D. Let $\mathbb{N}$ be the smallest set X s.t.
(1) $0 \in X$
and (2) if $n \in X$ then $Sn \in X$

$(A, \sqsubseteq)$

$\mathcal{P}(\mathcal{U})\ \subseteq$

$F(Y) = \{0\} \cup \underbrace{\{Sn \mid n \in Y\}}_{\text{"SY"}}$

$\bot = \emptyset$

$F(\bot) = \{0\}$

$F^2(\bot) = \{0, S0\}$

D. Let Σ be any finite set
Σ^* is the smallest set X st
(1) $\varepsilon \in X$
and (2) for all $a \in \Sigma$, $\underbrace{aX}_{aX = \{ax \mid x \in X\}} \subseteq X$

Σ^* is the set of finite words over Σ

$\Sigma = \{0,1\}$. 01001, ...

Notes.
– "def by induction" defines countable sets of finite members
– " is often written as rules:

$$\frac{}{0} \qquad \frac{n}{Sn}$$

if this is in the set
then this is in the set

$$\frac{}{\varepsilon} \qquad \frac{x}{ax}\ a \in \Sigma$$

– " allows the following proof rule ("proof by induction")

Show for all x in X, if x in Σ^*, then $G(x)$.
inductively defined

G
Σ^*

Example:

D: Let S be the largest set X s.t.
$$X \subseteq 01X \cup 10X \qquad (*)$$

Let $[0,1]$ be the largest set X st
$X \subseteq 0X \cup 1X \cup 2X \cup \dots \cup 9X$

$\mathcal{P}(\Sigma^\omega), \subseteq, F(X) = 01X \cup 10X$

set of infinite words over $\Sigma = \{0,1\}$

$T = \Sigma^\omega$

$F(T) = 01\Sigma^\omega \cup 10\Sigma^\omega$

$F^2(T) = 0101\Sigma^\omega \cup 0110\Sigma^\omega \cup 1001\Sigma^\omega \cup 1010\Sigma^\omega$

$\vdots$

$\prod F^i(T)$ = all inf words of 0s and 1s st every finite prefix of even length has same number of 0s and 1s

Notes.
– "def by coinduction" defines uncountable sets of infinite objects
– can again be written as "rules", but they are read "backwards".

$$\frac{x}{01x} \qquad \frac{x}{10x}$$

"For every y in S,
either y has the form $01x$ and $x \in S$,
or y " $10x$ and $x \in S$"

– Proof by coinduction:

Show for all x, if $K(x)$ then $x \in S$
coinductively defined set

(1) Consider arbitrary $\hat{x}$
Assume $K(01\hat{x})$
Show: $K(\hat{x})$

(2) Consider arbitrary $\hat{x}$
Assume $K(10\hat{x})$
Show $K(\hat{x})$

Coinduction hypothesis

S
K

Homework: prove by coinduction
For all $x \in \Sigma^\omega$,
$x \in S$ $(*)$ iff every finite prefix of even length has same # of 0s and 1s

$\Leftarrow$ coinduction on S $\Rightarrow$ induction on $\mathbb{N}$ (length of prefix) fixpoint $\quad x \sqsubseteq F(x)$

$F(x) \sqsubseteq x$

x pre and postfixpoint of F

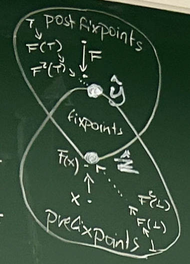
For a complete lattice, let $\hat{y} = \bigsqcup \{x \mid x \sqsubseteq F(x)\} = \text{gfp } F$

$\hat{z} = \bigsqcap \{x \mid F(x) \sqsubseteq x\} = \text{lfp } F$

Knaster-Tarski: If F is monotone, then (1) $\hat{y}$ and $\hat{z}$ are fixpoints

(2) for all fixpoints x of F,

$\hat{z} \sqsubseteq x \sqsubseteq \hat{y}$.



D. F $\sqcup$-continuous if for all increasing sequences $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$, $F(\bigsqcup \{x_i\}) = \bigsqcup \{F(x_i)\}$

F $\sqcap$ decreasing $x_0 \sqsupseteq x_1 \sqsupseteq x_2 \sqsupseteq \ldots$, $\sqcap$ $\sqcap$

(Both $\sqcup$- and $\sqcap$-continuity imply monotonicity of F.)

T. Let $(A, \sqsubseteq)$ be a complete lattice.

If F (from A to A) is $\sqcup$-continuous, then $\text{lfp } F = \bigsqcup \{F^i(\bot) \mid i \geq 0\}$

$\sqcap$. $\text{gfp } F = \bigsqcap \{F^i(\top) \mid i \geq 0\}$

Homework: prove this theorem

$\bot \sqsubseteq F(\bot) \sqsubseteq F^2(\bot) \sqsubseteq \ldots \to \bigsqcup \{F(\ldots$

$\top \sqsupseteq F(\top) \sqsupseteq F^2(\top) \sqsupseteq \ldots \to \sqcap$

D. Let $\mathbb{N}$ be the smallest set X s.t.

(1) $0 \in X$

and (2) if $n \in X$ then $Sn \in X$

$(A, \sqsubseteq)$

$\mathcal{P}(U)$ $\subseteq$

$\bot = \emptyset$

$\top = A$

$F(Y) = \{0\} \cup \underbrace{\{Sn \mid n \in Y\}}_{\text{"SY"}}$

$F(\bot) = \{0\}$

$F^2(\bot) = \{0, S0\}$

$F^3(\bot) = \{0, S0, SS0\}$

$\vdots$

$\bigsqcup \{F^i(\bot)\} = \{0, S0, SS0, \ldots\}$

D. Let Σ be any finite set.

Σ^* is the smallest set X st

(1) $\varepsilon \in X$

and (2) for all $a \in \Sigma$, $aX \subseteq X$

$aX = \{ax \mid x \in X\}$

Σ^* is the set of finite words over Σ

$\Sigma = \{0, 1\}$: $\quad 01001, \ldots$

$\mathbb{N} \cup \{\pi, S\pi, S^2\pi, \ldots\}$

Notes: – "def by induction" defines countable sets of finite members

– " is often written as rules.

$\dfrac{}{0} \qquad \dfrac{n}{Sn}$ ← if this is in the set; ← then this is in the set $\qquad \dfrac{}{\varepsilon} \qquad \dfrac{x}{ax}\ a \in \Sigma$

– " allows the following proof rule ("proof by induction")

Show for all x in X, if x in Σ^* (inductively defined), then $G(x)$.

Consider arb $\hat{x}$. $\quad$ (1) Show $G(\varepsilon)$ ← "base case"

Assume $\hat{x}$ in Σ^* $\quad$ (2) Consider arb $\hat{x}$ in Σ^*

Show $G(\hat{x})$. $\quad$ Consider arb $\hat{a}$ in Σ

Assume $G(\hat{x})$ ← "induction hypothesis"

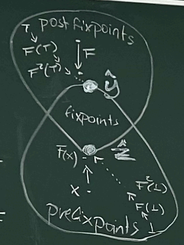
Show $G(\hat{a}\hat{x})$.

x prefixpoint of F if $x \sqsubseteq F(x)$
post
fixpoint $\qquad F(x) \sqsubseteq x$

$\qquad x$ pre and postfixpoint of F

For a complete lattice, let $\hat{y} = \bigsqcup \{x \mid x \sqsubseteq F(x)\} = \mathrm{gfp}\, F$

$\hat{z} = \bigsqcap \{x \mid F(x) \sqsubseteq x\} = \mathrm{lfp}\, F$



Knaster Tarski: If F is monotone, then (1) $\hat{y}$ and $\hat{z}$ are fixpoints
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F $\sqcap$ decreasing $x_0 \sqsupseteq x_1 \sqsupseteq x_2 \sqsupseteq \dots$, $\bigsqcap \quad \bigsqcap$

(Both $\sqcup$- and $\sqcap$-continuity imply monotonicity of F)

T: Let $(A, \sqsubseteq)$ be a complete lattice.
If F is $\sqcup$-continuous, then $\mathrm{lfp}\, F = \bigsqcup\{F^i(\bot) \mid i \geq 0\}$ $\qquad \bot \sqsubseteq F(\bot) \sqsubseteq F^2(\bot) \sqsubseteq \dots \to \bigsqcup\{F(x_i)\}$
$\sqcap$. $\qquad \mathrm{gfp}\, F = \bigsqcap\{F^i(\top) \mid i \geq 0\}$ $\qquad \top \sqsupseteq F(\top) \sqsupseteq F^2(\top) \sqsupseteq \dots \to \bigsqcap$

Homework: prove this theorem