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Value Iteration and Policy Iteration

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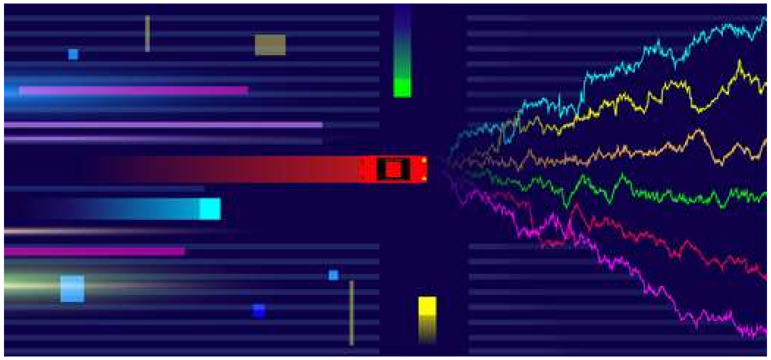

Infinite Horizon Discounted

An MDP is defined by: $\mathcal{M} = \{\mathcal{S}, \mathcal{A}, r, T, s_0, \gamma\}$

$\mathcal{S}$	State space
$\mathcal{A}$	Action space
$r : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$	Reward
$T : \mathcal{S} \times \mathcal{A} \rightarrow \Delta(\mathcal{S})$	Transitions
$\gamma \in [0, 1)$	Discount

Value Function

The Value Function is expected return for policy

$$V^\pi(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(S_h, A_h) \mid S_0 = s, A_h \sim \pi(S_h), S_{h+1} \sim T(\cdot \mid S_h, A_h)\right]$$

$$Q^\pi(s,a) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(S_h, A_h) \mid (S_0, A_0) = (s,a), A_h \sim \pi(S_h), S_{h+1} \sim T(\cdot \mid S_h, A_h)\right]$$

The Bellman Equations note that:

$$V^\pi(s) = \mathbb{E}_{A\sim\pi(s)}[r(s,A) + \gamma \mathbb{E}_{S'\sim T(\cdot|s,A)}[V^\pi(S')]]$$

$$Q^\pi(s,a) = r(s,a) + \gamma \mathbb{E}_{S'\sim T(\cdot|s,a)}[V^\pi(S')]$$

Main Question

Given an MDP, how do we find the optimal policy?

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Fully Known Model

- Reward function, transition distribution fully known
- Understand computational complexity to scale to large problems

Generative Model

- Sample from reward function / transition distribution from arbitrary (state,action)
- Understand statistical complexity to scale to large problems
- No issue of dynamic environment

Online Model

- Sample trajectory under current policy, update policy, repeat
- Understand statistical complexity
- *"Most complex"*, additional correlations in estimates

Main Question

Maybe a better model....

Exogenous MDP

- Unknown distribution over exogenous inputs (i.e. arrivals)
- Known reward and transition as function of exogenous trace
- Access to historical data of exogenous inputs

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Two Approaches

Value Iteration
Policy Iteration

Bellman Operator

Define the Bellman Operator, which given an arbitrary function:

$$(\mathscr{T} f)(s,a) = r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[\max_{a' \in \mathcal{A}} f(S', a')]$$

By Bellman Optimality we know: $\mathscr{T} Q^* = Q^*$

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If it was a contraction (it is), iterate!

Value Iteration

$$(\mathscr{T} f)(s,a) = r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[\max_{a' \in \mathcal{A}} f(S', a')]$$

Initialize: $Q^0(s,a) \in \left(0, \frac{1}{1-\gamma}\right)$

Iterate until convergence: $Q^{t+1} = \mathscr{T} Q^t$

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Couple notes:

- Explicitly using known model
- Storage/time scales with size of action + state space

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The Bellman operator is a $\boldsymbol{\gamma}$ contraction, so:

$$\|Q^t - Q^*\| \leq \gamma^t \|Q^0 - Q^*\|$$

Value Iteration

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$$\|Q^t - Q^*\| \leq \gamma^t \|Q^0 - Q^*\|$$

Not a guarantee on value of final policy, since $Q^t \neq Q^{\pi^t}$

$\pi^t(s) = \text{argmax}_{a \in \mathcal{A}} Q^t(s,a)$

$$V^{\pi^t}(s) \geq V^*(s) - \frac{2\gamma^t}{1-\gamma} \|Q^0 - Q^*\|_\infty$$

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Value Function

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$$Q^\pi(s,a) = r(s,a) + \gamma \mathbb{E}_{S'\sim T(\cdot|s,a)}[V^\pi(S')]$$

Via some linear algebra.... $T^\pi(s', s) = \sum_a \pi(a \mid s) T(s' \mid s, a)$

$$V^\pi = (I - \gamma T^\pi)^{-1} r$$
$$Q^\pi = r + \gamma V^\pi$$

Policy Iteration

Initialize: $\pi^0(s) : \mathcal{A} \to \Delta(\mathcal{A})$

Evaluate / solve Bellman Eqs for: $Q^{\pi^t}(s, a)$

Policy Improvement: $\pi^{t+1}(s) = \mathrm{argmax}_{a \in \mathcal{A}} Q^{\pi^t}(s, a)$

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Couple notes:

- Explicitly using known model
- Storage/time scales with size of action + state space (solving Bellman Eqs)

Natural Question

Value Iteration vs Policy Iteration

Which one is faster? How many iterations (computational complexity) are needed to find optimal policy?

Value Iteration

$$(\mathscr{T} f)(s,a) = r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[\max_{a' \in \mathcal{A}} f(S',a')]$$

Initialize: $Q^0(s,a) \in \left(0, \frac{1}{1-\gamma}\right)$

Iterate until convergence: $Q^{t+1} = \mathscr{T} Q^t$

Per iteration complexity: $S^2 A$

Policy Iteration

Initialize: $\pi^0(s) : \mathcal{A} \to \Delta(\mathcal{A})$

Evaluate / solve Bellman Eqs for: $Q^{\pi^t}(s,a)$

Policy Improvement: $\pi^{t+1}(s) = \mathrm{argmax}_{a \in \mathcal{A}} Q^{\pi^t}(s,a)$

Per iteration complexity: $S^3 + S^2 A$

Natural Question

Value Iteration vs Policy Iteration

Which one is faster? How many iterations (computational complexity) are needed to find optimal policy?

Neither are strongly polynomial time, but PI observed to be faster than VI

Natural Question

Can we design a polynomial time algorithm?

In comes linear programming.....

Primal LP

Can equivalently write Bellman Equation as a linear program

$$\begin{aligned} &\min V(s_0) \\ &\text{s.t.} V(s) \geq r(s,a) + \mathbb{E}_{S' \sim T(\cdot|s,a)}[V(S')] \end{aligned}$$

Primal LP

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Generic polytime LP solver gives polytime algorithm (interior point algorithm is strongly polynomial)

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VI ~ Fixed Point Algorithm
PI ~ Block Simplex Algorithm

Primal LP

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What is the dual?

Dual LP

The dual of the Bellman Equation LP:

$$\max \sum_{s,a} \nu(s,a) r(s,a)$$

$$\text{s.t.} \sum_{a} \nu(s,a) = (1-\gamma)\mathbb{I}[s = s_0] + \gamma \sum_{s',a'} T(s \mid s', a')\nu(s', a')$$

Flow constraints

$\nu(s,a)$ State-action visitation distribution for optimal policy

$$\nu(s,a) = (1-\gamma)\sum_{t=0}^{\infty} \gamma^t \Pr(S_t = S, A_t = A \mid s_0)$$

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