

# Calculus I: Derivatives

Day 3 AM

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# Day 3 Agenda

- Derivatives: Concept, notation, and how-to
- Fundamental Derivative Rules
- Partial Derivatives
- Advanced Rules of Derivatives

# Calculus

# What is calculus?

- So basically, calculus is the study of *instantaneous change of a function*.

*Well, what does that mean?*

- On the one hand, we could be interested in *discrete* change. Discrete change tells us the difference between two points on a graph, thus the difference between two observations modeled by a function.
- Finding discrete change means finding the slope of two points, or the *secant*. You all are probably familiar with  $\frac{\text{rise}}{\text{run}}$  from algebra.

# What is calculus?

- But, finding the secant has a limitation. Discrete change only tells about the functional behavior over *an interval*. Instead we might want to find the rate of change at a *very specific moment* in the function.
- Consequently, we might also want to find extrema (min and max) of a function, any other critical points, or understand its shape. As you may remember from other courses, these are difficult tasks with merely a function  $f(x)$ .
- Calculus gives us some tools to calculate instantaneous change and related quantities.

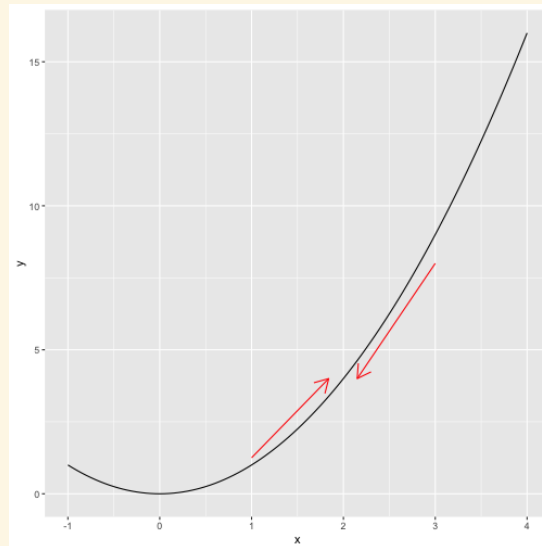
# Capturing Change

- In our last session we went over limits of functions.
  - As a reminder, the limit of a function  $f(x)$  at point  $x$  is the value of the function as it approaches  $x$ .
- To capture discrete change we calculate the secant between two points by taking the difference of the functional limits, i.e.  $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ .

# Capturing Change

- To capture the instantaneous or continuous rate of change of a function, we *could* take the difference of the limits over iterations, making the interval between each  $x$  smaller and smaller.

For example, we could take the difference of the limits of  $f(x) = x^2$  as  $x$  approaches 2:



# Tangents and Derivatives

- A **secant** is the slope of a line.
- A **tangent** is a line that touches the function at a given point. The tangent's slope tells us the instantaneous rate of change *at that particular point*.
- Therefore, if we find a tangent line for function at a specific point and determine its slope, we therefore have information about the instantaneous rate of change for the primary function.



# Notation

- There are a couple of ways to notate derivatives, they mean the same thing.

$f'(x)$ : Read: "f prime of x"

$\frac{dy}{dx}$ : Read: "dy-dx" or "dy over dx"

- There also possibilities for **higher order derivatives**, i.e. derivatives of derivatives and so on...
- Higher order derivatives will use the same types of notation, with details to indicate the order of the derivative

- $f''(x)$  or  $\frac{d^2y}{dx^2}$  for a second derivative;  $f'''(x)$  or  $\frac{d^3y}{dx^3}$  for a third derivative

# Calculating a Derivative

To calculate the derivative, begin with the secant formula. Use this formula to reduce the difference to some arbitrarily small value,  $h$ .

As  $h$  goes to zero, we go from discrete to instantaneous change.

Secant Formula:  $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$

# Example: Calculating the Derivative of $f(x) = 3x$

Secant Formula:  $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$

Example:  $3x$

$$\lim_{h \rightarrow 0} \frac{3(x+h) - 3x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3x + 3h - 3x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3h}{h}$$

$$= \lim_{h \rightarrow 0} 3$$

$$= 3$$

# Example: Calculating the Derivative of $f(x) = x^2$

Secant Formula:  $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$

Example:  $x^2$

$$\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + 2h^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2xh + 2h^2}{h}$$

$$= \lim_{h \rightarrow 0} 2x + 2h$$

$$= 2x + 0$$

$$= 2x$$

# What does it all mean?

- Derivatives give us information regarding the rate of change of given function at a value  $x$ . We can use this *information* to learn more about the primary function at that particular point.
- Think back to what the slope of a linear equation tells us from the formula  $y = mx + b$ .
- If  $m$  is positive, the slope is increasing; if  $m$  is negative, the slope is decreasing.
- The derivative gives us information that we interpret similarly, but oftentimes for more complex functions.

# Derivative as information: Rate of change

- Positive Derivative: Function is increasing
- Negative Derivative: Function is decreasing
- Zeroes: Maxima or minima (extrema)

# Example: Derivative as Information

Function A:  $f(x) = x + 2$ ;  $f'(x) = 1$

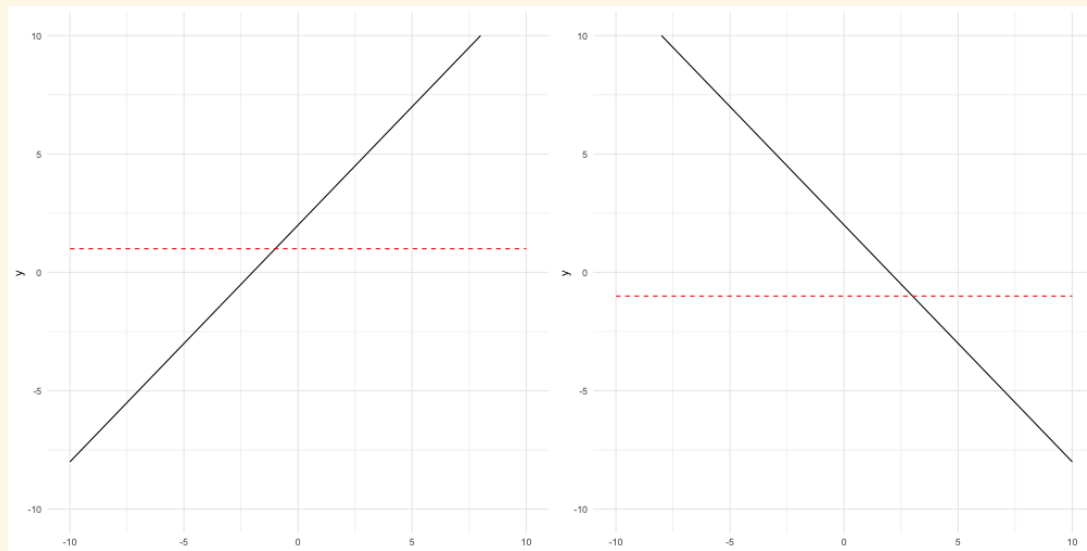
Function B:  $f(x) = -x + 2$ ;  $f'(x) = -1$



# Example: Derivative as Information

Function A:  $f(x) = x + 2$ ;  $f'(x) = 1$

Function B:  $f(x) = -x + 2$ ;  $f'(x) = -1$



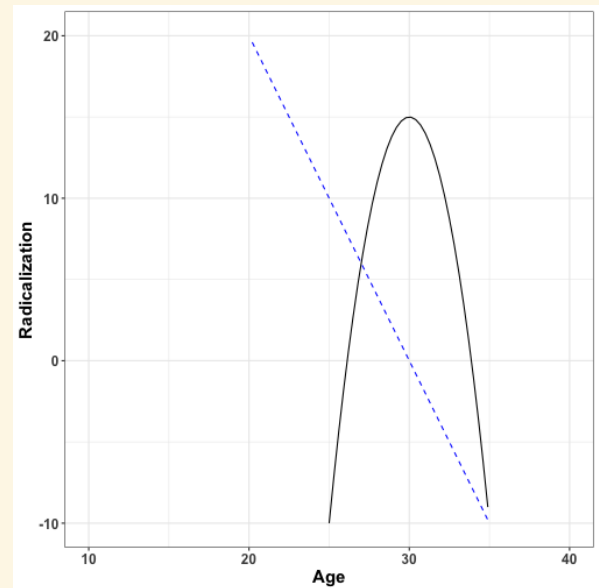


# Example: Age and Radical Politics

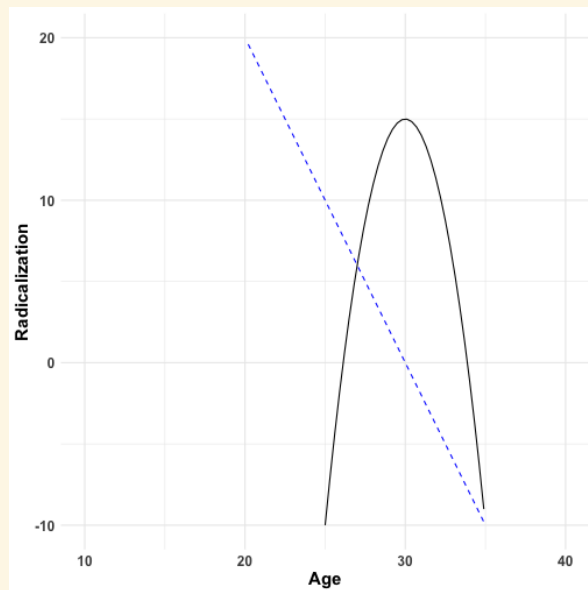
Function A:

$$f(x) = -x^2 + 60x - 885$$

Derivative:  $f'(x) = -2x + 60$



# Example: Age and Radical Politics



$$-2 * 25 + 60$$

### [1] 10

$$-2 * 30 + 60$$

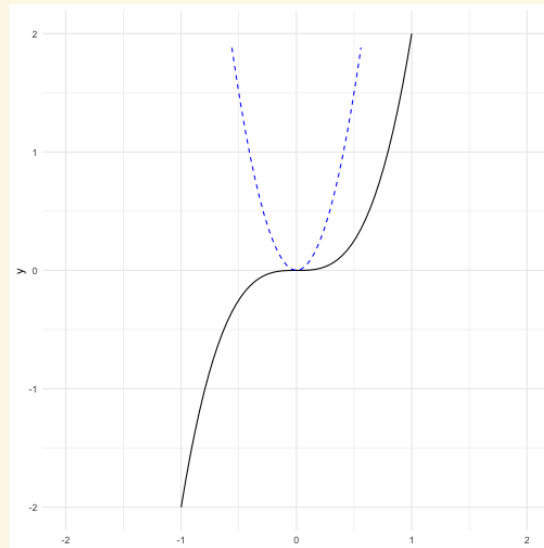
### [1] 0

$$-2 * 35 + 60$$

### [1] -10

# Derivatives and Extrema

**Extrema:** max or min of a function, i.e. where is the top-most or bottom-most value of the function?



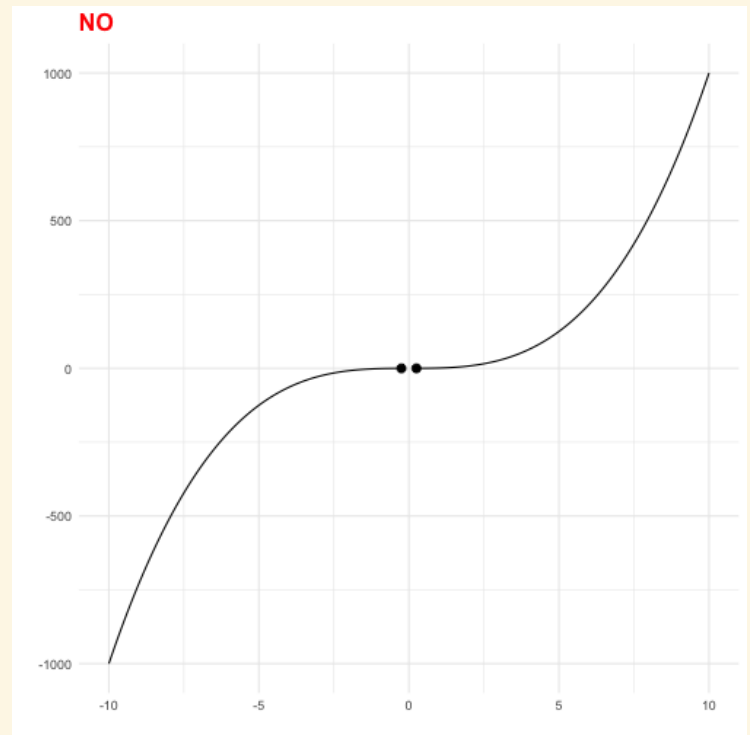
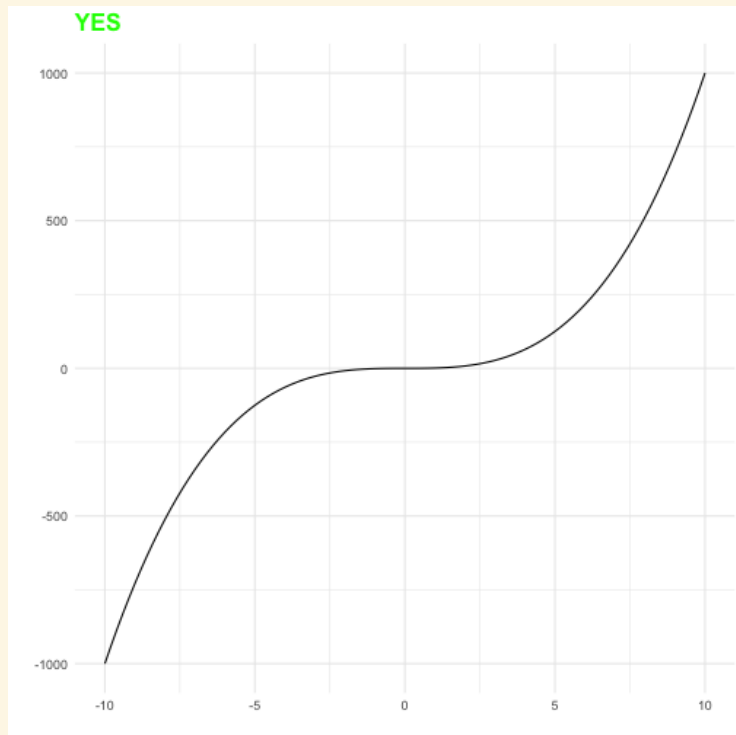
# Functional Behavior

The function must be *continuous* on the *interval* to be differentiable.

Some functions are not differentiable *at all* or are not differentiable at *a certain point*. Need to determine the continuity of the function.

# Continuity

Continuous function:



# Derivative: Calculation

While we can calculate the derivative using the formula from before, it's a bit tedious. Can't there be another way?

Think through the work it would take to differentiate  $4x^3 + 3x - 2$  using  $\frac{f(x_0+h)-f(x_0)}{h} \dots$

**Not fun!**

# Derivative Rules

Take the derivative  $f(x)$  as  $f'(x)$  below with constant  $k$ :

- $f(k * x) = k * f(x), f'(k * x) = k * f'(x)$
- $f(x) = k$  has derivative  $f'(x) = 0$
- $f(x) = x^n, f'(x) = n * x^{n-1}$
- $[f(x) + g(x)]' = f'(x) + g'(x)$
- $[f(x) - g(x)]' = f'(x) - g'(x)$

NOTE:  $[f(x) * g(x)]' = f'(x) * g'(x)$  EX:  $(3x * 10x)' = 30$

# Derivative Rules

Take the derivative  $f(x)$  as  $f'(x)$  below with constant  $k$ :

- $f(k * x) = k * f(x), f'(k * x) = k * f'(x)$

- $f(x) = 3x \rightarrow f'(x) = 3$

- $f(x) = k$  has derivative  $f'(x) = 0$

- $f(x) = 4 \rightarrow f'(x) = 0$



# Derivative Rules

Take the derivative  $f(x)$  as  $f'(x)$  below with constant  $k$ :

- $f(x) = x^n, f'(x) = n * x^{n-1}$

- $f(x) = x^3 \rightarrow f'(x) = 3x^2$

- $[f(x) + g(x)]' = f'(x) + g'(x)$

- $t(x) = 3x + 7 \rightarrow 3 + 0 = 3$

- $[f(x) - g(x)]' = f'(x) - g'(x)$

- $t(x) = x^3 - 7x \rightarrow 3x^2 - 7$

# Derivatives Two Ways

Find the derivative of  $f(x) = \frac{1}{x}$

Formal Definition:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Plug in our function :

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{x}{x(x+h)} - \frac{(x+h)}{x(x+h)}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)}$$

$$f'(x) = \frac{-1}{x(x+0)}$$

$$f'(x) = \frac{-1}{x^2}$$

$$f'(x) = -x^{-2}$$

# Using the Simpler Rules

Relevant Rules:  $f(x) = x^n, f'(x) = n * x^{n-1}$

$$f(x) = x^{-1}$$

$$f'(x) = -1 * x^{-1-1}$$

$$f'(x) = -x^{-2}$$

1.  $f(x) = 5$

2.  $f(x) = 3x - 7$

3.  $f(x) = 3x^2$

4.  $f(x) = \frac{x^2}{x}$

5.  $f(s) = s^{-2}$

6.  $f(y) = y(y + 7)(y - 3)$

7.  $f(z) = \frac{z^2 - 5z - 6}{z + 1}$

1.  $f'(x) = 0$

2.  $f'(x) = 3$

3.  $f'(x) = 6x$

4.  $f'(x) = 1$

5.  $f'(s) = -2s^{-3}$  (not  
continuous at (s=0))

6.  $f'(y) = 3y^2 + 8y - 21$

7.  $f'(z) = 1$  (not continuous  
at (z=-1))

# Higher Order Derivatives

Second derivatives (nth derivatives): take a derivative a second (nth) time

Rate of change of rate of change (e.g., velocity vs acceleration)

$$f(x) = x^5 + 3x^3 + 2x + 8$$

$$f'(x) = 5x^4 + 9x^2 + 2$$

$$f''(x) = 20x^3 + 18x$$

# (Additional) Rules for derivatives

So far, we've just had simple functions but what if we are multiplying, dividing, or have an otherwise-more-advanced function?

- Product Rule:  $f(x) * g(x)$
- Quotient Rule:  $\frac{f(x)}{g(x)}$
- Chain Rule:  $f(g(x))$
- Other: eg, exponentials:  $e^x$ ,  $\ln(x)$

# (Additional) Rules for derivatives

So far, we've just had simple functions but what if we are multiplying, dividing, or have an otherwise-more-advanced function?

- Product Rule:  $f(x) * g(x)$ ,  $x^3 * x^2$
- Quotient Rule:  $\frac{f(x)}{g(x)}$ ,  $\frac{x^4+3x}{x^2}$
- Chain Rule:  $f(g(x))$ ,  $(x^2 + 1)^3$  (composition!)
- Other: eg, exponentials:  $e^x$ ,  $\ln(x)$

# Product Rule

If we have two things multiplied together and need the derivative, we have two options: multiply everything and then take the derivative OR use the product rule.



# Product Rule

Example:  $(3x+4)(x+2)$

$$f'(x) * g(x) + g'(x)f(x)$$

- Identify constitutive functions:  $f(x) = 3x + 4$  and  $g(x) = x + 2$ . The derivatives are  $f'(x) = 3$  and  $g'(x) = 1$ , respectively.
- Substitute these in:

$$f'(x) * g(x) + g'(x)f(x) = 3(x + 2) + 1(3x + 4)$$

- Simplify to get:

$$3x + 6 + 3x + 4 = 6x + 10$$

# Product Rule: A Motivating Example

$$t(x) = (3x^2 + 3x + 4)(x^3 + 2x^2 + x + 2)$$

$$f(x) = (3x^2 + 3x + 4) \text{ and } g(x) = (x^3 + 2x^2 + x + 2)$$

$$f'(x) = 6x + 3 \text{ and } g'(x) = 3x^2 + 4x + 1.$$

We can substitute this into the formula:

$$f'(x) * g(x) + g'(x)f(x)$$

$$(6x + 3)(x^3 + 2x^2 + x + 2) + (3x^2 + 4x + 1)(3x^2 + 3x + 4)$$

# Quotient Rule

Example:  $\frac{3x^2}{x+2}$ .

Formula is  $\frac{f'(x)*g(x)-g'(x)f(x)}{(g(x))^2}$

So, we identify the following:  $f(x) = 3x^2$  and  $g(x) = x + 2$ ,  
therefore  $f'(x) = 6x$  and  $g'(x) = 1$ .

Plug in to get:

$$\frac{6x(x+2) - 1(3x^2)}{(x+2)^2} = \frac{6x^2 + 12x - 3x^2}{(x+2)^2} = \frac{3x^2 + 12x}{(x+2)^2} = \frac{3x(x+4)}{(x+2)^2}$$

# Chain Rule

Sometimes, you have a function to a power:  $f(g(x)) = (x + 3)^3$ . We can use the chain rule to evaluate this.

Take the derivative of the function and multiply it by the derivative of the inside:  $f'(g(x)) * g'(x)$ .

So, for our example:  $f(x) = x^3$  and  $g(x) = (x + 3)$ .

- The derivative of each is  $f'(x) = 3x^2$  and  $g'(x) = 1$ .
- We substitute in to get:  $3(x + 3)^2 * 1$ .

# More Chain Rule

Try:  $f(x) = (2x^2 + 8x)^4$

$$4(4x + 8)(2x^2 + 8x)^3$$

# Exponentials: $e$ and $\ln$

You can take the derivative of continuous functions -- including those with a log and/or  $e$  in them. The rules are a little hard, but once you learn them, it's not too bad:

1.  $f(x) = e^x$

2.  $f(x) = e^{g(x)}$

3.  $f(x) = a^x$

4.  $f(x) = \ln(x)$

5.  $f(x) = \ln(g(x))$

6.  $f(x) = \log_a(x)$

1.  $f'(x) = e^x$  (a favorite of mine)

2.  $f'(x) = e^{g(x)} * g'(x)$

3.  $f'(x) = a^x (\ln(a))$

4.  $f'(x) = \frac{1}{x}$

5.  $f'(x) = \frac{1}{x} * g'(x)$

6.  $f'(x) = \frac{1}{x \ln(a)}$

You can make these more complicated by including a function of  $x$ . How would we take the derivative in that case?

Example:  $\ln(3x)$

Chain Rule!

$$f'(x) = \frac{1}{3x} * 3 = \frac{1}{x}.$$

# Partial Derivatives

Similar to a "regular" derivative; treat additional variable(s) as constants. Written as  $\partial_x$  or  $\frac{\partial f}{\partial x}(x, \dots)$

Find  $\partial_x$

$$f(x, z) = 7xz + 4x^2 + z$$

$$\partial_x = 7z + 8x$$

$$f(x, y) = x + 4y$$

$$\partial_x = 1$$



# Derivatives in Review

There are a few more handy rules and techniques that are important, perhaps even on your homework:

Source on Stack Exchange

And if you really want to explore more, check out all [these techniques](#).