Algebra and Pre-Calculus

Day 2 AM

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Day 2 Agenda

- Functions and Relations
- Describing Functions and their Behavior
- Limits and Functional Continuity

Relations

Relations generally allow comparison of variables and expressions between a *range (y)* and *domain (x)*.

You may be familiar with the term functions, which are a type of relation. These are the most common relation you will encounter in statistical social science.

Correspondences are a type of relation more likely found in game theory.

Functions

- Functions assign exactly *one* element of the *range (y)* to each element of the *domain (x)*.

Correspondences

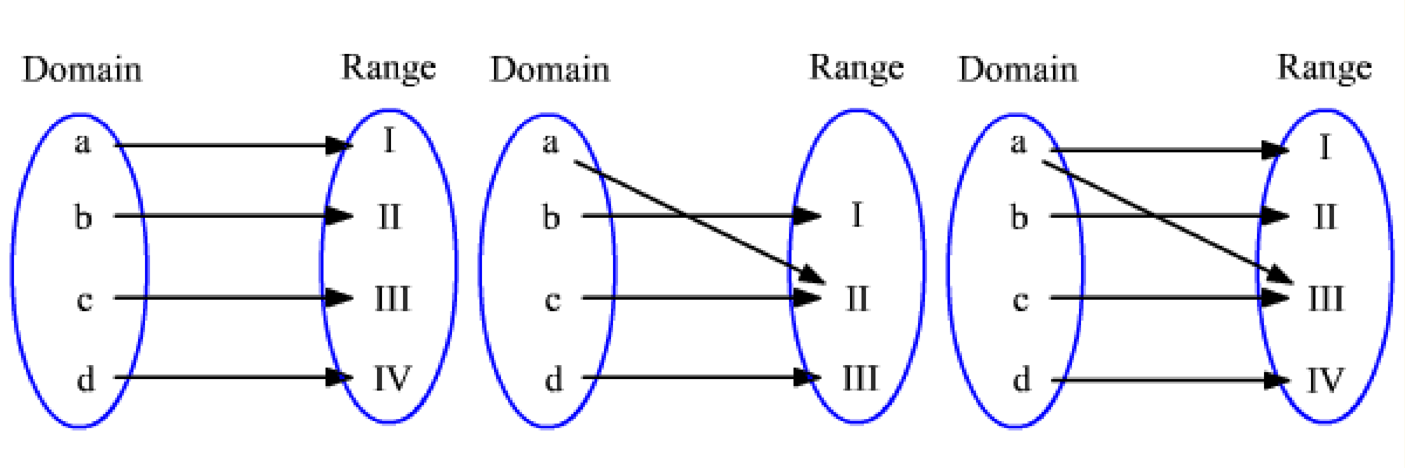
- Correspondences are relations which consist of a subset of two sets, *(a)* and *(b)*. This is a way of associating true/false with combinations of the two sets.

Relating Sets to Functions

Yesterday we went over the basics of sets and set operations. How can we relate that information to functions?

- Domain: "the set of elements over which the function is defined"
- Co-Domain: "the set from which values of f(x) may be drawn"
- Range: "the set of all values actually reached", a subset of the co-domain

Determining of something is a function



1) One-to-one, is a function

2) Many-to-one, is a function

3) One-to-many, is not a function

https://www.varsitytutors.com/hotmath/hotmath_help/topics/mapping-diagrams

Is it a function?

Vertical Line Test

If any vertical line will pass through more than 1 point on the graph of a relation, then the relation is not a function.

Vertical Line Test

Which of the following graphs is not of a function?

Types of Functions

1. Identity function: Elements in domain are mapped to identical elements in codomain.

2. Inverse function:Function that when composed with original function returns identity function.

3. Surjective: Every value in co-domain produced by value in domain

4. Injective: Each value in range comes from only one value in domain (one-to-one)

5. Bijective Both surjective and injective; function has an inverse (invertible)

Monotonic Function Terms

Term	Meaning
Increasing	Function increases on subset of domain
Decreasing	Function decreases on subset of domain
Strictly increasing	Function always increases on subset of domain
Strictly decreasing	Function always decreases on subset of domain
Weakly increasing	Function does not decrease on subset of domain
Weakly decreasing	Function does not increase on subset of domain
(Strict) monotonicity	Order preservation; function (strictly) increasing or decreasing over domain

Table 2.2 from Moore and Siegel (p. 151)

Why is understanding functional behavior important?

Obviously useful to understand a function's behavior mathematically-- i.e. to understand how the input relates to the output over the function.

BUT this is where we link some of the logic of math to theory construction. Ultimately you will choose what mathematical functions likely describe the relationship between x and y.

Therefore, we need to know how to interpret and model the functional relationship between concepts of theoretical interest.

Linearity

- *Linear functions* are a general category of functions that follow the rules of additivity and scaling:
  - Additivity: $f(x_1 + x_2) = f(x_1) + f(x_2)$
  - Scaling: $f(ax) = af(x)$ for all a

Nonlinear Functions

- A function is nonlinear if it does not meet the properties of scalability or additivity.

Why does this matter?

- What if we have theoretical inclination to believe that the relationship is *nonlinear*-- what if there is a ceiling effect to our x or diminishing returns?

- We need to be able to eventually account for these issues in our models, therefore, we must know how to model them mathematically.

Exponents

Exponents are where you take a variable to some power -- e.g. x^a where x is a variable and a is a constant. Typically, we focus on the numerical portion of the exponent--calling it `the exponent'.

Exponential

An exponential is the reverse of the exponent -- here it is a number to the power of a variable, e.g. a^x. To get the x `down', we need to use logarithms (aka logs).

The exponential function commonly has base, e, (where e is Euler's e and is approx 2.72.)

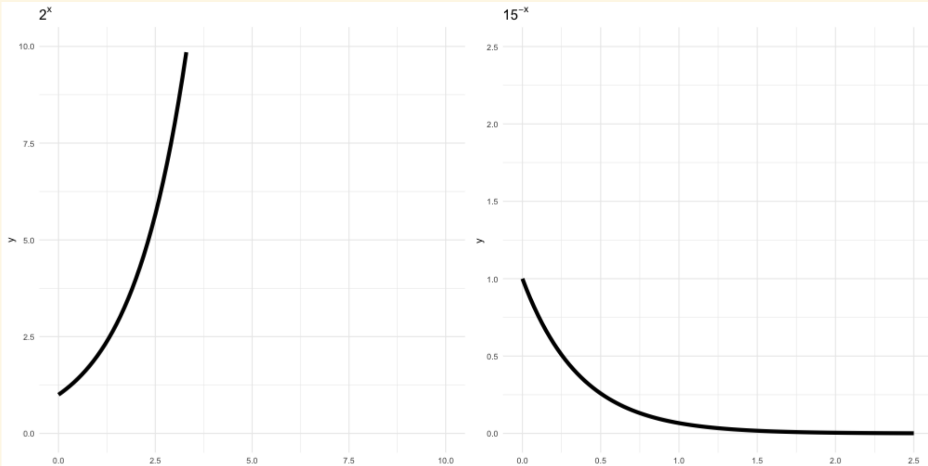
Properties of Exponential Functions

General Form: $y = a^x$

1) Differentiation between *exponential growth* versus *exponential decay* and their implications.

2) Domain: (- ∞, ∞)

3) Injective (one-to-one)

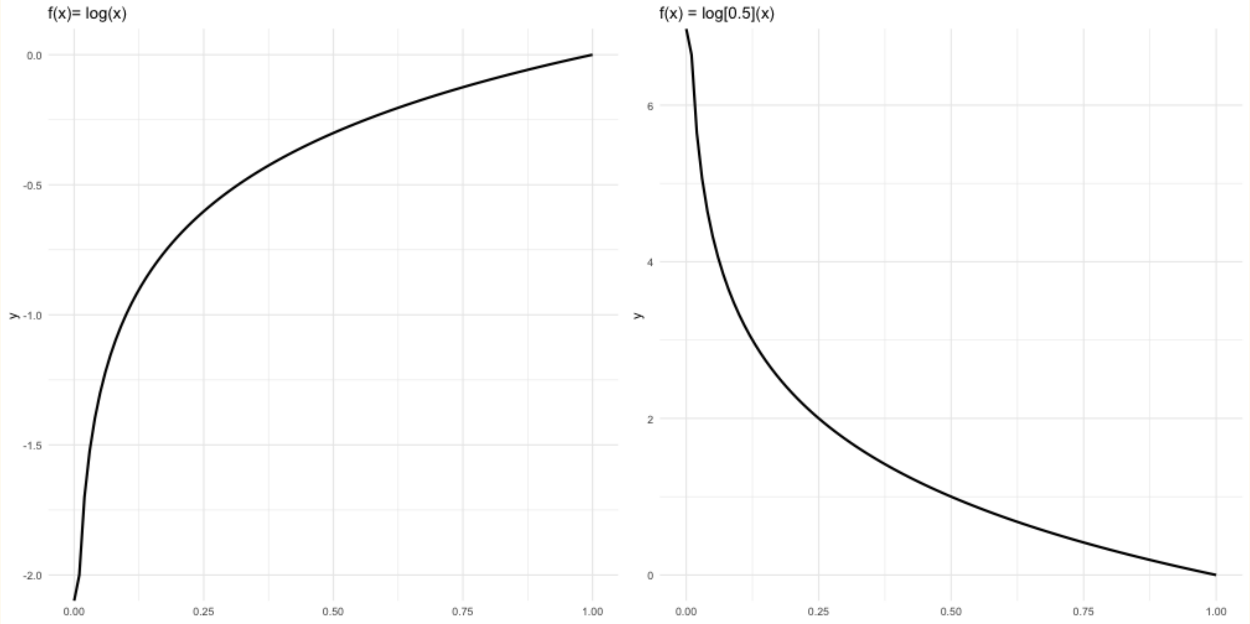

2^x
10.0
7.5
5.0
2.5
0.0
y
0.0
2.5
5.0
7.5
10.0
15^{-x}
2.5
2.0
1.5
1.0
0.5
0.0
y
0.0
0.5
1.0
1.5
2.0
2.5

Logarithmic Functions

General Form: $y = log_b(x)$

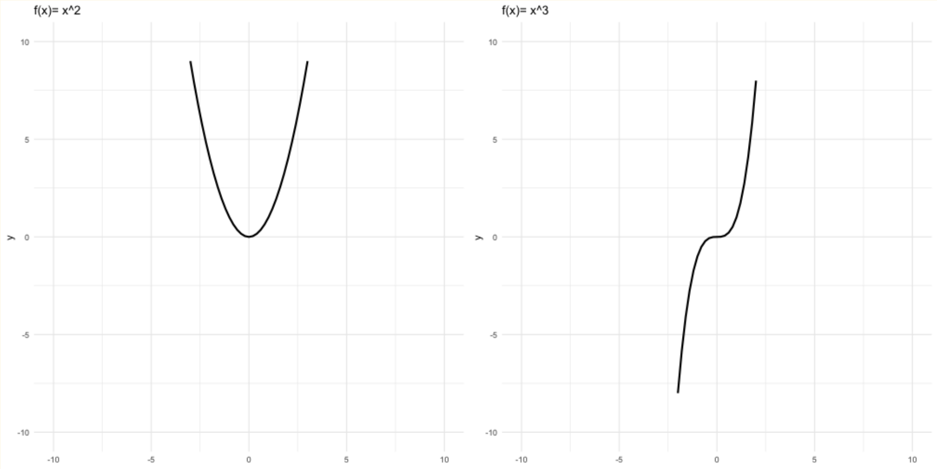
1) Domain is (0, ∞); Range is (-∞, ∞)

2) Increasing if $b > 1$; Decreasing if $0 < b < 1$



Higher Order Functions

- Higher order polynomials refer to functions whose degree is greater than 1, thus entail a greater degree than a linear polynomial.
- Quadratic polynomial: in the form $f(x) = ax^2 + bx + c$, wherein the highest power is 2
- Higher order polynomials: entail a degree higher than 2-- can be 3 or greater (unlikely, but could be there!)



Functional Continuity

- Some functions have certain points at which they *do not exist*.

Functional Continuity

$$y = \frac{1}{x}$$

Functional Continuity

$$y = log(x)$$

Limits

The sums, sequences, and functions that are not actually defined using basic math may nonetheless have *limits*.

Limits describe the behavior that a series, sum, or function approaches or converges toward as a certain quantity approaches some value that cannot be directly observed --- infinity, say, or a value where the function is undefined.

$$\boldsymbol{lim_{N \to \infty} \sum_{i=1}^{N} x_i = S}$$

Translation: The limit of the sum $\boldsymbol{x_i}$ from $\boldsymbol{i}$ to N as N approaches infinity is S.

We can talk in the same way about sequences $\boldsymbol{lim_{i \to \infty} x_i = L}$.

Limits

We are most concerned with the limits of functions.

$$lim_{i \to \infty} f(x) = L$$

We want to look at what happens to the value of a function as as the distance between points approaches zero. A function may or may not exist for a certain point x, we first determine if the limit does exist, and if it does what its value is.

Limits

For us, you should be able to:

- Plug a value in to the limit and see what you get out (check for dividing by zero).
- Recognize an indeterminate limit: $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Try:

$$lim_{x \to 4} \frac{x+2}{x+5}$$

$$lim_{x \to 4} \frac{x^2-4}{x+2}$$

Ans: 6/9, simplifies to 2/3; simplifies to x-2, limit is 2.

What is calculus?

- So basically, calculus helps understand *continuous change of a function*.

Well, what does that mean?

- On the one hand, we could be interested in *discrete* change. Discrete change tells us the difference between two points on a graph, thus the difference between two observations modeled by a function.
- Finding discrete change means finding the slope of two points, or the *secant*. You all are probably familiar with $\frac{rise}{run}$ from algebra.

What is calculus?

- But, finding the secant has a limitation. Discrete change only tells about the functional behavior *over an interval*. Instead we might want to find the rate of change at a *very specific moment* in the function.

- Consequently, we might also want to find extrema (min and max) of a function, any other critical points, or understand its shape. As you may remember from other courses, these are difficult tasks with merely a function $f(x)$.

- Calculus gives us some tools to calculate instantaneous change and other downstream quantities.

Capturing Change

- Today, we went over limits of functions. As a reminder, the limit of a function $f(x)$ at a given point x is the value of the function as it approaches the given x.
- Therefore, to capture discrete change we calculate the secant between two points by taking the difference of the functional limits, i.e. $\frac{f(x_2)-f(x_1)}{x_2-x_1}$.

Capturing Change

- To capture the instantaneous or continuous rate of change of a function, we *could* take the difference of the limits over iterations, making the interval between each x smaller and smaller.

- However, the concept of tangents, and consequentially derivatives, makes this easier, this is what we will go over tomorrow.

Differentiation

To differentiate a function, it must be *continuous* on the *interval*.

Some functions are not differentiable or not differentiable at a certain point.

A function is continuous if:

- Limit exists
- Limit from left equals limit from right
- These limits equal the value at the point