1}{L} \sum_{l=1,L} (1 - \frac{\tau^\nu_{\epsilon,l}}{\tau^\nu_{\sigma,l}}) \right), \tag{1}$$

where L is the number of Zener solids, M_ν the relaxed modulus, M_{u_ν} the relaxed modulus and $(\tau^\nu_{\epsilon,l}, \tau^\nu_{\sigma,l})_{l=1,L}$ the strain relaxation times for $\nu = 1$ and shear relaxation times for $\nu = 2$. These unrelaxed moduli appear in the stress-strain relationship as

$$\sigma_{ij} = \frac{1}{n}\delta_{ij}(M_{u_1} + \Phi^c_1 *)\epsilon_{kk} + (M_{u_2} + \Phi^c_2 *)\epsilon^d_{ij}, \tag{2}$$

where n is the spatial dimension, Φ^c_ν the response function, δ_{ij} the dirac distribution and $\epsilon_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i$. Then if you want to build Carcione's analytical solution you will need to compute the relaxed moduli M_ν that read, from eq. (A.27a)-(A.27b)

$$\begin{aligned} M_1 &= \rho(n v_p^2 - 2(n-1)v_s^2), \\ M_2 &= 2\rho v s^2, \end{aligned} \tag{3}$$

where v_p, v_s are the relaxed P and S wave velocities.

But in SPECFEM2D, by default, the velocities in the "Par_file" represent the unrelaxed velocities at $f = f_0$, where f_0 is the chosen peak attenuation frequency that matches the option f0_attenuation in the Par_file. Then, in order to compute the right M_ν for Carcione's analytical solution, you should use

$$\begin{aligned} M_1 &= \frac{\rho(nv_{u,p}^2-2(n-1)v_{u,s}^2)}{\left(1-\frac{1}{L}\sum_{l=1,L}(1-\frac{\tau_{\epsilon,1}^l}{\tau_{\sigma,1}^l})\right)}, \\ M_2 &= \frac{2\rho v_{u,s}^2}{\left(1-\frac{1}{L}\sum_{l=1,L}(1-\frac{\tau_{\epsilon,2}^l}{\tau_{\sigma,2}^l})\right)}, \end{aligned} \tag{4}$$

where $v_{u,p}, v_{u,s}$ are the unrelaxed velocities, for $f \to \infty$, that you used in the Par_file. Note that the option "READ_VELOCITIES_AT_f0 = .true." will change SPECFEM2D behavior by using wave velocities in the Par_file as unrelaxed velocities for $f = f_0$ instead of unrelaxed ones for $f \to \infty$. In that case, the relationship becomes,

$$\begin{aligned} M_1 &= \frac{\rho(nv_{u,p}^2-2(n-1)v_{u,s}^2)}{(1-\frac{1}{L}\sum_{l=1,L}\frac{1-\frac{\tau_{\epsilon,1}^l}{\tau_{\sigma,1}^l}}{1+\frac{1}{(\omega_0\tau_{\sigma,1}^l)^2}})}, \\ M_2 &= \frac{2\rho v_{u,s}^2}{(1-\frac{1}{L}\sum_{l=1,L}\frac{1-\frac{\tau_{\epsilon,2}^l}{\tau_{\sigma,2}^l}}{1+\frac{1}{(\omega_0\tau_{\sigma,2}^l)^2}})}, \end{aligned} \tag{5}$$

Then, for very high frequency attenuation peaks ($\omega_0 \to \infty$) you recover the relationship (4). And for low frequency attenuation peaks ω_0 one obtains, as $\omega_0 \to 0$

$$\begin{aligned} M_1 &\to \rho(nv_{u,p}^2 - 2(n-1)v_{u,s}^2), \\ M_2 &\to 2\rho v_{u,s}^2. \end{aligned} \tag{6}$$

References

Carcione, J. M., Kosloff, D., & Kosloff, R., 1988. Wave propagation simulation in a linear viscoelastic medium, *Geophys. J. Int.*, **95**, 597–611.