Value Iteration and Policy Iteration

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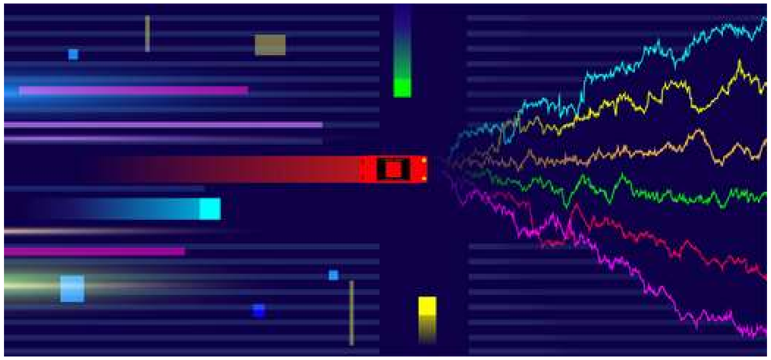

Infinite Horizon Discounted

A MDP is defined by: $\mathcal{M} = \{\mathcal{S}, \mathcal{A}, r, T, s_0, \gamma\}$

$\mathcal{S}$	State space
$\mathcal{A}$	Action space
$r : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$	Reward
$T : \mathcal{S} \times \mathcal{A} \rightarrow \Delta(\mathcal{S})$	Transitions
$\gamma \in [0, 1)$	Discount

Value Function

The Value Function is expected return for policy

$$V^\pi(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(S_h, A_h) \mid S_0 = s, A_h \sim \pi(S_h), S_{h+1} \sim T(\cdot \mid S_h, A_h)\right]$$

$$Q^\pi(s,a) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(S_h, A_h) \mid (S_0, A_0) = (s,a), A_h \sim \pi(S_h), S_{h+1} \sim T(\cdot \mid S_h, A_h)\right]$$

The Bellman Equations note that:

$$V^\pi(s) = \mathbb{E}_{A\sim\pi(s)}[r(s,A) + \gamma \mathbb{E}_{S'\sim T(\cdot|s,A)}[V^\pi(S')]]$$

$$Q^\pi(s,a) = r(s,a) + \gamma \mathbb{E}_{S'\sim T(\cdot|s,a)}[V^\pi(S')]$$

Main Question

Given a MDP, how do we find the optimal policy?

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Fully Known Model

- Reward function, transition distribution fully known
- Understand computational complexity to scale to large problems

Generative Model

- Sample from reward function / transition distribution from arbitrary (state,action)
- Understand statistical complexity to scale to large problems
- No issue of dynamic environment

Online Model

- Sample trajectory under current policy, update policy, repeat
- Understand statistical complexity
- *"Most complex"*, additional correlations in estimates

Main Question

Maybe a better model….

Exogenous MDP

- Unknown distribution over exogenous inputs (i.e. arrivals)
- Known reward and transition as function of exogenous trace
- Access to historical data of exogenous inputs

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Two Approaches

Value Iteration
Policy Iteration

Bellman Operator

Define the Bellman Operator, which given an arbitrary function:

$$(\mathscr{T} f)(s,a) = r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[\max_{a' \in \mathcal{A}} f(S',a')]$$

By Bellman Optimality we know: $\mathscr{T} Q^* = Q^*$

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If it was a contraction (it is), iterate!

Value Iteration

$$(\mathscr{T} f)(s,a) = r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[\max_{a' \in \mathcal{A}} f(S',a')]$$

Initialize: $Q^0(s,a) \in \left(0, \frac{1}{1-\gamma}\right)$

Iterate until convergence: $Q^{t+1} = \mathscr{T} Q^t$

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Couple notes:

- Explicitly using known model
- Storage/time scales with size of action + state space

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The Bellman operator is a $\boldsymbol{\gamma}$ contraction, so:

$$\|Q^t - Q^*\| \leq \gamma^t \|Q^0 - Q^*\|$$

Value Iteration

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Not a guarantee on value of final policy, since $Q^t \neq Q^{\pi^t}$

$\pi^t(s) = \mathrm{argmax}_{a \in \mathcal{A}} Q^t(s, a)$

$$V^{\pi^t}(s) \geq V^*(s) - \frac{2\gamma^t}{1-\gamma} \|Q^0 - Q^*\|_\infty$$

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$$Q^\pi(s,a) = r(s,a) + \gamma \mathbb{E}_{S'\sim T(\cdot|s,a)}[V^\pi(S')]$$

Via some linear algebra.... $T^\pi(s',s) = \sum_a \pi(a \mid s) T(s' \mid s,a)$

$$V^\pi = (I - \gamma T^\pi)^{-1} r$$
$$Q^\pi = r + \gamma V^\pi$$

Policy Iteration

Initialize: $\pi^0(s) : \mathcal{A} \to \Delta(\mathcal{A})$

Evaluate / solve Bellman Eqs for: $Q^{\pi^t}(s, a)$

Policy Improvement: $\pi^{t+1}(s) = \mathrm{argmax}_{a \in \mathcal{A}} Q^{\pi^t}(s, a)$

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Couple notes:

- Explicitly using known model
- Storage/time scales with size of action + state space (solving Bellman Eqs)

Natural Question

Value Iteration vs Policy Iteration

Which one is faster? How many iterations (computational complexity) are needed to find optimal policy?

Value Iteration

$$(\mathscr{T}f)(s,a) = r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[\max_{a' \in \mathcal{A}} f(S',a')]$$

Initialize: $Q^0(s,a) \in \left(0, \frac{1}{1-\gamma}\right)$

Iterate until convergence: $Q^{t+1} = \mathscr{T} Q^t$

Per iteration complexity: $S^2 A$

Policy Iteration

Initialize: $\pi^0(s) : \mathcal{A} \to \Delta(\mathcal{A})$

Evaluate / solve Bellman Eqs for: $Q^{\pi^t}(s,a)$

Policy Improvement: $\pi^{t+1}(s) = \mathrm{argmax}_{a\in\mathcal{A}} Q^{\pi^t}(s,a)$

Per iteration complexity: hard to quantify, need to evaluate Q value for current policy at each iteration

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Neither are strongly polynomial time, but PI observed to be faster than VI

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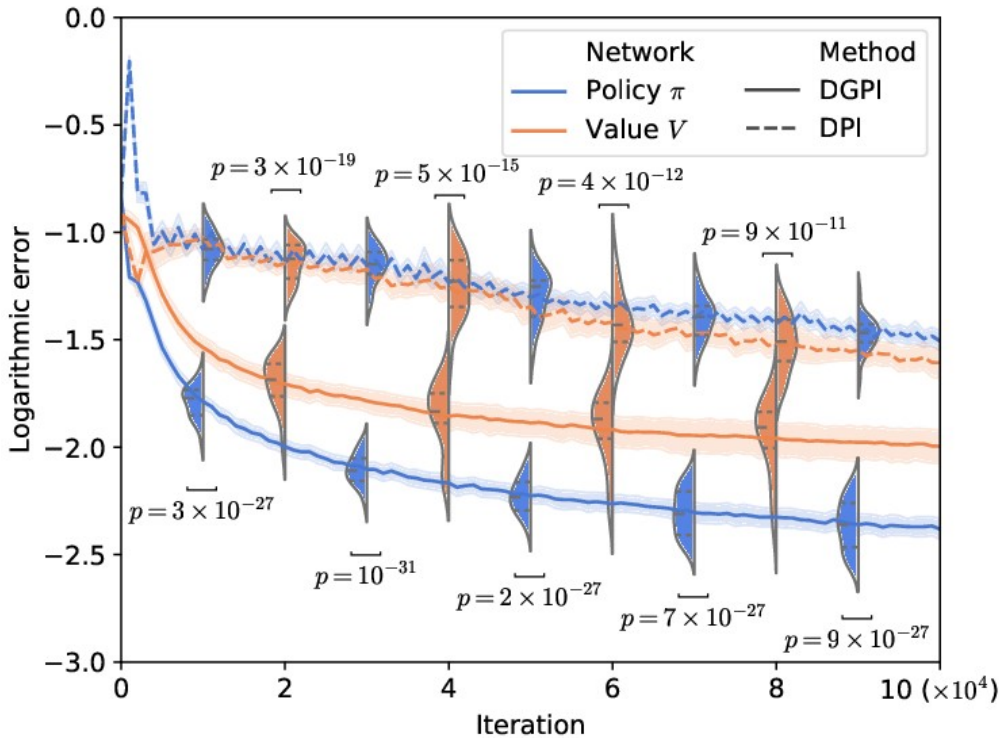

Natural Question

Can we design a polynomial time algorithm?

In comes linear programming.....

Primal LP

Can equivalently write Bellman Equation as a linear program

$$\min V(s_0)$$
$$\text{s.t.} V(s) \geq r(s,a) + \mathbb{E}_{S' \sim T(\cdot|s,a)}[V(S')]$$

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$$V^*(s) = \max_{a \in \mathcal{A}} r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[V^*(S')]$$

Encoding max operator through constraints

Primal LP

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Generic polytime LP solver gives polytime algorithm (interior point algorithm is strongly polynomial)

Primal LP

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VI ~ Fixed Point Algorithm
PI ~ Block Simplex Algorithm

Primal LP

Can equivalently write Bellman Equation as a linear program

$$
\begin{aligned}
&\min V(s_0) \\
&\text{s.t.} V(s) \geq r(s,a) + \mathbb{E}_{S' \sim T(\cdot|s,a)}[V(S')]
\end{aligned}
$$

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What is the dual?

Dual LP

The dual of the Bellman Equation LP:

$$\max \sum_{s,a} \nu(s,a) r(s,a)$$

$$\text{s.t.} \sum_{a} \nu(s,a) = (1-\gamma)\mathbb{I}[s = s_0] + \gamma \sum_{s',a'} T(s \mid s', a') \nu(s', a')$$

Flow constraints

$\nu(s,a)$ State-action visitation distribution for optimal policy

$$\nu(s,a) = (1-\gamma) \textstyle\sum_{t=0}^{\infty} \gamma^t \Pr(S_t = S, A_t = A \mid s_0)$$

Dual LP

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Lots of recent work on understanding and exploiting LP properties of RL formulation

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