TRANSFORMACIONES DE V.A (GUIA 4)

X V.A CONOZCO $F_X(x)$, $P_X(x)$, $F_X(x)$ Y $G: \mathbb{R} \to \mathbb{R}$ FUNCIÓN, $Y = G(X)$, COMO OBTENGO: $F_Y(y)$, $P_Y(y)$?

$$\boxed{F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(G(X) \leq y) = \mathbb{P}(\{x : G(x) \leq y\})}$$

EJEMPLOS

1) A) X CONTINUA, $G: \mathbb{R} \to \mathbb{R}$ Y ESTRICTAMENTE CRECIE-NTE ($\Rightarrow$ G ES INYECTIVA Y $\exists\, G^{-1}$, INVERSA QUE TAMBIEN ES ESTRICTAMENTE CRECIENTE)

APLICO G^{-1}

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(G(x) \leq y) = \mathbb{P}(\{x : G(x) \leq y\})$$

CALCULO

ENTONCES:

$$\boxed{\begin{aligned} F_Y(y) &= F_X(G^{-1}(y)) \\ F_Y(y) &= \frac{F_X(G^{-1}(y))}{G'(G^{-1}(y))} \end{aligned}}$$

B) X CONT, G CONT, DEBO ESTRICTAMENTE DECRECIENTE ($\exists\, Y^{-1}$ INVERSIBLE ESTRIC. DECRE) → CAMBIO LA DESIGUALDAD

$$\boxed{\begin{aligned} F_Y(y) &= 1 - F_X(G^{-1}(y)) \\ F_Y(y) &= \frac{F_X(G^{-1}(y))}{|G'(G^{-1}(y))|} \end{aligned}}$$

NOTA

CASO PARTICULAR

$$Y = AX + B \Rightarrow G(x) = Ax + B$$
$$G'(x) = A$$

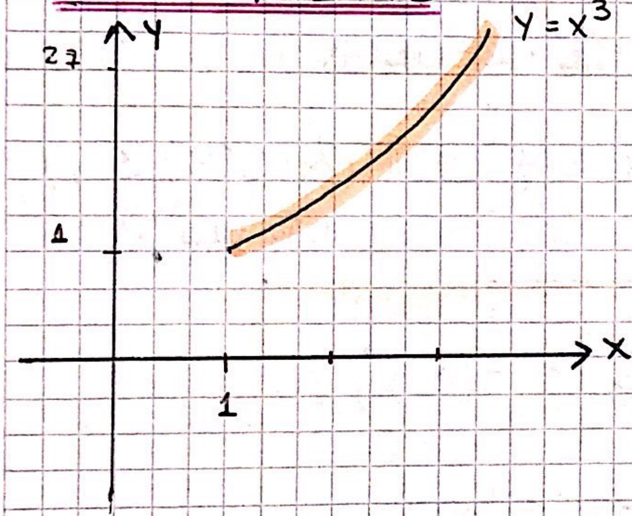
$$f_Y(y) = \frac{f_X\left(\frac{y-B}{A}\right)}{|A|}$$

EJEMPLO

$X \sim \mathcal{U}(1,3)$, $Y = X^3$ QUIERO HALLAR:

- $F_Y(y)$ y $f_Y(y)$ (II)
- CALCULAR $\mathbb{P}(1 \leq Y \leq 8)$ (I)

GRAFICAMENTE



I) $\mathbb{P}(1 \leq y \leq 8) = \mathbb{P}(1 \leq X^3 \leq 8)$

$\mathbb{P}(\sqrt[3]{1} \leq \sqrt[3]{x^3} \leq \sqrt[3]{8})$

$\mathbb{P}(1 \leq X \leq 2) = 1/2$

II) $R_Y = [1, 27]$

$y < 1: \ F_Y(y) = 0$

$y \geq 27: \ F_Y(y) = 1$

$1 \leq y < 27$, $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X \leq \sqrt[3]{y}) = F_X(\sqrt[3]{y})$

$\downarrow \mathcal{U} \quad F_X(x) = \frac{1}{2}x - \frac{1}{2}$

$$\frac{\overbrace{\sqrt[3]{y}}^{x} - 1}{3-1} = \frac{\sqrt[3]{y} - 1}{2}$$

$$F_Y(y) = \left(\frac{\sqrt[3]{y} - 1}{2}\right) \mathbb{1}\{1 \leq y < 27\} + \mathbb{1}\{y \geq 27\}$$

$$f_Y(y) = \frac{1}{2} \cdot \frac{1}{3} \cdot y^{1/3 - 1} \, \mathbb{1}\{1 \leq y < 27\} = \frac{1}{6\sqrt[3]{y^2}} \mathbb{1}\{1 \leq y < 27\}$$

NOTA

OTRA FORMA (USANDO ECUACIÓN)

$$F_X(x) = \left(\frac{x-1}{2}\right) \mathbb{1}\{1 \leq x < 3\} + \mathbb{1}\{x \geq 3\}$$

$$f_X(x) = \frac{1}{2} \mathbb{1}\{1 \leq x < 3\}$$

$$G(x) = x^3$$

$$G'(x) = 3x^2 \qquad G^{-1}(x) = \sqrt[3]{y}$$

$$F_Y(y) = F_X(G^{-1}(y)) = \left(\frac{\sqrt[3]{y} - 1}{2}\right) \mathbb{1}\{1 \leq \sqrt[3]{y} < 3\} + \mathbb{1}\{\underbrace{\sqrt[3]{y} \geq 3}_{y \geq 27}\}$$

$$f_Y(y) = \frac{f_X(G^{-1}(y))}{G'(G^{-1}(y))} = \frac{1/2\, \mathbb{1}\{1 \leq \sqrt[3]{y} < 3\}}{3(\sqrt[3]{y})^2} =$$

$$f_Y(y) = \frac{1}{6(\sqrt[3]{y})^2} \mathbb{1}\{1 \leq y < 27\}$$

30/09

X. V.A CONTINUA

1) F_X ESTRICTAMENTE MONOTONA (→ CRECIENTE O DECRECIENTE), $Y = G(X)$, G INYECTIVA.

• DISTRIBUCIÓN NORMAL DE PARAMETROS μ Y σ^2 (μ → E[X], σ^2 → VA(X))

$(\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}_{>0})$

RECORDAMOS $Z \sim N(0,1)$, $\mathbb{R}_Z = \mathbb{R}$

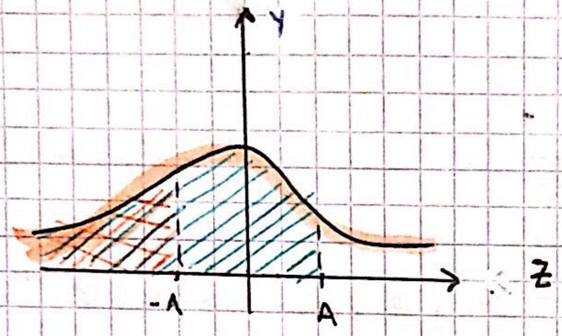
$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$\phi(Z) = F_Z(G) \rightarrow$ TABLA

$\mathbb{P}(-1 \leq Z \leq 1) = \phi(1) - \phi(-1) = 0,6827$

$\mathbb{P}(-2 \leq Z \leq 2) = \phi(2) - \phi(-2) = 0,9545$

$\mathbb{P}(-3 \leq Z \leq 3) = \phi(3) - \phi(-3) = 0,9973$



$\phi(-A) = 1 - \phi(A)$

- $\mathbb{E}[Z] = 0$

$VAR(Z) = 1$, $\sigma_Z = 1$

$\phi(1,35) = 0,91149$

SEA $X = \sigma Z + \mu \rightarrow$ CALCULAR VALORES CUANDO

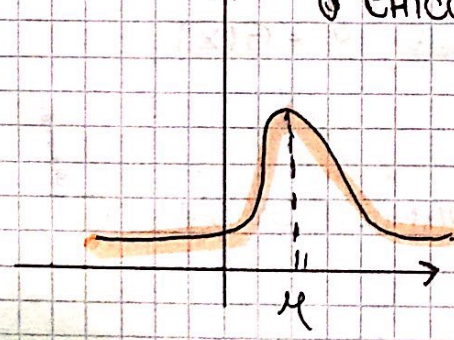
$F_X(x) = \dfrac{\varphi\left(\frac{x-\mu}{\sigma}\right)}{\gamma\sigma} \neq N(0,1) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$

$R_X = \mathbb{R}$

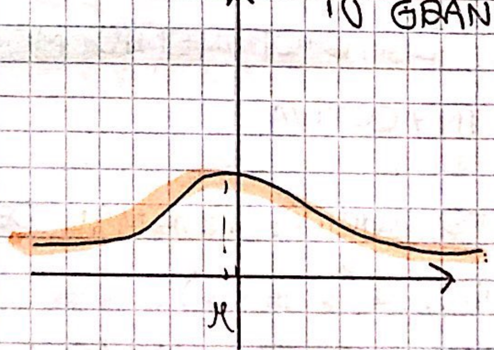
- $\mathbb{E}[X] = \mathbb{E}[\sigma Z + \mu] = \sigma\mathbb{E}[Z] + \mu = \mu$

- $VAR(X) = VAR(\sigma Z + \mu) = VAR(\sigma Z) = \sigma^2 VAR(Z) = \sigma^2$

'σ CHICO'



'σ GRANDE'



NOTA

NOMBRE: $X \sim N(\mu, \sigma^2)$

OBS! $Z = \dfrac{X-\mu}{\sigma}$ NORMAL STANDAR (ES X NORMALIZADA)

$$\boxed{\begin{aligned} \mathbb{P}(A \leq X \leq B) &= \mathbb{P}\left(\frac{A-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{B-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{B-\mu}{\sigma}\right) - \Phi\left(\frac{A-\mu}{\sigma}\right) \end{aligned}}$$

2) CUANDO G NO ES INYECTIVA

2.1 $X \sim F_X(x) = \dfrac{(x+1)^2}{9} \mathbb{1}\{-1 \leq x \leq 2\}$

$$Y = X^2$$

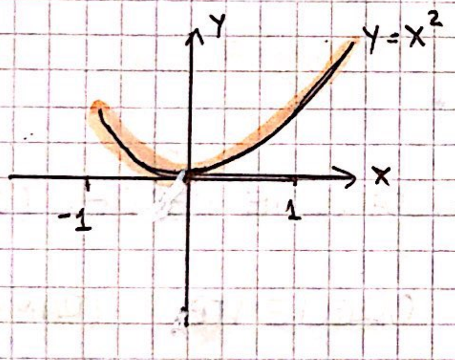
(I) $\mathbb{P}(0 \leq Y \leq 1/2)$

(II) $F_Y(y)$ y $f_Y(y)$

$R_X = [-1, 2]$ $\quad R_Y = [1, 4]$

NO VALE PORQUE ES DECRE Y CAMBIARIA EL SIGNO.

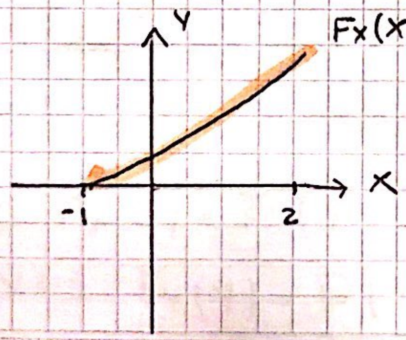
$\hookrightarrow \mathbb{1}\{(-1)^2 \leq x^2 \leq 2^2\}$



PERO GRAFICANDO $R_Y = [0, 4]$

(I) $\mathbb{P}(0 \leq Y \leq 1/2) = \mathbb{P}\left(-\frac{1}{\sqrt{2}} \leq X \leq \frac{1}{\sqrt{2}}\right)$

$= \displaystyle\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{(x+1)^2}{9} dx = \ldots$



(II) $y > 0$: $F_Y(y) = 0$

$y \geq 4$: $F_Y(y) = 1$

$0 \leq y < 4$ $\begin{cases} \bullet\ 0 \leq y \leq 1 \quad F_Y(y) = \mathbb{P}(\overset{\overset{x^2}{\uparrow}}{Y} \leq y) = \\ \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = \displaystyle\int_{-\sqrt{y}}^{\sqrt{y}} \frac{(x+1)^2}{9}dx \\ = \dfrac{(\sqrt{y}+1)^3 - (-\sqrt{y}+1)^3}{27} \\ \bullet\ 1 \leq y < 4 : \quad F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(-1 \leq X \leq \sqrt{y}) \\ = \displaystyle\int_{-1}^{\sqrt{y}} \frac{(x+1)^2}{9}dx = \frac{(\sqrt{y}+1)^3}{27} \end{cases}$

ENTONCES

$$F_Y(y) = \frac{(\sqrt{y}+1)^3 - (-\sqrt{y}+1)^3}{27} \mathbb{1}\{0 \leq y < 1\} + \frac{(\sqrt{y}+1)^3}{27} \mathbb{1}\{1 \leq y < 4\} + \mathbb{1}\{y \geq 4\}$$

$$F_Y(y) = F_Y'(y)$$

$G_1(x) = x^2 \, \mathbb{1}\{-1 \leq x < 0\}$ ESTRICTAMENTE DECRE. (RAMA 1)

$G_2(x) = x^2 \, \mathbb{1}\{0 \leq x \leq 2\}$ " CRECIENTE (RAMA 2)

$G_1(y)^{-1} = -\sqrt{y} \, \mathbb{1}\{0 \leq y \leq 1\}$

$G_2(y)^{-1} = \sqrt{y} \, \mathbb{1}\{0 \leq y \leq 4\}$

NOTA

$$F_Y(y) = \frac{F_X(G^{-1}(y))}{|G'(G^{-1}(y))|}$$

$$F_Y(y) = \frac{(-\sqrt{y}+1)^2}{9|-2\sqrt{y}|}\,\mathbb{1}\{0\le y\le 1\} + \frac{(\sqrt{y}+1)^2}{9(2\sqrt{y})}\,\mathbb{1}\{0<y<4\}$$

$$= \frac{(\sqrt{y}+1)^2+(-\sqrt{y}+1)^2}{18\sqrt{y}}\,\mathbb{1}\{0\le y<1\} + \frac{(\sqrt{y}+1)^2}{18\sqrt{y}}\,\mathbb{1}\{1<y<4\}$$

$$F_Y(y) = F_Y'(y)$$

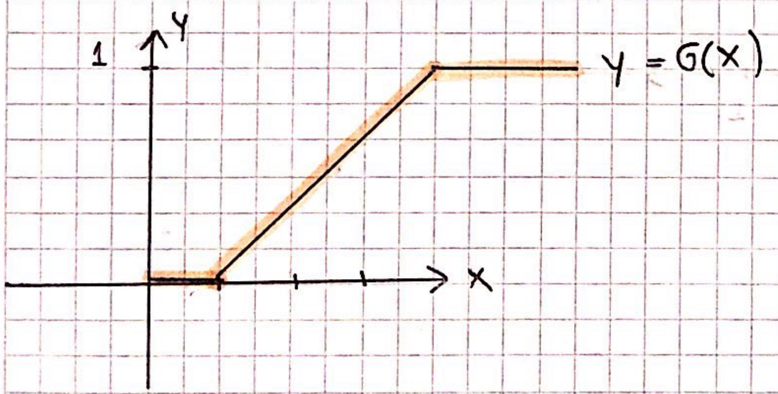
$$F_Y(y) = \left(\frac{\cancel{3}(\sqrt{y}+1)^2}{\cancel{27}\,2\sqrt{y}} + \cancel{3}\,\frac{(-\sqrt{y}+1)^3}{\cancel{27}\,2\sqrt{y}}\right)\mathbb{1}\{0\le y\le 1\} +$$
$$\frac{1}{A}\,\frac{(\sqrt{y}+1)^2}{2\sqrt{y}}\,\mathbb{1}\{1\le y\le 4\}$$

2.2)

$X \sim \mathcal{U}(1,5)$

$$Y = \frac{(X-2)}{2}\,\mathbb{1}\{2\le x<4\} + \mathbb{1}\{x\geq 4\}$$

CALCULAR $F_Y(y)$



- $R_Y = [0,1]$

$y<0 = F_Y(y)=0$

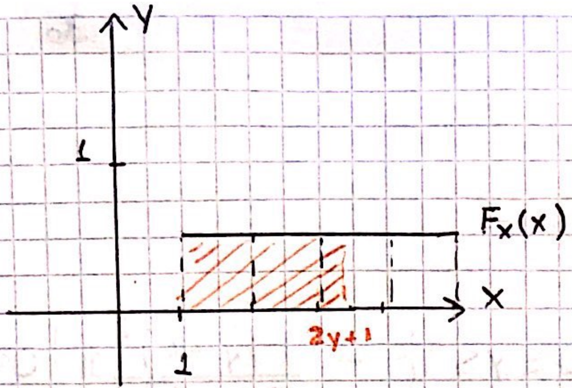
$y\geq 1 = F_Y(y) = 1$

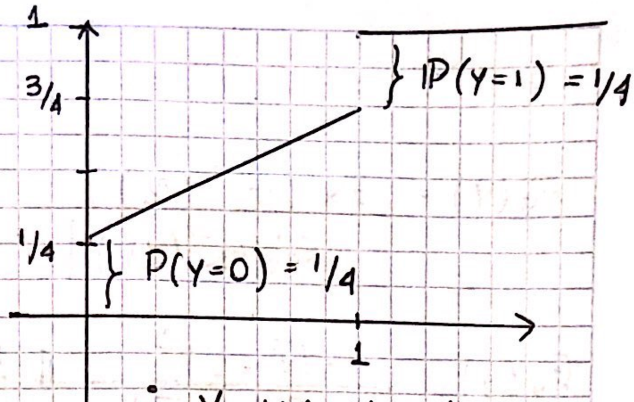
$y=0 \quad F_Y(0) = \mathbb{P}(Y\le 0) = \mathbb{P}(Y=0) = \mathbb{P}(1\le X\le 2) = 1/4$

$0<y<1 \quad F_Y(y) = \mathbb{P}(Y\le y) = \mathbb{P}(G(X)\le y) = \mathbb{P}(1\le X\le 2y+2)$

$$= \frac{2y+2-1}{4} = \frac{2y+1}{4} = 1/2$$

NOTA





$\therefore$ Y V.A MIXTA

$$F_Y(y) = \frac{(2y+1)}{4} \mathbb{1}\{0 \leq y < 1\} + \mathbb{1}\{y \geq 1\}$$

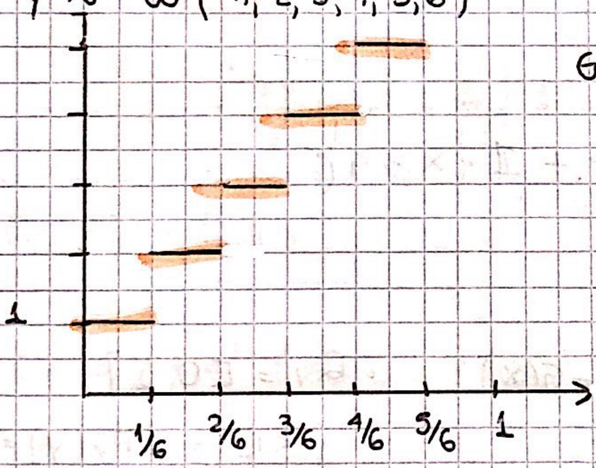
$$\mathbb{P}(Y=1) = \mathbb{P}(4 \leq X \leq 5) = 1/2$$

2.3) DE CONTINUA A DISCRETA

HALLAR UN CAMBIO DE VARIABLES ENTRE

$X \sim \mathcal{U}(0,1)$

$Y \sim \mathcal{U}(1,2,3,4,5,6)$



$$G(x) = \sum_{n=1}^{6} n \, \mathbb{1}\left\{\frac{n-1}{6} \leq x < \frac{n}{6}\right\}$$

NOTA

EJEMPLOS IMPORTANTES (DE C.V QUE SE USAN)

1) $U \sim \mathcal{U}(0,1) \Rightarrow V = 1-U$

$$F_V(v) = \mathbb{P}(V \leq v) = \mathbb{P}(1-U < v) = \mathbb{P}(U \geq 1-v) =$$
$$1 - \mathbb{P}(U \leq 1-v) = 1-(1-v) = v$$

2) $U \sim \mathcal{U}(0,1) \to V = AU + B \sim \mathcal{U}(\text{MÍN}\{B, A-B\}; \text{MAX}\{..\})$

$0 \to B$

$1 \to A+B$

3) $U \sim \mathcal{U}(0,1) \to X = \frac{1}{\lambda} \ln(1-U) \sim \text{EXP}(\lambda)$

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}\left(\frac{1}{\lambda} \ln(1-U) \leq x\right) \overset{(\lambda=0)}{=} \mathbb{P}(-\ln(1-U) \leq \lambda x)$$
$$= \mathbb{P}(\ln(1-U) \geq -\lambda x) = \mathbb{P}(1-U \geq e^{-\lambda x}) = \mathbb{P}($$
$$F_U(1-e^{-\lambda x}) = 1 - e^{-\lambda x}$$

4) $U \sim \mathcal{U}(0,1) \Rightarrow X = -\frac{1}{\lambda} \ln(U) \sim \text{EXP}(\lambda)$

$$X = \frac{1}{\lambda} \ln(U) = -\frac{1}{\lambda} \ln(1-(1-U)) \sim \text{EXP}(\lambda)$$

NOTA

TRANFORMACIÓN DE VECTORES ALEATORIOS

(X,Y) VECTOR ALEATORIO, CON F_{XY} ó P_{XY}. SEA $G:\mathbb{R}^2 \to \mathbb{R}$ UNA FUNCIÓN $Z = (G(X,Y))$, ES UNA VARIABLE ALEATORIA

F_Z, f_Z, P_Z ?

PROPIEDAD: POR EQUIVALENCIA DE EVENTOS, $A \subset \mathbb{R}$

$$\mathbb{P}(Z \in A) = \mathbb{P}(G(X,Y) \in A) = \mathbb{P}((x,y) : G(x,y) \in A)$$

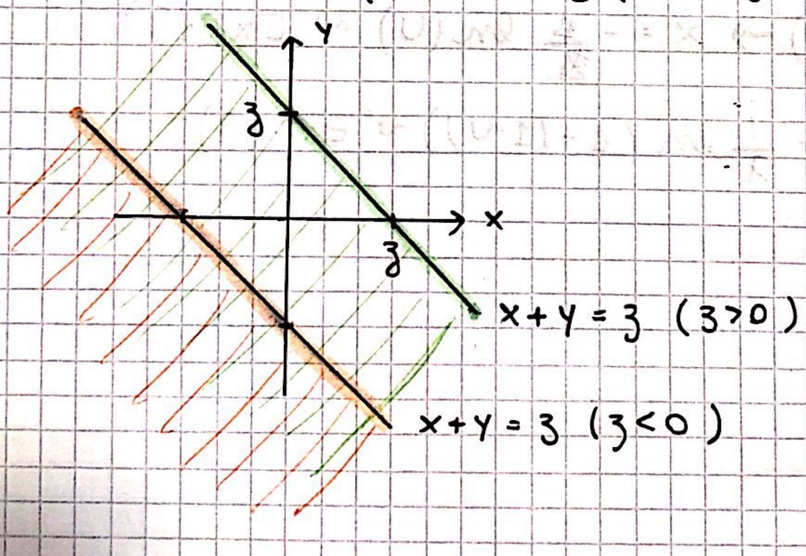
EN PARTICULAR, $F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(Z \in (-\infty; z])$

$$\mathbb{P}(G(X,Y) \leq z) = \mathbb{P}(\{(x,y) : G(x,y) \leq z\}) =$$

- $\displaystyle\iint_{\{G(x,y) \leq z\}} f_{XY}(x,y)\,dx\,dy$
- $\displaystyle\sum\sum_{\{G(x,y) \leq z\}} P_{XY}$

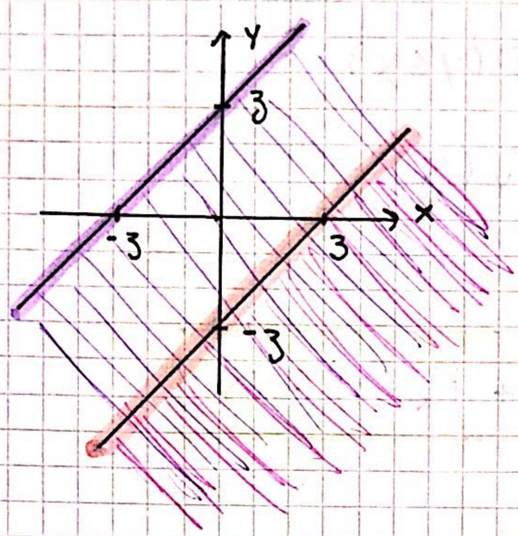
EJEMPLOS DE $\{(x,y) : G(x,y) \leq z\}$ PARA ALGUNAS FUNCIONES $G(x,y)$ PARTICULARES.

(1) CONJUNTO $\{x + y \leq z\}$ $\forall z \in \mathbb{R}$

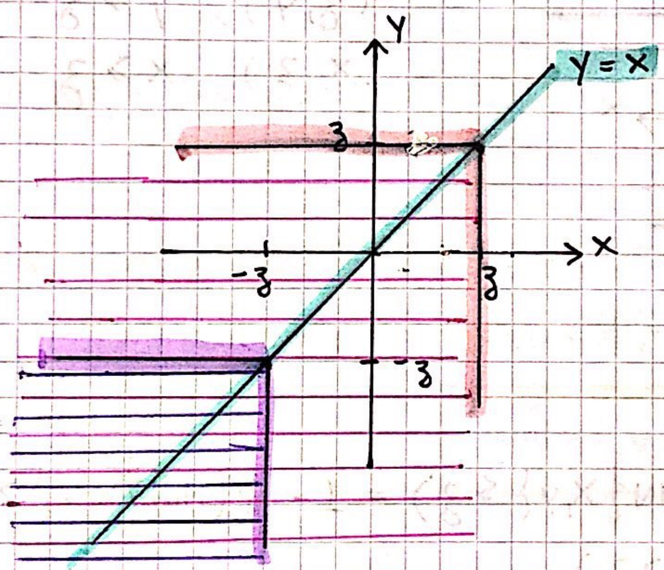


NOTA

(2) CONJUNTO $\{ y - x \leq z \}$



(3) CONJUNTO $\{ \max\{x, y\} \leq z \}$



BORDE : $\max\{x,y\} = z$

(z, y) CON $y \leq z$

(x, z) CON $x \leq z$

- $Z = \max\{x, y\}$

$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(\max\{x,y\} \leq z) = \mathbb{P}(x \leq z, y \leq z)$

ENTONCES,

$$\boxed{F_Z(z) = F_{xy}(z, z)}$$

SI Z ES CONTINUA: $F_Z(z) = F'_Z(z) = \dfrac{d^2 F_{XY}}{dx\,dy}$

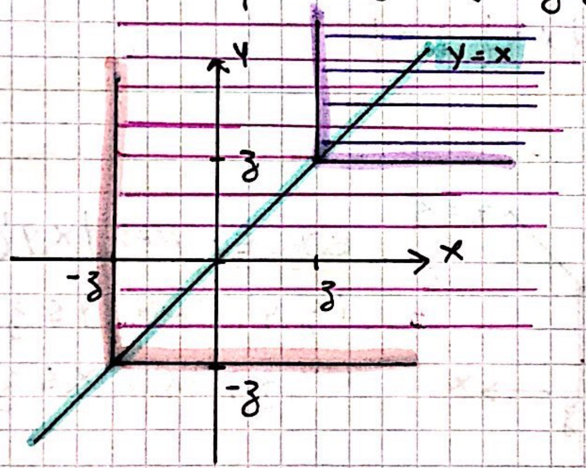
EN PARTICULAR, SI X E Y SON INDEPENDIENTES:

- $F_Z(z) = \mathbb{P}(X \leq z) \cdot \mathbb{P}(Y \leq z)$
- $F_Z(z) = F_X(z) \cdot F_Y(z)$

SI Z CONTINUA,

$$\boxed{f_Z(z) = f_X(z)\,F_Y(z) + F_X(z)\,f_Y(z)}$$

(4) CONJUNTO $\{\text{MÍN}\{x,y\} \leq z\}$



BORDES $\text{MÍN}\{x,y\} = z$

$(z, y),\ y \geq z$

$(x, z),\ x \geq z$

- $Z = \text{MÍN}\{X, Y\}$

$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(\text{MIN}\{X,Y\} \leq z) = 1 - \mathbb{P}(\text{MÍN}\{X,Y\} > z)$

$= 1 - \mathbb{P}(X > z,\ Y > z)$

SI X E Y SON INDEPENDIENTES, ENTONCES:

$$\boxed{F_Z(z) = 1 - \left[(1 - F_X(z))(1 - F_Y(z))\right]}$$

NOTA

SI Z ES CONTINUA:

$$\boxed{F_Z(z) = F_X(z)\,(1 - F_Y(z)) + F_Y(z)\,(1 - F_X(z))}$$

APLICACIÓN IMPORTANTE

SEAN $\left.\begin{array}{l} X_1 \sim EXP(\lambda_1) \\ X_2 \sim EXP(\lambda_2) \end{array}\right\}$ SI SON INDEPENDIENTES

$$\boxed{MÍN\{X_1, X_2\} \sim EXP(\lambda_1 + \lambda_2)}$$

DEMOSTRACIÓN: $Z = MÍN\{X_1, X_2\}$

$$F_Z(z) = 1 - [\,(1 - F_{X_1}(z))(1 - F_{X_2}(z)\,]$$
$$= 1 - e^{-\lambda_1 z} \cdot e^{-\lambda_2 z} = 1 - e^{(\lambda_1+\lambda_2)z}$$

OJO X, Y (VARIABLE ALEATORIA INDEP IGUALMENTE DISTRIBUIDA) $\sim EXP(\lambda)$:

$$\boxed{MÍN\{X, Y\} \sim EXP(2\lambda)}$$

NOTA

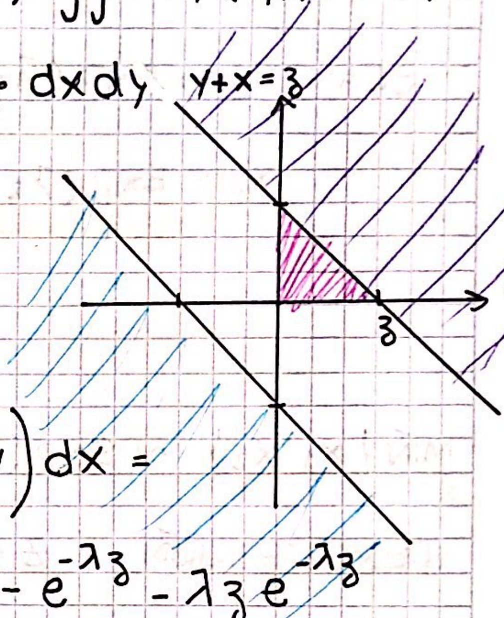
EJEMPLO

$X, Y \underset{\text{V.A.i.i}}{\sim} \text{EXP}(\lambda) \qquad Z = G(X,Y) = X + Y$

HALLAR $F_Z(z)$ Y $f(z)$

$$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(X+Y \leq z) = \iint f_{XY}(x,y)\,dx\,dy$$

$$= \iint_{\{x+y\leq z\}} \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y}\, \mathbb{1}\{x>0, y>0\}\, dx\,dy$$



si $z<0 \quad F_Z(z)=0$

$z \geq 0 \quad F_Z(z) = \mathbb{P}(X+Y \leq z)$

$$\mathbb{P}(X+Y \leq z) = \int_0^z \left(\int_0^{z-x} \lambda e^{-\lambda x} \lambda e^{-\lambda y}\, dy \right) dx =$$

$$\int_0^z \lambda e^{-\lambda x} \left(\int_0^{z-x} \lambda e^{-\lambda y}\, dy \right) dx = 1 - e^{-\lambda z} - \lambda z e^{-\lambda z}$$

$$F_Z(z) = (1 - e^{-\lambda z} - \lambda z e^{-\lambda z})\, \mathbb{1}\{z>0\}$$

$$f_Z(z) = F_Z'(z) = z\lambda^2 e^{-\lambda z}\, \mathbb{1}\{z>0\}$$

$Z \sim \Gamma(2,\lambda)$

EN GENERAL,

SI $X_1, \ldots, X_n \overset{\text{vaii}}{\sim} \text{EXP}(\lambda)$

ENTONCES $Z = X_1 + X_2 + \cdots + X_n \sim \Gamma(n,\lambda)$

- (X,Y) VECTOR CONTINUO CON $f_{XY}(x,y)$ DENSIDAD CONJUNTA.

  SEA $(U,V) = G(X,Y) \rightarrow G(x,y) = (U(x,y), V(x,y))$

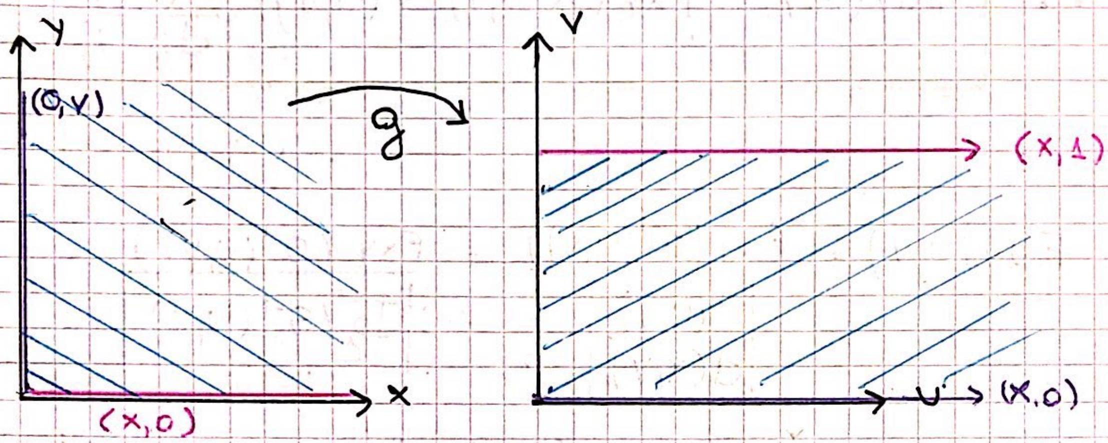
  $g: \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{C}^1$, $G(D) = \tilde{D}$ $\quad g(\partial D) = \partial\tilde{D}$ $\quad$ BIYECTIVA

NOTA

EJEMPLO

$X, Y \sim EXP(\lambda) \qquad (x>0,\ y>0)$

$$\begin{cases} U = X+Y \\ V = \dfrac{X}{X+Y} \to V \in (0,1) \end{cases}$$



$(x,0),\ x>0 \qquad (x,1) \qquad SOP(F_{UV}) = \{u>0,\ 0<v<1\}$

$(0,y),\ y>0 \qquad (x,0)$

$(x,y) \xrightarrow[g]{} (u,v)$

$$g' \begin{cases} x = u \cdot v \\ y = u(1-v) \end{cases} \qquad |J_g|G^{-1}(u,v)| = \begin{vmatrix} v & u \\ 1-v & u \end{vmatrix} = u$$

$$F_{UV} = F_{XY}(x(u,v), y(u,v)) \cdot \left| \frac{d(x,y)}{d(x,y)} \right|$$

$$F_{UV}(u,v) = \lambda^2 e^{-\lambda u} \cdot u\, \mathbb{1}\{u>0,\ 0<v<1\}$$

$$= \underbrace{(\lambda u)\lambda e^{-\lambda u}}_{F_U(u)} \cdot \mathbb{1}\{u>0\} \quad \underbrace{1\, \mathbb{1}\{0<v<1\}}_{F_V(v)}$$

CAMBIO DE VARIABLES

$$\iint_D F_{UV}(u,v)\,du\,dv = \iint_D F_{XY}(x(u,v), y(u,v)) \left| \frac{d(x,y)}{d(u,v)} \right| dx\,dy$$

$$\boxed{\mathbb{P}((u,v) \in \tilde{D}) = \mathbb{P}((x,y) \in D)}$$

$$\therefore \boxed{F_{UV}(x,y) = F_{XY}(x(u,v), y(u,v)) \left| \frac{d(x,y)}{d(u,v)} \right|}$$

$$F_{UV}(u,v) = \frac{F_{XY}(G^{-1}(u,v))}{\left| \frac{d(u,v)}{d(x,y)} \right|} = \frac{F_{XY}(G^{-1}(u,v))}{\left| J_g(G^{-1}(u,v)) \right|}$$

•/ EN GENERAL - Si $X_1, \dots X_N$ V.A.i :

$$X_i \sim N(\mu_i, \sigma_i^2)$$

ENTONCES $X_1 + \dots + X_N \sim N\left(\sum_1^n \mu_i, \sum_1^n \sigma_i^2\right)$

(2 EJEMPLOS MÁS QUE NO COPIE)

NOTA