MISC

Log

$\ln(mn) = \ln(m) + \ln(n)$

$\ln\left(\frac{m}{n}\right) = \ln(m) - \ln(n)$

$\ln(m^r) = r\ln(m)$

Exponent

$(ab)^x = a^x b^x$

$(a^x)^y = a^{xy}$

$a^x a^y = a^{x+y}$

Summation

$\sum_{i=1}^{n} ar^{i-1} = a\frac{1-r^n}{1-r}$

Integrals

$\int \frac{1}{x}dx = \ln|x|$

$\int e^{ax}dx = \frac{1}{a}e^{ax}$

$\int a^x dx = \frac{a^x}{\ln(a)}$

$\int \ln(x)dx = x\ln(x) - x$

$\int \cos(x)dx = \sin(x)$

$\int \sin(x)dx = -\cos(x)$

Derivatives

$(e^x)' = e^x$

$(\ln(x))' = \frac{1}{x}$

$\sin(x) = \cos(x)$

$\cos(x) = -\sin(x)$

$(fg)' = fg' + f'g$

$\frac{1}{f} = -\frac{f'}{f^2}$

$(f(g(x)))' = f'(g(x))g'(x)$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Median

Middle number in sorted. If discrete distro, check up to where we have $p < 0.5$ and then $p > 0.5$, the number we have to add to cross threshold is median (see here)

Mode

Most likely value/appears most often.

COUNTING

n choose k

nb of combinations (any order)

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

permutations

nb of ways of ordering n elements (order matters)

$n!$

subsets of n elements

2^n

partitions

n objects into r groups

$\frac{n!}{n_1!n_2!\ldots n_r!}$

PROB BASICS

Properties

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \le P(A) + P(B)$

Conditional

$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

$P(A \cap B) = P(B)P(A \mid B)$

Total Prob Theorem

$P(B) = P(A_1 \cap B) + \ldots + P(A_n \cap B)$

$= P(A_1)P(B \mid A_1) + \ldots$

Bayes

$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)}$

Independence

$P(A \mid B) = P(A)$

$P(A \cap B) = P(A)P(B)$

CONT. DISTROS

$P(a \le x \le b) = \int_a^b f_X(x)dx$

Disjoint

$P(1 \le x \le 3 \text{ or } 4 \le x \le 5)$

$= P(1 \le x \le 3) + P(4 \le x \le 5)$

Properties

$f(x) \ge 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$

CDF Properties

· $\to_{x\to\infty} 1$ and $\to_{x\to-\infty} 0$

·increasing/monotonic

·right-continuous ___*---

Uniform

$f_X(x) = \frac{1}{b-a}$ $\qquad \hat{b}^{\text{MLE}} = \max(X_i)$

$F_X(x) = \frac{x-a}{b-a}$

$E[X] = \frac{a+b}{2}$

$Var(X) = \frac{(b-a)^2}{12}$

Exponential

time to wait for something

$f_X(x) = \lambda e^{-\lambda x}$

$P(X \ge a) = \int_a^\infty \lambda e^{-\lambda x}dx = e^{-\lambda a}$

$F_X(x) = 1 - e^{-\lambda x}$

$E[X] = \frac{1}{\lambda}$ $\qquad I = \frac{1}{\lambda^2}$

$E\left[X^2\right] = \frac{2}{\lambda^2}$

$Var(X) = \frac{1}{\lambda^2}$

Beta

$f(x) = \frac{1}{k}x^{a-1}(1-x)^{b-1}\mathbf{1}\{x \in [0,1]\}$

$k = \int_0^1 t^{a-1}(1-t)^{b-1}dt$

$E[X] = \frac{a}{a+b}$

DISCRETE DISTROS

Bernouilli

$P(X=1) = p$ $\qquad I = \frac{1}{p(1-p)}$

$E[X] = p$ $\qquad \hat{p}^{\text{MLE}} = \overline{X_n}$

$Var(X) = p(1-p)$

Uniform

$p_X(x) = \frac{1}{b-a+1}$

$E[X] = \frac{a+b}{2}$

$Var(X) = \frac{(b-a+1)^2 - 1}{12}$

Binomial

k successes in n trials

$p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}$

$E[X] = np$

$Var(X) = np(1-p)$

Geometric

number of trials until success

$p_X(k) = (1-p)^{k-1}p$

$E[X] = \frac{1}{p}$

$Var(X) = \frac{1-p}{p^2}$

Poisson

how many occurrences k in τ given rate λ

$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$ $\qquad I = \frac{1}{\lambda}$

$E[N_\tau] = \lambda\tau$

$Var(N_\tau) = \lambda\tau$

NORMALS

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$E[X] = \mu$ $\qquad I(\mu,\sigma^2) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{pmatrix}$

$Var(X) = \sigma^2$

Linear Functions

$Y = aX + b$ with $X \sim N\left(\mu,\sigma^2\right)$

$Y = N\left(a\mu + b, a^2\sigma^2\right)$ $\qquad \hat{\mu}^{\text{MLE}} = \overline{X_n}$ $\qquad \widehat{\sigma^2}^{\text{MLE}} = S_n$

Indie Sum

$Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

Tables

$\Phi(-2) = P(Y \le -2) = 1 - P(Y \le 2) = 1 - \Phi(2)$

Standardising

$X \sim N\left(\mu,\sigma^2\right)$ and $Y \sim N(0,1)$

$Y = \frac{X-\mu}{\sigma}$ $\quad X = \mu + \sigma Y$

Moments

1	μ	0
2	$\mu^2+\sigma^2$	σ^2
3	$\mu^3+3\mu\sigma^2$	0
4	$\mu^4+6\mu^2\sigma^2+3\sigma^4$	$3\sigma^4$
5	$\mu^5+10\mu^3\sigma^2+15\mu\sigma^4$	0
6	$\mu^6+15\mu^4\sigma^2+45\mu^2\sigma^4+15\sigma^6$	$15\sigma^6$
7	$\mu^7+21\mu^5\sigma^2+105\mu^3\sigma^4+105\mu\sigma^6$	0

CONDITIONAL VARS

same for PDF

$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$

Multiplication Rule

$p_{X,Y}(x,y) = p_Y(y)p_{X|Y}(x \mid y)$

$p_{X,Y,Z}(x,y,z) = p_X(x)p_{Y|X}(y \mid x)p_{Z|X,Y}(z \mid x,y)$

$p_{X,Y|Z}(x,y \mid z) = \frac{p_{X,Y,Z}(x,y,z)}{p_Z(z)}$

MULTIPLE VARS

$\sum_x \sum_y p_{X,Y}(x,y) = 1$

$P((X,Y) \in B) = \int\int_{(x,y)\in B} f_{X,Y}(x,y)dxdy$

Marginals

$p_X(x) = \sum_y p_{X,Y}(x,y)$

$f_X(x) = \int f_Y(y)f_{X|Y}(x \mid y)dy$

⚠ ranges: what values can Y take when X = x?

$= \int f_{X,Y}(x,y)dx$

Expected Value Rule

$E[g(X,Y)] = \sum_x \sum_y g(x,y)p_{X,Y}(x,y)$

$E[g(X,Y)] = \int E[g(x,y) \mid Y = y]f_Y(y)dy$

$E[g(X,Y) \mid Y = y] = \int g(x,y)f_{X|Y}(x \mid y)dy$

CDF

$F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(s,t)dsdt$

VARIANCE

$Var(x) = E\left[(x-\mu)^2\right]$ and $\sigma = \sqrt{Var(X)}$

Properties

$Var(aX+b) = a^2Var(X)$

$Var(X) = E\left[X^2\right] - (E[X])^2$

Dependent Sum

$Var(X_1 + \ldots + X_n) = \sum_{i=1}^{n} Var(X_i) + \sum_{\{i,j\}:i\ne j} Cov(X_i,X_j)$

Independent Sum

$Var(X+Y) = Var(X-Y) = Var(X) + Var(Y)$

Law of Total Var

$Var(X) = E[Var(X \mid Y)] + Var(E[X \mid Y])$

Sample Variance

$S_n = \frac{1}{n}\sum^{n}\left(X_i - \overline{X_n}\right)^2$

$E[S_n] = \frac{n-1}{n}\sigma^2$

Unbiased Sample Variance

$\widetilde{S_n} = \frac{n}{n-1}S_n$

$E\left[\widetilde{S_n}\right] = \sigma^2$

STATISTICAL MODEL

$(E, (P_\theta)_{\theta\in\Theta})$

E: sample space $(X_1 \ldots)$

P: family of prob measures on E

Θ: Param set

well specified if $\theta^\star \in \Theta$

⚠ sample space must not depend on parameter

⚠ sample space must be the support for the distribution. i.e. $\left([0,\infty), \left\{N\left(\mu,\sigma^2\right)\right\}\right)$ is not valid because the sample space for a N is all R

EXPECTATIONS

Expected Value

$E[g(x)] = \sum_x g(x)p_X(x)$

$E[g(x)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$

Linearity of Expectations

$E[aX+b] = aE[X] + b$

$E[X+Y] = E[X] + E[Y]$

Total Expectation Th.

$E[X] = \sum_y p_Y(y)E[X \mid Y = y]$

$E[X] = \int_{-\infty}^{\infty} f_Y(y)E[X \mid Y = y]dy$

$E[X] = \sum_i P(A_i)E[X \mid A_i]$

Cond. Expectation

$E[g(x) \mid Y = y] = \sum_x g(x)p_{X|Y}(x \mid y)$

Iterated Expectation

$E[E[X \mid Y]] = E[X]$ (ex.)

INDEPENDENCE

If Indie

$E[XY] = E[X]E[Y]$

$Var(X+Y) = Var(X) + Var(Y)$

$p_{X,Y|Z}(x,y \mid z) = p_{X|Z}(x \mid z)p_{Y|Z}(y \mid z)$

$f_{X,Y}(x,y) = f_X(x)f_Y(y)$

$f_{X|Y}(x \mid y) = f_X(x)$

$Cov(X,Y) = 0$

MIXED RV

$X = Y$ (discrete) w.p p

Z (continuous) w.p (1-p)

$F_X(x) = pF_Y(x) + (1-p)F_Z(x)$

$E[X] = pE[Y] + (1-p)E[Z]$

DERIVED DISTROS

PMF function of discrete RV

$p_Y(y) = P(g(x) = y) = \sum_{x:g(x)=y} p_X(x)$

Linear Functions

$Y = aX + b$

$p_Y(y) = p_X\left(\frac{y-b}{a}\right)$

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

g is monotonic

$f_Y(y) = f_X(h(y))\left|\frac{dh}{fy}(y)\right|$

general case

1) find CDF: $F_Y(y) = P(g(x) \le y)$
2) derive CDF for PDF

CONVOLUTIONS

$Z = X + Y$

$p_Z(z) = \sum_x p_X(x) p_Y(z-x)$

$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$

COVARIANCE

$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$

Direction

$Cov(X,Y) > 0$ same sign

If Indie

$Cov(X,Y) = 0$

⚠ inverse not usually true but true for Gaussians:

$Cov(X,Y) = 0 \rightarrow$ X,Y ~ N indie

Properties

$Cov(X,X) = Var(X)$

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$Cov(aX+b, Y) = aCov(X,Y)$

$Cov(X, Y+Z) = Cov(X,Y) + Cov(Y,Z)$

CORRELATION COEF.

$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

FRESH START/MEMORYLESSNESS

Exponential

$f_{X|X>t}(x \mid x > t) = f_X(x)$

Bernouilli/Poisson

$P(A \mid B) = P(A)$

i.e. prob of two arrivals (A) after 1 arrival (B) = prob of 2 arrivals (A)

MISC

$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \exp(\lambda)$

e limits

$\lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n = e^{-t}$

$\lim_{n\to\infty}\left(1 + \frac{t}{n}\right)^n = e^{t}$

BERNOUILLI PROCESS

requires indie, time homogen.

Properties

$S = X_1 + ... + X_n$

$P(S = K) = \binom{n}{k} p^k (1-p)^{n-k}$

$E[S] = np$

$Var(S) = np(1-p)$

Time until 1st success

$T_1 = \min\{i : X_i = 1\}$

$P(T_1 = k) = (1-p)^{k-1}p$

Time of kth arrival

$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}$

$E[Y_k] = \frac{k}{p}$

$Var(Y_k) = \frac{k(1-p)}{p^2}$

Merging

$Z_t = g(X_t, Y_t) \sim Ber(p + q - pq)$

$\Rightarrow$ prob either or both have arrival at time t

Splitting

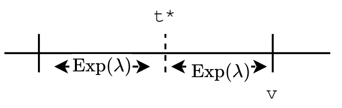
flip a coin with prob q

$A \sim Ber(qp)$

$B \sim Ber((1-q)p)$

⚠ these streams are not indie

INTER-ARRIVAL TIMES / R.INCIDENCE



we arrive at t*

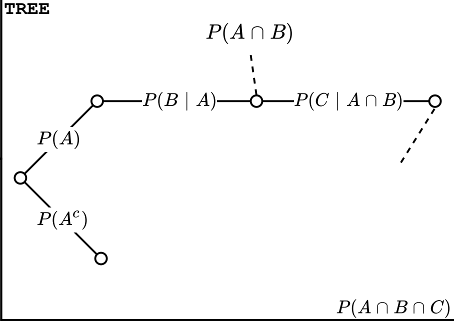
u,v are each $Exp(\lambda)$ away from t*

u

previous arrival

$\Rightarrow$ $E[V - U]$ is twice the expectation of $\mathrm{Exp}(\lambda)$

TREE



MIN/MAX

$P(\max > x) = 1 - P(\max < x)$

$= 1 - [P(X_i < x)]^n$

$P(\min > x) = [P(X_i > x)]^n$

$= [1 - P(X_i < x)]^n$

POISSON PROCESS

indie, time homogen. seq of exp

λ: arrival rate

$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$ $\qquad I = \frac{1}{\lambda}$

$E[N_\tau] = \lambda\tau$

$Var(N_\tau) = \lambda\tau$

$\lambda = \frac{E[N_\tau]}{\tau}$

Time of kth arrival / Erlang

$f_{Y_k} = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$

$= Erlang(k)$

$= Erlang\left(\frac{k}{2}\right) + Erlang\left(\frac{k}{2}\right)$

Sum

⚠ must be indie

M: $Poisson(\mu)$ N: $Poisson(v)$

M+N: $Poisson(\mu + v)$

Merging

A: λ_A B: λ_B

$\lambda = \lambda_A + \lambda_B$

$P\left(k^{th}\text{arrival is A}\right) = \frac{\lambda_A}{\lambda_A + \lambda_B}$

$P(\text{k arrivals are A})$ is $\text{Binomial}\left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)$

Splitting

flip a coin with prob q

⚠ these streams <u>are</u> indie

A: $\lambda_A = \lambda q$

B: $\lambda_B = \lambda(1-q)$

Multiple Engine Example

3 engines with death rate λ_e

rate until 1st dies is $\lambda = 3\lambda_e$

then rate until 2nd dies $\lambda = 2\lambda_e$

Min

$P(\min\{X,Y,Z\} \ge t)$

$= P(X \ge t, Y \ge t, Z \ge t)$

$= e^{-3\lambda t}$

$\Rightarrow$ have 3 merged Poissons and want to know first arrival

$\Rightarrow$ $\min\{X,Y,Z\}$ is $\mathrm{Exp}(3\lambda)$

$E[\min\{X,Y,Z\}] = \frac{1}{3\lambda}$

Max

$P(\max(T_1, T_2, T_3) \le t)$

$= P(T_1 \le t)P(T_2 \le t)P(T_3 \le t)$

$= \left(1 - e^{-\lambda t}\right)^3$

then derive this to get PDF

LLN

req. iid and $E[|X_i|] < -\infty$

$\overline{X_n} = \frac{1}{n}\sum^n X_i = E[X]$

$E\left[\overline{X_n^2}\right] = Var\left(\overline{X_n}\right) + \left(E\left[\overline{X_n}\right]\right)^2$

⚠ because $\overline{X_n}$ is a RV like any other

CONT MAPPING TH.

$T_n \to T$ then $f(T_n) \to f(T)$

COVARIANCE MATRIX AND MV STUFF

$\Sigma = \begin{pmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{pmatrix}$

$= E\left[(X - E[X])(Y - E[Y])^T\right]$

$Var(\mathbf{X}) = Cov(\mathbf{X})$

$Cov(\mathbf{AX} + \mathbf{B}) = Cov(\mathbf{AX}) = \mathbf{A}Cov(\mathbf{X})\mathbf{A}^T = \mathbf{A}\Sigma\mathbf{A}^T$

Gaussian vector

defined by $\boldsymbol{\mu}$ and Σ, $x \in R^d$

$f_X(x) = \frac{1}{\sqrt{(2\pi)^d \det\Sigma}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)$

MV CLT

$X_i \sim R^d$ $E[\mathbf{X}_i] = \boldsymbol{\mu}$ $Cov(\mathbf{X}_i) = \Sigma$

MV Delta

$\sqrt{n}(g(T_n) - g(\theta)) \to N\left(0, \nabla g(\theta)^T \Sigma \nabla g(\theta)\right)$

CLT

req. iid, $E[X_i] < \infty$ and $Var(X_i) < \infty$

$\sqrt{n}\frac{\overline{X_n} - \mu}{\sigma} \to N(0,1)$

$\sqrt{n}\left(\overline{X_n} - \mu\right) \to N\left(0, \sigma^2\right)$

QUANTILES

$P(X \le q_\alpha) = 1 - \alpha$

$\alpha = .1 \to q_\alpha$ is 90th percentile

$P(|Z| > 1.96) = 0.05$

α	2.5%	5%	7.5%	10%
q_α	1.96	1.65	1.44	1.28

SLUTSKY TH.

$T_n \to T$ and $U_n \to u$

T is r.v. and u is real

$T_n + U_n \to T + u$

$T_n U_n \to Tu$

$\frac{T_n}{U_n} \to \frac{T}{u}$

LIKELIHOODS

Bernouilli $p^{\sum^n X_i}(1-p)^{n - \sum^n X_i}$

Poisson $\frac{\lambda^{\sum X_i}}{x_1!...x_n!}\exp(-n\lambda)$

Gaussian $\frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}}\exp\left(-\frac{1}{2\sigma^2}\sum(x_i - \mu)^2\right)$

Exponential $\lambda^n \exp\left(-\lambda\sum X_i\right)$

Uniform $\frac{1}{b^n}\mathbf{1}\{\max X_i \le b\}$

⚠ a=0 here

IDENTIFIABILITY

θ identifiable iff mapping $\theta \in \Theta \to P_\theta$ is injective

(injective: $\theta \ne \theta' \Rightarrow P_\theta \ne P_{\theta'}$)

ESTIMATORS

Asym. normal if

$\sqrt{n}\left(\widehat{\theta_n} - \theta\right) \to N\left(0, \sigma^2\right)$

Consistency

$\widehat{\theta_n} \to \theta$ as $n \to \infty$

Bias

$bias\left(\widehat{\theta_n}\right) = E\left[\widehat{\theta_n}\right] - \theta$

Quadratic Risk

$R\left(\widehat{\theta_n}\right) = E\left[\left|\widehat{\theta_n} - \theta\right|^2\right]$

Confidence Interval level $1 - \alpha$

conf.int. can't depend on unknown

$P\left(\overline{X_n} - q_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X_n} + q_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$

UNBIASED ESTIMATOR

we want $bias\left[\widehat{\theta_n}'\right] = 0$

find $\widehat{\theta_n}$ and use linear property of expectations to create a new estimator such that

$E\left[\widehat{\theta_n}'\right] = cE\left[\widehat{\theta_n}\right] = \theta$

1D DELTA METHOD

g: cont. differentiable

$\sqrt{n}(Z_n - \theta) \to N\left(0, \sigma^2\right)$

$\sqrt{n}(g(Z_n) - g(\theta)) \to N\left(0, (g'(\theta))^2 \cdot \sigma^2\right)$

P-VALUE

⚠ is a level α

what is the probability of observing a result more extreme than this one under H_0?

⚠ low p-value is bad → H_0 is unlikely

i.e. $P(|\widehat{a}| \ge \widehat{a}_{\text{obs}})$

KL DIVERGENCE

$KL(P_\theta, P_{\theta'}) = \sum_{x\in E} p_\theta(x)\log\left(\frac{p_\theta(x)}{p_{\theta'}(x)}\right)$

$KL(P_\theta, P_{\theta'}) = \int_E f_\theta(x)\log\left(\frac{f_\theta(x)}{f_{\theta'}(x)}\right)dx$

Properties

<u>not</u> symmetric

not negative

definite

triangle ineq

TESTS

⚠ failing to reject H_0 does not mean accepting H_0

Errors

test / reality	H_0	H_1
H_0	✔	type 1 error (reject when shouldn't)
H_1	type 2 error (fail to reject when should)	✔

level α

max type 1 error rate

higher $\alpha \rightarrow$ more likely to reject H_0

power β

$\pi_\psi = \inf_{\theta \in \Theta n}(1 - \beta_\psi(\theta))$

example 2 sided

coin $H_0 : p = \frac{1}{2}$ and $H_1 : p \neq \frac{1}{2}$

$$\psi = 1\left\{\sqrt{n}\frac{\left|\overline{X_n} - \frac{1}{2}\right|}{\sqrt{\frac{1}{2}\left(1 - \frac{1}{2}\right)}}\right) > q_{\frac{\alpha}{2}}\right\}$$

stats diff between X and Y?

$\overline{X_n} \sim N(\mu_1, \sigma_1^2)$ and $\overline{Y_n} \sim N(\mu_2, \sigma_2^2)$

$H_0 : \mu_1 = \mu_2 \text{and} H_1 : \mu_1 \neq \mu_2$

$$\sqrt{n}\frac{\overline{X_n} - \overline{Y_n}}{\sqrt{\sigma_1^2 \sigma_2^2}} \sim N(0, 1)$$

single-sided

⚠ evaluate H_0 at boundary (see part c here)

$H_0 \mu \geq \sigma$ and $H_1 \mu < \sigma$

boundary is $\mu = \sigma$ for $g(\theta)$ or θ

TOTAL VARIATION DISTANCE

max dist between two distros

⚠ E is joint set of values of RVs

$$TV(P_\theta, P_{\theta'}) = \frac{1}{2}\sum_{x \in E}|p_\theta(x) - p_{\theta'}(x)|$$

$$TV(P_\theta, P_{\theta'}) = \frac{1}{2}\int_{-\infty}^{\infty}|f_\theta(x) - f_{\theta'}(x)|dx$$

Properties

symmetric: $TV(P_\theta, P_{\theta'}) = TV(P_{\theta'}, P_\theta)$

positive: $0 \leq TV \leq 1$

definite: if $TV(P_\theta, P_{\theta'}) = 0$ then $P_\theta = P_{\theta'}$

triangle ineq:

$TV(P_\theta, P_{\theta'}) \leq TV(P_\theta, P_{\theta''}) + TV(P_{\theta''}, P_{\theta'})$

if disjoint: $TV = 1$

if same: $TV = 0$

MAXIMIZATION

global extremes on range

test critical points and end points

min/max

$h''(x) \leq 0 \rightarrow$ concave, maximum

$h''(x) < 0 \rightarrow$ global max

$h''(x) \geq 0 \rightarrow$ convexe, minimum

MV min/max

$X^T Hh(\theta) X \leq 0$ concave, max

+1 top diag: convexe, minimum $\begin{pmatrix} +1 & ? \\ ? & ? \end{pmatrix}$

MLE

minimizes KL divergence

$\hat{\theta}_n^{MLE} = arg \max_{\theta \in \Theta} \log(L)$

⚠ function must be cont. diff. to use derivative to find extremums. use a plot and think if not

Consistency and Asym. Norm.

if:

- param is identifiable
- support of P_θ does not depend on θ
- $\theta^\star$ is not at boundary
- $I(\theta)$ is invertible
- more stuff

$\hat{\theta}_n^{MLE} \rightarrow \theta^\star$

$\sqrt{n}\left(\hat{\theta}_n^{MLE} - \theta^\star\right) \rightarrow N\left(0, I(\theta^\star)^{-1}\right)$

Process to find extremum

- get l_n
- find crits with $l_n{}'(\theta) = 0$
- check if crits are local min/max
- check values at endpoints

METHOD OF MOMENTS

$\widehat{m}_k = \overline{X_n^k} = \frac{1}{n}\sum X_i^k$

LLN $\widehat{m}_k \rightarrow m_k(\theta) = E_\theta[X_1^k]$

ASYM NORM $\sqrt{n}\left(\hat{\theta} - \theta\right) \rightarrow N(0, \Gamma(\theta))$

$$\Gamma(\theta) = \left[\frac{\delta M^{-1}}{\delta \theta}\right]^T \Sigma(\theta)\left[\frac{\delta M^{-1}}{\delta \theta}\right]$$

finding $\hat{\theta}$

write θ as function $E[X]$, $E\left[X^2\right]$...

then sub for $\overline{X_n}$, $\overline{X_n^2}$

M-ESTIMATION

Lecture 12, tab 2

FISHER INFORMATION

⚠ use ONE observation

not well defined if support depends on unknown (shifted exp)

⚠ $l''(\theta)$ must exist

$I(\theta) = Var(l'(\theta)) = -E[l''(\theta)]$

⚠ the E[] is of the observation X and not the unknown! $E[\theta X] = \theta E[X]$

χ^2 DISTRO

distro of sum of $Z_i \sim N(0,1)$

$E[V] = d$

$Var(V) = 2d$

COCHRAN'S TH.

$\frac{nS_n}{\sigma^2} \sim \chi_{n-1}^2$ or $nS_n \sim \frac{\sigma^2}{n}\chi_{n-1}^2$

t DISTRO

for small nb of Gaussian samples w/

$Z \sim N(0,1)$ and $V \sim \chi_d^2$ and $\text{SampleVar} = \frac{V}{d}$

$$\frac{Z}{\sqrt{\frac{V}{d}}}$$

t TEST

- requires Gaussian samples
- is pivotal (q in tables)
- test is non-asymptotic

one sample two-sided

$H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$

$$T_n = \frac{\sqrt{n}\overline{X_n}}{\sqrt{\widetilde{S_n}}} = \frac{\sqrt{n}\frac{\overline{X_n} - \mu_0}{\sigma}}{\sqrt{\frac{\widetilde{S_n}}{\sigma^2}}} \sim t_{n-1}$$

$\psi_\alpha = \mathbf{1}\left\{|T_n| > q_{\frac{\alpha}{2}}\right\}$

one sample one-sided

$H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$

$$T_n = \frac{\sqrt{n}(\overline{X_n} - \mu_0)}{\sqrt{\widetilde{S_n}}} \sim t_{n-1}$$

$\psi_\alpha = \mathbf{1}\{T_n > q_\alpha\}$

two sample

$\overline{X_n} \sim N\left(\Delta_d, \frac{\sigma_d^2}{n}\right)$ and $\overline{Y_m} \sim N\left(\Delta_c, \frac{\sigma_c^2}{m}\right)$

$$\frac{\overline{X_n} - \overline{Y_m} - (\Delta_d - \Delta_c)}{\sqrt{\frac{\hat{\sigma}_d^2}{n} + \frac{\hat{\sigma}_c^2}{m}}} \sim t_N$$

where N according to Welch-Satter:

$$N = \frac{\left(\frac{\hat{\sigma}_d^2}{n} + \frac{\hat{\sigma}_c^2}{m}\right)^2}{\frac{\hat{\sigma}_d^4}{n^2(n-1)} + \frac{\hat{\sigma}_c^4}{m^2(m-1)}} \geq \min(n, m)$$

WALD'S TEST

- test is asymptotic
- not invariant to change in rep of H_0
- only req est of unrestricted model, lower computation

$H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ for $\theta \in \mathbb{R}^d$

$$n\left(\hat{\theta}^{\text{MLE}} - \theta_0\right)^T I\left(\hat{\theta}^{\text{MLE}}\right)\left(\hat{\theta}^{\text{MLE}} - \theta_0\right) \rightarrow \chi_d^2$$

equivalently

$$T_n = \left\|\sqrt{n}I(\theta_0)^{\frac{1}{2}}\left(\hat{\theta}^{\text{MLE}} - \theta_0\right)\right\|^2 \rightarrow \chi_d^2$$

which gives test $\psi_\alpha = \mathbf{1}\{T_n > q_\alpha\}$

where q_α is the $(1-\alpha)$-quantile of χ_d^2

LIKELIHOOD RATIO TEST

- how diff is likelihood from null

X_i iid $\Theta \in \mathbb{R}^d$

$H_0 : (\theta_{r+1}, ..., \theta_d) = \left(\theta_{r+1}^{(0)}, ..., \theta_d^{(0)}\right)$

$$T_n = 2\left(l_n\left(\hat{\theta}_n^{\text{MLE}}\right) - l_n\left(\theta_n^{(0)}\right)\right)$$

⚠ same n for both likelihoods

Wilk Th.

assuming H_0 is true and MLE conditions

$T_n \rightarrow \chi_{d-r}^2$

$\psi_\alpha = \mathbf{1}\{T_n > q_\alpha\}$

CATEGORICAL LIKELIHOOD

i.e. are Zodiac signs uniformly distributed? $p_0 = \left(\frac{1}{12}, \frac{1}{12}...\right)$

$L_n = p_1^N{}_{-}1 \ldots p_k^N{}_{-}1$

$N_j = \#\{X_i = a_j\}$

$\overset{\rightarrow\text{MLE}}{\hat{p}}$: $\hat{p}_j = \frac{N_j}{n}$ prob of obs. outcome j

$p_j = P(X = a_j) = \prod p_i^{\mathbf{1}(a_i = a_j)}$

χ^2 TEST

$H_0 : \vec{p} = \vec{p}^0$ vs. $H1 : \vec{p} \neq \vec{p}^0$

$$T_n = n\sum^k\left[\frac{\left(\hat{p}_j - p_j^0\right)^2}{p_j^0}\right] \rightarrow \chi_{k-1}^2$$

where k is nb of categories

χ^2 TEST FOR FAMILY OF DIST

$H_0 : p \in \{\text{Bin}(k, \theta)\}_{\theta \in \Theta}$ vs $H_1 : p \notin \{\text{Bin}(k, \theta)\}_{\theta \in \Theta}$

$$T_n = n\sum_{j=0}^{k}\frac{\left(\frac{N_j}{n} - f_{\hat{\theta}}(j)\right)^2}{f_{\hat{\theta}}(j)} \rightarrow \chi_{(k+1)-d-1}^2$$

⚠ $k - d - 1$ if we start at j=1

$\Theta \in \mathbb{R}^d$

f_θ is PMF of $\text{Bin}(k, \theta)$

$\hat{\theta}$ is MLE here

EMPIRICAL CDF

$F_n(t) = \frac{1}{n}\sum \mathbf{1}\{X_i \leq t\}$

it is discontinuous

$\sqrt{n}(F_n(t) - F(t)) \rightarrow N(0, F(t)(1 - F(t)))$

DONSKER'S TH.

if F cont:

$\sqrt{n}\max_{t \in \mathbb{R}}|F_n(t) - F(t)| \rightarrow \max_{0 \leq t \leq 1}|B(t)|$

where B is Brownian bridge

KS TEST (example)

X_i: real RV with unk CDF

$H_0 : F = F^0$ vs $H1 : F \neq F^0$

$\delta_\alpha^{\text{KS}} = \mathbf{1}\{T_n > q_\alpha\}$

$= \mathbf{1}\left\{\max_{t \in \mathbb{R}}\sqrt{n}|F_n(t) - F(t)| > q_\alpha\right\}$

p-value $P(Z > T_n \mid T_n)$

computation

$$\frac{T_n}{\sqrt{n}}$$

$$= \max_{1 \leq i \leq n}\left[\max\left(\left|F^0(X_i) - F_n(X_i)\right|, \left|F^0(X_i) - \frac{i}{n}\right|\right)\right]$$

$$= \max_{1 \leq i \leq n}\left[\max\left(\left|F^0(X_i) - F_n(X_i)\right|, \left|F^0(X_i) - \frac{i-1}{n}\right|\right)\right]$$

CDF OF SAMPLE IS UNIFORM

$Y = F_X(x)$

$F_Y \sim Un$ if $(0, 1)$

KL TEST (example)

is my data Gaussian?

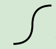
more likely to reject than KS test

$\max_{t \in \mathbb{R}}\left|F_n(t) - \Phi_{\hat{\mu}, \hat{\sigma}^2}(t)\right|$

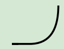
QQ PLOT (example 1, 2)

$F_n^{-1} = X_i$ (F_n is sample CDF)

points are $\left(F^{-1}\left(\frac{1}{n}\right), x_1\right), \left(F^{-1}\left(\frac{2}{n}\right), x_2\right)$...



lighter tails than normal



exp(+)

fatter tails than normal

exp(-)

BAYESIAN STATS

$$\pi(\theta \mid X_1...X_n) = \frac{\pi(\theta)L_n(X_1...X_n \mid \theta)}{\int_\Theta \pi(\theta)L_n(X_1...X_n \mid \theta)}$$

$$\propto \pi(\theta)L_n(X_1...X_n \mid \theta)$$

conjugate prior if post. distro. same as prior distro.

improper prior i.e. $\pi(\theta) = 1$, not a valid distro

Jeffrey's prior

non-informative prior, not always improper. reflects no prior belief, only stats model

$\pi_J(\theta) \propto \sqrt{\det I(\theta)}$

reparam. invariance

we have Jeff prior for θ, want $\eta = \Phi(\theta)$

- replace θ with $\Phi^{-1}(\eta)$
- multiply by $dth\frac{\eta}{\det a} = \frac{1}{\Phi'(\theta)}$

confidence region

$P(\theta \in \mathbb{R} \mid X_1...X_n) = 1 - \alpha$

Bayes estimator

mean of posterior

⚠ MUST USE ACTUAL POSTERIOR, not the prop. one

$\hat{\theta}^\pi = \int_\Theta \theta\pi(\theta \mid X_1...X_n)d\theta$

$\text{aVar} = I^{-1}(\theta)$ of distro sampled

MAP

$\hat{\theta}^{\text{MAP}} = arg \max_\theta \pi(\theta \mid X_1...X_n)$

$= arg \max_\theta L_n(X_1...X_n \mid \theta)\pi(\theta)$

⚠ look at Bayes estimator and ask "which actual possible values of θ make this result most likely

i.e. is $\left|\theta_1 - \hat{\theta}^{\text{Bayes}}\right| > \left|\theta_2 - \hat{\theta}^{\text{Bayes}}\right|$

⚠ if discrete, MAP is in set of possible values

CONSISTENCY AND ASYM NORMALITY

is estimator consistent?

check lim as $n \rightarrow \infty$ against estimator

is estimator asym. normal?

start with CLT definition, then put in the estimator. also get aVar like this. see examples.