Calculus I: Derivatives

Day 3 AM

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Day 3 Agenda

- Derivatives: Concept, notation, and how-to
- Fundamental Derivative Rules
- Advanced Rules of Derivatives

Calculus

What is calculus?

> - So basically, calculus is the study of *instantaneous change of a function*.

Well, what does that mean?

> - On the one hand, we could be interested in *discrete* change. Discrete change tells us the difference between two points on a graph, thus the difference between two observations modeled by a function.
>
> - Finding discrete change means finding the slope of two points, or the *secant*. You all are probably familiar with $\frac{rise}{run}$ from algebra.

What is calculus?

- But, finding the secant has a limitation. Discrete change only tells about the functional behavior *over an interval*. Instead we might want to find the rate of change at a *very specific moment* in the function.

- Consequently, we might also want to find extrema (min and max) of a function, any other critical points, or understand its shape. As you may remember from other courses, these are difficult tasks with merely a function $f(x)$.

- Calculus gives us some tools to calculate instantaneous change and related quantities.

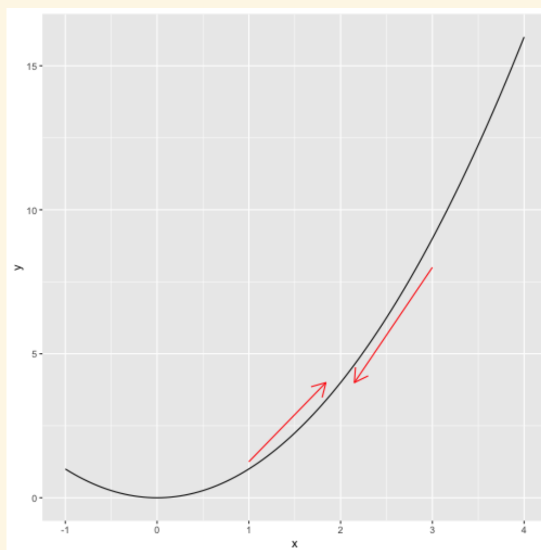
Capturing Change

- In our last session we went over limits of functions.
  - As a reminder, the limit of a function $f(x)$ at point x is the value of the function as it approaches x.
- To capture discrete change we calculate the secant between two points by taking the difference of the functional limits, i.e. $\frac{f(x_2)-f(x_1)}{x_2-x_1}$.

Capturing Change

- To capture the instantaneous or continuous rate of change of a function, we *could* take the difference of the limits over iterations, making the interval between each x smaller and smaller.

For example, we could take the difference of the limits of $f(x) = x^2$ as x approaches 2:



Tangents and Derivatives

- A secant is the slope of a line.
- A tangent is a line that touches the function at a given point. The tangent's slope tells us the instantaneous rate of change *at that particular point*.
- Therefore, if we find a tangent line for function at a specific point and determine its slope, we therefore have information about the instantaneous rate of change for the primary function.

Notation

- There are a couple of ways to notate derivatives, they mean the same thing.

$f'(x)$: Read: "f prime of x"

$\frac{dy}{dx}$: Read: "dy-dx" or "dy over dx"

- There also possibilities for higher order derivatives, i.e. derivatives of derivatives and so on...

- Higher order derivatives will use the same types of notation, with details to indicate the order of the derivative

> - $f''(x)$ or $\frac{d^2y}{dx^2}$ for a second derivative; $f'''(x)$ or $\frac{d^3y}{dx^3}$ for a third derivative

Calculating a Derivative

To calculate the derivative, begin with the secant formula. Use this formula to reduce the difference to some arbitrarily small value, h.

As h goes to zero, we go from discrete to instantaneous change.

Secant Formula: $lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

Example: Calculating the Derivative of $f(x) = 3x$

Secant Formula: $lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

Example: $3x$

$lim_{h \to 0} \frac{3(x+h)-3x}{h}$

$= lim_{h \to 0} \frac{3x+3h-3x}{h}$

$= lim_{h \to 0} \frac{3h}{h}$

$= lim_{h \to 0} 3$

$= 3$

Example: Calculating the Derivative of $f(x) = x^2$

Secant Formula: $lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

Example: x^2

$lim_{h \to 0} \frac{(x+h)^2-x^2}{h}$

$$= lim_{h \to 0} \frac{x^2+2xh+2h^2-x^2}{h}$$

$$= lim_{h \to 0} \frac{2xh+2h^2}{h}$$

$$= lim_{h \to 0} 2x + 2h$$

$$= 2x + 0$$

$$= 2x$$

What does it all mean?

- Derivatives give us information regarding the rate of change of given function at a value x. We can use this *information* to learn more about the primary function at that particular point.
- Think back to what the slope of a linear equation tells us from the formula $y = mx + b$.
- If m is positive, the slope is increasing; if m is negative, the slope is decreasing.
- The derivative gives us information that we interpret similarly, but oftentimes for more complex functions.

Derivative as information: Rate of change

- Positive Derivative: Function is increasing
- Negative Derivative: Function is decreasing
- Zeroes: Maxima or minima (extrema)

Example: Derivative as Information

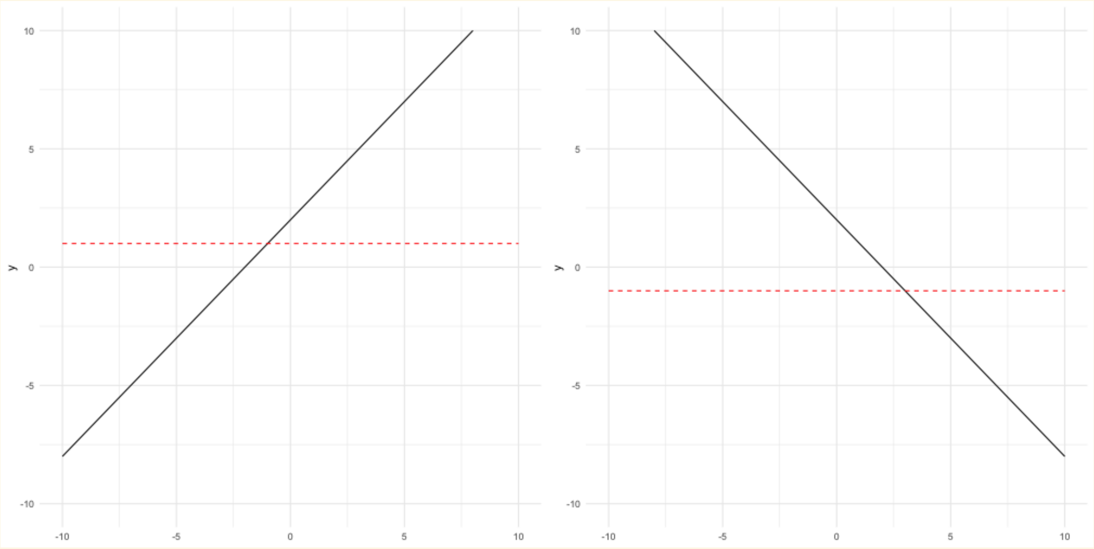
Function A: $f(x) = x + 2$; $f'(x) = 1$

Function B: $f(x) = -x + 2$; $f'(x) = -1$

Example: Derivative as Information

Function A: $f(x) = x + 2$; $f'(x) = 1$

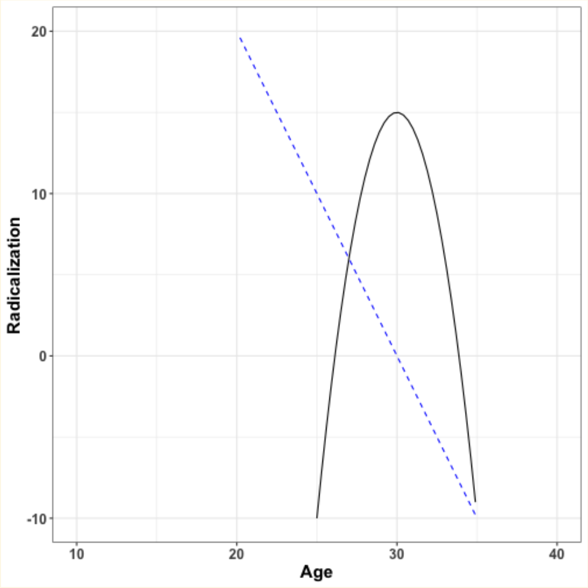
Function B: $f(x) = -x + 2$; $f'(x) = -1$



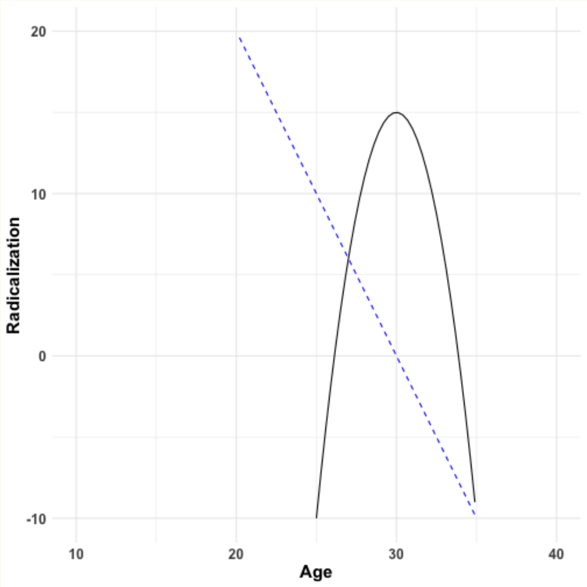
Example: Age and Radical Politics

Function A:
$f(x) = -x^2 + 60x - 885$

Derivative: $f'(x) = -2x + 60$



Example: Age and Radical Politics




```r
-2 * 25 + 60
```

```
## [1] 10
```

```r
-2 * 30 + 60
```

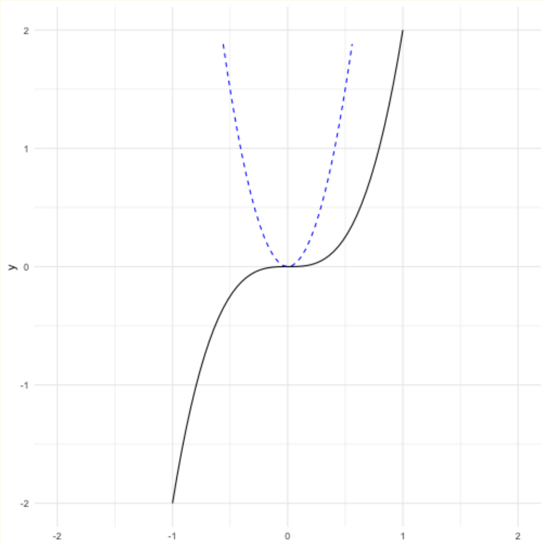
```
## [1] 0
```

```r
-2 * 35 + 60
```

```
## [1] -10
```

Derivatives and Extrema

Extrema: max or min of a function, i.e. where is the top-most or bottom-most value of the function?



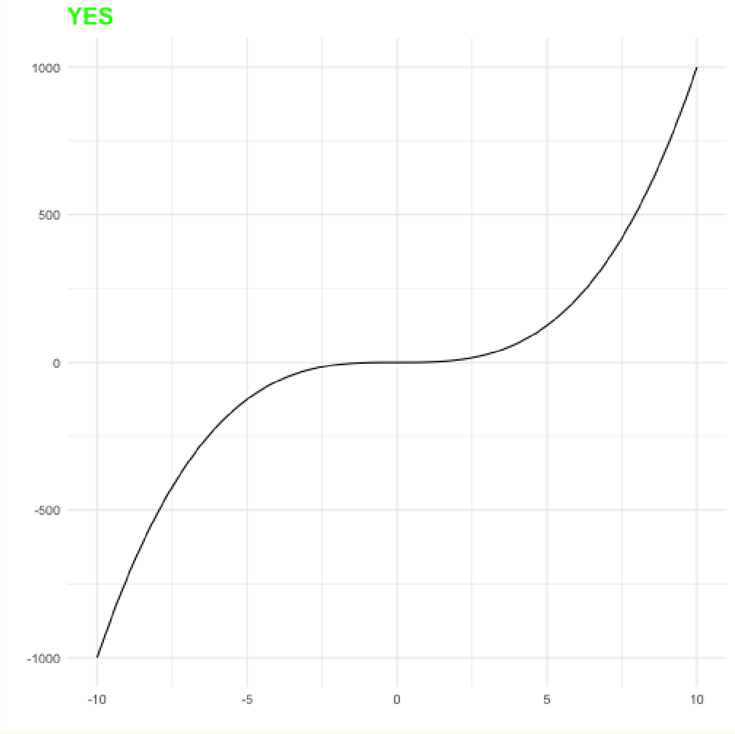
Functional Behavior

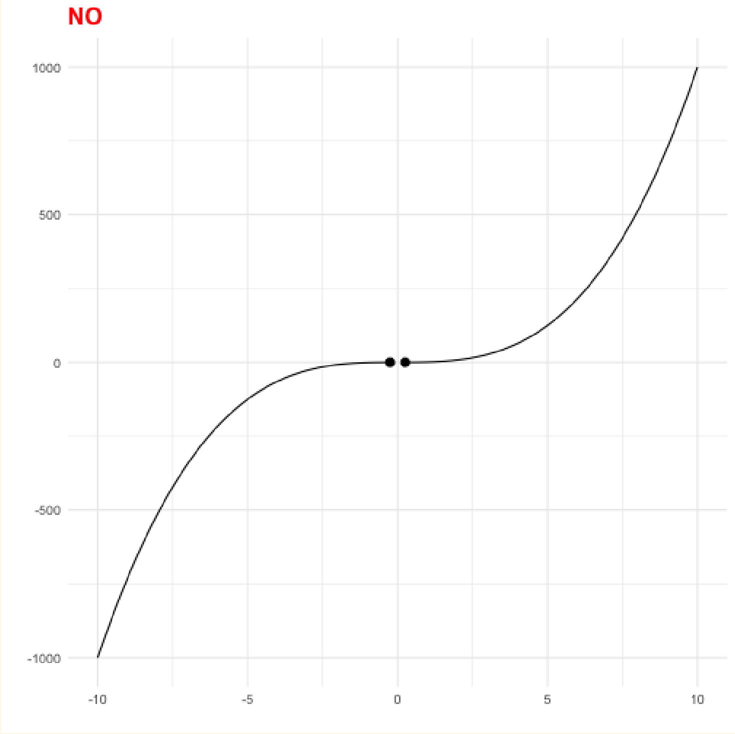
The function must be *continuous* on the *interval* to be differentiable.

Some functions are not differentiable *at all* or are not differentiable at *a certain point*. Need to determine the continuity of the function.

Continuity

Continuous function:





Derivative: Calculation

While we can calculate the derivative using the formula from before, it's a bit tedious. Can't there be another way?

Think through the work it would take to differentiate $4x^3 + 3x - 2$ using $\frac{f(x_0+h)-f(x_0)}{h}$...

Not fun!

Derivative Rules

Take the derivative $f(x)$ as $f'(x)$ below with constant k:

- $f(k * x) = k * f(x), f'(k * x) = k * f'(x)$
- $f(x) = k$ has derivative $f'(x) = 0$
- $f(x) = x^n, f'(x) = n * x^{n-1}$
- $[f(x) + g(x)]' = f'(x) + g'(x)$
- $[f(x) - g(x)]' = f'(x) - g'(x)$

NOTE: $[f(x) * g(x)]! = f'(x) * g'(x)$ Ex: $(3x * 10x)'! = 30$

Derivative Rules

Take the derivative $f(x)$ as $f'(x)$ below with constant k:

- $f(k*x) = k*f(x)$, $f'(k*x) = k*f'(x)$
    - $f(x) = 3x \rightarrow f'(x) = 3$
- $f(x) = k$ has derivative $f'(x) = 0$
    - $f(x) = 4 \rightarrow f'(x) = 0$

Derivative Rules

Take the derivative $f(x)$ as $f'(x)$ below with constant k:

- $f(x) = x^n, f'(x) = n * x^{n-1}$
  - $f(x) = x^3 \rightarrow f'(x) = 3x^2$
- $[f(x) + g(x)]' = f'(x) + g'(x)$
  - $t(x) = 3x + 7 \rightarrow 3 + 0 = 3$
- $[f(x) - g(x)]' = f'(x) - g'(x)$
  - $t(x) = x^3 - 7x \rightarrow 3x^2 - 7$

Derivatives Two Ways

Find the derivative of $f(x) = \frac{1}{x}$

Formal Definition:

$$f'(x) = lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Plug in our function :

$$f'(x) = lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$f'(x) = lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{(x+h)}{x(x+h)}}{h}$$

$$f'(x) = lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h}$$

$$f'(x) = lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$f'(x) = \frac{-1}{x(x+0)}$$

$$f'(x) = \frac{-1}{x^2}$$

$$f'(x) = -x^{-2}$$

Using the Simpler Rules

Relevant Rules: $f(x) = x^n, f'(x) = n * x^{n-1}$

$$f(x) = x^{-1}$$

$$f'(x) = -1 * x^{-1-1}$$

$$f'(x) = -x^{-2}$$

1. $f(x) = 5$
2. $f(x) = 3x - 7$
3. $f(x) = 3x^2$
4. $f(x) = \frac{x^2}{x}$
5. $f(s) = s^{-2}$
6. $f(y) = y(y+7)(y-3)$
7. $f(z) = \frac{z^2-5z-6}{z+1}$

1. $f'(x) = 0$
2. $f'(x) = 3$
3. $f'(x) = 6x$
4. $f'(x) = 1$
5. $f'(s) = -2s^{-3}$ (not continuous at (s=0))
6. $f'(y) = 3y^2 + 8y - 21$
7. $f'(z) = 1$ (not continuous at (z=-1))

Higher Order Derivatives

Second derivatives (nth derivatives): take a derivative a second (nth) time

Rate of change of rate of change (e.g., velocity vs acceleration)

$f(x) = x^5 + 3x^3 + 2x + 8$

$f'(x) = 5x^4 + 9x^2 + 2$

$f''(x) = 20x^3 + 18x$

(Additional) Rules for derivatives

So far, we've just had simple functions but what if we are multiplying, dividing, or have an otherwise-more-advanced function?

- Product Rule: $f(x) * g(x)$
- Quotient Rule: $\frac{f(x)}{g(x)}$
- Chain Rule: $f(g(x))$
- Other: eg, exponentials: e^x, $ln(x)$

(Additional) Rules for derivatives

So far, we've just had simple functions but what if we are multiplying, dividing, or have an otherwise-more-advanced function?

- Product Rule: $f(x) * g(x)$, $x^3 * x^2$
- Quotient Rule: $\frac{f(x)}{g(x)}$, $\frac{x^4+3x}{x^2}$
- Chain Rule: $f(g(x))$, $(x^2+1)^3$ (composition!)
- Other: eg, exponentials: e^x, $ln(x)$

Product Rule

If we have two things multiplied together and need the derivative, we have two options: multiply everything and then take the derivative OR use the product rule.

Product Rule

Example: (3x+4)(x+2)

$$f'(x) * g(x) + g'(x)f(x)$$

- Identify constitutive functions: $f(x) = 3x + 4$ and $g(x) = x + 2$. The derivatives are $f'(x) = 3$ and $g'(x) = 1$, respectively.
- Substitute these in:

$$f'(x) * g(x) + g'(x)f(x) = 3(x + 2) + 1(3x + 4)$$

- Simplify to get:

$$3x + 6 + 3x + 4 = 6x + 10$$

Product Rule: A Motivating Example

$$t(x) = (3x^2 + 3x + 4)(x^3 + 2x^2 + x + 2)$$

$f(x) = (3x^2 + 3x + 4)$ and $g(x) = (x^3 + 2x^2 + x + 2)$

$f'(x) = 6x + 3$ and $g'(x) = 3x^2 + 4x + 1$.

We can substitute this into the formula:

$$f'(x) * g(x) + g'(x)f(x)$$

$$(6x + 3)(x^3 + 2x^2 + x + 2) + (3x^2 + 4x + 1)(3x^2 + 3x + 4)$$

Quotient Rule

Example: $\frac{3x^2}{x+2}$.

Formula is $\frac{f'(x)*g(x)-g'(x)f(x)}{(g(x))^2}$

So, we identify the following: $f(x) = 3x^2$ and $g(x) = x + 2$, therefore $f'(x) = 6x$ and $g'(x) = 1$.

Plug in to get:

$$\frac{6x(x+2) - 1(3x^2)}{(x+2)^2} = \frac{6x^2 + 12x - 3x^2}{(x+2)^2} = \frac{3x^2 + 12x}{(x+2)^2}$$

$$= \frac{3x(x+4)}{(x+2)^2}$$

Chain Rule

Sometimes, you have a function to a power: $f(g(x)) = (x+3)^3$
. We can use the chain rule to evaluate this.

Take the derivative of the function and multiply it by the derivative of the inside: $f'(g(x)) * g'(x)$.

So, for our example: $f(x) = x^3$ and $g(x) = (x+3)$.

- The derivative of each is $f'(x) = 3x^2$ and $g'(x) = 1$.
- We substitute in to get: $3(x+3)^2 * 1$.

More Chain Rule

Try: $f(x) = (2x^2 + 8x)^4$

$4(4x + 8)(2x^2 + 8x)^3$

Exponentials: e and ln

You can take the derivative of continuous functions -- including those with a log and/or e in them. The rules are a little hard, but once you learn them, it's not too bad:

1. $f(x) = e^x$
2. $f(x) = e^{g(x)}$
3. $f(x) = a^x$
4. $f(x) = ln(x)$
5. $f(x) = ln(g(x))$
6. $f(x) = log_a(x)$

1. $f'(x) = e^x$ (a favorite of mine)
2. $f'(x) = e^{g(x)} * g'(x)$
3. $f'(x) = a^x(ln(a))$
4. $f'(x) = \frac{1}{x}$
5. $f'(x) = \frac{1}{x} * g'(x)$
6. $f'(x) = \frac{1}{xln(a)}$

You can make these more complicated by including a function of x. How would we take the derivative in that case?

Example: $ln(3x)$

Chain Rule!

$f'(x) = \frac{1}{3x} * 3 = \frac{1}{x}$.

Partial Derivatives

Similar to a "regular" derivative; treat additional variable(s) as constants. Written as ∂_x or $\frac{\partial f}{\partial x}(x, \dots)$

Find ∂_x

$f(x, z) = 7xz + 4x^2 + z$

$\partial_x = 7z + 8x$

$f(x, y) = x + 4y$

$\partial_x = 1$

Derivatives in Review

There are a few more handy rules and techniques that are important, perhaps even on your homework:

Source on Stack Exchange

And if you really want to explore more, check out all these techniques.