1 Preliminaries

1.1 Commutative algebra

We begin with some algebraic prerequisites. For more see [**atiyah2018introduction**].

Definition 1.1.1. A *commutative ring* $(R, +, \cdot, 0, 1)$ is an abelian group $(R, +, 0)$ along with an associative binary operator $\cdot$ and a unit element 1 such that for all $x, y, z \in R$

1. $x \cdot y = y \cdot x$,
2. $x \cdot 1 = 1$, and
3. $x \cdot (y + z) = x \cdot y + x \cdot z$

We call a commutative ring $(R, +, \cdot, 0, 1)$ simply a *ring* and write it as R when there is no ambiguity. Also we denote $x \cdot y$ by xy.

Definition 1.1.2. An *ideal* of a ring R is a subset $I \subset R$ that is closed under addition $(+)$ and "absorbs" multiplication from elements of R: for all $x \in R$ and $y \in I$, $xy \in I$.

Proposition 1.1.1. *The images and preimages of a prime ideal under ring homomorphisms are prime.*

1.2 Categories and functors

Definition 1.2.1. A *category* $\mathcal{C}$ consists of a collection of *objects*, a set $\mathcal{C}(A, B)$ of *morphisms* between any two objects, an *identity* morphism $id_A \in \mathcal{C}(A, A)$ for each object A, and a composition law

$$\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C) \tag{1.1}$$

for each triple of objects A, B, C. Composition must be associative, and identity morphisms must behave as their names indicate: $h \circ (g \circ f) = (h \circ g) \circ f, id \circ f = f$, and $f \circ id = f$ whenever the the composites are defined.

Definition 1.2.2. A *terminal object* of category $\mathcal{C}$ is an object T to which there is a unique morphism from each object of $\mathcal{C}$.

Definition 1.2.3. A *functor* $F : \mathcal{C} \to \mathcal{D}$ assigns an object $F(A)$ of $\mathcal{D}$ to each object A of $\mathcal{C}$ and a morphism $F(f) : F(A) \to F(B)$ of $\mathcal{D}$ to each morphism $f : A \to B$ of $\mathcal{C}$ such that

$$F(id_A) = id_{F(A)} \text{ and } F(g \circ f) = F(g) \circ F(f). \tag{1.2}$$

Definition 1.2.4. A *natural transformation* $\alpha : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of a morphism $\alpha_A : F(A) \to G(A)$ for each object A of $\mathcal{C}$ such that the following diagram commutes for each morphism $f : A \to B$ of $\mathcal{C}$:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$