Assignment 2: Discrete Time Markov Chains

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February 19, 2015

1 The Question

Assume a finite buffer of size 2 connecting a sender to a receiver. The system operates synchronously so that during one cycle, provided the buffer is not full, the sender produces a packet with probability l and stores it in the buffer. Additionally, provided the buffer is not empty, the receiver removes a packet from the buffer with probability m. Note that both actions can take place during one cycle (e.g., a packet in the buffer is removed and another packet is inserted leaving the buffer in the same state).

1.1 Markov Chain

The states below represent the number of packets in the buffer at any time. In state 0, you cannot consume because the buffer is empty. Similarly, in state 2, you cannot produce because the buffer is full. Finally, in state 1, you have to multiply the probability of not consuming when a packet is produced, as a production and consumption can occur during one cycle. The same is true in reverse for transitioning to state 0.

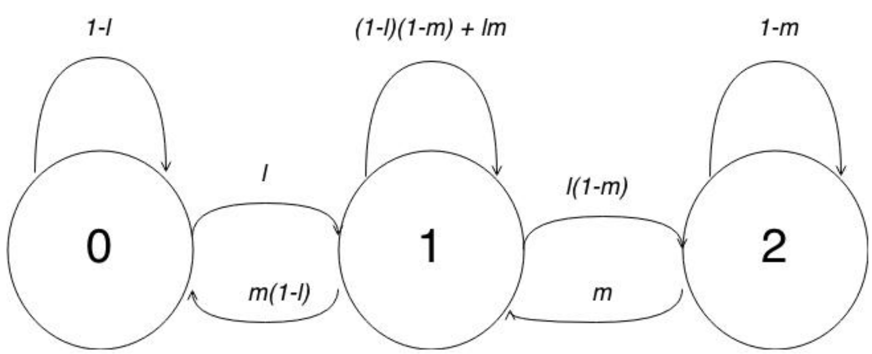

Figure 1: Sender/receiver finite buffer Markov Chain

1.2 Probability Matrix

$$
\begin{array}{cc}
\begin{array}{ccc} \quad 0 \quad & \quad\quad 1 \quad\quad & \quad 2 \quad \end{array} & \\
\begin{bmatrix} 1-l & l & 0 \\ m(1-l) & (1-l)(1-m)+lm & l(1-m) \\ 0 & m & 1-m \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \end{array}
\end{array}
$$

1.3 Irreducibility

This markov chain is irreducible if l,m≠0 and l,m≠ 1.

1.4 Invariant Probability Distribution

The system of equations based on the probability matrix in section 1.2 is below:

$\pi(0) = (1-l)\pi(0) + m(1-l)\pi(1)$
$\pi(1) = l\pi(0) + ((1-l)(1-m) + lm)\pi(1) + m\pi(2)$
$\pi(2) = l(1-m)\pi(1) + (1-m)\pi(2)$

The system of equations was inputted into a solver at http://quickmath.com. This comes out to an invariant probability distribution of the following:

$$
\begin{cases}
x = \dfrac{(l-1)\, m^2}{(l-1)\, m^2 + (l^2 - l)\, m - l^2} \\
y = -\dfrac{l\, m}{(l-1)\, m^2 + (l^2 - l)\, m - l^2} \\
z = \dfrac{l^2\, m - l^2}{(l-1)\, m^2 + (l^2 - l)\, m - l^2}
\end{cases}
$$

Figure 2: Sender/receiver finite buffer Markov Chain