Introduction to Digital Communications

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Chapter 1

Introduction and background

Modern telecommunication systems are formed by terminals exchanging *information* by transmitting *signals* through some transmission resource called *channel*. The ensemble of rules allowing this exchange of information is called *protocol*. The ensemble of terminals, channels and protocols forms a *telecommunication network*.

Protocols have been organized by the International Standard Organization into a reference hierarchical system, called Open System Interconnection (ISO-OSI). The goal of the ISO-OSI reference model is to provide a layered structure where upper layers can be built on top of existing lower layers without redesigning the whole network from scratch, and where existing higher levels can use new lower layers in a transparent way. The ISO-OSI is composed by 7 layers: Physical, Data Control Link, Network, Transport, Session, Presentation, Application. In this course, we are concerned with the Physical Layer. Moreover, we restrict our study to the simple case of one-way point-to-point transmission.

1.1 Reference model for one-way point-to-point transmission

We consider the simple idealized one-way point-to-point communication system represented in Fig. 1.1. This is composed by: information source, source encoder, channel encoder, signal modulator, physical channel, signal demodulator, channel decoder, source decoder, end user.

Information source. What is information? Intuitively, an event brings us some information if its occurrence is unexpected. In 1948, Claude Shannon defined in a mathematically rigorous way the theory of information *contained* in random variables and processes, and the problem of its reliable transmission over communication channels [1]. His seminal work is now considered as the birth of *Information Theory*, and more in general of modern communication theory [2].

Without getting into the details, we can say that an information source (e.g., speech, video) is characterized by its *rate-distortion function* $R(D)$, that gives the theoretical minimum number of bit/s (rate) necessary to represent the source with distortion D, according to some distortion measure. In the case of discrete sources (e.g., a data file), the minimum rate necessary to represent the source with zero distortion is called *entropy rate*.

Source encoder. The task of the source encoder is to translate the source signal into a sequence of bits at rate $R_b \geq R(D)$, for a given desired distortion level D. Examples of source

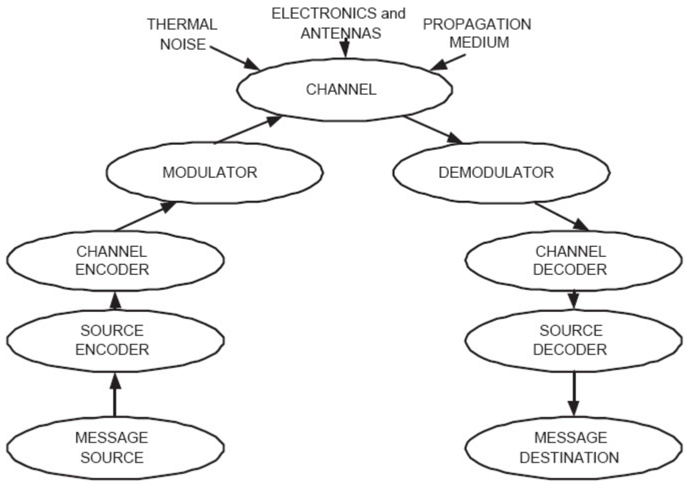


Figure 1.1: Block diagram of a one-way point-to-point communication system.

encoders for analog sources are: PCM quantizers, vector quantizers, vocoders [3], still images and video compression schemes (e.g., JPEG, MPEG), subband coding schemes for audio. Examples of zero-distortion discrete source encoders are: Huffman encoders, Lempel-Ziv encoders [2] and various practical file compression schemes.

Channel encoder. The channel encoder takes the sequence of bits produced by the source encoder (ideally i.i.d.) and map them onto a sequence of channel symbols belonging to a given channel input alphabet $\mathcal{X}$. The set of all possible code sequences $\mathcal{C} \subseteq \mathcal{X}^N$ (where N denotes the sequence length) is called *codebook*, or simply *code*. Each code sequence $\mathbf{x} \in \mathcal{C}$ is called *codeword*. The code is constructed in order to "protect" the input information sequence from possible transmission errors occurring on the channel.

The code rate, expressed in bit per channel symbol, is given by $R = K/N$, where K is the length of the input information sequence (in bits). The symbol rate, i.e., the number of symbols per second, necessary to transmit the bit rate R_b is given by $R_s = R_b/R$.

Perhaps the main result of Shannon Theory is that nearly every physically meaningful communication channel is characterized by a rate C bit/s, called *channel capacity*, such that if $R_b < C$, there exist channel codes with arbitrarily small error probability, while if $R_b > C$, there are no such codes, and in many cases of interest the probability of error is close to 1.

A source is said to be transmissible with distortion D over a channel with capacity C if $R(D) < C$.

Signal modulator. The task of the signal modulator is to turn the codeword that must be transmitted over the channel into a signal able to travel over the physical channel without too much distortion. Three examples are given in Figure **??**. The first is was used in 300 bit/s (Baud) telephone MODEMs and is an example of *Binary Frequency Shift Keying (FSK)*, which we will revisit later in the course. The signals conveying 0 and 1 are two sinusoids of different frequency. The second example is used in T1 transmission systems for transmitting 1.5 Mbit/s over a copper wire. It is based on *Alternate Mark Inversion*, where a 1 is transmitted by a positive or negative polarity depending on whether the previous bit was a 1. Adjacent 1s use opposite polarity in order to avoid a DC component in the transmitted signal. 0s are conveyed by not transmitting any signal. The third example is a *Quarternary Pulse-Position Modulation* which is common in short-range optical transmission (e.g. television remote control) and some evolving ultrawideband short range radio systems. Here two bits are conveyed every 25 ns by modulating the position (among 4) of a pulse in the interval of 25ns.

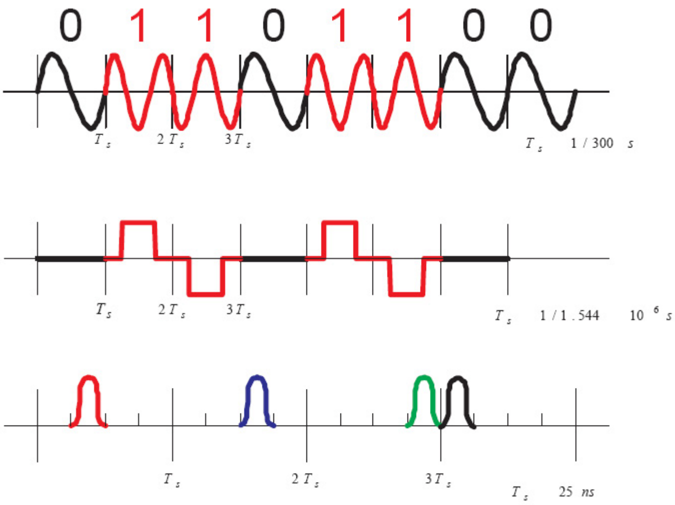


Figure 1.2: Block diagram of a one-way point-to-point communication system.

Physical channel. We include as part of the physical channel all electronic devices (filters, amplifiers, antennas, free space propagation, cables, wires) and all external disturbances (thermal noise, interference) that may distort the signal produced by the modulator.

We restrict our treatment to linear time-invariant (LTI) additive-noise channels, whose relationship between the input signal $x(t)$ and the output signal $y(t)$ is given by

$$y(t) = h(t) \otimes x(t) + n(t) \tag{1.1}$$

where $h(t)$ is the channel impulse response, $n(t)$ is additive noise and $\otimes$ denotes convolution.

Demodulator. The task of the demodulator is to process the received signal output by the channel and to produce an *observation* from which the channel decoder can recover the transmitted codeword.

Channel decoder. The channel decoder decides (or *detects*) which codeword was transmitted, based on the observation of the received signal produced by the demodulator. The detected codeword is then mapped back to the corresponding information bit sequence.

Source decoder. The source decoder reproduces the original source signal from the bit sequence recovered by the channel decoder. If the sequence does not contain errors, it is able to represent the source with the desired level of distortion.

1.2 Signals and systems

In this section we recall some basic facts about deterministic and stochastic signals and systems.

1.2.1 Finite-energy and finite-power deterministic signals

The signal $x(t)$ is said to have finite energy if

$$\mathcal{E}_x \triangleq \int |x(t)|^2 dt < \infty$$

and it is said to have finite power if

$$\mathcal{P}_x \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt < \infty$$

For signals of the first type, we define the autocorrelation function of $x(t)$ as

$$\phi_x(\tau) \triangleq \int x(t)x(t-\tau)^* dt$$

For signals of the second type, we define the time-averaged autocorrelation function

$$\phi_x(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)^* dt$$

Let $\mathcal{F}[\cdot]$ denote the Fourier transform operator, such that $X(f) = \mathcal{F}[x] = \int x(t)e^{-j2\pi ft}dt$. For a finite-energy signals $x(t)$, $|X(f)|^2 = \mathcal{F}[\phi_x]$ is called *energy spectral density* (ESD). In fact, because of Parseval identity,

$$\int |X(f)|^2 df = \phi_x(0) = \mathcal{E}_x$$

For a finite-power signals $x(t)$, we define the *power spectral density* (PSD) $S_x(f) \triangleq \mathcal{F}[\phi_x]$. In fact, $\int S_x(f)df = \phi_x(0) = \mathcal{P}_x$.

The output of an LTI system with impulse response $h(t)$ to the input $x(t)$ is given by the convolution integral

$$y(t) = h(t) \otimes x(t) \triangleq \int h(\tau)x(t-\tau)d\tau$$

In the frequency domain, we have $Y(f) = H(f)X(f)$, where $H(f) = \mathcal{F}[h]$ is the system *transfer function*. The ESD (resp. PSD) of $y(t)$ and $x(t)$ are related by: $|Y(f)|^2 = |H(f)|^2|X(f)|^2$ (resp. $S_y(f) = |H(f)|^2 S_x(f)$), where $|H(f)|^2$ is the system energy (resp. power) transfer function. In the time domain, we have

$$\phi_y(\tau) = \phi_h(\tau) \otimes \phi_x(\tau)$$

1.2.2 Random processes

Roughly speaking, a random process $x(t)$ can be seen either as a sequence of random variables $x(t_1), x(t_2), \ldots, x(t_n)$, indexed by the "time" index $t = t_1, t_2, \ldots$, or as a collection of signals $x(t;\omega)$, where ω is a random experiment taking on values in a certain event space Ω. The full statistical characterization of a random process $x(t)$ is given by the collection of all joint probability cumulative distribution function (cdf)

$$\Pr(x(t_1) \leq x_1, x(t_2) \leq x_2, \ldots, x(t_n) \leq x_n)$$

for all $n = 1, 2, \ldots$ and for all instants $t_1, t_2, \ldots, t_n$.

Complex processes and random variables are characterized by the joint statistics of its real and imaginary parts. For example, a random variable $X = X_1 + jX_2$ is characterized by the joint cdf $\Pr(X_1 \leq x_1, X_2 \leq x_2)$. A complex random variable is said to be *circularly-symmetric* if its real and imaginary parts satisfy

$$\text{cov}(X_1, X_2) = 0, \quad \text{var}(X_1) = \text{var}(X_2)$$

The first and second order statistics of $x(t)$ are defined by its mean

$$\mu_x(t) = E[x(t)]$$

and by its autocorrelation function

$$\phi_x(t_1, t_2) = E[x(t_1)x(t_2)^*]$$

The average energy content of a random process $x(t)$ is given by

$$\begin{aligned}
\mathcal{E}_x &= E\left[\int |x(t)|^2 dt\right] \\
&= \int \phi_x(t,t)dt
\end{aligned}$$

The average power content of a random process $x(t)$ is given by

$$\begin{aligned}
\mathcal{P}_x &= E\left[\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt\right] \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \phi_x(t,t)dt
\end{aligned}$$

The PSD of a random process $x(t)$ is defined in general by

$$S_x(f) \triangleq E\left[\lim_{T\to\infty} \frac{|X_T(f)|^2}{T}\right]$$

where $X_T(f) = \int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt$. We have the following result [3]:

Wiener-Khinchin theorem. If for all finite τ and any interval $\mathcal{I}$ of length $|\tau|$ the autocorrelation function of the random process $x(t)$ satisfies the condition

$$\left|\int_{\mathcal{I}} \phi_x(t+\tau, t)dt\right| \leq \infty$$

the PSD of $x(t)$ is the Fourier transform of $\bar{\phi}_x(\tau)$, given by

$$\bar{\phi}_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \phi_x(t+\tau, t)dt$$

$\square$

For two random processes $x(t)$ and $y(t)$, defined on a joint probability space, we define the *cross-correlation* function

$$\phi_{xy}(t_1, t_2) = E[x(t_1)y(t_2)^*]$$

and the cross-spectrum

$$S_{xy}(f) \triangleq E\left[\lim_{T\to\infty} \frac{X_T(f)Y_T(f)^*}{T}\right]$$

WSS and WSC processes. A random process $x(t)$ is said to be wide-sense stationary (WSS) if

1. $\mu_x(t) = \mu_x$ is constant with t.
2. $\phi_x(t_1, t_2)$ depends only on the difference $\tau = t_1 - t_2$ (for brevity, we shall use the notation$\phi_x(\tau) \triangleq \phi_x(t+\tau, t)$).

Two random processes $x(t)$ and $y(t)$ are said to be jointly WSS if both $x(t)$ and $y(t)$ are individually WSS and if their cross-correlation function $\phi_{xy}(t_1, t_2)$ depends only on the difference $t_1 - t_2$ (again, we shall use the notation $\phi_{xy}(\tau)$ in this case).

A random process $x(t)$ is said to be wide-sense cyclostationary (WSC) if

1. $\mu_x(t)$ is periodic of period T, i.e., $\mu_x(t) = \mu_x(t + kT)$, for $k \in \mathbb{Z}$.
2. $\phi_x(t_1, t_2)$ is periodic of period T, i.e., $\phi_x(t_1, t_2) = \phi_x(t_1 + kT, t_2 + kT)$, for $k \in \mathbb{Z}$.

For WSS processes, we have that the Wiener-Khinchin theorem holds and yields

$$S_x(f) = \mathfrak{F}[\phi_x] = \int \phi_x(\tau)e^{-j2\pi f\tau}d\tau$$

For WSC processes, if $|\int_0^T \phi_x(t+\tau, t)dt| < \infty$, then the Wiener-Khinchin theorem yields

$$S_x(f) = \mathfrak{F}[\bar{\phi}_x]$$

where $\bar{\phi}_x(\tau)$ is obtained by time-averaging the autocorrelation function over one period, i.e.,

$$\bar{\phi}_x(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \phi_x(t+\tau, t)dt$$

For jointly WSS processes, the cross-spectrum is given by $S_{xy}(f) = \mathfrak{F}[\phi_{xy}(\tau)]$.

The output of a LTI system with impulse response $h(t)$ to the WSS input $x(t)$ is the WSS process given by the convolution integral

$$y(t) = h(t) \otimes x(t) \triangleq \int h(\tau)x(t-\tau)d\tau$$

The two processes $x(t)$ and $y(t)$ are jointly WSS. The mean and autocorrelation of $y(t)$ and the cross-correlation between $x(t)$ and $y(t)$ are given by

$$\begin{aligned}
\mu_y &= \mu_x \int h(t)dt \\
\phi_y(\tau) &= \phi_h(\tau) \otimes \phi_x(\tau) \\
\phi_{xy}(\tau) &= h(-\tau)^* \otimes \phi_x(\tau)
\end{aligned}$$

In the frequency domain we have

$$\begin{aligned}
\mu_y &= \mu_x H(0) \\
S_y(f) &= |H(f)|^2 S_x(f) \\
S_{xy}(f) &= H(f)^* S_x(f)
\end{aligned}$$

Since $\phi_{yx}(\tau) = \phi_{xy}(-\tau)^*$, we have $S_{yx}(f) = S_{xy}(f)^*$, that yields $S_{yx}(f) = H(f)S_x(f)$, since $S_x(f)$ is real.

1.2.3 Sampling bandlimited signals and processes

Let $x(t)$ be a bandlimited signal, i.e., such that $X(f)$ is zero for $f \notin [-W/2, W/2]$. Then, $x(t)$ is defined uniquely by its samples $x[i] = x(i/W)$ taken at rate W, by the interpolation formula

$$x(t) = \sum_i x[i]\text{sinc}(W(t - i/W)) \tag{1.2}$$

where $\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$. Sampling can be seen as the multiplication of $x(t)$ by a sequence of Dirac delta functions spaced by $1/W$. In the frequency domain, the sampled signal has a periodic transform. In fact, the transform of $\sum_i \delta(t - i/W)$ is given by $W \sum_i \delta(f - iW)$. Therefore, the transform of $x(t) \sum_i \delta(t - i/W)$ is given by

$$X(f) \otimes W \sum_i \delta(f - iW) = W \sum_i X(f - iW) \tag{1.3}$$

If W is larger than the maximum frequency for which $X(f)$ is non-zero, the replicas of $X(f)$ translated in integer multiples of W in the above periodic spectrum are non-overlapping. Therefore,

$$X(f) = \frac{1}{W}\Pi(f/W)W \sum_i X(f - iW) \tag{1.4}$$

where $\Pi(f)$ is the ideal brickwall low-pass filter with amplitude 1 for $f \in [-1/2, 1/2]$ and zero elsewhere. By applying inverse Fourier transform to the above equation, and by recalling that

$$\mathcal{F}^{-1}[\Pi(f)] = \text{sinc}(t) \tag{1.5}$$

we obtain the interpolation formula (1.2). The steps in (1.2,1.3,1.4,1.5) are shown in Figure 1.3.

The discrete-time Fourier transform of the signal $x[i]$ is defined by

$$\mathcal{X}(\lambda) = \sum_i x[i] \exp(-j2\pi\lambda i) \tag{1.6}$$

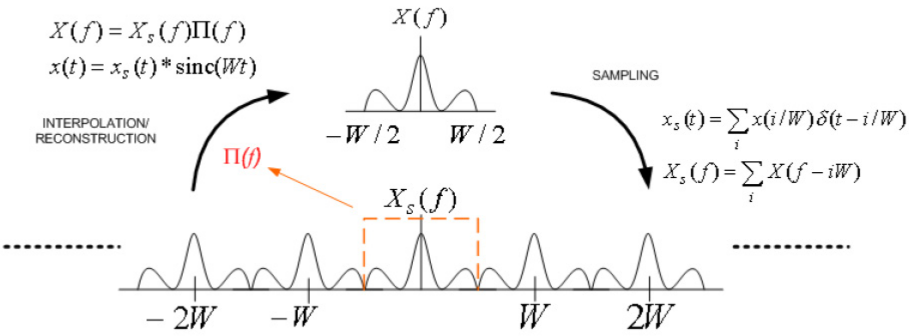


Figure 1.3: The Sampling and Reconstruction Processes.

for $\lambda \in [-1/2, 1/2]$. The relation between the continuous-time Fourier transform $X(f)$ of $x(t)$ and the discrete-time Fourier transform $\mathcal{X}(\lambda)$ of the sampled version $x[i]$ of $x(t)$ is called *spectral folding* and it is obtained as follows

$$\begin{aligned}
\mathcal{X}(\lambda) &= \sum_i x[i] \exp(-j2\pi\lambda i) \\
&= \sum_i \int X(f) \exp(j2\pi f i/W) df \exp(-j2\pi\lambda i) \\
&= \int X(f) \sum_i \exp(j2\pi(f - W\lambda)i/W) df \\
&= \int X(f) W \sum_i \delta(f - (\lambda + i)W) df \\
&= W \sum_i X((\lambda + i)W) \qquad (1.7)
\end{aligned}$$

where we have used the identity

$$\sum_i \exp(j2\pi a i/W) = W \sum_i \delta(a - iW)$$

A WSS process is said to be bandlimited in $[-W/2, W/2]$ if its PSD is zero outside this interval. In this case, the autocorrelation function of $x(t)$ can be written as

$$\phi_x(\tau) = \sum_i \phi_x[i] \text{sinc}(W(\tau - i/W))$$

and the interpolation formula (1.2) holds in the mean-square sense, i.e.,

$$E\left[\left| x(t) - \sum_i x[i] \text{sinc}(W(t - i/W)) \right|^2\right] = 0$$

1.2.4 Representation of finite-energy signals by orthonormal expansions

The set of all signals $x(t)$ with finite energy, defined over the interval $-T/2, T/2$ (where T can be infinite), is a Hilbert space, i.e., a closed complete vector space with inner product

$$(x|y) \triangleq \int x(t)^* y(t) dt$$

The norm of this space is given by $\|x\|^2 = (x|x)$, and the squared distance between two elements of the space is given by

$$\|x(t) - y(t)\|^2 = \int |x(t) - y(t)|^2 dt = \|x\|^2 + \|y\|^2 - 2\text{Re}\{(x|y)\}$$

Two signal are said to be orthogonal if $(x|y) = 0$. A set of signals $\mathcal{B} = \{\xi_1(t), \ldots, \xi_n(t)\}$ is said to be orthonormal if

$$(\xi_i|\xi_j) = \delta_{i,j}$$

In general Hilbert spaces, we have the following fundamental results

Cauchy-Schwarz inequality. For all x, y,

$$|(x|y)| \leq \|x\| \|y\|$$

with equality if and only if $x(t) = \lambda y(t)$, for some $\lambda \in \mathbb{C}$. □

Projection theorem. Let S be a subspace of a Hilbert space, and x an arbitrary element of the Hilbert space. Then, there is a unique element $\widetilde{x} \in S$ minimizing $\|x - y\|$ over all $y \in S$. Moreover, $\widetilde{x}$ must satisfy the necessary and sufficient condition $(x - \widetilde{x}|y) = 0$ for all $y \in S$. $\widetilde{x}$ is called orthogonal projection of x on S. □

Gram-Schmidt orthonormalization. Let $x_1, \ldots, x_n$ be elements of a Hilbert space, and let $S = \text{span}(x_1, \ldots, x_n)$ be the subspace generated by $x_1, \ldots, x_n$. Then, there exists an orthonormal set (basis) $\mathcal{B} = \{\xi_1, \ldots, \xi_m\}$ with $m \leq n$, such that $S = \text{span}(\xi_1, \ldots, \xi_m)$. □

The proof of the above statement is constructive and it is known as the *Gram-Schmidt algorithm*. It allows to find an orthonormal basis for the subspace $S = \text{span}(x_1, \ldots, x_n)$ starting from the elements x_i. Let $j = 0$ and define $s_i = \sum_{\ell=1}^{j-1} (\xi_\ell | x_i) \xi_\ell$. Then, for $i = 1, \ldots, n$

1. Let $z_i = x_i - s_i$.
2. If $z_i \neq 0$, then let $j \leftarrow j + 1$ and $\xi_j = z_i / \|z_i\|$.

In general, the basis obtained by the above algorithm depends on the order in which the elements x_i are considered. Different orderings yield different bases for the same vector space.

Minimum distance approximation. We want to approximate an arbitrary element x in a Hilbert space by a linear combination of elements $x_1, \ldots, x_n$. In other words, we look for the element $\widetilde{x} \in S = \text{span}(x_1, \ldots, x_n)$ such that $\|x - \widetilde{x}\|$ is minimum. In general, we can write $\widetilde{x}$ as the linear combination

$$\widetilde{x} = \sum_{i=1}^{n} \alpha_i x_i$$

By applying the projection theorem we get

$$\left(y \left| x - \sum_{i=1}^{n} \alpha_i x_i \right. \right) = 0$$

for all $y \in S$. The above equation holds in particular for all x_i, and since any $y \in S$ can be written as a linear combination of the x_i's, by the linearity of the inner product we get that if it holds for all x_i, then it holds for all $y \in S$. Eventually, we obtain the system of equations

$$\begin{aligned} r_{1,1}\alpha_1 + r_{1,2}\alpha_2 + \cdots + r_{1,n}\alpha_n &= d_1 \\ r_{2,1}\alpha_1 + r_{2,2}\alpha_2 + \cdots + r_{2,n}\alpha_n &= d_2 \\ &\vdots \\ r_{n,1}\alpha_1 + r_{n,2}\alpha_2 + \cdots + r_{n,n}\alpha_n &= d_n \end{aligned} \tag{1.8}$$

where we let $r_{i,j} = (x_i|x_j)$ and $d_i = (x_i|x)$. The above equations are called *normal equations* and can be written in compact form as $\mathbf{R}\boldsymbol{\alpha} = \mathbf{d}$, where $\mathbf{R} = [r_{i,j}]$ is the matrix of inner products of the elements x_i, and it is called *Gram matrix*, and $\mathbf{d}$ is the vector of the inner products between the x_i's and the desired element x. The determinant of $\mathbf{R}$ is non-zero if and only if the elements x_i are linearly independent. If they are not, there exist multiple solutions. However, a possibly non-unique solution always exists.

Infinite series of orthogonal elements. Let $\{\xi_1, \xi_2, \ldots\}$ be an infinite set of orthonormal elements in a Hilbert space. The series $\sum_{i=1}^{n} \alpha_i \xi_i$ converges to an element x in the space if and only if $\sum_{i=1}^{n} |\alpha_i|^2 \leq \infty$. In that case, we have $\alpha_i = (\xi_i|x)$. □

Bessel inequality. Let x be an arbitrary element of a Hilbert space, and let $\{\xi_1, \xi_2, \ldots\}$ be an orthonormal sequence in the space, then

$$\sum_{i=1}^{\infty} |(\xi_i|x)|^2 \leq \|x\|^2$$

□

From the projection theorem, we have that in general the series $\widetilde{x} = \sum_{i=1}^{\infty} (\xi_i|x)\xi_i$ exists and it is the element of the closed subspace generated by the orthonormal sequence $\{\xi_1, \xi_2, \ldots\}$ at minimum distance from x. Moreover, the *approximation error* $x - \widetilde{x}$ is orthogonal to this subspace.

Complete orthonormal sequences. A sequence of orthonormal elements $\{\xi_1, \xi_2, \ldots\}$ in a given Hilbert space is said to be complete if the closed subspace generated by it is the whole Hilbert space. Moreover, a sequence of orthonormal elements is complete if and only if the only element of the space orthogonal to each element of the sequence is 0. □

Example: Fourier series. The sequence of orthonormal functions $\{\xi_i(t) = \frac{1}{\sqrt{T}} e^{j2\pi it/T} : i \in \mathbb{Z}\}$ is complete for the Hilbert space of finite-energy signals over the interval $[-T/2, T/2]$. The expansion of a given signal $x(t)$ in terms of the $\xi_i(t)$ is the well-known exponential Fourier series

$$x(t) = \sum_i \alpha_i e^{j2\pi it/T}$$

where the i-th Fourier coefficient is given by

$$\alpha_i = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi it/T} dt$$

Geometrical representation of finite-dimensional signals. Let $x_1(t), \ldots, x_n(t)$ be a set of signals (not necessarily linearly independent) and let $\mathcal{B} = (\xi_1(t), \ldots, \xi_m(t))$ be an orthonormal basis for $S = \text{span}(x_1(t), \ldots, x_n(t))$. Then, any signal $y(t) \in S$ is represented in a unique way as

$$y(t) = \sum_{i=1}^{m} y_i \xi_i(t)$$

where $y_i = (\xi_i | y)$. Therefore, there is a one-to-one correspondence between the signals in S and the complex m-dimensional vectors in $\mathbb{C}^m$. In other words, once we fix a basis of the signal set, we can represent signals as finite-dimensional vectors.

1.3 Baseband complex equivalent system

In general, a communication channel is characterized by a range of frequencies where its transfer function has non-negligible magnitude. The channel frequency band is specified by giving its center frequency (carrier) f_c and its one-sided bandwidth B. A signal $s(t)$ suited to travel along the channel must have his Fourier transform (spectrum) concentrated around f_c, over the bandwidth B. Therefore, it can be written as

$$\begin{aligned} s(t) &= a(t)\cos(2\pi f_c t + \theta(t)) \\ &= x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) \\ &= \text{Re}\left\{x(t)e^{j2\pi f_c t}\right\} \end{aligned} \tag{1.9}$$

where $a(t), \theta(t), x_I(t), x_Q(t)$ and $x(t)$ are called amplitude, phase, in-phase component, quadrature component and complex envelope of the signal $s(t)$, respectively.

In the frequency domain, by letting $S(f) = \mathcal{F}[s(t)]$ and $X(f) = \mathcal{F}[x(t)]$, we have

$$\begin{aligned} S(f) &= \int \text{Re}\left\{x(t)e^{j2\pi f_c t}\right\} e^{-j2\pi f t} dt \\ &= \frac{1}{2}\int \left(x(t)e^{j2\pi f_c t} + x(t)^* e^{-j2\pi f_c t}\right) e^{-j2\pi f t} dt \\ &= \frac{1}{2}\left(X(f - f_c) + X(-f - f_c)^*\right) \end{aligned} \tag{1.10}$$

Since the frequency content of $s(t)$ is concentrated around $\pm f_c$, the frequency content of $x(t)$ is concentrated around frequency zero (baseband). That is why the complex envelope is also called *baseband complex equivalent* signal.

A linear system with real impulse response $h(t)$ has the symmetry property

$$H(-f)^* = H(f)$$

If the system is pass band with center frequency f_c, let

$$C(f - f_c) = H(f)u(f) \tag{1.11}$$

and

$$C(-f - f_c)^* = H(-f)^* u(-f) \tag{1.12}$$

where $u(f)$ is the unit step function. From the symmetry property of $H(f)$, we have that

$$H(f) = C(f - f_c) + C(-f - f_c)^*$$

that in the time-domain becomes

$$\begin{aligned} h(t) &= c(t)e^{j2\pi f_c t} + c(t)^* e^{-j2\pi f_c t} \\ &= 2\mathrm{Re}\left\{c(t)e^{j2\pi f_c t}\right\} \end{aligned} \tag{1.13}$$

The complex signal $c(t)$ is called *complex baseband equivalent* channel impulse response. Let $r(t) = h(t) \otimes s(t)$. In the frequency domain this becomes

$$\begin{aligned} R(f) &= \frac{1}{2}(X(f - f_c) + X(-f - f_c)^*)(C(f - f_c) + C(-f - f_c)^*) \\ &= \frac{1}{2}(X(f - f_c)C(f - f_c) + X(-f - f_c)^* C(-f - f_c)^*) \\ &= \frac{1}{2}(V(f - f_c) + V(-f - f_c)^*) \end{aligned} \tag{1.14}$$

where $V(f) = X(f)C(f)$ and where we let $r(t) = \mathrm{Re}\{v(t)e^{j2\pi f_c t}\}$. We conclude that the complex envelope $v(t)$ of the output $r(t)$ can be obtained by convolving the complex envelope $x(t)$ of the input by the channel baseband equivalent impulse response $c(t)$.

This fact allows us to work on complex baseband equivalent signals and systems, and neglect the carrier term. All what stated in terms of complex envelopes, can be easily translated in terms of the corresponding real modulated signals. A special case is represented by baseband signals and systems (for which $f_c = 0$). In this case, the real signal is equal to its in-phase component, and the quadrature component is zero. The complex envelope coincides with the signal itself and it is also real.

1.3.1 Passband WSS processes

Let $z(t)$ be a bandpass real WSS process. Then, we can write

$$\begin{aligned} z(t) &= \mathrm{Re}\left\{n(t)e^{j2\pi f_c t}\right\} \\ &= n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned} \tag{1.15}$$

It can be shown [3] that the quadrature components n_I and n_Q satisfy

$$\begin{aligned} \phi_{n_I}(\tau) &= \phi_{n_Q}(\tau) \\ \phi_{n_I,n_Q}(\tau) &= -\phi_{n_Q,n_I}(\tau) \end{aligned} \tag{1.16}$$

By using these symmetry properties, we obtain

$$\begin{aligned} \phi_z(\tau) &= \phi_{n_I}(\tau)\cos(2\pi f_c \tau) - \phi_{n_I,n_Q}(\tau)\sin(2\pi f_c \tau) \\ \phi_n(\tau) &= 2(\phi_{n_I}(\tau) + j\phi_{n_I,n_Q}(\tau)) \end{aligned} \tag{1.17}$$

that yield

$$\phi_z(\tau) = \frac{1}{2}\mathrm{Re}\left\{\phi_n(\tau)e^{j2\pi f_c \tau}\right\}$$

In the frequency domain, by applying Fourier transform to the above relation we obtain

$$S_z(f) = \frac{1}{4}(S_n(f - f_c) + S_n(-f - f_c))$$

From the fact that for every two jointly WSS real processes, the cross-correlation function satisfies $\phi_{n_Q,n_I}(\tau) = \phi_{n_I,n_Q}(-\tau)$, and by using the above symmetry relation, we get that $\phi_{n_I,n_Q}(\tau)$ must be an odd function, i.e.,

$$\phi_{n_I,n_Q}(\tau) = -\phi_{n_I,n_Q}(-\tau)$$

Consequently, $\phi_{n_I,n_Q}(0) = 0$, i.e., the in-phase and quadrature components of a WSS bandpass process are uncorrelated for zero relative delay. This does not mean that $n_I(t_1)$ and $n_Q(t_2)$ are uncorrelated for all t_1 and t_2.

1.3.2 White Gaussian noise

A particularly important noise model is the *white circularly-symmetric complex Gaussian* noise $n(t) = n_I(t) + jn_Q(t)$, where $n_I(t)$ and $n_Q(t)$ are jointly Gaussian zero-mean (real) processes, such that

$$\begin{aligned} E[n_I(t)n_I(t-\tau)] &= \frac{N_0}{2}\delta(\tau) \\ E[n_Q(t)n_Q(t-\tau)] &= \frac{N_0}{2}\delta(\tau) \\ E[n_I(t)n_Q(t-\tau)] &= 0 \end{aligned} \tag{1.18}$$

The autocorrelation function of the complex noise $n(t)$ is given by $E[n(t)n(t-\tau)^*] = N_0\delta(\tau)$, and its PSD is constant and equal to N_0 over all frequencies.

White noise does not exist in nature, since it would have infinite power. However, it is a good model when the bandwidth of the noise introduced by the channel is much larger than the bandwidth of the signal. In any case, white noise is always "seen" through a filter, so that the filtered noise does have finite power, and it is physically meaningful.

In particular, we shall always deal with the following two cases:

Low-pass filtered and sampled noise. Let $\Psi(f)$ be a low-pass filter such that $\int |\Psi(f)|^2 df < \infty$, and let $\psi(t)$ the corresponding impulse response. A discrete-time circularly-symmetric Gaussian process can be obtained by passing $n(t)$ through the filter $\Psi(f)$ and by sampling the output at a given rate W. We have

$$\nu[i] = \int \psi(i/W - \tau)n(\tau)d\tau \tag{1.19}$$

The autocorrelation function of $\nu[i]$ is given by

$$\begin{aligned} \phi_\nu[i-j] &= E[\nu[i]\nu[j]^*] \\ &= E\left[\int\int \psi(i/W-\tau)n(\tau)\psi(j/W-\theta)^*n(\theta)^*d\tau d\theta\right] \\ &= N_0\int\int \psi(i/W-\tau)\psi(j/W-\theta)^*\delta(\tau-\theta)d\tau d\theta \\ &= N_0\int \psi(\tau)\psi(\tau-(i-j)/W)^*d\tau \\ &= N_0\phi_\psi((i-j)/W) \end{aligned} \tag{1.20}$$

The variance of the sampled noise is given by $\sigma_\nu^2 = N_0 \|\psi(t)\|^2$. In particular, if $\psi(t)$ has unit energy and its autocorrelation function satisfies $\phi_\psi(k/W) = 0$ for all $k \neq 0$, then $\nu[i]$ is a complex Gaussian discrete-time process such that $\nu_I[i] = \text{Re}\{\nu[i]\}$ and $\nu_Q[i] = \text{Im}\{\nu[i]\}$ are independent identically distributed Gaussian processes, with mean 0, cross-correlation 0 and autocorrelation function $N_0/2\delta_{k,0}$. We say that $\nu[i]$ is a *white* (i.e., uncorrelated) discrete-time complex circularly-symmetric Gaussian process with mean 0 and variance $E[|\nu[i]|^2] = N_0$. The term circularly-symmetric is due to the fact that the probability density function (pdf) of $\nu[i]$ and of $\nu[i]e^{j\theta}$ is the same, for any θ independent of $\nu[i]$. In other words, the pdf of $\nu[i]$ is invariant with respect to phase rotations.

Projected noise. Consider the projection of $n(t)$ over the set of signals $\{\xi_1(t), \dots, \xi_M(t)\}$. Let

$$\nu_i = \int \xi_i(t)^* n(t) dt$$

The vector $\boldsymbol{\nu} = (\nu_1, \dots, \nu_M)^T$ is circularly-symmetric complex Gaussian with mean zero and covariance matrix $\mathbf{R}$ whose (i,j)-th element is given by

$$\begin{aligned}
[\mathbf{R}]_{i,j} &= E[\nu_i \nu_j^*] \\
&= E\left[\int\int \xi_i(\tau)^* n(\tau) \xi_j(\theta) n(\theta)^* d\tau d\theta\right] \\
&= N_0 \int\int \xi_i(\tau)^* \xi_j(\theta) \delta(\tau - \theta) d\tau d\theta \\
&= N_0 \int \xi_i(\tau)^* \xi_j(\tau) d\tau
\end{aligned} \tag{1.21}$$

If $\{\xi_i(t)\}$ is an orthonormal set, then $\mathbf{R} = N_0 \mathbf{I}$ (i.e., $\boldsymbol{\nu}$ is white vector).

The proper complex Gaussian distribution. The joint probability density function (pdf) of the random variables ν_I and ν_Q, corresponding to the real and imaginary parts of a Gaussian noise sample ν, is given by

$$f_{\nu_I,\nu_Q}(x,y) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{1}{N_0}x^2\right) \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{1}{N_0}y^2\right) \tag{1.22}$$

We can define the complex variable $z = x + jy$, and use the short-hand notation

$$f_\nu(z) = \frac{1}{\pi N_0} \exp\left(-\frac{1}{N_0}|z|^2\right) \tag{1.23}$$

The probability distribution corresponding to the above pdf is called the one-dimensional *proper complex Gaussian distribution*. As already said above, we notice that the pdf of ν is invariant to rotations, i.e., $\nu' = e^{j\phi}\nu$ has the same distribution of ν for any deterministic phase ϕ. For this reason, the sampled (or projected) baseband-equivalent narrowband white Gaussian noise is said to be circularly-symmetric.

The proper complex Gaussian distribution with mean μ and variance σ^2 is indicated by $\mathcal{N}_\mathbb{C}(\mu, \sigma^2)$. Then, we use the notation $\nu \sim \mathcal{N}_\mathbb{C}(0, N_0)$ to indicate that ν is proper complex Gaussian with mean zero and variance N_0.

More in general, an n-dimensional complex random vector $\mathbf{x}$ with joint pdf

$$f_{\mathbf{x}}(\mathbf{z}) = \frac{1}{\pi^n \det(\mathbf{R})} \exp\left(-(\mathbf{z}-\boldsymbol{\mu})^H \mathbf{R}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right) \tag{1.24}$$

is said to be proper Gaussian, with mean $\boldsymbol{\mu} = E[\mathbf{x}]$ and covariance matrix $\mathbf{R} = E[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^H]$, and it is indicated as $\mathbf{x} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{\mu}, \mathbf{R})$. Let $\mathbf{x} = \mathbf{a} + j\mathbf{b}$ with $\mathbf{a}$ and $\mathbf{b}$. Then, the joint pdf of $(\mathbf{a}, \mathbf{b})$ is Gaussian with mean $(\mathrm{Re}\{\boldsymbol{\mu}\}, \mathrm{Im}\{\boldsymbol{\mu}\})$ and covariance matrix

$$\widetilde{\mathbf{R}} = \frac{1}{2}\left[\begin{array}{cc} \mathrm{Re}\{\mathbf{R}\} & -\mathrm{Im}\{\mathbf{R}\} \\ \mathrm{Im}\{\mathbf{R}\} & \mathrm{Re}\{\mathbf{R}\} \end{array}\right]$$

Notice that since $\mathbf{R}$ is Hermitian symmetric, then $\mathrm{Re}\{\mathbf{R}\}$ is symmetric and $\mathrm{Im}\{\mathbf{R}\}$ is antisymmetric, therefore $\widetilde{\mathbf{R}}$ above is symmetric.

In general, any random vector $\mathbf{x}$ (also non-Gaussian) with covariance matrix satisfying the above structure is said to be *proper*. A proper random vector has zero pseudo-covariance matrix, i.e.,

$$E\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T\right] = \frac{1}{2}\left(\mathrm{Re}\{\mathbf{R}\} - \mathrm{Re}\{\mathbf{R}\}\right) - \frac{j}{2}\left(\mathrm{Im}\{\mathbf{R}\} + \mathrm{Im}\{\mathbf{R}\}^T\right) = \mathbf{0}$$

1.4 Bayesian hypothesis testing

In this section we study the following problem:

Bayesian hypothesis testing. Consider two sets $\mathcal{X} = \{1, \ldots, M\}$ and $\mathcal{Y}$, and a joint probability assignment defined on $\mathcal{X} \times \mathcal{Y}$. Let $(X, Y) \in \mathcal{X} \times \mathcal{Y}$ be a pair of joint random variables, and consider the M hypotheses:

$$\begin{aligned} H_1 &: Y \sim P_1(y) \\ H_2 &: Y \sim P_2(y) \\ &\vdots \\ H_M &: Y \sim P_M(y) \end{aligned} \tag{1.25}$$

where if $\mathcal{Y}$ is discrete, $P_x(y) \triangleq \Pr(Y = y | X = x)$, and if $\mathcal{Y}$ is continuous, $P_x(y) = \Pr(Y \leq y | X = x)$ with density $p_x(y)$.

By observing the event $\{Y = y\}$ we want to determine the best hypothesis $\widehat{x}$ in order to minimize the *Bayes risk*

$$R = E[r(X, Y)]$$

where $r(x, y)$ is a given cost function of (x, y). $\Diamond$

For simplicity, in the following we use the notation for the case of continuous $\mathcal{Y}$. The operation of "guessing" the hypothesis by observing $\{Y = y\}$ is expressed mathematically by a function

$$g : \mathcal{Y} \to \{\mathcal{X}, \epsilon\}$$

defined on the whole $\mathcal{Y}$ called *decision rule*. The range of $g(y)$ is the set of hypotheses plus an extra symbol ϵ, corresponding to the "I don't know" answer (we assume that the cost function

$r(x, y)$ is defined also for $x = \epsilon$). Since $\{\mathcal{X}, \epsilon\}$ is a discrete set of $M + 1$ elements, any decision rule $g(y)$ defines the *decision regions*

$$\mathcal{D}_i = \{y \in \mathcal{Y} : g(y) = i\} \quad \text{for } \ i = 1, \ldots, M, e$$

Vice versa, any *partition* of $\mathcal{Y}$ into at most $M + 1$ regions $\mathcal{D}_i$ defines a decision rule

$$g(y) = i \quad \text{if } \ y \in \mathcal{D}_i$$

Fig. 1.4 illustrates the concept of the decision regions.

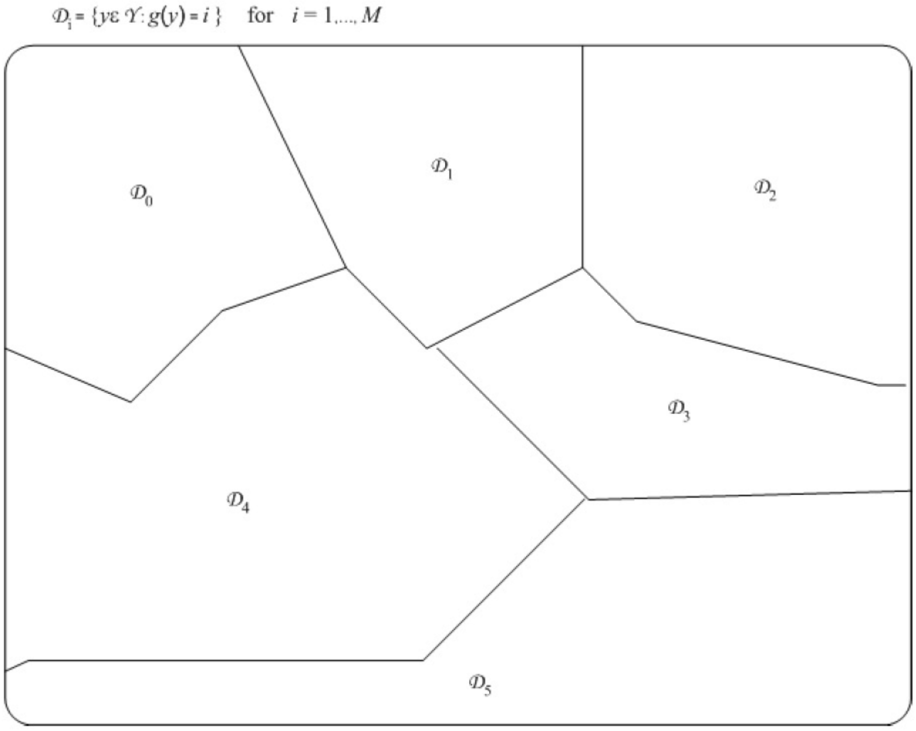


Figure 1.4: Partitioning of the observation set $\mathcal{Y}$ by the decision regions.

The most important Bayes risk in classical communications problems is the average error probability, given by

$$P(e) = \Pr(g(Y) \neq X) \tag{1.26}$$

which corresponds to the choice of the cost function

$$r(x, y) = 1\{g(y) \neq x\}$$

for $x = 1, \ldots, M$. It follows immediately that any decision rule minimizing $P(e)$ has necessarily $\mathcal{D}_\epsilon = \emptyset$. In other words, the answer "I don't know" never helps, if the goal is to minimize the average error probability, since its associated cost is always 1 error.

In order to determine the optimal decision rule, we can write

$$\begin{aligned}
P(e) &= \Pr(g(Y) \neq X) \\
&= 1 - \Pr(g(Y) = X) \\
&= 1 - \sum_{x=1}^{M} \Pr(X = x) \Pr(g(Y) = x | X = x) \\
&= 1 - \sum_{x=1}^{M} \int_{\mathcal{D}_x} p_x(y)\pi(x) dy
\end{aligned} \tag{1.27}$$

where we let $\pi(x) = \Pr(X = x)$ be the *a priori* probability of the hypothesis x.

The optimal decision rule $g(y)$ has decision regions $\mathcal{D}_x$ maximizing the sum of integrals in the last line of (1.27). Since $\{\mathcal{D}_x\}$ is a partition of $\mathcal{Y}$, the same differential element $p_x(y)\pi(x)dy$ cannot be assigned to more than one region. Therefore, the optimal choice for the regions is to assign to $\mathcal{D}_x$ all points $y \in \mathcal{Y}$ such that $p_x(y)\pi(x)$ is not smaller than any $p_i(y)\pi(i)$, for all $i = 1, \ldots, M$. Eventually, we obtain the optimal decision regions given by

$$\mathcal{D}_x = \left\{ y \in \mathcal{Y} : p_x(y)\pi(x) = \max_i \{p_i(y)\pi(i)\} \right\} \tag{1.28}$$

The boundary between two regions $\mathcal{D}_x$ and $\mathcal{D}_{x'}$ is given by the set of points y such that

$$p_x(y)\pi(x) = p_{x'}(y)\pi(x') = \max_i \{p_i(y)\pi(i)\}$$

and can be assigned to either $\mathcal{D}_x$ or $\mathcal{D}_{x'}$ without affecting the value of $P(e)$. Also, randomized decision rules choosing at random between x and x' if y belongs to the boundary yield the same average error probability.

The optimal decision rule defined by (1.28) can be written in a more concise form as

$$g(y) = \arg \max_{x \in \mathcal{X}} \{p_x(y)\pi(x)\} \tag{1.29}$$

This is sometimes referred to as *Maximum A Posteriori probability* (MAP) rule, since maximizing $p_x(y)\pi(x)$ with respect to x is equivalent to maximizing the a posteriori probability $\Pr(X = x|Y = y)$ of X given the the observation Y, in fact, from the Bayes rule we can write

$$\Pr(X = x|Y = y) = \frac{p_x(y)\pi(x)}{p(y)} = \frac{p_x(y)\pi(x)}{\sum_{i=1}^{M} p_i(y)\pi(i)}$$

(since the denominator in the RHS of the above equation is independent of x, maximization of $\Pr(X = x|Y = y)$ and of $p_x(y)\pi(x)$ yields the same decision regions).

A particularly important special case occurs when $\pi(x) = 1/M$ for all $x \in \mathcal{X}$ (uniform a priori probability). In this case, the term $\pi(x)$ can be neglected from maximization in (1.29), and the optimal rule consists of maximizing the conditional probability density $p_x(y)$ over $x \in \mathcal{X}$. In statistical estimation, the conditional density of the observation given the "parameter" that we want to estimate is called *likelihood function*, and the decision rule

$$g(y) = \arg \max_{x \in \mathcal{X}} \{p_x(y)\} \tag{1.30}$$

is called *Maximum-Likelihood* (ML) rule. In many practical cases, if the a priori probabilities are not known, they are artificially assumed to be uniform and the ML instead of the MAP rule is used.

Example: binary symmetric channel. Let $\mathcal{X} = \mathcal{Y} = \{0, 1\}$. Fig. 1.5 represents the diagram of a binary symmetric channel (BSC) with cross-over probability $p < 1/2$. Assuming uniform prior probabilities, we have

$$P_0(y) = \begin{cases} 1-p & y = 0 \\ p & y = 1 \end{cases}$$
$$P_1(y) = \begin{cases} p & y = 0 \\ 1-p & y = 1 \end{cases}$$

The MAP (or, equivalently, the ML) rule is given by

$$g(y) = y$$

Obviously, if $p > 1/2$, the rule would be $g(y) = 1 \oplus y$. □

Example: location testing with Gaussian error. Let $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y} = \mathbb{R}$. Assume that $Y \sim \mathcal{N}(\mu_x, \sigma^2)$, where $\mu_0 > \mu_1$ are given real numbers and where X has uniform prior probabilities. The MAP (or, equivalently, the ML) rule is defined by the decision region

$$\mathcal{D}_0 = \{y \in \mathbb{R} : p_0(y) \geq p_1(y)\}$$

The inequality defining $\mathcal{D}_0$ can be written as

$$\begin{aligned}
p_0(y) &\geq p_1(y) \\
\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{|y-\mu_0|^2}{2\sigma^2}} &\geq \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{|y-\mu_1|^2}{2\sigma^2}} \\
|y-\mu_0|^2 &\leq |y-\mu_1|^2 \\
-2\mu_0 y + \mu_0^2 &\leq -2\mu_1 y + \mu_1^2 \\
2(\mu_0-\mu_1)y &\geq \mu_0^2 - \mu_1^2 \\
y &\geq \qquad\qquad\qquad (\mu_0+\mu_1)/2
\end{aligned}$$

Therefore, the optimal decision rule consists of comparing the observation y with the threshold $(\mu_0 + \mu_1)/2$, which is equal to the middle point between μ_1 and μ_0. □

1.5 Upper and lower bounds to ML error probability

In this section we consider uniform prior probabilities and we find upper and lower bounds to $P(e)$ resulting from the (optimal) ML decision rule.

For any given pair of symbols $x \neq x'$, we define the pairwise error event

$$\{x \to x'\} \triangleq \{y \in \mathcal{Y} : p_x(y) \leq p_{x'}(y)\}$$

and the pairwise error probability (PEP) as

$$P(x \to x') \triangleq \Pr(\{x \to x'\}|X = x)$$

In words, $P(x \to x')$ is the probability that the ML decision rule chooses for x' when the true hypothesis was x, and as if x and x' were the two only possible hypotheses.

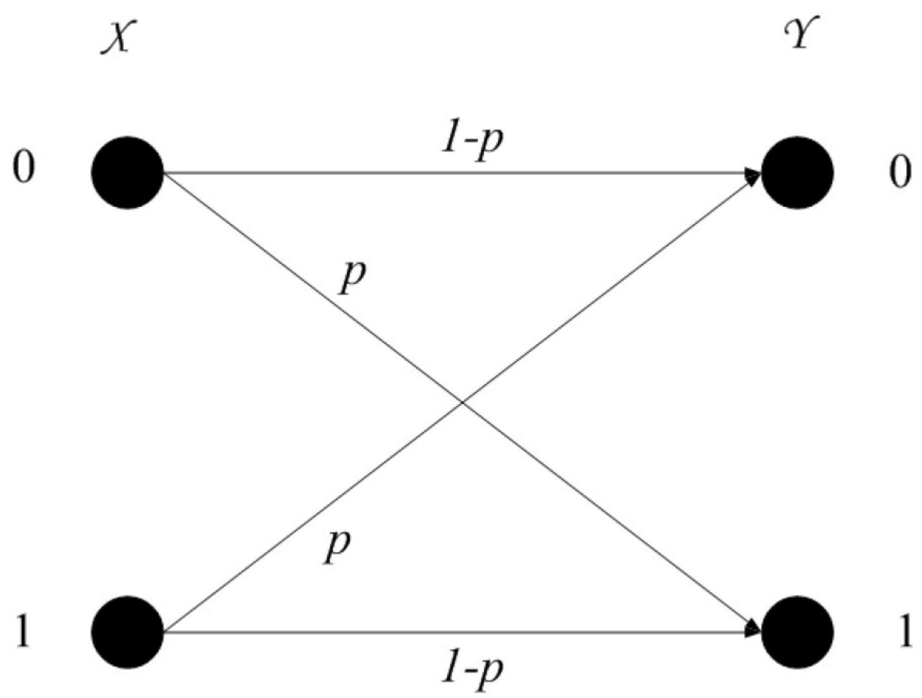


Figure 1.5: The diagram of a BSC with cross-over probability p.

The event $\{g(y) \neq x\}$ (the complement set of $\mathcal{D}_x$) can be written as the union of pairwise error events

$$\{g(y) \neq x\} = \bigcup_{x' \neq x} \{x \to x'\}$$

Then, by using the fact that the probability of the union of events is not larger than the sum of the probabilities of all individual events, we can write the upper bound

$$\begin{aligned}
\Pr(g(Y) \neq x | X = x) &= \Pr\left(\bigcup_{x' \neq x} \{x \to x'\} | X = x\right) \\
&\leq \sum_{x' \neq x} \Pr(\{x \to x'\} | X = x) \\
&= \sum_{x' \neq x} P(x \to x') \qquad (1.31)
\end{aligned}$$

By averaging over all possible x (with respect to the uniform prior probability), we obtain the upper *union bound* on $P(e)$,

$$P(e) \leq \frac{1}{M} \sum_{x=1}^{M} \sum_{x' \neq x} P(x \to x') \qquad (1.32)$$

We can obtain a looser but sometimes easier to calculate upper bound by upperbounding further all PEP terms in (1.32). Suppose that we find a function $f(y)$ such that

$$f(y) \geq 1\{p_x(y) \leq p_{x'}(y)\}$$

Then, we have

$$\begin{aligned} P(x \to x') &= \Pr(\{x \to x'\}|X = x) \\ &= E[1\{p_x(Y) \leq p_{x'}(Y)\}|X = x] \\ &\leq E[f(Y)|X = x] \end{aligned} \tag{1.33}$$

A particularly interesting choice for the function $f(y)$ is the following:

$$f(y) = \exp\left(-\lambda L_{x,x'}(y)\right) \tag{1.34}$$

where we define the log-likelihood ratio (LLR) between the hypotheses x and x' as

$$L_{x,x'}(y) \triangleq \log \frac{p_x(y)}{p_{x'}(y)}$$

and where $\lambda \geq 0$ is a parameter that can be further optimized in order to obtain a bound as tight as possible. The resulting bound is referred to as *Chernoff bound*, and it is given by

$$P(x \to x') \leq \min_{\lambda \geq 0} E\left[e^{-\lambda L_{x,x'}(Y)} \middle| X = x \right] \tag{1.35}$$

Interestingly, we can define a random variable $L \triangleq L_{x,x'}(Y)$ with probability distribution induced by $Y \sim p_x(y)$, and the expectation in (1.35) is the characteristic function

$$\Phi_L(s) \triangleq E\left[e^{-sL}\right]$$

of L evaluated over the real axis $s = \lambda \in \mathbb{R}$. In many important cases, this characteristic function is easier to calculate than the exact PEP.

From the trivial inclusion relation

$$\{g(y) \neq x\} \supseteq \{x \to x'\}$$

we obtain

$$\begin{aligned} \Pr(g(Y) \neq x|X = x) &\geq \max_{x' \neq x} \Pr(\{x \to x'\}|X = x) \\ &= \max_{x' \neq x} P(x \to x') \end{aligned} \tag{1.36}$$

By averaging over all possible x (with respect to the uniform prior probability), we obtain the lower bound on $P(e)$,

$$P(e) \geq \frac{1}{M} \sum_{x=1}^{M} \max_{x' \neq x} P(x \to x') \tag{1.37}$$

Unfortunately, we cannot use in the above lower bound the Chernoff upper bound on the PEP, since an upper bound of a lower bound is no longer a bound.

In several cases of interest, the PEP is very easy to calculate while $P(e)$ is difficult if not impossible. Therefore, the upper bound (1.32) and the lower bound (1.37) are useful tools to estimate the error probability.

1.6 Sufficient statistics

Suppose that there exist a random variable Z such that the distribution of the observation Y conditioned on Z no longer depends on X, i.e., that

$$P(x, y, z) = P(y|x, z)P(x, z) = P(y|z)P(x, z)$$

where $P(x, y, z)$ is the joint probability assignment of X, Y, Z. Then, all the information that we can obtain on X by observing Y, is *captured* by Z. We say that Z is a *sufficient statistic* for X. A particular important case is when $Z = T(Y)$, i.e., when Z is a function of the observation. In other words, we can transform the observation Y by a function $T(y)$ without loosing any useful information for the determination of X, provided that $Z = T(Y)$ is sufficient for X. We have the following result [4]:

Factorization theorem. Let $p_x(y)$ be the conditional density of Y given $X = x$. A statistic $T(y)$ is sufficient for X if an only if there exist functions $f_x(y)$ and $h(y)$ such that

$$p_x(y) = f_x(T(y))\, h(y)$$

□

Example: sufficiency of the LLR in binary hypothesis testing. Consider the binary hypothesis testing problem with conditional probability densities $p_1(y)$ and $p_2(y)$. We define the LLR

$$L_{2,1}(y) = \log \frac{p_2(y)}{p_1(y)}$$

From the factorization theorem, we see immediately that $T(Y) = L_{2,1}(Y)$ is sufficient for X, in fact we are able to write

$$p_x(y) = \begin{cases} p_1(y) & x = 1 \\ e^{L_{2,1}(y)} p_1(y) & x = 2 \end{cases}$$

which is in the form $p_x(y) = f_x(T(y))h(y)$ with the choice $h(y) = p_1(y)$ and

$$f_x(t) = \begin{cases} 1 & x = 1 \\ e^t & x = 2 \end{cases}$$

1.7 Problems

Problem 1. Define $p_T(t)$ as

$$p_T(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

and assume $f_c T \gg 1$.

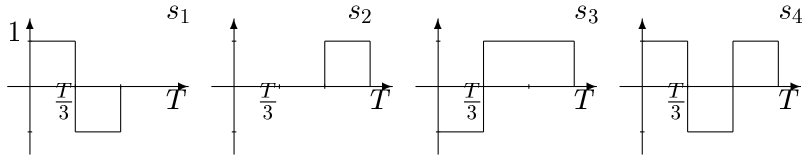
1) Find an orthonormal basis for the real signal set $\mathcal{S} = \{s_m(t) : m = 0, \ldots, 7\}$ defined by

$$s_m(t) = \begin{cases} \text{Re}\{p_T(t) \exp(j2\pi(f_c t + m/4)\} & m = 0, 1, 2, 3 \\ \sqrt{2}\text{Re}\{p_T(t) \exp(j2\pi(f_c t + (m-4)/4 + 1/8)\} & m = 4, 5, 6, 7 \end{cases}$$

2) Represent the signal constellation corresponding to $\mathcal{S}$ in the real Euclidean space with respect to the basis found.

3) Find the average energy per signal, assuming that all signals are equiprobable.

Problem 2. Find an orthonormal basis for the four real signals given below:



and represent them as points in the Euclidean space generated by the basis found.

Then, compute the average energy per signal (assume that the signals are used with equal probability) and the minimum squared Euclidean distance between the signals.

Problem 3. Give a geometrical representation of the following set of signals

$$s_m(t) = \begin{cases} \mathrm{Re}\{Ap_T(t)e^{j(\frac{\pi}{2}m+\frac{\pi}{4})}e^{j2\pi f_0 t}\} & \text{for } m = 0,1,2,3 \\ \mathrm{Re}\{\frac{1+\sqrt{3}}{\sqrt{2}}Ap_T(t)e^{j\frac{\pi}{2}(m-4)}e^{j2\pi f_0 t}\} & \text{for } m = 4,5,6,7 \end{cases}$$

where A is real positive and $p_T(t)$ is a unit energy rectangular pulse of support $[0,T]$. Assume $f_0 T \gg 1$.

Problem 4. Represent the following signals $s_1(t), s_2(t), s_3(t)$ as points in a Euclidean geometrical space, by choosing an appropriate orthonormal basis.

$$s_1(t) = \begin{cases} e^{-t} & t \in [0,1] \\ 0 & \text{elsewhere} \end{cases}$$

$$s_2(t) = \begin{cases} +1 & t \in [0,1/2] \\ -1 & t \in (1/2,1] \\ 0 & \text{elsewhere} \end{cases}$$

$$s_3(t) = \begin{cases} -1 & t \in [0,1/4] \\ +1 & t \in (1/4,1] \\ 0 & \text{elsewhere} \end{cases}$$

Problem 5. Consider an additive-noise channel $y = x+n$ where x takes on the values $\pm A$ with equal probability and where n is a Laplace random variable with probability density function

$$p_n(z) = \frac{1}{\sqrt{2}\sigma}e^{-|z|\sqrt{2}/\sigma}$$

Determine the decision regions of the MAP detector. Compare the decision regions found with those of the MAP detector for $n \sim \mathcal{N}(0,\sigma^2)$. Compute the error probability in the two cases of Laplace and Gaussian noise and compare the resulting error probabilities for the same SNR (note: the SNR here is defined as the average energy per symbol divided by the noise variance). What is the worst-case noise?

Problem 6. Consider a real binary input Laplacian noise channel $y = x + z$ with input alphabet $\{-A, +A\}$ and noise pdf

$$p(z) = \frac{b}{2}\exp(-b|z|)$$

The prior input probability distribution is $\Pr(x = A) = 0.8$, $\Pr(x = -A) = 0.2$. Find the MAP decision regions and compute the minimum average decision error probability.

Problem 7. Consider the signal $y = \sqrt{E}x + z + \nu$ where $x \in \{\pm 1\}$ represents the transmitted signal, $z \in \{\pm 1\}$ an interfering signal and ν is a real noise variable (independent of x and z) with distribution given by

$$p_\nu(u) = \begin{cases} 0 & \text{for } |u| \geq 1 \\ u+1 & \text{for } -1 < u < 0 \\ -u+1 & \text{for } 0 \leq u < 1 \end{cases}$$

x and z are independent and takes on the values +1 and -1 with equal probability.

Find the optimal MAP decision regions and the corresponding minimum symbol error probability for the detection of x with observation y.

Problem 8. Let $p(y|x)$ and $p(y|x')$ be the conditional pdfs of the observation y given the two equiprobable hypotheses x and x'. Prove the upper bound to the PEP

$$P(x \to x') \leq E\left[\sqrt{\frac{p(y|x')}{p(y|x)}}\,\middle|\, x\right] = \int \sqrt{p(y|x)p(y|x')}dy$$

Then, show that in the case where $y = x + \nu$, with $\nu \sim \mathcal{N}_{\mathbb{C}}(0, N_0)$ and x, x' are points in the complex plane, the above bound yields the same result of the Chernoff bound optimized with respect to the parameter λ for the hypothesis testing problem of detecting x by observing y.

Problem 9. A discrete communication channel with input x and output y has input alphabet $\{0, 1\}$, output alphabet $\{a, b, c, d\}$ and transition probability $P(y|x)$ given by

$$\begin{aligned} &P(a|0) = 0.3, \quad P(b|0) = 0.1, \quad P(c|0) = 0.5, \quad P(d|0) = 0.1, \\ &P(a|1) = 0.1, \quad P(b|1) = 0.5, \quad P(c|1) = 0.1, \quad P(d|1) = 0.3, \end{aligned}$$

The a priori probability of the input is $Q(x = 0) = 0.6$, $Q(x = 1) = 0.4$. Compute the maximum a posteriori (MAP) detection rule and the resulting average error probability.

Chapter 2

Signal detection in AWGN

In this chapter we study the elementary communication system described by the block diagram of Fig. 2.1. The transmitter (modulator) has a set $\mathcal{S}$ of M waveforms

$$\mathcal{S} = \{s_m(t) : m = 1, \ldots, M\}$$

of finite energy and duration T. It generates an information message m at random with a given a priori probability distribution $\pi(m)$, and transmits the corresponding waveform $s_m(t)$. The transmission channel is distortionless and introduces additive white Gaussian noise (AWGN) $\nu(t)$ with power spectral density N_0 and a phase rotation θ. The phase rotation is assumed to be independent of the transmitted signal and of the noise, random but constant over the duration T of the waveform. The phase rotation represents the phase offset between the local oscillator of the transmitter and of the receiver (including the effect of the propagation through the channel). The received signal over the observation interval $[0, T]$ is

$$r(t) = s_m(t)e^{j\theta} + \nu(t) \tag{2.1}$$

The receiver must estimate with minimal probability of error the value of the transmitted message by observing the channel output $r(t)$, $t \in [0, T]$.

We shall distinguish the cases of *coherent* and *non-coherent* detection. In the first case, the receiver has perfect knowledge of the phase rotation θ. Therefore, it can apply an inverse rotation by multiplying $r(t)$ by $e^{-j\theta}$. Since the noise is rotationally invariant, the coherent detection channel model reduces to

$$r(t) = s_m(t) + \nu(t) \tag{2.2}$$

In practice, coherent detection can be obtained by estimating the signal phase θ. This can be achieved by some phase synchronization technique.

In the case where phase estimation is not possible or, in order to keep the receiver as simple as possible, it is not implemented, we are in the presence of non-coherent detection. In this case, we shall assume that the phase θ is completely unknown, and model it as an independent random variable uniformly distributed on $[-\pi, \pi)$.

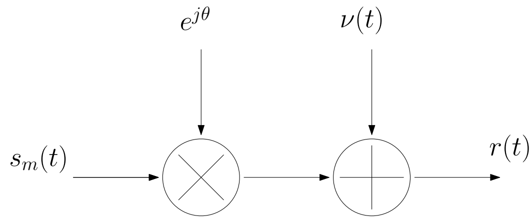


Figure 2.1: Waveform transmission over the AWGN channel.

2.1 Coherent detection of binary signal sets

Before considering the general case of M arbitrary waveforms, we consider the simpler example of $M = 2$, where

$$s_m(t) = a_m \psi(t) \qquad m = 1, 2$$

where $a_1 = +1$ and $a_2 = -1$ are called *modulation symbols* and $\psi(t)$ is an elementary waveform with energy $\mathcal{E} = \|\psi(t)\|^2$, used to convey information. This type of modulation is called binary (since there are two waveforms), and antipodal (since the two waveforms are mutually opposite). Moreover, since all signals in $\mathcal{S}$ are proportional to the same elementary waveform (they are linearly dependent), this type of modulation belongs to the class of so called *linear modulations*. The two messages are transmitted with probabilities $\pi(1)$ and $\pi(2)$, respectively.

In order to find the optimal detector yielding minimum error probability, we need to apply the MAP decision rule derived for the general hypothesis testing problem. However, it is not intuitive how to apply this to a continuous-time observation (see [4] for a complete and rigorous treatment). Therefore, we first find a discrete sufficient statistic for m and then apply the MAP detector to this statistics.

Let $\xi_1(t) = \psi(t)/\sqrt{\mathcal{E}}$ and let $\mathcal{B} = \{\xi_i(t) : i = 1, 2, \ldots\}$ be a complete orthonormal basis for the finite-energy signals defined on $[0, T]$. The received signal can be written as

$$r(t) = \sum_{i=1}^{N} r_i \xi_i(t)$$

where

$$r_i = (\xi_i(t)|r(t)) = \int_0^T \xi_i(t)^* r(t) dt \tag{2.3}$$

Because of the construction of the basis $\mathcal{B}$, and from the result on projected white noise, we can write the coefficients r_i as

$$r_i = \begin{cases} a_m\sqrt{\mathcal{E}} + \nu_1 & i = 1 \\ \nu_i & i > 1 \end{cases}$$

where the projected noise samples ν_i are i.i.d. circularly symmetric complex Gaussian with variance N_0. Obviously, the sequence of samples $\mathbf{r} = (r_1, r_2, \ldots)$ is conditionally independent of the message m given its first sample r_1. Therefore, we define $y = T(\mathbf{r}) = r_1$, which is sufficient for the detection of m.

The resulting discrete channel model is given by

$$y = a_m\sqrt{\mathcal{E}} + \nu \tag{2.4}$$

where $\nu \sim \mathcal{N}_{\mathbb{C}}(0, N_0)$.

The conditional pdf of the observation y given m is

$$p(y|m) = \frac{1}{\pi N_0} \exp\left(-\frac{1}{N_0}|y - a_m\sqrt{\mathcal{E}}|^2\right) \tag{2.5}$$

Since we have to do with a binary hypothesis testing problem, the optimal decision rule is determined simply by the decision region for one of the hypotheses (the other is given by the complement region). Without loss of generality, we focus on the region $\mathcal{D}_1$. This is given by all $y \in \mathbb{C}$ satisfying the inequality

$$\pi(1)\frac{1}{\pi N_0} \exp\left(-\frac{1}{N_0}|y - \sqrt{\mathcal{E}}|^2\right) \geq \pi(2)\frac{1}{\pi N_0} \exp\left(-\frac{1}{N_0}|y + \sqrt{\mathcal{E}}|^2\right)$$

For simplicity, we let $\pi(1) = q$ and $\pi(2) = 1 - q$, and we take the logarithm of both sides. We obtain

$$\begin{aligned} \log q - \frac{1}{N_0}(|y|^2 - 2\sqrt{\mathcal{E}}\mathrm{Re}\{y\} + \mathcal{E}) \geq & \qquad \log(1-q) - \frac{1}{N_0}(|y|^2 + 2\sqrt{\mathcal{E}}\mathrm{Re}\{y\} + \mathcal{E}) \\ \mathrm{Re}\{y\} \geq & \qquad \frac{N_0}{4\sqrt{\mathcal{E}}} \log\frac{1-q}{q} \end{aligned}$$

Then, the optimal decision rule corresponds to comparing $\mathrm{Re}\{y\}$ with the threshold $\tau = \frac{N_0}{4\sqrt{\mathcal{E}}} \log\frac{1-q}{q}$. If the prior probability is uniform (i.e., $q = 1 - q = 1/2$), the threshold is $\tau = 0$.

The resulting minimum probability of error is given by

$$P(e) = qP(e|m = 1) + (1 - q)P(e|m = 2)$$

where the conditional probability of error is given by

$$P(e|m = i) = \begin{cases} \Pr(\sqrt{\mathcal{E}} + \mathrm{Re}\{\nu\} < \tau) & i = 1 \\ \Pr(-\sqrt{\mathcal{E}} + \mathrm{Re}\{\nu\} > \tau) & i = 2 \end{cases}$$

First, we examine the conditional probability of error for message 1. We have

$$\begin{aligned} P(e|m = 1) &= \Pr(\sqrt{\mathcal{E}} + \mathrm{Re}\{\nu\} < \tau) \\ &= \int_{-\infty}^{\tau-\sqrt{\mathcal{E}}} \frac{1}{\sqrt{\pi N_0}} e^{-z^2/N_0} dz \\ &= \int_{\sqrt{\mathcal{E}}-\tau}^{+\infty} \frac{1}{\sqrt{\pi N_0}} e^{-z^2/N_0} dz \\ &= Q\left(\frac{\sqrt{\mathcal{E}} - \tau}{\sqrt{N_0/2}}\right) \end{aligned} \tag{2.6}$$

where we define the Gaussian tail function

$$Q(z) \triangleq \int_z^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

Analogously, the conditional probability of error for message 2 is given by

$$
\begin{aligned}
P(e|m=2) &= \Pr(-\sqrt{\mathcal{E}} + \mathrm{Re}\{\nu\} > \tau) \\
&= \int_{\sqrt{\mathcal{E}}+\tau}^{+\infty} \frac{1}{\sqrt{\pi N_0}} e^{-z^2/N_0} dz \\
&= Q\left(\frac{\sqrt{\mathcal{E}}+\tau}{\sqrt{N_0/2}}\right) \qquad (2.7)
\end{aligned}
$$

The average error probability is given by

$$
P(e) = qQ\left(\frac{\sqrt{\mathcal{E}}-\tau}{\sqrt{N_0/2}}\right) + (1-q)Q\left(\frac{\sqrt{\mathcal{E}}+\tau}{\sqrt{N_0/2}}\right)
$$

which for uniform prior probability boils down to

$$
P(e) = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}}\right)
$$

The derivation of the above optimal MAP detector for binary antipodal signals shows that the detector can be divided into two distinct operations:

1. Projection of the received signal onto a basis of the signal space of the transmitter.
2. Decision rule based on the projected (sufficient) statistics.

We shall see later that this partitioning holds in general. The extraction of y from the received signal $r(t)$ is obtained simply by the projection (inner product) $y = (\xi_1(t)|r(t))$. Equivalently, this can be seen as a filtering and sampling operation, where $r(t)$ is passed through a filter with impulse response $h(t) = \xi_1(T-t)^*$, and the filter output is sampled at time T. In fact, we have

$$
\begin{aligned}
r(t) \otimes h(t)|_{t=T} &= \left. \int h(t-\tau) r(\tau) d\tau \right|_{t=T} \\
&= \left. \int \xi_1(T+\tau-t)^* r(\tau) d\tau \right|_{t=T} \\
&= \int \xi_1(\tau)^* r(\tau) d\tau \\
&= (\xi_1(t)|r(t))
\end{aligned}
$$

The filter $h(t) = \xi_1(T-t)^*$ is said to be *matched* to the waveform $\xi_1(t)$. The properties of the matched filter will be seen later in this chapter. Figs. **??**(a) and **??**(b) represent the block diagrams of the optimal MAP detector for binary antipodal signals based on correlation and on matched filtering, respectively.

2.1.1 Matched filtering

We have seen before that a sufficient statistic for ML (or MAP) detection of m is obtained as the matched filter output. In the detection of signals in white noise, matched filtering plays a very important role [4]. In particular, we consider the signal model

$$
r(t) = s(t) + \nu(t) \quad t \in [0, T]
$$

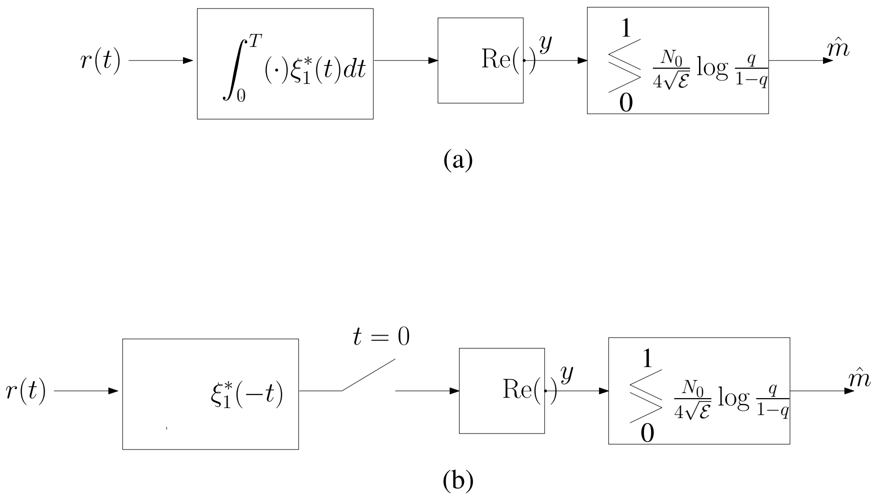


Figure 2.2: MAP detector for binary antipodal signals based on a signal correlator.

where $\nu(t)$ is a non-necessarily Gaussian white noise with autocorrelation function $E[\nu(t_1)\nu(t_2)^*] = N_0\delta(t_1 - t_2)$, and we constrain the receiver to be formed by a linear filter $h(t)$ whose output is sampled at time T. The output sample is given by

$$y = \int_0^T h(T-\tau)r(\tau)d\tau$$

We define the signal-to-noise ratio (SNR) as the ratio between the squared magnitude of the signal component and the variance of the noise component at the filter output. Because of linearity, we have $y = x + z$, where

$$x = \int_0^T h(T-\tau)s(\tau)d\tau$$

and

$$z = \int_0^T h(T-\tau)\nu(\tau)d\tau$$

Therefore, the SNR is given by

$$\begin{aligned} \text{SNR} &= \frac{|x|^2}{E[|z|^2]} \\ &= \frac{\left|\int_0^T h(T-\tau)s(\tau)d\tau\right|^2}{E\left[\left|\int_0^T h(T-\tau)\nu(\tau)d\tau\right|^2\right]} \end{aligned}$$

Consider the denominator of the above expression. We can write

$$
\begin{aligned}
E\left[\left|\int_0^T h(T-\tau)\nu(\tau)d\tau\right|^2\right] &= E\left[\int_0^T\int_0^T h(T-\tau)h(T-\theta)^*\nu(\tau)\nu(\theta)^* d\tau d\theta\right] \\
&= N_0\int_0^T\int_0^T h(T-\tau)h(T-\theta)^*\delta(\tau-\theta)d\tau d\theta \\
&= N_0\int_0^T |h(T-\tau)|^2 d\tau
\end{aligned}
$$

We have the following result:

SNR optimality of the matched filter. The maximum SNR over all possible filters $h(t)$ is SNR= $\frac{E_s}{N_0}$ where E_s is the energy of $s(t)$, and it is obtained by the matched filter $h(t) = s(T-t)^*$.

Proof. We write the output SNR as

$$
\text{SNR} = \frac{\left|\int_0^T h(T-\tau)s(\tau)d\tau\right|^2}{N_0\int_0^T |h(T-\tau)|^2 d\tau}
$$

From the Cauchy-Schwartz inequality we have

$$
\left|\int_0^T h(T-\tau)s(\tau)d\tau\right|^2 \leq \left(\int_0^T |h(T-\tau)|^2 d\tau\right)\left(\int_0^T |s(\tau)|^2 d\tau\right)
$$

with equality if an only if $h(T-\tau) \propto s(\tau)^*$. Then, by letting $h(t) = s(T-t)^*$ we obtain the maximum SNR as

$$
\text{SNR} = \frac{\left|\int_0^T |s(\tau)|^2 d\tau\right|^2}{N_0\int_0^T |s(\tau)|^2 d\tau} = \frac{E_s}{N_0}
$$

$\square$

2.2 Coherent detection of M-ary signal sets

We are ready to consider the general problem where the modulator is equipped with an M-ary signal set $\mathcal{S} = \{s_1(t), \ldots, s_M(t)\}$. Let $\{\xi_1(t), \ldots, \xi_N(t)\}$ be an orthonormal basis for the signal subspace generated by $\mathcal{S}$, i.e., for the space of all signals obtained as linear combinations of the $s_m(t)$'s. From the Gram-Schmidt theorem, we know that this basis exists (not unique) and that $N \leq M$. Then, let $\mathcal{B} = \{\xi_1(t), \ldots, \xi_N(t), \xi_{N+1}(t), \ldots\}$ be a complete orthonormal basis for the set of finite energy signals over $[0, T]$. As before, the discrete representation of the received signal with respect to $\mathcal{B}$ is given by the sequence $\mathbf{r} = (r_1, r_2, \ldots)$ such that

$$
\begin{aligned}
r_1 &= s_{m,1} + \nu_1 \\
r_2 &= s_{m,2} + \nu_2 \\
&\vdots \\
r_N &= s_{m,N} + \nu_N \\
r_{N+1} &= \nu_{N+1} \\
r_{N+2} &= \nu_{N+2} \\
&\vdots
\end{aligned}
$$

where r_i is given by (2.3), and where $\mathbf{s}_m = (s_{m,1}, \ldots, s_{m,N})$ is the vector representation of the signal $s_m(t)$ with respect to the basis $\{\xi_1(t), \ldots, \xi_N(t)\}$. By construction, all signals $s_m(t)$ can be represented exactly in such way, and they are orthogonal to all the elements $\xi_i(t)$ of the basis $\mathcal{B}$ for $i > N$. Since all components r_i for $i > N$ are independent of the transmitted message m, the desired sufficient statistics is given by the vector $\mathbf{y}$ corresponding to the first N components of $\mathbf{r}$.

The resulting N-dimensional discrete channel model is given by

$$\mathbf{y} = \mathbf{s}_m + \boldsymbol{\nu} \tag{2.8}$$

where $\boldsymbol{\nu} \sim \mathcal{N}_{\mathbb{C}}(0, N_0\mathbf{I})$.

The conditional pdf of the observation $\mathbf{y}$ given m is the N-variate complex Gaussian pdf

$$p(\mathbf{y}|m) = \frac{1}{(\pi N_0)^N} \exp\left(-\frac{1}{N_0}|\mathbf{y} - \mathbf{s}_m|^2\right) \tag{2.9}$$

In order to determine the MAP decision rule, we have to find the decision regions $\mathcal{D}_m$, for all $m = 1, \ldots, M$. The region $\mathcal{D}_m$ is given by all $\mathbf{y} \in \mathbb{C}^N$ satisfying the inequalities

$$\pi(m)\frac{1}{(\pi N_0)^N} \exp\left(-\frac{1}{N_0}|\mathbf{y} - \mathbf{s}_m|^2\right) \geq \pi(m')\frac{1}{(\pi N_0)^N} \exp\left(-\frac{1}{N_0}|\mathbf{y} - \mathbf{s}_{m'}|^2\right)$$

for all $m' \neq m$. Each pair of symbols (m, m') determines one inequality, the solution of which is a *pairwise decision region* in $\mathbb{C}^N$. The desired decision region $\mathcal{D}_m$ is obtained as the intersection of all those pairwise decision regions. Let us consider the pair (m, m'). The associated pairwise decision region is given by all $\mathbf{y}$ satisfying

$$|\mathbf{y} - \mathbf{s}_m|^2 - |\mathbf{y} - \mathbf{s}_{m'}|^2 \leq N_0 \log\frac{\pi(m)}{\pi(m')} \tag{2.10}$$

It is immediate to see that the boundary of the above region is the hyperplane

$$2\mathrm{Re}\left\{(\mathbf{s}_{m'} - \mathbf{s}_m)^H \mathbf{y}\right\} = N_0 \log\frac{\pi(m)}{\pi(m')} - |\mathbf{s}_m|^2 + |\mathbf{s}_{m'}|^2 \tag{2.11}$$

i.e., the affine variety obtained by translating the subspace perpendicular to the difference vector $\mathbf{d} = \mathbf{s}_{m'} - \mathbf{s}_m$ by a quantity that depends on the difference of signal energies, on the a priori probabilities and on the noise variance.

The MAP decision rule can be written compactly in the form

$$\widehat{m} = g(\mathbf{y}) = \arg \max_{m \in \{1,\ldots,M\}} \left\{\log \pi(m) - \frac{1}{N_0}|\mathbf{y} - \mathbf{s}_m|^2\right\} \tag{2.12}$$

In the important case of uniform prior probabilities, the decision region (2.10) becomes

$$|\mathbf{y} - \mathbf{s}_m|^2 \leq |\mathbf{y} - \mathbf{s}_{m'}|^2 \tag{2.13}$$

This means that the detector chooses m instead of m' if the received vector $\mathbf{y}$ is closer to $\mathbf{s}_m$ than to $\mathbf{s}_{m'}$. The resulting ML decision rule is the intuitive *minimum squared Euclidean distance (SED) rule*

$$\widehat{m} = g(\mathbf{y}) = \arg \min_{m \in \{1,\ldots,M\}} \left\{|\mathbf{y} - \mathbf{s}_m|^2\right\} \tag{2.14}$$

As in the case of binary antipodal signals, we have that the detector can be divided into two distinct operations: a projection onto the signal space, in order to extract the sufficient statistic, followed by a decision based on the sufficient statistics. The components y_i of the observation $\mathbf{y}$ can be obtained either by a bank of N correlators, calculating $y_i = (\xi_i(t)|r(t))$ for $i = 1, \dots, N$, or by a bank of matched filters, with impulse responses $h_i(t) = \xi_i(T-t)^*$, whose output is sampled at time T. Figs. **??** and 2.4 represent the block diagrams of the optimal MAP detector for general M-ary signals based on correlation and on matched filtering, respectively.

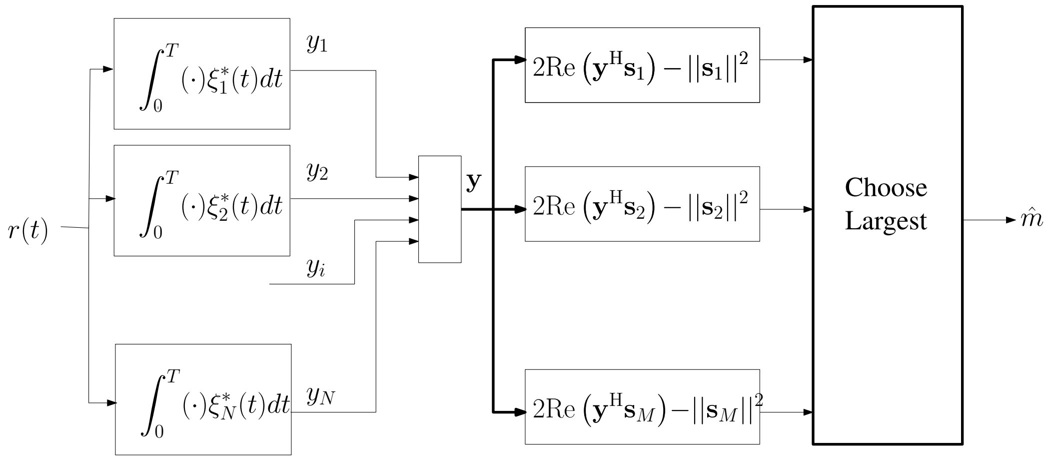


Figure 2.3: MAP detector for M-ary signals based on a bank of signal correlators.

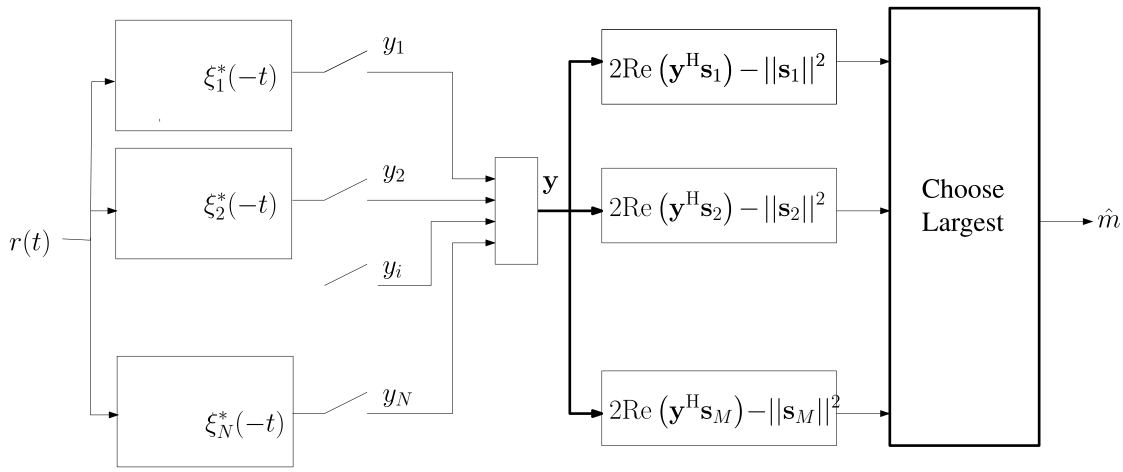


Figure 2.4: MAP detector for M-ary signals based on a bank of matched filters.

As an alternative, it is sometimes convenient to work directly with the signals $s_m(t)$ rather than with an orthonormal basis. By writing $|\mathbf{y} - \mathbf{s}_m|^2 = |\mathbf{y}|^2 - 2\mathrm{Re}\{\mathbf{s}_m^H \mathbf{y}\} + |\mathbf{s}_m|^2$, we notice that the received signal energy term $|\mathbf{y}|^2$ does not play any role in the decision, since it is common to all hypotheses. Moreover, the vector inner product $\mathbf{s}_m^H \mathbf{y}$ can be written in terms of the continuous-time signals as

$$\mathbf{s}_m^H \mathbf{y} = (s_m(t)|r(t)) = \int s_m(t)^* r(t) dt$$

Therefore, the minimum distance rule (2.14) can be replaced by the maximization

$$\widehat{x} = g(\mathbf{y}) = \arg \max_{m\in\{1,\dots,M\}} \left\{ 2\mathrm{Re}\left\{ \int s_m(t)^* r(t)\,dt \right\} - |\mathbf{s}_m|^2 \right\} \tag{2.15}$$

The corresponding detector, implemented by a bank of M correlators (or equivalently, by a bank of M matched filters), is represented in Fig. 2.5.

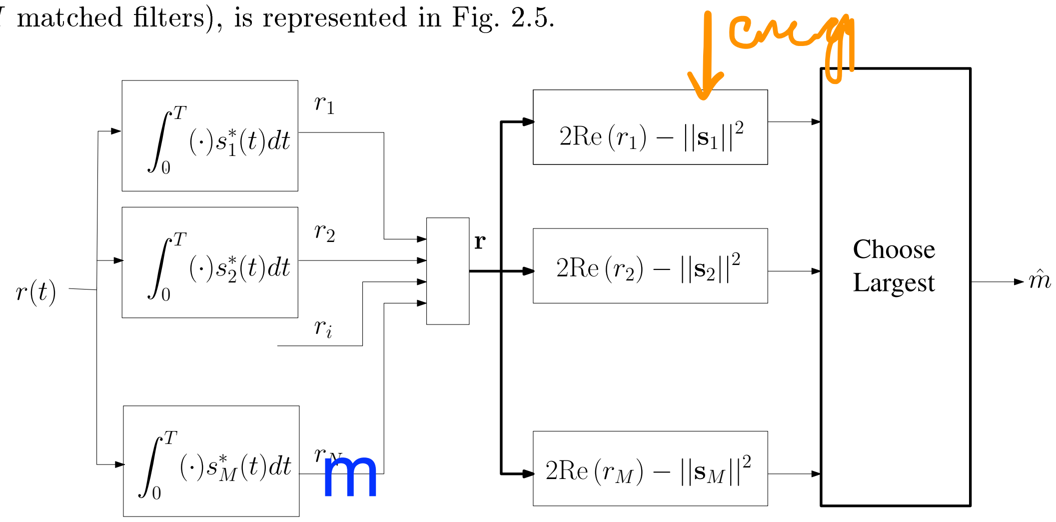


Figure 2.6

Figure 2.5: ML detector for M-ary signals based on a bank of signal correlators.

We have seen that there are several different equivalent formulations of the same optimal detector. In general, we say that a real function $\mathcal{M}(\mathbf{y}, m)$ is a *decision metric* for the MAP detection of m given the observation $\mathbf{y}$ if the outcome of

$$\widehat{m} = \arg \ \mathrm{sel}_{m\in\{1,\dots,m\}} \mathcal{M}(\mathbf{y}, m)$$

where "sel" denotes either the "min" or the "max" selection operation, is the same of the MAP rule (2.12).

2.3 Performance of coherent detection

For a general signal set $\mathcal{S} = \{\mathbf{s}_m : m = 1, \dots, M\}$, the average symbol error probability (assuming uniform prior probability) is given by

$$P(e) = \Pr(\widehat{m} \neq m) = \frac{1}{M} \sum_{m=1}^{M} \Pr(\widehat{m} \neq m|m)$$

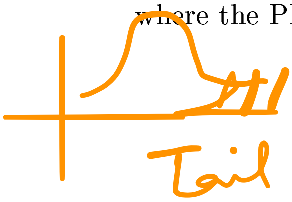
In general, the above probability is difficult if not impossible to compute in closed form. Therefore, we shall resort to the union upper bound (1.32) and to the lower bound (1.37). Next, we shall see some important particular cases for which $P(e)$ finds a closed form expression or for which it is possible to find tighter upper bounds than the simple union bound.

The average symbol error probability is upperbounded by

$$P(e) \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{m' \neq m} P(m \to m') \tag{2.16}$$

where the PEP is given by

distance



$$\begin{aligned} P(m \to m') &= \Pr(|\mathbf{y} - \mathbf{s}_{m'}|^2 \leq |\mathbf{y} - \mathbf{s}_m|^2 | m) \\ &= \Pr(2\mathrm{Re}\{(\mathbf{s}_{m'} - \mathbf{s}_m)^H \boldsymbol{\nu}\} \geq |\mathbf{s}_{m'} - \mathbf{s}_m|^2) \\ &= \Pr(\eta \geq d^2(m, m')) \end{aligned} \tag{2.17}$$

Tail gaussian

distance between signals

where we have defined the SED between the signals $\mathbf{s}_m$ and $\mathbf{s}_{m'}$ as

$$d^2(m, m') = |\mathbf{s}_{m'} - \mathbf{s}_m|^2$$

and where η is a real Gaussian random variable $\sim \mathcal{N}(0, 2N_0 d^2(m, m'))$. Recalling the definition of the Gaussian tail function $Q(u)$, we can write

$$P(m \to m') = Q\left(\sqrt{\frac{d^2(m, m')}{2N_0}}\right) \tag{2.18}$$

The resulting general expression for the union bound on the average symbol error probability is

$$P(e) \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{m' \neq m} Q\left(\sqrt{\frac{d^2(m, m')}{2N_0}}\right) \tag{2.19}$$

In order to provide a fair comparison of different signal sets, it is useful to express their symbol error probability as a function of the ratio E_b/N_0, where E_b is the average energy per transmitted bit. To this purpose, we define the average energy per transmitted symbols as

$$E_s = \frac{1}{M} \sum_m |\mathbf{s}_m|^2$$

Since the symbols are transmitted with uniform probability, each symbol carries $\log_2 M$ bits of information (in order words, we can label each signal in $\mathcal{S}$ by binary labels of $\log_2 M$ bit each). Therefore, E_b is given by

$$E_b = E_s / \log_2 M$$

We can define a normalized *unit-energy* signal set $\widetilde{\mathcal{S}}$ obtained from $\mathcal{S}$ by dividing all its vectors by $\sqrt{E_s}$. Obviously, for any pair of symbols (m, m'), the distance $d^2(m, m')$ is related to the corresponding distance $\widetilde{d}^2(m, m')$ between signals in $\widetilde{\mathcal{S}}$ by

$$d^2(m, m') = E_s \widetilde{d}^2(m, m')$$

Finally, the union bound can be conveniently expressed as a function of E_b/N_0 by using the normalized distances in (2.19). We obtain

$$P(e) \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{m' \neq m} Q\left(\sqrt{\frac{\log_2 M \widetilde{d}^2(m, m')}{2} \frac{E_b}{N_0}}\right) \tag{2.20}$$

always less than ?

Notice that if the signal set has the *uniform distance property*, i.e., if the list of distances $\{d^2(m, m') : m' = 1, \ldots, M\}$ is independent of the reference symbol m, then the outer average with respect to m is not needed, and the bound can be calculated by taking any suitable m as reference.

The Chernoff union bound to the symbol error probability is obtained by noticing that the LLR $L_{m,m'}(\mathbf{y}) = \log p(\mathbf{y}|\mathbf{s}_m)/p(\mathbf{y}|\mathbf{s}_{m'})$ is given by

$$\begin{aligned} L_{m,m'}(\mathbf{y}) &= \frac{1}{N_0}\left(-|\mathbf{y} - \mathbf{s}_m|^2 + |\mathbf{y} - \mathbf{s}_{m'}|^2\right) \\ &= \frac{1}{N_0}\left(-|\boldsymbol{\nu}|^2 + |\boldsymbol{\nu}|^2 - 2\mathrm{Re}\{(\mathbf{s}_{m'} - \mathbf{s}_m)^H \boldsymbol{\nu}) + d^2(m, m')\right) \\ &= \frac{1}{N_0}\left(-2\mathrm{Re}\{(\mathbf{s}_{m'} - \mathbf{s}_m)^H \boldsymbol{\nu}) + d^2(m, m')\right) \end{aligned} \tag{2.21}$$

i.e., it is a Gaussian real random variable $\sim \mathcal{N}(d^2(m, m')/N_0, 2d^2(m, m')/N_0)$. Its characteristic function on the real positive axis is given by

$$\Phi_L(\lambda) = \exp\left((\lambda^2 - \lambda)\frac{d^2(m, m')}{N_0}\right)$$

By minimizing this with respect to λ, we get that the minimum is achieved by $\lambda = 1/2$ and it is given by

$$\exp\left(-\frac{d^2(m, m')}{4N_0}\right)$$

Finally, the resulting union Chernoff bound is given by

$$P(e) \leq \frac{1}{M}\sum_{m=1}^{M}\sum_{m' \neq m} \exp\left(-\frac{\log_2 M \tilde{d}^2(m, m')}{4}\frac{E_b}{N_0}\right) \tag{2.22}$$

We notice that this upper bound could have been obtained directly by applying the upper bound

$$Q(u) \leq \exp(-u^2/2)$$ ← Wikipedia

(valid for $u \geq 0$) to the union bound (2.20).

From the general lower bound (1.37), we obtain a lower bound on the symbol error probability as

$$P(e) \geq \frac{1}{M}\sum_{m=1}^{M}\max_{m' \neq m} Q\left(\sqrt{\frac{\log_2 M \tilde{d}^2(m, m')}{2}\frac{E_b}{N_0}}\right) \tag{2.23}$$

Since $Q(u)$ is monotonically decreasing, the maximum over all $m' \neq m$ of $Q\left(\sqrt{\frac{\log_2 M \tilde{d}^2(m,m')}{2}\frac{E_b}{N_0}}\right)$ is obtained for the signal $\mathbf{s}_{m'}$ at minimum distance from $\mathbf{s}_m$. Therefore, we can also write the lower bound as

$$P(e) \geq \frac{1}{M}\sum_{m=1}^{M} Q\left(\sqrt{\frac{\log_2 M \min_{m' \neq m} \tilde{d}^2(m, m')}{2}\frac{E_b}{N_0}}\right) \tag{2.24}$$

Lower bound

The same observation regarding the uniform distance property done for the union upper bound applies here.

2.3.1 Simplifications and importance of the minimum SED

The upper and lower bounds (2.20) and (2.24) require the evaluation of all distances between any pair of signals $\mathbf{s}_m, \mathbf{s}_{m'}$. Looser upper and lower bounds can be obtained in terms of the minimum distance of the normalized signal set, defined by

$$d^2_{\min}(\mathcal{S}) = \min_{m \neq m'} \tilde{d}^2(m, m') \tag{2.25}$$

Because of the monotonicity of $Q(u)$, each term in (2.20) is upperbounded by substituting $d^2_{\min}(\mathcal{S})$ to the corresponding distance $\tilde{d}^2(m, m')$. Then, we obtain the bound

$$P(e) \leq (M-1)Q\left(\sqrt{\frac{\log_2 M d^2_{\min}(\mathcal{S})}{2} \frac{E_b}{N_0}}\right) \tag{2.26}$$

In order to simplify the lower bound, we can trivially lower bound each term in (2.24) for which $\min_{m' \neq m} \tilde{d}^2(m, m') > d^2_{\min}(\mathcal{S})$ by zero. The resulting looser lower bound is

$$P(e) \geq f_{\min} Q\left(\sqrt{\frac{\log_2 M d^2_{\min}(\mathcal{S})}{2} \frac{E_b}{N_0}}\right) \tag{2.27}$$

lower bound
less than T

where $0 < f_{\min} < 1$ is the fraction of signals having at least a *nearest neighbor* at distance $d^2_{\min}(\mathcal{S})$.

Finally, we can obtain an approximation of $P(e)$ (neither an upper nor a lower bound) by simply eliminating from the union bound all terms with distance larger than the minimum distance. We obtain

$$P(e) \approx \bar{N}_{\min} Q\left(\sqrt{\frac{\log_2 M d^2_{\min}(\mathcal{S})}{2} \frac{E_b}{N_0}}\right) \tag{2.28}$$

where $\bar{N}_{\min}$ is the average number of nearest neighbors at minimum distance, given by

$$\bar{N}_{\min} = \frac{1}{M} \sum_{m=1}^{M} N_{\min}(m)$$

where $N_{\min}(m)$ is the number of nearest neighbors at distance $d^2_{\min}(\mathcal{S})$ for the signal $\mathbf{s}_m$.

We observe that, both the upper and lower bound on $P(e)$ (as well as the above approximation) have the same exponential behavior with respect to E_b/N_0. We conclude that $P(e) = O(e^{-0.25\gamma(\mathcal{S})E_b/N_0})$ where the coefficient in the exponent is given by

$$\gamma(\mathcal{S}) = \log_2 M d^2_{\min}(\mathcal{S}) \tag{2.29}$$

(recall that the minimum distance is defined for a signal set with normalized unit average energy).

The *asymptotic gain* of a signal set $\mathcal{S}$ over another signal set $\mathcal{S}'$ is defined by the ratio, expressed in dB, of the coefficients $\gamma(\mathcal{S})/\gamma(\mathcal{S}')$. This corresponds, for small values of $P(e)$, to the increment of E_b/N_0 needed by $\mathcal{S}'$ in order to have the same symbol error probability of $\mathcal{S}$ (notice that this increment, or "gain", can be negative).

2.4 Signal sets for coherent detection

In this section, the most common signal sets used in digital communications systems based on coherent detection are presented and their error probability is calculated or approximated. In many cases, by exploiting the particular geometry of the signal sets, it is possible to find exact error probability expressions and upper bounds tighter than the general union bound.

In general, a signal set $\mathcal{S} = \{s_m(t) : m \in \{1, \ldots, M\}\}$ where $s_m(t)$ are all proportional to a single basic waveform $\psi(t)$, i.e., they are obtained by modulating the amplitude and the phase of a basic waveform as

$$s_m(t) = a_m \psi(t) \tag{2.30}$$

via complex amplitudes $a_m \in \mathbb{C}$, is called (somewhat improperly) a "linear modulation scheme".

In this case, the signal space has dimension one, and it is generated by the versor $\xi_1(t) = \psi(t)/\|\psi(t)\|$. Without loss of generality, we assume that

$$\frac{1}{M} \sum_{m=1}^{M} |a_m|^2 = 1$$

so that $E_s = \|\psi(t)\|^2$ is the average symbol energy. Signals in $\mathcal{S}$ are represented as points in the complex plane (one-dimensional complex vectors)

$$s_m = \sqrt{E_s} a_m$$

As usual, we denote by $\mathcal{S}$ both the set of signals and the corresponding set of points, that in this case is also referred to as *signal constellation.*

The most common signal sets of this type are pulse-amplitude modulation (PAM), quadrature amplitude modulation (QAM) and phase shift keying (PSK).

2.4.1 Pulse-Amplitude Modulation

The signal constellation for M-PAM is formed by M points on the real axis. Usually, these points are equally spaced and symmetrically placed with respect to the origin (antipodal PAM). In this case, we have

$$\mathcal{S} = \{\Delta(2m - M + 1) : m = 0, \ldots, M - 1\}$$

where Δ is a real parameter. The relation between Δ and E_s is given by

$$\begin{aligned} E_s &= \frac{\Delta^2}{M} \sum_{m=0}^{M-1} (2m - M + 1)^2 \\ &= \Delta^2 \frac{M^2 - 1}{3} \end{aligned} \tag{2.31}$$

and the constellation minimum SED is $d^2_{\min}(\mathcal{S}) = 4\Delta^2 = 12 \log_2 M E_b/(M^2 - 1)$. Fig. 2.6 shows an example for the 4-PAM constellation.

The symbol error probability of M-PAM is easily upperbounded by

$$P(e) \leq 2Q\left(\sqrt{\frac{6 \log_2 M}{M^2 - 1} \frac{E_b}{N_0}}\right) \tag{2.32}$$

Figure 2.6: 4-PAM signal constellation.

since every signal has at most two other signals at minimum distance, and since all pairwise error regions between signals at distance larger than the minimum distance are included into regions of signals at minimum distance.

We can find an exact expression for $P(e)$ as follows. We notice that the decision regions for all signals are congruent, but for the two extremal signals. Then, we can write,

$$P(e) = \frac{1}{M}\left[\sum_{m=1}^{M-2} \Pr(\mathbf{y} \notin \mathcal{D}_m|m) + \Pr(\mathbf{y} \notin \mathcal{D}_0|m=0) + \Pr(\mathbf{y} \notin \mathcal{D}_{M-1}|m = M-1)\right]$$

By symmetry, we see that all inner points have the same conditional error probability and the two outer points have the same conditional error probability (we need to compute only two terms, and not M). For the inner points we have

$$\begin{aligned}\Pr(\mathbf{y} \notin \mathcal{D}_m|m) &= \Pr(\{\mathrm{Re}\{\boldsymbol{\nu}\} < -\Delta\} \cup \{\mathrm{Re}\{\boldsymbol{\nu}\} > \Delta\}) \\ &= 2Q(\sqrt{2\Delta^2/N_0})\end{aligned}$$

For the outer points we have

$$\begin{aligned}\Pr(\mathbf{y} \notin \mathcal{D}_0|m=0) &= \Pr(\mathrm{Re}\{\boldsymbol{\nu}\} > \Delta) \\ &= Q(\sqrt{2\Delta^2/N_0})\end{aligned}$$

Finally, we obtain

$$\begin{aligned}P(e) &= \frac{2(M-1)}{M} Q(\sqrt{2\Delta^2/N_0}) \\ &= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right)\end{aligned} \tag{2.33}$$

We observe that the factor 2 of the upper bound (2.32) is changed into the factor $2(M-1)/M$ in the true error probability expression.

2.4.2 Quadrature-Amplitude Modulation

The signal constellation for M-QAM is formed by M points on the complex plane. Usually, these points belongs to a square lattice and are symmetrically placed with respect to the origin (squared QAM). When M is a square, the squared M-QAM constellation can be seen as two $\sqrt{M}$-PAM constellation in phase quadrature, with points of coordinates

$$\mathcal{S} = \{\Delta[(2m - M' + 1) + j(2n - M' + 1)] : m = 0, \dots, M'-1, n = 0, \dots, M'-1\}$$

where $M' = \sqrt{M}$ and where Δ is a real parameter. The relation between Δ and E_s is given by

$$\begin{aligned}
E_s &= \frac{\Delta^2}{M} \sum_{m=0}^{M'-1} \sum_{n=0}^{M'-1} (2m - M' + 1)^2 + (2n - M' + 1)^2 \\
&= 2\Delta^2 \frac{M-1}{3} \qquad (2.34)
\end{aligned}$$

and the constellation minimum SED is $d^2_{\min}(\mathcal{S}) = 4\Delta^2 = 6 \log_2 M E_b/(M-1)$. Fig. 2.7 shows an example for the 16-QAM constellation.

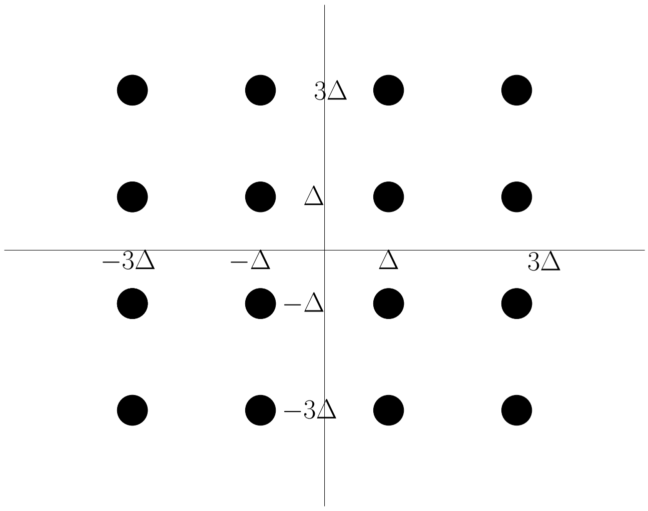


Figure 2.7: 16-QAM signal constellation.

In general, we can upperbound the symbol error probability of squared QAM by noticing that the decision region of each constellation point is larger or equal to a square of side 2Δ, centered in the point. Then, we have

$$\begin{aligned}
P(e) &\leq 1 - \left(1 - 2Q\left(\sqrt{\frac{2\Delta^2}{N_0}}\right)\right)^2 \\
&= 4Q\left(\sqrt{\frac{2\Delta^2}{N_0}}\right)\left(1 - Q\left(\sqrt{\frac{2\Delta^2}{N_0}}\right)\right) \\
&= 4Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{E_b}{N_0}}\right)\left(1 - Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{E_b}{N_0}}\right)\right) \qquad (2.35)
\end{aligned}$$

In the case where M is a square, $P(e)$ can be calculated exactly by noticing that a symbol error occurs either when the in-phase (I) or when the quadrature (Q) symbol components are erroneously detected. Since the real and imaginary parts of the noise are i.i.d. and the decision regions have boundaries parallel to the real and imaginary axes, detection errors in the I and Q components are statistically independent. Then, the exact probability of error can be written as

$$P(e) = 1 - (1 - P_{\mathrm{PAM}}(e))^2$$

where $P_{\mathrm{PAM}}(e)$ is the symbol error probability of the corresponding $\sqrt{M}$-PAM in the I and Q components. By using (2.33) we obtain

$$P(e) = 4\frac{\sqrt{M}-1}{\sqrt{M}}Q\left(\sqrt{\left(\frac{3\log_2 M}{M-1}\right)\frac{E_b}{N_0}}\right)\left(1-\frac{\sqrt{M}-1}{\sqrt{M}}Q\left(\sqrt{\left(\frac{3\log_2 M}{M-1}\right)\frac{E_b}{N_0}}\right)\right) \tag{2.36}$$

2.4.3 Phase-Shift Keying

In M-PSK, the information messages are encoded in the phase of the signal. The corresponding signal constellation is formed by M points equally spaced on a circle in the complex plane centered in the origin, with radius Δ, i.e.,

$$\mathcal{S} = \{\Delta \exp(j2\pi m/M) : m = 0, \ldots, M-1\}$$

Notice that 2-PSM (also denoted by BPSK) coincides with antipodal 2-PAM, and 4-PSK (also denoted by QPSK) coincides with 4-QAM. The average energy of M-PSK is given simply by $E_s = \Delta^2$, and the minimum distance is $d^2_{\min}(\mathcal{S}) = 4\Delta^2\sin^2(\pi/M) = 4\log_2 M \sin^2(\pi/M)E_b$. Fig. 2.8 shows an example for the 8-PSK constellation.

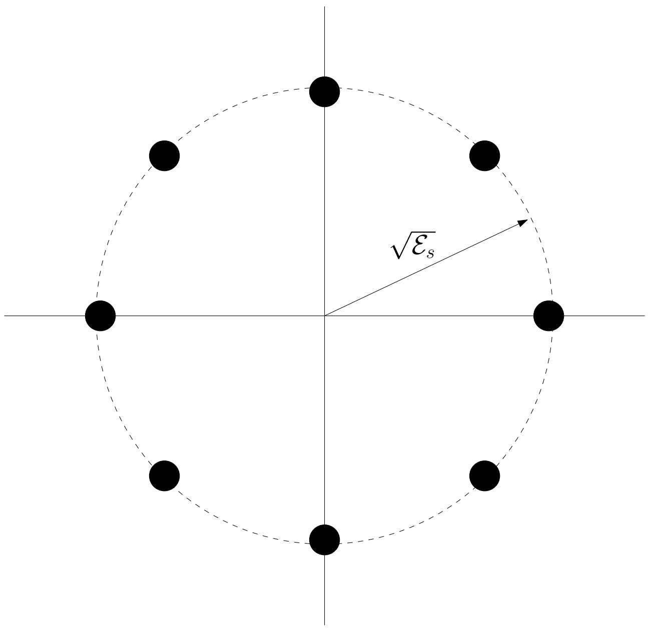


Figure 2.8: 8-PSK signal constellation.

The symbol error probability for BPSK and QPSK is given exactly by using (2.33) and (2.36) for $M = 2$ and for $M = 4$, respectively. For $M > 4$ it is not possible to find an exact expression of $P(e)$. However, a tight upper bound is obtained by noticing that the error region for any reference symbol is included in the union of the two pairwise regions given by the two symbols at minimum distance with respect to the reference symbol. This implies that the union bound can be limited to only two terms at minimum distance. The resulting upper bound is given by

$$P(e) \leq 2Q\left(\sqrt{2\log_2 M \sin^2(\pi/M)\frac{E_b}{N_0}}\right) \tag{2.37}$$

2.5 Non-coherent detection

We assume that the modulation waveforms belong to an N-dimensional signal set $\mathcal{S}$ of size M, generated by an orthonormal basis $\mathcal{B} = \{\xi_1(t), \ldots, \xi_N(t)\}$. With respect to the basis $\mathcal{B}$, the modulation signals are represented as N-dimensional complex vectors $\mathbf{s}_m = (s_{m,1}, \ldots, s_{m,N})^T$, so that

$$s_m(t) = \sum_{i=1}^{N} s_{m,i}\xi_i(t) \tag{2.38}$$

The received signal is given by (2.2), i.e., $y(t) = s_m(t)e^{j\theta} + \nu(t)$, for $t \in [0, T]$. By projecting $y(t)$ onto the versor $\xi_i(t)$ and letting $\mathbf{y} = (y_1, \ldots, y_N)^T$ be the projected vector, we obtain the vector channel model

$$\mathbf{y} = \mathbf{s}_m e^{j\theta} + \boldsymbol{\nu} \tag{2.39}$$

where $\boldsymbol{\nu} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, N_0\mathbf{I})$. As already said at the beginning of this chapter, a non-coherent receiver does not estimate explicitly the carrier phase and treats it as random variables, uniformly distributed over $[-\pi, \pi)$.

The signal vector $\mathbf{y}$ in (2.39) is Gaussian conditionally on the transmitted symbol m and on the phase θ. Its conditional pdf is given by

$$p(\mathbf{y}|m,\theta) = \frac{1}{(\pi N_0)^N} \exp\left(-\frac{1}{N_0}|\mathbf{y} - \mathbf{s}_m e^{j\theta}|^2\right) \tag{2.40}$$

The non-coherent ML decision rule for the detection symbol m with observation $\mathbf{y}$ is given by

$$\widehat{m} = \arg \max_{m\in\{1,\ldots,M\}} p(\mathbf{y}|m) \tag{2.41}$$

where $p(\mathbf{y}|m)$ is obtained from $p(\mathbf{y}|m,\theta)$ by averaging with respect to the phase θ.

We have

$$\begin{aligned}
p(\mathbf{y}|m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\mathbf{y}|m,\theta)d\theta \\
&= \frac{1}{(\pi N_0)^N} \exp\left(-\frac{|\mathbf{y}|^2 + |\mathbf{s}_m|^2}{N_0}\right) \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(\frac{2}{N_0}\mathrm{Re}\{\mathbf{y}^H\mathbf{s}_m e^{j\theta}\}\right) d\theta \\
&= \frac{1}{(\pi N_0)^N} \exp\left(-\frac{|\mathbf{y}|^2 + |\mathbf{s}_m|^2}{N_0}\right) \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(\frac{2}{N_0}|\mathbf{y}^H\mathbf{s}_m|\cos(\theta+\phi)\right) d\theta \\
&= \frac{1}{(\pi N_0)^N} \exp\left(-\frac{|\mathbf{y}|^2 + |\mathbf{s}_m|^2}{N_0}\right) I_0\left(\frac{2}{N_0}|\mathbf{y}^H\mathbf{s}_m|\right)
\end{aligned} \tag{2.42}$$

non linear

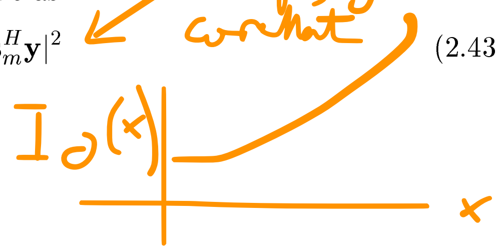
where we used the integral definition of the modified Bessel function [5] $I_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(z\cos\theta)d\theta$ and we observed that if θ is uniformly distributed over $[-\pi, \pi)$, also $\theta + \phi$ is uniformly distributed over the same interval, where $\phi = \angle\, \mathbf{y}^H\mathbf{s}_m$ is a phase independent of θ.

The maximization of the above likelihood function for a general signal set $\mathcal{S}$ does not lead to particularly appealing structures. However, in the special case of equal energy signals (i.e., signal sets $\mathcal{S}$ where $|\mathbf{s}_m|^2 = E_s$ for all $m = 1, \ldots, M$), we notice that $p(\mathbf{y}|m)$ is an increasing function of the squared magnitude of the correlation $|\mathbf{s}_m^H\mathbf{y}|^2$. From now on, we make the assumption of equal energy signals and we rewrite the ML decision rule as

energy content?

$$\widehat{m} = \arg \max_{m\in\{1,\ldots,M\}} |\mathbf{s}_m^H\mathbf{y}|^2 \tag{2.43}$$

no euclidian geometry



This rule is called *square-law detector* and can be implemented by a bank of correlators, which correlate the received signal vector $\mathbf{y}$ with all possible transmitted signal vectors $\mathbf{s}_m \in \mathcal{S}$, followed by a comparator that selects the signal whose correlation has maximum squared magnitude (squared envelope). Obviously, the squared magnitude of the signal correlation does not depend any longer on the signal phase, therefore, this ML detector is insensitive to the phase rotation introduced by the channel. Fig. 2.9 shows the structure of a non-coherent ML detector.

correlato

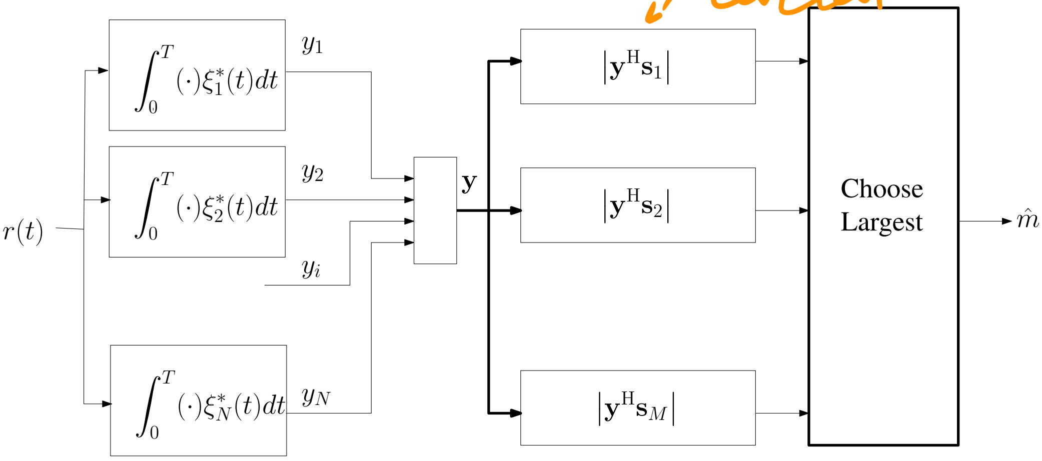


Figure 2.9: Block diagram of a non-coherent ML detector for equal energy signals.

2.6 Performance of non-coherent detection

In this section we consider only equal-energy signals, for which the ML non-coherent detector coincides with the squared-envelope detector. As for the case of coherent detection, we compute the decision metric difference between the correct signal $\mathbf{s}_m$ and a wrong signal $\mathbf{s}_{m'}$. Without loss of generality, we let $\theta = 0$ since the receiver is independent of the carrier phase. Then,

difference 2 magnitude squares

$$\begin{aligned}\Delta &= |\mathbf{s}_m^H \mathbf{y}|^2 - |\mathbf{s}_{m'}^H \mathbf{y}|^2 \\ &= |\mathbf{s}_m^H(\mathbf{s}_m + \boldsymbol{\nu})|^2 - |\mathbf{s}_{m'}^H(\mathbf{s}_m + \boldsymbol{\nu})|^2 \\ &= |E_s + v|^2 - |\rho E_s + v'|^2 \\ &= \mathbf{z}^H \mathbf{F} \mathbf{z}\end{aligned} \tag{2.44}$$

complex gaussian

rho

where we define the proper Gaussian vector $\mathbf{z} = (E_s + v, \rho E_s + v')^T$, with $v = \mathbf{s}_m^H \boldsymbol{\nu}$, $v' = \mathbf{s}_{m'}^H \boldsymbol{\nu}$, the signal energy $E_s = |\mathbf{s}_m|^2 = |\mathbf{s}_{m'}|^2$, the signal correlation coefficient $\rho = \frac{1}{E_s}\mathbf{s}_{m'}^H \mathbf{s}_m$ and the Hermitian 2×2 matrix

quadratic form of gaussian rv

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.45}$$

The vector $\mathbf{z}$ has mean $\bar{\mathbf{z}} = (E_s, \rho E_s)^T$ and covariance matrix

$$\mathbf{R} = N_0 E_s \begin{bmatrix} 1 & \rho^* \\ \rho & 1 \end{bmatrix}$$

Pairwise Error Probability

The PEP is given by $P(m \to m') = P(\Delta \leq 0)$. The pdf of Δ cannot be found in closed form. However, we are able to find the characteristic function of Δ, defined by

Characteristic function

$$\Phi_\Delta(s) = E[\exp(-s\Delta)] = \int \exp(-sz)p_\Delta(z)dz$$

Then, the PEP can be evaluated by the Laplace transform inversion formula

$$P(\Delta \leq 0) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{1}{s}\Phi_\Delta(s)ds \tag{2.46}$$

where the vertical integration path $s = \sigma + j\omega$, $\omega \in \mathbb{R}$ must belong to the region of convergence (ROC) of $\Phi_\Delta(s)/s$. The ROC is the vertical strip $0 < \sigma < \alpha_+$, where α_+ is the smallest positive real part of singularities of $\Phi_\Delta(s)$ (see [6, 7, 8] and references therein). Several efficient numerical techniques are available for Laplace inversion [9] and, more in general, for the evaluation of $P(\Delta \leq 0)$ from the characteristic function $\Phi_\Delta(s)$. A simpler upper bound on the PEP, that often yields enough qualitative information, is the Chernoff bound

$$P(\Delta \leq 0) \leq \min_{0<\lambda<\alpha_+} \Phi_\Delta(\lambda) \tag{2.47}$$

In our case, (2.44) turns out to be a *Hermitian Quadratic Form of complex Gaussian Random Variables* (briefly, HQF-GRV), for which a general expression of the characteristic function is known [10]. The following result is proved in Appendix 2.9

Proposition. The characteristic function of the HQF-GRV $\Delta = \mathbf{z}^H\mathbf{F}\mathbf{z}$, where $\mathbf{z} \sim \mathcal{N}_\mathbb{C}(\bar{\mathbf{z}}, \mathbf{R})$ is given by

$$\Phi_\Delta(s) = \frac{\exp\left(-s\bar{\mathbf{z}}^H\mathbf{F}(\mathbf{I}+s\mathbf{R}\mathbf{F})^{-1}\bar{\mathbf{z}}\right)}{\det(\mathbf{I}+s\mathbf{R}\mathbf{F})} \tag{2.48}$$

□

After some algebra and the change of variable $(1-|\rho|^2)E_s^2 s \to s$, we obtain

$$\Phi_\Delta(s) = \frac{\exp\left(-\frac{s(1-as)}{1-bs^2}\right)}{1-bs^2} \tag{2.49}$$

where the coefficients a and b are given by

$$a = \frac{N_0}{E_s}\frac{1}{1-|\rho|^2}$$
$$b = \left(\frac{N_0}{E_s}\right)^2\frac{1}{1-|\rho|^2}$$

In the case where $\Phi_\Delta(s)$ has only poles (i.e., no essential singularities), we can use the Laplace inversion formula and compute exactly the PEP by using the *residue theorem* [11]:

$$\frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{1}{s}\Phi_\Delta(s)ds = \begin{cases} -\sum_{s_i\in\text{RHPpoles}}\text{Res}(\Phi_\Delta(s)/s, s_i) \\ 1+\sum_{s_i\in\text{LHPpoles}}\text{Res}(\Phi_\Delta(s)/s, s_i)\end{cases} \tag{2.50}$$

where RHP and LHP denote the right half-plane ($\text{Re}\{s\} > 0$) and left half-plane ($\text{Re}\{s\} < 0$) of the complex plane, and where $\text{Res}(f(s), s_i)$ denotes the residue of a function $f(s)$ in s_i. For poles of multiplicity k, we have the following lemma:

Lemma: residue formula. Let $f(s) = g(s)/(s - s_0)^k$ with $g(s)$ analytic in s_0. Then, the residue of $f(s)$ in $s = s_0$ is given by

$$\text{Res}(f(s), s_0) = \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} g(s)\bigg|_{s=s_0}$$

□

For example, in the case of orthogonal signals ($\rho = 0$), we obtain the PEP in closed form. In fact, by letting $\rho = 0$ in (2.49) we obtain

$$\Phi_\Delta(s) = \frac{\exp\left(-s/(1 + sN_0/E_s)\right)}{(1 + sN_0/E_s)(1 - sN_0/E_s)}$$

This function has two poles of order 1, located in $\pm E_s/N_0$. By applying the residue formula we obtain

$$P(m \to m') = \frac{1}{2} e^{-\frac{1}{2}\frac{E_s}{N_0}} \tag{2.51}$$

non coherent

With some more involved calculation (see [3]), it is possible to obtain a general result for non-orthogonal equal energy signals in the following form

$$P(m \to m') = Q(u, v) - \frac{1}{2} e^{-(u^2+v^2)/2} I_0(uv) \tag{2.52}$$

where we define

$$u = \sqrt{\frac{E_s}{2N_0}(1 - \sqrt{1 - |\rho|^2})}$$
$$v = \sqrt{\frac{E_s}{2N_0}(1 + \sqrt{1 - |\rho|^2})}$$

and where $Q(u, v)$ is the Marcum "Q" function, defined by the integral

$$Q(u, v) = \int_v^\infty z e^{-(z^2+u^2)/2} I_0(uz) dz \tag{2.53}$$

Notice that for $\rho = 0$ we have $u = 0$ and $v = \sqrt{E_s/N_0}$, $Q(0, v) = e^{-v^2/2}$, therefore we find again that for orthogonal signals $P(m \to m') = \frac{1}{2} e^{-E_s/(2N_0)}$, as found above in a direct way.

2.6.1 Performance of M-ary orthogonal signals

Orthogonal M-ary signals deserve a special treatment because it is possible to derive directly the *exact* error probability without resorting to the PEP and union bound. This section is dedicated to the calculation of error probability of M-ary orthogonal signals.

Let $\mathbf{s}_1, \ldots, \mathbf{s}_M$ be M orthogonal N-dimensional signals, and assume, without loss of generality, that $\mathbf{s}_1$ is transmitted, so that the received signal is given by

$$\mathbf{y} = \mathbf{s}_1 + \boldsymbol{\nu}$$

(without loss of generality, we can consider $\theta = 0$ since the receiver does not depend on θ). Let $u_i = |\mathbf{s}_i^H \mathbf{y}|$ be the magnitude of the output of the i-th correlator, then

$$u_1 = |E_s + v_1|, \qquad u_i = |v_i|, \quad \text{for} \quad i = 2, \ldots, M$$

where $v_i = \mathbf{s}_i^H \boldsymbol{\nu}$ is the projection of the noise on the i-th signal vector. The noise variables v_i are i.i.d. $\sim \mathcal{N}_{\mathbb{C}}(0, N_0 E_s)$. It follows that u_1 has a Rice distribution, while the u_i's for $i = 2, \ldots, M$ have a Rayleigh distribution, given by:

$$u_1 \sim p_{u_1}(z) = \frac{z}{2E_s N_0} \exp\left(-\frac{z^2 + 4E_s^2}{4E_s N_0}\right) I_0(z/N_0)$$

$$u_i \sim p_{u_i}(z) = \frac{z}{2E_s N_0} \exp\left(-\frac{z^2}{4E_s N_0}\right)$$

The probability that $\mathbf{s}_1$ is correctly detected, conditionally on $u_1 = z$, can be written as

$$P(\text{correct}|u_1 = z) = P(u_2 \leq z, u_3 \leq z, \ldots, u_M \leq z) = \prod_{i=2}^{M} P(u_i \leq z)$$

The i-th term in the product is evaluated from the Rayleigh pdf

$$P(u_i \leq z) = \int_0^z p_{u_i}(z') dz' = 1 - \exp\left(-\frac{z^2}{4E_s N_0}\right)$$

The product can be expanded by using Newton binomial formula

$$\prod_{i=2}^{M} P(u_i \leq z) = \left(1 - e^{-\frac{z^2}{4E_s N_0}}\right)^{M-1} = \sum_{m=0}^{M-1} \binom{M-1}{m} (-1)^m e^{-\frac{mz^2}{4E_s N_0}}$$

Finally, we average the above expression with respect to the Rician distribution of u_1 and we obtain

$$P(\text{correct}) = \sum_{m=0}^{M-1} \binom{M-1}{m} \frac{(-1)^m}{m+1} e^{-\frac{m}{m+1} E_s/N_0}$$

where we have used the result

$$\int_0^\infty \frac{z}{2ab} e^{-\frac{(m+1)z^2 + 4a^2}{4ab}} I_0\left(\frac{z}{b}\right) dz =$$
$$\frac{1}{m+1} e^{-\frac{m}{m+1}\frac{a}{b}} \int_0^\infty \frac{y}{2ab} e^{-\frac{y^2 + 4a^2/(m+1)}{4ab}} I_0\left(\frac{y}{b\sqrt{m+1}}\right) dy =$$
$$\frac{1}{m+1} e^{-\frac{m}{m+1}\frac{a}{b}}$$

obtained by the change of integration variable $y = \sqrt{m+1}z$.

The exact error probability is given by $P(e) = 1 - P(\text{correct})$, that is

$$P(e) = \sum_{m=1}^{M-1} \binom{M-1}{m} \frac{(-1)^{m+1}}{m+1} e^{-\frac{m}{m+1} E_s/N_0} \tag{2.54}$$

Again, by letting $M = 2$ we obtain the exact error probability with orthogonal binary signals as $P(e) = \frac{1}{2} e^{-E_s/(2N_0)}$.

2.7 Signal sets for non-coherent detection

In this section we present some examples of equal-energy signal sets suited to non-coherent detection.

2.7.1 Frequency-Shift Keying

Among the signals suited to non-coherent detection a prominent role is played by frequency-shift keying (FSK). In FSK, information is encoded in the signal frequency. Typically, an FSK modulator can be seen as the concatenation of a PAM modulator with a *Voltage-Controlled Oscillator* (VCO), with central frequency f_0. In the time domain, M-ary FSK signals are given by

$$s_m(t) = A \exp(j2\pi a_m t)\, p_T(t)$$

where, for $m = 0, \ldots, M-1$,

$$a_m = \Delta(2m - M + 1)/2$$

and where $p_T(t)$ is a rectangular pulse of duration T and amplitude 1. The spacing between frequencies corresponding to different symbols is Δ. Since all signals have the same energy, the average energy per symbol is given by

$$E_s = \int_0^T |s_m(t)|^2 dt = A^2 T$$

The distance properties and error probability of FSK depend on the normalized frequency spacing ΔT. In general, the distance between two signals can be better calculated in the time domain and it is given by

$$\begin{aligned} d^2(m, m') &= A^2 \int_0^T |\exp(j2\pi a_m t) - \exp(j2\pi a_{m'} t)|^2 dt \\ &= 2E_s(1 - \mathrm{Re}\{\rho(m, m')\}) \end{aligned} \tag{2.55}$$

where we define the correlation coefficient

$$\begin{aligned} \rho(m, m') &= \frac{1}{T}\int_0^T \exp(j2\pi\Delta(m - m')t)dt \\ &= \exp(j\pi(m - m')\Delta T)\mathrm{sinc}((m - m')\Delta T) \end{aligned} \tag{2.56}$$

The two signals are said to be *coherently orthogonal* if $\mathrm{Re}\{\rho(m, m')\} = 0$, and *non-coherently orthogonal* if $\rho(m, m') = 0$. From (2.56) and from the fact that the minimum (in absolute value) of the difference $m - m'$ is 1, we obtain immediately that the condition for coherent orthogonality is that $\Delta T = \frac{k}{2}$ for $k \in \mathbb{Z}$, and the condition for non-coherent orthogonality is that $\Delta T = k$ for $k \in \mathbb{Z}$.

2.7.2 Hadamard orthogonal codes

We can obtain an orthogonal signal set from any unitary $N \times N$ matrix $\mathbf{U}$. We define the signals

$$s_m(t) = \sum_{i=1}^{N} u_{m,i}\psi(t - (i-1)T/N)$$

where $\psi(t)$ is a rectangular pulse of energy E_s and support $[0, T/N]$, and where $u_{m,i}$ are the elements of $\mathbf{U}$. Since the above are linearly modulated signals, an optimal receiver front-end is obtained by taking N samples at rate N/T at the output of a matched filter with impulse response $\sqrt{\frac{1}{E_s}}\psi(-t)^*$. Equivalently, the orthonormal basis $\mathcal{B}$ for this signal set is formed by the versors

$$\xi_i(t) = \sqrt{\frac{1}{E_s}}\psi(t - (i-1)T/N)$$

for $i = 1, \ldots, N$. The resulting vector channel has the form (2.39) where $\mathbf{s}_m = \sqrt{E_s}(u_{m,i}, \ldots, u_{N,i})^T$ In particular, if the unitary matrix is given by $\mathbf{U} = \frac{1}{\sqrt{N}}\mathbf{H}$, where $\mathbf{H}$ is a Hadamard matrix, this signal set is said to be a *Hadamard orthogonal code.* Hadamard matrices are $N \times N$ matrices with elements ± 1 and mutually orthogonal row and columns. If N is a power of 2, the $N \times N$ Hadamard matrix is said of the *Sylvester type* and can be constructed by the following recursion

$$\mathbf{H}_n = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \mathbf{H}_{n-1}$$

initialized by $\mathbf{H}_0 = 1$. [1]

2.7.3 Differential M-PSK

In the presence of a random phase rotation θ, the receiver is not able to detect information symbols encoded in the signal phase. Therefore, M-PSK with symbol-by-symbol detection is useless in a non-coherent system. In order to use M-PSK with non-coherent detection, we need to detect blocks of consecutive symbols. In particular, consider a simple scheme that transmits blocks of PSK symbols $\mathbf{x} = \sqrt{E_s}(1, x_1, x_2, \ldots, x_{N-1})^T$ (with $|x_i| = 1$). The first symbol is always set to 1, in order to create a phase reference for the receiver. Notice that in this way, none of the blocks $\mathbf{x}$ can be obtained from another by a phase rotation. Therefore, the signal blocks are distinguishable even without knowledge of the phase θ.

This scheme copes with the phase ambiguity of the non-coherent receiver but wastes one out of N symbols. In order to avoid this wasting, *differential encoding* can be used [3]. The PSK information symbols $a_1, a_2, \ldots, a_{N-1}$ are encoded into the PSK channel symbols $x_1, x_2, \ldots, x_{N-1}$ via the recursive mapping

$$x_i = x_{i-1}a_i \tag{2.57}$$

and where x_0 is set equal to the last symbol of the previously transmitted block. This method is called "differential" since the phase of a_i determines the phase shift (difference) between x_{i-1} and x_i. It is clear that even if the sequence of symbols x_i rotated by $e^{j\theta}$ the information symbols a_i can still be recovered (in the absence of noise) by

$$a_i = \left(x_{i-1}e^{j\theta}\right)^* x_i e^{j\theta} = x_{i-1}^* x_i$$

[1] $\otimes$ denotes the Kronecker product, given by

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

Also, no explicit phase reference symbol is needed, since the phase reference is provided by the last symbol of the previous block.

Generalizing the idea of differential PSK to blocks of length N, we have that the sequence of N-dimensional blocks

$$(\dots, \underbrace{(1, a_{1,n}, a_{2,n}, \dots, a_{N-1,n})}_{N}, \underbrace{(1, a_{1,n+1}, a_{2,n+1}, \dots, a_{N-1,n+1})}_{N}, \dots)$$

is transmitted as

$$(\dots, \underbrace{(x_{1,n}, x_{2,n}, \dots, x_{N-1,n})}_{N-1}, \underbrace{(x_{1,n+1}, x_{2,n+1}, \dots, x_{N-1,n+1})}_{N-1}, \dots)$$

where $x_{i,n} = a_{i,n} x_{N-1,n-1}$. For detection, the n-th block of $N-1$ symbols is considered together with the leading additional symbol, in order to carry out non-coherent detection over a block of N components. All signals in the n-th block are rotated by $x_{N-1,n-1}$, and since detection is non-coherent, this rotation is irrelevant, i.e., detection must be carry out by considering the original blocks of length N with the leading symbol equal to 1. Therefore, the corresponding signal block in the vector channel model (2.39) is given by $\mathbf{x} = \sqrt{\mathcal{E}}(1, x_1, \dots, x_{N-1})^T$, even though the number of PSK symbols actually sent on the channel is only $N-1$ per block (apart from the very first block of the sequence, that in any case has no influence on the transmission rate if the number of transmitted blocks is large).

2.8 Problems

Problem 1. A binary baseband communication system makes use of the two signals

$$s_0(t) = \begin{cases} At/T & t \in [0, T] \\ 0 & \text{elsewhere} \end{cases}$$

$$s_1(t) = \begin{cases} A - At/T & t \in [0, T] \\ 0 & \text{elsewhere} \end{cases}$$

The signals are transmitted with equal probability over a (real) AWGN channel with power spectral density $N_0/2$.

Determine the optimal detector that minimizes the probability of symbol error and represent its block diagram. Also, compute the symbol error probability as a function of the SNR (defined as usual by the average symbol energy over N_0).

Problem 2. Define $p_T(t)$ as

$$p_T(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$$

and consider the signal set

$$s_1(t) = \sqrt{2\mathcal{E}} \cos(2\pi f_1 t) p_T(t)$$
$$s_2(t) = \sqrt{2\mathcal{E}} \cos(2\pi f_2 t) p_T(t)$$
$$s_3(t) = -\sqrt{2\mathcal{E}} \cos(2\pi f_1 t) p_T(t)$$
$$s_4(t) = -\sqrt{2\mathcal{E}} \cos(2\pi f_2 t) p_T(t)$$

where $f_1 = k/T$ and $f_2 = (k+1)/T$, and k is an integer. Consider transmission over a (real) AWGN channel. The received signal is $y(t) = x(t) + n(t)$ where $x(t)$ can be one of the signals $s_1(t), s_2(t), s_3(t), s_4(t)$, with uniform probability, and where $n(t)$ is a white Gaussian noise with power spectral density $N_0/2$.

1) Find the Maximum-Likelihood detector structure that employs the minimum possible number of correlators, and represent its block diagram.

2) Find the symbol error probability as a function of E_b/N_0 for ML detection (either an exact expression or an upper bound. In the latter case, make the bound as tight as possible).

Problem 3. Consider the signal constellation defined by the points

$$s_m = \sqrt{\mathcal{E}_1}\exp(j(\pi(m-1)/2 + \pi/4)) \quad \text{for } m = 1, \dots, 4$$
$$s_m = \sqrt{\mathcal{E}_2}\exp(j\pi(m-5)/2) \quad \text{for } m = 5, \dots, 8$$

1) Find the average energy per symbol E_s (symbols are equiprobable) as a function of $\mathcal{E}_1$ and $\mathcal{E}_2$.

2) For fixed E_s, find the ratio $\rho = \mathcal{E}_2/\mathcal{E}_1$ that maximizes the minimum squared Euclidean distance of the constellation (assume $\mathcal{E}_2 \geq \mathcal{E}_1$, i.e., consider the interval $\rho \geq 1$).

3) For the optimal ratio ρ found, determine an upperbound to the symbol error probability as a function of E_b/N_0.

Problem 4. Compute the symbol error probability of a 2-FSK system with coherent detection, $\Delta T = 0.7$ over an AWGN channel with $E_s/N_0 = 10$ dB. *Note: Δ denotes the spacing between the two frequencies of the 2-FSK modulation and T denotes the symbol interval.*

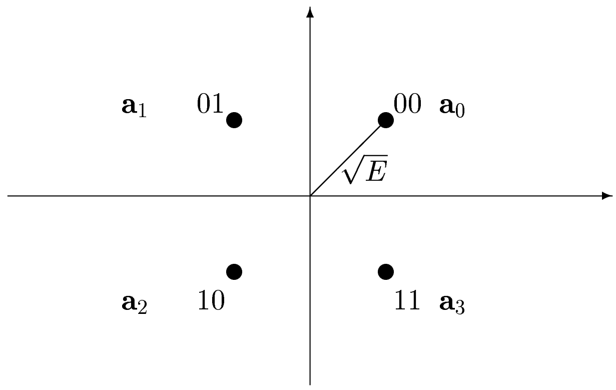
Problem 5. Repeat the previous problem in the case of non-coherent detection.

Problem 6. A digital modulator gets as input a sequence of independent and uniformly distributed bits and produces symbols a labeled by pairs of bits (b_1, b_2) according to the labeling rule

b_1b_2	a
00	$-3\frac{d}{2}$
01	$-\frac{d}{2}$
11	$\frac{d}{2}$
10	$3\frac{d}{2}$

The received signal output by a (real) AWGN channel is given by $y = a + n$, where $n \sim \mathcal{N}(0, N_0/2)$. The receiver is based on the optimal decision rule that minimizes the symbol error probability. Compute the average bit-error probability.

Problem 7. In a 4-PSK modulator, each symbol is labeled by the pair of bits (b_1b_2) according to the following labeling rule:



The bits b_1 e b_2 are statistically independent with probability distribution

$$\begin{cases} P(b_i = 0) & = \varepsilon \\ P(b_i = 1) & = 1 - \varepsilon \end{cases} \qquad i = 1, 2$$

(with $\varepsilon < 0.5$) The signal is modulated and transmitted over an AWGN channel with power spectral density N_0.

Determine the decision regions of the MAP detector (i.e., the detector that minimizes the symbol-error probability), and determine the decision regions of a detector that minimizes the bit-error probability.

Problem 8. The constellation with points

$$\begin{aligned} s_1 &= L/2 \\ s_2 &= -L/2 \\ s_3 &= j\sqrt{3}L/2 \\ s_4 &= -j\sqrt{3}L/2 \\ s_5 &= L + j\sqrt{3}L/2 \\ s_6 &= L - j\sqrt{3}L/2 \\ s_7 &= -L + j\sqrt{3}L/2 \\ s_8 &= -L - j\sqrt{3}L/2 \end{aligned}$$

is used for transmission over an AWGN channel (note: L is a scaling factor that determines the average energy per symbol).

1) Find an upper bound on the symbol error probability for this constellation as a function of the ratio E_b/N_0, where E_b is the average energy per bit and N_0 is the noise power spectral density.

2) Find the asymptotic gain (expressed in dB) of the above constellation with respect to an 8PSK constellation.

Problem 9. A signal set of M orthogonal signals is constructed by varying the position of a rectangular impulse. The signals are

$$s_m(t) = p(t - mT/M) \quad m = 0, 1, 2, \ldots, M - 1$$

where

$$p(t) = \begin{cases} A & 0 \le t \le T/M \\ 0 & \text{elsewhere} \end{cases}$$

Determine the peak power level A^2 as a function of M for which this system has the same symbol error probability of an orthogonal M-FSK (with non-coherent detection).

Problem 10. Consider the on-off modulation with signals $s_0(t) = 0$ and $s_1(t) = \sqrt{2E_s} p_T(t)$, where $p_T(t)$ is a unit-energy rectangular pulse of support $[0, T]$. Find the ML non-coherent decision rule. For this rule, assuming equiprobable symbols, give an expression for the average error probability.

Problem 11. Find an exact expression of the symbol error probability for differential BPSK (non-coherent detection is performed over blocks of length $N = 2$ of adjacent symbols).

Problem 12. Compute the union bound on the symbol error probability of differential BPSK with non-coherent block detection performed on blocks of length $N = 3$ of adjacent symbols.

Problem 13. A 4-PSK modulation is used for transmission over AWGN. Due to a phase synchronization error in the receiver, the discrete-time signal at the input of the detector is given by

$$y = \sqrt{E_s} e^{j\theta} x + z$$

where z is a complex Gaussian circularly-symmetric noise with mean zero and variance N_0, x is a symbol of the unit-energy 4-PSK constellation and θ is the phase error, assumed to be unknown and deterministic.

Compute an upper bound to the symbol error probability as a function of θ and E_s/N_0 assuming that the detector uses the same decision regions of the ML detector that ignores the phase error (i.e., that assumes $\theta = 0$).

Problem 14. Gray labeling is a binary labeling of the constellation points of a signal set such that the label of any tow signals at minimum distance differ by only one bit. Consider a 8PSK constellation and find a Gray labeling. Sketch the constellation with the associated binary labels. Find an upper bound (as tight as possible) to the symbol error probability P_e with ML detection in AWGN, as a function of the SNR E_s/N_0. Find an *exact* expression for the average bit-error probability P_b as a function of E_s/N_0 and verify that, with Gray labeling, we have $P_b \approx \frac{1}{3} P_e$ for large E_s/N_0.

2.9 Appendix: the characteristic function of HQF-GRV

In this appendix we derive the expression (2.48) for the characteristic function of the HQF-GRV $\Delta = \mathbf{z}^H \mathbf{F} \mathbf{z}$, where $\mathbf{z} \sim \mathcal{N}_{\mathcal{C}}(\bar{\mathbf{z}}, \mathbf{R})$.

Since $\mathbf{R}$ is Hermitian positive definite, it can be decomposed as

$$\mathbf{R} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^H$$

where $\mathbf{U}$ is unitary and its columns are the (normalized) eigenvector of $\mathbf{R}$, and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ contains the eigenvalues of $\mathbf{R}$. Since $\lambda_i > 0$, we can write $\mathbf{\Lambda} = \mathbf{\Psi}^2$, where $\mathbf{\Psi} = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. The new Gaussian vector $\mathbf{w} = \mathbf{\Psi}^{-1}\mathbf{U}^H\mathbf{z}$ has i.i.d. unit covariance components, in fact

$$\begin{aligned} E[(\mathbf{w} - \bar{\mathbf{w}})(\mathbf{w} - \bar{\mathbf{w}})^H] &= \mathbf{\Psi}^{-1}\mathbf{U}^H\mathbf{R}\mathbf{U}\mathbf{\Psi}^{-1} \\ &= \mathbf{\Psi}^{-1}\mathbf{U}^H\mathbf{U}\mathbf{\Lambda}\mathbf{U}^H\mathbf{U}\mathbf{\Psi}^{-1} \\ &= \mathbf{I} \end{aligned}$$

We have the inverse relation $\mathbf{z} = \mathbf{U}\mathbf{\Psi}\mathbf{w}$, so that

$$\Delta = \mathbf{w}^H\mathbf{\Psi}\mathbf{U}^H\mathbf{F}\mathbf{U}\mathbf{\Psi}\mathbf{w} = \mathbf{w}^H\mathbf{T}\mathbf{w}$$

The Hermitian matrix $\mathbf{T} = \mathbf{\Psi}\mathbf{U}^H\mathbf{F}\mathbf{U}\mathbf{\Psi}$ can also be written as

$$\mathbf{T} = \mathbf{S}\mathbf{\Phi}\mathbf{S}^H$$

where $\mathbf{S}$ is unitary and $\mathbf{\Phi}$ =diag$(\phi_1, \ldots, \phi_n)$. We define the new Gaussian vector $\mathbf{v} = \mathbf{S}^H\mathbf{w}$ (notice that $\mathbf{v}$ has also covariance $\mathbf{I}$ since $\mathbf{S}$ is unitary). Finally, the quadratic form can be diagonalized and written as the sum of n independent terms as follows:

$$\begin{aligned} \Delta &= \mathbf{w}^H\mathbf{T}\mathbf{w} \\ &= \mathbf{w}^H\mathbf{S}\mathbf{\Phi}\mathbf{S}^H\mathbf{w} \\ &= \mathbf{v}^H\mathbf{\Phi}\mathbf{v} \\ &= \sum_{i=1}^{n} \phi_i|v_i|^2 \end{aligned}$$

The characteristic function $\Phi_\Delta(s)$ is obtained in product form as

$$\begin{aligned} \Phi_\Delta(s) &= E\left[e^{-s\Delta}\right] \\ &= E\left[\exp\left(-s\sum_{i=1}^{n}\phi_i|v_i|^2\right)\right] \\ &= \prod_{i=1}^{n}\Phi_{|v_i|^2}(s) \end{aligned}$$

where we define

$$\Phi_{|v_i|^2}(s) = E\left[\exp\left(-s\phi_i|v_i|^2\right)\right]$$

Since $v_i \sim \mathcal{N}_{\mathcal{C}}(\bar{v}_i, 1)$ and it is circularly symmetric, we have

$$|v_i|^2 = a_i^2 + b_i^2$$

where $a_i \sim \mathcal{N}(\bar{a}_i, 1/2)$ and $b_i \sim \mathcal{N}(\bar{b}_i, 1/2)$ are independent, and where $\bar{v}_i = \bar{a}_i + j\bar{b}_i$. Then, in order to complete our calculation it is sufficient to evaluate the characteristic function of the square of a real Gaussian random variable x with variance $1/2$ and mean $\bar{x}$. We have

$$\begin{aligned} f_{x^2}(s) &= E\left[\exp\left(-s\phi x^2\right)\right] \\ &= \frac{\exp\left(-s\frac{\phi}{1+s\phi}\bar{x}^2\right)}{\sqrt{1+s\phi}} \end{aligned} \tag{2.58}$$

Then, by using the above result we obtain

$$\begin{aligned}\Phi_{|v_i|^2}(s) &= f_{a_i^2}(s) f_{b_i^2}(s) \\ &= \frac{\exp\left(-s\frac{\phi_i}{1+s\phi_i}|\bar{v}_i|^2\right)}{1+s\phi_i}\end{aligned} \tag{2.59}$$

The desired result is given by

$$\Phi_\Delta(s) = \frac{\exp\left(-s\sum_{i=1}^n \frac{\phi_i}{1+s\phi_i}|\bar{v}_i|^2\right)}{\prod_{i=1}^n(1+s\phi_i)} \tag{2.60}$$

Finally, we re-write the above characteristic function directly in terms of the matrices $\mathbf{R}$ and $\mathbf{F}$. Notice that

$$\begin{aligned}\prod_{i=1}^n(1+s\phi_i) &= \det(\mathbf{I}+s\mathbf{T}) \\ &= \det\left(\mathbf{U\Psi}(\mathbf{I}+s\mathbf{T})\mathbf{\Psi}^{-1}\mathbf{U}^H\right) \\ &= \det(\mathbf{I}+s\mathbf{RF})\end{aligned}$$

In fact, $\mathbf{U\Psi T\Psi}^{-1}\mathbf{U}^H = \mathbf{U\Lambda U}^H\mathbf{F} = \mathbf{RF}$. Also, we can write

$$\sum_{i=1}^n \frac{\phi_i}{1+s\phi_i}|\bar{v}_i|^2 = \bar{\mathbf{v}}^H\mathbf{\Phi}(\mathbf{I}+s\mathbf{\Phi})^{-1}\bar{\mathbf{v}}$$

Recalling that $\mathbf{\Phi} = \mathbf{S}^H\mathbf{\Psi U}^H\mathbf{FU\Psi S}$, and that $\bar{\mathbf{v}} = \mathbf{S}^H\mathbf{\Psi}^{-1}\mathbf{U}^H\bar{\mathbf{z}}$, we obtain

$$\bar{\mathbf{v}}^H\mathbf{\Phi}(\mathbf{I}+s\mathbf{\Phi})^{-1}\bar{\mathbf{v}} = \bar{\mathbf{z}}^H\mathbf{F}(\mathbf{I}+s\mathbf{RF})^{-1}\bar{\mathbf{z}}$$

so that (2.48) follows.

Chapter 3

Digitally modulated signals and ML sequence detection

In the previous chapter we considered the problem of the transmission and reception of a single waveform defined on a finite time interval $[0, T]$. This setting is general, since it encompasses any practical situation by suitably defining the modulator waveforms and the transmission duration T. In this section we shall show that, without loss of generality, waveforms of arbitrary duration T can be constructed by concatenating elementary waveforms modulated by complex amplitudes. We shall refer to these signals as "digitally modulated signals", or also as "signal sequences". The modulation and demodulation of these signal sequences is obtained by simple linear filtering and sampling operations and ML detection can be obtained in a very efficient way, by using the *Viterbi Algorithm*. Finally, we examine the performance of a simple suboptimal symbol-by-symbol detector and we find that, is the elementary waveform satisfies the *Nyquist condition*, the optimal ML sequence detector coincides with the simple symbol-by-symbol detector.

3.1 Modulated signal sequences

In this section we examine the structure of signals of duration approximately T and bandwidth approximately $[-W/2, W/2]$. Clearly, strictly bandlimited and time-limited signals do not exist. However, for large product WT, it can be proved (see [**?**] and references therein) that the signal space of such signals has $N \approx \lfloor WT \rfloor$ dimensions, i.e., that any such signal $x_T(t)$ can be represented as

$$x_T(t) = \sum_{n=0}^{N-1} x_n \xi_n(t), \quad t \in [0, T]$$

for some suitable signal basis $\mathcal{B} = \{\xi_n(t)\}$.

Although the rigorous theory yielding this result and the basis $\mathcal{B}$ is fairly involved, we can get an inutitive explanation of this fact by using the sampling theorem. Let $x(t)$ be a signal of infinite duration, such that $X(f) = \mathcal{F}[x(t)]$ is strictly bandlimited in $[-W/2, W/2]$. Then, we can write

$$x(t) = \sum_n x[n] \mathrm{sinc}(W(t - n/W))$$

where $x[n] = x(n/W)$ are samples of $x(t)$ taken at sampling frequency W.

Now, consider the restriction $x_T(t)$ of $x(t)$ over $[0, T]$. Since the sinc function is mostly concentrated around $t = 0$ and its magnitude decreases as $1/t$, this can be approximated by

$$x_T(t) \approx \sum_{n=-D}^{N+D-1} x[n]\mathrm{sinc}(W(t - n/W)), \qquad t \in [0, T] \tag{3.1}$$

for some suitable integer D. Notice that the LHS of (3.1) is strictly time-limited and hence has infinite bandwidth, while the RHS of (3.1) is strictly bandlimited and hence has infinite duration. However, for large product WT the two signals are "close" both in the time domain and in the frequency domain in the time-requency region $[0, T] \times [-W/2, W/2]$.

The above consideration suggests that, for any practical purpose, a modulator signal set defined over the interval $[0, T]$ and the bandwidth $[-W/2, W/2]$ can be generated, without loss of generality, as a sequence of elementary pulses multiplied by suitable complex coefficients, and by relaxing the strictly time-limited and strictly band-limited assumptions by allowing some guard time and guard frequency intervals.

The resulting general expression for such modulated signal sequences is given by

$$s(t; \mathbf{a}) = \sum_{n=0}^{N-1} a_n \psi(t - nT_s) \tag{3.2}$$

where

1. $\mathbf{a} = (a_0, \ldots, a_{N-1})$ is a sequence of complex "modulation" symbols defined over a modulation alphabet $\mathcal{A} \subseteq \mathbb{C}$. For example, $\mathcal{A}$ can be a PSK or a QAM constellation. We assume that $\frac{1}{|\mathcal{A}|}\sum_{a\in\mathcal{A}} |a|^2 = 1$, i.e., that the constellation has normalized average energy per symbol equal to 1.

2. $\psi(t)$ is the elementary modulation waveform, of energy $E_s = \|\psi\|^2 = \int |\psi(t)|^2 dt$. The elemntary waveform $\psi(t)$ is assumed to be well concentrated in time and frequency. Ideally, its energy should be mostly concentrated in the time-frequency region $[-T_s/2, T_s/2] \times [-1/(2T_s), 1/(2T_s)]$, whose time-frequency product is equal to 1 (i.e., it spans one dimension in the signal space). In practice, we assume that the energy of $\psi(t)$ is mostly concentrated in a slightly wider region

$$\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right] \times \left[-\frac{W}{2}, \frac{W}{2}\right]$$

such that $1 \leq W\Delta \leq L_\psi$, where L_ψ is a (small) integer.

3. T_s is the symbol interval, and its reciprocal $R_s = 1/T_s$ is referred to as the symbol rate (measured in "baud", i.e., symbols per second). Notice that since $\Psi(t)$ has duration larger than T_s, then the successive translated waveforms forming $s(t; \mathbf{a})$ overlap in time.

The modulator signal set, defined over the time interval $[-\Delta/2, (N-1)T_s + \Delta/2]$, is formed by all waveforms $s(t; \mathbf{a})$ for all possible symbol sequences $\mathbf{a}$, i.e.,

$$\mathcal{S} = \{s(t; \mathbf{a}) : \mathbf{a} \in \mathcal{A}^N\} \tag{3.3}$$

Therefore, we can identify each symbol sequence $\mathbf{a}$ with a "message" m to be transmitted. The total number of messages is exponential in the sequence lengh, in fact, it is given by $M = |\mathcal{A}|^N$.

3.1.1 Statistical Characterization of Modulated Signal Sequences

In order to verify the time-bandwidth property of signals $s(t; \mathbf{a})$, we examine the random process

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \psi(t - nT_s) \tag{3.4}$$

obtained by letting $N \to \infty$ and by selecting at random a signal $s(t; \mathbf{a}) \in \mathcal{S}$. The power spectral density of such process can be regarded as an approximation of the power spectral density of the signals generated by the modulator, for large N, i.e., when the signal sequence is very long (which is the case of most practical interest).

The mean signal is given by

$$E[x(t)] = \sum_n E[a_n]\psi(t - nT_s)$$

and the autocorrelation function is given by

$$\begin{aligned}
\phi_x(t+\tau, t) &= E[x(t+\tau)x(t)^*] \\
&= \sum_n \sum_m E[a_n a_m^*]\psi(t+\tau-nT_s)\psi(t-mT_s)^* \\
&= \sum_n \sum_m R_a[n-m]\psi(t+\tau-nT_s)\psi(t-mT_s)^* \\
&= \sum_m R_a[m] \sum_n \psi(t+\tau-nT_s)\psi(t-nT_s+mT_s)^*
\end{aligned}$$

where we have assumed that the symbol sequence $\{a_n\}$ is discrete-time WSS, with autocorrelation function $R_a[m]$ and mean $\mu_a = E[a_n]$. We observe that both mean and autocorrelation are periodic in t with period T_s. Therefore, when $\{a_n\}$ is discrete-time WSS, $x(t)$ is a WSC continuous-time process.

In order to obtain the power spectrum of a WSC process, we compute the Fourier transform of the time-averaged autocorrelation function

$$\overline{\phi}_x(\tau) = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \phi_x(t+\tau, t)dt$$

By substituting the expression obtained before, we get

$$\begin{aligned}
\overline{\phi}_x(\tau) &= \frac{1}{T_s}\sum_m R_a[m] \sum_n \int_{-T_s/2}^{T_s/2} \psi(t+\tau-nT_s)\psi(t-nT+mT_s)^* dt \\
&= \frac{1}{T_s}\sum_m R_a[m] \sum_n \int_{-T_s/2-nT_s}^{T_s/2-nT_s} \psi(t+\tau)\psi(t+mT_s)^* dt \\
&= \frac{1}{T_s}\sum_m R_a[m] \int_{-\infty}^{\infty} \psi(t+\tau)\psi(t+mT_s)^* dt \\
&= \frac{1}{T_s}\sum_m R_a[m]\phi_\psi(\tau - mT_s)
\end{aligned}$$

By applying Fourier transform to the last line of the above equation, we obtain

$$S_x(f) = \frac{1}{T_s}|\Psi(f)|^2 \mathcal{S}_a(fT_s)$$

where $\Psi(f) = \mathcal{F}[\psi(t)]$ and where $\mathcal{S}_a(\lambda) = \sum_m R_a[m]e^{-j2\pi\lambda m}$ is the power spectrum of the WSS discrete-time sequence $\{a_n\}$.

Define the autocovariance function $\sigma_a[m] = R_a[m] - |\mu_a|^2$. By substituting into the expression of $S_x(f)$ we get

$$S_x(f) = \frac{1}{T_s}|\Psi(f)|^2 \Sigma_a(fT_s) + \frac{|\mu_a|^2|\Psi(f)|^2}{T_s}\sum_m e^{-j2\pi f T_s m}$$

where $\Sigma_a(\lambda) = \sum_m \sigma_a[m]e^{-j2\pi\lambda m}$ is the Fourier transform of the autocovariance function. Now, we use the identity $\sum_m e^{-j2\pi f T_s m} = \frac{1}{T_s}\sum_m \delta(f - m/T_s)$, and we obtain

$$S_x(f) = \frac{1}{T_s}|\Psi(f)|^2 \Sigma_a(fT_s) + \frac{|\mu_a|^2}{T_s^2}\sum_m \left|\Psi\left(\frac{m}{T_s}\right)\right|^2 \delta(f - m/T_s)$$

From the above expression, we see that the power spectrum of $x(t)$ contains *spectral lines* only if $\mu_a \neq 0$. Moreover, the correlation of the data sequence $\{a_n\}$ modifies the spectral shape of the original elementary waveform $\psi(t)$ used for modulation. In particular, when $\{a_n\}$ is a zero-mean uncorrelated sequence, we have $S_x(f) = \sigma_a[0]|\Psi(f)|^2/T_s$, i.e., the modulated signal has PSD proportional to the energy specturm of the pulse-shaping waveform. By noticing that $\sigma_a[0] = E[|a_n|^2] = 1$ and that $\int |\Psi(f)|^2 df = \|\psi(t)\|^2 = E_s$ is the energy of the pulse-shaping waveform, we get that the average energy per symbol interval is given by $\mathcal{P}_x = E_s/T$, which obviously coincides with the signal power. In fact, by intergrating $S_x(f)$ we obtain exactly the power $\mathcal{P}_x$, as expected.

In conclusions, we see from the expression of the power spectral density $S_x(f)$ that the bandwidth of $x(t)$ is essentially determined by the bandwidth of $\psi(t)$. This means that if this is approximately limited in $[-W/2, W/2]$, the whole digitally modulated signal will be approximately bandlimited in such interval. Moreover, as we said before, the time duration of such signal for a sequence of length N symbols is $\approx NT_s$. Hence, for waveforms $\psi(t)$ with time-bandwidth product ≈ 1, the "WT" theorem is well satisfied in the sense that for large WT we have indeed $N \approx WT$.

Example: the AMI code. In this example, we see how introducing correlation in the symbol sequence $\{a_n\}$ might modify the spectral shape of the digitally modulated signal process.

The Alternate-Mark Inversion (AMI) "code" is a simple baseband PAM linear modulation scheme designed to transmit over channels that do not pass DC [3]. The elementary waveform used is the simple rectangular pulse $\psi(t) = p_{T_s}(t)$. In order to create a spectral zero in the signal PSD, the modulator introduces correlation in the symbol sequence. The symbol sequence $\{a_n\}$ is produced by a finite-state machine whose state diagram is shown in Fig. **??**.

The modulator input is the binary sequence $\{b_n\}$. The modulator state space is $\{0, 1\}$, and the state equation is given by

$$\sigma_{n+1} = \sigma_n + b_n \mod 2$$

where σ_n denotes the modulator state. The modulator output alphabet is the ternary alphabet $\{-A, 0, A\}$, and the output equation is given by

$$a_n = \begin{cases} -A & \text{if } \sigma_n = 1, b_n = 1 \\ A & \text{if } \sigma_n = 0, b_n = 1 \\ 0 & \text{if } b_n = 0 \end{cases}$$

The input symbols b_n are binary, i.i.d. and equiprobable. Then, the state sequence is a Markov chain with transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The state *stationary probability distribution* [3] is the solution of the system

$$\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$$

with the constraints that all elements of the vector $\boldsymbol{\pi}$ are non-negative and sum to 1. In this case, it is immediate to se that $\boldsymbol{\pi} = (1/2, 1/2)$ is the stationary distribution. Then, there is 1/2 probability that $a_n = 0$, 1/4 that $a_n = +A$ and 1/4 that $a_n = -A$. We have to compute μ_a and $\sigma_a[m]$. For the mean we get obviously $\mu_a = 0$. In order to calculate $\sigma_a[m]$, we consider separately $m = 0, 1, \ldots$ (notice that $\sigma_a[-m] = \sigma_a[m]$, since $\{a_n\}$ is a real sequence). For $m = 0$ we have $\sigma_a[0] = E[|a_n|^2] = A^2/2$. For $m = 1$, we have

$$\begin{aligned} \sigma_a[1] &= E[a_n a_{n-1}] \\ &= \frac{1}{4}E[a_n|a_{n-1} = +A] - \frac{1}{4}E[a_n|a_{n-1} = -A]] \\ &= -\frac{A^2}{8} - \frac{A^2}{8} = -\frac{A^2}{4} \end{aligned}$$

For all $m > 1$, $\sigma_a[m] = 0$ since the probability of an even number of ones between time n and $n - m$ is equal to the probability of an odd number of ones.

By using the result obtained before, we get

$$\Sigma_a(\lambda) = \sum_m \sigma_a[m]e^{-j2\pi\lambda m} = \frac{A^2}{2}(1 - \cos(2\pi\lambda))$$

We have also

$$\Psi(f) = T_s \mathrm{sinc}(fT_s)$$

Finally, the power spectrum of the AMI code is given by

$$S_x(f) = \frac{A^2 T_s}{2}\mathrm{sinc}(fT_s)^2(1 - \cos(2\pi f T_s))$$

It can be noticed that $S_x(0) = 0$, therefore, even if the channel does not pass DC, it will not cause distortion. ◊

3.2 Symbol-by-symbol detection and ISI

Consider again the transmission of a signal sequence based on the elementary pulse $\psi(t)$ and on the symbol alphabet $\mathcal{A}$ with transmitted signal

$$s(t; \mathbf{a}) = \sum_{n=0}^{N-1} a_n \psi(t - nT_s) \tag{3.5}$$

and channel models given as in Fig. **??** or in Fig. **??** depending on what type of matched filter is used. For the sake of clarity, we shall write again the equivalent discrete-time channel outputs corresponding to the MF and WMF cases. These are given respectively by

$$y_n = \sum_{\ell=-L}^{L} g_\ell a_{n-\ell} + v_n$$

where v_n has autocorelation function $N_0 g_n$ and w_n has autocorrelation function $N_0 \delta_n$.

We can rewrite the first line of (3.2) by putting in evidence three components: the useful term, the intersymbol interference (ISI) and the noise term. We obtain

$$y_n = g_0 a_n + \sum_{\ell \neq 0} g_\ell a_{n-\ell} + v_n \tag{3.6}$$

In particular, the ISI term contains the contribution of the symbols transmitted in the other symbol intervals that create interference for the detection of the symbol transmitted in the n-th interval.

A generally suboptimal detection strategy consists of making a decision for symbol a_n based uniquely on the observation y_n. Next, we investigate the effect of ISI on the performance of the suboptimal symbol-by-symbol detector and a condition, known as the Nyquist criterion, for which symbol-by-symbol detection is indeed optimal. For the sake of simplicity we restrict our treatment to the simple case of antipodal 2-PAM signals, assuming g_ℓ real. We consider the matched filter output sample y_n and the threshold detector based on the rule

$$\widehat{a}_n = \mathrm{sgn}\{y_n\}$$

which is the optimal ML decision rule in the absence of ISI. Without loss of generality, we can neglect the index n and rewrite (3.6) as

$$y = g_0 a + I + v \tag{3.7}$$

define the ISI random variable $I = \sum_{\ell \neq 0} a_\ell g_\ell$ where the a_ℓ's are i.i.d. random variables taking on values in $\{-1, +1\}$ with uniform probability, and $v \sim \mathcal{N}_{\mathbb{C}}(0, N_0 g_0)$.

The error probability $P(e) = \Pr(\mathrm{sgn}\{y\} \neq \mathrm{sgn}\{a\})$ can be evaluated as follows. First, we condition with respect to the ISI variable, and obtain

$$P(e|I) = \frac{1}{2}\left(Q\left(\frac{|g_0| + I}{\sqrt{|g_0| N_0/2}}\right) + Q\left(\frac{|g_0| - I}{\sqrt{|g_0| N_0/2}}\right)\right)$$

Then, we average with respect to I. Since I is symmetrically distributed, I and $-I$ are identically distributed and we obtain

$$P(e) = E_I \left[Q \left(\frac{|g_0| + I}{\sqrt{|g_0| N_0/2}} \right) \right]$$

Since there are at most $2L$ interfering symbols, I takes on at most 2^{2L} values and we can write

$$P(e) = \frac{1}{2^{2L}} \sum_{\mathbf{b} \in \{\pm 1\}^{2L}} Q \left(\frac{|g_0| + \sum_{\ell=-L, \ell \neq 0}^{L} b_\ell |g_\ell|}{\sqrt{|g_0| N_0/2}} \right)$$

A simple upper bound on $P(e)$ is obtained by considering the worst-case ISI, when all interfering symbols combine constructively with opposite sign with respect to the useful signal component. We obtain

$$\begin{aligned} P(e) &\leq Q \left(\frac{|g_0| - \sum_{\ell=-L, \ell \neq 0}^{L} |g_\ell|}{\sqrt{|g_0| N_0/2}} \right) \\ &= Q \left(\sqrt{\frac{2|g_0|(1 - D_p)^2}{N_0}} \right) \end{aligned} \tag{3.8}$$

where we define the *peak-distortion*

$$D_p = \frac{\sum_{\ell \neq 0} |g_\ell|}{|g_0|} \tag{3.9}$$

The above bound is meaningful if $D_p < 1$. This condition is referred to as the "open-eye" condition. The reason for this "hystorical" terminology is the following. The continous-time received signal in the absence of noise, after the continuous-time matched filter, can be observed modulo $2T_s$ on the display of an oscilloscope. This corresponds to wrapping the received signal with period $2T_s$. The resulting diagram looks like an eye, and it is called *eye diagram*. The thickness of the superposition of all trajectories provides a visualization of the peak distortion. If $D_p > 1$ (closed-eye condition), the system makes decision errors even in the absence of noise (systematic errors). If $D_p < 1$ (open-eye condition) the error probability can be made small by increasing the transmitted signal power. Clearly, any reasonable system based on symbol-by-symbol detection should work in open eye conditions.

Another method for simple evaluation of the error probability in the presence of ISI consists of approximating I with a Gaussian random variable with mean zero and variance

$$\sigma_I^2 = E[|I|^2] = \sum_{\ell \neq 0} |g_\ell|^2$$

In this case, we obtain the approximation

$$P(e) \approx Q \left(\sqrt{\frac{|g_0|^2}{\frac{N_0}{2} g_0 + (\sum_{\ell \neq 0} |g_\ell|^2)}} \right)$$

The above approximation yields meaningful results in open-eye conditions and low or moderate SNR. For large SNR, this approximation is meaningless since, as $N_0 \to 0$, the true error probability in open-eye conditions vanishes, while the above Gaussian approximation converges to a positive value.

A lower bound on the error probability of any detector (including the ML sequence detector) is provided by the following Matched Filter Bound (MFB) [3].

Matched filter bound. Let $E_s = g_0$ be the average received energy per symbol coresponding to the useful term only. Then, the symbol-error probability resulting from any detector is always larger or equal to the symbol-error probability resulting from a ML detector at the output of the matched filter in the assumption that ISI is not present (i.e., assuming that only one symbol at time 0 is transmitted). This probability, referred to as the MFB, is given by the error probability of the underlying constellation $\mathcal{A}$, with SNR equal to E_s/N_0. □

3.2.1 Mismatched symbol-by-symbol receiver

In some cases, either because of complexity issues or because non-ideal implementation (e.g., the receiver local clock is not alligned with the transmitter clock, or some component in the RF receiving chain introduced uncompensated linear or non-linear distortion), the receiver is *mismatched* with respect to the transmitted signal. This means that the matched filter $\psi(-t)^*$ in Fig. **??** is replaced by some filter $h(t)$, mismatched with respect to $\psi(t)$. The error probability analysis carried out for the standard matched filter case carries out almost unchanged for the mismatched case too.

Assume that the receiver takes samples at rate $1/T_s$ at the output of the mismatched filter. The corresponding discrete-time channel is given by

$$y_n = \sum_{\ell=-L}^{L} \widetilde{g}_\ell a_{n-\ell} + v_n \tag{3.10}$$

where

1. $\widetilde{g}_\ell = \int h(\ell T_s - \tau)\psi(\tau)d\tau$;
2. v_n has autocorelation function $N_0 \phi_h(nT_s)$, where

$$\phi_h(t) = \int h(\tau)h(\tau - t)^* d\tau$$

is the autocorrelation function of $h(t)$.

Notice that in this case the useful term coefficient $\widetilde{g}_0$ is not necessarily real and positive, therefore, some phase compensation might be needed. After compensating for the phase of $\widetilde{g}_0$ and by using the fact that the statistics of the noise is invariant with respect to phase rotations, we can write the n-th receiver sample as

$$\begin{aligned} y_n &= |\widetilde{g}_0| a_n + \sum_{\ell \neq 0} \frac{\widetilde{g}_0^* \widetilde{g}_\ell}{|\widetilde{g}_0|} a_{n-\ell} + v_n \\ &= \beta_0 a_n + \sum_{\ell \neq 0} \beta_\ell a_{n-\ell} + v_n \end{aligned} \tag{3.11}$$

where the coefficient β_0 is real and positive, and the coefficients β_ℓ for $\ell \neq 0$ are the so called *leakage coefficients*.

We consider again the simple threshold detector for 2-PAM signals based on the rule

$$\widehat{a}_n = \mathrm{sgn}\{y_n\}$$

Without loss of generality, we can neglect the index n and rewrite (3.11) as

$$y = \beta_0 a + I + v \tag{3.12}$$

define the ISI random variable $I = \sum_{\ell \neq 0} a_\ell \beta_\ell$ where the a_ℓ's are i.i.d. random variables taking on values in $\{-1, +1\}$ with uniform probability, and $v \sim \mathcal{N}_{\mathbb{C}}(0, N_0 \phi_h(0))$. Assume that the number of non-zero leakage coefficients is finite, and denoted by L'. The exact probability of error can be written as

$$P(e) = \frac{1}{2^{L'}} \sum_{\mathbf{b} \in \{\pm 1\}^{L'}} Q\left(\frac{\beta_0 + \sum_{\ell=1}^{L'} b_\ell |\beta_\ell|}{\sqrt{\phi_h(0) N_0/2}}\right)$$

The peak distortion upper bound is given by

$$\begin{aligned} P(e) &\leq Q\left(\frac{\beta_0 - \sum_{\ell=1}^{L'} |\beta_\ell|}{\sqrt{\phi_h(0) N_0/2}}\right) \\ &= Q\left(\sqrt{\frac{2|\beta_0|^2 (1 - D_p)^2}{\phi_h(0) N_0}}\right) \end{aligned} \tag{3.13}$$

where

$$D_p = \frac{\sum_{\ell=1}^{L'} |\beta_\ell|}{\beta_0} \tag{3.14}$$

The corresponding Gaussian approximation is given by

$$P(e) \approx Q\left(\sqrt{\frac{|\beta_0|^2}{\frac{N_0}{2}\phi_h(0) + \sum_{\ell=1}^{L'} |\beta_\ell|^2}}\right)$$

Again, the above approximation yields meaningful results only in open-eye conditions and low or moderate SNR.

Fig. **??** and **??** show the eye diagrams corresponding to a mismatched system where $\psi(t)$ is a rectangular pulse of duration $T_s = 1$ and the receiving filter $h(t)$ is a first-order (RC) low-pass filter with time constant large and small (with respect to T_s), respectively.

3.2.2 The Nyquist condition

In this section we obtain a condition on $\psi(t)$ known as *Nyquist condition* such that the sequence of sampled matched filter outputs is ISI-free (i.e., for which $D_p = 0$). This also implies that the noise sequence at the output of the sampled matched filter is white, and that symbol-by-symbol detection is optimal, i.e., it coincides with ML sequence detection.

In order to have $D_p = 0$, the following condition must be satisfied

$$g_\ell = \begin{cases} E_s \neq 0 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

Cosider the discrete-time Fourier transform $\mathcal{F}[g_\ell] = \mathcal{G}(\lambda)$ given in (3.51). Then, the zero-ISI condition in the frequency domain can be written as

$$\mathcal{G}(\lambda) = E_s = \text{constant}$$

This relation can be stated in terms of the continous-time Fourier transform $G(f)$ by using the spectral folding relation (1.7) as

$$\mathcal{G}(\lambda) = \frac{1}{T_s} \sum_m G\left(\frac{\lambda + m}{T_s}\right) = E_s$$

Since the g_ℓ are samples taken at ℓT_s of the autocorrelation function $\phi_\psi(t)$, we have that $G(f) = \mathcal{F}[\phi_\psi(t)] = |\Psi(f)|^2$, where $\Psi(f) = \mathcal{F}[\psi(t)]$. Finally, the condition for zero ISI is given in the continuous-time domain by

$$\sum_m \left|\Psi\left(f + \frac{m}{T_s}\right)\right|^2 = \text{constant} \tag{3.15}$$

The procedure for the design of the elementary waveform filter $\psi(t)$ (and the corresponding matched filter $\psi(-t)^*$) satisfying the above condition is the following:

1. Choose an energy spectrum $G(f)$ (i.e., a non-negative symmetric function of frequency) such that $\sum_m G(f + m/T_s)$ =constant.
2. Let $\Psi(f) = \sqrt{G(f)}$ and $\psi(t) = \mathcal{F}^{-1}[\Psi(f)]$.
3. By using truncation and introducing delay, approximate the resulting $\psi(t)$ and matched filter $\psi(\Delta T_s - t)^*$ are by physically realizable (i.e., causal) waveforms.

It is evident that with this choice the error probability of the threshold detector is the same as in the case of a single pulse, since the ISI is zero and we are using a matched filter receiver.

Example: Nyquist bandwidth and raised-cosine pulses. In order to illustrate the above procedure, consider the choice $G(f) = T_s \Pi(T_s f)$, where $\Pi(f) = 1$ for $f \in [-1/2, 1/2]$ and 0 elsewhere. We have $\Psi(f) = \sqrt{T_s}\Pi(T_s f)$ and

$$\psi(t) = \frac{1}{\sqrt{T_s}} \text{sinc}(t/T_s)$$

This is strctly bandlimited in $[-1/(2T_s), 1/(2T_s)]$ and has infinite support in time. This means that $\psi(t)$ is not physically realizable. In reality, all strictly bandlimited waveforms are truncated in time and delayed, in order to make them causal. Actual system design seeks the best trade-off between the length of the transmit and receiver filters and the spourious sidelobes that are generated by truncation. The bandwidth $W = 1/T_s$ is the minimum bandwidth necessary to satisfy the Nyquist criterion (Nyquist bandwidth) for the symbol rate $R_s = 1/T_s$.

Normally, real systems use larger bandwidth $W = (1 + \alpha)/T_s$, where α is called **excess bandwidth** or **roll-off factor**. This is because in the case of $\alpha = 0$ a little error in the sampling time causes closed-eye condition. In order to see this, let $D_p(\epsilon)$ be the peak distortion when sampling with error ϵ, i.e., when the output of the matched filter is sampled at epochs $(n + 1)T_s + \epsilon$. Then, the peak distortion resulting from the rectangular-spectrum pulse is

$$D_p(\epsilon) = \sum_{m \neq 0} |\text{sinc}(\epsilon/T_s + m)| = \begin{cases} 0 & \epsilon = 0 \\ \infty & \epsilon \neq 0 \end{cases}$$

because $|\text{sinc}(\epsilon/T_s + m)| = O(1/m)$ decreases too slowly for all $\epsilon \neq 0$.

A pulse shape that has found wide use in digital transmission is the so called **raised-cosine** (RC) [3], whose spectum is given by

$$G_{\mathrm{RC}}(f) = \begin{cases} T_s & 0 \leq |f| \leq (1-\alpha)/(2T_s) \\ \frac{T_s}{2}\left[1 - \sin\left(\frac{\pi T_s}{\alpha}(f - 1/(2T_s))\right)\right] & (1-\alpha)/(2T_s) \leq |f| \leq (1+\alpha)/(2T_s) \end{cases}$$

Its Fourier inverse transform is

$$g(t) = \mathrm{sinc}(t/T_s)\frac{\cos(\alpha\pi t/T_s)}{1 - 4\alpha^2 t^2/T_s^2}$$

Since this decreases as $1/|t|^3$ for $|t| \to \infty$, the peak distortion is always limited even if there is a little timing error. The pulse-shaping filter obtained by inverse Fourier transform of $\sqrt{G_{\mathrm{RC}}(f)}$ is referred to as root-raised-cosine (RRC) pulse.

Fig. **??** shows a RRC pulse for roll-off $\alpha = 0.5$. Fig. **??** shows the eye diagram resulting from the RRC pulse of Fig. **??**.

3.3 Spectral efficiency

From the "WT" theorem we know that the number of independent symbols (dimensions) that can be transmited over a bandwidth W and time T is $N \approx WT$, for large time-frequency product WT. Consider a digitally modulated signal that transmits η bits per dimension (for example, for a M-QAM constellation we have $\eta = \log_2 M$ and for an orthogonal M-FSK signal set we have $\eta = \frac{\log_2 M}{M}$. The quantity η is referred to as the "spectral efficiency" of the digital modulation, and it is measured in bit/s/Hz, since the number of dimensions N is roughly given by the number of s$\times$Hz of the product WT.

For a given bit-rate R_b (bit/s), and bandwidth W, we have

$$\eta = \frac{R_b}{W} \tag{3.16}$$

Different modulation schemes can be compared in the efficiency plane. We fix a desired (target) bit-error probability $P_b(e)$. Then, each modulation scheme can be represented as a point of coordinates $(E_b/N_0, \eta)$, where E_b/N_0 is the SNR necessary to achieve the target $P_b(e)$ and η is the corresponding spectral efficiency.

Notice that the product $\eta E_b/N_0$ yields immediately the ratio between the transmit signal power and the total noise power, in fact we have

$$\frac{P}{N} = \frac{E_b R_b}{N_0 W} = \frac{E_b}{N_0}\eta$$

Then, η can be seen also as the factor relating the signal-to-noise power ratio and the quantity E_b/N_0.

Fig. **??** shows some well-known modulation schemes for $P_b(e) = 10^{-5}$ on the spectral efficiency plane. The solid line represent the maximum achievable spectral efficiency for any signal set on AWGN with arbitrarily small error probability, given by Shannon's capacity formula [2]

$$C/W = \log_2(1 + (C/W)E_b/N_0)$$

From this formula, by letting $E_b/N_0 = (2^{C/W} - 1)/(C/W))$ we obtain the minimum E_b/N_0 for which reliable communication is possible as the limit

$$(E_b/N_0)_{\min} = \lim_{C/W \to 0} \frac{2^{C/W} - 1}{C/W} = \log 2 \approx -1.6 \text{ dB}$$

3.4 Orthogonal Frequency Division Multiplexing(OFDM)

OFDM is a quasi-memoryless transmission strategy for transmission over bandlimited dispersive channels. It allows for high data rates over wideband wired (e.g. ADSL) and wireless (e.g. WIFI, WiMax) communication media without the need for sophisticated equalization techniques. It is the modulation (and multiple-access) method chosen for the evolution of the UMTS downlink channel. Consider the general modulated signal with symbol rate $1/T$ given by

$$s(t) = \sum_k s_{u_k}(t - kT) \tag{3.17}$$

where u_k belongs to some N-dimensional signal set in the field of complex vectors, $\mathcal{S} \in \mathbb{C}^N$. Here we transmit the waveforms for each symbol spaced by T seconds. If the waveforms last more than T seconds, then they overlap in time and, therefore, they are not necessarily orthogonal at transmission. The information signal, $s(t)$, is transmitted over a dispersive AWGN channel $h(t)$ yielding the complex baseband signal at the receiver

$$r(t) = s(t) * h(t) + z(t)$$

3.4.1 "Bandlimited" Signal Generation and Receiver Sampling

We briefly describe here the notion of generation and reception of bandlimited signals. The information-bearing signal for a particular OFDM symbol, u_k, is generated using Nyquist interpolation as

$$s_{u_k}(t) = \sum_{n=0}^{N_s-1} s_{u_k}[n] \text{sinc}\left(W\left(t - \frac{n}{W}\right)\right) \tag{3.18}$$

where $s_{u_k,n} = s_{u_k}(n/W)$ are the samples of $s_{u_k}(t)$ with sampling rate W Hz, and N_s is the number of samples needed to represent the information signal (note that $N_s > N$). We note that the samples for negative $n < 0$ and $n \geq N_s$ are zero in writing (3.18). We see immediately that the symbols are not necessarily orthogonal when shifted in time by $T = N_s/W$ s as in (3.17). To get an idea of the degradation in assuming orthogonality consider the energy of the signal outside $[-\frac{\delta N_s}{W}, T + \frac{\delta N_s}{W})$ given by

$$\begin{aligned}
E_{\text{ISI}} &= \sum_{n=0}^{N_s-1} \text{E}|s_{u_k}[n]|^2 \left(\int_{T+\frac{\delta N_s}{W}}^{\infty} \left(\text{sinc}\left(\pi W\left(t - \frac{n}{W}\right)\right)\right)^2 dt + \int_{-\infty}^{-\frac{\delta N_s}{W}} \left(\text{sinc}\left(\pi W\left(t - \frac{n}{W}\right)\right)\right)^2 dt \right) \\
&= 2 \sum_{n=0}^{N_s-1} \text{E}|s_{u_k}[n]|^2 \int_{T+\frac{\delta N_s}{W}}^{\infty} \left(\frac{\sin\left(\pi W\left(t - \frac{n}{W}\right)\right)}{\pi W\left(t - \frac{n}{W}\right)} \right)^2 dt \\
&\leq \sum_{n=0}^{N_s-1} \text{E}|s_{u_k}[n]|^2 \frac{1}{\pi^2 W^2 \left(T + \frac{\delta N_s}{W} - \frac{n}{W}\right)}
\end{aligned} \tag{3.19}$$

The ratio of total signal energy to that outside $[0, T + \frac{N_s}{W})$ is therefore lower-bounded by

$$\gamma(N_s, \delta) = \frac{N_s}{\sum_{n=0}^{N_s-1} \frac{1}{\pi^2((1+\delta)N_s - n)}} \tag{3.20}$$

if $\mathrm{E}|s_{u_k,n}|^2 = E_s \forall k, n$. For the sake of numerical justification, consider that for $\delta = 0.01$, with $N_s = 64$ we have a ratio of 22dB at transmission, and for $N_s = 1024$, 33 dB. In practice this means that for signal-to-noise ratios at the receiver significantly less than these transmitted energy ratios, the time-shifted modulation signals in (3.17) can be considered to be orthogonal from a practical perspective, since the amount of "distortion" energy arising from intersymbol interference is significantly less than the energy of thermal noise.

Turning now to reception, if $s(t)$ is bandlimited to $[-W/2, W/2]$ Hz we may work directly with the samples of $r(t)$ sampled at W samples/s yielding

$$r\left(\frac{n}{W}\right) = s\left(\frac{n}{W}\right) * h'(n) + z\left(\frac{n}{W}\right) \tag{3.21}$$

with

$$\begin{aligned} h'[n] &= \int_{-W/2}^{W/2} H(f) e^{-j\frac{2*\pi f k}{W}} df \\ &= \int_{-\infty}^{\infty} h(t) \frac{\sin\left(\pi W \left(t - \frac{n}{W}\right)\right)}{\pi W \left(t - \frac{n}{W}\right)} dt \end{aligned}$$

We note that the latter is just the projection of the channel reponse on the signal sub-space (i.e. the space of bandlimited signals). Another way of interpreting this is that the relevant components of the channel are those which lie in the signal sub-space.

In practice, the duration of the projected channel ($h'(n)$) will be limited to a small number of samples, say L', in the sense that the great majority of its total energy will be contained in these samples. In reality the duration of $h'(n)$ is infinite, as for any bandlimited signal.

3.4.2 Cyclic Extension

In order to assure quasi-memoryless behaviour (in the sense of the orthogonality arguments in the previous section), OFDM systems insert a *cyclic prefix* (see below) of length L samples at the beginning of each symbol period. L should at least be equal to the worst-case channel duration. Note that the channel duration should include sufficient padding to guarantee quasi-orthogonality at reception. Suppose that the symbol contains N information samples and L samples are required for the cyclic prefix, we have that $T = \frac{N+L}{W}$ (i.e. $N_s = N + L$). If we denote the $N + L$ samples of the k^{th} symbol by $\mathbf{s}_k^{N+L}$, the samples of the cyclic prefix are chosen such that $s_k^{N+L}[i \bmod N] = s_k^{N+L}[i], i = 0, 1, \cdots, N + L - 1$. As an example $N = 48$ and $L = 16$ in 802.11a/g. The information-bearing component, $\mathbf{s}_k^N$ is in positions $\{L, L+1, \cdots, N+L-1\}$, or equivalently $\mathbf{s}_k^{N+L} = [\mathbf{s}_{k,p}^L | \mathbf{s}_k^N]$, where $\mathbf{s}_{k,p}^L$ is the cyclic prefix.

Typically, in the receiver, the samples corresponding to the cyclic prefix at the beginning of each OFDM-symbol are ignored since they contain interference from the previous symbol. This is shown in Figure 3.1.The efficiency of OFDM is therefore related to N and L.

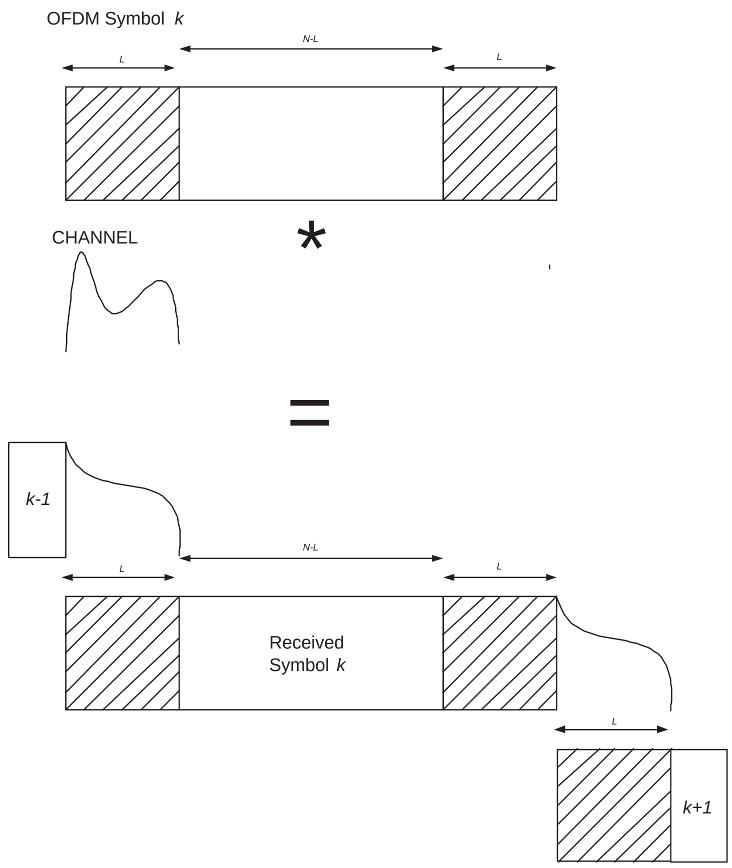


Figure 3.1: OFDM symbol overlap

3.4.3 Circulant Representation of OFDM

We will now develop the circulant representation for the convolution with the channel. The received signal in a particular symbol epoch (excluding the samples of the cyclic prefix) is given by

$$r[n] \approx \sum_{l=0}^{L-1} h'[l] s_{u_k}[n-l] + z[n], \quad n = L + k(N+L), \cdots, (k+1)(N+L) - 1$$

We now write this as

$$r[n + k(N+L)] = \sum_{l=0}^{L-1} h'[l] s_k^{N+L}[n-l] + z[n + k(N+L)], \quad n = L, \cdots, N+L-1,$$

or

$$r[n+L+k(N+L)] = \sum_{l=0}^{L-1} h'[l] s_k^N[(n-l) \mod N] + z[n+L+k(N+L)], \quad n = 0, \cdots, N-1. \quad (3.22)$$

We immediately realize that (3.22) is a *circular discrete-time convolution* which can be written as

$$\mathbf{r}_k^N = \mathbf{h}^N \circledast \mathbf{s}_k^N + \mathbf{z}_k^N$$

where $\circledast$ corresponds to the above circular convolution, and $\mathbf{h}^N = [h'(0) h'(1) \cdots h'(L-1) 0 \cdots 0]^t$. We can write equivalently in the Fourier domain,

$$\mathbf{R}_k^N = \mathrm{DFT}(\mathbf{r}_k^N) = \mathbf{H}^N \mathbf{S}_k^N + \mathbf{Z}_k^N$$

where the N-dimensional DFT operator for a vector $\mathbf{x}^N$ is given by

$$X^N[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x^N[n] e^{-\frac{2\pi j k n}{N}}$$

and $\mathbf{H}^N = \mathrm{diag}(\mathrm{DFT}(\mathbf{h}^N))$ is a diagonal matrix whose diagonal components contain the DFT of the disrete-time channel response. This follows from the property that a circular convolution is deconvolved by a discrete Fourier transform operator.

3.4.4 Transceiver Structures based on the FFT/IFFT

Through the use of the DFT, OFDM symbol ($\mathbf{S}_k^N$) can be created in the frequency domain and transfered to the time domain via the IDFT. The n^{th} component of the k^{th} symbol, $S_k^N[n]$, is typically a symbol from a traditional alphabet (e.g. QPSK, 16-QAM, etc.), $\mathcal{S}$, so that the overall symbol is simply and N-dimensional extension of the basic symbol alphabet, $\mathcal{S}^N$. The transmitted signal is simply $\mathbf{s}_k^{N+L} = [\mathbf{s}_{k,p}^L | \mathbf{s}_k^N]$ where $\mathbf{s}_k^N = \mathrm{IDFT}(\mathbf{S}_k^N)$ and $\mathbf{s}_{k,p}^L$ is the cyclic prefix corresponding to the last L samples of $\mathbf{s}_k^N$.

As we saw in the last section, through the use of the cyclic prefix, the dispersive channel is deconvolved and results in a complex channel gain for each element of the parallel channel decomposition ($H^N[n]$). The receiver is therefore simply a DFT to yield the parallel channel decomposition in the frequency domain and then a classical receiver for each channel. The OFDM transceiver based on the FFT/IFFT is shown in Figure 3.2.

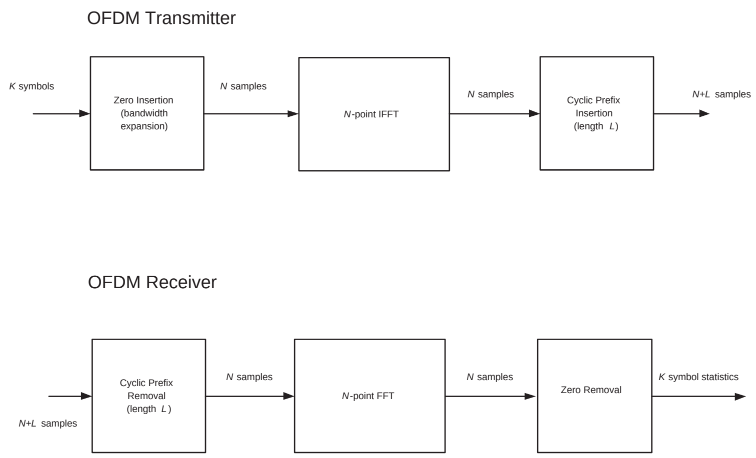


Figure 3.2: OFDM transceiver

In order to control the spectral characteristics of the output waveform, a certain number of carriers in $\mathbf{S}_k^N$ are used to transmit information and the rest are set to zero. This is a type of pulse-shaping analagous to Nyquist pulses, but now created in the frequency-domain. As an example, in 802.11a/g, the number of carriers, N, is 64 and the number of non-zero carriers is 48. Very often (especially in radio systems), the carriers around DC are nulled out along with a portion in the high-end of the spectrum. An example is shown in Figure 3.3.

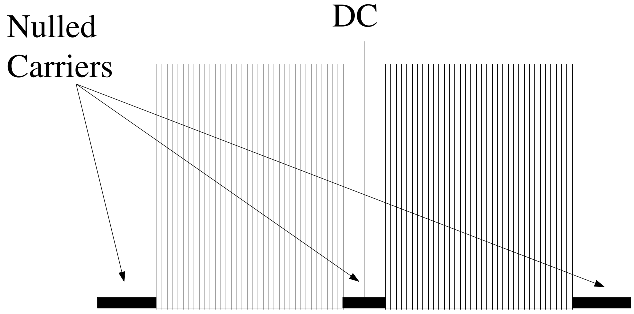


Figure 3.3: Spectral Shaping with OFDM

3.4.5 Data Detection

Once the channel is estimated (see next section), the received signal must be processed prior to error decoding. If we consider a maximum-likelihood receiver with perfect channel state information (i.e. knowledge of $\mathbf{H}^N$, we have that the likelihood function for the received signal in the frequency-domain is

$$p(\mathbf{R}_k^N|\mathbf{S}_k^N) = \frac{1}{(\pi N_0)^N}\exp\left(-\frac{1}{N_0}||\mathbf{R}_k^N - \mathbf{H}^N\mathbf{S}_k^N||^2\right) \tag{3.23}$$

On a per-carrier basis we have

$$p(R_k^N[n]|S_k^N[n]) = \frac{1}{(\pi N_0)}\exp\left(-\frac{1}{N_0}|R_k^N[n] - H^N[n]S_k^N[n]|^2\right) \tag{3.24}$$

so that the detection rule for the n^{th} carrier of the k^{th} OFDM symbol is

$$\hat{S}_k^N[n] = \underset{S_k^N[n]\in\mathcal{S}}{\text{argmax}}\, 2\text{Re}\left(H^N[n]^* S_k^N[n]^* R_k^N[n]\right) - |H^N[n]|^2 |S_k^N[n]|^2 \tag{3.25}$$

The only difference with respect to isolated symbol transmission considered in the last chapter is the weighting of the symbol energy by the channel response on each carrier. Using a union-bound the performance can be bounded similarly as

$$P_e \leq (|\mathcal{S}| - 1) \sum_{n=0}^{N-1} Q\left(\sqrt{\log_2 |\mathcal{S}| \tilde{d}_{\min}^2(\mathcal{S}) |H^N[n]|^2 \frac{E_b}{N_0}}\right) \tag{3.26}$$

3.4.6 Channel Estimation

Determination of the $\mathbf{H}$ at the receiver is known as *channel estimation*. One common procedure for performing channel estimation is through the use of a pilot or training sequence, which is a sequence of symbols that the receiver uses to deduce (or deconvolve) the channel response. Suppose that the transmitter periodically sends a whole OFDM symbol of pilots to the receiver for the purpose of channel estimation. Call this $\mathbf{S}_{\text{p}}$. The receiver sees $R_{\text{I},k} = \tilde{H}_k S_{\text{p},k} + Z_k$. Without any *a priori* information on the statistical properties of $\tilde{\mathbf{H}}$, the least-squares estimate of $\tilde{\mathbf{H}}$, or $\hat{\mathbf{H}}$, assuming constant-modulus pilot information is

$$\hat{H}_k = S_{\text{p},k} R_{\text{I},k} \tag{3.27}$$

To exploit the *a priori* time-limited structure of the channel response (i.e. limited to N_c samples) some interpolation can be performed. This can either be done in the frequency-domain or in the time-domain via the IDFT. In the latter approach the channel estimation procedure becomes

$$\hat{\mathbf{H}} = \text{DFT}(\text{IDFT}(\mathbf{S}_{\text{p}} \otimes \mathbf{R}_{\text{I}}) \otimes \mathbf{u}_{N_c}) \tag{3.28}$$

where the operator $\otimes$ refers to componentwise product of the vectors. The vector $\mathbf{u}_{N_c}$ is a perfect time-windowing function which performs the interpolation defined as

$$u_{N_c,n} = \begin{cases} 1, & (N_d - N_c - l) \leq n \leq N_d - 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.29}$$

From intuition, the channel is time-limited to $(N_d - N_c - l) \leq n \leq N_d - 1$ (remember the shift due to the cyclic extension!), so we can disgard any noisy samples outside of this window. Recall that a window is a sinc-function (here aliased because of the finite block-length) in the frequency-domain which is a canonical interpolation filter.

3.5 ML coherent signal sequence detection in AWGN

In this section we show how the coherent signal detection in AWGN is applied to the detection of signal sequences in an efficient way. The necessity of an efficient algorithm to implement signal setection is easily seen through the following simple example. Consider the case where the modulated signal sequences are binary antipodal, with i.i.d. symbols. The modulator signal set is formed by the 2^N possible signal sequences $s(t;\mathbf{a}) = \sum_{n=0}^{N-1} a_n \psi(t - nT_s)$, for $a_n \in \{-1, +1\}$. The received signal is given by $r(t) = s(t;\mathbf{a}) + \nu(t)$.

A "brute-force" implementation of the ML detection rule (minimum distance), that computes exhaustively the distance between the received signal and all possible transmitted signals, would require the computation of 2^N distances. This becomes immediately impossible for practical block lengths (N can be easily larger than 100).

Fortunately, there exists a general algorithm known as the *Viterbi Algorithm* (VA) [**?**] that solves a class of maximization problems to which the ML sequence detection rule belongs to. We shall present first the Viterbi algorithm in general terms, and then apply it to the problem of ML sequence detection in two different (but equivalent) forms, known as the Ungerboeck and the Forney forms.

3.5.1 The Viterbi algorithm

A finite-state dynamical system is defined by a sequence of finite sets $\Sigma_0, \ldots, \Sigma_N$, and a set of possible *trajectories* given by sequences of states $(\sigma_0, \sigma_1, \ldots, \sigma_N)$ such that $\sigma_n \in \Sigma_n$. We assume that the system state depends on an input sequence $(a_0, \ldots, a_{N-1})$ and evolves according to a given state evolution law, expressed by

$$\sigma_{n+1} = f_n(a_n, \sigma_n) \tag{3.30}$$

A trellis is a graph $\mathcal{T} = (\Sigma, \mathcal{E})$, where the set of nodes is $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_N$ and the set of edges $\mathcal{E}$ includes only edges $e = (\sigma, \sigma')$ such that $\sigma \in \Sigma_n$, $\sigma'_{n+1} \in \Sigma_{n+1}$ for some $0 \leq n \leq N-1$. A path in the trellis $\mathcal{T}$ is a sequence of N connected edges containing exactly one state for every subset Σ_n. We say that a trellis $\mathcal{T}$ is a representation of the dynamical system if the paths are in one-to-one correspondence with the state trajectories of the system.

The bipartite subgraph containing Σ_n, Σ_{n+1} and all edges $e = (\sigma, \sigma')$ with $\sigma \in \Sigma_n$ and $\sigma' \in \Sigma_{n+1}$ is called n-th *trellis section*, and it is denoted by $\mathcal{T}_n$. Then, $\mathcal{T} = \bigcup_{n=0}^{N-1} \mathcal{T}_n$.

Assume that some reward, or "branch metric", $m_n(\sigma, \sigma')$ is associated to every state transition $(\sigma, \sigma') \in \mathcal{T}_n$. A classical problem in control is to find the input sequence $\widehat{\mathbf{a}}$ for a given initial state $\sigma_0 \in \Sigma_0$ that maximizes the total metric

$$\mathcal{M}_N = \sum_{n=0}^{N-1} m_n$$

over all possible trajectories.

For a fixed initial state σ_0, let $\mathcal{C}$ indicate the set of all state trajectories and let $\mathcal{C}_n(\sigma)$ be the subset of all trajectories that pass through state σ at time n. Because of the one-to-one correspondence between state trajectories, trellis paths and input sequences, we shall not make explicit distinction. Let $\mathbf{a} \in \mathcal{C}_n(\sigma)$. Then, we define the "past" and the "future" of $\mathbf{a}$ as

$$\begin{aligned} \mathcal{P}_n(\mathbf{a}) &= (a_0, \ldots, a_{n-1}) \\ \mathcal{F}_n(\mathbf{a}) &= (a_n, \ldots, a_{N-1}) \end{aligned}$$

Clearly, for any $\mathbf{a}, \mathbf{a}' \in \mathcal{C}_n(\sigma)$, also $\mathbf{a}'' = (\mathcal{P}_n(\mathbf{a}), \mathcal{F}_n(\mathbf{a}'))$ and $\mathbf{a}''' = (\mathcal{P}_n(\mathbf{a}'), \mathcal{F}_n(\mathbf{a}))$ are paths in $\mathcal{C}_n(\sigma)$. In other words, we can concatenate any past with any future of two paths, and still obtain a valid path in the trellis.

Now, we fix n, σ and $\mathbf{a} \in \mathcal{C}_n(\sigma)$, and we write the corresponding metric as the sum of the contributions of its past and its future, i.e.,

$$\mathcal{M}_N(\mathbf{a}) = \sum_{i=0}^{n-1} m_i(a_i) + \sum_{i=n}^{N-1} m_i(a_i) \tag{3.31}$$

Assume that there exists $\mathbf{a}' \in \mathcal{C}_n(\sigma)$ such that

$$\sum_{i=0}^{n-1} m_i(a_i) < \sum_{i=0}^{n-1} m_i(a'_i) \tag{3.32}$$

Hence, there exists $\mathbf{a}'' = (\mathcal{P}_n(\mathbf{a}'), \mathcal{F}_n(\mathbf{a}))$ such that $\mathcal{M}_N(\mathbf{a}'') > \mathcal{M}_N(\mathbf{a})$. Then, if the condition (3.32) is verified, the path $\mathbf{a}$ can be eliminated from the maximization of the total metric without changing the final result.

By extending this reasoning, it follows that at every step n only at most $|\Sigma_n|$ paths should be kept in the list of possible candidates for the maximization. These candidates are called "survivors", and that the survivor $\mathbf{a}$ corresponding to state $\sigma \in \Sigma_n$ must satisfy the local maximum condition

$$\sum_{i=0}^{n-1} m_i(a_i) \geq \max_{\mathbf{a}' \in \mathcal{C}_n(\sigma)} \left\{ \sum_{i=0}^{n-1} m_i(a'_i) \right\} \tag{3.33}$$

Based on this idea, the VA eliminated from the list of possible candidates to achieve the maximum of the total metric all paths that do not meet the survivor condition, for all $n = 1, \ldots, N$.

We define the "path metric" at time n and state σ as the metric accumulated up to time n by the survivor path terminating at state σ, i.e.,

$$\mathcal{M}_n(\sigma) = \max_{\mathbf{a} \in \mathcal{C}_n(\sigma)} \sum_{i=0}^{n-1} m_i(a_i) \tag{3.34}$$

Then, the VA updates recursively the path metrics as follows.

1. Initialization: $\mathcal{M}_0(\sigma_0) = 0$.

2. Add Compare and Select (ACS) recursion: for $n = 1, 2, \ldots, N$, and for all states $\sigma \in \Sigma_n$, let

$$\mathcal{M}_n(\sigma) = \max_{\sigma' \in \Pi_n(\sigma)} \left\{ \mathcal{M}_{n-1}(\sigma') + m_{n-1}(a(\sigma', \sigma)) \right\} \tag{3.35}$$

   where $\Pi_n(\sigma)$ is the set of "parent" states of σ, i.e., the set of states $\sigma' \in \Sigma_{n-1}$ for which there exists a transition $(\sigma', \sigma) \in \mathcal{T}_{n-1}$, and where $a(\sigma', \sigma)$ is the input symbol corresponding to the state transition (σ', σ) in trellis section $\mathcal{T}_{n-1}$. The symbol $\widehat{a}$ achieving the maximum in (3.35) is stored in the survivor path terminating in state σ at time n.

3. Final decision (trace-back): the maximum of the total metric is obtained as $\max_{\sigma \in \Sigma_N} \mathcal{M}_N(\sigma)$. The optimal input sequence $\widehat{\mathbf{a}}$ is the corresponding survivor path.

3.5.2 ML sequence detection: Ungerboeck's formulation

The received signal (assuming coherent detection) is given by

$$r(t) = s(t; \mathbf{a}) + \nu(t) \tag{3.36}$$

where $s(t; \mathbf{a})$ is in the form (3.2). In this case it is convenient to use the ML decision metric given by (see (2.15)

$$\mathcal{M}(r, \mathbf{a}) = 2\text{Re}\left\{\int r(t)s(t; \mathbf{a})^* dt\right\} - \int |s(t; \mathbf{a})|^2 dt \tag{3.37}$$

This can be further expanded as

$$\begin{aligned}
\mathcal{M}(r, \mathbf{a}) &= 2\text{Re}\left\{\int r(t) \sum_{n=0}^{N-1} a_n^* \psi(t - nT_s)^* dt\right\} - \int \sum_{n=0}^{N-1} \sum_{\ell=0}^{N-1} a_n^* a_\ell \psi(t - nT_s)^* \psi(t - \ell T_s) dt \\
&= 2\text{Re}\left\{\sum_{n=0}^{N-1} a_n^* \int r(t)\psi(t - nT_s)^* dt\right\} - \sum_{n=0}^{N-1} \sum_{\ell=0}^{N-1} a_n^* a_\ell g_{n-\ell} \\
&= 2\text{Re}\left\{\sum_{n=0}^{N-1} a_n^* y_n\right\} - g_0 \sum_{n=0}^{N-1} |a_n|^2 - 2\text{Re}\left\{\sum_{n=0}^{N-1} a_n^* \sum_{\ell=1}^{n} a_{n-\ell} g_\ell\right\}
\end{aligned} \tag{3.38}$$

where we define

1. The projection of the received signal onto the n-th translated version of the elementary modulation waveform $\psi(t - nT_s)$, also obtained by taking the n-th sample at sampling rate $1/T_s$ of the matched filter $\psi(-t)^*$, given by $y_n = \int r(t)\psi(t - nT_s)^* dt$.

2. The discrete-time autocorrelation function obtained by sampling the autocorrelation function of $\psi(t)$ at rate $1/T_s$, given by $g_\ell = \int \psi(t)\psi(t - \ell T_s)^* dt$.

From (3.38) it is clear that a sufficient statistics for the detection of $s(t; \mathbf{a})$ in AWGN is obtained by sampling at rate $1/T_s$ the output of the matched filter $\psi(-t)^*$, matched to the modulation elementary waveform, and by taking the corresponding N samples $y_0, y_1, \ldots, y_{N-1}$. By noticing that the modulated signal sequence $s(t; \mathbf{a})$ can be obtained by filtering the sequence of impulses $\sum_{n=0}^{N-1} a_n \delta(t - nT_s)$ by the *transmit filter* $\psi(t)$, we can include both the transmit filter and the corresponding matched filter as part of the channel, and find the continuous-time reference model for transmission and detection of signal sequences given in Fig. **??**

For any practical purpose, we can assume that the autocorrelation of the elementary waveform $\psi(t)$ is zero outside the interval $[-LT_s, LT_s]$, for some suitable integer L, and that $N \gg L$. We re-write the decision metric (3.38) in a more compact matrix form as follows. Define the column vectors $\mathbf{y} = (y_0, \ldots, y_{N-1})^T$, $\mathbf{a} = (a_0, \ldots, a_{N-1})^T$ and the Toeplitz matrix

$$\mathbf{G} = \begin{bmatrix}
g_0 & g_{-1} & \cdots & g_{-L} & 0 & \cdots & 0 \\
g_1 & g_0 & g_{-1} & \cdots & g_{-L} & \ddots & \vdots \\
\vdots & & \ddots & & & \ddots & 0 \\
g_L & \cdots & g_1 & g_0 & g_{-1} & \cdots & g_{-L} \\
0 & \ddots & & & \ddots & & \vdots \\
\vdots & \ddots & g_L & \cdots & g_1 & g_0 & g_{-1} \\
0 & \cdots & 0 & g_L & \cdots & g_1 & g_0
\end{bmatrix} \tag{3.39}$$

(notice that, by definition, $g_0 = E_s$). Hence, we can write (3.38) as

$$\mathcal{M}(r, \mathbf{a}) = 2\text{Re}\left\{\mathbf{a}^H\mathbf{y} - \frac{1}{2}\mathbf{a}^H\mathbf{G}\mathbf{a}\right\} \tag{3.40}$$

The ML detection rule is given by

$$\widehat{\mathbf{a}} = \arg \max_{\mathbf{a}\in\mathcal{A}^N} \mathcal{M}(r, \mathbf{a}) \tag{3.41}$$

As already pointed out, a "brute-force" implementation of the above decision rule is not possible in any situation of practical interest. Fortunately, the maximization of $\mathcal{M}(r, \mathbf{a})$ falls in the class of problems that can be solved by the VA.

In order to show this fact, we write $\mathcal{M}(r, \mathbf{a})$ in an additive recursive form, as the metric accumulated over the trajectories of a finite-state dynamical system. We let

$$\begin{aligned}
\mathcal{M}_n(r, a_0, \ldots, a_n) &= 2\text{Re}\left\{\sum_{i=0}^{n} a_i^*\left(y_i - \frac{1}{2}g_0 a_i - \sum_{\ell=1}^{L} g_\ell a_{i-\ell}\right)\right\} \\
&= \mathcal{M}_{n-1}(r, a_0, \ldots, a_{n-1}) + 2\text{Re}\left\{a_n^*\left(y_n - \frac{1}{2}g_0 a_n - \sum_{\ell=1}^{L} g_\ell a_{n-\ell}\right)\right\} \\
&= \mathcal{M}_{n-1}(r, a_0, \ldots, a_{n-1}) + m_n(y_n, a_n, (a_{n-1}, \ldots, a_{n-L}))
\end{aligned} \tag{3.42}$$

Then, we identify the dynamical system state $\sigma_n = (a_{n-1}, \ldots, a_{n-L})$, and the state evolution given by the shift-register state equation

$$\sigma_{n+1} = (a_n, \text{right-shift}(\sigma_n))$$

For fixed (known) initial state σ_0, the set of all sequences $\mathbf{a} \in \mathcal{A}^N$ is in one-to-one correspondence with the set of paths in the trellis defined by the above state evolution. Fig. **??** shows an example of shift-register trellis for a binary alphabet $\mathcal{A} = \{-1, +1\}$ and "memory" $L = 3$.

The VA branch metric at time n, for the transition (σ, σ') is given by

$$m_n = 2\text{Re}\left\{a_n^*\left(y_n - \frac{1}{2}g_0 a(\sigma, \sigma') - \sum_{\ell=1}^{L} g_\ell a_\ell(\sigma)\right)\right\}$$

where $a(\sigma, \sigma')$ is the symbol corresponding to the transition (σ, σ') and $a_\ell(\sigma)$, for $\ell = 1, \ldots, L$ indicates the ℓ-th component of the state σ (seen as a vector in $\mathcal{A}^L$).

3.5.3 ML sequence detection: Forney's formulation

The sequence of samples y_n at the matched filter output is given by

$$\begin{aligned}
y_n &= \int \psi(t - nT_s)^* r(t) dt \\
&= \sum_{\ell=0}^{N-1} a_\ell \int \psi(t - nT_s)^* \psi(t - \ell T_s) + v_n \\
&= \sum_{\ell=0}^{N-1} g_{n-\ell} a_\ell + v_n
\end{aligned} \tag{3.43}$$

where $v_n = \int \psi(t - nT_s)^* \nu(t) dt$ is $\sim \mathcal{N}_{\mathbb{C}}(0, N_0 E_s)$. The corresponding discrete-time reference model for transmission and detection of signal sequences given in Fig. **??**

In matrix form, we have

$$\mathbf{y} = \mathbf{G}\mathbf{a} + \mathbf{v} \tag{3.44}$$

where $\mathbf{y}$, and $\mathbf{a}$ and $\mathbf{G}$ have already been defined in (3.40), and where $\mathbf{v} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, N_0\mathbf{G})$.

Therefore, in general the output of the sampled matched filter contains correlated noise, and the noise covariance matrix is given by the Toeplitz matrix defined by the autocorrelation of the modulation elementary waveform $\psi(t)$.

It is often convenient to deal with white Gaussian noise. We notice that $\mathbf{y}$ provides a sufficient statistics for the detection of the sequence of symbols $\mathbf{a}$. However, any other statistics obtained from $\mathbf{y}$ via an invertible transformation is also a sufficient statistics. Hence, we look for an invertible *noise whitening* transformation, in order to produce a sufficient statistics $\mathbf{y}^{(w)}$ containing white Gaussian noise. This is given by the following Cholesky factorization of the matrix $\mathbf{G}$.

Cholesky factorization. Let $\mathbf{G} \in \mathbb{C}^{N\times N}$ be Hermitian symmetric and positive definite. Then, there exists a lower triangular positive definite matrix $\mathbf{L}$ such that $\mathbf{G} = \mathbf{L}^H\mathbf{L}$. □

The noise whitening transformation is given by $(\mathbf{L}^H)^{-1}$, briefly denoted by $\mathbf{L}^{-H}$. We have

$$\mathbf{y}^{(w)} = \mathbf{L}^{-H}\mathbf{y} = \mathbf{L}\mathbf{a} + \mathbf{w} \tag{3.45}$$

where $E[\mathbf{w}\mathbf{w}^H] = \mathbf{L}^{-H}E[\mathbf{v}\mathbf{v}^H]\mathbf{L}^{-1} = N_0\mathbf{I}$, hence, $\mathbf{v} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, N_0\mathbf{I})$.

At this point, we can apply the ML detection rule to the noise-whitened sufficient statistics (3.45). The conditional distribution of $\mathbf{y}^{(w)}$ given $\mathbf{a}$ is the multivariate complex circularly symmetric Gaussian pdf $\mathcal{N}_{\mathbb{C}}(\mathbf{L}\mathbf{a}, N_0\mathbf{I})$, then we obtain the ML decision rule

$$\widehat{\mathbf{a}} = \arg \min_{\mathbf{a}\in\mathcal{A}^N} |\mathbf{y}^{(w)} - \mathbf{L}\mathbf{a}|^2 \tag{3.46}$$

Again, brute-force implementation of the above rule is impossible in any practical case. However, also in this case the VA can be used in order to implement (3.46) with complexity linear in N and exponential in the correlation length L. First, we notice that if $\mathbf{G}$ is Toeplitz, in the form given by (3.39), then $\mathbf{L}$ contains only L non-zero diagonals below the main diagonal, i.e.,

$$\mathbf{L} = \begin{bmatrix} l_{0,0} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ l_{1,1} & l_{1,0} & 0 & \cdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & & & \ddots & 0 \\ l_{L,L} & \cdots & l_{L,1} & l_{L,0} & 0 & \cdots & 0 \\ 0 & \ddots & & & \ddots & & \vdots \\ \vdots & \ddots & l_{N-2,L} & \cdots & l_{N-2,1} & l_{N-2,0} & 0 \\ 0 & \cdots & 0 & l_{N-1,L} & \cdots & l_{N-1,1} & l_{N-1,0} \end{bmatrix} \tag{3.47}$$

Hence, we define the additive decision metric

$$\begin{aligned}\mathcal{M}(r,\mathbf{a}) &= -|\mathbf{y}^{(w)} - \mathbf{L}\mathbf{a}|^2 \\ &= -\sum_{n=0}^{N-1}\left|y_n^{(w)} - l_{n,0}a_n - \sum_{\ell=1}^{L} l_{n,\ell}a_{n-\ell}\right|^2 \end{aligned} \tag{3.48}$$

The path metric at time n is given by

$$\mathcal{M}_n(r,a_0,\ldots,a_n) = \mathcal{M}_{n-1}(r,a_0,\ldots,a_{n-1}) + m_n(y_n,a_n,(a_{n-1},\ldots,a_{n-L})) \tag{3.49}$$

where the branch metric at time n, for the transition (σ,σ'), is given by

$$m_n = -\left|y_n^{(w)} - l_{n,0}a(\sigma,\sigma') - \sum_{\ell=1}^{L} l_{n,\ell}a_\ell(\sigma)\right|^2$$

where $a(\sigma,\sigma')$ is the symbol corresponding to the transition (σ,σ') and $a_\ell(\sigma)$, for $\ell = 1,\ldots,L$ indicates the ℓ-th component of the state σ (seen as a vector in $\mathcal{A}^L$).

The dynamical system state is again defined as $\sigma_n = (a_{n-1},\ldots,a_{n-L})$, and the state evolution is given by the shift-register state equation

$$\sigma_{n+1} = (a_n, \text{right-shift}(\sigma_n))$$

exactly as for the ML sequence detection in Ungerboeck's form.

3.5.4 Spectral factorization and whitened matched filtering

In practice, it is not convenient to compute explicitly the Cholesky decomposition and obtaining the whitened observation $\mathbf{y}^{(w)}$ via the matrix-vector multiplication (3.45). On the contrary, assuming $N \to \infty$ and working with discrete-time WSS processes it is possible to find a receiver filter that produces directly the noise-whitened sufficient statistics. This filter is referred to as the *Whitened Matched Filter* (WMF). For finite but large block length N, this can be regarded as an approximation of the Cholesky factorization and noise-whitening transformation given by the matrix multiplication by $\mathbf{L}^{-H}$.

In order to handle discrete-time WSS sequences, it is useful to introduce the D-transform representation [**?**].

D-transform: definition and basic properties. The D-transform of a sequence s_n over the complex numbers is defined by

$$s(D) = \sum_n s_n D^n$$

where D is a complex variable.

- The convolution of two sequences is given by the product of their D-transforms

$$z_n = \sum_m x_m y_{n-m} \to z(D) = x(D)y(D)$$

- The sequence s^*_{-n} has D-transform $s(1/D^*)^*$.
- An autocorrelation sequence satisfies $s^*_{-n} = s_n$, therefore its D-transform satisfies the symmetry condition $s(1/D^*)^* = s(D)$.
- The power spectrum of a random WSS sequence x_n with autocorrelation $\phi_x(D)$ is given by
$$\mathcal{S}_x(\lambda) = \phi_x(D)|_{D=e^{-j2\pi\lambda}}$$
$\mathcal{S}_x(\lambda)$ is real and non-negative for all $-1/2 \leq \lambda \leq 1/2$ (i.e., for all λ).
- Let $z(D) = x(D)y(D)$, with $x(D)$ deterministic and $y(D)$ random WSS. Then, the autocorrelation sequence of $z(D)$ is given by $\phi_z(D) = x(D)x(1/D^*)^*\phi_y(D)$ and the cross-correlations of $z(D)$ and $y(D)$ are given by $\phi_{yz}(D) = x(1/D^*)^*\phi_y(D)$ and by $\phi_{zy}(D) = x(D)^*\phi_y(D)$.

□

The following theorem and corollary yield the desired WMF.

Spectral factorization theorem. Let x_n be a random WSS sequence. If its power spectral density $\mathcal{S}_x(\lambda)$ satisfies the *Paley-Wiener* condition:

$$c_0 = \int_{-1/2}^{1/2} \log \mathcal{S}_x(\lambda) d\lambda > -\infty \tag{3.50}$$

then $\mathcal{S}_x(\lambda)$ can be written as the product of two functions $\mathcal{S}_x^+(\lambda)$ and $\mathcal{S}_x^-(\lambda)$ such that

1. $|\mathcal{S}_x^+(\lambda)|^2 = |\mathcal{S}_x^-(\lambda)|^2 = \mathcal{S}_x(\lambda)$.
2. $\mathcal{S}_x^+(\lambda)$ is the Fourier transform of a causal sequence, and $\mathcal{S}_x^-(\lambda)$ is the Fourier transform of an anticausal sequence.
3. $1/\mathcal{S}_x^+(\lambda)$ is the Fourier transform of a causal sequence, and $1/\mathcal{S}_x^-(\lambda)$ is the Fourier transform of an anticausal sequence.

Sketch of proof. First, notice that since $\log(\cdot)$ is a concave function, then by Jensen's inequality

$$c_0 \leq \log \int_{-1/2}^{1/2} \mathcal{S}_x(\lambda) d\lambda = \log E[|x_n|^2] < \infty$$

Therefore, $\log \mathcal{S}_x(\lambda)$ can be written as a Fourier transform,

$$\log \mathcal{S}_x(\lambda) = \sum_n c_n e^{-j2\pi\lambda n}$$

This implies that

$$\begin{aligned}
\mathcal{S}_x(\lambda) &= \exp\left(\sum_n c_n 2^{-j2\pi\lambda n}\right) \\
&= \exp\left(c_0/2 + \sum_{n=1}^{\infty} c_n 2^{-j2\pi\lambda n}\right) \exp\left(c_0/2 + \sum_{n=-\infty}^{-1} c_n 2^{-j2\pi\lambda n}\right) \\
&= \mathcal{S}_x^+(\lambda)\mathcal{S}_x^-(\lambda)
\end{aligned}$$

Now, we have to verify that the two functions $\mathcal{S}_x^+(\lambda)$ and $\mathcal{S}_x^-(\lambda)$ satisfy the properties of the spectral factorization lemma.

We notice that $\mathcal{S}_x(\lambda)$ is real and even. Therefore, also $\log \mathcal{S}_x(\lambda)$ is real and even. This implies that $c_{-n} = c_n = c_n^*$, which in turns enable us to write

$$
\begin{aligned}
\mathcal{S}_x^-(\lambda) &= \exp\left(c_0/2 + \sum_{n=-\infty}^{-1} c_n 2^{-j2\pi\lambda n}\right) \\
&= \exp\left(c_0/2 + \sum_{n=1}^{\infty} c_n 2^{j2\pi\lambda n}\right) \\
&= \exp\left(c_0/2 + \sum_{n=1}^{\infty} c_n 2^{-j2\pi\lambda n}\right)^* \\
&= \mathcal{S}_x^+(\lambda)^*
\end{aligned}
$$

Therefore, $|\mathcal{S}_x^+(\lambda)|^2 = |\mathcal{S}_x^-(\lambda)|^2 = \mathcal{S}_x^+(\lambda)\mathcal{S}_x^-(\lambda) = \mathcal{S}_x(\lambda)$. In order to show the causality property, we expand the exponential in Taylor series and obtain

$$
\mathcal{S}_x^+(\lambda) = e^{c_0/2}\left[\sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{n=1}^{\infty} c_n e^{-j2\pi\lambda n}\right)^k\right]
$$

By inspection, we see that the above series contains only non-negative powers of $e^{-j2\pi\lambda}$. Therefore, the n-th coefficient of the inverse Fourier transform of $\mathcal{S}_x^+(\lambda)$ is zero for all $n < 0$. The same reasoning applies to $\mathcal{S}_x^-(\lambda)$ and to $1/\mathcal{S}_x^+(\lambda)$ and $1/\mathcal{S}_x^-(\lambda)$. □

In terms of the D-transform, the spectral factorization theorem is formulated as follows: let $\phi_x(D)$ the autocorrelation function of x_n. Then,

$$
\phi_x(D) = \phi_x^+(D)\phi_x^-(D)
$$

where $\phi_x^+(D)$ is causal and $\phi_x^-(D) = \phi_x^+(1/D^*)^*$.

Corollary: whitening filter. Let $x(D)$ be a WSS sequence with autocorrelation sequence $\phi_x(D)$. Then, $w(D) = (1/\phi_x^-(D))x(D)$ and $w'(D) = (1/\phi_x^+(D))x(D)$ are white sequences.

Proof. The autocorrelation sequence of $w(D)$ is given by

$$
\phi_w(D) = \frac{1}{\phi_x^-(D)\phi_x^-(1/D^*)^*}\phi_x(D) = 1
$$

and the autocorrelation of $w'(D)$ is given by

$$
\phi_w'(D) = \frac{1}{\phi_x^+(D)\phi_x^+(1/D^*)^*}\phi_x(D) = 1
$$

□

In our case, the autocorrelation function of the noise at the output of the matched filter is given by $N_0 g(D)$, where $g(D) = \sum_{\ell-L}^{L} g_\ell D^\ell$. If the Fourier transform

$$
\mathcal{G}(\lambda) = \sum_{\ell=-L}^{L} g_\ell e^{-j2\pi\ell\lambda} \tag{3.51}
$$

satisfies in Paley-Wiener condition (3.50), in particular, if $\mathcal{G}(\lambda) > 0$ for all $\lambda \in [-1/2, 1/2]$, then there exists an anti-causal filter $f(D) = 1/g^-(D)$ such that the noise process at its output has autocorrelation $\Phi_w(D) = N_0$, i.e., the noise samples w_n form a white discrete-time noise process.

The concatenation of the (continuous-time) matched filter $\psi(-t)^*$, sampling at the symbol rate and whitening filter $1/g^-(D)$ forms the so called sampled whitened matched filter front-end. By noticing that $g(D)/g^-(D) = g^+(D)$ is causal, we can write the resulting discrete-time observation as

$$y_n^{(w)} = \sum_{\ell=0}^{L} g_\ell^+ a_{n-\ell} + w_n \tag{3.52}$$

The corresponding discrete-time reference model for transmission and detection of signal sequences including the WMF given in Fig. **??**

Finally, by comparing this with (3.45) and by using the expression (3.47) for the Cholesky factor $\mathbf{L}$, we see that the equivalent discrete-time causal filter $\mathbf{g}^+ = (g_0^+, \ldots, g_L^+)$ is given by the $L+1$ non-identically zero elements of any row of $\mathbf{L}$, provided that the dimension N of the block is very large. Indeed, it can be checked that in the limit for large N the Cholesky factor $\mathbf{L}$ tends to become Toeplitz, i.e., tends to the limit

$$\mathbf{L} \to \begin{bmatrix} g_0^+ & 0 & \cdots & 0 & 0 & \cdots & 0 \\ g_1^+ & g_0^+ & 0 & \cdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & & & \ddots & 0 \\ g_L^+ & \cdots & g_1^+ & g_0^+ & 0 & \cdots & 0 \\ 0 & \ddots & & & \ddots & & \vdots \\ \vdots & \ddots & g_L^+ & \cdots & g_1^+ & g_0^+ & 0 \\ 0 & \cdots & 0 & g_L^+ & \cdots & g_1^+ & g_0^+ \end{bmatrix} \tag{3.53}$$

3.6 Problems

Problem 1. Consider the linearly modulated signal

$$x(t) = \sum_n a[n]\psi(t - nT)$$

where $a[n] = b[n] + b[n-1]$, and where the $b[n]$'s are independent and identically distributed random variables that take on the values $+1$ and -1 with equal probability. The modulation basic waveform is given by

$$\psi(t) = \begin{cases} 0 & t < -T \\ t/T + 1 & -T \le t \le 0 \\ 1 - t/T & 0 \le t \le T \\ 0 & t > T \end{cases}$$

Find the power spectral density of $x(t)$.

Problem 2. Consider the linearly modulated signal

$$x(t) = \sum_n a[n]\psi(t - nT)$$

where $a[n] = b[n] - 2b[n-1] + b[n-2]$, and where the $b[n]$'s are independent and identically distributed random variables that take on the values $+1$ and -1 with equal probability. The modulation basic waveform is given by

$$\psi(t) = \begin{cases} 0 & t < 0 \\ A & 0 \le t < T/2 \\ -A & T/2 \le t < T \\ 0 & t \ge T \end{cases}$$

where A is a positive real number. Find the power spectral density of $x(t)$.

Problem 3. Consider a baseband 2-PAM transmission system, where the transmitted signal is given by

$$x(t) = \sum_n a_n p_T(t - nT)$$

where

$$p_T(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [0,\, T] \\ 0 & \text{elsewhere} \end{cases}$$

and where a_n are i.i.d. random variables that take on the values $\pm A$ with the same probability. A zero-mean AWGN $n(t)$ with variance N_0 adds up to the signal $x(t)$ and the sum $r(t) = x(t)+n(t)$ passes through a linear time-invariant filter with impulse response equal to

$$h(t) = \begin{cases} 1 & t \in [0, T+\tau] \\ 0 & \text{elsewhere} \end{cases}$$

where $0 < \tau < T$. The receiver samples the signal $y(t) = \mathrm{Re}\{h(t) \otimes r(t)\}$ at the time instants t_n and makes an estimate $\hat{a}_n$ of the transmitted symbol a_n according to the symbol-by-symbol decision rule

$$\widehat{a}_n = \mathrm{sgn}(y(t_n))A$$

1. Determine the values of t_n such that the useful signal sample (i.e., the coefficient that multiplies a_n in the expression of $y(t_n)$) is maximum. For this value determine the peak distortion D_p.

2. Determine the values of t_n in order to minimize the energy associated with the inter-symbol interference (i.e., minimize the mean square value of the interfering samples).

3. Assume that $(n+1)T \le t_n \le (n+1)T + \tau$. Then determine the value of t_n that minimizes the error probability $P(e) = \Pr(\hat{a}_n \ne a_n)$.

   Hint: Define $\theta = t_n - (n+1)T$ and consider that the function $Q(x)$ is convex for $x \ge 0$. Therefore given x_1 and x_2 non-negative it results

$$\frac{1}{2}[Q(x_1) + Q(x_2)] \ge Q\left(\frac{x_1 + x_2}{2}\right)$$

Problem 4. Consider a baseband 2-PAM transmission system, where the transmitted signal $x(t)$ is given by as in Problem 3. A zero-mean AWGN $n(t)$ with variance N_0 adds up to the signal $x(t)$ and the sum $r(t) = x(t) + n(t)$ is sent to a receiver which consists of a filter with impulse response $h(t) = p_T(t)$. The output signal $y(t) = \mathrm{Re}\{r(t) \otimes h(t)\}$ is sampled at time instants t_n and the corresponding samples $y_n = y(t_n)$ are used to make a decision on the variable a_n according to the rule symbol-by-symbol decision rule

$$\widehat{a}_n = \mathrm{sgn}(y_n)A$$

1. Determine the optimal sampling time $t_n = \bar{t}_n$ minimizing the error probability.
2. For the optimal value $\bar{t}_n$ assume that the system is not perfectly synchronized so that the sampling time is $t_n = \bar{t}_n + \varepsilon$ where ε is a random variable uniformly distributed in the range $[-T/M, T/M]$ with $M > 1$. Then find an expression for the peak distortion as a function of ε and find its average $\mathrm{E}\{D_p\}$.

Problem 5. Consider a baseband 2-PAM transmission system, where the transmitted signal is given by

$$x(t) = \sum_n a_n \psi_T(t - nT)$$

where

$$\psi_T(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [-T/2, T/2] \\ 0 & \text{elsewhere} \end{cases}$$

and where a_n are i.i.d. random variables that take on the values $\pm A$ with the same probability. The signal $x(t)$ is transmitted through a linear time-invariant channel with impulse response

$$h(t) = \delta(t) + 0.5\delta(t - T/2)$$

The received signal is given by

$$r(t) = h(t) \otimes x(t) + n(t)$$

where $n(t)$ is an AWGN with zero-mean and variance N_0.

The receiver consists of a filter $g(t)$ matched to the pulse $\psi_T(t)$. The receiver takes the real part of the signal at the output of the matched filter $y(t) = \mathrm{Re}\{r(t) \otimes g(t)\}$ and samples it at the time instants $t_n = nT$ for integer n. Then a threshold decision device estimates the variables a_n according to the rule

$$\widehat{a}_n = \mathrm{sgn}(y(t_n))A$$

Compute the resulting exact bit error probability, the upper bound based on the peak distortion and the approximation based on the Gaussian approximation of the ISI and the MFB lower bound.

Problem 6. Consider the discrete-time reference model of Fig. **??**, where the symbol sequence a_n is binary antipodal, taking on values in $\{-1, +1\}$, and the discrete-time channel response is given by $g(D) = 1 + 0.5D + 0.5D^{-1}$.

Consider the sequence of received matched filter output samples $y_0 = 1.2, y_1 = -2.3, y_2 = -0.5, y_3 = 0.4, y_4 = 2.5$. Assuming that the initial state is $\sigma_0 = +1$, find the sequence resulting from ML sequence detection (Ungerboeck form) by applying the Viterbi algorithm. Represent the evolution of the path metrics of the VA on the trellis representing the system.

Problem 7. Consider the discrete-time reference model of Fig. **??**, where the symbol sequence a_n is binary antipodal, taking on values in $\{-1, +1\}$, and the discrete-time channel response is given by $g^{+}(D) = 1 + 0.5D$.

Consider the sequence of received samples at the output of the whitened matched filter $y_0^{(w)} = 1.2, y_1^{(w)} = -2.3, y_2^{(w)} = -0.5, y_3^{(w)} = 0.4, y_4^{(w)} = 2.5$. Assuming that the initial state is $\sigma_0 = +1$, find the sequence resulting from ML sequence detection (Forney form) by applying the Viterbi algorithm. Represent the evolution of the path metrics of the VA on the trellis representing the system.

Problem 8. Prove that if the elementary waveform $\psi(t)$ of a digitally modulated signal satisfies the Nyquist condition, then the ML sequence detector coincides with the symbol-by-symbol detector (thus in this case symbol-by-symbol detection is optimal).

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