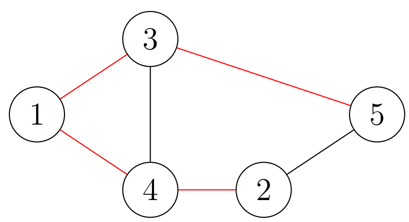
# Homework 8: Spanning Trees and Shortest Paths

Due: Friday, April 3 at 11:59 pm on Canvas

**Concepts:** minimum spanning trees, breadth-first search, shortest paths

1. (6 points) Prove that if a graph $G = (V, E)$ has unique edge weights (i.e. $w_{e1} \neq w_{e_2}$ for any two edges $e_1, e_2 \in E$), then there is a single minimum spanning tree. In other words, there is only one optimal solution. **Hint:** Use a proof by contradiction and suppose that there are two optimal spanning trees $T_1^*$ and $T_2^*$.

2. (15 points) Suppose that you are given an unweighted graph $G = (V, E)$ and a node $i \in V$. One way to find the shortest path from $i$ to all other nodes $j \in V$ is to run breadth-first search from $i$. In particular, for every node $v$ added to the queue when processing $u$ in BFS, we mark $u$ as $v$'s parent. This induces a tree of shortest paths from $i$, as shown below in red for $i = 1$. In this case, the shortest path from 1 to 5 is 1-3-5.




   (a) (6 points) Use a proof by contradiction to show that BFS keeps track of the shortest path from $i$ to all other nodes. **Hint:** Consider the closest node $v$ to the source $i$ whose BFS path is not a shortest path.

   (b) (6 points) Given a weighted graph $G = (V, E)$ in which each edge weight $w_e \geq 0$ is an integer, explain how to convert $G$ into an unweighted graph $G' = (V', E')$ such that finding the shortest path from $i$ to $j$ in $G'$ gives you the shortest path in $G$. How many vertices and edges does this new graph have?

   (c) (3 points) Using your construction above, explain how to find the shortest path from $i$ to all other nodes $j$ in a weighted graph $G = (V, E)$ using BFS. What is the runtime of your overall algorithm? How does it compare to the runtime of Dijkstra's algorithm?