p_trajectory-optimization

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- consider optimal control problem

$$(\text{OCP}) \quad \min_{u} c(x,u) \text{ s.t. } x(s+1) = F(s, x(s), u(s)) \qquad x(s) \in \mathbb{R}^n,\ u(s) \in \mathbb{R}^m$$

where $c(x,u) = \ell(t, x(t)) + \sum_{s=0}^{t-1} \mathcal{L}(s, x(s), u(s))$

- Bellman's principle reduces this to solving a sequence of parameterized NLP:
  - letting $v_s^*(x_s)$ denote optimal value of state x_s,

    we know $v_s^*(x_s) = \min_{u_s \in \mathbb{R}^m} \left[\mathcal{L}(s, x_s, u_s) + v_{s+1}^*(x_{s+1}) \right]$
  - if we solved this parameterized NLP, we'd obtain

    optimal policy $u^* : [0,t) \times \mathbb{R}^n \to \mathbb{R}^m$ that solves v_s^* NLP at every s

    $: (s, x_s) \mapsto u_s^*(x_s)$

$\longrightarrow$ generally intractable — requires solution for all x_s

- what if we tried to solve (OCP) for a given $x_0 \in \mathbb{R}^n$

  to obtain optimal $u_{x_0}^* : [0,t) \to \mathbb{R}^m$?

$\longrightarrow$ show that (OCP) can be rewritten as

$$(\text{NLP}) \quad \min_{\bar{u} \in \mathbb{R}^{\bar{m}}} \bar{c}(\bar{x}, \bar{u}) \text{ s.t. } \bar{f}(\bar{x}, \bar{u}) = 0$$

i.e. determine $\bar{u}, \bar{x}, \bar{c}, \bar{f}$

- note that $(u : [0,t) \to \mathbb{R}^m) \in (\mathbb{R}^m)^t = \mathbb{R}^{mt} \ni \bar{u}$

  i.e. define $\bar{u} \in \mathbb{R}^{t \cdot m}$ be $[\bar{u}]_s = u(s)$

  and similarly $\bar{x} \in \mathbb{R}^{t \cdot n}$, $[\bar{x}]_s = x_u(s)$

  where x_u is generated by u from x_0

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- then $\bar{c}(\bar{x},\bar{u}) = c(x,u)$, $\bar{f}(\bar{x},\bar{u}) = 0 \iff x_{s+1} - F_s(x_s,u_s) = 0$

- for $u_0 : [0,t) \to \mathbb{R}^m$, $x_0 : [0,t) \to \mathbb{R}^m$ to be optimal, they must be stationary: $\underbrace{D\bar{c}(\bar{x},\bar{u})}_{(1\times t\cdot n \times t\cdot m)} + \bar{\lambda}\cdot D\bar{f}(\bar{x},\bar{u}) = 0$

$\to$ start at step t, use stationarity wrt $x_t \in \mathbb{R}^n$ to determine $\lambda_t \in \mathbb{R}^{1\times n}$

- $D_{x_t}\bar{c} = D_{x_t} l_t$

- $D_{x_t}\bar{f} = D_{x_t}\left[x_t - F_{t-1}(x_{t-1}, u_{t-1})\right] = I_n$

- so $D_{x_t}\bar{c} + \lambda_t \cdot D_{x_t}\bar{f} = D_{x_t} l_t + \lambda_t \cdot I_n$, so $\boxed{\lambda_t = -D_{x_t} l_t}$

$\to$ start at step s, use stationarity wrt x_s to determine λ_s

- $D_{x_s}\bar{c} = D_{x_s}\mathcal{L}_s$

- $D_{x_s}\bar{f}$: $D_{x_s}\left[x_s - F_{s-1}(x_{s-1}, u_{s-1})\right] = I_n$

$$D_{x_s}\left[x_{s+1} - F_s(x_s, u_s)\right] = -D_{x_s} F_s$$

- so $D_{x_s}\bar{c} + [\lambda_s, \lambda_{s-1}] \cdot D_{x_s}\bar{f} = 0$

$$= D_{x_s}\mathcal{L}_s + \lambda_{s-1}\cdot I_n - \lambda_s \cdot D_{x_s} F_s = 0$$

$$\iff \boxed{\lambda_{s-1} = -D_{x_s}\mathcal{L}_s + \lambda_s \cdot D_{x_s} F_s}$$

- to summarize: if $(x^*, u^*) : [0,t) \to \mathbb{R}^n \times \mathbb{R}^m$ optimal for (OCP)

then necessarily $\lambda^* : (0,t] \to \mathbb{R}^n$

• In summary: if $(x^*, u^*) : [0,t) \to \mathbb{R}^n \times \mathbb{R}^m$ optimal

then necessarily $\lambda^* : (0,t] \to \mathbb{R}^n$

defined by $\lambda_t^* = -D_{x_t} \ell_t(x_t^*)$

$$\lambda_s^* \cdot A_s - b_s = \lambda_{s-1}^* = \lambda_s^* \cdot D_{x_s} F_s(x_s^*, u_s^*) - D_{x_s} \mathcal{L}_s(x_s^*, u_s^*)$$

ensure $\underbrace{D\bar{c}(\bar{x}^*, \bar{u}^*) + \lambda^* \cdot D\bar{f}(\bar{x}^*, \bar{u}^*)}$

$$D_{u_s} \mathcal{L}_s(x_s^*, u_s^*) + \lambda_s^* \cdot D_{u_s} F_s(x_s^*, u_s^*) = 0$$

ex: $\mathcal{L}_s(x_s, u_s) = g_s(x_s) + \frac{1}{2} u^T R_s u$

$\Rightarrow D_{u_s} \mathcal{L}_s = u^T R_s$,

so $\boxed{u_s^+ = u_s - \gamma \cdot R^{-1} \lambda_s \cdot D_{u_s} F_s(x_s, u_s)}$