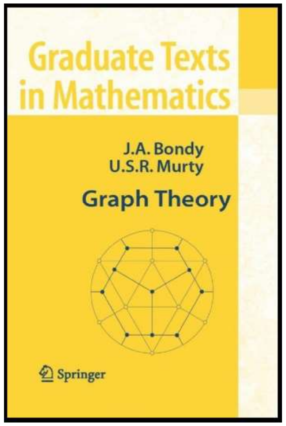
Graph Theory

Chapter 3. Connected Graphs

3.3. Euler Tours—Proofs of Theorems



Lemma 3.3.A

Lemma 3.3.A. If G is an Eulerian graph then G is even.

Proof. Let the walk $W = v_0 e_1 v_1 e_2 v_2 \cdots v_{m-1} e_m v_0$ be an Euler tour of G. The internal vertices of W, namely $v_1, v_2, \ldots, v_{m-1}$, are incident with edges in W 2 at a time; for $i = 1, 2, \ldots, m-1$ we have v_i is incident with edges e_i and e_{i+1}. Since the edges are distinct in an Euler tour then, each internal vertex in an Euler tour (i.e., each internal vertex of W) is of even degree. Now v_0 may also appear in the set of vertices $\{v_1, v_2, \ldots, v_{m-1}\}$ and it will be incident to an even number of edges in the counting process used for these vertices. But v_0 is also incident to edges e_1 and e_m, so its total degree is even as well. That is (since G is connected, by the definition of "tour"), each vertex of G is of even degree and G is an even graph, as claimed. □

Theorem 3.4

Theorem 3.4. If G is a connected even graph, then the walk W returned by Fleury's Algorithm is an Euler tour of G.

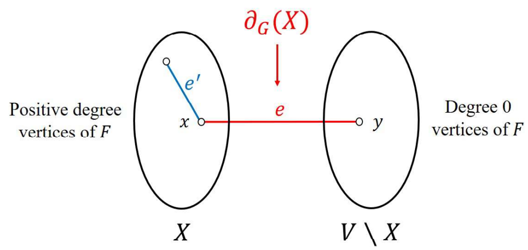
Proof. Since the algorithm chooses an edge to add to the walk W under construction and then deletes that edge (when replacing F by $F \setminus e$) from those which may be chosen in subsequent steps, then the edges of walk W must be distinct and so the walk is a trail throughout the procedure. The algorithm starts with initial vertex u of W, so in graph F we have $d_F(u) = d_G(u) - 1$ initially and remains so unless the walk returns to u and continues on, so that $d_F(u)$ drops by 2 and $d_F(u)$ remains odd. Since G is an even graph by hypothesis, the algorithm cannot terminate at some $x \neq u$ since such vertex x is of even degree in G and when W has x as its terminal vertex we then have $d_F(x)$ odd so that $\partial_F(x) \neq \varnothing$ and the algorithm does not end. So the algorithm and the walk W produced can only terminate at vertex u. Hence the algorithm produces a closed trail of G with vertex u as its initial and terminal vertex. We now need to confirm that W includes all edges of G. We do so with a proof by contradiction.

Theorem 3.4 (continued 1)

Proof (continued). ASSUME that W, the walk produced by Fleury's Algorithm, is not an Euler tour of G. Let X be the set of vertices of positive degree in subgraph F at the stage when the algorithm terminates. Then $X \neq \varnothing$ (since W is assumed to omit some edge(s) of G), G is even by hypothesis, and W determines an even induced subgraph of G, so the induced subgraph $F[X]$ is an even subgraph of G. As described above, the algorithm must terminate at vertex u so $d_F(u) = 0$, $u \notin X$, and so $u \in V \setminus X$ so that $V \setminus X \neq \varnothing$. Now $\partial_G(X) \neq \varnothing$, or else X and $V \setminus X$ would form a "separation" of G, but G is hypothesized to be connected. But $\partial_F(X) = \varnothing$ since each vertex of $V \setminus X$ has degree 0 in F. Since the algorithm selects edges for inclusion in W (and then deletes those edges in the creation of F), all of the edges of $\partial_G(X)$ must have been chosen for W since, when the algorithm ended, we had $\partial_F(X) = \varnothing$. Let $e = xy$ be the last edge of $\partial_G(X)$ chosen for inclusion in W, where $x \in X$ and $y \in V \setminus X$.

Theorem 3.4 (continued 2)

Proof (continued). At the step when e was chosen, graph F must have included edge e and edge e must be a cut edge of F (since this is the step at which the last edge of $\partial_G(X)$ was added to W so that $c(F \setminus e) = c(F) + 1$ because $F \setminus e$ has a "new" component which is a subset of $V \setminus X$ and includes y; see the figure below).



But when the algorithm ended, the degree of x in the final graph F was positive so that there was another choice of an edge e' to add to W when cut edge e was chosen, since $d_F(x) > 0$ when the algorithm ends.

Theorem 3.4 (continued 3)

Theorem 3.4. If G is a connected even graph, then the walk W returned by Fleury's Algorithm is an Euler tour of G.

Proof (continued). As argued above, all vertices of X are of even positive degree in $F[X]$ when the algorithm ends. So by Exercise 3.2.3(a), the connected component of $F \setminus e$ containing vertex x has no cut edges so that e' is not a cut edge of $F \setminus e$ (and so not of F itself before edge e was chosen). But this violates the algorithm since it will not add cut edge e to W since non-cut edge e' is available for inclusion in W, a CONTRADICTION. So the assumption that the walk produced by the algorithm is not an Euler tour is false. Hence, an Euler tour of G is produced by the algorithm, as claimed. $\square$