KURATOWSKI'S THEOREM

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Abstract. This paper introduces basic concepts and theorems in graph theory, with a focus on planar graphs. On the foundation of the basics, we state and present a rigorous proof of Kuratowski's theorem, a necessary and sufficient condition for planarity.

Contents

1. Introduction

The planarity of a graph, whether a graph can be drawn on a plane in a way that no edges intersect, is an interesting property to investigate. With a few simple theorems it can be seen that K_5 (see figure 1) and $K_{3,3}$ (see figure 2) are nonplanar. Kuratowski pushes this nearly effortless observation into a powerful theorem exposing the sufficient and necessary condition of planarity. Simple as the theorem appears to be, to prove this we need a significant amount of preparations. In this paper, we start with basic graph theory and proceed into concepts and theorems related to planar graphs. In the last section we will give a proof of Kuratowski's theorem, which in general corresponds with that in *Graph Theory with Applications* (see [1] in the list of references) but provides more details and hopefully more clarity.

2. Basic Graph Theory

We need several definitions as a start.

Definition 2.1. A *graph* G is an ordered pair $(V(G), E(G))$, consisting of a nonempty set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, each edge a two-element subset of V. Denote $|V(G)|$, the number of vertices in the vertex set, by ν and $|E(G)|$ by ϵ. Note that (i) $E(G)$ can be empty and (ii) an edge can link a vertex to itself.

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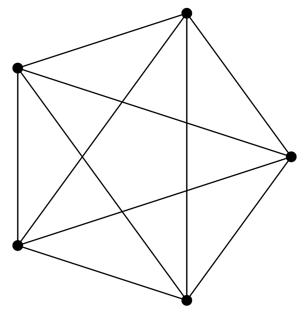

FIGURE 1. a diagram of K_5

Definition 2.2. An edge whose ends are identical is a *loop*. Otherwise, the edge is a *link*. A graph is *simple* if it has no loops and no two links have the same pair of unordered vertices.

Definition 2.3. A vertex is *incident* to an edge if the vertex is one of the ends of the edge; two vertices are *adjacent* to each other if they are connected by an edge. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges incident to v, with a loop counted as two edges. We let $\delta := min_{v\in V}(\deg v)$ and $\Delta := max_{v\in V}(\deg v)$.

Definition 2.4. A *walk* W is a set of alternating vertices and edges, denoted by $W{=}v_0e_1v_1e_2...e_kv_k$ where e_i ($i \in [1, k]$, $i \in \mathbb{N}$) links v_{i-1} with v_i. A walk where all e_i's are distinct is a *trail*. In addition, if all vertices are distinct, then W is a *path*. A graph G is *connected* if there exists a path between every pair of vertices in G.

A walk is *closed* if it has positive length and identical ends. A closed trail with distinct vertices is a *cycle*.

A family of walks is *internally disjoint* if no vertex is an internal vertex of more than one walk in the family.

Definition 2.5. H is a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and endpoints of all edges in $E(H)$ are included in $V(H)$. In addition, if H is a maximally connected subgraph, H is a *component* of G. The number of components in G is denoted by $\omega(G)$. An edge e is a *cut edge* if $\omega(G - e) > \omega(G)$.

Definition 2.6. A *complete* graph is a graph where each pair of vertices is connected by a unique edge. We let K_m denote the complete graph on m vertices.

A graph G is *bipartite* if its vertex set V can be divided into two nonempty subsets X and Y such that every edge in G connects one vertex in X to another one in Y.

A graph G is *complete bipartite* if for all $x \in X, y \in Y$, x is connected to y by a unique edge. When X contains m vertices and Y contains n vertices, G is denoted by $K_{m,n}$.

Theorem 2.7. *For a graph G with vertex set V and edge set E,*

$$\sum_{v\in V} \deg(v) = 2\epsilon$$

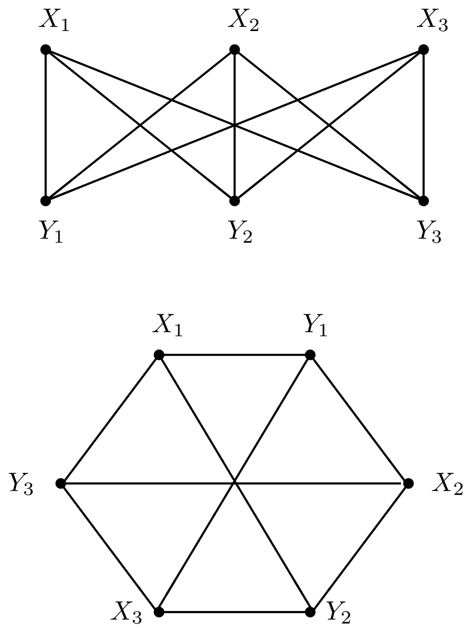

FIGURE 2. two diagrams of $K_{3,3}$

Proof. For $v \in V$ and $e \in E$, we let $n(v)$ denote the set of edges incident to v and $m(e)$ the set of endpoints of e. Then

$$\begin{aligned}\sum_{v\in V}\deg(v) &= \sum_{v\in V}\Big(\sum_{e\in n(v)} 1\Big)\\ &= \sum_{e\in E}\Big(\sum_{v\in m(e)} 1\Big)\\ &= 2\epsilon.\end{aligned}$$

□

3. PLANAR GRAPHS

Definition 3.1. A way to draw the graph, representing vertices by points and edges by lines connecting points, is called a *diagram* of the graph. A diagram is embedded in the plane. A graph that has a diagram whose edges do not intersect anywhere besides their ends (i.e., vertices) is called a *planar* graph. The diagram is then called the *planar embedding* of a planar graph, or simply, a *plane graph.*

Definition 3.2. Closures of regions partitioned by a plane graph are *faces*, the number of which is denoted by ϕ. In a plane graph, the degree of a face f, denoted by $\deg(f)$, is the number of edges incident to f, with cut edges being counted twice.

Definition 3.3. A *dual* of a plane graph G, denoted by G^*, can be constructed as follows: every vertex v^* in G^* corresponds to a face f in G and every edge e^* in G^* corresponds to an edge e in G. Two vertices in G^*, v^* and w^*, are joined by the edge e^* if and only if their corresponding faces in G, f and g, are separated by e.

Theorem 3.4. *For G a plane graph, let $F(G)$ denote the set of faces in G. The following holds:*

$$\sum_{f\in F(G)} \deg(f) = 2\epsilon.$$

Proof. Consider the dual, G^*, of G. By Theorem 2.13,

$$\sum_{v^*\in V^*} \deg(v^*) = 2\epsilon^*.$$

By definition of a dual graph, $\forall f \in F(G)$, $\deg(f)$=$\deg(v^*)$, and ϵ=ϵ^*.Then

$$\sum_{f\in F(G)} \deg(f) = \sum_{v^*\in V^*} \deg(v^*) = 2\epsilon^* = 2\epsilon.$$

□

Theorem 3.5. *(Euler's Formula) For G a connected plane graph, the following relationship holds:*

$$\nu - \epsilon + \phi = 2.$$

Proof. We prove this by induction on ϵ.

(1) Basic step: when $\epsilon = 0$, since G is connected, G must be trivial. Then $\nu = 1$, $\epsilon = 0$, $\phi = 1$.

$$\nu - \epsilon + \phi = 1 - 0 + 1 = 2.$$

Clearly the formula holds.

(2) Inductive step: suppose for G with $\epsilon(G) = k$,

$$\nu(G) - \epsilon(G) + \phi(G) = 2.$$

Let H be a planar graph such that G is a subgraph and $\epsilon(H) = k+1$. There are 3 cases:

Case(a): e is a loop added on some $v \in V$. Then $\nu(H) = \nu(G)$. Since H needs to remain a plane graph, the loop does not intersect with any other edge in G. Therefore the loop constitutes a new face, $\phi(H) = \phi(G) + 1$, which then gives us

$$\begin{aligned}\nu(H) - \epsilon(H) + \phi(H) &= \nu(G) - (\epsilon(G) + 1) + (\phi(G) + 1)\\ &= \nu(G) - \epsilon(G) + \phi(G)\\ &= 2.\end{aligned}$$

Case(b): e adds a link between two vertices v_1, v_2 in G. Since G is connected, there is a path between v_1 and v_2. The addition of e creates a new cycle, and hence a new face (if there are more than one path between v_1 and v_2, there is necessarily one that borders the new face). Therefore $\nu(H) = \nu(G)$, $\phi(H) = \phi(G)+1$, $\epsilon(H) = \epsilon(G) + 1$. Similar to (a), the formula holds.

Case(c): e links some $v_1 \in V(G)$ to a new vertex v_2. Then $\nu(H) = \nu(G) + 1$. Since v_2 is a new vertex, e lies in some face bordering v_1, without creating any new face. Then $\phi(H) = \phi(G)$. In this case,

$$\begin{aligned}\nu(H) - \epsilon(H) + \phi(H) &= (\nu(G) + 1) - (\epsilon(G) + 1) + \phi(G)\\ &= \nu(G) - \epsilon(G) + \phi(G)\\ &= 2.\end{aligned}$$

Therefore in all 3 cases, the formula holds in H. By the principle of induction, the formula holds for all connected plane graphs. □

Corollary 3.6. *For G a simple planar graph with $\nu \geq 3$,*

$$\epsilon \leq 3\nu - 6.$$

Proof. Since G is a simple graph with $\nu(G) \geq 3$, the planar embedding of G, G', is also simple with $\nu(G') \geq 3$. For $f \in F(G')$, there exist 3 cases:

(1) if f is bounded by a cycle, since G' is simple, the minimum size of a cycle is 3 and $\deg_{G'}(f) \geq 3$;

(2) if f is incident to some edges in addition to a cycle, $\deg_{G'}(f) > 3$;

(3) if f is incident to no cycle at all, then there exists at least two cut edges on the boundary of f—if not, the rest of the boundary must be a cycle, which contradicts the condition—which means $\deg_{G'}(f) \geq 4 > 3$.

Therefore, for every $f \in F(G')$,

$$\sum \deg_{G'}(f) \geq 3\phi(G')$$

then by Theorem 3.7,

$$\sum_{f \in F(G')} \deg(f) = 2\epsilon(G') \geq 3\phi(G')$$

which yields

$$\phi(G') \leq \frac{2}{3}\epsilon(G').$$

Then by Theorem 3.8,

$$\begin{aligned} \nu(G') - \epsilon(G') + \phi(G') = 2 &\leq \nu(G') - \epsilon(G') + \frac{2}{3}\epsilon(G') \\ &= \nu(G') - \frac{1}{3}\epsilon(G') \\ &= \nu(G) - \frac{1}{3}\epsilon(G). \end{aligned}$$

Therefore,

$$\nu(G) - \frac{1}{3}\epsilon(G) \geq 2$$

and

$$\epsilon \leq 3\nu - 6.$$

□

Corollary 3.7. *K_5 is nonplanar.*

Proof. We have

$$\epsilon(K_5) = \binom{5}{2} = 10 > 3\nu(K_5) - 6 = 9,$$

by Corollary 3.9, K_5 is nonplanar. □

Corollary 3.8. *$K_{3,3}$ is nonplanar.*

Proof. We are going to prove this by contradiction. Suppose there exists a planar embedding of $K_{3,3}$. In a simple bipartite graph, the minimum size of a cycle is 4, which means for $f \in F(K_{3,3})$,

$$\deg(f) \geq 4$$

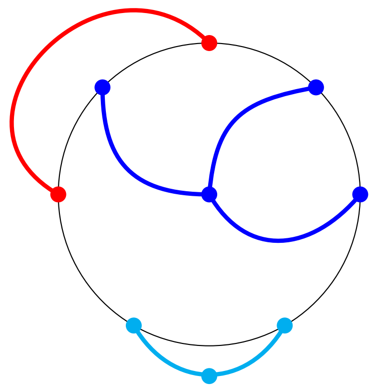

FIGURE 3. an example of 3 bridges on a cycle (the red one is skew to the blue one; the cyan one avoids the other two; red and cyan outer bridges, blue an inner bridge)

(the argument similar to that in Corollary 3.9). By Theorem 3.7,

$$4\phi \leq \sum_{f\in F(G)} \deg(f) = 2\epsilon = 18$$

then since $\phi \in \mathbb{N}$,

$$\phi \leq 4.$$

By Theorem 3.8,

$$\nu - \epsilon + \phi = 2 \leq 6 - 9 + 4 = 1,$$

which is impossible. Therefore, $K_{3,3}$ is nonplanar. $\square$

Definition 3.9. Let H be a subgraph of a graph G. Define an equivalence relation $\sim$ on $E(G) \setminus E(H)$ as follows: $a \sim b$ if there is a walk W such that a and b are the first and last edge in W respectively, and that no internal vertex of W is in $V(H)$.

A *bridge* of H in G is a subgraph of $G - E(H)$ induced by an equivalent class of $\sim$ (a bridge containing e is the subgraph containing every e', $e' \sim e$, e and e' edges of G). For a bridge B of H, we define *vertices of attachment* of B to H as the vertices in the set $V(B) \cap V(H)$.

Let C be a cycle. Then two bridges of C, B_1 and B_2, are *skew* if two vertices of attachment of B_1, say u_1 and v_1, and two of B_2, u_2 and v_2, appear in the order of u_1, u_2, v_1, v_2 on C.

Definition 3.10. Suppose C is a cycle in a planar embedding of a planar graph G. Then for some bridge B of C, B is contained entirely in either $Int(C)$ (the region inside C) or $Ext(C)$ (the region outside C). A bridge in $Int(C)$ is an *inner bridge*, while one in $Ext(C)$ is an *outer bridge*.

In the planar embedding, inner (or outer) bridges *avoid* each other: for all B_1, B_2 two inner(outer) bridges, all vertices of attachment in B_1 lie on the arc uv of C which contains no vertices of attachment of B_2 other than u and v.

Definition 3.11. In some planar embedding G_1 of a planar graph G, an inner bridge B of C (a cycle in G) is *transferrable* if there exists another planar embedding G_2 of G, where B is an outer bridge but everything else remains the same as in G_1.

Theorem 3.12. *Let G be a plane graph and C a cycle in G. An inner bridge B of C is transferrable if B avoids every outer bridge of C.*

Proof. Find an inner bridge B that avoids every outer bridge. Then we can find a face in $Ext(C)$ whose boundary contains all vertices of attachment of B. Drawing B on the new face gives us another planar embedding, which means B is transferrable. □

4. Kuratowski's Theorem

In 1930, Kuratowski published the theorem giving a necessary and sufficient condition for planarity. Kuratowski's Theorem states that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. To prove this theorem, we first need some simple lemmas.

Lemma 4.1. *Every subgraph of a planar graph is planar.*

Proof. If G is planar, then there exists a planar embedding of G. For every subgraph H of G, we can find the vertices and edges of H in the planar embedding of G. This is how we can construct a planar embedding of H. □

Definition 4.2. A *subdivision* of an edge is the operation where the edge is replaced by a path of length 2, the internal vertex added to the original graph. A *subdivision* of a graph G is a graph achieved by a sequence of edge-subdivisions on G.

Lemma 4.3. *Every subdivision of a nonplanar graph is nonplanar.*

Proof. Suppose for G, there exists a planar embedding of its subdivision, G'. When we remove the vertices created in edge-subdivisions, and reconstruct the original edge (without changing the shape and position of the path), we get a planar embedding of G and find G planar. Therefore, if G is nonplanar, every subdivision of G is nonplanar. □

From the two lemmas above the necessity easily follows. Then it suffices to prove that the condition is also sufficient. In order to show if G contains no subdivisions of K_5 or $K_{3,3}$, G is planar, it is equivalent to show that if G is nonplanar, G has to contain some subdivision of K_5 or $K_{3,3}$. Two definitions are necessary before we get to the strategy of proving the equivalent statement.

Definition 4.4. For a graph G, a *vertex cut* V' is a subset of V whose removal renders $G - V'$ disconnected (when we *remove* a vertex, we remove the vertex as well as all edges incident to it). The connectivity of G, denoted by $\kappa(G)$, is the minimum size of the vertex cut V'. A graph G is said to be *k-connected* if $k \leq \kappa(G)$.

Definition 4.5. For a graph G, H is a *proper subgraph* of G if $V(H) \subsetneq V(G)$ and $E(H) \subsetneq E(G)$. A *minimal nonplanar graph* is a nonplanar graph that does not have any nonplanar proper subgraph.

Clearly, it suffices to prove the statement for all G some minimal nonplanar graph. The strategy is as follows:

(1) Show that if minimal nonplanar graphs without any subdivision of K_5 or $K_{3,3}$

as subgraphs did exist, they would be 3-connected and simple.
(2) Show that every 3-connected graph with no subdivision of K_5 or $K_{3,3}$ as subgraphs is in fact planar. This is how we arrive at a contradiction, forcing the original statement to be true.

To show **(1)**, we need to establish a few more lemmas.

Lemma 4.6. *A minimal nonplanar graph is 2-connected.*

Proof. First show that a minimal nonplanar graph is 1-connected (connected). Suppose G is disconnected and nonplanar, but all of its components are planar. Without loss of generality, suppose G has two components, G_1 and G_2. Since G_1 and G_2 are both planar, we can add a planar embedding of G_1 to one of the faces of a planar embedding of G_2 (the infinite face for example), which yields a planar embedding of G, a contradiction.

Then we show that it is 2-connected. Suppose G is nonplanar, with $\kappa(G) = 1$. By definition of connectivity, there exists a vertex v such that $G - v$ is disconnected. Without loss of generality, suppose $G - v$ has two components, H_1 and H_2. We know that $H_1 \cup v$ and $H_2 \cup v$ are both planar. In the planar embedding of each, we can find a face f whose boundary contains v. With stereographic projection, we can get a planar embedding for each of $H_1 \cup v$ and $H_2 \cup v$ where v lies on the boundary of the unbounded face, by placing the point at infinity on the sphere inside f. Then we can combine $H_1 \cup v$ and $H_2 \cup v$ by merging v and get a planar embedding of G, a contradiction. Therefore if G is a minimal nonplanar graph, G is 2-connected. $\square$

Lemma 4.7. *If G is a graph that has the fewest edges possible among all connected nonplanar graphs with no subdivision of K_5 or $K_{3,3}$, then G is 3-connected.*

Proof. The hypothesis suggests G is a minimal nonplanar graph. By the previous lemma, G is 2-connected. Suppose $\kappa(G) = 2$. Then there exists a vertex cut $\{u, v\}$ such that $G - \{u, v\}$ is disconnected. Name the components of $G - \{u, v\}$ H_1, H_2,..., H_k. Construct M_1, M_2,..., M_k, where M_i is $H_i \cup \{u, v\}$ with the addition of a new edge uv. We claim here that among M_i, $1 \leq i \leq k$, there exists at least one M_i that is nonplanar. Below is a proof for the claim:

Suppose all M_i's are planar for $1 \leq i \leq k$. Then there is a planar embedding for each. Since $\{u, v\}$ and the edge uv are the only part M_i's share, we can merge the planar embeddings of M_i's and get a planar embedding of $G + uv$ ($G \cup \{uv\}$), which means $G + uv$ is planar. Then by Lemma 4.1, G is planar, a contradiction. Therefore, there exists some M_j where $1 \leq j \leq k$ that is nonplanar.

It is clear that $\epsilon(M_j) < \epsilon(G)$. But since G, by the original hypothesis, is the smallest connected nonplanar graphs with no subdivision of K_5 or $K_{3,3}$, M_j must have some subdivision of K_5 or $K_{3,3}$. Moreover, since G contains no such subdivision, M_j is not a subgraph of G, which means G does not have an edge uv. Now we combine $M_j - uv$ with $M_p - uv$ where $p \neq j, 1 \leq p \leq k$ by merging the vertices u and v and get a subgraph of G. Since $M_p - uv$ is connected, there exists a path between u and v. When we combine this path with $M_j - uv$, we get a subdivision of K_5 or $K_{3,3}$. This means G contains such a subdivision, a contradiction. Therefore, G has to be 3-connected. $\square$

Now **(1)** has been shown. To complete **(2)**, we again need some lemmas.

Lemma 4.8. *(Whitney's Theorem) Let G be a graph with $\nu \geq 3$. Then G is 2-connected if and only if for all $u, v \in V(G)$, there are at least two internally-disjoint paths between them.*

Proof. ($\Leftarrow$) If any two vertices in G are connected by at least two internally-disjoint paths, then clearly there exists no 1-vertex cut (since no matter which vertex is removed, between every two vertices that remain, there still exists at least one path between them). Hence G is 2-connected.

($\Rightarrow$) Suppose G is 2-connected. We shall prove this direction by induction. Take two vertices $u, v \in V(G)$. Denote the number of edges in the shortest paths between them by $d(u, v)$.

(a) Basic step: when $d(u, v) = 1$. Since G is 2-connected, there exists another path connecting u, v, which does not contain the edge uv.

(b) Inductive step: Suppose there exist at least two internally-disjoint paths for all u, v with $d(u, v) \leq k$. For x, y with $d(x, y) = k + 1$, find a path P_0 of length d(x,y) between x, y and a vertex z that is closest to y ($d(y, z) = 1$) in P_0. Then $d(x, z) = d(x, y) - 1$. By the inductive hypothesis, there exist two internally-disjoint paths P_1 and P_2 between x, z. Since G is 2-connected, there exists another path Q between x, y that does not contain z (or $\{w\}$ would be a 1-vertex cut). Let w be the vertex in $(Q \cap (P_1 \cup P_2))$ that is closest to y on Q. Without loss of generality we suppose w is contained in P_1. Then we can find two internally-disjoint paths between x, y: the first one would be the part from x to w on P_1 combined with the part from w to y on Q; the second one would be P_2 combined with the edge zy. $\square$

Lemma 4.9. *If G is simple and 3-connected and uv is an edge in G, then $G - uv$ is 2-connected.*

Proof. We want to show that for all $a, b \in V(G - uv)$, there exist at least two internally-disjoint paths between them. In other words, we want to show for every two vertices of $G - uv$, there exists a cycle they both lie on. We prove this by discussing 3 cases.

(1) $\{a, b\} = \{u, v\}$. Clearly $\nu(G) \geq 4$. Pick another two vertices c and d in $G - uv$. Without loss of generality, assume $u = a$. Now consider u and c. Since G is 3-connected, G does not contain any 2-vertex cut, which means when v and d are removed, u and c are still connected. In other words, there exists a path P_1 between u and c which does not contain v and d. Similarly, there exists a path P_2 between c and v that avoids u and d, a P_3 between v and d that avoids u and c, and finally a P_4 between d and u that avoids c and v. But then u and v lie on the same cycle u-P_1-c-P_2-v-P_3-d-P_4-u.

(2) One and only one of $\{a, b\}$ is u or v. Without loss of generality, let $a = u$ and $b \neq v$. Find $c \neq b$ that is neither u nor v. Then following the similar argument as in (1), we can find a path P_1 avoiding c and v between u and b, a P_2 avoiding u and v between c and b, and a P_3 avoiding v between c and u. Again u-P_1-b-P_2-c-P_3-u is a cycle containing u, b.

(3) Neither of $\{a, b\}$ equals u or v. Once again following the same argument, we can find a path P_1 avoiding u, v between a, b, a P_2 avoiding u, a between v, b, and a P_3 avoiding u, b between a, v. Then a-P_1-b-P_2-v-P_3-a is a cycle containing a, b.

Since in all 3 cases, we construct a cycle without the edge uv where both a, b lie, the same cycles can be constructed in $G - uv$, which means $G - uv$ must be 2-connected. $\square$

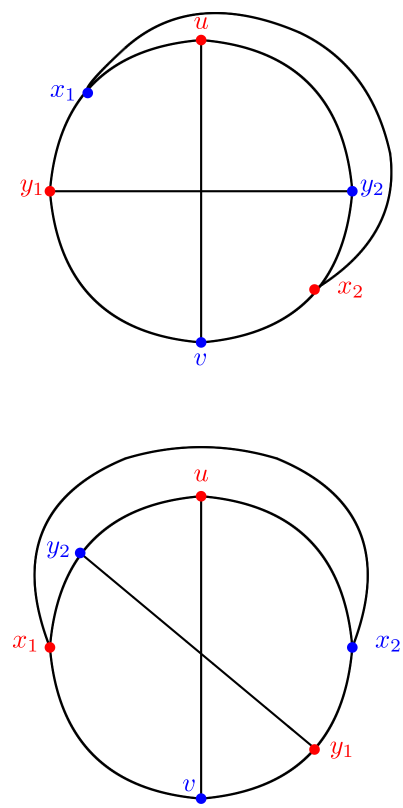

FIGURE 4. Case (1) and (2), with colors indicating bipartition

Now we are ready for the actual proof.

Theorem 4.10. *(Kuratowski's Theorem) A graph is planar if and only if it does not contain any subdivision of K_5 or $K_{3,3}$.*

Proof. ($\Rightarrow$) It is true by the first two lemmas in this section.

($\Leftarrow$) Suppose there exists a nonplanar graph that does not contain any subdivision of K_5 or $K_{3,3}$. Without loss of generality, let G be a nonplanar graph that contains no subdivision K_5 or $K_{3,3}$ and has the fewest edges possible. Then G is a minimal nonplanar graph. By Lemma 4.7, G is 3-connected (and simple). Take two adjacent vertices $u, v \in V(G)$. Consider the subgraph $G - uv$. By minimality, $G - uv$ is planar.

By Lemma 4.9, $G - uv$ is 2-connected. By Lemma 4.8, there are at least two internally-disjoint paths between u and v. In other words, u and v lie on some common cycle. Among all cycles containing u and v in a planar embedding of $G - uv$, find C_0 with the most edges in $Int(C_0)$.

Now consider the bridges of C_0 in $G - uv$ (if $G - uv$ does not contain any bridge of C_0, then it is clear that with the addition of edge uv, the graph is still planar, which means G is planar, a contradiction). Suppose there exists a bridge with only one vertex of attachment v_1. Then v_1 is a one-vertex cut of $G - uv$, which

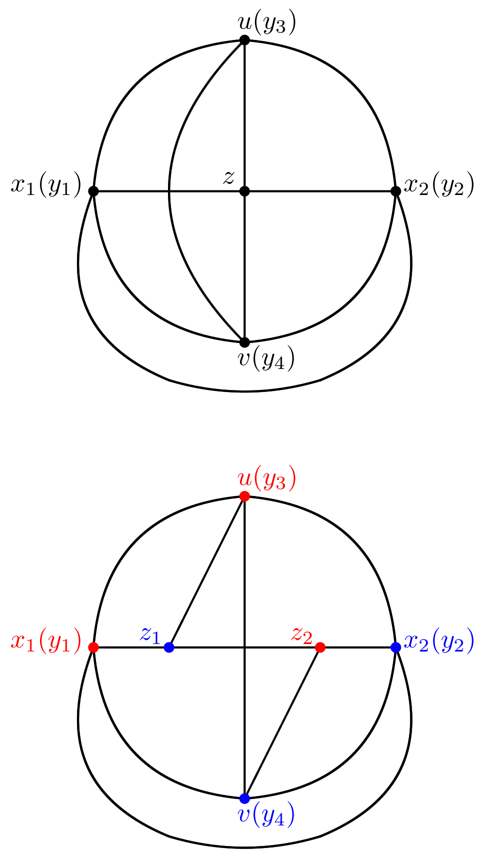

FIGURE 5. Case (3) and (4)

contradicts the condition that $G - uv$ is 2-connected. Therefore, all bridges of C_0 in $G - uv$ have at least two vertices of attachment. Moreover, if an outer bridge of C_0 has more than 2 vertices of attachment, we can always find a new cycle that contains parts of the outer bridge and has more edges in its interior. Therefore all outer bridges of C_0 have exactly 2 vertices of attachment. Following the same argument, if an outer bridge avoids the arc uv, then there would be another cycle with more edges in the interior. Hence all outer bridges overlap the arc uv, that is, for any outer bridge, not all vertices of attachment lie on the same arc uv. Also, if the size of an outer bridge is more than one, there exists a vertex that is not on C_0 in the bridge. Then the two vertices of attachment form a 2-vertex cut of both $G - uv$ and G, which contradicts the condition of G being 3-connected. Therefore, we can conclude that all outer bridges of C_0 have 2 vertices of attachment, have the size 1, and overlap the arc uv.

Find an outer bridge B_1 and an inner bridge B_2 that overlap. Justification for finding such B_1, B_2 is as follows. If all bridges of C_0 are inner (outer) bridges, then we can draw the edge uv in the exterior (interior) of C_0 and achieve a planar embedding of G, which contradicts the hypothesis. Hence C_0 has to have both inner and outer bridges. The reason why there exists a pair that overlap is that if not, then every inner bridge of C_0 avoids every outer bridge, and by Theorem 3.16, all inner bridges of C_0 are transferrable. We can then find a planar embedding of $G - uv$ where C_0 have only outer bridges, which again contradicts the hypothesis.

Let the vertices of attachment of B_1 be x_1, x_2, and those of B_2 be $y_1, y_2, y_3, \ldots$. We know that B_2 overlaps the arc uv, and is skew to B_1. We consider 4 cases in terms of the relative position of B_1 and B_2. Without loss of generality, we assume that u, x_2, v, x_1 lie on the cycle in a clockwise order.

(1) Among all vertices of attachment of B_2, there exist y_1, y_2 such that y_1 lies between x_1 and v, y_2 between u and x_2. Then G contains a subdivision of $K_{3,3}$, which is against our assumption.

(2) There exist y_1, y_2 such that y_1 lies between x_2 and v, y_2 between x_1 and u. Still G contains a subdivision of $K_{3,3}$, a contradiction.

(3) There exist $\{y_1, y_2, y_3, y_4\} = \{x_1, x_2, u, v\}$ such that the u-v path P_1 and the x_1-x_2 path P_2 have one and only one vertex z in common (P_1 and P_2 must have some vertices in common because of the planarity of $G - uv$). Then G contains a subdivision of K_5, a contradiction.

(4) There exist $\{y_1, y_2, y_3, y_4\} = \{x_1, x_2, u, v\}$ such that P_1 and P_2 have more than one vertex in common. Then G again contains a subdivision of $K_{3,3}$.

By now we have covered every possible case and derived a contradiction from each of them. Therefore, the theorem is true. □

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References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications.* The Macmillan Press Ltd., 1982, page 1-15, 143-156.

[2] Walter Klotz. "A constructive proof of Kuratowski's theorem." *https://www.researchgate.net/publication/256078009.*