A Geometric Approach to Pursuit-Evasion Games

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To the joy of discovery and the pleasure in proving

Abstract

Pursuit-Evasion games are a class of differential games which have received considerable attention from mathematicians, engineers and biologists over the last sixty years. Owing to its applications in both civilian and military domains, a lot of effort has been put into developing this exciting field. Early work focused on providing analytical solutions to the games, while the current research is more focused on the developing numerical techniques to tackle the same. While developing a more complete set of numerical tools for pursuit and evasion games is still a developing topic of research, the following work focuses on providing intuitive and geometric solutions to extend the current body of analytical solutions and methods.

First, feedback strategies are derived from the existing results for pursuer and evader having double integrator dynamics and then extended to include systems having non-zero and rational eigenvalues. The efficacy of this approach and the derived strategies is illustrated by solving multi-agent consensus tracking problem on graphs.

Second, a multi-player game involving three pursuers and one evader on a plane, both having coupled single integrator systems on the two axes, is solved using the notion of Voronoi diagrams and the very effective Parallel Navigation strategy. Simulation results verify the simplicity of the above approach.

Publications

1. Mulla, A. ; Bhatt, S. ; Patil, D. ; Chakraborty, D. ; *“Min-max Time Consensus Tracking Over Directed Trees*", accepted in Mathematical Theory of Networks and Systems (MTNS), 2014.

Contents

1. Introduction

1.1. Overview

Pursuit and Evasion, as Paul J.Nahin says, defines the very nature of human existence - from the romantic pursuit of a future spouse to the more *serious* pursuit of an enemy aircraft. The essential concept of being the hunter or the hunted is so deeply embedded in nature that we can't help but reflect on it every once in a while. Take for example, a predator pursuing its prey for survival, or a person (me) pursuing (chasing) his dreams, although, I hope my dreams are not intelligently evading me - "dumb" dreams. In today's materialistic dog-eat-dog world, a person who does not go after something is considered a social outcast or as a mature person would say, a *saint*. Saints, albeit belong to a set of measure *zero*. Metallica have said in one of their songs, "New blood joins this earth and quickly he's subdued", meaning, if the search is not there in one's blood, they are injected by the society. Several other examples in military like guiding a missile towards an enemy aircraft or a vessel and in sports like pursuit of the football to intercept it by the goalie and the evasion of the players in the old but popular game of Tag, only serves to exemplify the extent to which the field of pursuit evasion (PE) games has pervaded this world.

Given the importance and prevalence, it should come as no surprise that many people have tried to wrap their minds around it. Pierre Bouger in 1732 provided an analytical approach to the classical pursuit problem, where the pursuer's instantaneous velocity vector is always pointed towards the evader during the pursuit, so called *Pure Pursuit*. In

1870, the invention of a guided boat by Werner von Siemens saw the dawn of applications of guidance in weaponry. We shall fast forward 80 years to 1950's (refer [Nah07] for a very good historical perspective and the work done pre-1950), where the work done by eminent mathematician Rufus Isaacs spawned a huge body of research and interest in pursuit evasion games we know today. Rufus Issacs proposed differential game theory [Isa99], which is a branch of game theory which models the interaction between the players as a continuous process described by a set of differential equations, to study the interactions between a single pursuer and evader. The players then analyze the outcome of a continuously evolving problem to manipulate the situation to their advantage. He considered pursuit evasion games in full generality, in the sense that both pursuer and evader attempt to maximize their (conflicting) objective - *min-max* formulation. While it is possible to obtain analytical solutions to simple games using the techniques of differential game theory, for complicated (either the dynamics or the cost function) games, numerical methods need to be called upon to solve them. Another approach to solving pursuit evasion games was formulated and solved by Kelendzheridze by using Pontryagin's *Maximum Principle* [Kel61]. Pontryagin himself made important contributions in differential games [MP67, Pon80, Pon65].This was followed by positional differential games by Krasovskii and his co-workers [KSK11] to obtain feedback solutions. This continuously growing field prompted researchers to consider general pursuit-evasion games involving multiple pursuers and multiple evaders. Initial effort in this direction has been limited to discrete time PE games which are tackled in adhoc manner. Algorithmic approach to multi-player games was given in [BS11]. An approach based on the notion of reachability set was considered in [CF08]. A comprehensive mathematical treatment using an approach similar to Pontryagin's method known as Method of Resolving functions was given by Chikrii and his co-workers [Chi97]. The increasing use of unmanned assets and robots in modern military operations necessitates the study of multi-agent scenarios. However, due to the difficulty in formulation and rigorous treatment, the literature in this field is limited.

1.2. Outline of the thesis

First half of the work deals with *time-optimal* PE games. The approaches outlined above either need the computation of the so-called *Value* function, a non-smooth function (for bounded controls) from which the optimal feedback control strategies are constructed, to solve the differential game or require solving a two-point boundary value problem (TBVP). There is abundant literature on synthesizing feedback controls by considering the notion of weak solutions, commonly known as viscosity solutions , and $v-$and $u-$stable functions. Numerical methods have been employed to compute the feedback strategies by solving Two-point boundary value problem (TBVP).

In Chapter 2, we provide a new way of constructing the feedback strategies using techniques from algebraic geometry. This sidesteps the need to explicitly construct the value function and also the need to numerically solve TBVP. The state-feedback control laws proposed in [PC13] for LTI systems with rational, distinct and non-zero eigenvalues, make use of the implicit representation of switching surfaces computed using Gröbner basis based elimination method. Using these switching surfaces, feedback control strategies are proposed, which albeit some off-line computation, provides elegant and easily implementable expressions for the control laws. These laws are applied to consensus tracking over directed trees, which substantiates the simplicity and usefulness of this approach.

Next half of the work deals with multi-player games. We consider a game of three pursuers and one evader on an unbounded plane and provide a justification for the same.

In Chapter 3, using techniques from simple geometry and calculus and using the concept of Voronoi diagrams, we solve the multi-player PE game for simple motion games, where the evader is not actively participating. We make use of the familiar *Parallel Navigation* strategy and provide mathematical justification for employing it.

We conclude the thesis by citing possible pathways and relevance to extend the results of chapter 3.

2. Two player time-optimal PE games

2.1. Overview

Time-optimal games form an important class of PE games wherein the pursuer not only has to capture the evader, but he has to do in minimum possible time while the evader is actively trying to maximize it. Owing to the non-smoothness of the value function (time), we attempt to obtain the feedback strategies by making use of the Maximum Principle [PBGM]. We first introduce the general setting used in all the subsequent sections of this chapter.

The following assumptions are made to simplify the PE game.

1. The models of pursuer and evader are accurate and identical.

2. It is assumed that one knows the model of the other.

3. It is assumed that both pursuer and evader are able measure instantaneously and accurately the relative state of the other with respect to their own.

4. It is assumed that the pursuer has more capability compared to evader ($\beta > \alpha$) to facilitate the capture.

The pursuer and evader states are governed by the following linear time invariant differential equations:

$$\begin{aligned} \text{Evader:} \quad \dot{x}_e(t) &= Ax_e(t) + Bu_e(t) \quad |u_e(t)| \leq \alpha & (2.1.1) \\ \text{Pursuer:} \quad \dot{x}_p(t) &= Ax_p(t) + Bu_p(t) \quad |u_p(t)| \leq \beta & (2.1.2) \end{aligned}$$

The admissible controls of both the pursuer and evader are given by measurable functions $u_p : \mathbb{R}^+ \to \mathscr{P}$ and $u_e : \mathbb{R}^+ \to \mathscr{E}$ respectively where $\mathscr{E} = [-\alpha, \alpha]$ and $\mathscr{P} = [-\beta, \beta]$. Using the pursuer (2.1.2) and the evader dynamics (2.1.1), we define following *difference system* which captures the dynamics of the relative states i.e. $x(t) = x_p(t) - x_e(t)$ between the pursuer and the evader:

$$\dot{x}(t) = Ax(t) + Bu_{pe}(t) \tag{2.1.3}$$

where $u_{pe}(t) = u_p(t) - u_e(t)$ and $x(t) = x_p(t) - x_e(t)$. We shall this henceforth deal with only the difference system. We say the game terminates or the pursuer captures the evader when the following condition is satisfied :

$$x(t) \in \mathscr{N}_\varepsilon$$

where $\varepsilon > 0$, $\mathscr{N}_\varepsilon$ is the $\varepsilon -$*neighborhood* of the origin and $x(t)$ is the solution of the difference system.

Before we set out to derive the strategies for general (rational eigenvalues) systems, we shall consider systems modeled by single and double integrator dynamics to get an intuition behind the strategies.

2.2. Single Integrator systems

PE games in which the Pursuer and Evader are modeled by single integrator are one of the simplest and easiest games to analyze. This is the most common model along with double integrator used to study the interaction of multi-agents on graphs, the reason being the ease of implementability and natural correspondence of the states to the physical position and velocity of the body being modeled.

The differential equation of the relative dynamics or the difference system is given by :

$$\dot{x}(t) = u_p(t) - u_e(t) \tag{2.2.1}$$

where $A = 0$, $B = 1$ and $x(t) = x_p(t) - x_e(t)$.

Let $x_p(0)$ and $x_e(0)$ be the initial conditions, therefore $x(0) = x_p(0) - x_e(0)$. Since $\beta > \alpha$, the system can be forced to the origin. Let $T_{p,e} = T$ be the time of completion. The solution of the first order equation (2.2.1) for constant inputs is given by :

$$x(T) = (u_p - u_e)T + x(0) \tag{2.2.2}$$

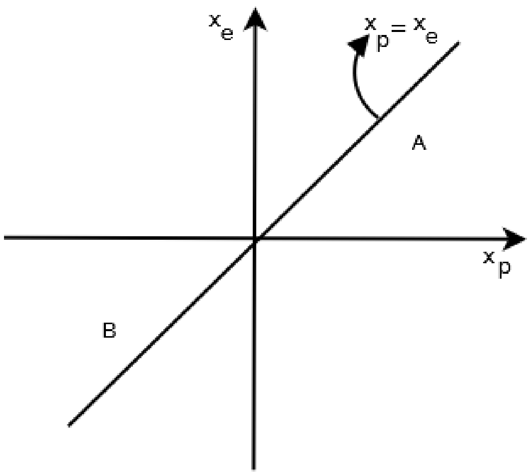

Figure 2.2.1.: Single Integrator

In Fig.2.2.1, if the initial condition is at the point identified by A then pursuer chooses $-\beta$ to move closer to the target set $x_p = x_e$. The evader in order to maximize the time of capture chooses $-\alpha$. If the initial condition is at the point identified by B then pursuer chooses β and the evader chooses α. As $x(0)$ is arbitrary, the strategy for the pursuer is $-\beta sgn(x)$. The evader being rational player, chooses his strategy to be $-\alpha sgn(x)$.

The value of the game, which is the time taken to reach the origin is easily calculated as :

$$T_{p,e} = \frac{|x(0)|}{\beta - \alpha} \tag{2.2.3}$$

From the above feedback laws we can infer that, on the one-dimensional line on which the game is taking place, the pursuer is heading with its maximum velocity towards the evader, while the evader is attempting to move away from the pursuer with maximum possible velocity - a reasonable and logical thing to do.

2.3. Double Integrator systems

We next consider the double integrator model for the Pursuer and Evader, which is one of the most fundamental models in control applications, representing single-degree-of-freedom translational and rotational motion. Applications of the double integrator include single-axis spacecraft rotation and rotary crane motion. It was extensively used to illustrate the minimum time control laws during the formative years of optimal control theory. Feedback strategies are proposed for both pursuer and evader in terms of certain *switching* surfaces which are the surfaces in the state space corresponding to minimum-time surfaces of optimal control problems [PBGM].

The differential equation of the relative dynamics or the difference system is given by :

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u_p(t) - u_e(t)) \tag{2.3.1}$$

where $x(t) = x_p(t) - x_e(t)$.

Let us regard (2.3.1) as a control system with single input and solve the time-optimal control problem. The reason for doing so will be justified later when we solve the PE game for the general case. It is well know that the minimum-time trajectories (or surfaces) for double integrators are parabolas in the state space. We provide the derivation here for completeness.

The switching surface is derived as follows :

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_{pe} \\ \frac{dx_2}{dx_1} &= \frac{u_{pe}}{x_2} \\ \therefore x_2 dx_2 &= u_{pe} dx_1\end{aligned}$$

Integrating on both sides, we get for constant inputs :

$$\frac{x_2^2}{2} = u_{pe}x_1 + c$$

The above equation corresponds to different set of parabolas in the state space for different c and sign of u_{pe}. The only parabolas with which origin is reached are those that correspond to $c = 0$ as shown in the figure. Let us call this surface as the *worst-case* switching surface. The reason behind this is due to the fact that the evader is actively trying to escape the pursuer and so from the pursuer's perspective it is the "worst-case" possible.

Theorem 2.1. Let S be the worst-case switching surface, i.e, the one that corresponds to $u_{pe} = |\beta - \alpha|$ and $c = 0$. For the PE game with double integrator dynamics, the optimal strategies for the pursuer and evader are respectively

$$u_e^* = -\alpha \; sgn(S) \; and \; u_p^* = -\beta \; sgn(S)$$

Proof. Proof of this will be given in a more general setting in the next section. □

2.3.1. Is the pursuer strategy reasonable.?

Let us clarify what we mean by the strategy being reasonable. We want the pursuer control (or feedback law or strategy) to force the difference system to origin, no matter what the evader chooses to do in *finite* time - a property we shall term as *stability*.

Before we go to the main results of this subsection, we shall define a few terminologies which are used throughout the section.

Let $\Phi(t)$ denote the fundamental matrix of both pursuer and evader.

$$\Phi(t) = e^{tA}$$

Let $\Psi(t) = \Phi(t)B$ and let the function $\Gamma_E(\lambda)$ be defined by

$$\Gamma_E(\lambda) = \sup_{u_e \in E}(\lambda u_e)$$

where λ is any unit row vector.

Now, let $\tilde{u}_e(t) = \Gamma_E(\lambda\Psi(T-t)) = \sup_{u_e}(\lambda\Psi(T-t)u_e)$ denote the supremum over the compact set E for some $T < \infty$.

Let Θ denote the interval in $\mathbb{R}$ given by :

$$\Theta_\gamma = \{a : a \in \mathbb{R},\ |a| \le \gamma, \gamma \in \mathbb{R}^2\}$$

For $T > 0$, define

$$\mathscr{U}[0,T] = \{\ u_{pe}(t) : u_{pe}(t) \in \Theta_{\beta-\alpha}\ \textit{and}\ u_{pe}(t)\ \textit{measurable on}\ [0,T]\ \}$$
$$\mathscr{U}_{\mathscr{B}\mathscr{B}}[0,T] = \{\ u_{pe}(t) : u_{pe}(t) \in \mathscr{U}[0,T]\ \textit{and}\ |u_{pe}(t)| = \beta - \alpha\ \textit{on}\ [0,T]\ \}$$

The reachable set at time t is defined by :

$$\mathscr{K}_{\mathscr{B}\mathscr{B}}(t;x_0) = \{\ x(t;x_0,u_{pe}) :\ u_{pe} \in \mathscr{U}_{\mathscr{B}\mathscr{B}}\ \}$$

The boundary of the figure Fig.2.3.1 shows all the states that can be reached with “one" switch. It is known that $\mathscr{K}_{\mathscr{B}\mathscr{B}}$ for a fixed T is compact, convex and exactly the entire closed region inside this boundary [MS82].

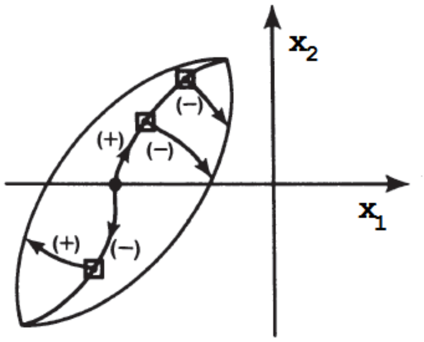

Figure 2.3.1.: Reachable set at time t

Let $T(x_0)$ denote the time taken for the system to reach origin from x_0 along the worst-case switching surface. This is easily determined from the system dynamics as both input, initial and final states are known. We shall establish the *finite-time* stability of the pursuer control against all actions of the evader in two parts :

1. The pursuer control has the ability to drive the system state from any point in the state space to the worst-case switching surface.

2. It has the ability to drive the system from any point on the surface to the origin.

Theorem 2.2. *For the system* $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u_p(t) - u_e(t))$*, the pursuer strategy* $u_p = -\beta sgn(S)$*, where S is the worst-case switching surface, ensures the convergence to the switching surface in finite time against all actions of the evader.*

Proof. Consider the arcs S_1 and S_2 (Fig. 2.3.2) whose algebraic characterizations are

$$S_1 = x_2^2 + 2(\alpha - \beta)x_1$$
$$S_2 = x_2^2 - 2(\alpha - \beta)x_1$$

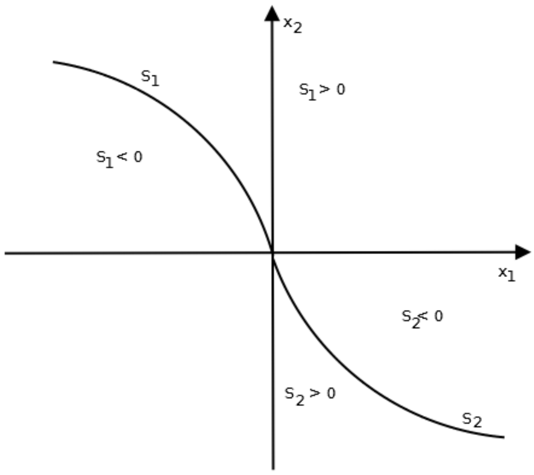

Figure 2.3.2.: Worst-case switching surfaces

Let $\mathscr{V}$ be the Lyapunov function defined by

$$\mathscr{V} = \frac{S_1^2}{2} \tag{2.3.2}$$

Differentiating with respect to t we have,

$$\begin{aligned} \dot{\mathscr{V}} &= S_1 \dot{S}_1 \\ &= S_1(2x_2\dot{x}_2 + 2(\beta - \alpha)x_1) \\ &= 2x_2 S_1(u_p - u_e + (\beta - \alpha)) \end{aligned}$$

We shall prove the convergence for only one case : $x_2 > 0$ and $S_1 < 0$ using the proposed pursuer strategy.

For $x_2 > 0$ and $S_1 < 0$,

$$\begin{aligned} \dot{\mathscr{V}} &= 2x_2S_1(-\alpha sgn(S_1)) + 2x_2S_1(\beta - \alpha) - 2x_2S_1u_e \\ &= 2\alpha x_2S_1 + 2x_2S_1(\beta - \alpha) - 2x_2S_1u_e \\ &= 4x_2S_1\alpha - 2x_2S_1\beta - 2x_2S_1u_e \\ &= 4x_2S_1\alpha - 2x_2S_1(\beta + u_e) \\ \therefore \dot{\mathscr{V}} &\leq 4x_2S_1(\beta - \alpha) \end{aligned} \tag{2.3.3}$$

Now, $x_2(t) = x_2(0) + \alpha t - \int_0^t u_e(s)ds$. Let $f(t) = \int_0^t u_e(s)ds$. For any measurable control $u_e(t)$, we have

$$f(t) = \int_0^t u_e(s)ds \leq \beta t$$

Therefore,

$$\begin{aligned}
x_2(t) &\geq x_2(0) + (\beta - \alpha)t \\
\dot{\mathscr{V}} &\leq 4x_2 S_1(\beta - \alpha) \\
\dot{\mathscr{V}} &\leq -4\sqrt{2\mathscr{V}}(\beta - \alpha)(x_2(0) + (\beta - \alpha)t) \\
\frac{d\mathscr{V}}{\mathscr{V}} &\leq -4\sqrt{2}(\beta - \alpha)(x_2(0) + (\beta - \alpha)t)dt
\end{aligned}$$

Integrating on both sides, we have

$$\begin{aligned}
2\sqrt{\mathscr{V}(t)} &\leq 2\sqrt{\mathscr{V}(0)} - 4\sqrt{2}(\beta - \alpha)\{(\beta - \alpha)\frac{t^2}{2} + x_2(0)t\} \\
\sqrt{\mathscr{V}(t)} &\leq \sqrt{\mathscr{V}(0)} - \sqrt{2}(\beta - \alpha)^2 t^2 - 2\sqrt{2}x_2(0)t
\end{aligned}$$

Since $\mathscr{V}(t)$ is a Lyapunaov function, $\mathscr{V}(t) \geq 0$ and hence $\sqrt{\mathscr{V}(t)} \geq 0$. Now, if the equation $\sqrt{\mathscr{V}(0)} - \sqrt{2}(\alpha - \beta)^2 t^2 - 2\sqrt{2}x_2(0)t = 0$ has a positive root then we are done. Consider $\sqrt{2}(\alpha - \beta)^2 t^2 + 2\sqrt{2}x_2(0)t - \sqrt{\mathscr{V}(0)} = 0$,

$$\begin{aligned}
\Delta &= (2\sqrt{2}x_2(0))^2 + 4\sqrt{2}(\alpha - \beta)^2\sqrt{\mathscr{V}(0)} \\
\Delta &= 8x_2(0)^2 + 4\sqrt{2}(\alpha - \beta)^2\sqrt{\mathscr{V}(0)} > 0
\end{aligned}$$

Since $\Delta > 0$, and $\sqrt{\Delta} - 2\sqrt{2}x_2(0) > 0$, $\exists$ a positive and finite time after which that state trajectory hits the switching surface. □

We thus established that the pursuer control drives the state trajectory to the switching surface. We shall now show that once the system trajectory hits the switching surface, the pursuer can always force the evader to the origin in finite time. The following lemma is required to establish the result.

Lemma 2.3. *Let λ be an arbitrary unit row vector and $t \geq 0$. For the evader control $u_e(t) = \tilde{u}_e(t)$ and any initial point x_0 on the switching surface, there exists a measurable pursuer control $u_{p_{\tilde{u}_e}}(t)$ which ensures the completion of the game in time $T(x_0)$.*

Proof. The solution of the state equation is :

$$x(t) = \Phi(t)x_0 + \int_0^t \Psi(t-s)(u_p(s) - u_q(s))ds \tag{2.3.4}$$

Taking the inner product with λ on both sides of the equation (2.3.4), we have

$$\lambda x(t) = \lambda \Phi(t)x_0 + \int_0^t \lambda \Psi(t-s)(u_p(s) - u_e(s))ds \tag{2.3.5}$$

Let the evader choose the control $\tilde{u}_e(t)$, which is the bang-bang control. For a given λ the number of sign changes is at most $n-1$ [PBGM], where n is the number of states. Here $n = 2$. So the sign of the evader control changes at most once and the switching instances are per-determined. Without loss of generality we can assume the initial point x_0 on $\alpha - \beta$ surface. The pursuer chooses the control $u_{p_{\tilde{u}_e}}(t)$ as follows.

$$u_{p_{\tilde{u}_e}}(t) - \tilde{u}_e(t) = (\alpha - \beta)$$
$$u_{p_{\tilde{u}_e}}(t) = -a\beta + (1-a)\beta$$

where a = 1 as long as $sgn(\tilde{u}_e(t))$ is negative and if $sgn(\tilde{u}_e(t))$ is positive, then the pursuer switches infinitely fast about the surface which results in a measurable function $u_p(t) \in \mathscr{P}_{\mathscr{B}\mathscr{B}}$ with a = 0.5. The infinitely fast switching results in a sliding behavior about the switching surface.

With the pursuer control so chosen, the state of the control system follows the $\alpha - \beta$ switching surface. The termination is now identified by the condition $x(T(x_0)) \in \mathscr{N}_\varepsilon$ or equivalently $\lambda x(T(x_0)) \leq \varepsilon$, where the finite-time $T(x_0)$ is easily calculated from the initial condition. $\square$

Theorem 2.4. *Let λ be an arbitrary unit row vector and $t \geq 0$. For any measurable* *control* $u_e(t) \in \mathscr{E}_{\mathscr{B}\mathscr{B}}$ *and any initial condition* x_0 *on the switching surface there exists a measurable control* $u_{p_{u_e}}(t) \in \mathscr{P}_{\mathscr{B}\mathscr{B}}$ *such that the game is completed in time* $T(x_0)$ *if and only if*

$$-\varepsilon \leq \lambda\Phi(T(x_0))x_0 + \int_0^{T(x_0)} \lambda\Psi(T(x_0)-t)u_{p_{\tilde{u}_e}}(t)dt - \int_0^{T(x_0)} \Gamma_E(\lambda\Psi(T(x_0)-t))dt \tag{2.3.6}$$

Proof. First, suppose that for any measurable control $u_e(t) \in \mathscr{E}_{\mathscr{B}\mathscr{B}}$ and any initial condition x_0 on the switching surface there exists a measurable control $u_{p_{u_e}}(t) \in \mathscr{P}_{\mathscr{B}\mathscr{B}}$ such that the game is completed in finite time $T(x_0)$. By multiplying $-\lambda$ on both sides of the equation (2.3.4) we have,

$$-\lambda x(t) = -\lambda\Phi(t)x_0 - \int_0^t \lambda\Psi(t-s)(u_p(s)-u_e(s))ds \tag{2.3.7}$$

Evaluating the above integral at time $T(x_0)$ we obtain

$$-\lambda x(T(x_0)) = -\lambda\Phi(t)x_0 - \int_0^{T(x_0)} \lambda\Psi(T(x_0)-s)(u_p(s)-u_e(s))ds \tag{2.3.8}$$

Since equation (2.3.8) holds for all $u_e(t)$, it must hold for $u_e(t) = \tilde{u}_e(t)$. Equation (2.3.6) now quickly follows from *Lemma 2.3*. Next, suppose equation (2.3.6) holds. Now, suppose there exists a quarry control $u_e(t)$ for which there exists no pursuer control $u_p(t) \in \mathscr{P}_{\mathscr{B}\mathscr{B}}[0, T(x_0)]$ such that $x(T(x_0)) \in \mathscr{N}_\varepsilon$. This means that the reachable set defined by

$$\mathscr{K}_{\mathscr{B}\mathscr{B}}(T(x_0);x_0) = \left\{ \int_0^{T(x_0)} \Psi(T(x_0)-t)u_p(t)dt \,:\, u_p(t) \in \mathscr{P}_{\mathscr{B}\mathscr{B}} \right\}$$

which is compact and convex [MS82], does not intersect the compact sphere

$$-\Phi(T(x_0))x_0 + \int_0^{T(x_0)} \Psi(T(x_0)-t)u_e(t)dt \;+\; \mathscr{N}_\varepsilon$$

By Separating Hyperplane Theorem, there exists a unit row vector λ such that the inequality

$$-\Phi(T(x_0))x_0+\int_0^{T(x_0)}\Psi(T(x_0)-t)u_e(t)dt\ +\ \lambda g>\int_0^{T(x_0)}\Psi(T(x_0)-t)u_p(t)dt \tag{2.3.9}$$

is true for all $t\in[0,T(x_0)]$ and all $g\in\mathscr{N}_\varepsilon$.

This can be visualized as pursuer starting from the origin and the evader shifting the ε sphere away from the reachable set of the pursuer. Since the equation (2.3.9) is true for all $u_p(t)$, it must hold for $u_p(t)=u_{p_{\tilde{u}_e}}(t)$. Now taking $g=-\varepsilon\lambda'$, the inequality (2.3.9) becomes

$$-\Phi(T(x_0))x_0+\int_0^{T(x_0)}\Psi(T(x_0)-t)u_e(t)dt\ -\varepsilon>\int_0^{T(x_0)}\Psi(T(x_0)-t)u_{p_{\tilde{u}_e}}(t)dt \tag{2.3.10}$$

Since for all $u_e(t)\in\mathscr{E}$, we have by definition

$$\int_0^{T(x_0)}\Gamma_E(\lambda\Psi(T(x_0)-t))dt\geq\int_0^{T(x_0)}\Psi(T(x_0)-t)u_e(t)dt \tag{2.3.11}$$

rearranging equation (2.3.10) after substituting (2.3.11) in (2.3.10), we get

$$-\varepsilon>\lambda\Phi(T(x_0))x_0+\int_0^{T(x_0)}\lambda\Psi(T(x_0)-t)u_{p_{\tilde{u}_e}}(t)dt-\int_0^{T(x_0)}\Gamma_E(\lambda\Psi(T(x_0)-t))dt \tag{2.3.12}$$

which contradicts equation (2.3.6) and this completes the proof. □

The above proof uses the methods described in [Ant63, Sak70]. We thus showed that the pursuer control drives the system to origin in finite-time against all disturbances.

2.4. General systems

We saw how feedback strategies were derived for single and double integrator systems. We shall now extend the result to systems with rational, distinct and nonzero eigenvalues. This sort of peculiar extension is due to the fact that the switching surface computation has been known to be possible only for such systems [PC13]. For a quick review of the computation of switching surfaces and the value of the game refer to [MBPC14]. Detailed discussion is presented in [PC13]. Bryson and Ho [Bry75] develop necessary conditions for the two player differential game that parallel those derived for the optimal control problem.

Using difference system dynamics given by equation (2.1.3), we form the Hamiltonian H given by $H = \lambda^T(Ax + B(u_p - u_e)) + 1$.

Theorem 2.5. *[Bry75] The necessary conditions on u_p and u_e for the stationarity of $T(u_p, u_e)$ are given by*

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -A^T\lambda \qquad \lambda(0) = \lambda_0 \tag{2.4.1}$$

$$H^* = \min_{u_p}\max_{u_e}(\lambda^T(Ax + B(u_p - u_e)) + 1) \tag{2.4.2}$$

The equation (2.4.1) describes the co-state dynamics and its solution is $\lambda(t) = e^{-A^T t}\lambda_0$. Thus, the optimal inputs u_e^* and u_p^* for pursuer and evader satisfying the above necessary conditions are given by

$$u_e^*(t) = \arg\max_{u_e} H(u_p, u_e) = -\alpha sign(\lambda_0^T e^{-At} B) \tag{2.4.3}$$

$$u_p^*(t) = \arg\min_{u_p} H(u_p, u_e^*) = -\beta sign(\lambda_0^T e^{-At} B) \tag{2.4.4}$$

From (2.4.3) and (2.4.4), it is inferred that under the maximizing strategy (2.4.3) played by the evader, the pursuer should play (2.4.4) to minimize the capture time.

Using similar arguments as in the min-max case, the optimal inputs for pursuer and evader respectively for the max-min situation are given by

$$u_e^*(t) = -\alpha sign(l_0^T e^{-At} B) \tag{2.4.5}$$

$$u_p^*(t) = -\beta sign(l_0^T e^{-At} B) \tag{2.4.6}$$

where l_0^T is the corresponding multiplier. From (2.4.5) and (2.4.6), it is inferred that under the minimizing strategy (2.4.6) played by the pursuer, the evader should play (2.4.5) to maximize the capture time.

Theorem 2.6. *Let T_{min} be Value of the game. The following holds :*

$$T_{min} = \max_{|u_e(t)| \leq \alpha} \min_{|u_p(t)| \leq \beta} T(u_p, u_e) = \min_{|u_p(t)| \leq \beta} \max_{|u_e(t)| \leq \alpha} T(u_p, u_e) \tag{2.4.7}$$

Proof. It is clear that from equations (2.4.3),(2.4.4) and (2.4.5),(2.4.6) the pursuer and evader strategies have same switching function (for both min-max and max-min objectives). Therefore, we conclude that when both evader and pursuer are playing optimal strategies, the input to (2.1.3) is constrained by $|u_{ep}| = |u_p - u_e| \leq \beta - \alpha$. Hence, the order in which players initiate the game doesn't matter and as a result (2.4.7) follows. □

Here the equality of the Hamiltonians (min-max and max-min) imply equality of the value function (min-max and max-min). It should however be noted that this is not true in general. The implication holds because capturability condition ($\beta > \alpha$) and the fact that the function $Ax + B(u_p - u_e)$ is a Lipschitz function for stable systems. Refer to [BS89] for details.

2.5. Application of feedback strategies

For those of us who are not always hooked to watching channels like AXN or HBO, occasionally will get witness on DISCOVERY or NATIONAL GEOGRAPHIC situations such ducklings *following* their mother to learn to survive - what foods not to eat, safety from predictors, to fly and to eventually migrate. Also, a group of ants *following* a "dude" ant who just happened to have stumbled upon a source of food. Those of us who watch ESPN would have at one point or another seen those red-flagged Formula One races, where race is suspended because of an accident or poor track conditions and red flags are shown around the circuit. All cars should *follow* the safety car without pitting or overtaking and the safety car will lead the field for one lap before pulling into the pits. These situations will be termed as *consensus tracking*.

We shall analyze such situations mathematically and make use of the feedback strategies we derived in the following sections. To do this, let us model the situations using agents (ants, ducklings or the race drivers) communicating over a graph and tracking the root (momma duck, dude ant or the safety car). For the sake of simplicity, let us assume we have a chain of $n-1$ agents communicating over a rooted directed spanning tree. A rooted directed spanning tree is defined as a graph which connects, without any cycle, all nodes with $n-1$ arcs, i.e., each node, except the root, has one and only one incoming arc. The derived *min-max* strategies for the pursuer are used in the following way :

1. The root is following a reference trajectory and its input is less than the rest in the chain.
2. Each agent (other than root) has limited information (directed tree), the information from the agent in the chain it is trying to track and has the same capability.
3. Each agent (other than root) employs the min-max strategy, assuming that the leading agent is trying to evade it.

As seen in the previous sections, the pursuer can catch an active evader in time T_{min} by employing the min-max strategy. If by any stroke of luck the evader is not actively participating, like in the model we have, the time $T < T_{min}$. So all the agents in the chain reach their respective leaders in minimum possible time. Complete mathematical treatment and simulation results are provided in [MBPC14].

3. Multi-player PE games

3.1. Overview

Multi-player as the name suggests, involves more than two players. There can be more than one pursuer and more than one evader. Current differential game theory is in some sense inadequate to handle such scenarios. The accepted methods for solving differential games are closely related to optimal control theory, which includes Dynamic Programming (HJI) and the Maximum Principle. However, in a multi-player PE game, the techniques encounter tremendous difficulty both in the problem formulation and in specifying terminal states. Thus, we give a suboptimal solution for a multi-player PE differential game by making use of Voronoi diagrams, the notion of attainability sets and the law of Parallel Navigation.

3.2. Preliminaries

We provide a quick review of some of the required concepts and swiftly move on to the main results.

3.2.1. Voronoi diagrams

We consider only Voronoi diagrams in a plane ($\mathbb{R}^2$). The Voronoi diagram of a set of locations is a division of the plane into regions such that each region corresponds to one of the location, and all of the points in one region are closer to the corresponding location than to any other location. Where there is not one closest point, there is a boundary. They have a plethora of applications. To name a few - used in the field of robotics for creating a protocol for avoiding detected obstacles, in the study of plant competition and to analyze the patterns of urban settlements. Let us formalize the definition to suit our needs.

Let $P = \{P_1, P_2, P_3, ..P_n\}$ be the set of n pursuers who are attempting to the capture the evader denoted by E. Let $p_i = (p_{i1}, p_{i2}) \in \mathbb{R}^2$ denote the location of the i^{th} pursuer with respect to the standard co-ordinate axes. Let $e = (e_1, e_2) \in \mathbb{R}^2$ denote the position of the evader. We define the *Voronoi region of the evader* as :

$$V(e) = \{x \in \mathbb{R}^2 : ||x - e|| \leq ||x - p_i||, \forall i \leq n\}$$

where $||.||$ denotes the usual Euclidean distance. It is very clear from the definition that Voronoi region of the evader and one pursuer is a half-space containing the location of the evader. Since there are finitely many pursuers, the Voroni region of the evader is the intersection of finitely many half-spaces and hence it is *convex*.

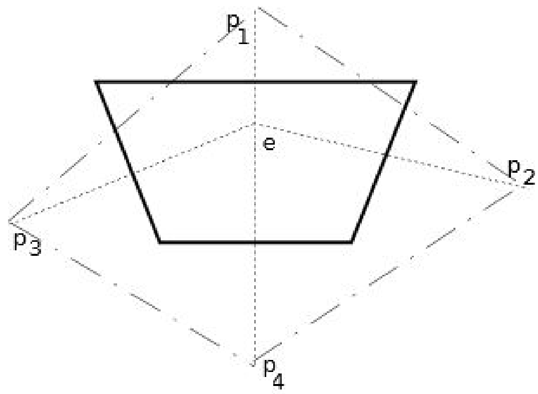

Figure 3.2.1.: Voronoi region

3.2.2. Parallel Navigation (PN) law

The parallel navigation law is a geometrical rule where the direction of the line-of-sight (LOS) is kept constant relative to inertial space, i.e., the LOS is kept *parallel* to the initial

LOS. It has been employed mostly by mariners, hence the term ‘navigation’. It has a very simple implementation and requires a single measurement, LOS rate. Moreover, it is a closed-loop control law. Consider the figure shown.

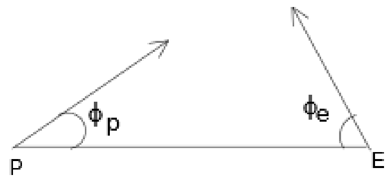

Figure 3.2.2.: PN law

If $\phi_e \leq 90^0$, then we say that there is a possibility of capture. If $\phi_e > 90^0$, then we say that there is no possibility of capture. The law (for each pursuer) is as follows :

1. If $\phi_e \leq 90^0$, then choose $\phi_p = \phi_e$.
2. If $\phi_e > 90^0$, then choose $\phi_p = 180 - \phi_e$.

Let us define a few terminologies which will be used later.

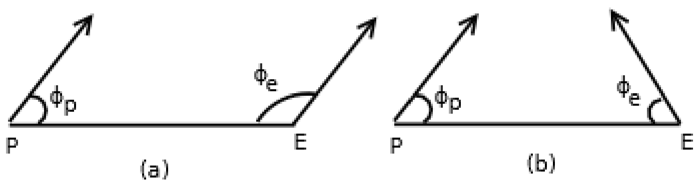

Figure 3.2.3.: Motions of the evader

Definition 3.1. An *away motion* for ‘a’ pursuer is the motion of the evader such that there is no possiblity of capture when the pursuer is using PN law. This is shown in figure (a).

Definition 3.2. A *toward motion* for ‘a’ pursuer is the motion of the evader such that there is a possiblity of capture when the pursuer is using PN law. This is shown in figure (b).

3.2.3. Attainability Set

Let $x(t)$ denote the state of the system (3.2.1) with input $u(t)$.

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.2.1}$$

The attainability set is the set of all possible values that solutions of the system (3.2.1) can asssume using all admissible controls. To state it formally,

$$\mathscr{A}(t) = \{x(t;u) \,:\, u\, is\, measurable\,, u \in U\; for\; \tau \in [0,t]\}$$

where U is any compact set.

3.3. Model

The following model is used for the pursuers and evader.

$$\begin{aligned} \dot{x}_1(t) &= u_1(t) \\ \dot{x}_2(t) &= u_2(t) \end{aligned} \tag{3.3.1}$$

where $u_1^2 + u_2^2 = r^2$. The speed of the players is the same and is r. They effectively control the angle.

The following assumptions are made for the multi-player game :

1. Each pursuer knows the knows the instantaneous state of the evader.
2. Also, each pursuer knows the velocity of the evader.
3. Evader knows the position and velocity of all the pursuers.

The game terminates when *atleast* one pursuer intercepts the evader.

3.4. Pursuit Strategy

Pursuit strategy is the strategy profile of all the participating pursuers. In this work, we choose to use PN law for all the pursuers. This strategy is local in the sense that each pursuer doesn't require any information about the state of other pursuers. The choice of the pursuit strategy is shown to be reasonable i.e, it assures finite-time capture of the evader for all possible choice of its actions and it is nearly time-optimal if the evader is moving along straight lines.

Theorem 3.3. *Evader lies in the interior of the convex hull of three pursuers iff the Voronoi region of the evader is bounded.*

Proof. ($\Longrightarrow$) Let evader lie in the interior of the convex hull. Without loss of generality we can consider a situation as shown in the figure.

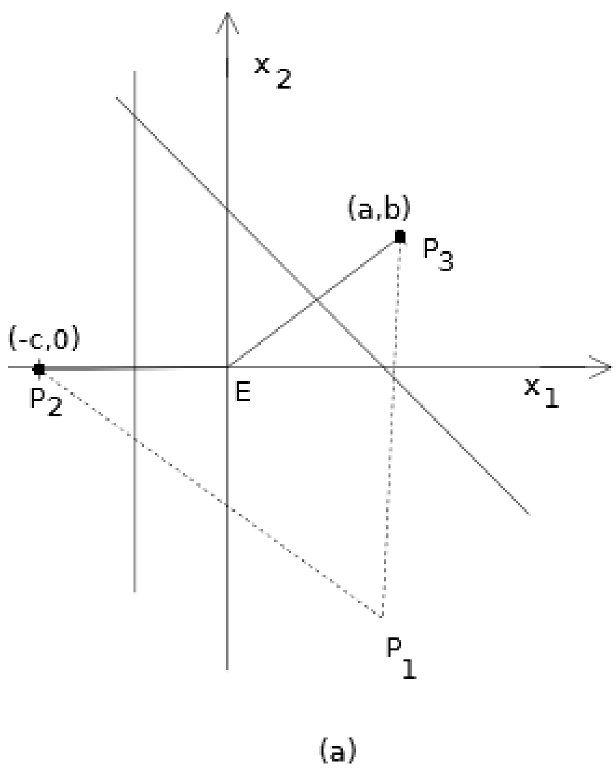

Figure 3.4.1.: Bounded Region

Let $\overline{EP_1}$, $\overline{EP_2}$ and $\overline{EP_3}$ denote the three line segments as shown. We know that Voronoi region of the evader is an intersection of halfspaces. Here the equation of the hyperplanes are the equations of perpendicular bisectors of the two line segments. Consider the two line segments $\overline{EP_2}$ and $\overline{EP_3}$. The equations of their perpendicular bisectors are : $x_1 = \frac{-c}{2}$ and $x_2 = \frac{-a}{b}x_1 + (\frac{b^2-a^2}{2b})$. The perpendicular bisectors always intersect at some point (circumcentre of $\triangle P_2P_3E$) "*outside*" the quadrilateral formed by the sides EP_2, EP_3, P_2P_1 and P_3P_1. Similarly, the perpendicular bisectors obtained from pursuers P_2,P_1 and the evader intersect at a point *outside* the quadrilateral EP_2, EP_1, P_2P_3 and P_3P_1. And, the perpendicular bisectors obtained from pursuers P_3,P_1 and the evader intersect at a point *outside* the quadrilateral EP_3, EP_1, P_2P_1 and P_3P_2. Hence, the Voronoi region of the evader, $V(e) = \{x \in \mathbb{R}^2 \,:\, ||x-e|| \leq ||x-p_i||, \, for\, i = 1,2,3\}$, forms a triangle and therefore is bounded - the intersection points being the extreme points.

($\Longleftarrow$) Let the Voronoi region of the evader be bounded. By definition e is inside this region. Let l_1, l_2 and l_3 be the three sides of the triangle (Voronoi region). Now, sides l_1, l_2 and l_3 are the perpendicular bisectors and therefore each of p_1, p_2 and $p_3$is a reflection of the point e about l_1, l_2 and l_3 respectively. Hence e must lie inside the convex hull (polygon) formed by p_1, p_2 and p_3. □

Let $e(0)$, $p_1(0)$, $p_2(0)$ and $p_3(0)$ denote the inital positions of the evader and three pursuers respectively in $\mathbb{R}^2$. Let $Hull^0(a,b,c)$ denote the interior of the convex hull formed by the points (a,b,c). Let $w_1(0)$, $w_2(0)$ and $w_3(0)$ be the co-ordinates of the Voronoi region of the evader. Let $d_{T_{min}} = \max||w_i(0) - e(0)||$ where $i = 1,2,3$.

Conjecture 3.4. *Let $e(0) \in Hull^0(p_1(0), p_2(0), p_3(0))$. Let $\widetilde{t} = d_{T_{min}}$. Then the following holds :*

$$\mathscr{A}_e(\tilde{t}) \subset \mathscr{A}_{p_1}(\tilde{t}) \cup \mathscr{A}_{p_2}(\tilde{t}) \cup \mathscr{A}_{p_3}(\tilde{t})$$

The following corollaries are a direct consequence of 3.4.

Corollary 3.5. *The minimum number of pursuers required to capture the evader, when all of them have the same capability is 3.*

Corollary 3.6. *There exists a strategy for the pursuers so that no matter how the evader responds, it is captured in a time which is atmost $d_{T_{min}}$.*

We show that the parallel navigation strategy when employed by all pursuers is a sub-optimal strategy i.e, it ensures capture for all possible actions of the evader in finite time. Consider the Voronoi region of the evader as shown in figure

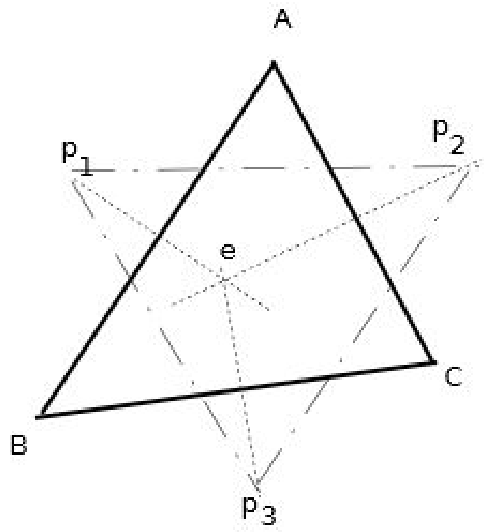

Figure 3.4.2.: Voronoi region of evader

Let us refer to the Voronoi region by ΔABC.

Proposition 3.7. *Let $e(0) \in Hull^0(p_1(0), p_2(0), p_3(0))$. The interior angles of the ΔABC remains constant if pursuers employ PN law.*

Proof. Given that the evader is initially inside ΔABC, any motion by it will be a 'toward motion' for atleast one of the three pursuers. Their Voronoi edge (perpendicular bisector) remains the same. For the other pursuers, it will be an 'away motion'. This motion by the evader shifts the Voronoi edges, with respect to these pursuers, inside the ΔABC i.e, towards the evader. Since the Voronoi edges move in a parallel fashion, the angles are corresponding angles and hence are equal. □

Theorem 3.8. *Let $e(0) \in Hull^0(p_1(0), p_2(0), p_3(0))$ and let the pursuers employ the parallel navigation strategy. The area of the Voronoi region of the evader decreases monotonically to zero in finite time.*

Proof. Let the pursuers employ the parallel navigation strategy. By proposition 3.7, we have that the interior angles remain constant. Let $l_{min}(t)$ be the length of the smallest side of ΔABC.The area of the triangle is given by :

$$
\begin{aligned}
A(t) &= \frac{l_{min}^2(t)\, sin\gamma sin\delta}{2\, sin(\gamma+\delta)} \\
\therefore l_{min}^2(t) &= \frac{2\, sin(\gamma+\delta)A(t)}{sin\gamma sin\delta}
\end{aligned}
$$

Now, since the angles δ and γ remain constant, we have $l_{min}(t) = q\sqrt{A(t)}$. Here $q = \sqrt{\frac{2sin(\gamma+\delta)}{sin\gamma sin\delta}}$. Now, as the Voronoi edge shifts inside as argued in proposition 3.7, trapezoids are removed from the Voronoi region as shown in the figure.

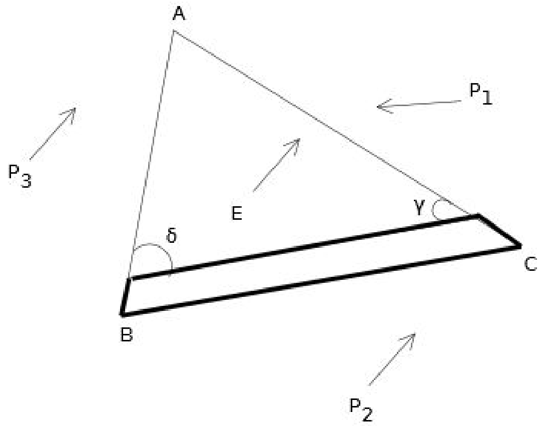

Figure 3.4.3.: Trapezoid

The area of the smallest possible trapezoid is easily approximated as $l_{min}(t)k(\Delta t)$, where k is a constant and is a proportion of speed (refer to Appendix for a calculation of k). Clearly, we have the following relations :

$$
\begin{aligned}
\frac{-dA(t)}{dt} &\geq l_{min}(t)k \\
\frac{-dA(t)}{dt} &\geq qk\sqrt{A(t)} \qquad (3.4.1) \\
\frac{-dA}{\sqrt{A}} &\geq kdt \qquad (3.4.2)
\end{aligned}
$$

Integrating equation 3.4.1 on both sides, we have

$$
\sqrt{A(t)} \leq \sqrt{A(0)} - \frac{qkt}{2} \qquad (3.4.3)
$$

Since $A(t) \geq 0$, we have $\sqrt{A(t)} \geq 0$. Equation 3.4.3 suggests that the area becomes equal to zero in a finite time $t' = \frac{2\sqrt{A(0)}}{qk}$. $\square$

Theorem 3.9. *Let the pursuers employ PN strategy. Evader can be captured by the three pursuers iff* $e(0) \in Hull^0(p_1(0), p_2(0), p_3(0))$.

Proof. If the evader intially lies inside the convex hull of the pursuers, we have by Theorem 3.8 the Voronoi region decreases in finite time monotonically and hence the evader is captured. On the other hand, if the evader initially lies outside the convex hull, then by Separating Hyperplane theorem we have a hyperplane H which separates the evader from the convex hull formed by the pursuers i.e, H divides $\mathbb{R}^2$ into two regions - one containg the pursuers and the other containing the evader (denoted by $\mathscr{R}_e$). Now, consider the region which the intersection of $\mathscr{R}_e$and the unbounded Voronoi region of the evader. This region by definition is the set of all points the evader can move into before the pursuer. Evader simply moves in a direction perpendicular to H in this region and escapes. $\square$

4. Conclusion and Future Work

We began this thesis by giving a quick overview of pursuit evasion games. In Chapter 2, we developed strategies for pursuer and evader dynamics described by single integrators and then obtained similar results for the double integrator dynamics case. We then extended the method for systems with rational, non-zero and distinct eigenvalues. We demonstrated the simplicity and usefulness of these strategies by applying them to obtain consensus tracking of agents over a directed tree.

In Chapter 3, the problem of multi-pursuer and single evader for the case of equal capabilities was considered. It was shown that one or two pursuers was not sufficient for the capture in this setting and necessary and sufficient conditions conditions for capture were derived using elementary geometry and algebra.

The sub-optimality of the PN law is evident by the absence of the time-optimality. Finding a closed loop feedback time-optimal strategy for the pursuers and finding an 'intelligent' evader strategy may be considered for future work.

Conjecture 3.4 is verified to be true using simulation and paper work. Owing to its importance, as noted by the corollaries, some time and energy may be expended in this regard.

The multi-player game is solved only for the case where all the participants are having coupled single integrator dynamics. An extension to at least double integrator case may be helpful.

A. Appendix

A.1. Calculation of the proportion of speed k

Since the Voronoi region is bounded, the evader can move 'away' from atmost two pursuers. Consider the figure shown. The smallest trapezoidal area chipped off from the Voronoi region should have the smallest edge of the Voronoi region as one of the shifting egdes.

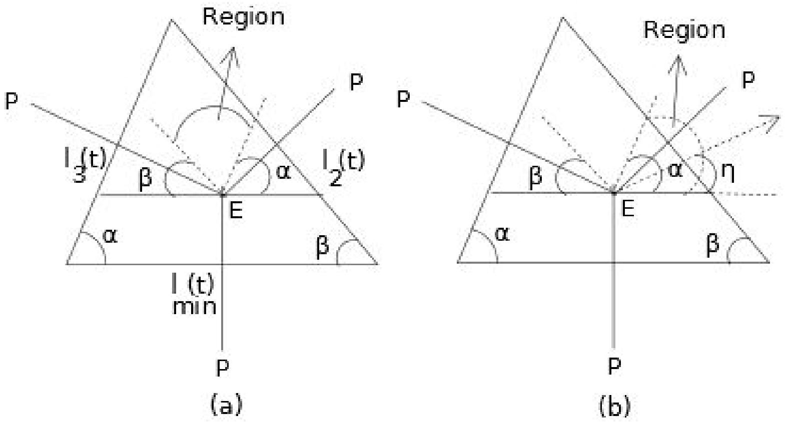

Figure A.1.1.: Region of consideration

For calculating k, we consider the following cases. It is clear that $\alpha, \beta \neq 0$.

- **Only the smallest edge is shifting :**

For this to happen, it is clear that the evader should move in the region depicted in the figure (a). Let $\theta = min\{\alpha, \beta\}$. A possible proportional constant is $\gamma_1 = sin\theta$.

- **The smallest egde and any other edge say $l_2(t)$ is shifting :**

  For this to happen, it is clear that the evader should move in the region as shown in the figure (b)

$$\begin{aligned} f(\eta) &= rl_{min}(t)sin(\alpha-\eta)+rl_1(t)sin\eta \\ f(\eta) &\geq rl_{min}(t)sin(\alpha-\eta)+rl_{min}(t)sin\eta \end{aligned} \tag{A.1.1}$$

  Differentiating the right hand side of (A.1.1) w.r.t η, we have

$$0 = -rl_{min}(t)cos(\alpha-\eta)+rl_{min}(t)cos\eta$$

  After straightforward manipulation we get,

$$\hat{\eta} = tan^{-1}(\frac{1-cos\alpha}{sin\alpha})$$

  A possible proportional constant is $\gamma_2 = sin(\alpha-\hat{\eta})+sin\hat{\eta}$.

- **The smallest egde and the edge $l_3(t)$ is shifting :**

  After similar calculations with respect to the other pursuer as above we get,

$$\hat{\zeta} = tan^{-1}(\frac{1-cos\beta}{sin\beta})$$

  A possible proportional constant is $\gamma_3 = sin(\beta-\hat{\zeta})+sin\hat{\zeta}$.

The proportion of speed is now selected as $k = min\{\gamma_1,\gamma_2,\gamma_3\}$.

A.2. Matlab Codes

We provide all the matlab codes used in the verification of the theoretical results. The simulation was carried out on Matlab-Simulink and the following snippets of code were used as inputs to drive the dynamical systems.

```
%% Code for the double integrator system dynamics
function u =doub_int(x,y)
x=x-y;
% For double integrator
 if x(2)>0
   s=-(2*x(1)+x(2)^2);   % Switching surface
 else
   s=-(2*x(1)-x(2)^2);
 end
%% Input or strategy
u=sign(s);
```

```
%% PN law for pursuers
%% p1 - current pursuer state
%% e - current evader state
function u_p = pursuer_ip(p1,e,u_e)
delta=0.01;                      %% time step
e_next = e + delta * u_e;        %% next state of the evader
w = e_next - e;                  %% Vector joining current state and next state of evader
q_p1 = p1 - e;                   %% Vector joining current state of the evader and the current state of the pursuer
p1_de = vec_an(w,q_p1);
if (p1_de > 90)
u_p= u_e;
else
u_p = rot_pur(p1,e,u_e);
end
```

```
%% Rotation of the pursuer angle
function [y] = rot_pur(p0,e0,u);
a=(p0-e0)/norm(p0-e0); b=orth(a); y=(eye(2)-2*a*a')*u;
end
```

```
%% Evader is heading to the farthest Voronoi vertex at every
   instant, a reasonable heuristic
function[u_e] = vor(e_0,p_1,p_2,p_3);
x = [e_0';p_1';p_2';p_3'];
y = e_0';
[V,C] = voronoin(x);
[n,m] = size(V);
dist_i = 0;
for k = 2 : n
temp = V(k,:);
dist = norm((temp-y),2);
if (dist_i < dist)
dist_i = dist;
e_dest = temp;
end
end
e_1 = e_dest';
temp1 = e_1 - e_0;
temp2 = temp1/(norm(temp1,2));
dir_e = theta_fn(temp2);
u_e = [cosd(dir_e); sind(dir_e)];
```

```
 %% Function to calculate theta
function [y] = theta_fn(w);
thet = abs(w(2)/w(1));
if w(1) < 0 && w(2) > 0
y = 180 - atand(thet);
elseif w(1) > 0 && w(2) < 0
y = 270 + atand(thet);
elseif w(1) < 0 && w(2) < 0
y = 180 + atand(thet);
else
y = atand(thet);
end
```

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