Calculus I: Derivatives

Day 3 AM

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Day 3 Agenda

- Derivatives: Concept, notation, and how-to
- Fundamental Derivative Rules
- Advanced Rules of Derivatives

Calculus

What is calculus?

> - So basically, calculus is the study of *instantaneous change of a function*.

Well, what does that mean?

> - On the one hand, we could be interested in *discrete* change. Discrete change tells us the difference between two points on a graph, thus the difference between two observations modeled by a function.
>
> - Finding discrete change means finding the slope of two points, or the *secant*. You all are probably familiar with —— from algebra.

What is calculus?

- But, finding the secant has a limitation. Discrete change only tells about the functional behavior *over an interval*. Instead we might want to find the rate of change at a *very specific moment* in the function.

- Consequently, we might also want to find extrema (min and max) of a function, any other critical points, or understand its shape. As you may remember from other courses, these are difficult tasks with merely a function .

- Calculus gives us some tools to calculate instantaneous change and related quantities.

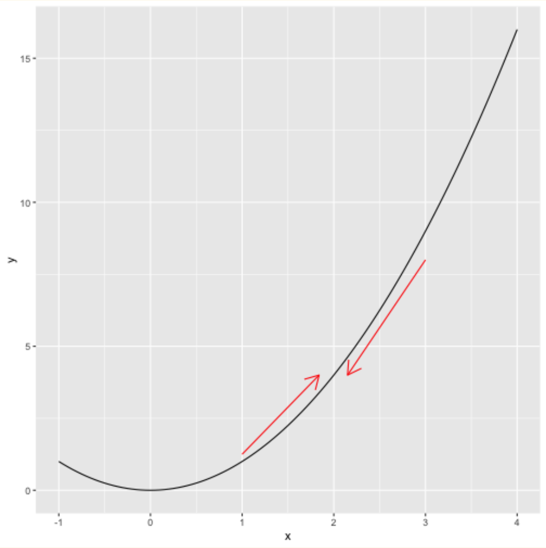
Capturing Change

- In our last session we went over limits of functions.
  - As a reminder, the limit of a function at point is the value of the function as it approaches .
- To capture discrete change we calculate the secant between two points by taking the difference of the functional limits, i.e. $\frac{\quad}{\quad}$.

Capturing Change

- To capture the instantaneous or continuous rate of change of a function, we *could* take the difference of the limits over iterations, making the interval between each smaller and smaller.

For example, we could take the difference of the limits of as approaches 2:



Tangents and Derivatives

- A secant is the slope of a line.
- A tangent is a line that touches the function at a given point. The tangent's slope tells us the instantaneous rate of change *at that particular point*.
- Therefore, if we find a tangent line for function at a specific point and determine its slope, we therefore have information about the instantaneous rate of change for the primary function.

Notation

- There are a couple of ways to notate derivatives, they mean the same thing.

  : Read: "f prime of x"

—: Read: "dy-dx" or "dy over dx"

- There also possibilities for higher order derivatives, i.e. derivatives of derivatives and so on…
- Higher order derivatives will use the same types of notation, with details to indicate the order of the derivative

> - or — for a second derivative; or — for a third derivative

Calculating a Derivative

To calculate the derivative, begin with the secant formula. Use this formula to reduce the difference to some arbitrarily small value, .

As goes to zero, we go from discrete to instantaneous change.

Secant Formula:

Example: Calculating the Derivative of

Secant Formula:

Example:

Example: Calculating the Derivative of

Secant Formula:

Example:

What does it all mean?

- Derivatives give us information regarding the rate of change of given function at a value . We can use this *information* to learn more about the primary function at that particular point.

- Think back to what the slope of a linear equation tells us from the formula .

- If is positive, the slope is increasing; if is negative, the slope is decreasing.

- The derivative gives us information that we interpret similarly, but oftentimes for more complex functions.

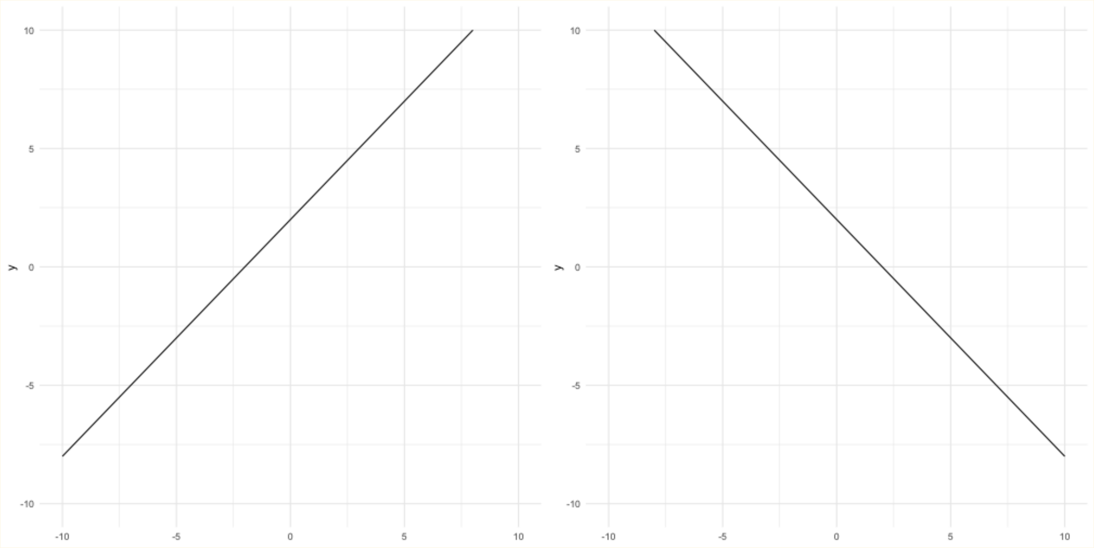
Derivative as information: Rate of change

- Positive Derivative: Function is increasing
- Negative Derivative: Function is decreasing
- Zeroes: Maxima or minima (extrema)

Example: Derivative as Information

Function A: $f(x) = x + 2$; $f'(x) = 1$

Function B: $f(x) = -x + 2$; $f'(x) = -1$



Example: Derivative as Information

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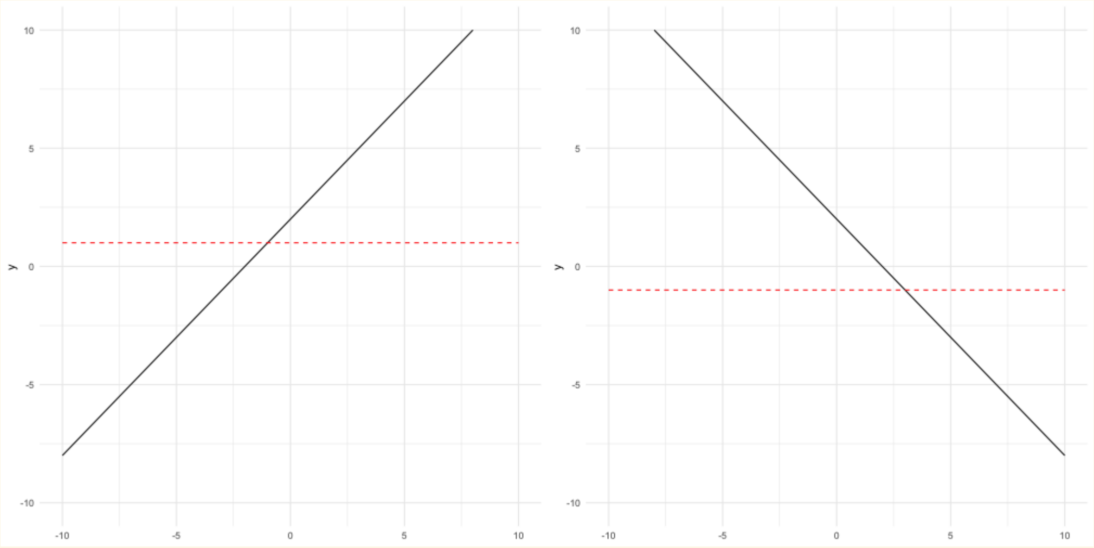
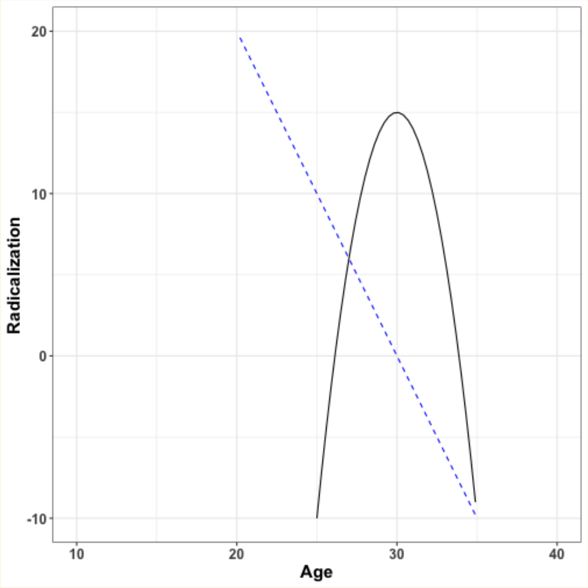


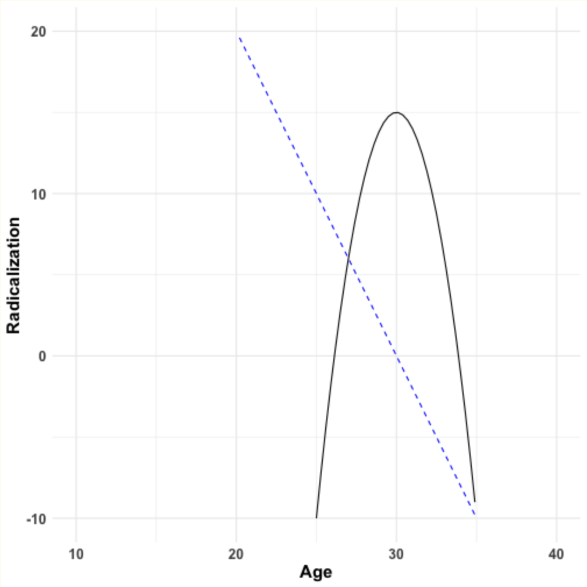
Example: Age and Radical Politics

Function A:

Derivative:



Example: Age and Radical Politics




```r
-2 * 25 + 60
```

```
## [1] 10
```

```r
-2 * 30 + 60
```

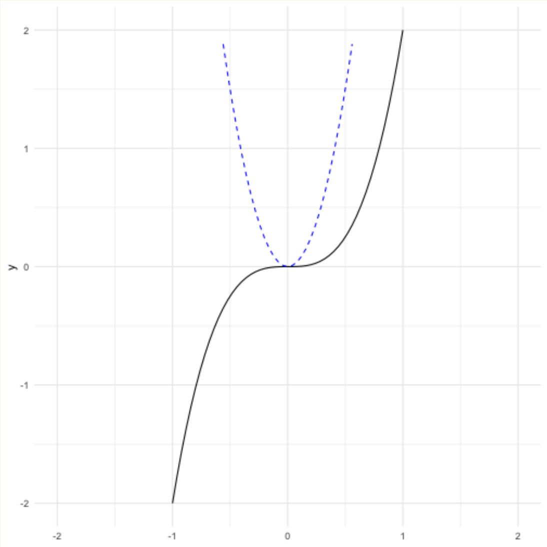
```
## [1] 0
```

```r
-2 * 35 + 60
```

```
## [1] -10
```

Derivatives and Extrema

Extrema: max or min of a function, i.e. where is the top-most or bottom-most value of the function?



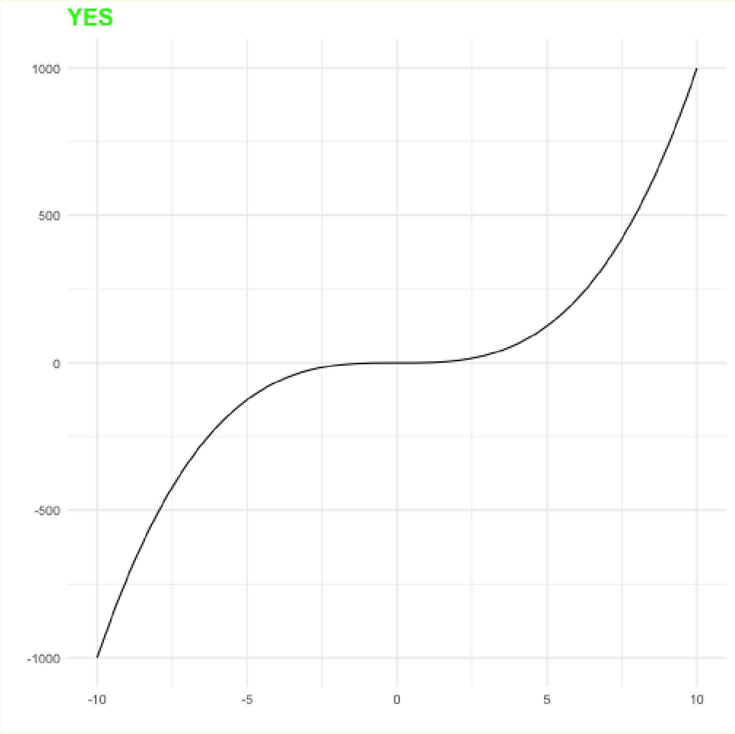
Functional Behavior

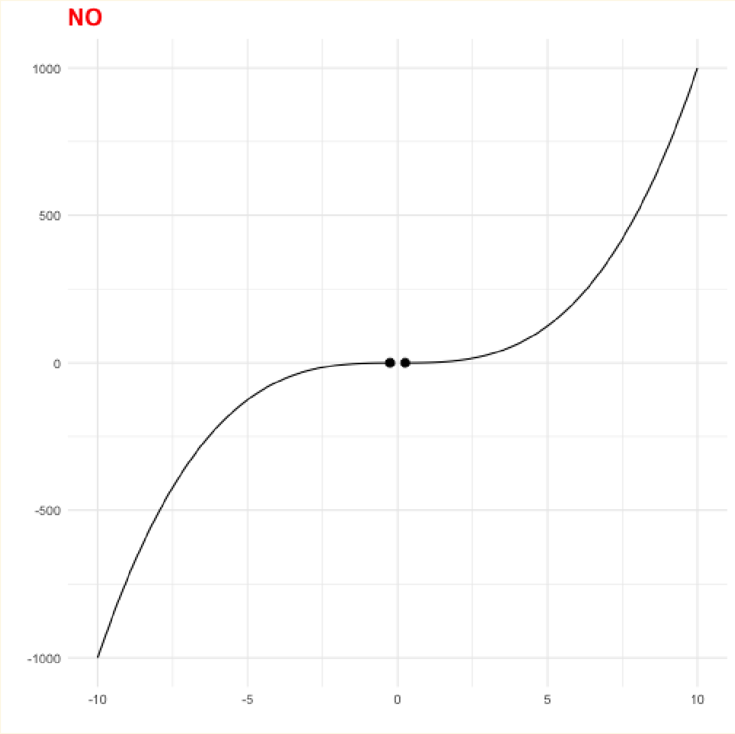
The function must be *continuous* on the *interval* to be differentiable.

Some functions are not differentiable *at all* or are not differentiable at *a certain point*. Need to determine the continuity of the function.

Continuity

Continuous function:





Derivative: Calculation

While we can calculate the derivative using the formula from before, it's a bit tedious. Can't there be another way?

Think through the work it would take to differentiate

using …

Not fun!

Derivative Rules

Take the derivative as below with constant :

-
- has derivative
-
-
-

NOTE: Ex:

Derivative Rules

Take the derivative as below with constant :

- ,
    -
- has derivative
    -

Derivative Rules

Take the derivative as below with constant :

-
    -
-
    -
-
    -

Derivatives Two Ways

Find the derivative of

Formal Definition:

Plug in our function :

Using the Simpler Rules

Relevant Rules:

1.
2.
3.
4.
5.
6.
7.

1.
2.
3.
4.
5. (not continuous at (s=0))
6.
7. (not continuous at (z=-1))

Higher Order Derivatives

Second derivatives (nth derivatives): take a derivative a second (nth) time

Rate of change of rate of change (e.g., velocity vs acceleration)

(Additional) Rules for derivatives

So far, we've just had simple functions but what if we are multiplying, dividing, or have an otherwise-more-advanced function?

- Product Rule:
- Quotient Rule: ——
- Chain Rule:
- Other: eg, exponentials: ,

(Additional) Rules for derivatives

So far, we've just had simple functions but what if we are multiplying, dividing, or have an otherwise-more-advanced function?

- Product Rule: ,
- Quotient Rule: —, —
- Chain Rule: , (composition!)
- Other: eg, exponentials: ,

Product Rule

If we have two things multiplied together and need the derivative, we have two options: multiply everything and then take the derivative OR use the product rule.

Product Rule

Example: (3x+4)(x+2)

- Identify constitutive functions: and .
  The derivatives are and , respectively.
- Substitute these in:
- Simplify to get:

Product Rule: A Motivating Example

and

and .

We can substitute this into the formula:

Quotient Rule

Example: .

Formula is

So, we identify the following: and ,
therefore and .

Plug in to get:

Chain Rule

Sometimes, you have a function to a power: .
We can use the chain rule to evaluate this.

Take the derivative of the function and multiply it by the derivative of the inside: .

So, for our example: and .

- The derivative of each is and .
- We substitute in to get: .

More Chain Rule

Try:

Exponentials: and

You can take the derivative of continuous functions -- including those with a log and/or in them. The rules are a little hard, but once you learn them, it's not too bad:

1.
2.
3.
4.
5.
6.

1. (a favorite of mine)
2.
3.
4.
5.
6.

You can make these more complicated by including a function of x. How would we take the derivative in that case?

Example:

Chain Rule!

$$ \frac{}{} \qquad \frac{}{}. $$

Partial Derivatives

Similar to a "regular" derivative; treat additional variable(s) as constants. Written as or —

Find

Derivatives in Review

There are a few more handy rules and techniques that are important, perhaps even on your homework:

Source on Stack Exchange

And if you really want to explore more, check out all these techniques.