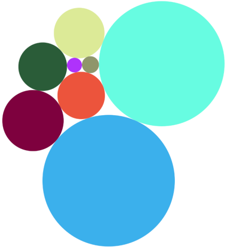
Planar graphs, circle packings, and conformal maps

Brice Loustau (HITS & Heidelberg University)



HITS Lab Meeting

07.09.2020

Planar graphs, circle packings, and conformal maps

Outline.

1. Planar graphs
2. Circle packings
3. Conformal maps
4. Beyond

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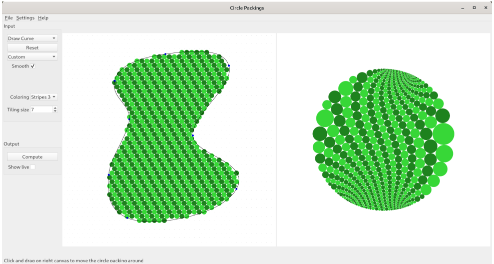

The software: *Circle Packings* (with B. Beeker)
brice.loustau.eu/circlepackings

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- A set of *vertices* (or *nodes*)
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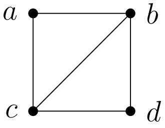

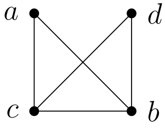

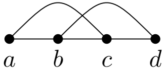

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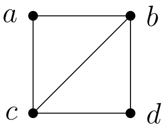

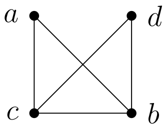

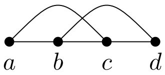

Applications of graph theory:

- Computer science (www, social media)
- Physics (Feynman diagrams)
- Chemistry (molecular graphs)
- Biology (neuroscience, genome graphs, evolutionary trees)
- Linguistics (syntax tree)
- Economy (economic networks)
- Sociology (social network analysis)

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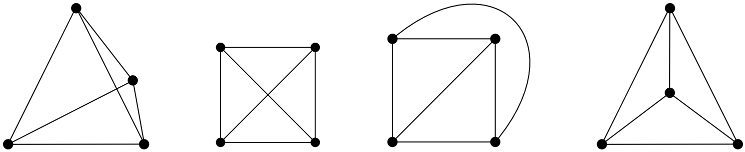

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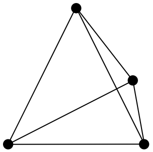
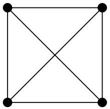
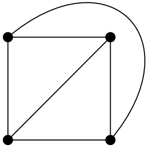
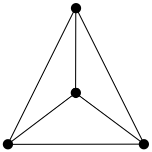
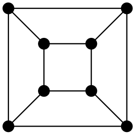

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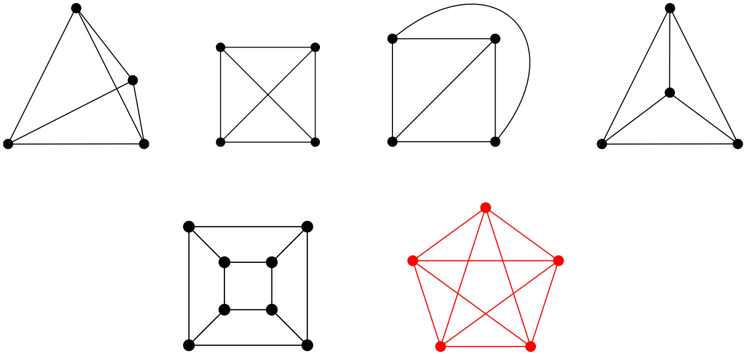
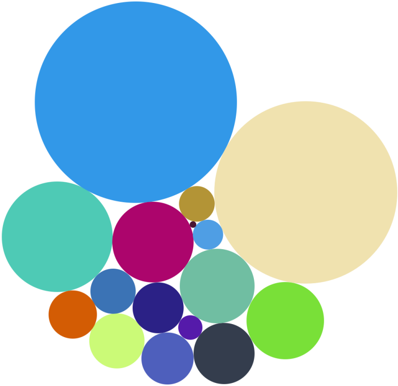
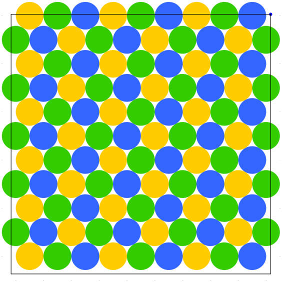


A ***circle packing*** is a collection of circles that are either disjoint or tangent.





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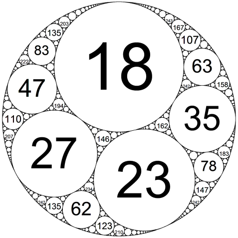

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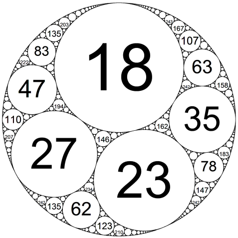

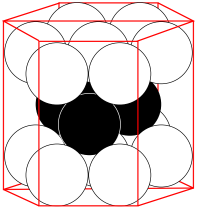

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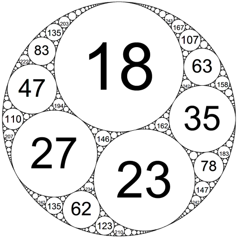
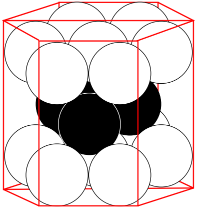
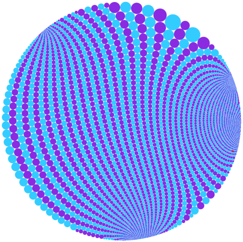
- Graphs and discrete conformal structures (this talk!)







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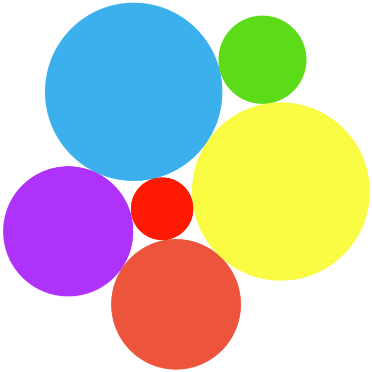

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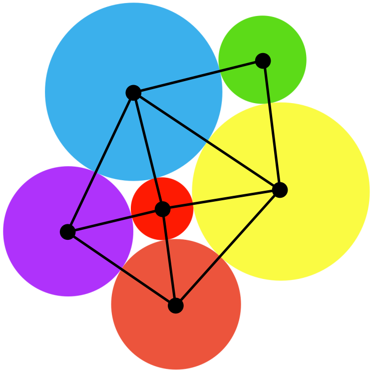

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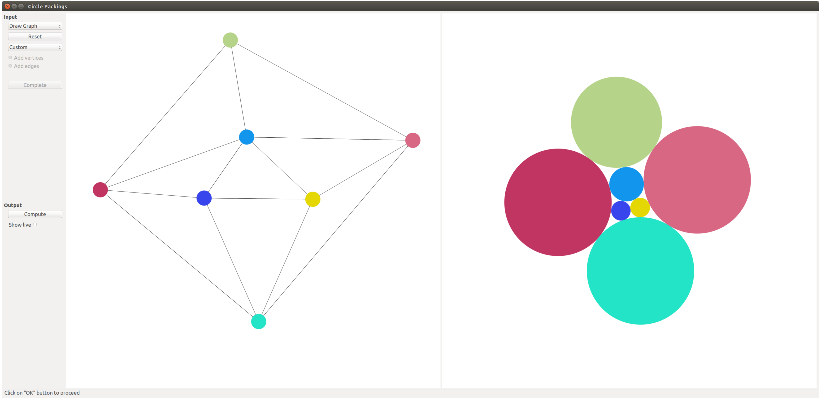

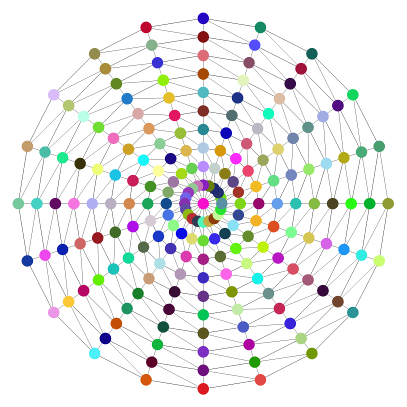

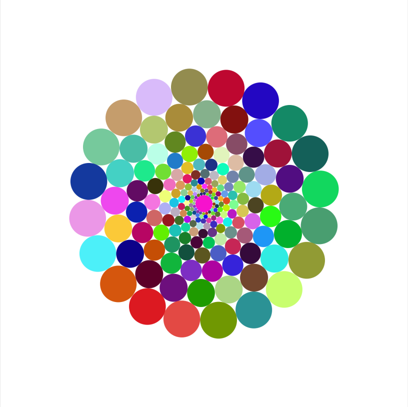

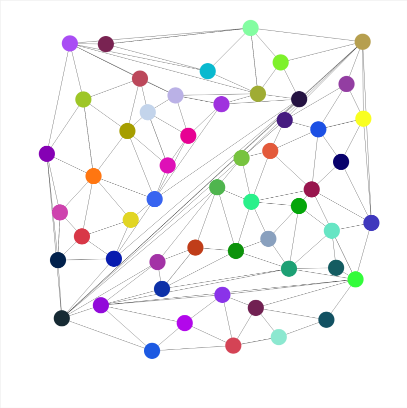

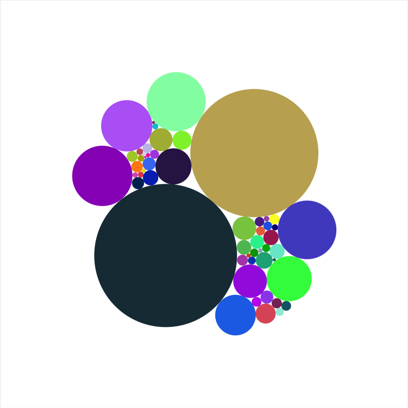

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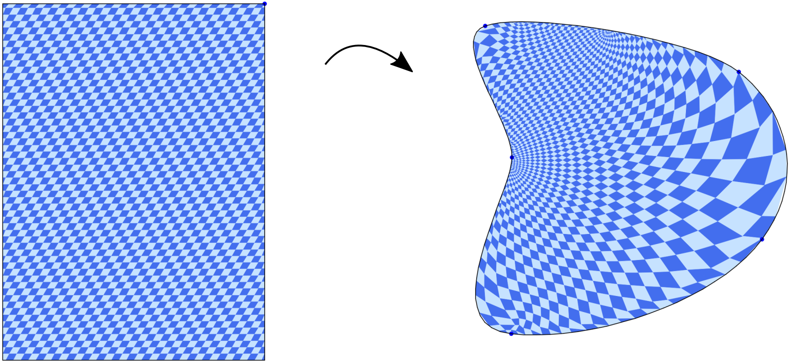

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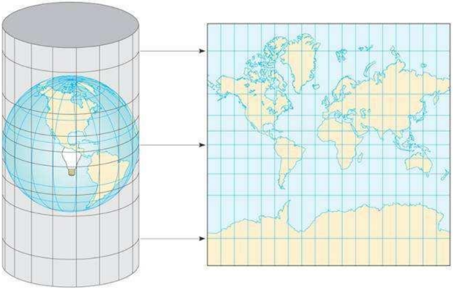

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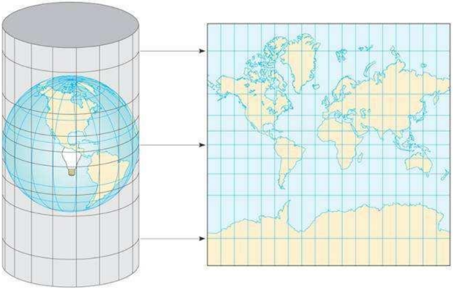

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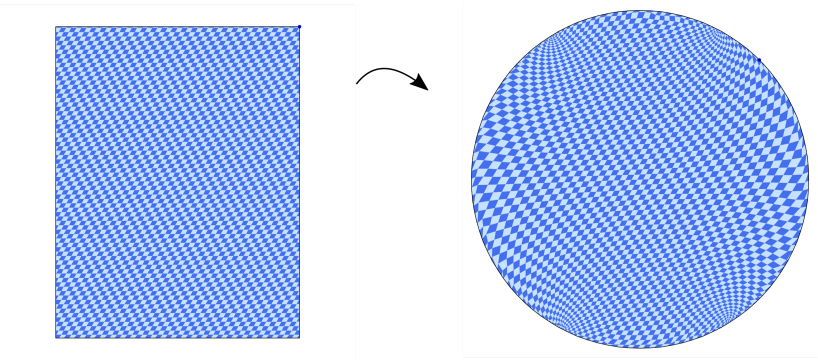

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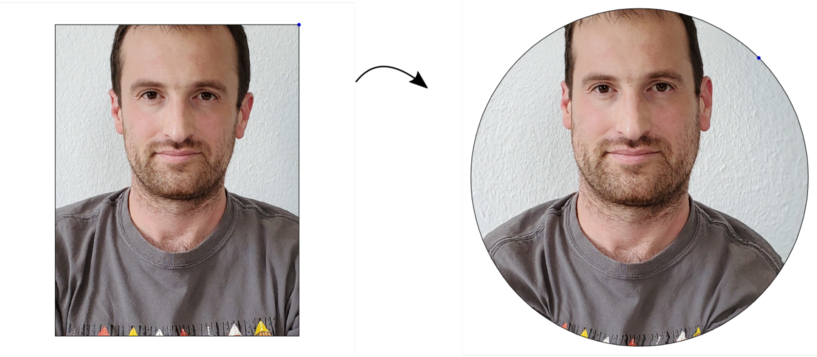

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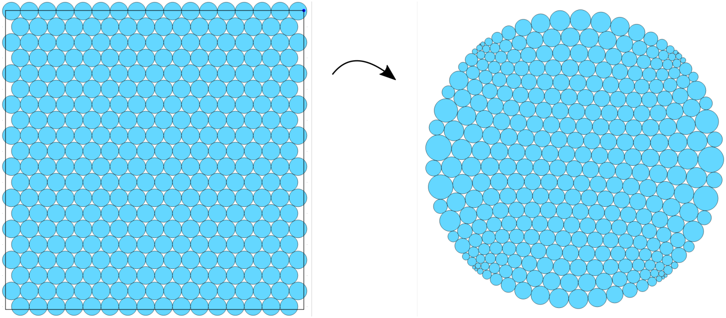

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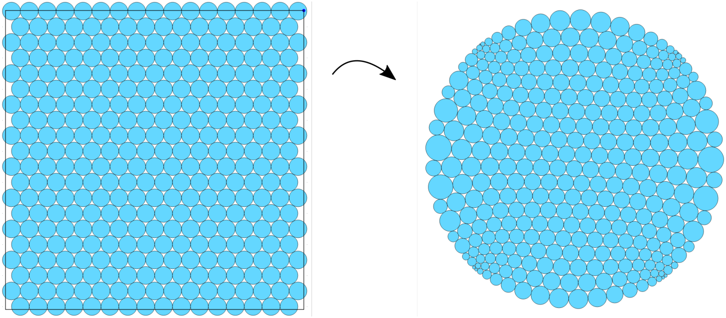

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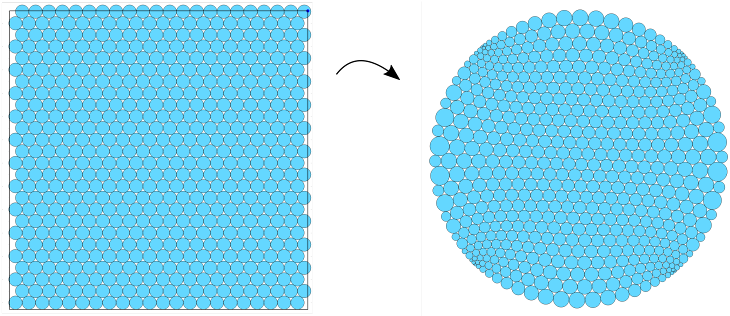

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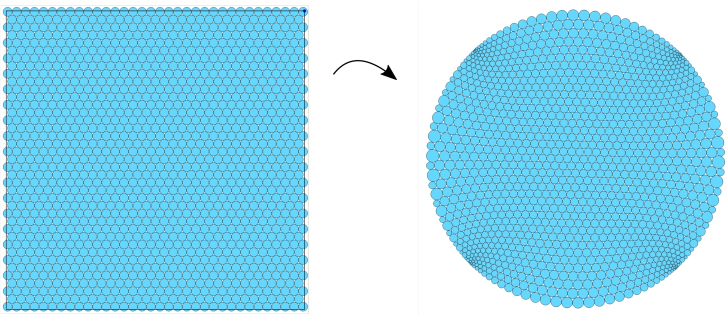

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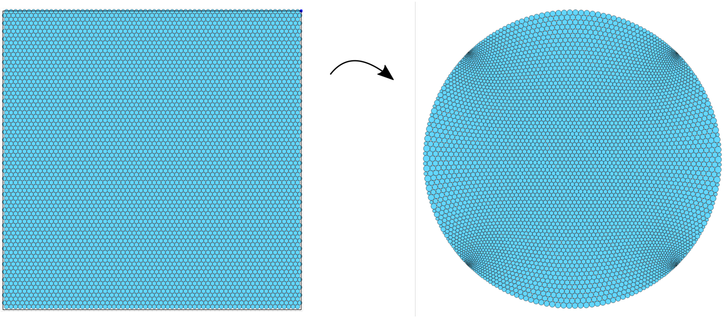

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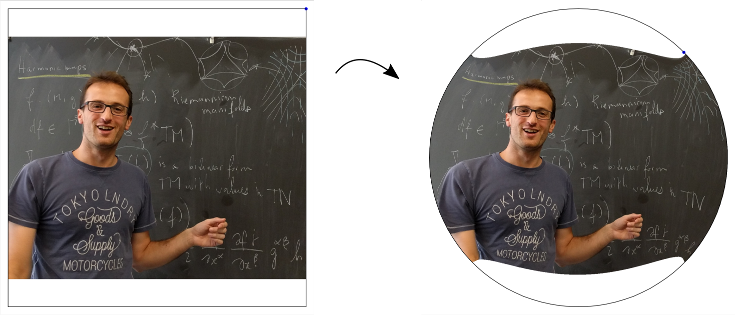

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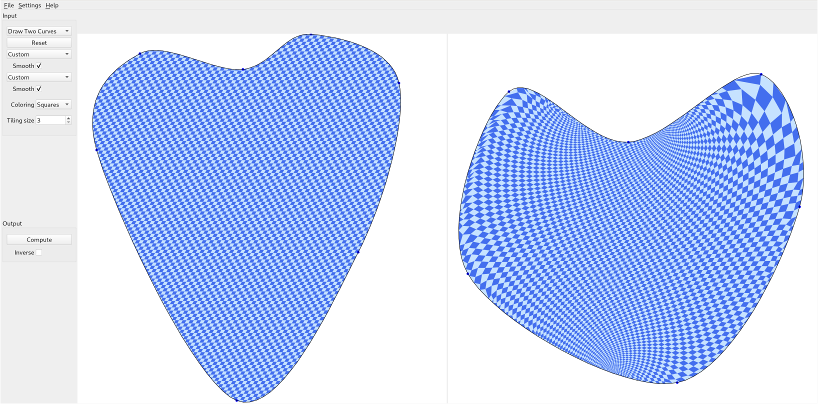

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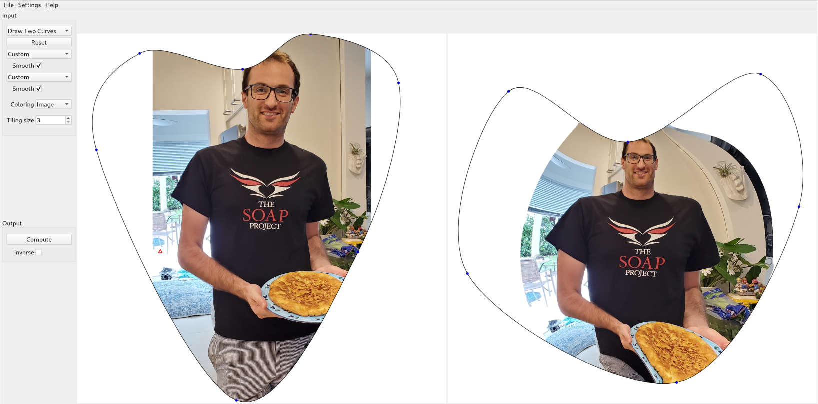

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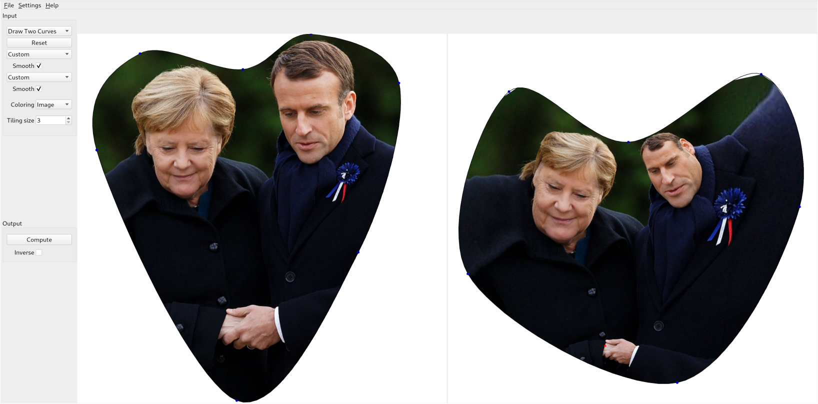

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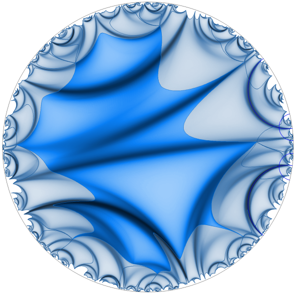

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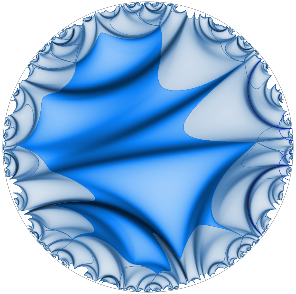

Thank you!