EE698G - Probabilistic Mobile Robotics Assignment

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1

Let 'E' be the event that the first coin (biased one), on flipping it twice, shows heads both times.
Let 'F' be the event theat the second coin (unbiased one), on flipping it twice, shows heads both times.
Let 'B' be the event that the biased coin is chosen and 'C' be the event that the unbiased coin is chosen.
Let 'H' be the event that on selecting a coin at random and flipping it, heads are observed in both tosses.

$$
\begin{aligned}
We\ have,\ P(B) = P(C) &= 0.5 \\
\because\ P(Heads) &= 0.6\ for\ the\ biased\ coin\ : \\
P(E) &= 0.6 \times 0.6 \\
&= 0.36 \\
Similarily,\ for\ the\ unbiased\ coin\ : \\
P(F) &= 0.5 \times 0.5 \\
&= 0.25 \\
Now,\ P(B|H) &= \frac{P(B \cap H)}{P(H)} \\
P(B \cap H) &= P(B \cap E) && \because\ B \cap H = B \cap E \\
&= P(B) \times P(E) && \because\ events\ B\ and\ E\ are\ independent \\
&= 0.5 \times 0.36 \\
&= 0.18 \\
Now,\ P(H) &= P(H \cap B) + P(H \cap C) && Using\ Total\ Probability\ Theorem \\
&&& \because\ B\ \&\ C\ are\ mutually \\
&&& exclusive\ and\ exhaustive. \\
&= P(B \cap E) + P(C \cap F) && \because\ B \cap H = B \cap E\ \&\ C \cap H = C \cap F \\
&= 0.18 + P(C) \times p(F) && \because\ events\ C\ and\ F\ are\ independent \\
&= 0.18 + 0.5 \times 0.25 \\
&= 0.18 + 0.125 \\
&= 0.305 \\
\therefore\ P(B|H) &= \frac{P(B \cap H)}{P(H)} \\
&= \frac{0.18}{0.305} \\
&= 0.59
\end{aligned}
$$

2

Let S_B be the event that the car appears blue and S_G be the event that the car apprears to be green.
Let G be the event that the car is green and B be the event that the car is blue.

$$Now,\ P(B|S_B) = \frac{P(S_B|B)P(B)}{P(S_B)}$$
$$\&\ P(G|S_G) = \frac{P(S_G|G)P(G)}{P(S_B)}$$

If we do not know the value of P (B) and P (G), it is not possible to calculate the most likely color of the taxi.

With the information, **'9 out of 10 Athenian taxi are green'**, we have,
$P(B) = 0.1$ & $P(G) = 0.9$

It is clear to me that the statement **'discrimination between blue and green is 75% reliable'** implies that $P(B \cap S_B) + P(G \cap S_G) = 0.75$.

Case 1

If the statement, **'discrimination between blue and green is 75% reliable' does not mean** that $P(S_B|B) = P(S_G|G)$, then it is not possible to calculate the most likely color of the taxi.

Case 2

However, if the statment, **'discrimination between blue and green is 75% reliable', does imply** that $P(S_B|B) = P(S_G|G)$, then it becomes possible to calculate the most likely color of the taxi.

$$Now,\ P(B \cap S_B) + P(G \cap S_G) = 0.75.$$
$$\implies P(S_B|B)P(B) + P(S_G|G)P(G) = 0.75$$
$$\implies P(S_B|B)(P(B) + P(G)) = 0.75$$
$$\implies P(S_B|B) \times 1 = 0.75$$
$$\implies P(S_B|B) = 0.75\ \&\ P(S_G|G) = 0.75$$

$$\begin{aligned} Also,\ P(S_B|G) &= \frac{P(S_B \cap G)}{P(G)} \\ &= \frac{P(G) - P(S_G \cap G)}{P(G)} \qquad \because [G = (S_G \cap G) \cup (S_B \cap G)\ \&\ \phi = (S_G \cap G) \cap (S_B \cap G)] \\ &= 1 - \frac{P(S_G \cap G)}{P(G)} \\ &= 1 - P(S_G|G) \\ &= 0.25 \end{aligned}$$

$$
\begin{aligned}
Now,\ P(B|S_B) &= \frac{P(S_B|B)P(B)}{P(S_B)} \\
&= \frac{0.75 \times 0.1}{P(S_B)} \\
&= 0.075\eta,\ where\ \eta = \frac{1}{P(S_B)} \\
Also,\ P(G|S_B) &= \frac{P(S_B|G)P(G)}{P(S_B)} \\
&= \frac{0.25 * 0.9}{P(S_B)} \\
&= 0.225\eta \\
We\ have, P(B|S_B) + P(G|S_B) &= \frac{P(B \cap S_B) + P(G \cap S_B}{P(S_B)} \\
&= \frac{P(S_B)}{P(S_B)} \\
&= 1. \\
\therefore\ 0.225\eta + 0.075\eta &= 1 \\
\implies\ 0.3\eta &= 1 \\
\eta &= 3.33 \\
\therefore\ P(B|S_B) &= 3.33 \times 0.075 \\
&= 0.25 \\
\&\ P(G|S_B) &= 3.33.225 \\
&= 0.75 \\
Hence,\ P(G|S_B) > P(B|S_B)
\end{aligned}
$$

$\therefore$ The most likely color of the taxi is green.

3

$$
\begin{aligned}
P(X,Y) &= \frac{1}{2\pi\sqrt{|\Sigma|}} exp\left\{-\tfrac{1}{2}\begin{bmatrix}X-\mu_X\\ Y-\mu_y\end{bmatrix}^T \Sigma^{-1}\begin{bmatrix}X-\mu_X\\ Y-\mu_y\end{bmatrix}\right\}\\
Now,\ P(Y=y) &= \int_{-\infty}^{\infty} P(X=x, Y=y)dx\\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{|\Sigma|}} exp\left\{-\tfrac{1}{2}\begin{bmatrix}x-\mu_X\\ y-\mu_y\end{bmatrix}^T \Sigma^{-1}\begin{bmatrix}x-\mu_X\\ y-\mu_y\end{bmatrix}\right\}dx
\end{aligned}
$$

$$Taking\ x' = x-\mu_X\ \&\ y' = y-\mu_Y,$$

$$
\begin{aligned}
P(Y=y) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{|\Sigma|}} exp\left\{-\tfrac{1}{2}\begin{bmatrix}x'\\ y'\end{bmatrix}^T \Sigma^{-1}\begin{bmatrix}x'\\ y'\end{bmatrix}\right\}dx'\\
Let\ E(x',y') &= \frac{1}{2}\begin{bmatrix}x'\\ y'\end{bmatrix}^T \Sigma^{-1}\begin{bmatrix}x'\\ y'\end{bmatrix}\\
Now,\ \Sigma^{-1} &= \frac{1}{|\Sigma|}\begin{bmatrix}\sigma_Y^2 & -\sigma_{XY}\\ -\sigma_{XY} & \sigma_X^2\end{bmatrix}\\
\therefore\ E(x',y') &= \frac{1}{2}\begin{bmatrix}x'\\ y'\end{bmatrix}^T \frac{1}{|\Sigma|}\begin{bmatrix}\sigma_Y^2 & -\sigma_{XY}\\ -\sigma_{XY} & \sigma_X^2\end{bmatrix}\begin{bmatrix}x'\\ y'\end{bmatrix}\\
&= \frac{\sigma_Y^2 x'^2 - 2\sigma_{XY}x'y' + \sigma_X^2 y'^2}{2|\Sigma|}\\
Now,\ \frac{\partial E}{\partial x'} &= \frac{2\sigma_Y^2 x' - 2\sigma_{XY}y'}{2|\Sigma|}\\
\therefore\ \mu_{x'} &= \frac{\sigma_{XY}y'}{\sigma_Y^2}\\
Also,\ \frac{\partial^2 E}{\partial x'^2} &= \frac{\sigma_Y^2}{|\Sigma|}\\
\therefore\ \Sigma_{x'} &= \frac{|\Sigma|}{\sigma_Y^2}\\
Taking\ E(x',y') &= E_1(x',y') + E_2(y'),\\
\&\ E_1(x',y') &= \frac{(x'-\mu_{x'})^2}{2\Sigma_{x'}}\\
We\ have,\ E_2(y') &= E(x',y') - E_1(x',y')\\
&= \frac{\sigma_X^2 y'^2 - \frac{(\sigma_{XY})^2}{\sigma_Y^2}y'^2}{2|\Sigma|}\\
&= \frac{(\sigma_X^2\sigma_Y^2 - (\sigma_{XY})^2)y'^2}{2|\Sigma|\sigma_Y^2}\\
&= \frac{|\Sigma|y'^2}{2|\Sigma|\sigma_Y^2}\\
&= \frac{y'^2}{2\sigma_Y^2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ P(Y=y) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{|\Sigma|}} exp\left\{-E(x',y')\right\} dx' \\
&= \frac{1}{2\pi\sqrt{|\Sigma|}} e^{-E_2(y')} \int_{-\infty}^{\infty} e^{-E_1(x',y')} dx' \\
&= \frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{y'^2}{2\sigma_Y^2}} \times \int_{-\infty}^{\infty} e^{(-\frac{(x'-\mu_{x'})^2}{2\Sigma_{x'}})} dx' \\
&= \frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{y'^2}{2\sigma_Y^2}} \times \sqrt{2\pi\Sigma_{x'}} \\
&= \frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{y'^2}{2\sigma_Y^2}} \times \sqrt{2\pi\frac{|\Sigma|}{\sigma_Y^2}} \\
&= \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{y'^2}{2\sigma_Y^2}} \\
&= \frac{1}{\sqrt{2\pi\sigma_Y^2}} exp\left\{-\tfrac{(y-\mu_Y)^2}{2\sigma_Y^2}\right\} \\
\therefore\ P(X|Y=y) &= \frac{P(X,Y=y)}{P(Y=y)} \\
&= \frac{P(X,Y=y)}{P(Y=y)} \\
&= \frac{\frac{1}{2\pi\sqrt{|\Sigma|}} exp\left\{-\frac{1}{2}\begin{bmatrix} X-\mu_X \\ y-\mu_y \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} X-\mu_X \\ y-\mu_y \end{bmatrix}\right\}}{\frac{1}{\sqrt{2\pi\sigma_Y^2}} exp\left\{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right\}} \\
&= \frac{\sqrt{2\pi\sigma_Y^2}}{2\pi\sqrt{|\Sigma|}} exp\left\{-\frac{1}{2}\begin{bmatrix} X-\mu_X \\ y-\mu_y \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} X-\mu_X \\ y-\mu_y \end{bmatrix} + \frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right\} \\
&= \frac{\sqrt{2\pi\sigma_Y^2}}{2\pi\sqrt{|\Sigma|}} exp\left\{-E(X-\mu_X, y-\mu_Y) + \tfrac{(y-\mu_Y)^2}{2\sigma_Y^2}\right\} \\
&= \frac{\sqrt{2\pi\sigma_Y^2}}{2\pi\sqrt{|\Sigma|}} exp\left\{-E_1(X-\mu_X, y-\mu_Y) - E_2(y-\mu_y) + E_2(y-\mu_y)\right\} \\
&= \frac{1}{\sqrt{2\pi\frac{|\Sigma|}{\sigma_Y^2}}} exp\left\{-E_1(X-\mu_X, y-\mu_Y)\right\} \\
We\, have,\ E_1(x',y') &= \frac{(x'-\mu_{x'})^2}{2\Sigma_{x'}}\ where, \\
\mu_{x'} &= \frac{\sigma_{XY}y'}{\sigma_Y^2}\ \&\ \Sigma_{x'} = \frac{|\Sigma|}{\sigma_Y^2} \\
Hence,\ E_1(X-\mu_X, y-\mu_Y) &= \frac{(X-\mu_X-\frac{\sigma_{XY}(y-\mu_Y)}{\sigma_Y^2})^2}{\frac{|\Sigma|}{\sigma_Y^2}} \\
&= \frac{(X-\mu_*)^2}{\sigma_*^2}\ where,
\end{aligned}
$$

$$
\begin{aligned}
\mu_* &= \mu_X + \frac{\sigma_{XY}}{\sigma_Y^2}(y-\mu_Y) \\
\sigma_*^2 &= \frac{|\Sigma|}{\sigma_Y^2} \\
&= \frac{(\sigma_X^2\sigma_Y^2 - (\sigma_{XY})^2)}{\sigma_Y^2} \\
&= \sigma_X^2 - \frac{(\sigma_{XY})^2}{\sigma_Y^2} \\
\therefore\ P(X|Y=y) &= \frac{1}{\sqrt{2\pi\frac{|\Sigma|}{\sigma_Y^2}}} exp\left\{-E_1(X-\mu_X, y-\mu_Y)\right\} \\
&= \frac{1}{\sqrt{2\pi\frac{|\Sigma|}{\sigma_Y^2}}} exp\left\{-\frac{(X-\mu_*)^2}{\sigma_*^2}\right\} \\
&= \frac{1}{\sqrt{2\pi\sigma_*^2}} exp\left\{-\frac{(X-\mu_*)^2}{\sigma_*^2}\right\}\ where \\
\mu_* &= \mu_X + \frac{\sigma_{XY}}{\sigma_Y^2}(y-\mu_Y)\ \&\ \sigma_*^2 = \sigma_X^2 - \frac{(\sigma_{XY})^2}{\sigma_Y^2}
\end{aligned}
$$

$q.e.d.$

4

On using Recuresice Bayes Filter, in order to determine the probability that Day 5 is indeed sunny :
Let x_t be the random variable of the weather on the t^{th} day.
Let $Bel(x_t) = \begin{bmatrix} P(sunny\ on\ t^{th}\ day) \\ P(cloudy\ on\ t^{th}\ day) \\ P(rainy\ on\ t^{th}\ day) \end{bmatrix}$ be the probabity distribution of the weather on the t^{th} day.
Let $\bar{Bel}(x_t) = \begin{bmatrix} \bar{P}(sunny\ on\ t^{th}\ day) \\ \bar{P}(cloudy\ on\ t^{th}\ day) \\ \bar{P}(rainy\ on\ t^{th}\ day) \end{bmatrix}$ be the poseterior belief distribution of the weather on the t^{th} day after the time update.
Let z_t be the random variable which is the measurement of the sensor on the t^{th} day.

$$
\begin{aligned}
We\ have,\ \bar{Bel}(x_t) &= \Sigma_{x_{t-1}} P(x_t|x_{t-1}) Bel(x_{t-1}) \\
&= TBel(x_t),\ where \\
The\ Transition\ matrix,\ T &= \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix} \\
Also,\ Bel(x_t) &= \eta P(Z_t|x_t)\bar{Bel}(x_{t-1}) \\
Now,\ Bel(x_1) &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

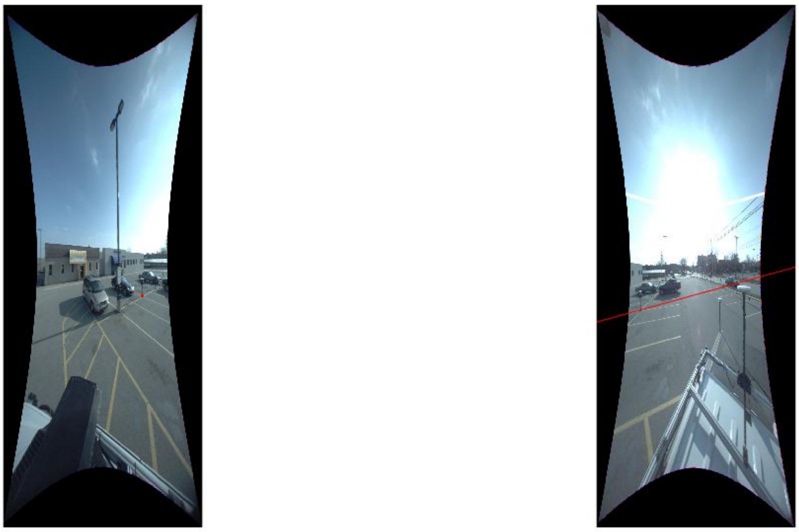
$$
\begin{aligned}
Now,\ \bar{Bel}(x_2) &= TBel(x_1) \\
&= \begin{bmatrix} 0.8 \\ 0.2 \\ 0 \end{bmatrix} \\
\therefore\ Bel(x_2) &= \eta P(Z_2|x_2)\bar{Bel}(x_2) \\
&= \eta P(Z_2 = cloudy|x_2)\bar{Bel}(x_2) \\
&= \eta \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \\ 0 \end{bmatrix} \\
&= \eta \begin{bmatrix} 0.32 \\ 0.14 \\ 0 \end{bmatrix} \\
On\ normalizing,\ \eta &= \frac{1}{0.32 + 0.14} \\
&= 2.1739 \\
\therefore\ Bel(x_2) &= \begin{bmatrix} 0.6956 \\ 0.3044 \\ 0 \end{bmatrix} \\
Now,\ \bar{Bel}(x_3) &= TBel(x_2) \\
&= \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0.6956 \\ 0.3044 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.6782 \\ 0.2609 \\ 0.0609 \end{bmatrix} \\
\therefore\ Bel(x_3) &= \eta P(Z_3|x_3)\bar{Bel}(x_3) \\
&= \eta P(Z_3 = cloudy|x_3)\bar{Bel}(x_3) \\
&= \eta \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.6782 \\ 0.2609 \\ 0.0609 \end{bmatrix} \\
&= \eta \begin{bmatrix} 0.2713 \\ 0.1826 \\ 0 \end{bmatrix} \\
On\ normalizing,\ \eta &= \frac{1}{0.2713 + 0.1826} \\
&= 2.2031 \\
\therefore\ Bel(x_3) &= \begin{bmatrix} 0.5977 \\ 0.4023 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Now,\ \bar{Bel}(x_4) &= TBel(x_3) \\
&= \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5977 \\ 0.4023 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.6391 \\ 0.2805 \\ 0.0805 \end{bmatrix} \\
\therefore\ Bel(x_4) &= \eta P(Z_4|x_4)\bar{Bel}(x_4) \\
&= \eta P(Z_4 = rainy|x_4)\bar{Bel}(x_4) \\
&= \eta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6391 \\ 0.2805 \\ 0.0805 \end{bmatrix} \\
&= \eta \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0805 \end{bmatrix} \\
On\ normalizing,\ \eta &= \frac{1}{0.0805} \\
&= 1.2422 \\
\therefore\ Bel(x_4) &= \begin{bmatrix} 0.0000 \\ 0.0000 \\ 1.0000 \end{bmatrix} \\
Now,\ \bar{Bel}(x_5) &= TBel(x_4) \\
&= \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0.0000 \\ 0.0000 \\ 1.0000 \end{bmatrix} \\
&= \begin{bmatrix} 0.2000 \\ 0.6000 \\ 0.2000 \end{bmatrix} \\
\therefore\ Bel(x_5) &= \eta P(Z_5|x_5)\bar{Bel}(x_5) \\
&= \eta P(Z_5 = sunny|x_5)\bar{Bel}(x_5) \\
&= \eta \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.2000 \\ 0.6000 \\ 0.2000 \end{bmatrix} \\
&= \eta \begin{bmatrix} 0.1200 \\ 0.1800 \\ 0.0000 \end{bmatrix}
\end{aligned}
$$

$$On\ normalizing,\ \eta = \frac{1}{0.12 + 0.18}$$
$$= 3.3333$$
$$\therefore\ Bel(x_5) = \begin{bmatrix} 0.4 \\ 0.6 \\ 0.0 \end{bmatrix}$$

$\therefore$ The probability that day 5 is indeed sunny is 0.4.

5



Plot of the epipolar line correspoing to the bottom of the parking meter in camera 1.

6

(a)

Let $x_n = \begin{bmatrix} h_n,\ \textit{height of the object at time } n\Delta t \\ v_n,\ \textit{upward velocity of the object at time } n\Delta t \end{bmatrix}$ be the state of the object at time, $t = n\Delta t$.

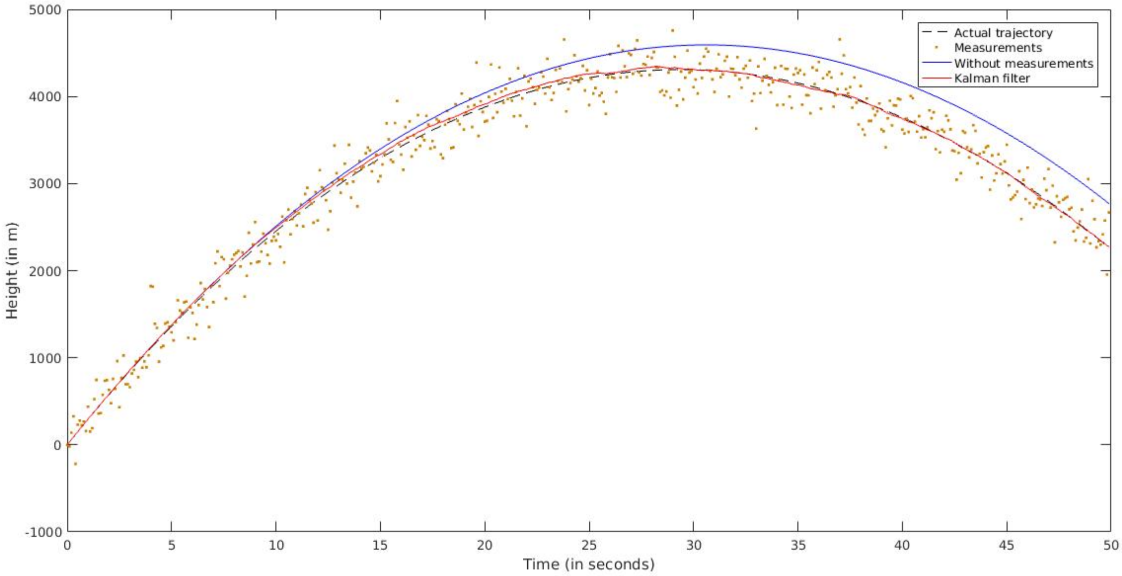
Now, the state equations to model the object's state are :

$$v_n = v_0 - gn\Delta t$$
$$\implies \quad v_n = v_{n-1} - g\Delta t$$
$$Also,\ h_n = v_0 n\Delta t - \frac{1}{2}g(n\Delta t)^2$$
$$\implies \quad h_n = h_{n-1} + v_{n-1}\Delta t - \frac{1}{2}g\Delta t^2$$
$$\implies \quad x_{n+1} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} x_n + \begin{bmatrix} -\frac{1}{2}g\Delta t^2 \\ -g\Delta t \end{bmatrix} + \varepsilon_{n+1} \qquad where\ \varepsilon_{n+1} \sim N(0, R_n)$$

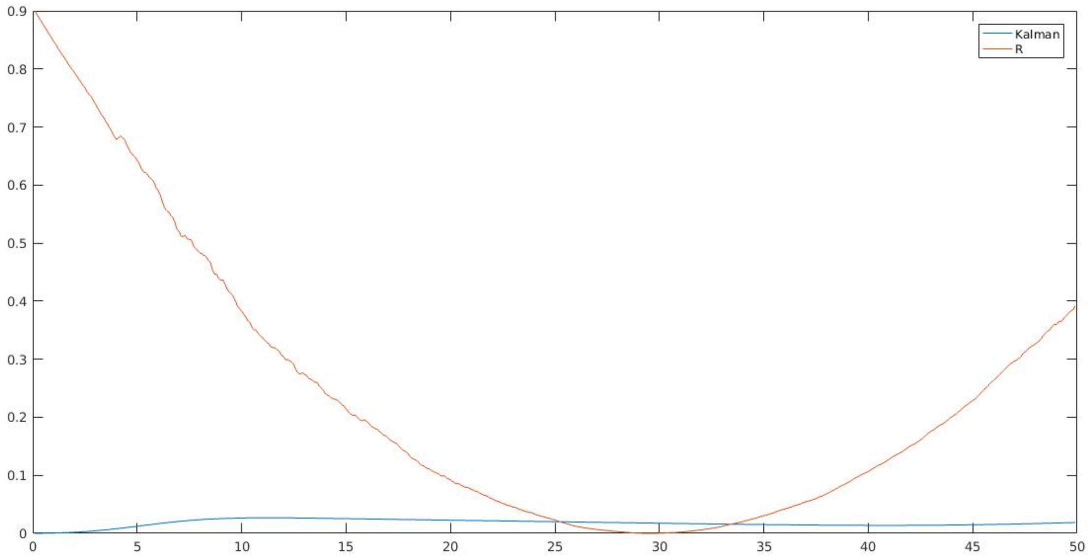
The measurement model is given by, where z_n is the measurement at time nΔt :

$$z_n = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_n + \delta_{n+1} \qquad\qquad where\ \delta_{n+1} \sim N(0, Q_n)$$

(b)



(c)



Observation :

I've calculated the RMSE (Root Mean Square Error) of both the orginal (with constant R) and modified (with R as a function of $||v||^2$) Kalman filters.

RMSE of Original = 29.4621

RMSE of Modifed = 16.9210

From this, one can conclude that modeling R as a function of $||v||^2$ provides a better state estimate than with the model with constant R.