Difference-in-Differences

Mixtape Session

Roadmap

- Covariates
  - Simple case, no covariates
  - IPW
  - DRDiD

Introducing DiD

- DiD in its modern form dates back to Orley Ashenfelter and David Card at Princeton in the late 70s and mid 80s (Card says they invent the term in mid 1980s)
- Unclear when key identifying assumptions like parallel trends are worked out (as originally there is no potential outcomes in play)
- Attractive elements included natural experimentation, panel data, and did not require randomization
- US context may also have been such that it had a lot of upside
- But let's start at the beginning with story and example

John Snow and cholera

- John Snow, epidemiologist in 19th century, usually credited with first use of DiD
- Believed cholera was spread through the Thames water supply which contradicted dominant theory about “dirty air” transmission
- Grand experiment: Lambeth moves its pipe between 1849 and 1854; Southwark and Vauxhall delay
- How can he use this event to test his hypothesis? Three ways: simple comparisons, interrupted time series of the difference in differences (DiD)

Simple cross-sectional design

Table: Lambeth and Southwark and Vauxhall, 1854

Company	**Cholera mortality**
Lambeth	$Y = L + D$
Southwark and Vauxhall	$Y = SV$

$$\widehat{\delta}_{cs} = D + (L - SV)$$

Interrupted time series design

Table: Lambeth, 1849 and 1854

Company	Time	Cholera mortality
Lambeth	1849	$Y = L$
	1854	$Y = L + (T + D)$

$$\widehat{\delta}_{its} = D + T$$

Difference-in-differences

Table: Lambeth and Southwark and Vauxhall, 1849 and 1854

Companies	**Time**	**Outcome**	D_1	D_2
Lambeth	Before	$Y = L$		
	After	$Y = L + T_L + D$	$T_L + D$	
				D
Southwark and Vauxhall	Before	$Y = SV$		
	After	$Y = SV + T_{SV}$	T_{SV}	

$$\widehat{\delta}_{did} = D + (T_L - T_{SV})$$

Sample averages

$$\widehat{\delta}_{kU}^{2x2} = \left(\overline{y}_k^{post(k)} - \overline{y}_k^{pre(k)} \right) - \left(\overline{y}_U^{post(k)} - \overline{y}_U^{pre(k)} \right)$$

Population expectations

$$\widehat{\delta}_{kU}^{2x2} = \Big(E[Y_k|Post] - E[Y_k|Pre] \Big) - \Big(E[Y_U|Post] - E[Y_U|Pre] \Big)$$

Potential outcomes and the switching equation

$$\widehat{\delta}_{kU}^{2x2} = \underbrace{\Big(E[Y_k^1|Post] - E[Y_k^0|Pre]\Big) - \Big(E[Y_U^0|Post] - E[Y_U^0|Pre]\Big)}_{\text{Switching equation}}$$

$$+ \underbrace{E[Y_k^0|Post] - E[Y_k^0|Post]}_{\text{Adding zero}}$$

Parallel trends bias

$$\widehat{\delta}^{2x2}_{kU} = \underbrace{E[Y^1_k|Post] - E[Y^0_k|Post]}_{\text{ATT}} + \underbrace{\Big[E[Y^0_k|Post] - E[Y^0_k|Pre]\Big] - \Big[E[Y^0_U|Post] - E[Y^0_U|Pre]\Big]}_{\text{Non-parallel trends bias in 2x2 case}}$$

OLS Specification

- Properly specified OLS model will also identify the ATT when there is only two groups and no covariates
- Often preferred because
  - → OLS estimates the ATT under parallel trends
  - → Easy to calculate the standard errors
  - → Easy to include multiple periods
- But some issues emerge with differential timing, time varying covariates and continuous treatments
- This literature continues to evolve, so I will focus on fundamentals but which may still be advanced depending on baseline

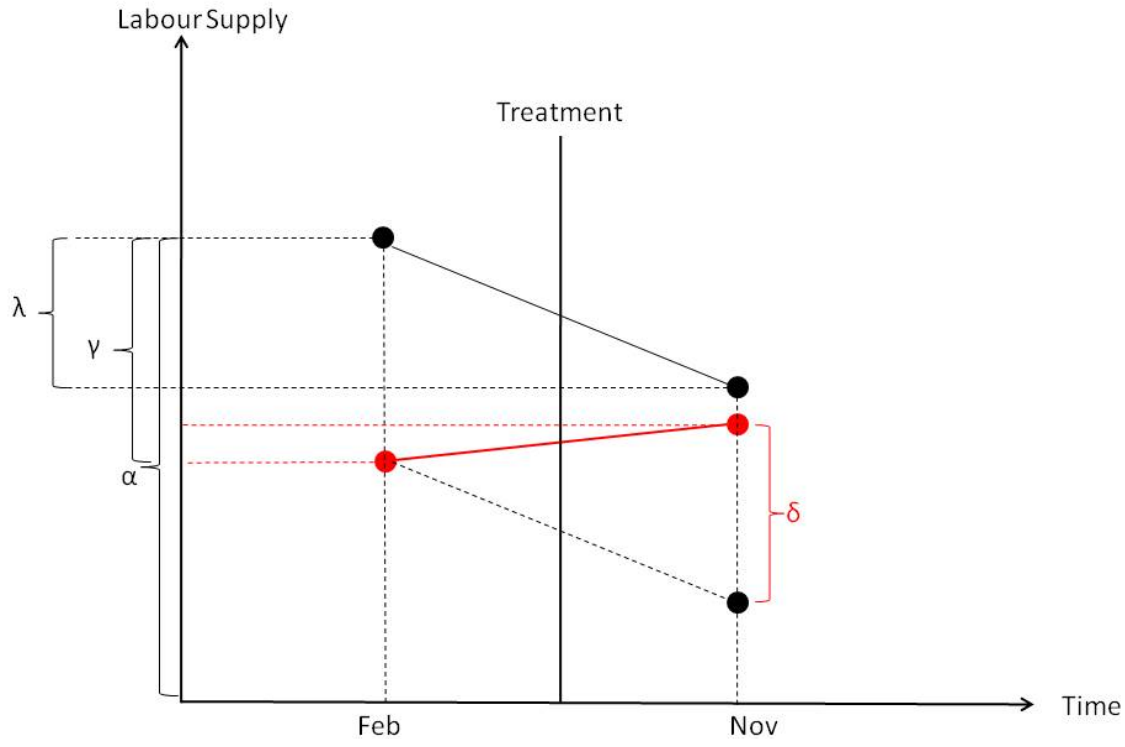
Regression DiD - Card and Krueger

- The equivalent regression includes time and group fixed effects:

$$Y_{its} = \alpha + \gamma NJ_s + \lambda d_t + \delta(NJ \times d)_{st} + \varepsilon_{its}$$

  - → NJ is a dummy equal to 1 if the observation is from NJ
  - → d is a dummy equal to 1 if the observation is from November (the post period)
- This equation takes the following values
  - → PA Pre: α
  - → PA Post: $\alpha + \lambda$
  - → NJ Pre: $\alpha + \gamma$
  - → NJ Post: $\alpha + \gamma + \lambda + \delta$
- DiD equation: (NJ Post - NJ Pre) - (PA Post - PA Pre) $= \delta$

$$Y_{ist} = \alpha + \gamma NJ_s + \lambda d_t + \delta(NJ \times d)_{st} + \varepsilon_{ist}$$



OLS with twoway fixed effects

Under parallel trends, OLS estimates the ATT. Researchers often will use OLS with time-varying covariates, but this is not advised as it is only unbiased under more restrictive assumptions which we discuss next (though see new working paper by Cattaneo, et al. 2022 on controlling for time-varying covariates).

“A good way to do econometrics is to look for good natural experiments and use statistical methods that can tidy up the confounding factors that nature has not controlled for us. – Daniel McFadden

Inverse probability weighting DiD

- Abadie (2005) proposed a DiD estimator that could incorporate covariates and get an unbiased estimate of the ATT
- Researcher needs treatment and comparison group observed before and after treatment
- If treatment group units are selected based on their (observed) covariates, then baseline covariates are also needed

Time varying versus time invariant covariates

- In a DiD, we may need to control for X because treatment is only conditional on X
- But in TWFE, all time invariant covariates are absorbed by the unit fixed effects – only time varying covariates will survive TWFE
- But time varying covariates place restrictions, as we will see, on the DGP and run the threat of conditioning on outcomes if they were changed by the treatment
- Abadie proposes using only the covariates at baseline to form weights in the simple DiD formula

Three step method

1. Compute each unit's "after minus before" which is the DD part
2. Then estimate a propensity score which you'll use to weight each unit
3. Finally, compare weighted changes in "after minus before" for treatment versus comparison groups

You can have heterogeneous treatment effects, but not differential timing

Terms

- t is year of treatment which doesn't vary across units (so no differential timing)
- Y^1 and Y^0 are potential outcomes (counterfactual versus actual)
- D is 1 or 0 based on group and time
- b is the "baseline" which is similar to CS using g as the one year pre-treatment
- X are "baseline" covariates **only** – they do not vary over time, which means propensity scores are estimated off the b period **only**

Assumptions

Kind of common for this propensity score literature to only have two assumptions. But usually the first conditional independence. Now it is parallel trends because this is DD

1. Conditional parallel trends

$$E[Y_t^0 - Y_b^0 | D = 1, X_b] - E[Y_t^0 - Y_t^0 | D = 0, X_b]$$

   (Notice the b subscript. What is that you think?)

2. Common support

$$Pr(D = 1) > 0; Pr(D = 1|X) < 1$$

   Let's see a picture of common support that I drew. Apologies it's horrible

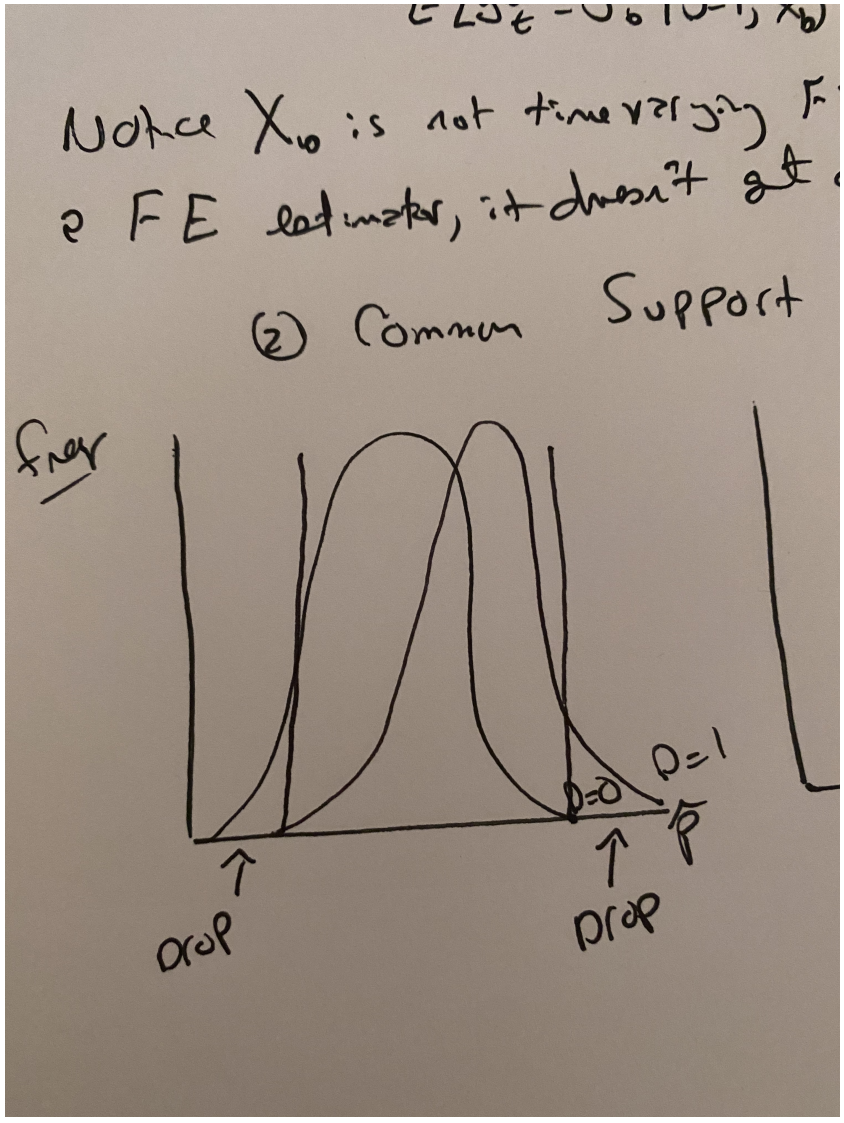
Common support

As we are identifying the ATT, we only need common support with respect to treated units

Your identify assumptions are always with respect to the missing covariates in other words and for the ATT, you are missing Y^0 for the treatment group

If we were estimating ATU, we'd be missing Y^1 for controls and need common support (Y in treatment for all ranges of control), and for ATE we'd need both

Visualizing propensity score to get common support



Definition and estimation

Defining the ATT parameter of interest

$$ATT = E[Y_t^1 - Y_t^0 | D_t = 1] \tag{1}$$

Abadie's estimator

$$E\left[\frac{Y_t - Y_b}{Pr(D_t = 1)} \times \frac{D_t - Pr(D = 1|X_b)}{1 - Pr(D = 1|X_b)}\right] \tag{2}$$

Propensity scores

- Paper is titlted “Semi-parametric DiD” because Abadie imposes structure on the polynomials used to construct the propensity score
- You can use OLS linear probability models or series logit estimation

Estimating propensity scores

It's common to hear people say that we don't know the propensity score; we can only estimate it. Same here – we approximate it with regressions

$$\widehat{Pr}(X_b) = \widehat{\gamma_0} + \widehat{\gamma_1} X + \widehat{\gamma_2} X^2 + \ldots \varepsilon \tag{3}$$

$$\widehat{Pr}(X_b) = F(\widehat{\gamma_0} + \widehat{\gamma_1} X + \widehat{\gamma_2} X^2 + \ldots) \tag{4}$$

Commentary on paper's influence

Abadie has a great year in 2003: synthetic control in AER, semiparametric DiD in Restud, semiparametric IV (his JMP) in JoE

Semiparametric DiD has over 2,000 cites

But I'm not sure it was widely adopted by the *applied* community. It seems like the *applied community* starts paying attention to the econometric contributions much later, and Abadie (2003) is gaining a renaissance at the moment because of the next paper

Nonetheless, there is code in Stata called -absdid- but an unsatisfying part of the semiparametric piece is that the results change with respect to not just the estimation method (OLS or logit), but also with respect to the order in which the covariates appear

Doubly Robust Difference-in-differences

- DR models control for covariates twice – once using the propensity score, once using outcomes adjusted by regression – and are unbiased so long as:
  - → The regression specification for the outcome is correctly specified
  - → The propensity score specification is correctly specified
- Sant'Anna and Zhao (2020) incorporated DR into DiD by combining inverse probability weighting and outcome regression into a single DiD model
- It's in the engine of Callaway and Sant'Anna (2020) that we discuss later so it merits close study
- One of my favorite lesser known of the new DiD papers

Defining the target parameter

Major part of the new econometrics is to always start with the target parameter and build to it using estimation and identification that "works"

$$\delta = E[Y_{it}^1 - Y_{it}^0 | D_i = 1]$$

Basic assumptions of DiD

Assumption 1: Assume panel data or repeated cross-sectional data

Handling repeated cross-sectional data is possible but assumes modularity which is a kind of stability assumption, but I'll use panel representation

Basic assumptions of DD

Assumption 2: Conditional parallel trends

Counterfactual trends for the treatment group are the same as the control group for all values of X

$$E[Y_1^0 - Y_0^0 | X, D = 1] = E[Y_1^0 - Y_0^0 | X, D = 0]$$

Basic assumptions of DD

Assumption 3: Common support or overlap

For some $e > 0$, the probability of being in the treatment group is greater than e and the probability of being in the treatment group conditional on X is $\leq 1 - e$.

Intuition of assumption 3: Called overlap or common support. Means there is at least a small fraction of the population that is treated and that for every value of the covariates X there is at least a small chance that the unit is not treated. It's called common support when it's a propensity score but it's just about the distribution of treatment and control across values of X.

Estimating DD with Assumptions 1-3

- Assumptions 1-3 gives us a couple of options of estimating the DiD
- We can either use the outcome regression (OR) approach of Heckman, et al 1997
- Or we can use the inverse probability weighting (IPW) approach of Abadie (2005)

Outcome regression

This is the Heckman, et al. (1997) approach where the outcome evolution is modeled with a regression

$$\widehat{\delta}^{OR} = \overline{Y}_{1,1} - \left[\overline{Y}_{1,0} + \frac{1}{n^T} \sum_{i|D_i=1} (\widehat{\mu}_{0,1}(X_i) - \widehat{\mu}_{0,0}(X_i))\right]$$

where $\overline{Y}$ is the sample average of Y among units in the treatment group at time t and $\widehat{\mu}(X)$ is an estimator of the true, but unknown, $m_{d,t}(X)$ which is by definition equal to $E[Y_t|D = d, X = x]$.

Outcome regression

$$\widehat{\delta}^{OR} = \overline{Y}_{1,1} - \left[\overline{Y}_{1,0} + \frac{1}{n^T} \sum_{i|D_i=1} (\widehat{\mu}_{0,1}(X_i) - \widehat{\mu}_{0,0}(X_i))\right]$$

1. Regress changes ΔY on X among untreated groups using baseline covariates only
2. Get fitted values of the regression using all X from $D = 1$ only. Average those
3. Calculate change in this fitted Y among treated with the average fitted values

Inverse probability weighting

This is the Abadie (2005) approach where we use weighting

$$\widehat{\delta}^{ipw} = \frac{1}{E_N[D]} E\left[\frac{D - \widehat{p}(X)}{1 - \widehat{p}(X)}(Y_1 - Y_0)\right]$$

where $\widehat{p}(X)$ is an estimator for the true propensity score. Reduces the dimensionality of X into a single scalar.

These models cannot be ranked

- Outcome regression needs $\widehat{\mu}(X)$ to be correctly specified, whereas
- Inverse probability weighting needs $\widehat{p}(X)$ to be correctly specified
- It's hard to "rank" these two in practice with regards to model misspecification because each is inconsistent when their own models are misspecified
- Well why don't we just use TWFE? I've never heard anyone complain about including covariates in TWFE and I've been doing it my entire adult life, so we're good right?
- Depends on if you want to assume three more things.

TWFE

Here's the TWFE specification:

$$Y_{it} = \alpha_1 + \alpha_2 T_t + \alpha_3 D_i + \delta(T_i \times D_t) + \varepsilon_{it}$$

Just add in covariates then right?

$$Y_{it} = \alpha_1 + \alpha_2 T_t + \alpha_3 D_i + \delta(T_i \times D_t) + \theta \cdot X_{it} + \varepsilon_{it}$$

Sure! If you're willing to impose three *more* assumptions

Decomposing TWFE with covariates

TWFE places restrictions on the DGP. Previous TWFE regression under assumptions 1-3 implies the following:

$$E[Y_1^1|D=1,X] = \alpha_1 + \alpha_2 + \alpha_3 + \delta + \theta X$$

Conditional parallel trends implies

$$\begin{aligned}
&E[Y_1^0 - Y_0^0|D=1,X] = E[Y_1^0 - Y_0^0|D=0,X] \\
&E[Y_1^0|D=1,X] - E[Y_0^0|D=1,X] = E[Y_1^0|D=0,X] - E[Y_0^0|D=0,X] \\
&E[Y_1^0|D=1,X] = E[Y_0^0|D=1,X] + E[Y_1^0|D=0,X] - E[Y_0^0|D=0,X] \\
&E[Y_1^0|D=1,X] = E[Y_0|D=1,X] + E[Y_1|D=0,X] - E[Y_0|D=0,X]
\end{aligned}$$

Last line from the switching equation. This gives us:

Collecting terms

$$
\begin{aligned}
&E[Y_1^1|D=1,X] = \alpha_1 + \alpha_2 + \alpha_3 + \delta + \theta_1 X \\
&E[Y_1^0|D=1,X] = \alpha_1 + \alpha_2 + \alpha_3 + \theta_2 X \\
&E[Y_1^1|D=1,X] - E[Y_1^0|D=1,X] \\
&= (\alpha_1 + \alpha_2 + \alpha_3 + \delta + \theta_1 X) - (\alpha_1 + \alpha_2 + \alpha_3 + \theta_2 X) \\
&= \delta + (\theta_1 X - \theta_2 X)
\end{aligned}
$$

By allowing for the possibility that $\theta_1 X \neq \theta_2 X$, we open up the possibility of bias from TWFE which is zero under three additional assumptions.

Assumption 4: Homogeneous treatment effects in X

TWFE requires homogenous treatment effects in X (i.e., the treatment effect is the same for all X)

If X is sex, then effects are the same for males and females.

If X is continuous, like income, then the effect is the same whether someone makes $1 or $1 million.

X-specific trends

TWFE also places restrictions on covariate trends for the two groups too. Take conditional expectations of our TWFE equation.

$$
\begin{aligned}
E[Y_1|D=1] &= \alpha_1 + \alpha_2 + \alpha_3 + \delta + \theta X_{11} \\
E[Y_0|D=1] &= \alpha_1 + \alpha_3 + \theta X_{10} \\
E[Y_1|D=0] &= \alpha_1 + \alpha_2 + \theta X_{01} \\
E[Y_0|D=0] &= \alpha_1 + \theta X_{00}
\end{aligned}
$$

X-specific trends

Now take the DiD formula:

$$\delta^{DD} = \Big((\alpha_1 + \alpha_2 + \alpha_3 + \delta + \theta X_{11}) - (\alpha_1 + \alpha_3 + \theta X_{10})\Big) - \Big((\alpha_1 + \alpha_2 + \theta X_{01}) - (\alpha_1 + \theta X_{00})\Big)$$

Eliminating terms, we get:

$$\delta^{DD} = \delta + (\theta X_{11} - \theta X_{10}) - (\theta X_{01} - \theta X_{00})$$

Second line requires that trends in X for treatment group equal trends in X for control group.

Assumption 5 and 6

We need “no X-specific trends” for the treatment group (assumption 5) and comparison group (assumption 6)

Intuition: No X-specific trends means the evolution of potential outcome Y^0 is the same regardless of X. This would mean you cannot allow rich people to be on a different trend than poor people, for instance.

Without these six, in general TWFE will not identify ATT.

Why not both?

- Let's review the problem. What if you claim you need X for conditional parallel trends?
- You have three options:
  1. Outcome regression (Heckman, et al. 1997) – needs Assumptions 1-3
  2. Inverse probability weighting (Abadie 2005) – needs Assumptions 1-3
  3. TWFE (everybody everywhere all the time) – needs Assumptions 1-6
- Problem is 1 and 2 need the models to be correctly specified
- Doubly robust combines them to give us insurance; we now get two chances to be wrong, as opposed to just one
- I'm going to only stick to the panel data expressions bc all repeated cross-section does is add in some terms (and I've not written up semiparametric bounds yet)

Notation

$p(x)$: propensity score model

$\Delta Y = Y_1 - Y_0 = Y_{post} - Y_{pre}$

$\mu_{d,\Delta} = \mu_{d,1}(X) - \mu_{d,0}(X),$ where $\mu(X)$ is a model for

$m_{d,t} = E[Y_t | D = d, X = x]$

So that means $\mu_{0,\Delta}$ is just the control group's change in average Y for each $X = x$

Population DR DiD model for panel data

$$\delta^{dr} = E\left[\left(\frac{D}{E[D]} - \frac{\frac{p(X)(1-D)}{(1-p(X))}}{E\left[\frac{p(X)(1-D)}{(1-p(X))}\right]}\right)(\Delta Y - \mu_{0,\Delta}(X))\right]$$

Notice how the model controls for X: you're weighting the adjusted outcomes using the propensity score

The reason you control for X twice is because you don't know which model is right. DR DiD frees you from making a choice without making you pay too much for it

Efficiency

- Authors exploit all the restrictions implied by the assumptions to construct semiparametric bounds
- This is where the influence function comes in, which those who have studied the DID code closely may have noticed
- One of the main results of the paper is that the DR DiD estimator is also DR for inference
- Let's skip to Monte Carlos

Monte Carlo details

- Compare DR with TWFE, OR and IPW
- Sample size is 1,000
- 10,000 Monte Carlo experiments
- Propensity score estimated with logit; OR estimated using linear specification

Table: Monte Carlo Simulations, DGP1, Both OR and Propensity score correct

	Bias	**RMSE**	**SE**	**Coverage**	**CI length**
TWFE	-20.9518	21.1227	2.5271	0.000	9.9061
OR	-0.0012	0.1005	0.1010	0.9500	0.3960
IPW	0.0257	2.7743	2.6636	0.9518	10.4412
DR	-0.0014	0.1059	0.1052	0.9473	0.4124

Figure 1: Monte Carlo for DID estimators, DGP1: Both pscore and OR are correctly specified

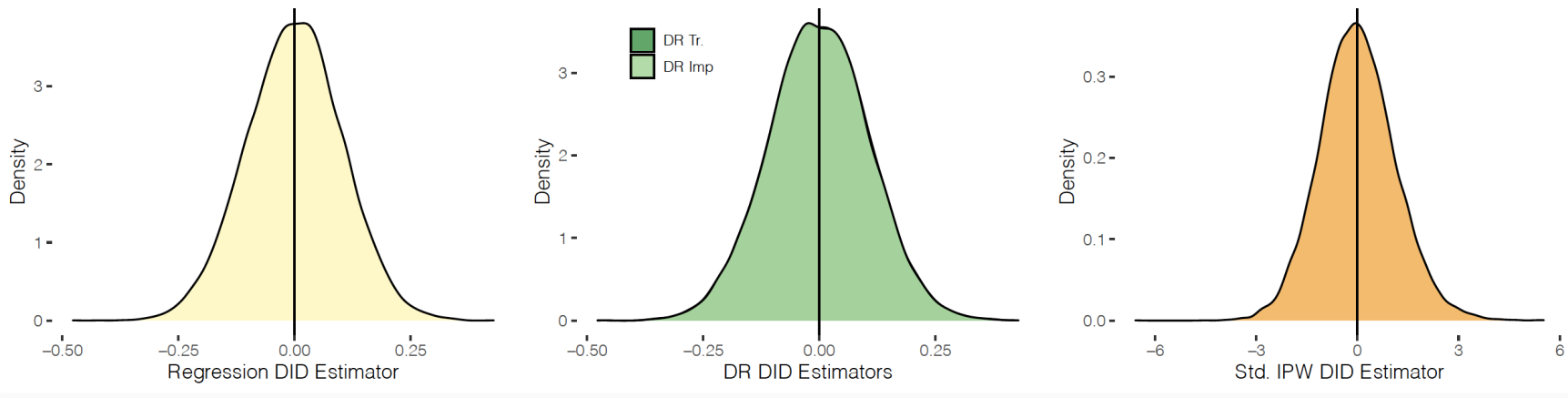
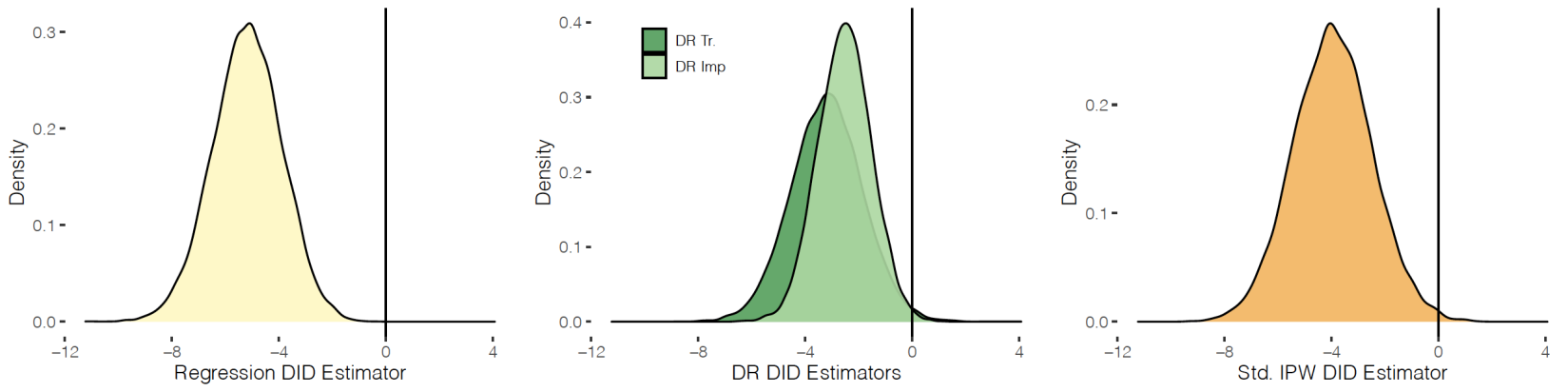


Table: Monte Carlo Simulations, DGP4, Neither OR and Propensity score correct

	Bias	**RMSE**	**SE**	**Coverage**	**CI length**
TWFE	-16.3846	16.5383	3.6268	0.000	14.2169
OR	-5.2045	5.3641	1.2890	0.0145	5.0531
IPW	-1.0846	2.6557	2.3746	0.9487	9.3084
DR	-3.1878	3.4544	1.2946	0.3076	5.0749

Figure 4: Monte Carlo for DID estimators, DGP4: Both OR and PS are misspecified



Code

There is code in R and Stata

- Stata: **drdid**
- R: **drdid**

Remember – it's for 2x2 with covariates (i.e., one treatment group)

Concluding remarks

- These two papers mark a different approach than is often the case for applied researchers who simply estimate regression models and hope they recover “reasonably weighted” causal effects
- These new DiD start with target parameter and identification then build estimation
- TWFE, as it turns out, is not mostly harmless