4.1) $p_X(x) = \frac{8x}{36} = \frac{2}{9}x$ $\quad$ $x = 0, \frac{1}{8}, \frac{2}{8}, \dots, 1$

discreto Σ

a) $Y = 2X - 1$

$P(Y=y) = P(2X-1=y) = P(2X = y+1) = P\left(X = \frac{y+1}{2}\right)$

$= p_X\left(\frac{y+1}{2}\right) = \frac{2}{9}\left(\frac{y+1}{2}\right) = \frac{y+1}{9} = p_Y(y)$

para $y = -1; -\frac{3}{4}; -\frac{1}{2}; -\frac{1}{4}; 0; \frac{1}{4}; \frac{1}{2}; \frac{3}{4}; 1$

b) $Y = 128X^2$

$p_Y(y) = P(Y=y) = P(128X^2 = y) = P\left(X^2 = \frac{y}{128}\right) =$

$= P\left(X = \sqrt{\frac{y}{128}}\right) = p_X\left(\sqrt{\frac{y}{128}}\right) = \frac{2}{9}\sqrt{\frac{y}{128}}$

para $y = 0, 2, 8, 18, 32, 50, 72, 98, 128$

c) $Y = -64X^2 + 64X + 2$

$p_Y(y) = P(Y=y) = P(-64X^2 + 64X + 2 = y)$

* $p_Y(2) = p_X(0) + p_X(1)$ $\quad$ * $p_Y(9) = p_X(1/8) + p_X(7/8)$

* $p_Y(14) = p_X(2/8) + p_X(6/8)$

* $p_Y(17) = p_X(3/8) + p_X(5/8)$

* $p_Y(18) = p_X(1/2)$

$y = 2, 9, 14, 17, 18, 17, 14, 9, 2$

$\Rightarrow f_X(x) = \frac{\ }{\pi(1+x^2)}$

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d) $y = 64X^2 - 96X + 128$

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c) 4.2) $y = \left|\operatorname{sen}\left(\frac{1}{2}\pi X\right)\right|$ $\quad X \sim Po(2)$

) y toma valores 0 y 1 → amplitud

si X es par $\Rightarrow y = 0$

si x es impar $\Rightarrow y = 1$

$P(x = \text{impar}) = \operatorname{senh}(2)\, e^{-2}$

$P(X = \text{par}) = \cosh(2)\, e^{-2}$

(4.3) $F_X(x) = \dfrac{12x}{\pi^2(e^x + 1)} \mathbb{1}\{x > 0\}$

a) $y = aX + b \; (a \neq 0, b \in \mathbb{R}) \Rightarrow F_Y(y) =$

$= P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$

$\Rightarrow f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = \dfrac{12\left(\frac{y-b}{a}\right)}{\pi^2\left(e^{\left(\frac{y-b}{a}\right)} + 1\right) a} \mathbb{1}\left\{\frac{y-b}{a} > 0\right\}$

$\Downarrow$

$\mathbb{1}\{y > b\}$

b) $Y = -X^3 \Rightarrow X = \sqrt[3]{-Y}$ $F_Y(y) = F_X(g^{-1}(y))$ / $|y'_x|$

$$\boxed{F_Y(y) = \frac{F_X(\sqrt[3]{-y})}{-3X^2}} = \frac{F_X(\sqrt[3]{-y})}{-3(\sqrt[3]{-y})^2} = \frac{12\sqrt[3]{-y}}{-\pi^2(e^{\sqrt[3]{-y}}+1)\,3\sqrt[3]{-y}^2}$$

$\mathbb{1}\{y \leq 0\}$

c) $Y = X + X^{-1} \Rightarrow YX = X^2 + 1 \Rightarrow X^2 - Xy + 1 = 0$

$$X_1 = \frac{y + \sqrt{y^2-4}}{2} \qquad X_2 = \frac{y - \sqrt{y^2-4}}{2}$$

$$\boxed{F_Y(y) = \left.\frac{F_X(x)}{1-x^{-2}}\right|_{x=\frac{y+\sqrt{y^2-4}}{2}} + \left.\frac{F_X(x)}{1-x^{-2}}\right|_{x=\frac{y-\sqrt{y^2-4}}{2}}}$$

$\mathbb{1}\{2 < y < \infty\}$

d) $Y = X^2 - 3X \rightarrow X^2 - 3X - Y = 0$

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4.4) $\theta \sim U(-\pi/2, \pi/2)$ $tg(\theta) = X$

$F_\theta(t) = \frac{1}{\pi} \mathbb{1}\{-\pi/2 < t < \pi/2\}$

integro

$F_\theta(t) = \left(\frac{t}{\pi} + \frac{1}{2}\right) \mathbb{1}\{-\pi/2 < t < \pi/2\} + \mathbb{1}\{t > \pi/2\}$

$F_X(x) = P(tg\,\theta \leq x) = P(\theta \leq tg^{-1}(x)) = F_\theta(tg^{-1}(x)) = \frac{tg^{-1}(x)}{\pi} + \frac{1}{2}$

$\Rightarrow F_X(x) = \frac{1}{\pi(1+x^2)} \mathbb{1}\{x \in \mathbb{R}\}$

4.5) $\phi \sim U(-\pi, \pi) \rightarrow f_{\phi}(t) = \frac{1}{2\pi} \mathbb{1}\{-\pi < t < \pi\}$

a) $C = \cos\phi$

$$F_{\phi}(t) = \int_{-\pi}^{t} \frac{1}{2\pi}\, du = \frac{t}{2\pi} + \frac{1}{2} \mathbb{1}\{-\pi < t < \pi\} + \mathbb{1}\{t > \pi\}$$

$F_C(c) =$

$$= P(C \leq c) = P(\cos\phi \leq c) = P(\phi \leq \cos^{-1}(c)) =$$

$$= F_{\theta}(\cos^{-1}(c)) = \frac{\cos^{-1}(c)}{2\pi} + \frac{1}{2}$$

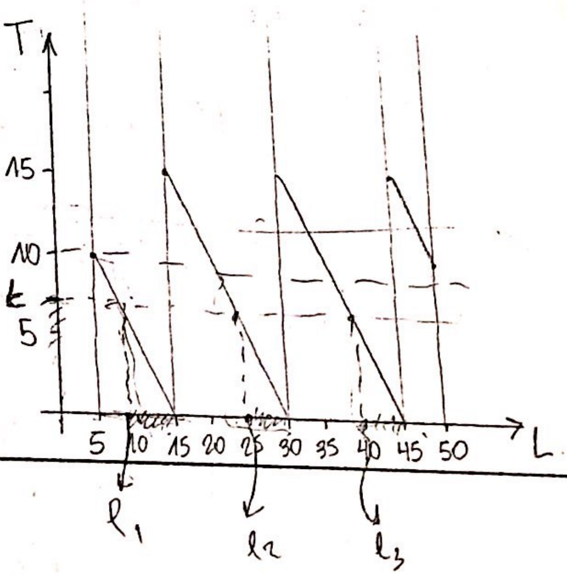
$$\Rightarrow f_C(c) = \frac{-1}{2\pi\sqrt{1-c^2}} + \frac{1}{2} \mathbb{1}\{-1 < c < 1\}$$

b) $$P(|C| < 0{,}5) = \int_{-0{,}5}^{0{,}5} \left(\frac{-1}{2\pi\sqrt{1-c^2}} + \frac{1}{2} \right) dc = \frac{1}{3}$$

4.6) $L \sim U(5, 50) \rightarrow f_L(l) = \frac{1}{45} \mathbb{1}\{5 < l < 50\}$

el tren llega 7:15, 7:30, 7:45, 8:00

$$T = \begin{cases} 15 - L & \text{si } 5 < l \leq 15 \\ 30 - L & \text{si } 15 < l \leq 30 \\ 45 - L & \text{si } 30 < l \leq 45 \\ 60 - L & \text{si } 45 < l \leq 50 \\ 0 & \text{si } l < 5 \wedge l > 50 \end{cases}$$



$$F_T(t) = P(T \leq t) = \begin{cases} 0 \text{ si } t<0 \\ 1 \text{ si } t>15 \\ * \text{ si } t\in(0,10) \\ ** \text{ si } t\in(10,15) \end{cases}$$

$$* P(T\leq t) = P(15-L\leq t) + P(30-L\leq t) + $$
$$+ P(45-L\leq t) =$$
$$= P(L\geq 15-t) + P(L\geq 30-t) +$$
$$+ P(L\geq 45-t) =$$
$$= \int_{15-t}^{15} \frac{1}{45}\, dl + \int_{30-t}^{30} \frac{1}{45}\, dl + \int_{45-t}^{45} \frac{1}{45}\, dl =$$
$$= \frac{15-(15-t)}{45} + \frac{30-(30-t)}{45} + \frac{45(45-t)}{45} =$$
$$= \frac{t}{45} + \frac{t}{45} + \frac{t}{45} = \frac{t}{15}$$

$$** P(T\leq t) = P(5<L<15) + P(30-L\leq t) +$$
$$+ P(45-L\leq t) + P(60-L\leq t) =$$
$$= \int_{5}^{15} \frac{1}{45}\, dl + \int_{30-t}^{30} \frac{1}{45}\, dl + \int_{45-t}^{45} \frac{1}{45}\, dl + \int_{60-t}^{50} \frac{1}{45}\, dl =$$

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$$\mathbb{P} = \frac{10}{45} + \frac{t}{45} + \frac{t}{45} + \frac{t-10}{45} = \frac{t}{15}$$

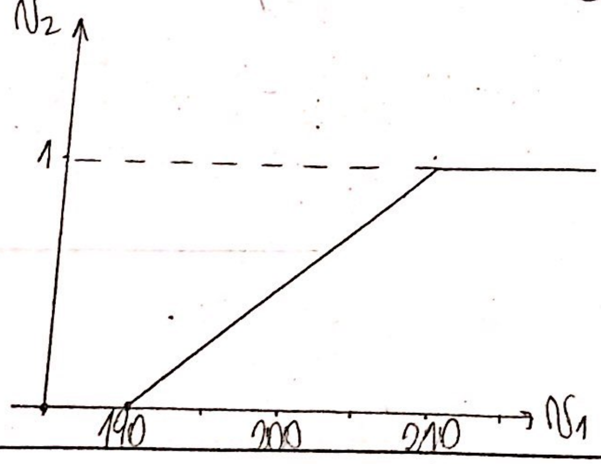
$$\Rightarrow F_T(t) = \begin{cases} 0 & \text{si } t<0 \\ t/15 & \text{si } 0\leq t<15 \\ 1 & \text{si } t\geq 15 \end{cases}$$

4.9 $\Rightarrow f_T(t) = \frac{1}{15}\mathbb{1}\{0<t<15\}$

4.7) $V_1 \sim U(180,220) \rightarrow f_{V_1}(v_1) = \frac{1}{40}\mathbb{1}\{180<v_1<220\}$

$$g(v_1) = \frac{v_1-190}{20}\mathbb{1}\{190\leq v_1\leq 210\} + \mathbb{1}\{v_1>210\}$$

$$F_{V_2}(v_2) = P(V_2\leq v_2) = \begin{cases} 0 & \text{si } v_2<0 \quad ** \text{ si } v_2=0 \\ * & \text{si } 0<v_2<1 \\ 1 & \text{si } v_2>1 \quad *** \text{ si } v_2=1 \end{cases}$$



$$* \; P(V_2\leq v_2) = P\left(\frac{v_1-190}{20}\leq v_2\right)$$

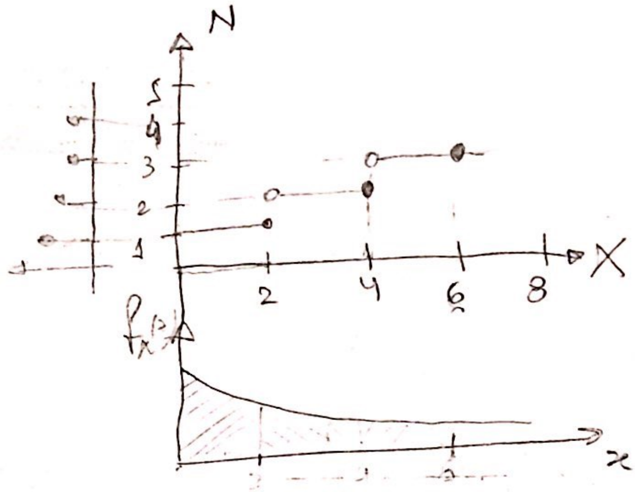
$$= P(v_1 \leq 20v_2+190) =$$

$$= \int_{190}^{20v_2+190} 1/40 \, dv_1 = \frac{20v_2}{40} = \frac{v_2}{2}$$

$** \; P(180<v_1<190) = \frac{10}{40}$ $\quad *** \; P(210<v_1<220) = \frac{10}{40}$

4.8) $X \sim EXP(1/8)$

N: # picos



$$\mathbb{P}(N=n) = \mathbb{P}\left(2(n-1) < X \leq 2n\right)$$

$$= \int_{2(n-1)}^{2n} \frac{1}{8} e^{-x/8} \, dx =$$

$$= e^{-\frac{2(n-1)}{8}} - e^{-\frac{2n}{8}} = e^{-\frac{n-1}{4}} - e^{-\frac{n}{4}}, \quad n = 1, 2, 3, \dots$$

4.9) a) $U = X \quad V = X + Y$

U \ V	-4	-3	-2	-1	0	1	2	3	4
-2	1/16	1/16	0	1/8	1/8	0	0	0	0
-1	0	1/16	1/16	0	1/8	1/8	0	0	0
1	0	0	0	0	0	0	1/16	1/16	0
2	0	0	0	0	0	0	0	1/16	1/16

$p_{UV}(-2,-4) = p_{XY}(-2,-2) = 1/16 \qquad p_{UV}(-2,-3) = p_{XY}(-2,-1) = 1/16$

$p_{UV}(-2,-2) = p_{XY}(-2,0) = 0 \qquad p_{UV}(-2,-1) = p_{XY}(-2,1) = 1/8$

$p_{UV}(-2,0) = p_{XY}(-2,2) = 1/8 \qquad p_{UV}(-1,-4) = 0$

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b) $U = \min(X,Y)$ $V = \max(X,Y)$

U \ V	-2	-1	1	2
-2	1/16	2/16	1/8	1/8
-1	2/16	1/16	1/8	1/8
1	1/8	1/8	1/16	2/16
2	1/8	1/8	2/16	1/16

$p_{UV}(-2,-2) = p_{XY}(-2,-2) = 1/16$

$p_{UV}(-2,-1) = p_{XY}(-2,-1) + p_{XY}(-1,-2)$

c) $U = X^2 + Y^2$ $V = \frac{Y}{X}$

U \ V	-1	-1/2	1/2	1
2	1/8	0	0	2/16
5	0	1/8	2/16	0
8	1/8	0	0	2/16

$p_{UV}(2,-1) = p_{XY}(1,-1) + p_{XY}(-1,1)$

$p_{UV}(2,-1/2) = 0$

U \ V	-2	2
2	0	0
5	1/8	2/16
8	0	0

4.10) a) $(U,V) = A(X,Y)^T + B$

$$(U,V) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{cases} U = a_{11}X + a_{12}Y + b_1 \\ V = a_{21}X + a_{22}Y + b_2 \end{cases} \Rightarrow Y = \frac{V - \frac{a_{21}U}{a_{11}} + \frac{a_{21}b_1}{a_{11}} - b_2}{a_{22} - \frac{a_{21}a_{12}}{a_{11}}} \quad *$$

$$F_{UV}(u,v) = \frac{F_{XY}(x(u,v), y(u,v))}{\left|\frac{\partial(U,V)}{\partial(X,Y)}\right|}$$

$$* \; X = \frac{U - b_1 - a_{12}\left(\frac{V - \frac{a_{21}U}{a_{11}} + \frac{a_{21}b_1}{a_{11}} - b_2}{a_{22} - \frac{a_{21}a_{12}}{a_{11}}}\right)}{a_{11}}$$

$$\Rightarrow \left|\frac{\partial(U,V)}{\partial(X,Y)}\right| = \left|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right| = a_{11}a_{22} - a_{12}a_{21} \neq 0$$ por ser inversible

b) $(U,V)=(\min(X,Y),\max(X,Y))$

$$(U,V)=\begin{cases}(X,Y), & X<Y \quad ①\\ (Y,X), & X\geq Y \quad ②\end{cases}$$

$$f_{UV}(u,v)=\frac{f_{XY}(x(u,v),y(u,v))}{|J_1|}+\frac{f_{XY}(x(u,v),y(u,v))}{|J_2|}$$

$$J_1=\left|\frac{\partial(U,V)}{\partial(X,Y)}\right|=\left|\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=1 \qquad J_2=\left|\begin{pmatrix}0&1\\1&0\end{pmatrix}\right|=-1$$

$$\Rightarrow f_{UV}(u,v)=2\,f_{XY}(x(u,v),y(u,v))$$

c) $(U,V)=(X^2+Y^2,\ Y/X)$

$$\left.\begin{matrix}U=X^2+Y^2\\ V=\dfrac{Y}{X}\end{matrix}\right. \Rightarrow \left.\begin{matrix}(-X,Y)\\(X,-Y)\end{matrix}\right\rangle (U,V) \text{ NO ES UNO A UNO}$$

① $U=(-X)^2+Y^2$, $V=Y/(-X)$ $\Rightarrow$ $|J_1|=\left|\dfrac{\partial(U,V)}{\partial(X,Y)}\right|=\left|\begin{pmatrix}2X & 2Y\\ Y/X^2 & -1/X\end{pmatrix}\right|=|-2-2V^2|$

② $U=X^2+(-Y)^2$, $V=-Y/X$

$|J_2|=\left|\begin{pmatrix}2X & 2Y\\ Y/X^2 & -1/X\end{pmatrix}\right|=|-2-2V^2|$

$$\Rightarrow f_{UV}(u,v)=\frac{f_{XY}(x(u,v),y(u,v))}{2+2v^2}+\frac{f_{XY}(x(u,v),y(u,v))}{2+2v^2}$$

4.11) $Z_1 \sim N(0,1)$ independientes
$Z_2 \sim N(0,1)$

a) I) $U = Z_1 + Z_2$
$V = Z_1 - Z_2$ $\Rightarrow \left|\frac{\partial(U,V)}{\partial(Z_1,Z_2)}\right| = \left|\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right| = 2$

$$F_{Z_1Z_2}(z_1,z_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z_2^2}{2}} = \frac{1}{2\pi} e^{-\frac{(z_1^2+z_2^2)}{2}} \mathbb{1}\{z_1,z_2 \in \mathbb{R}\}$$

$$F_{UV}(u,v) = \frac{\frac{1}{2\pi} e^{-\frac{\left(\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right)}{2}}}{2} = \frac{1}{4\pi} e^{-\frac{\left(\frac{u^2}{2}+\frac{v^2}{2}\right)}{2}} \mathbb{1}\{u,v \in \mathbb{R}\}$$

II) $(U,V) = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ $\theta \in (0, 2\pi)$

$U = Z_1 \cos\theta - Z_2 \operatorname{sen}\theta$
$V = Z_1 \operatorname{sen}\theta + Z_2 \cos\theta$ $\Rightarrow U^2 + V^2 = Z_1^2 + Z_2^2$

$$F_{UV}(u,v) = \frac{1}{2\pi} e^{-\frac{(u^2+v^2)}{2}} \times 1 \quad \mathbb{1}\{u,v \in \mathbb{R}\}$$

$$\left|\frac{\partial(U,V)}{\partial(Z_1,Z_2)}\right| = \left|\begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix}\right| = 1$$

$* \; U=(-Z_1)^2+Z_2^2$

$V = Z_2/(-Z_1)$

$|J_1| = 2+2V^2$

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$** \; U = Z_1^2+(-Z_2)^2$

$V = -Z_2/Z_1$

$|J_2| = 2+2V^2$

III) $U = Z_1^2+Z_2^2$ $\Rightarrow$ $\left.\begin{matrix}(Z_1,-Z_2)\\(-Z_1,Z_2)\end{matrix}\right\rangle (U,V)$ NO ES UNO A UNO

$V = \dfrac{Z_2}{Z_1}$

$$\begin{matrix}*\\ **\end{matrix} \Rightarrow f_{UV}(u,v) = \frac{2 f_{XY}(x(u,v),y(u,v))}{2+2v^2} = \frac{1}{\pi(2+2v^2)} e^{-\frac{u}{2}} \mathbb{1}\{u>0, v\in\mathbb{R}\}$$

b) I) $$f_{UV}(u,v) = \frac{1}{4\pi} e^{-\frac{\left(\frac{u^2}{2}+\frac{v^2}{2}\right)}{2}} = \underbrace{\frac{1}{2\sqrt{\pi}} e^{-\frac{u^2}{4}}}_{f_U(u)} \times \underbrace{\frac{1}{2\sqrt{\pi}} e^{-\frac{v^2}{4}}}_{f_V(v)}$$

$\Rightarrow$ son independientes

II) $$f_{UV}(u,v) = \frac{1}{2\pi} e^{-\frac{(u^2+v^2)}{2}} = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}}_{f_U(u)} \times \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}}_{f_V(v)}$$

$\Rightarrow$ son independientes

III) $$f_{UV}(u,v) = \frac{1}{\pi(2+2v^2)} e^{-\frac{u}{2}} = \underbrace{\frac{1}{\pi(1+v^2)}}_{f_V(v)} \times \underbrace{\frac{1}{2} e^{-\frac{u}{2}}}_{f_U(u)}$$

$\Rightarrow$ son independientes

2}

$$P(Z_1^2+Z_2^2>4) \overset{\text{(III)}}{=} P(U>4) = \int_4^{+\infty} \frac{1}{2} e^{-\frac{u}{2}} du = e^{-\frac{4}{2}} =$$

$$= e^{-2} = 0{,}135$$

$$P(Z_2>\sqrt{3}Z_1) = P\left(Z_2/Z_1 > \sqrt{3}\right) \overset{\text{(III)}}{=} P(V>\sqrt{3}) = \int_{\sqrt{3}}^{+\infty} \frac{1}{\pi(1+v^2)} dv =$$

$$= 0{,}167$$

4.12) $X_1 \sim U(0,2)$, $X_2 \sim U(0,2)$ $\Rightarrow$ independientes $\quad U = \text{MIN}(X_1,X_2)$, $V = \text{MAX}(X_1,X_2)$

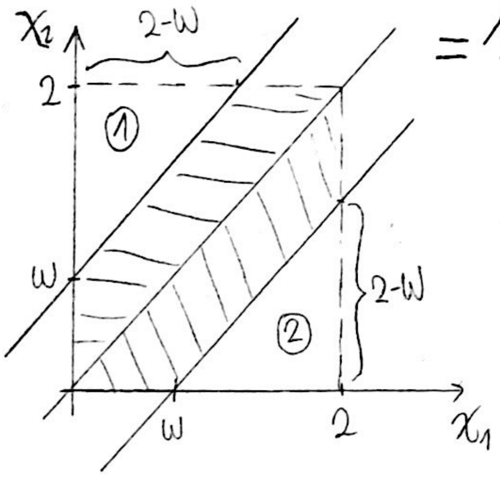
$$f_{X_1X_2}(x_1,x_2) = \frac{1}{4}$$

a)

$$f_{UV}(u,v) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\,\mathbb{1}\{0<u<v<2\}$$

b) $W = V - U$

$$P(W \le w) = P(X_1 - X_2 \le w,\ X_1 > X_2) + P(X_2 - X_1 \le w,\ X_1 \le X_2)$$



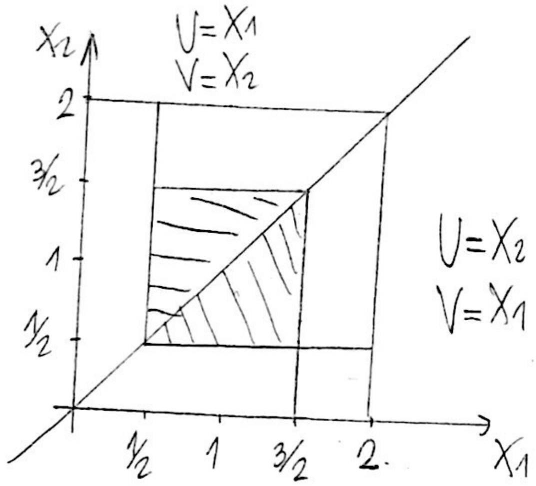
$$= 1 - \left(\frac{\text{AREA ①} + \text{AREA ②}}{4}\right) =$$

$$= 1 - \frac{(2-w)^2}{4}$$

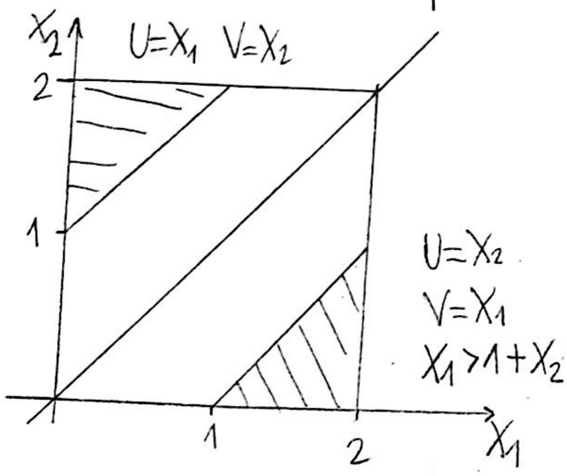
$$\Rightarrow f_W(w) = -\frac{2(2-w)}{4} \times (-1) =$$

$$= \frac{2-w}{2} = 1 - \frac{w}{2}\,\mathbb{1}\{0<w<2\}$$

c) $P(U>1/2, V<3/2) = \frac{1}{4}$

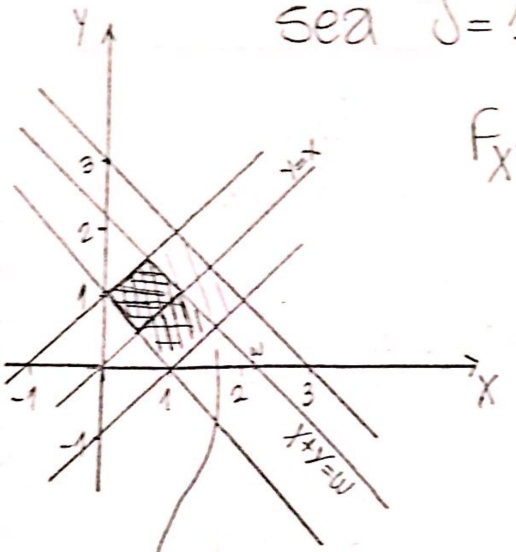


$P(V>1+U) = \frac{1}{4}$



4.13) $(X,Y) \sim U(A)$

sea $J = \mathbb{1}\{X<Y\} \rightarrow P(J=1) = \frac{1}{2}$



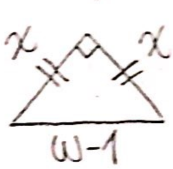
$F_{XY}(x,y) = \frac{1}{2}$

$P(X+Y \leq w) = P(Y \leq w-X) =$

$= \sqrt{2} \times \frac{w-1}{2}$

$P(X+Y \leq w, J=1) = \frac{P(X+Y\leq w)}{2} =$

$= \frac{\sqrt{2} \times (w-1)}{4}$



$(w-1)^2 = x^2 + x^2$

$(w-1)^2 = 2x^2$

$\frac{(w-1)^2}{2} = x^2$

$\frac{w-1}{\sqrt{2}} = x$

$\Rightarrow$ son independientes

4.14) $X_1 \sim EXP(\lambda_1) \quad X_2 \sim EXP(\lambda_2)$

$U = MIN(X_1, X_2) \quad V = MAX(X_1, X_2) \quad W = V - U$

$J = \mathbb{1}\{U = X_1\} + 2\,\mathbb{1}\{U = X_2\}$.

a) $U \sim EXP(\lambda_1 + \lambda_2)$

$f_U(u) = (\lambda_1 + \lambda_2) e^{-(\lambda_1+\lambda_2)u} \mathbb{1}\{u>0\}$

b) $p_J(1) = P(J=1) = P(X_1 < X_2) = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$

$p_J(2) = P(J=2) = P(X_1 > X_2) = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$

c) $f_W(w) = f_{W|J=1}(w) \times P(J=1) + f_{W|J=2}(w) \times P(J=2)$

$= \lambda_2 e^{-\lambda_2 w} \dfrac{\lambda_1}{\lambda_1+\lambda_2} + \lambda_1 e^{-\lambda_1 w} \dfrac{\lambda_2}{\lambda_1+\lambda_2} \mathbb{1}\{w>0\}$

d) probar $P(J=1, U \leq u) = P(J=1) \times P(U \leq u)$

$\Rightarrow$ U y J son independientes

e) probar $f_{WU}(w,u) = f_W(w) \times f_U(u)$

$\Rightarrow$ U y W son independientes

4.15) $X \sim EXP \sim (5)$: "tiempo entre llamadas de Juan"
$Y \sim EXP \sim (10)$: " " " " Pedro"

$5 - 60\,min$ $\quad$ $10 - 60\,min$

$X - 1\,min \Rightarrow \frac{1}{12}$ $\quad$ $X - 1\,min \Rightarrow \frac{1}{6}$

$$f_X(x) = \frac{1}{12} e^{-\frac{x}{12}} \mathbb{1}\{x>0\}$$

$$f_Y(y) = \frac{1}{6} e^{-\frac{y}{6}} \mathbb{1}\{y>0\}$$

$W = \max(x,y) - \min(x,y)$

$\hookrightarrow f_W(w) = \underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}}_{\mathbb{P}(x<y)} \underbrace{\lambda_2 e^{-\lambda_2}}_{f_{y|x}} + \underbrace{\frac{\lambda_2}{\lambda_1+\lambda_2}}_{\mathbb{P}(y<x)} \underbrace{\lambda_1 e^{-\lambda_1}}_{f_{x|y}}$

densidad

a) $P(U \leq 5) \underset{\substack{\uparrow \\ U \sim EXP(\lambda_1+\lambda_2)}}{=} 1 - e^{-\left(\frac{1}{12}+\frac{1}{6}\right)5} = 0{,}713$

b) $P(X<Y) = \dfrac{1/12}{1/12 + 1/6} = \dfrac{1}{3}$

c) $P(U \leq 5 | X<Y) = P(U \leq 5) = 0{,}713$

U y J son independientes

d) $P(X<Y | U>5) = P(X<Y) = \dfrac{1}{3}$

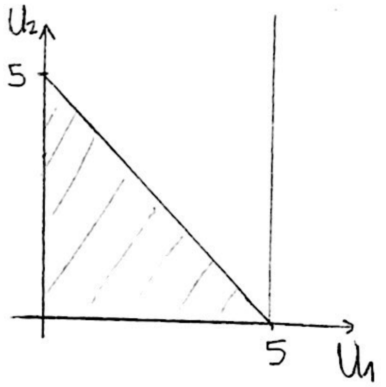
e) $P(\overbrace{U_2-U_1}^{W}>5)=P(U_2>5)=1-P(U\leq 5)=$

$$=1-0{,}713=0{,}287$$

F) $P(U<5)=0{,}713$

g) $\underbrace{P(U_2<5|U_1>5)}_{0{,}713}\times\underbrace{P(U_1>5)}_{0{,}287}+P(U_1+U_2<5|U_1<5)\times\underbrace{P(U_1<5)}_{0{,}713}$

$$P(U_1+U_2<5|U_1<5)=\frac{P(U_1+U_2<5 \wedge U_1<5)}{P(U_1<5)}$$



$$=\frac{\int_0^5\int_0^{5-u_2}\frac{1}{4}e^{-\frac{u_1}{4}}\times\frac{1}{4}e^{-\frac{u_2}{4}}\,du_1\,du_2}{0{,}713}=$$

$$=\frac{0{,}355}{0{,}713}=0{,}498$$

$$\Rightarrow 0{,}713\times 0{,}287+0{,}498\times 0{,}713=0{,}6$$

4.16) $X \sim EXP(\lambda)$ $\quad U = X+Y$

$Y \sim EXP(\lambda)$ $\quad V = \frac{X}{X+Y}$

a)

$$\left|\frac{\partial(U,V)}{\partial(X,Y)}\right| = \left|\begin{pmatrix} 1 & 1 \\ \frac{(X+Y)-X}{(X+Y)^2} & \frac{-X}{(X+Y)^2}\end{pmatrix}\right| = \frac{-X}{(X+Y)^2} - \frac{(X+Y)-X}{(X+Y)^2} = \frac{-(X+Y)}{(X+Y)^2} =$$

$$= \frac{-1}{X+Y} = \frac{-1}{u}$$

$$\Rightarrow f_{UV}(u,v) = \frac{\lambda^2 e^{-\lambda(x+y)}}{1/u} = \lambda^2 e^{-\lambda u} u \; \mathbb{1}\{u \in \mathbb{R}, |v| \leq 1\}$$

b) $U \sim GAMMA(2,\lambda)$

$$f_U(u) = \lambda^2 u e^{-\lambda u} = f_{UV}(u,v)$$

$\Rightarrow V \sim U(1,0)$

$\Rightarrow$ son independientes

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4.17) $X \sim N(0,1)$, $Y \sim N(0,1)$ $\Rightarrow f_{XY}(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} \mathbb{1}\{x,y \in \mathbb{R}\}$

$X = r\cos\theta$, $Y = r\,\mathrm{sen}\,\theta$ $\Rightarrow \left|\frac{\partial(X,Y)}{\partial(r,\theta)}\right| = r$

$$\Rightarrow f_{R\Theta}(r,\alpha) = \frac{1}{2\pi} e^{-\frac{r^2}{2}} r \; \mathbb{1}\{r>0, \alpha \in (0,2\pi)\}$$

$$f_R(r) = \int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{r^2}{2}} r \, d\theta = e^{-\frac{r^2}{2}} r \; \mathbb{1}\{r>0\}$$

$$\Rightarrow f_\Theta(\alpha) = \frac{1}{2\pi} \mathbb{1}\{\alpha \in (0,2\pi)\}$$

4.18) $U_1 \sim U(0,1)$, $U_2 \sim U(0,1)$ $\quad (Z_1, Z_2) = (R\cos\theta, R\,\mathrm{sen}\,\theta)$ $\quad R = \sqrt{-2\ln(U_1)}$, $\theta = 2\pi U_2$

a)

$$F_R(r) = P(R \leq r) = P(\sqrt{-2\ln(U_1)} \leq r) = P(U_1 \geq e^{-\frac{r^2}{2}}) =$$
$$= 1 - P(U_1 \leq e^{-\frac{r^2}{2}}) = 1 - F_{U_1}(e^{-\frac{r^2}{2}}) = 1 - e^{-\frac{r^2}{2}}$$

$$\Rightarrow f_R(r) = r e^{-\frac{r^2}{2}} \; \mathbb{1}\{r>0\}$$

$$F_\Theta(\varphi) = P(\Theta \leq \varphi) = P(2\pi U_2 \leq \varphi) = P(U_2 \leq \varphi/2\pi) = \frac{\varphi}{2\pi}$$

$$\Rightarrow f_\Theta(\varphi) = \frac{1}{2\pi} \mathbb{1}\{\varphi \in (0,2\pi)\}$$

4.24) b) $f_{Z_1Z_2}(z_1,z_2) = \dfrac{f_{R\Theta}(r,\varphi)}{\left|\dfrac{\partial(Z_1,Z_2)}{\partial(R,\Theta)}\right|} = \dfrac{\frac{1}{2\pi} r e^{-\frac{r^2}{2}}}{r} =$

a)

$$= \frac{1}{2\pi} e^{-\frac{r^2}{2}} = \frac{1}{2\pi} e^{-\frac{(z_1^2+z_2^2)}{2}} \mathbb{1}\{z_1,z_2>0\}$$

c) $f_{Z_1Z_2}(z_1,z_2) = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{z_1^2}{2}}}_{Z_1 \sim N(0,1)} \times \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{z_2^2}{2}}}_{Z_2 \sim N(0,1)} \Rightarrow$ son independientes

* 4.19) X, Y independientes

$P(X=1) = P(X=2) = \dots = P(X=36) = \frac{1}{36}$

* $P(Y=1) = P(Y=2) = P(Y=3) = P(Y=4) = \frac{1}{4}$

hallar $p_{X+Y}(x+y)$

Y \ X	1	2	3	...	35	36
1	2	3	4		36	37
2	3	4	5		37	38
3	4	5	6		38	39
4	5	6	7		39	40

$p_{X+Y}(2) = P(X=1, Y=1) = \frac{1}{4} \times \frac{1}{36} = p_{X+Y}(40)$

$p_{X+Y}(3) = P(X=1, Y=2) + P(X=2, Y=1) =$
$= 2\left(\frac{1}{4} \times \frac{1}{36}\right) = p_{X+Y}(39)$

$p_{X+Y}(4) = 3\left(\frac{1}{4} \times \frac{1}{36}\right) = p_{X+Y}(38)$

$p_{X+Y}(5) = p_{X+Y}(6) = \dots = p_{X+Y}(37) = 4\left(\frac{1}{4} \times \frac{1}{36}\right)$

$X \sim U$
Y
$U = \frac{X}{2}$

si $\frac{5}{}$

$P(($

$*$ si

$*1$

4.20) $L \sim Poisson(2)$
$M \sim Poisson(8)$ independientes

a) $L+M \sim Poisson(10)$

$$p_{L+M}(n) = \frac{10^n e^{-10}}{n!}, \quad n \geq 0$$

b) $M \mid L+M=10 \sim BINOMIAL(10, 8/10)$

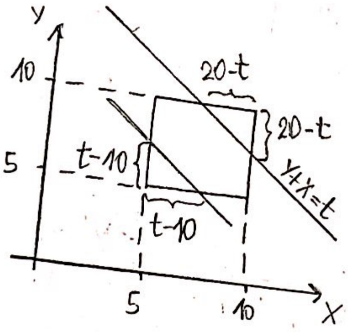
$$p_{M\mid L+M=10}(m) = \binom{10}{m}\left(\frac{8}{10}\right)^m \left(\frac{2}{10}\right)^{10-m}, \quad m \geq 0$$

c) $P(M>2 \mid L+M=10) = 1 - P(M \leq 2 \mid L+M=10) =$

$$= 1 - \left(\binom{10}{0}\left(\frac{8}{10}\right)^0\left(\frac{2}{10}\right)^{10} + \binom{10}{1}\left(\frac{8}{10}\right)^1\left(\frac{2}{10}\right)^9 + \binom{10}{2}\left(\frac{8}{10}\right)^2\left(\frac{2}{10}\right)^8\right)$$

$$= 1$$

4.21) $X \sim U(5,10)$
$Y \sim U(5,10)$



a) $T = X+Y$

$$P(T \leq t) = P(X+Y \leq t) = \begin{cases} 0 & \text{si } t < 10 \\ 1 & \text{si } t \geq 20 \\ * & \text{si } 10 \leq t < 15 \\ ** & 15 \leq t < 20 \end{cases}$$

$$* \quad \frac{(t-10)^2}{2} \times \frac{1}{25} = \frac{(t-10)^2}{50}$$

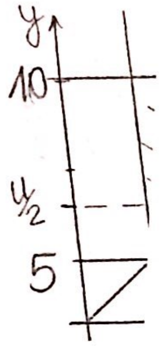
$$** \quad 1 - \frac{(20-t)^2}{2} \times \frac{1}{25} = 1 - \frac{(20-t)^2}{50}$$

1) $X \sim U(5,10)$

Y

$U = \frac{X}{2}$

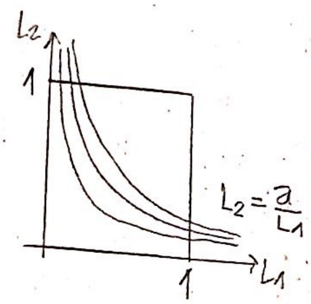
$\Rightarrow F_{X+Y}(t) = \frac{t-10}{25} \mathbb{1}\{10 \leq t < 15\} + \frac{20-t}{25} \mathbb{1}\{15 \leq t < 20\}$



b) $P(X+Y<16) = \int_{10}^{15} \frac{t-10}{25} dt + \int_{15}^{16} \frac{20-t}{25} dt = 0{,}68$

4.22) $L_1 \sim U(0,1)$
$L_2 \sim U(0,1)$
independientes

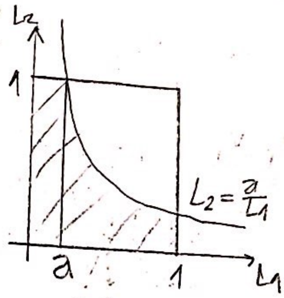
$A = L_1 \times L_2$



si $\frac{5}{2}$

a) $P(A \leq a) = P(L_1 \times L_2 \leq a) =$

$P(U \leq$



$$= \begin{cases} 1 \text{ si } a \geq 1 \\ 0 \text{ si } a < 0 \\ * \text{ si } 0 \leq a < 1 \end{cases}$$

$*$ si 5

$P($

$$* \quad a + \int_a^1 \int_0^{a/L_1} dL_2 \, dL_1 = a + \int_a^1 \frac{a}{L_1} = a + a\ln\left(\frac{1}{a}\right) =$$

$$= a - a\ln(a)$$

$***$

$P($

$$\Rightarrow F_A(a) = 1 - (\ln(a) + 1) = -\ln(a) \, \mathbb{1}\{0 \leq a < 1\}$$

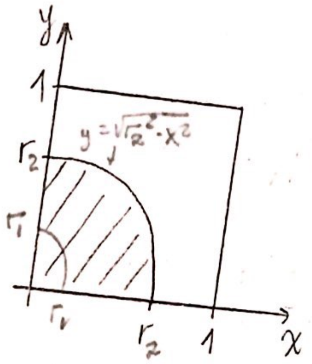
F_U' b) $P(A > 1/4) = \int_{1/4}^{1} -\ln(a) = 0{,}403$

4.21

4.23) $X \sim U(0,1)$, $Y \sim U(0,1)$ $\quad R=\sqrt{X^2+Y^2}$

a) $P(R\leq r)=P(\sqrt{X^2+Y^2}\leq r)= \begin{cases} 1 \text{ si } r\geq\sqrt{2} \\ 0 \text{ si } r<0 \\ * \text{ si } 0\leq r<1 \\ ** \text{ si } 1\leq r<\sqrt{2} \end{cases}$

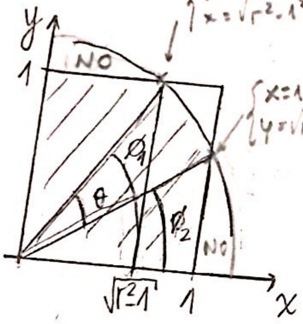
*



$$P(R\leq r)=\int_0^r\int_0^{\sqrt{r^2-x^2}} dy\,dx=\frac{\pi r^2}{4} \quad \forall r\leq 1$$

(4to de círculo)

**



$$P(R\leq r)=\sqrt{r^2-1}+\int_{\sqrt{r^2-1}}^{1}\int_0^{\sqrt{r^2-x^2}} dy\,dx=$$

forman un □ / $A(\square)=\sqrt{r^2-1}$

$$=A(\triangledown)+A(\triangleleft)+A(\lhd)=$$

$$=\frac{\sqrt{r^2-1}}{2}+\frac{\sqrt{r^2-1}}{2}+\pi r^2\frac{\theta}{2\pi}, \quad \theta=\phi_1-\phi_2 \quad ⊛$$

densidad

$$\Rightarrow F_R(r)=\frac{\pi r}{2}\mathbb{1}\{0\leq r<1\}+\bigcirc\,\mathbb{1}\{1\leq r<\sqrt{2}\}$$

b) $P(R>1/2)=1-P(R\leq 1/2)=1-\frac{\pi(1/2)^2}{4}=0{,}804$

⊛

$\phi_1=\cancel{sen^{-1}\left(\frac{1}{r}\right)}\ tg^{-1}\left(\frac{\sqrt{r^2-1^2}}{1}\right)$

derivar esto ⚠ recordar que $[tg(u)]^{-1'}=\frac{u'}{1+u^2}$

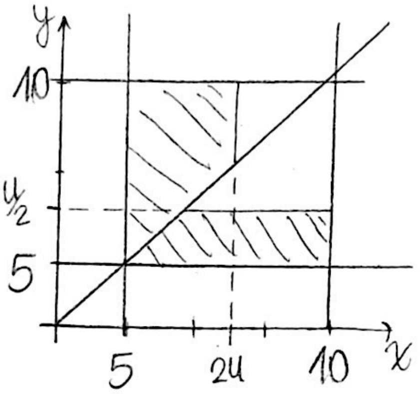
$\Rightarrow P(R\leq r)=\sqrt{r^2-1}+\frac{r^2}{2}\left(\cancel{sen^{-1}\left(\frac{1}{r}\right)}-\cancel{cos^{-1}\left(\frac{1}{r}\right)}\right)$

$\qquad\qquad\qquad\qquad\qquad\qquad\theta_2 \quad - \quad \theta_1$

$\phi_2=\cancel{cos^{-1}\left(\frac{1}{r}\right)}\ tg^{-1}\left(\frac{1}{\sqrt{r^2-1^2}}\right)$

4.24) $X \sim U(5,10)$
$Y \sim U(5,10)$

a) $U = \frac{X}{2} \mathbb{1}\{X \leq Y\} + 2Y \mathbb{1}\{X > Y\}$



$$P(U \leq u) = P\left(\frac{X}{2} \leq u, X \leq Y\right) + P(2Y \leq u, X > Y) =$$
$$= P(X \leq 2u, X \leq Y) + P\left(Y \leq \frac{u}{2}, X > Y\right)$$

$$\Rightarrow F_U(u) = \begin{cases} 0 & \text{si } u \leq 5/2 \\ 1 & \text{si } u \geq 20 \\ * & \text{si } 5/2 < u < 5 \\ ** & \text{si } 5 \leq u < 10 \\ *** & \text{si } 10 \leq u < 20 \end{cases}$$

$*$ si $\frac{5}{2} < u < 5$

$$P(U \leq u) = 1 - \left(\frac{(10-2u)^2}{2} + \frac{25}{2}\right) \times \frac{1}{25} = \frac{1}{2} - \frac{(10-2u)^2}{50}$$

$**$ si $5 \leq u < 10$

$$P(U \leq u) = \frac{1}{2}$$

$***$ si $10 \leq u < 20$

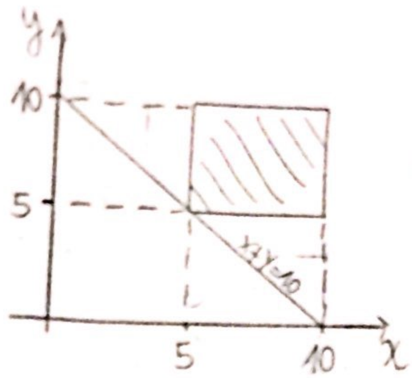
$$P(U \leq u) = 1 - \left(\frac{(10-u/2)^2}{2}\right) \times \frac{1}{25} = 1 - \frac{(10-u/2)^2}{50}$$

derivada de $F_U(u)$

$$f_U(u) = \frac{2(10-2u)}{25} \mathbb{1}\left\{\frac{5}{2} \leq u < 5\right\} + \frac{(10-u/2)}{50} \mathbb{1}\{10 \leq u < 20\}$$

b) $V = \mathbb{1}\{X+Y \leq 10\}$

$p_V(1) = P(V=1) = P(X+Y \leq 10) = 0$



$p_V(0) = P(V=0) = P(X+Y > 10)$
$= 1$

c)