9.7 Sea $\mathbf{X}_n$ una muestra aleatoria de tamaño n de la distribución uniforme sobre el intervalo $[0, \theta]$.

(**a**) Hallar un estadístico suficiente para θ basado en $\mathbf{X}_n$.

(**b**) Hallar el estimador de máxima verosimilitud de θ basado en $\mathbf{X}_n$.

(**c**) Sea $\hat{\theta}_n$ el estimador de máxima verosimilitud de θ hallado en el inciso anterior. Mostrar que $\mathbf{E}_\theta[\hat{\theta}_n] = \frac{n}{n+1}\theta$ y $\mathbf{var}_\theta(\hat{\theta}_n) = \frac{n\theta^2}{(n+1)^2(n+2)}$ para concluir que $\hat{\theta}_n$ converge en media cuadrática a θ cuando $n \to \infty$.

(a)

$$\underline{X} = (X_1, \dots, X_n) \sim \mathcal{U}(0, \theta), \quad f_\theta(x_i) = \frac{1}{\theta} \mathbb{1}\{0 < x_i < \theta\}$$

$$f_\theta(\underline{x}) = \prod_{i=1}^{n} f_\theta(x_i) = \left(\frac{1}{\theta}\right)^n \cdot \underbrace{\prod_{i=1}^{n} \mathbb{1}\{0 < x_i < \theta\}}$$

$$\mathbb{1}\{\max(x_1, \dots, x_n) < \theta\}$$

$$\mathbb{1}\{\min(x_1, \dots, x_n) > 0\}$$

$$L(\theta) = f_\theta(\underline{x}) = \underbrace{\frac{1}{\theta^n} \mathbb{1}\{\max(x_1, \dots, x_n) < \theta\}}_{g(\max(x_1, \dots, x_n), \theta)} \underbrace{\mathbb{1}\{\min(x_1, \dots, x_n) > 0\}}_{h(\underline{x})}$$

Por lo tanto, por teorema de factorización

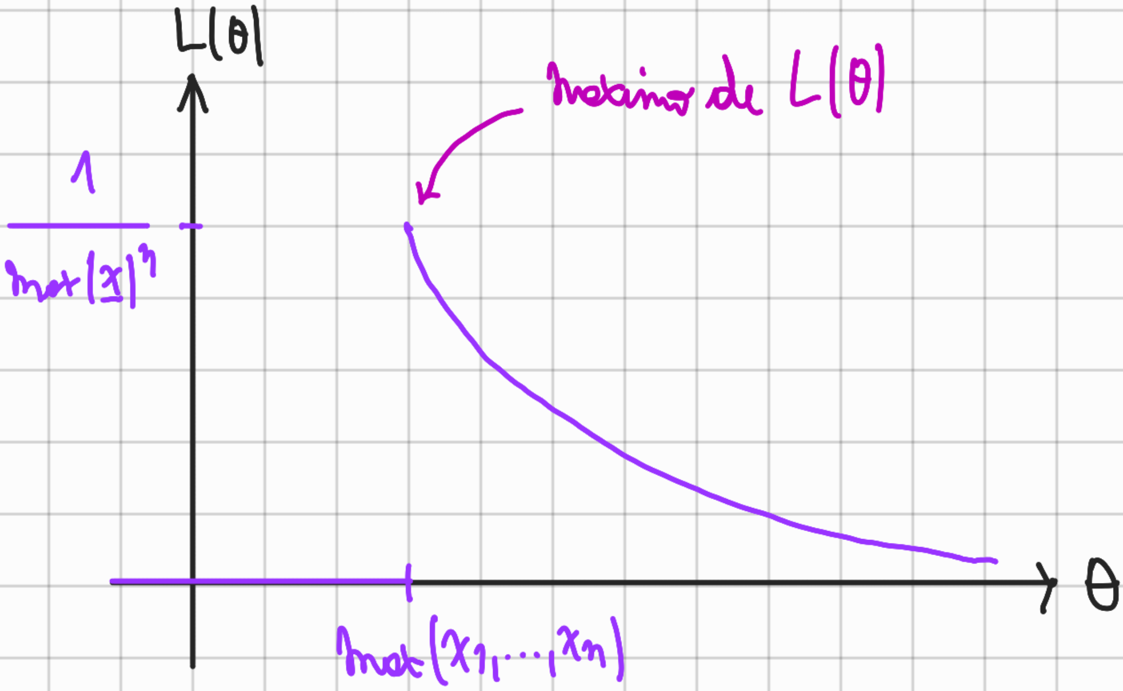
$$\Rightarrow \boxed{T = \max(X_1, \dots, X_n)}$$

(b)

$$\Rightarrow L(\theta) = \frac{1}{\theta^n} \mathbb{1}\{\max(x_1,\dots,x_n) < \theta\} \mathbb{1}\{\min(x_1,\dots,x_n) > 0\}, \quad \theta > 0$$

no puedo derivar (como derivo una $\mathbb{1}$) ni aplicar ln (la $\mathbb{1}$ vale 0 o 1)

Miro el gráfico



$$\Longrightarrow \boxed{\hat{\theta} = \max(X_1,\dots,X_n)}$$

(c)

Como se distribuye $\hat{\theta}$?

$$F_{\hat{\theta}}(t) = \mathbb{P}(\hat{\theta} \leq t) = \mathbb{P}(\max\{X_1, \dots, X_n\} \leq t)$$

$$= \mathbb{P}(X_1 \leq t) \cdot \mathbb{P}(X_2 \leq t) \dots \mathbb{P}(X_n \leq t)$$

$$= \mathbb{P}(X_1 \leq t)^n$$

$$= F_{X_1}(t)^n$$

$$= \left(\frac{1}{\theta} t\right)^n \mathbb{1}\{t < \theta\} + 1\, \mathbb{1}\{t \geq \theta\}$$

$$f_{\hat{\theta}}(t) = \frac{d}{dt} F_{\hat{\theta}} = n \left(\frac{1}{\theta} t\right)^{n-1} \cdot \frac{1}{\theta}\, \mathbb{1}\{0 < t < \theta\}$$

$$= \frac{n}{\theta^n} t^{n-1}\, \mathbb{1}\{0 < t < \theta\} \longrightarrow \int_0^\theta \frac{n}{\theta^n} t^{n-1} dt = 1 \checkmark$$

$$E[\hat{\theta}] = \int_0^\theta t \frac{n}{\theta^n} t^{n-1} dt = \frac{n}{\theta^n} \int_0^\theta t^n dt = \frac{n}{\theta^n} \left(\frac{t^{n+1}}{n+1} \Big|_0^\theta \right)$$

$$= \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1}$$

$$\Rightarrow \boxed{E[\hat{\theta}] = \frac{n}{n+1}\theta}$$

$$Var(\hat{\theta}) = E(\theta^2) - E(\theta)^2$$

$$E(\theta^2) = \int_0^\theta t^2 \frac{n}{\theta^n} t^{n-1} dt = \frac{n}{\theta^n}\int_0^\theta t^{n+1} dt = \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2}$$

$$= \frac{n}{n+2}\theta^2$$

$$E(\theta)^2 = \left(\frac{n}{n+1}\theta\right)^2 = \frac{n^2}{(n+1)^2}\theta^2$$

$$Var(\hat{\theta}) = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}\theta^2$$

$$= \frac{n^3+2n^2+n-n^3-2n}{(n+2)(n+1)^2}\theta^2$$

$$\Longrightarrow \boxed{\quad Var(\hat{\theta}) = \frac{n\,\theta^2}{(n+2)(n+1)^2} \quad}$$

Convergencia en media cuadrática:

$$\lim_{n\to\infty} ECM(\hat{\theta}) = 0 \,,\ \forall \theta \in \Theta$$

Error cuadrático medio : $ECM(\hat{\theta}) = B(\hat{\theta})^2 + V(\hat{\theta})$

Sesgo : $B(\hat{\theta}) = E[\hat{\theta}] - \theta$

$$B(\hat{\theta}) = \frac{n}{n+1}\theta - \theta = \left(\frac{n}{n+1} - 1\right)\theta = \frac{-1}{n+1}\theta$$

$$ECM(\hat{\theta}) = \frac{1}{(n+1)^2}\theta^2 + \frac{n\,\theta^2}{(n+2)(n+1)^2}$$

$$\lim_{n\to\infty} ECM(\hat{\theta}) = \lim_{n\to\infty} \overbrace{\frac{1}{(n+1)^2}}^{\to 0}\theta^2 + \overbrace{\frac{n\,\theta^2}{(n+2)(n+1)^2}}^{\to 0} = 0$$

$\Rightarrow$ $\hat{\theta}$ converge en media cuadrática a θ cuando $n \to \infty$