9.13

(a) Mostrar que la familia de distribuciones $\mathcal{N}(\mu, \sigma^2)$ puede expresarse en la forma

$$f_\theta(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2\right),$$

donde $\theta = (\mu, \sigma^2)$.

(b) Sea $X_1, \dots, X_n$ una muestra aleatoria de tamaño n de la distribución $\mathcal{N}(\mu, \sigma^2)$. Hallar la expresión de la densidad conjunta y mostrar que $T = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ es un estadístico suficiente para θ.

(c) Sea $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Mostrar que $\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2$ y deducir que $T' = (\bar{X}, S^2)$, donde

$$S^2 := \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2,$$

es un estadístico suficiente para θ.

(d) Hallar el estimador de máxima verosimilitud para θ basado en la muestra aleatoria $X_1, \dots, X_n$.

(a) $X \sim \mathcal{N}(\mu, \sigma^2)$

$$f_{\underbrace{(\mu,\sigma^2)}_{=\theta}}(x) = \frac{1}{\sqrt{2\pi}\,(\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \mathbb{1}\{x \in \mathbb{R}\}$$

$$= \frac{1}{\sqrt{2\pi}\,(\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}x^2 + \frac{1}{\sigma^2}\mu x - \frac{1}{2\sigma^2}\mu^2} \mathbb{1}\{x \in \mathbb{R}\}$$

$$\Longrightarrow \boxed{f_\theta(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\mu^2}{2\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x}}$$

(b) $X_1, X_2, \dots X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$

$$f_\theta(\hat{x}) = \prod_{i=1}^{n} F_\theta(x) = \underbrace{\left(\frac{1}{\sqrt{2\pi}\ \sigma}\right)^n e^{-\frac{n\mu^2}{2\sigma^2}}}_{A(\mu,\theta^2)} \underbrace{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2 + \frac{\mu}{\sigma^2}\sum_{i=1}^{n} x_i}}_{\circledast}$$

$\circledast$ $C_1 = -\frac{1}{2\sigma^2}$; $C_2 = \frac{\mu}{\sigma^2}$; $r_1(\underline{x}) = \sum_{i=1}^{n} x_i$, $r_2(\underline{x}) = \sum_{i=1}^{n} x_i^2$

$h(\underline{x}) = 1$

$$\Rightarrow \boxed{T = \left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)}$$

(c)

Propiedad:

$\swarrow$ no dinvade

si $T(\underline{x})$ es suficiente para θ y $T'(\underline{x}) = w(T(\underline{x}))$ con w inyectiva

$\Rightarrow T'$ es suficiente para θ

$$T(\underline{X}) = \Big(\underbrace{\sum_{i}^{n} X_i}_{a} \, , \, \underbrace{\sum_{i}^{n} X_i^2}_{b} \Big)$$

$$T'(\underline{X}) = \left(\frac{\sum X_i}{n} \, , \, \frac{\sum X_i^2 - n\left(\frac{\sum X_i}{n}\right)^2}{n-1} \right)$$

Veo que $T' = \left(\frac{a}{n} \, , \, \frac{b - a^2/n}{n-1} \right) = w(a,b) = (c,d)$

$$\begin{cases} c = a/n \\ d = \dfrac{b - a^2/n}{n-1} \end{cases} \rightsquigarrow \begin{cases} a = nc \\ b = (n-1)d + \dfrac{c^2}{n} \end{cases}$$

se ve que es inyectiva

$$\Longrightarrow \boxed{T' = (\hat{X}, S^2) \text{ es suficiente para } \theta}$$

(d) $X \sim N(\mu, \sigma^2)$; $\mu \in \mathbb{R}$, $\sigma^2 > 0$

$$\underbrace{L(\mu,\sigma^2)} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,(\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}x_i^2 + \frac{\mu}{\sigma^2}x_i - \frac{\mu^2}{2\sigma^2}}$$

$$= \frac{1}{(\sqrt{2\pi})^n (\sigma^2)^{n/2}} e^{-\frac{n\mu^2}{2\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2}\sum x_i^2 + \frac{\mu}{\sigma^2}\sum x_i}$$

cte

→ 2 parametros ⇒ maximizo función de 2 variables

Primero aplico el logaritmo:

$$\log L(\mu,\sigma^2) = \log \text{cte} - \frac{n}{2}\log(\sigma^2) - \frac{n\mu^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum x_i^2 + \frac{\mu}{\sigma^2}\sum x_i$$

$$\begin{cases} \dfrac{\partial \log L}{\partial \mu} = -\dfrac{n}{2\sigma^2}2\mu + \dfrac{\sum x_i}{\sigma^2} = 0 \\[2ex] \dfrac{\partial \log L}{\partial \sigma^2} = -\dfrac{n}{2}\dfrac{1}{\sigma^2} + \dfrac{n\mu^2}{2(\sigma^2)^2} + \dfrac{\sum x_i^2}{2(\sigma^2)^2} - \dfrac{\mu\sum x_i}{(\sigma^2)^2} = 0 \end{cases}$$

De la primera ecuación $\rightsquigarrow \mu = \dfrac{\sum x_i}{n}$

Ahora voy a la segunda

$$\to \frac{-n\sigma^2 + n\mu^2 + \sum x_i^2 - 2\mu \sum x_i}{2(\sigma^2)^2} = 0$$

$$-n\sigma^2 + n\frac{(\sum x_i)^2}{n^2} + \sum x_i^2 - 2\frac{\sum x_i}{n}\sum x_i = 0$$

$$n\sigma^2 = \frac{(\sum x_i)^2}{n} + \sum x_i^2 - \frac{2}{n}(\sum x_i)^2 = \sum x_i^2 - \frac{n^2}{n}\underbrace{\left(\frac{\sum x_i}{n}\right)}_{\bar{X}}^2$$

$$\sigma^2 = \frac{1}{n}\left[\sum X_i^2 - n\bar{X}^2\right] = \frac{\sum (X_i - \bar{X})^2}{n}$$

$$\Rightarrow \hat{\Theta}_{MV} = (\hat{\mu}, \hat{\sigma}^2) = \left(\bar{X} ; \frac{\sum_1^n (X_i - \bar{X})^2}{n}\right) = \left(\bar{X}, \frac{(n-1)S^2}{n}\right)$$

Ahora veo las derivadas segundas:

$$\frac{\partial^2 \log L}{\partial^2 \mu} = -\frac{n}{\sigma^2} \qquad \frac{\partial^2 \log L}{\partial \sigma^2 \partial \mu} = \frac{n\mu}{\sigma^4} - \frac{\sum_1 x_i}{\sigma^4}$$

$$\begin{cases} \dfrac{\partial \log L}{\partial \mu} = -\dfrac{n}{2\sigma^2} 2\mu + \dfrac{\sum_1 x_i}{\sigma^2} = 0 \\[2ex] \dfrac{\partial \log L}{\partial \sigma^2} = -\dfrac{n}{2}\dfrac{1}{\sigma^2} + \dfrac{n\mu^2}{2(\sigma^2)^2} + \dfrac{\sum_1 x_i^2}{2(\sigma^2)^2} - \dfrac{\mu \sum_1 x_i}{(\sigma^2)^2} = 0 \end{cases}$$