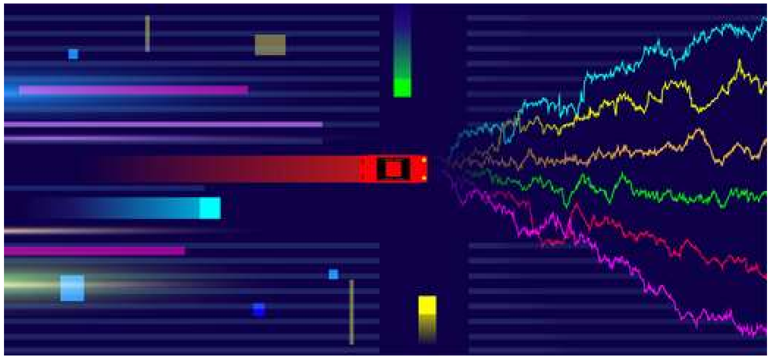
RL for Operations
Day 1: MDP Basics, VI+PI, Deep RL

Sean Sinclair, Sid Banerjee, Christina Yu
Cornell University



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Plan for Today

MDP Basics

- Basic framework for Markov Decision Processes
- Tabular RL Algorithms with policy iteration + value iteration
- DeepRL algorithms (and their "tabular" counterparts)

Simulation Packages

- OpenAI Framework for simulation design
- Existing packages and code-bases for RL algorithm development

Simulation Implementation

- Develop simulator for problem using OpenAI Gym API

Tabular RL Algorithms

- Implement basic tabular RL algorithms to understand key algorithmic design aspects of *value estimates + value iteration*, *policy iteration*

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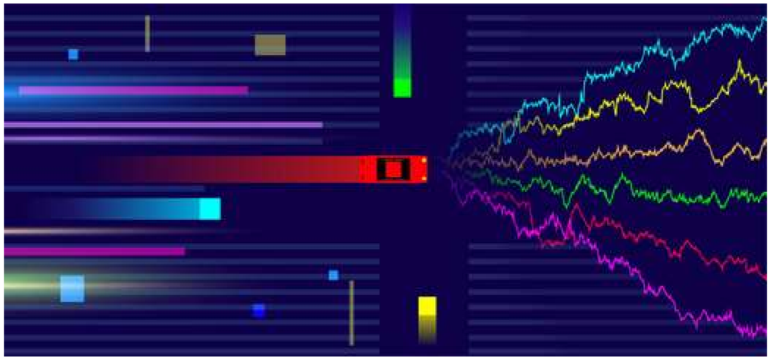
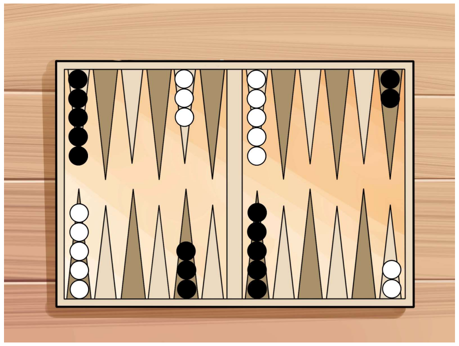
Basic MDPs

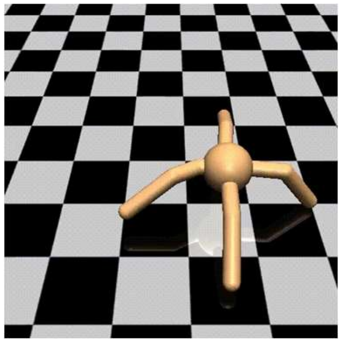
Siddhartha Banerjee
Cornell University



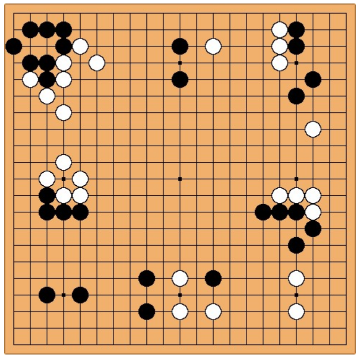
Success of RL



Backgammon



MuJoCo Simulator



AlphaGo Zero

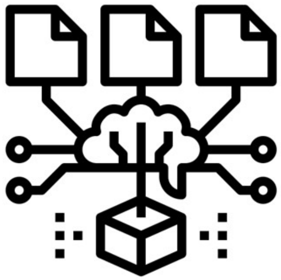
Focused on game playing + robotics

[Silver2017, Tesauro1995]

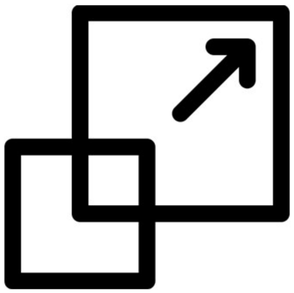
This Workshop

This workshop focuses on RL for Operations

We care about:



OR Models

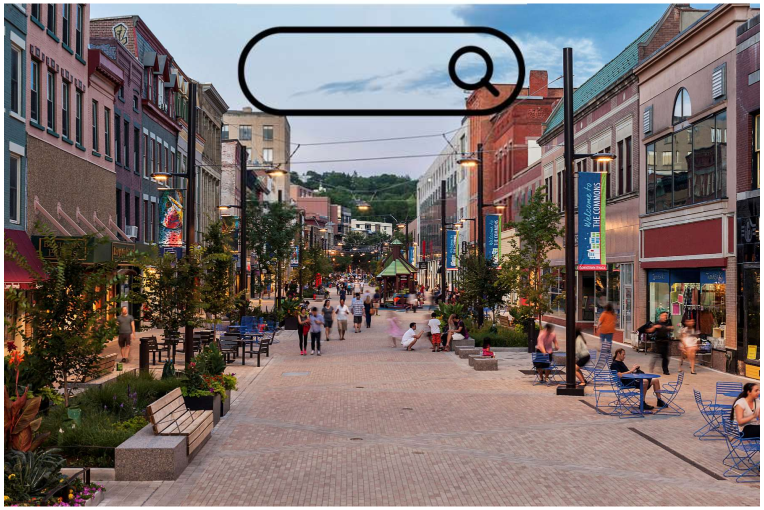


Computation + Scale

Impact

A Story

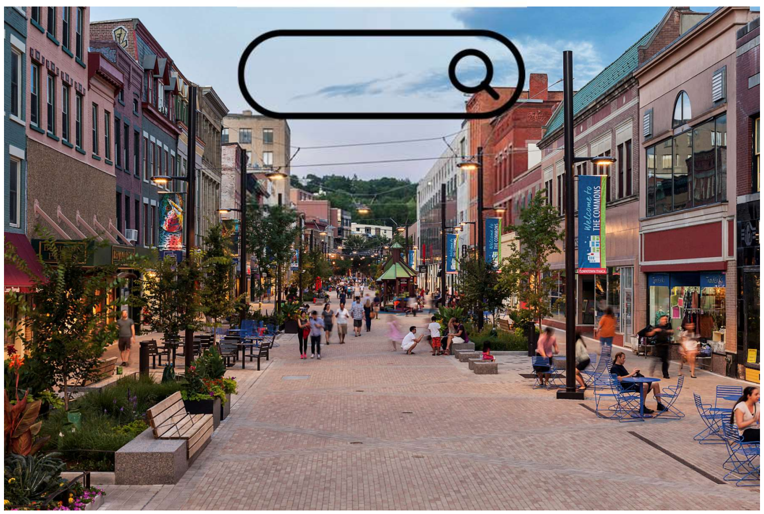
Typical question: “which decision is better”



City of Ithaca homepage photo

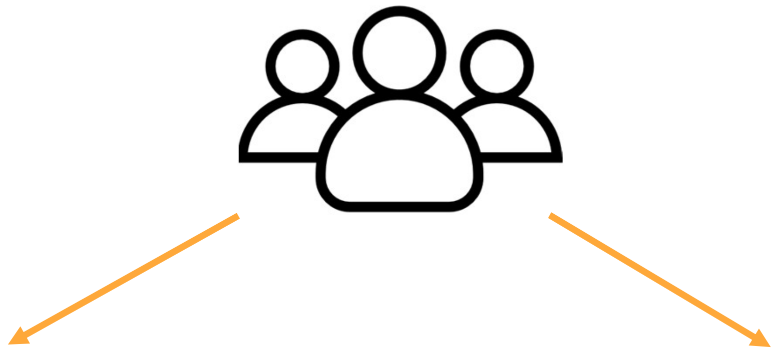
A/B Test

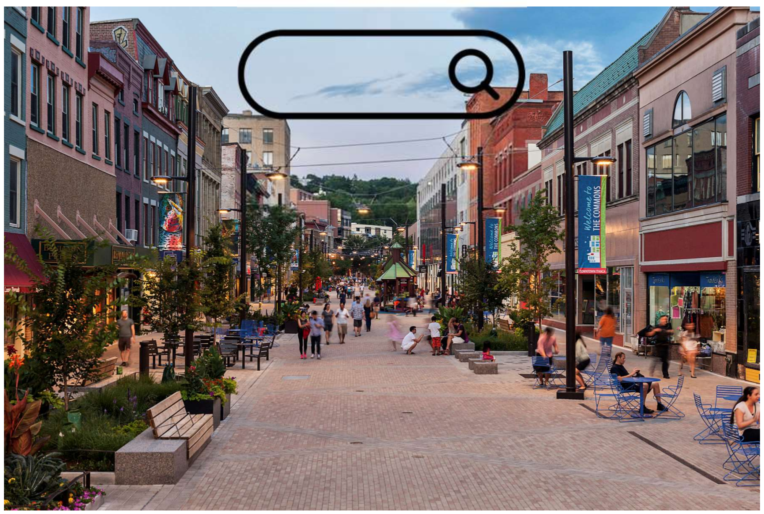
50% 50%



Take users, divide randomly, observe which has longer visit times

Bandit Algorithms





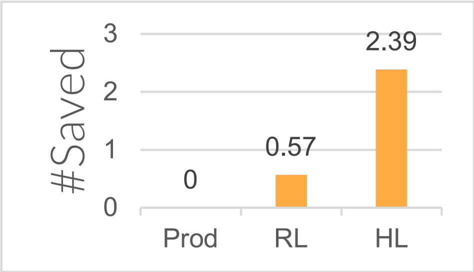
Adaptively partition users based on observed feedback thus far

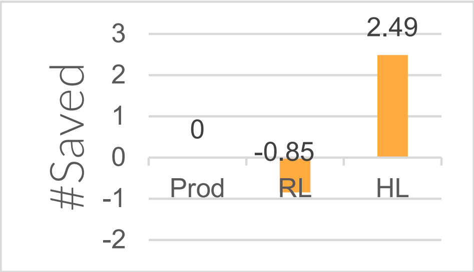
Bandit Algorithms

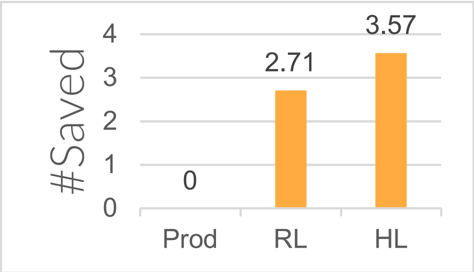
Renaissance of use in industry, motivated new theory + practice research

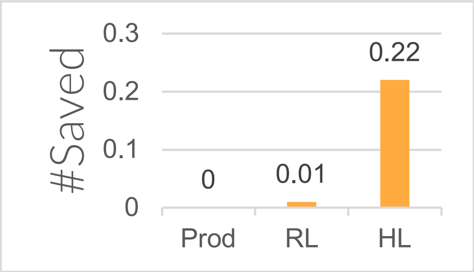
Bandit Algorithms

Bandit problems are a dynamic "supervised learning" model, no feedback effects









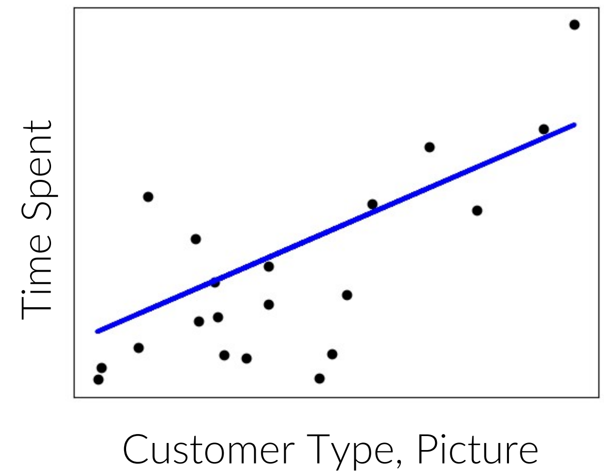
Developing algorithms for RL in Operations can be the *next big success* in industry

Supervised Learning

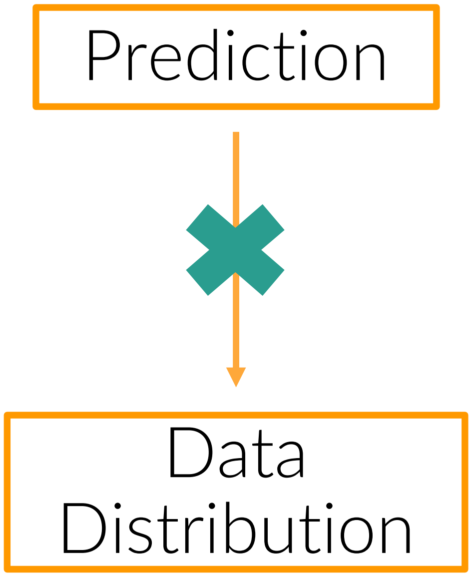
( ,  , 14 mins)

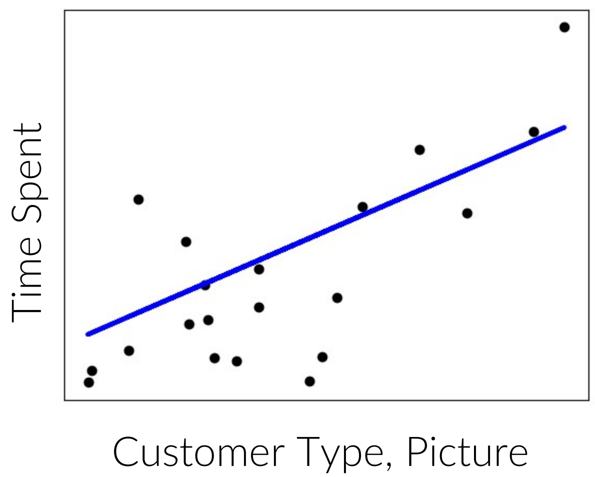
( ,  , 8 mins)

( ,  , 24 mins)



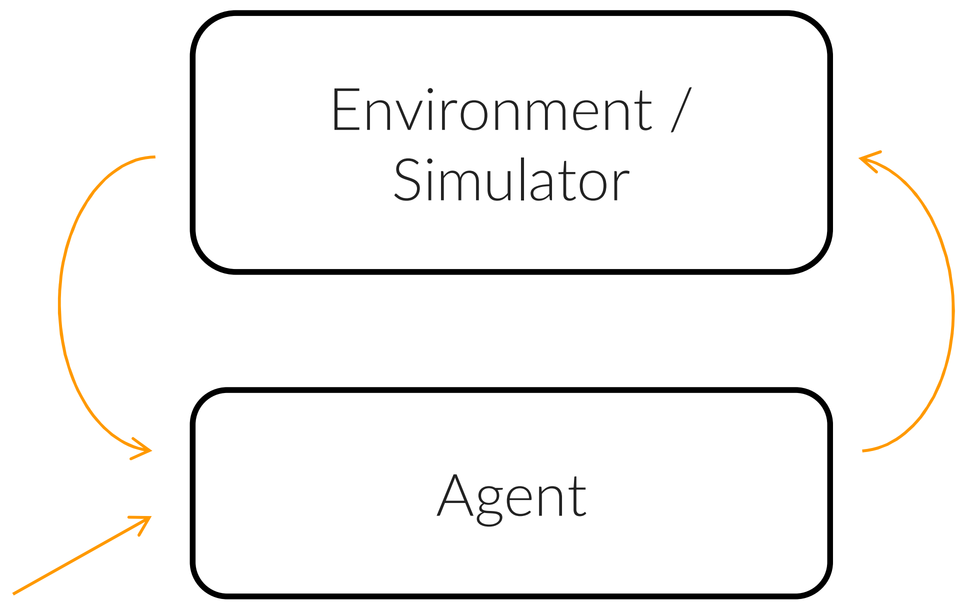
Supervised Learning





Theory and practice relies on prediction not affecting data distribution

Markov Decision Process (MDP)



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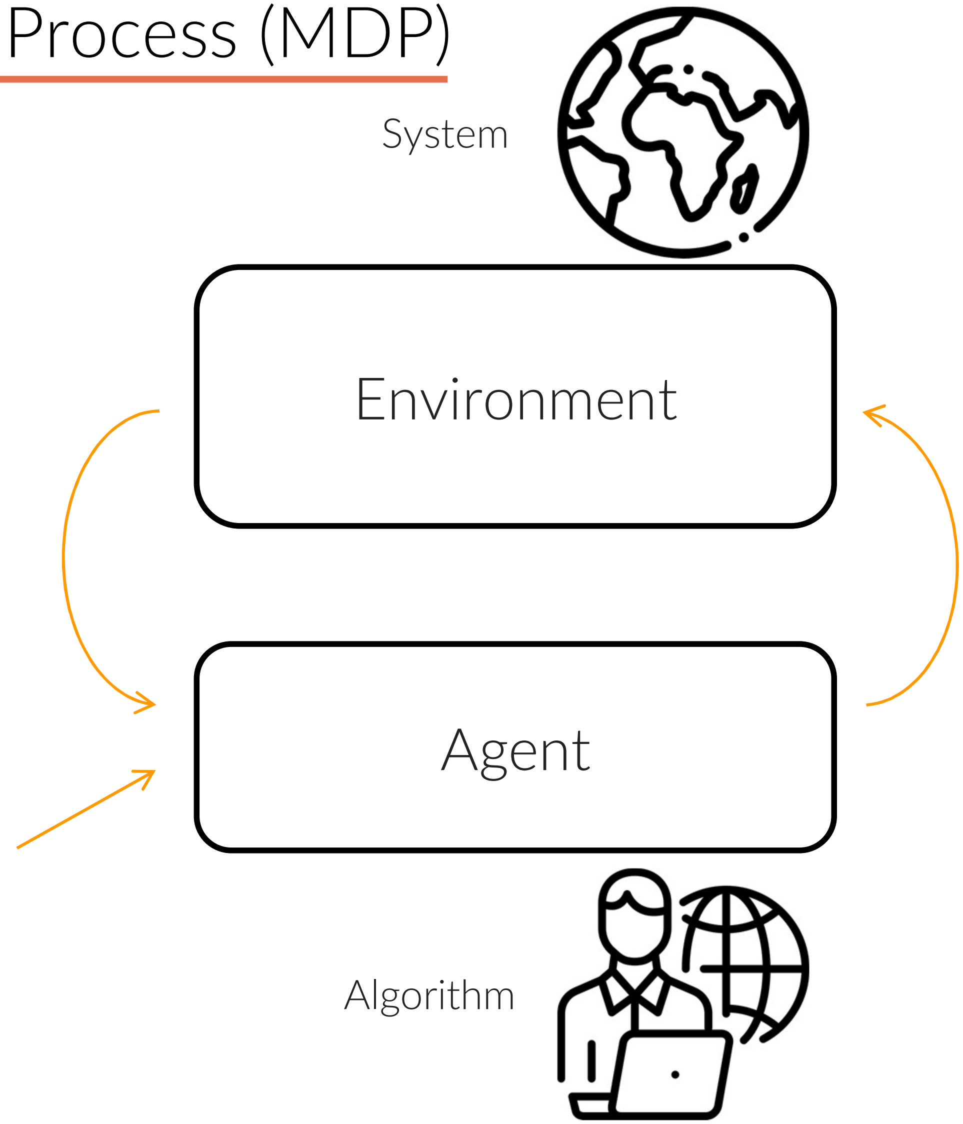

Markov Decision Process (MDP)

$$\pi(s) \rightarrow \Delta(\mathcal{A})$$

Policy: Determine action based on state

Markov Decision Process (MDP)

Environment: Determine reward and new state

$$r(s,a), s' \sim T(\cdot \mid s, a)$$

$$\pi(s) \to \Delta(\mathcal{A})$$

Policy: Determine action based on state

Markov Decision Process (MDP)

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MDP vs Supervised Learning

	Learn from Experience	Generalize	Interactive	Exploration	Credit Assignment
Supervised Learning	✓	✓	✗	✗	✗
Reinforcement Learning	✓	✓	✓	✓	✓

MDP vs Supervised Learning

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Reinforcement Learning	✓	✓	✓	✓	✓

Infinite Horizon Discounted

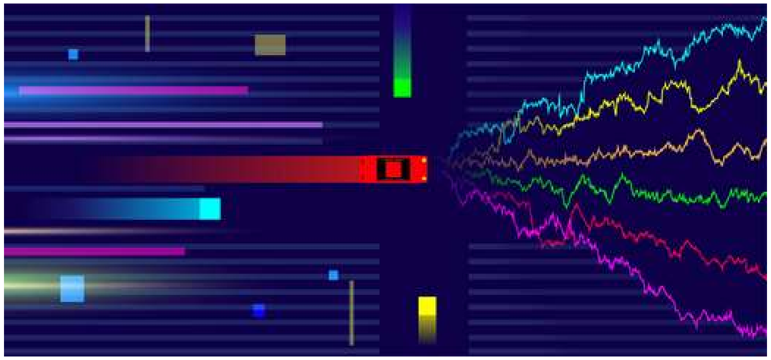
An MDP is defined by: $\mathcal{M} = \{\mathcal{S}, \mathcal{A}, r, T, s_0, \gamma\}$

$\mathcal{S}$	State space
$\mathcal{A}$	Action space
$r : \mathcal{S} \times \mathcal{A} \to [0, 1]$	Reward
$T : \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})$	Transitions
$\gamma \in [0, 1)$	Discount

Infinite Horizon Discounted

An MDP is defined by: $\mathcal{M} = \{\mathcal{S}, \mathcal{A}, r, T, s_0, \gamma\}$

$$r_1 + \gamma r_2 + \gamma^3 r_3 + \gamma^4 r_4 \ldots$$



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$T : \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})$	Transitions
$\gamma \in [0, 1)$	Discount
$\pi : \mathcal{S} \to \Delta(\mathcal{A})$	Policy

Policies are non-stationary

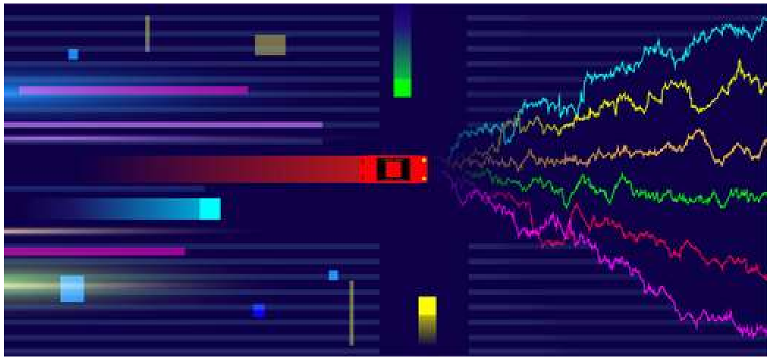
Finite Horizon

An MDP is defined by: $\mathcal{M} = \{\mathcal{S}, \mathcal{A}, r, T, s_0, \gamma\}$

$\mathcal{S}$	State space
$\mathcal{A}$	Action space
$r_h : \mathcal{S} \times \mathcal{A} \to [0, 1]$	Reward
$T_h : \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})$	Transitions
H	Time horizon
$\pi_h : \mathcal{S} \to \Delta(\mathcal{A})$	Policy

Finite Horizon

$$r_1 + r_2 + r_3 + \ldots + r_H$$



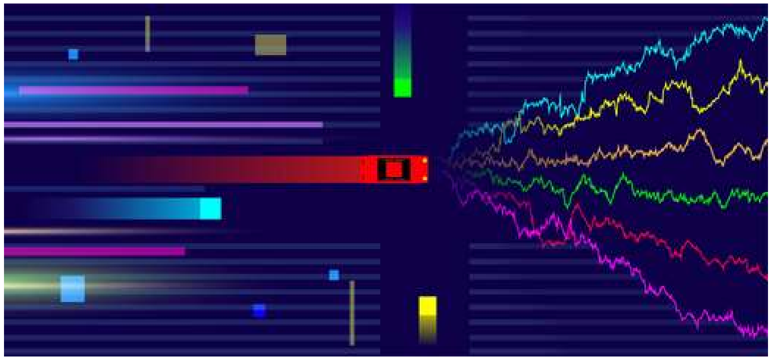
H Time horizon

Discounted?

Infinite Horizon

- Future rewards discounted at rate of γ
- "Less" importance on future rewards
- Transition, rewards, policy, not allowed to depend on timestep

$$r_1 + \gamma r_2 + \gamma^3 r_3 + \gamma^4 r_4 \ldots$$



Finite Horizon

- Future rewards are *not* discounted
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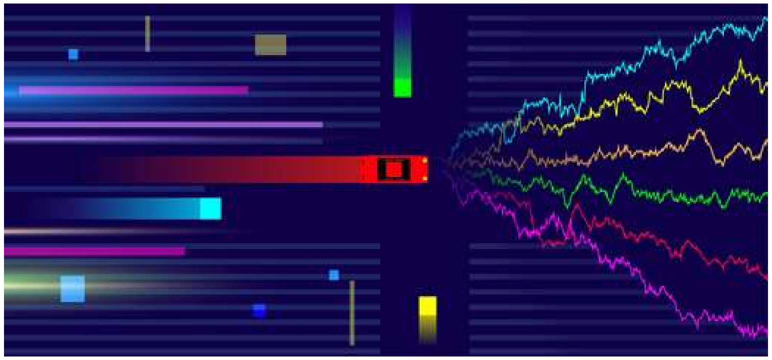
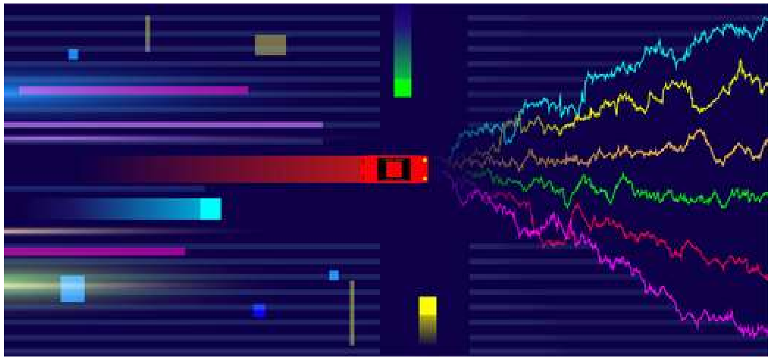
$$r_1 + r_2 + r_3 + \ldots + r_H$$



Discounted?

$$\frac{1}{T}(r_1 + r_2 + r_3 + r_4 \ldots)$$

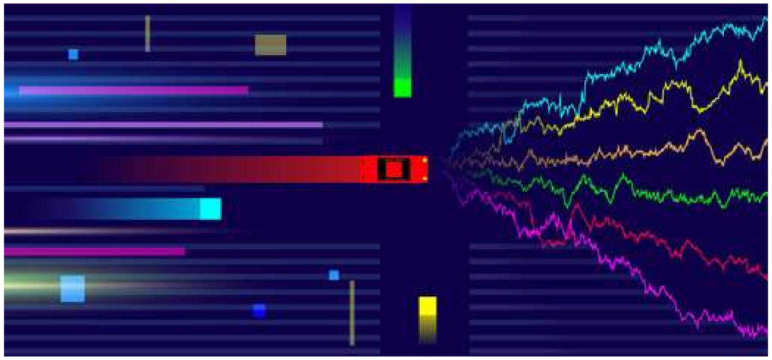


Average Cost

- Future rewards are *not* discounted
- Transition, rewards, policy, not allowed to depend on timestep
- Typically studied in pure OR literature

Discounted?

$$\frac{1}{T}(r_1 + r_2 + r_3 + r_4 \ldots)$$



People tune discount factor in practice

Average Cost

- Future rewards are *not* discounted
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- Typically studied in pure OR literature

Informal Theorem: There exists a discount such that the optimal policy in infinite horizon discounted problem is optimal under average cost

Discounted?

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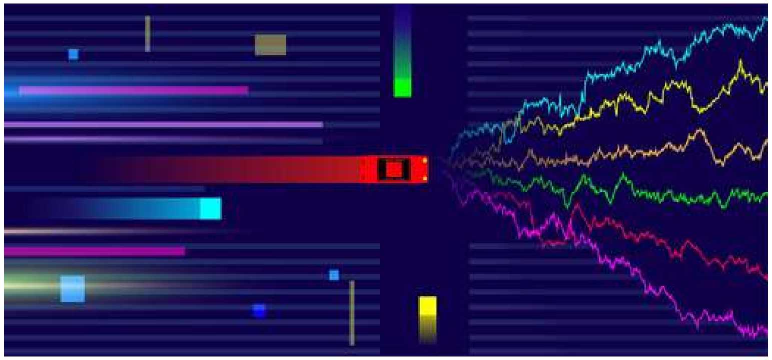
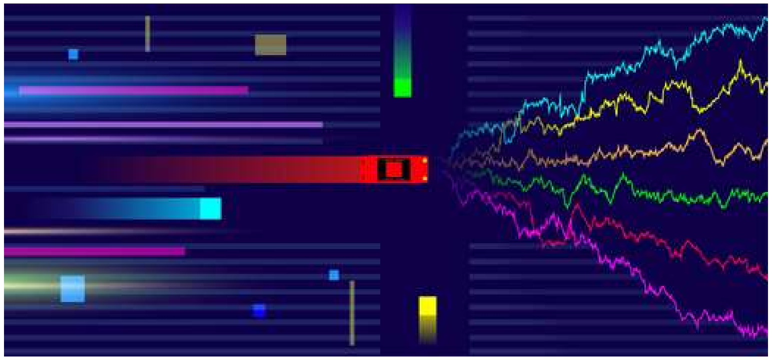


$$r_1 + r_2 + r_3 + \ldots + r_H$$



Value Function

The Value Function is expected return for policy

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(S_h, A_h) \mid S_0 = s, A_h \sim \pi(S_h), S_{h+1} \sim T(\cdot \mid S_h, A_h)\right]$$

$$Q^{\pi}(s,a) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(S_h, A_h) \mid (S_0, A_0) = (s,a), A_h \sim \pi(S_h), S_{h+1} \sim T(\cdot \mid S_h, A_h)\right]$$

Expectation over randomness in policy and transitions

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Starting State

Actions by policy

Next state by environment

Expectation over randomness in policy and transitions

Bellman Equation

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{h=0}^{\infty} \gamma^h r(S_h, A_h) \mid S_0 = s, A_h \sim \pi(S_h), S_{h+1} \sim T(\cdot \mid S_h, A_h)\right]$$

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The Bellman Equations note that:

$$V^{\pi}(s) = \mathbb{E}_{A \sim \pi(s)}[r(s,A) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,A)}[V^{\pi}(S')]]$$

$$Q^{\pi}(s,a) = r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[V^{\pi}(S')]$$

Optimal Policy

For an infinite horizon discounted MDP, there exists a deterministic stationary policy:

$$\pi^* : \mathcal{S} \to \mathcal{A}, \ \text{s. t. } V^{\pi^*}(s) \geq V^{\pi}(s) \ \ \forall s, \pi$$

See [Puterman1994]

Our goal is to find this policy, either looking at:
- Sample complexity (statistics)
- Optimization complexity

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See [Puterman1994]

Denote $\quad V^* = V^{\pi^*}, Q^* = Q^{\pi^*}$

Bellman Optimality

The optimal policy satisfies Bellman Optimality equation:

$$V^*(s) = \max_{a \in \mathcal{A}} r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[V^*(S')]$$

Q-greedy policy: $\pi^*(s) = \mathrm{argmax}_a\, Q^*(s,a)$

See [Puterman1994]

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$$V^\pi(s) = \mathbb{E}_{A \sim \pi(s)}[r(s,A) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,A)}[V^\pi(S')]]$$

Fixed Point Uniqueness

If $V(s) = \max_{a \in \mathcal{A}} r(s,a) + \gamma \mathbb{E}_{S' \sim T(\cdot|s,a)}[V(S')]$

then $V(s) = V^*(s) \forall s$

See [Puterman1994]

Highlights uniqueness of fixed point of the Bellman equation

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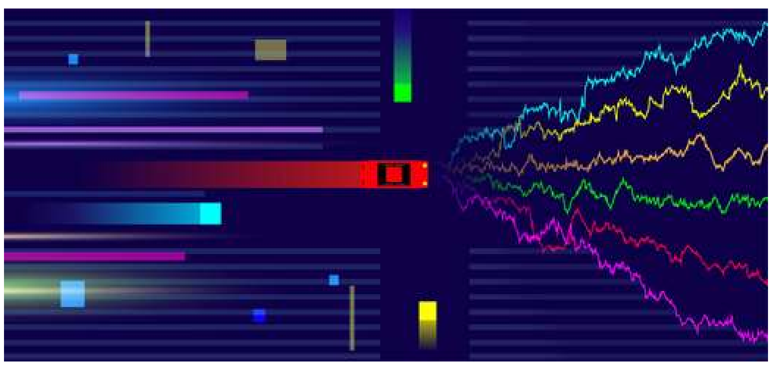

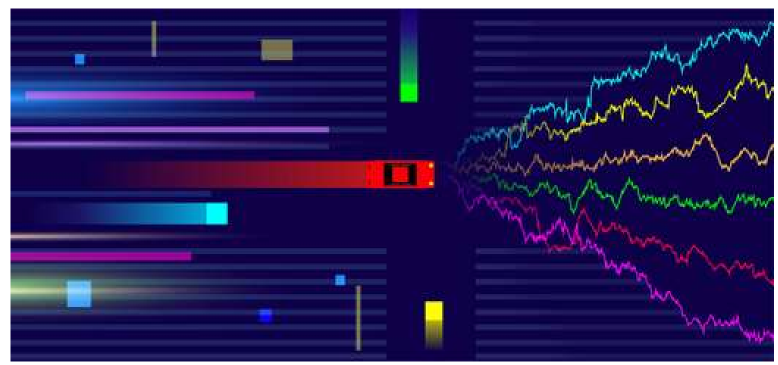

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Expectation over randomness in policy and transitions

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$$V_h^{\pi}(s) = \mathbb{E}_{A \sim \pi_h(s)}[r_h(s, A) + \mathbb{E}_{S' \sim T_h(\cdot|s,A)}[V_{h+1}^{\pi}(S')]]$$

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$$\pi_h^* : \mathcal{S} \to \mathcal{A}, \text{ s. t. } V_h^{\pi^*}(s) \geq V_h^{\pi}(s) \forall s, \pi$$

See [Puterman1994]

Our goal is to find this policy, either looking at:

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Denote $V_h^* = V_h^{\pi^*}, Q_h^* = Q^{\pi^*}$

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Highlights uniqueness of fixed point of
the Bellman equation

References

[Puterman1994] Martin Puterman. “Markov Decision Processes: Discrete Stochastic Dynamic Programming”. *John Wiley + Sons*, 1994.

[Sutton2018] Richard Sutton. “Reinforcement Learning: An Introduction.” *MIT Press*, 2018.

[Agarwal2021] Alekh Agarwal, Nan Jiang, Sham M. Kakade, Wen Sun. “Reinforcement Learning: Theory and Algorithms”. 2021.

[Slivkins2019] Alexsandrs Slivkins. “Introduction to Multi-Armed Bandits.” *Foundations and Trends in ML*, 2019.

[Powell2021] Warren Powell. “Reinforcement Learning and Stochastic Optimization.” 2021.

[Meyn2021] Sean Meyn. “Control Systems and Reinforcement Learning”. *Cambridge University Press*, 2021.

Course Slides

Cornell CS6789: Foundations of Reinforcement Learning
https://wensun.github.io/CS6789_fall_2021.html
Stanford CS 234: Reinforcement Learning
https://web.stanford.edu/class/cs234/
UCL COMPM050: Course on RL
https://www.davidsilver.uk/teaching/