# NOTES

Edited by: **John Duncan**

# A Proof of Dilworth's Chain Decomposition Theorem

**Fred Galvin**

Let $P$ be a finite partially ordered set: a *chain* (*antichain*) in $P$ is a set of pairwise comparable (incomparable) elements; the *width* of $P$ is the maximum cardinality of an antichain in $P$. According to a celebrated theorem of Dilworth [**2**], the width of $P$ is also equal to the minimum number of chains needed to cover $P$. The wider combinatorial significance of Dilworth's theorem, especially as regards matching theory, is discussed by Bogart, Greene, and Kung [**1**], Mirsky [**3**], and Reichmeider [**5**]. Bogart, Greene, and Kung survey various proofs of Dilworth's theorem; the proofs of Perles [**4**] and Tverberg [**6**] are especially simple and elegant. The proof given here seems to me to be as simple as any. This proof is probably well-known folklore; still, as far as I know, it has never appeared in print.

**Theorem.** *A finite partially ordered set $P$ is the union of $m$ chains, where $m$ is the width of $P$.*

*Proof:* We use induction on the cardinality of $P$. Let $a$ be a maximal element of $P$, and let $P' = P \setminus \{a\}$ have width $n$. Then $P'$ is the union of $n$ disjoint chains $C_1, \dots, C_n$. We have to show that $P$ either contains an $(n+1)$-element antichain, or else is the union of $n$ chains. Now, every $n$-element antichain in $P'$ consists of one element from each $C_i$. Let $a_i = \max\{x \in C_i\colon x$ belongs to some $n$-element antichain in $P'\}$. It is easy to see that $A = \{a_1, \dots, a_n\}$ is an antichain. If $A \cup \{a\}$ is an antichain, we are done. Otherwise, we have $a > a_i$ for some $i$. Then $K = \{a\} \cup \{x \in C_i\colon x \le a_i\}$ is a chain, and there are no $n$-element antichains in $P \setminus K$, whence $P \setminus K$ is the union of $n-1$ chains.

REFERENCES


1. Kenneth P. Bogart, Curtis Greene, and Joseph P. S. Kung, The impact of the chain decomposition theorem on classical combinatorics, in: Kenneth P. Bogart, Ralph Freese, and Joseph P. S. Kung, Eds., *The Dilworth Theorems*, Birkhäuser, Boston, 1990, 19–29.
2. R. P. Dilworth, A decomposition theorem for partially ordered sets, *Ann. of Math.* 51 (1950), 161–166.
3. L. Mirsky, *Transversal Theory*, Academic Press, New York, 1971.
4. Micha A. Perles, A proof of Dilworth's decomposition theorem for partially ordered sets, *Israel J. Math.* 1 (1963), 105–107.
5. Philip F. Reichmeider, *The Equivalence of Some Combinatorial Matching Theorems*, Polygonal Publishing House, Washington, New Jersey, 1984.

6. Helge Tverberg, On Dilworth's decomposition theorem for partially ordered sets, *J. Combin. Theory* 3 (1967), 305–306.

*Department of Mathematics*
*University of Kansas*
*Lawrence, KS 66045-2142*

# On Intervals, Transitivity = Chaos.

**Michel Vellekoop and Raoul Berglund**

In an earlier article in the American Mathematical Monthly a redundancy was found in the definition of chaos by Devaney [**1**]:

Let $V$ be a set. $A$ continuous map $f\colon V \to V$ is said to be chaotic on $V$ if

(1) $f$ is *topologically transitive*: for any pair of open non-empty sets $U, W \subset V$ there exists a $k > 0$ such that $f^k(U) \cap W \neq \emptyset$.
(2) the *periodic points* of $f$ are *dense* in $V$.
(3) $f$ has *sensitive dependence on initial conditions*: there exists a $\delta > 0$ such that, for any $x \in V$ and any neighbourhood $N$ of $x$, there exists a $y \in N$ and an $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

In [**2**], Banks et al. prove that (1) and (2) imply (3) in any metric space $V$, and in [**3**] Assaf IV and Gadbois show that for *general* maps this is the only redundancy: (1) and (3) do not imply (2), and (2) and (3) do not imply (1). But if we restrict our attention to maps on an interval a stronger result can be obtained:

**Proposition.** *Let $I$ be a, not necessarily finite, interval and $f\colon I \to I$ a continuous and topologically transitive map. Then* (1) *the periodic points of $f$ are dense in $I$ and* (2) *$f$ has sensitive dependence on initial conditions.*

The first result (1) can be found in [**4**] (Chapter IV.5, Lemma 41) but the proof uses a lot of other highly non-trivial results. Since Devaney's text is being used by so many students, we think that it is interesting to give a very short, intuitive proof of this proposition.

We will need the following lemma, which can be found in [**4**] (Chapter IV.1, Corollary 10) in a more general form:

**Lemma.** *Suppose that $I$ is a, not necessarily finite, interval and $f\colon I \to I$ is a continuous map. If $J \subset I$ is an interval which contains no periodic points of $f$ and $z$, $f^m(z)$ and $f^n(z) \in J$ with $0 < m < n$, then either $z < f^m(z) < f^n(z)$ or $z > f^m(z) > f^n(z)$.*

*Proof of the lemma:* Suppose we can find such a $z \in J$ with $z < f^m(z)$ and $f^m(z) > f^n(z)$. Define the function $g(x) = f^m(x)$. Then we know that $z < g(z)$ and this implies $z < g(z) < g^{k+1}(z)$ for all natural numbers $k \geq 1$ by induction. Because, if $g^{k+1}(z) < g(z)$ for a certain $k$ then the function $g^k(x) - x$ has a