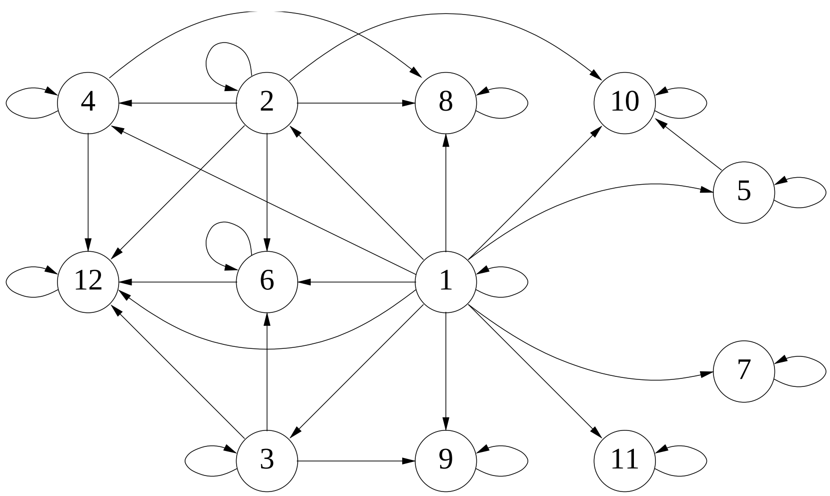
Solutions to In-Class Problems Week 5, Mon.

Problem 1.
If a and b are distinct nodes of a digraph, then a is said to *cover* b if there is an edge from a to b and every path from a to b traverses this edge. If a covers b, the edge from a to b is called a *covering edge*.

(a) What are the covering edges in the following DAG?



Solution. TBA ■

(b) Let covering (D) be the subgraph of D consisting of only the covering edges. Suppose D is a finite DAG. Explain why covering (D) has the same positive path relation as D.

Hint: Consider *longest* paths between a pair of vertices.

Solution. What we need to show is that if there is a path in D between vertices $a \neq b$, then there is a path consisting only of covering edges from a to b. But since D is a finite DAG, there must be a *longest* path from a to b. Now every edge on this path must be a covering edge or it could be replaced by a path of length 2 or more, yielding a longer path from a to b. ■

(c) Show that if two *DAG*'s have the same positive path relation, then they have the same set of covering edges.

Solution. *Proof.* Suppose C and D are DAG's with the same positive path relation and that $a \to b$ is a covering edge of C. We want to show that $a \to b$ must also be a covering edge of D.

Since $a \to b$ itself defines a (length one) positive length path in C, there must be a positive length path in D from a to b. If this positive length path in D is of length greater than one, then the path must consist of a positive length path from a to c followed by a positive length path from c to b for some vertex, c. Also, since D is a DAG, c cannot be a or b.

This means there must also be positive length paths in C from a to c and from c to b, and neither of these paths can traverse $a \to b$ or there would be a cycle. Hence the path from a to c to b is a path in C that does not traverse $a \to b$, contradicting the fact that $a \to b$ is a covering edge of C.

In sum, there is a length one path from a to b in D, namely $a \to b$, and this is the *only* path from a to b in D, which proves that $a \to b$ is a covering edge in D. ■

(d) Conclude that covering (D) is the *unique* DAG with the smallest number of edges among all digraphs with the same positive path relation as D.

Solution. By part (c), any DAG with the same positive path relation as D must contain all the edges of covering (D). By part (b), covering (D) has this same positive path relation. It follows immediately that covering (D) is the unique minimum-size DAG with the same positive path relation as D. ■

The following examples show that the above results don't work in general for digraphs with cycles.

(e) Describe two graphs with vertices $\{1, 2\}$ which have the same set of covering edges, but not the same positive path relation (*Hint:* Self-loops.)

Solution. Let one graph have edges $\{(1, 2), (1, 1)\}$ and the other $\{(1, 2), (2, 2)\}$. They have the same set of covering edges, namely, $(1, 2)$. But in the second there is a positive length path from 2 to 2, namely a path of length one but there is no positive length path from 2 to 2 in the first graph. ■

(f) (i) The *complete digraph* without self-loops on vertices $1, 2, 3$ has edges between every two distinct vertices. What are its covering edges?

(ii) What are the covering edges of the graph with vertices $1, 2, 3$ and edges $1 \to 2, 2 \to 3, 3 \to 1$?

(iii) What about their positive path relations?

Solution. (i) There are no covering edges, since there is always a length two path from a to b that does not use the edge $a \to b$.

(ii) All three edges are the covering edges.

(iii) They have the same positive path relation, namely, each vertex is connected to all the vertices, including itself, by positive length paths.

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Problem 2. **(a)** Give an example showing that two vertices in a digraph may be on the same cycle, but *not* necessarily on the same *simple* cycle.

Solution. Let the vertices be a, b, c and edges be $(a, b), (b, a), (b, c), (c, b)$. Now a and c are on the cyle a, b, c, b, a, but every cycle from a to c must go through b at least twice, and so will not be simple. ■

(b) Prove that if two vertices in a digraph are connected, then they are connected by a simple path. *Hint:* the shortest path.

Solution. Consider a shortest path from a to $b \neq a$:

$$a = a_0, a_1, \ldots, a_i, \ldots, a_j, \ldots, a_k = b,$$

and suppose this path is not simple. That is, suppose $a_i = a_j$ for some i, j. Then

$$a = a_0, a_1, \ldots, a_i, a_{j+1}, \ldots, a_k = b.$$

is a shorter path from a to b, a contradiction.

■

Problem 3.
In an n-player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner —there are no ties. The results of such a tournament can then be represented with a *tournament digraph* where the vertices correspond to players and there is an edge $x \rightarrow y$ iff x beat y in their game.

(a) Explain why a tournament digraph cannot have cycles of length 1 or 2.

Solution. There are no self-loops in a tournament graph since no player plays himself, so no length 1 cycles. Also, it cannot be that x beats y and y beats x for $x \neq y$, since every pair competes exactly once and there are no ties. This means there are no length 2 cycles.

■

(b) Is the "beats" relation for a tournament graph always/sometimes/never:

- asymmetric?
- reflexive?
- irreflexive?
- transitive?

Explain.

Solution. No self-loops implies the relation is irreflexive. It is also asymmetric since it is irreflexive and for every pair of distinct players, exactly one game is played and results in a win for one of the players. Some tournament graphs represent transitive relations and others don't. ■

(c) Show that a tournament graph represents a total order iff there are no cycles of length 3.

Solution. As observed in the previous part, the "beats" relation whose graph is a tournament is asymmetric and irreflexive. Since every pair of players is comparable, the relation is a total order iff it is transitive.

"Beats" is transitive iff for any players x, y and z, $x \to y$ and $y \to z$ implies that $x \to z$ (and consequently that there is no edge $z \to x$). Therefore, "beats" is transitive iff there are no cycles of length 3. ■

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