ESPERANZA – (GUÍA 3)

LA ESPERANZA DE UNA V.A X

- X DISCRETA : $\boxed{E[X] = \sum_{x \in R_X} x \, p_X(x)}$

TMB PUEDE LLAMARSE CENTRO DE MASA Ó PROMEDIO PONDERADO.

EJEMPLOS

1) $X \sim Bi(2, 1/2)$

x	0	1	2
$p_X(x)$	1/4	1/2	1/4

$$\mathbb{E}[X] = 0\frac{1}{4} + 1\frac{1}{2} + 2\frac{1}{4} = 1$$

2) $X \sim \mathcal{U}(1, 2, .., 6)$

$$\mathbb{E}[X] = \frac{1}{6} + 2\frac{1}{6} + 3\frac{1}{6} + \dots + 6\frac{1}{6} = \frac{1}{6}\sum i$$

$$= \frac{1}{6}\,\frac{6.7}{2} = \frac{7}{2} = 3{,}5$$

3) $X \sim B(p)$

$$\mathbb{E}[X] = 0(1-p) + 1p = p$$

4) $X \sim G(p) \rightarrow \mathbb{E}[X] = 1/p$

GUÍA: 3

• X CONTINUA :

$$\boxed{\mathbb{E}[X] = \int_{-\infty}^{+\infty} x\, F_X(x)\, dx}$$

EJEMPLOS

1) $X \sim \mathcal{U}(a,b)$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \frac{1}{B-A} \mathbb{1}\{A<x<b\}\, dx = \int_a^b \frac{x}{B-A}\, dx =$$

$$= \frac{1}{b-a} \frac{x^2}{2}\Big|_a^b = \frac{a+b}{2}$$

2) $X \sim EXP(\lambda)$

$$F_X(x) = \lambda e^{-\lambda x} \cdot \mathbb{1}\{x>0\}$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, F_X(x)\, dx = \int_{-\infty}^{+\infty} x\lambda e^{-\lambda x} \cdot \mathbb{1}\{x>0\}\, dx =$$

$$0 - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^{+\infty} = \frac{1}{\lambda} - 0 = 1/\lambda$$

CASO V.A MIXTA:

X. V.A MIXTA, $R_X = D \cup C$

$$\boxed{\mathbb{E}[X] = \sum_{x \in D} x \cdot \mathbb{P}(X=x) + \int_C x\, F_X'(x)\, dx}$$

TEOREMA: SEA X V.A (CUALQUIERA), $X>0$, $R_X = R_{>0}$

ENTONCES

$$\boxed{\mathbb{E}[X] = \int_0^{+\infty} \mathbb{P}(X>x)\, dx = \int_0^{+\infty} (1 - F_X(x))\, dx}$$

NOTA

EJEMPLOS

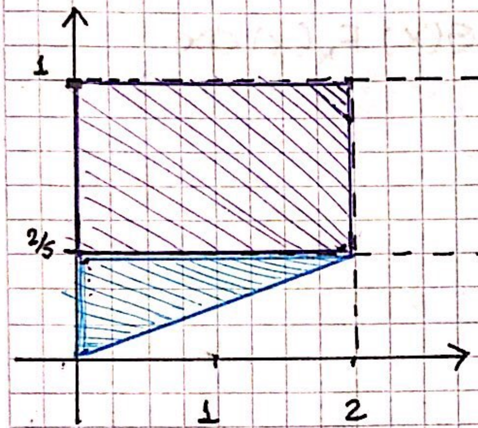
1) $X \sim EXP(\lambda)$, $F_X(x) = 1 - e^{-\lambda x}$

$$E[X] = \int_0^{+\infty} (1 - F_X(x))\,dx = \int_0^{+\infty} 1 - 1 + e^{-\lambda x}\,dx = \int_0^{+\infty} e^{-\lambda x}\,dx$$

$$= -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^{+\infty} = 1/\lambda$$

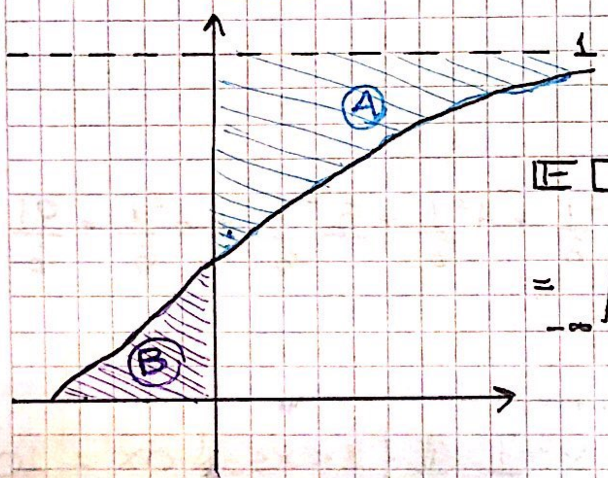
2) $X \sim \mathcal{U}(0,5)$

$$Y \sim \text{MÍN}\{2, X\} = \begin{cases} X & \text{sii } X < 2 \\ 2 & \text{sii } X \geq 2 \end{cases}$$



$$E[Y] = 2 \cdot \frac{3}{5} + 2\,\frac{2}{5} \cdot \frac{1}{2} = \frac{8}{5}$$

PROPIEDAD:



$$E[X] = \text{AREA } A - \text{AREA } B$$

$$= -\int_{-\infty}^{0} F_X(x)\,dx + \int_0^{+\infty} (1 - F_X(x))\,dx$$

NOTA

X.V.A $\mathbb{E}[X]$ CENTRO DE MASA UNIDIMENSIONAL

25/09

$$\mathbb{E}[X] \quad \sum_{x_i \in R_x} x_i \, P_X(x_i)$$

$$\int_{-\infty}^{+\infty} F_X(x)\,dx$$

PROPIEDAD Si $G: \mathbb{R} \to \mathbb{R}$ UNA FUNCIÓN, y

$Y = G(X)$ ES TMB UNA V.A.

$$\mathbb{E}[Y] = \mathbb{E}[G(X)] = \sum_{x_i \in R_x} G(x) P_X(x)$$

$$\int_{-\infty}^{+\infty} G(x)\, F_X(x)\, dx$$

OJO NO SE NECESITA
CAMBIAR LA VARIABLE
PARA HALLAR LA $\mathbb{E}[Y]$

EJEMPLOS

1) $X \sim \mathcal{U}(1,2,3,4,5,6)$

$$\mathbb{E}[X^2] = \sum_{n=1}^{6} N^2 P_X(n)$$

$$\mathbb{E}[X^2] = 1^2\frac{1}{6} + 2^2\frac{1}{6} + 3^2\frac{1}{6} + 4^2\frac{1}{6} + 5^2\frac{1}{6} + 6^2\frac{1}{6} = \frac{91}{6}$$

2) $X \sim \mathcal{U}(1,3)$

$$\mathbb{E}[X^3] = \int_{-\infty}^{+\infty} x^3 F_X(x)\,dx = \int_{-\infty}^{+\infty} x^3 \frac{1}{\ } \mathbb{1}\{1 < x \le 3\}\,dx = 10$$

NOTA

FECHA

CASOS PARTICULARES IMPORTANTES:

i) SI $G(x) = K$ CTE

$$\mathbb{E}[K] = \mathbb{E}[G(x)] = \int_{-\infty}^{+\infty} K\, F_x(x)\,dx = K\int_{-\infty}^{+\infty} F_x(x)\,dx = K$$

ii) $G(x) = \alpha x + \beta$

$$\mathbb{E}[\alpha x + \beta] = \int_{-\infty}^{+\infty} (\alpha x + \beta)\, F_x(x)\,dx = \alpha\int_{-\infty}^{+\infty} x\, F_x(x)\,dx + \beta$$

$$\mathbb{E}[\alpha x + \beta] = \alpha\,\mathbb{E}[x] + \beta$$

iii) $G(x) = \mathbb{1}_A\{(x)\}$

$$\mathbb{E}[\mathbb{1}_A(x)] = \mathbb{E}[\mathbb{1}\{x\in A\}] = \int_{-\infty} \mathbb{1}\{x\in A\}\, F_x(x)\,dx$$

$$\mathbb{E}[\mathbb{1}_A(x)] = \mathbb{P}(A)$$

ESPERANZA DE V.A TRUNCADAS

X V.A, $\mathbb{P}(A) > 0$, $X|A \to$ V.A

$P_{X|A}(x)$

$F_{X|A}(x)$

$$\mathbb{E}[X|A] = \sum_{x\in R_x} x\, P_{X|A}(x)$$

$$\int_{-\infty}^{+\infty} x\, F_{X|A}(x)\,dx$$

$$\boxed{\mathbb{E}[X|A] = \frac{\mathbb{E}[X]\mathbb{1}\{x\in A\}}{\mathbb{P}(A)}}$$

PROPIEDAD FORMULA DE LA PROBABILIDAD TOTAL PARA LA ESPERANZA.

$\{A_i\}$ PARTICIÓN DE Ω (O DE R_X), TAL QUE $\mathbb{P}(A_i) > 0$.

$$\mathbb{E}[X] = \mathbb{E}[X|A_1]\,\mathbb{P}(X \in A_1) + \dots + \mathbb{E}[X|A_m]\,\mathbb{P}(X \in A_m)$$

ENTONCES $\sum_{i=1}^{m} \mathbb{E}[X|A_i]\,\mathbb{P}(A_i)$

TMB VALE QUE:

$$\mathbb{E}[G(X)] = \sum_{i=1}^{m} \mathbb{E}[G(X) \,|\, A_i]\,\mathbb{P}(A_i)$$

VARIANZA DE UNA VARIABLE ALEATORIA.

$\mathbb{E}[X] = \mu$ (NOTACIÓN)

$$VA(X) \quad \sum_{x \in R_X} (x_i - \mathbb{E}[X])^2 P_X(x) = \sum (x-\mu)^2 P_X(x)$$

$$\int_{-\infty}^{+\infty} (x - \mathbb{E}[X]^2)\, F_X(x)\,dx = \int_{-\infty}^{+\infty} (x-\mu)^2 F_X(x)\,dx$$

$$VA(X) = \mathbb{E}[(X-\mu)^2] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

OBS! $VA(X) \geq 0$ $\quad (VA(X) = 0 \rightarrow X = \mu)$

NOTA

Escaneado con CamScanner

PROPIEDADES

(i) $VA(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

(ii) $VA(X) = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[(X^2 - 2\mu X + \mu^2)]$

$= \mathbb{E}[X^2] - 2\mu\,\mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mu^2$

EJEMPLOS

1) $X \sim BER(P)$

$\mathbb{E}[X] = P$

$\mathbb{E}[X^2] = 0^2 P_X(0) + 1^2 P_X(1) = P$

$VAR(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = P - P^2 = P(1-P)$

2) $X \sim \mathcal{U}(A,B)$

$$\mathbb{E}[X^2] = \int \frac{x^2}{B-A} \mathbb{1}\{A<x<B\}\, dx = \frac{A^2+AB+B^2}{3}$$

$VA(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$$= \frac{A^2+AB+B^2}{3} - \left(\frac{A+B}{2}\right)^2 = \frac{(B-A)^2}{12}$$

(NOTACIÓN $VA(X) = \sigma^2$)

DESVIO DE UNA V.A X

$$\boxed{\sigma_X = \sqrt{VA(X)} = \sqrt{\sigma^2}}$$

OBS! $\sigma_X > 0$ ($\sigma_X = 0$ ENTONCES $X =$)

PROPIEDADES

X, $\mathbb{E}[X] = \mu$, $VA(X) = \sigma^2$, $\alpha, \beta \in \mathbb{R}$

(i) $VA(X+\beta) = \mathbb{E}[(X+\beta)^2] - \mathbb{E}[(X+\beta)]^2 = \dots =$
$= VA(X)$

(ii) $VA(\alpha X) = \mathbb{E}[(\alpha X)^2] - \mathbb{E}[(\alpha X)]^2 = \dots =$
$= \alpha^2 \, VA(X)$

$$\therefore \boxed{\begin{array}{c} VA(\alpha X + \beta) = \alpha^2 \, VA(X) \\ \sigma_{\alpha X + \beta} = |\alpha| \, \sigma_X \end{array}}$$

OBS! $\mathbb{E}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma} \mathbb{E}[X-\mu] = \frac{1}{\sigma}(\mathbb{E}[X] - \mu)$

$VA\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} VA(X-\mu) = \frac{1}{\sigma^2} VA(X) = 1$

$\sigma_{X-\mu} = 1$ $\qquad \mathbb{E}[X^*] = 0$

$X^* = \frac{X-\mu}{\sigma}$ V.A X NORMALIZADA

NOTA

FECHA

DESIGUALDAD DE MARKOV

X V.A , $X > 0$, VALE QUE:

$$\boxed{\mathbb{P}(X \geq A) \leq \frac{\mathbb{E}[X]}{A}}$$

EJ PARCIAL (ö)

LA DURACION (hs) DE UN CORTE DE LUZ DE MEDIA = 5.
¿COMO DEBEN SER LAS DURACIONES DE UNA LAMPARA t PARA TENER LA GARANTIA QUE DURA TODO EL CORTE?

($t > 0$) ES TIEMPO

T : DURACIÓN DEL CORTE

$\mathbb{E}[T] = 5$

$$\mathbb{P}(T \leq t) \geq 1/2 \iff \mathbb{P}(T > t) < 1/2$$

POR MARKOV $\mathbb{P}(T > t) \leq \frac{\mathbb{E}[T]}{t} \leq 1/2$

$$\frac{5}{t} \leq 1/2 \Rightarrow t \geq 10$$

DESIGUALDAD DE CHEBYCHOV

X V.A $\mathbb{E}[X] = \mu$, $VA(X) = \sigma^2$ $\forall_{\varepsilon} > 0$

$$\boxed{\mathbb{P}(|X - \mu| > \varepsilon) \leq \frac{VA(X)}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}}$$

NOTA

PROPIEDAD

X, Y V.A DEFINIDAS EN $(\Omega, \mathcal{A}, \mathbb{P})$; $g: \mathbb{R}^2 \to \mathbb{R}$

ENTONCES $Z = G(X,Y)$ ES UNA V.A DEFINIDA EN $(\Omega, \mathcal{A}, \mathbb{P})$.

¿? $F_Z(G)$ ó $P_Z(G)$ ... (GUIA 4).

$$\mathbb{E}[Z] = \mathbb{E}[G(X,Y)] = \sum_{y \in R_Y} \sum_{x \in R_X} G(x,y)\, P_{XY}(x,y)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x,y)\, F_{XY}(x,y)\, dx\, dy$$

PROPIEDAD

EN PARTICULAR, $G(X,Y) = X$ ó $G(X,Y) = Y$ ENTONCES:

$$\boxed{\begin{aligned} \mathbb{E}[X] &= \iint x\, F_{XY}(x,y)\, dx\, dy \\ \mathbb{E}[Y] &= \iint y\, F_{XY}(x,y)\, dx\, dy \end{aligned}}$$

IDEM PARA $\sum\sum$ PARA LAS DISCRETAS.

(1) $G(X,Y) = \alpha X + \beta Y$

$$\mathbb{E}[\alpha X + \beta Y] = \alpha\, \mathbb{E}[X] + \beta\, \mathbb{E}[Y]$$

(2) SI X, Y SON V.A INDEPENDIENTES ENTONCES,

$\mathbb{E}[X, Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (NO VALE LA VUELTA)

COVARIANZA

X, Y V.A, $\mathbb{E}[X] = \mu_X$, $\mathbb{E}[Y] = \mu_Y$ SE DEFINE LA COVARIANCIA ENTRE X, Y COMO:

$$COV(X,Y) = \mathbb{E}[(X - E[X])(Y - \mathbb{E}[Y])] =$$
$$= \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$

$$\sum\sum (x - \mu_x)(y - \mu_y) P_{XY}(x\,y)$$

$$\iint (x - \mu_x)(y - \mu_y) F_{XY}(x,y)\,dx\,dy$$

(NOTACIÓN $COV(X,Y) = \sigma_{XY}$

PROPIEDAD

(i) $COV(X,Y) = \mathbb{E}[X,Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$

$$= \mathbb{E}[X,Y] - \mu_X \mu_Y$$

COEFICIENTE DE CORRELACIÓN ENTRE X E Y.

$$\rho(X,Y) = COV\left(\frac{X - \mu_X}{\sigma_X}, \frac{Y - \mu_Y}{\sigma_Y}\right) = \frac{COV(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$-1 \leq \rho(X,Y) \leq 1$$

COROLARIO: SI X E Y SON INDEPENDIENTES
ENTONCES $COV(X,Y) = 0$
(NO VALE LA VUELTA)

NOTA

PROPIEDADES DE COV(X,Y)

1) $COV(X,X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = VA(X)$

2) $COV(X+Y, Z) = COV(X,Z) + COV(Y,Z)$

3) $COV(X,Y) = COV(Y,X)$

4) $VAR(\alpha X + \beta Y) = \mathbb{E}[(\alpha X + \beta Y)^2] - \mathbb{E}[\alpha X + \beta Y]^2$

$= \dots$

$VAR(\alpha X + \beta Y) = \alpha^2 VA(X) + \beta^2 VA(Y) + 2\alpha\beta\, COV(X,Y)$

COROLARIO SI X E Y SON DESCORRELACIONADAS $(COV(X,Y) = 0)$

$VAR(X\alpha + \beta Y) = \alpha^2 VA(X) + \beta^2 VA(Y)$

OBS! EN PARTICULAR SI X E Y SON INDEPENDIENTES

$VAR(\alpha X + \beta Y) = \alpha^2 VA(X) + \beta^2 VA(Y)$

EJEMPLO IMPORTANTE

X E Y SON IDEP. $\Rightarrow$:

$VA(X+Y) = VA(X) + VA(Y)$

$VA(X-Y) = VA(X) + VA(Y)$

ENTONCES $VA\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} VA(X_i)$

$\rho(X,Y) = \frac{COV(X,Y)}{\sigma_X \sigma_Y} = \cos(\alpha(X,Y))$

ANGULO ENTRE VARIABLES.

NOTA

RECTA DE REGRESIÓN DE Y SOBRE X

$$\boxed{Y = \frac{COV(X,Y)}{VA(X)}\left(X - \mathbb{E}[X]\right) + \mathbb{E}[Y]}$$