descent algorithms

topic: (deterministic) gradient descent algorithms for unconstrained optimization

refs:

OPTIMAL CONTROL AND ESTIMATION

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Section 3.6

Nonlinear Programming

SECOND EDITION

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Chapter 1

- we've seen why derivatives can be useful to characterize (local) optima in

$$\text{(NLP)} \quad \min_{u \in \mathbb{R}^m} c(u)$$

← assume c is "smooth", i.e. continuously differentiable as many times as needed

- suppose we start with a "guess" $u \in \mathbb{R}^m$ and find $\underbrace{Dc(u)}_{\text{"gradient" of } c} \neq 0$

→ how should we modify our guess to get closer to a (local) min?

– regarding "$Dc(u)$" as a function $Dc(u) : \mathbb{R}^m \to \mathbb{R}$
$: v \mapsto \langle Dc(u), v \rangle$

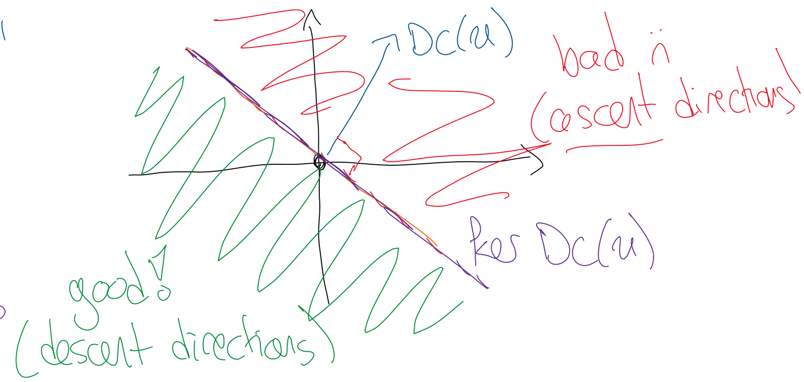
we see that the gradient defines a function that tells us, for any given direction $v \in \mathbb{R}^m$, how quickly c changes in the v direction:

$$c(u + \gamma \cdot v) \simeq c(u) + \gamma \cdot \langle Dc(u), v \rangle$$

* any $v \in \mathbb{R}^m$ s.t. $\langle Dc(u), v \rangle < 0$ is a valid descent direction

e.g. $v = -Dc(u) \Rightarrow \langle Dc(u), -Dc(u) \rangle = -\|Dc(u)\|^2 < 0$

but there's a whole "half-space" of descent directions!



among the infinite volume of possibilities

- to select a descent direction, we can formulate another optimization problem

→ solve $\min_{v \in \mathbb{R}^m} \langle Dc(u), v \rangle$ ← find the steepest / most rapid descent direction

s.t. $\|v\|_2 \leq \|Dc(u)\|_2$

– recalling that $\langle x, y \rangle = \|x\|_2 \|y\|_2 \cos\theta$,

θ = angle between x & y

$\Rightarrow$ solution is $v^* = -Dc(u)^T \in \mathbb{R}^m$

algorithm (gradient descent): $(*)\ u^+ = u - \gamma \cdot Dc(u)^T$, step size $\gamma > 0$

* note that $(*)$ defines a difference equation (DE)

→ observe/show that local minima of (NLP) are equilibria of (DE)

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* let's use stability analysis of (DE) to find largest choice for $\gamma > 0$

→ linearize (DE) about a local min of (NLP)

- differentiating (*) wrt u: $D_u\left[u - \gamma \cdot Dc(u)^T\right] = \underbrace{I - \gamma \cdot D^2c(u)}_{A(\gamma)}$

  evaluating at u^* yields $(u-u^*)^+ \simeq A(\gamma)\cdot(u-u^*)$, $A(\gamma) =$

* need $|\lambda| < 1$ for all $\lambda \in \operatorname{spec} A(\gamma) = \operatorname{spec} I - \gamma \cdot D^2c(u)$

- spectral mapping theorem says that:

$$\text{if } \sigma \in D^2c(u) \text{ then } \lambda = 1 - \gamma\cdot\sigma \in \operatorname{spec} I - \gamma \cdot D^2c(u)$$

→ so we need $-1 < 1 - \gamma\cdot\sigma < +1 \iff 0 < \gamma < \frac{2}{\sigma}$, all $\sigma \in \operatorname{spec} D^2c(u)$

CHALLENGE

* → solve
$$\min_{v \in \mathbb{R}^m} \langle Dc(u), v\rangle$$
$$\text{s.t. } \|v\|_{D^2c(u)} \leq \|Dc(u)\|_{D^2c(u)}$$
$$\text{where } \|v\|_S := \sqrt{\tfrac{1}{2} v^T S v}$$

- let $g = Dc(u)$, $H = D^2c(u)$ so we're solving $\min_{v\in\mathbb{R}^m} \langle g, v\rangle$ s.t. $\|v\|_H \leq \|g\|_H$
$$\Updownarrow$$
$$\tfrac{1}{2} v^T H v \leq \tfrac{1}{2} g H g^T$$

- with $\tilde{c}(v,\lambda) = g\cdot v + \frac{\lambda}{2}\left(v^T H v - g H g^T\right)$

  then necessarily $D_v\tilde{c} = 0$, $D_\lambda \tilde{c} = 0$:

  - $D_v \tilde{c} = g + \lambda \cdot v^T H = 0 \iff v_0 = -\frac{1}{\lambda_0} H^{-1} g^T$

  - $D_\lambda \tilde{c} = \frac{1}{2}\left(v^T H v - g H g^T\right) = 0 \iff \|v_0\|_H^2 = \|g\|_H^2$

$$-D_x \tilde{C} = \frac{1}{2}\left(v^T H v - g H g^T\right) = 0 \iff \|v_0\|_H = \|g\|_H$$

$$\text{but } v_0 = -\frac{1}{\lambda_0} H^{-1} g^T \text{ so } \|v_0\|_H^2 = \frac{1}{\lambda_0^2} g H^{-1} H H^{-1} g^T = \frac{1}{\lambda_0^2} g H g^T = \frac{1}{\lambda_0^2} \|g\|_H^2$$

$$\text{so } \lambda_0^2 = 1 \iff |\lambda_0| = 1 \iff \lambda_0 = \pm 1$$

* sign of λ_0 determines whether v_0 is direction of <u>ascent</u> (−) or <u>descent</u> (+)

<u>note:</u> $u^+ = u - [D^2 c(u)]^{-1} Dc(u)^T$ is called Newton-Raphson