

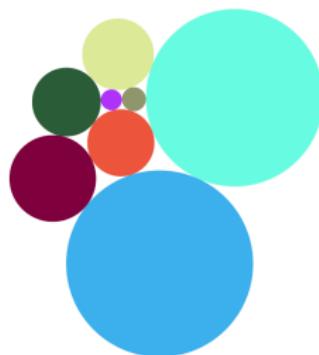


**HITS**

Heidelberg Institute for  
Theoretical Studies

## Planar graphs, circle packings, and conformal maps

Brice Loustau (HITS & Heidelberg University)



HITS Lab Meeting

07.09.2020

## **Planar graphs, circle packings, and conformal maps**

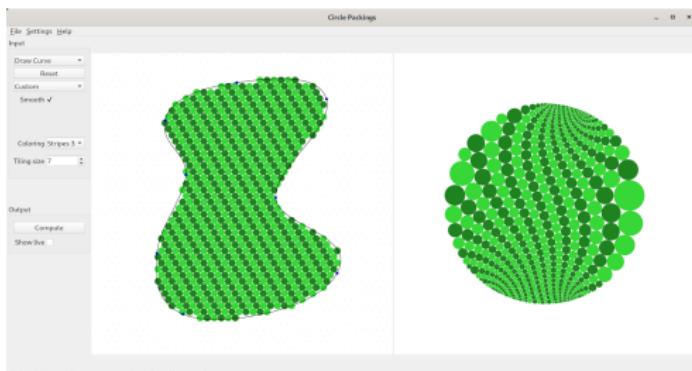
*Outline.*

1. Planar graphs
2. Circle packings
3. Conformal maps
4. Beyond

## Planar graphs, circle packings, and conformal maps

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The software: *Circle Packings* (with B. Beeker)  
[brice.loustau.eu/circlepackings](http://brice.loustau.eu/circlepackings)

## 1. Planar graphs

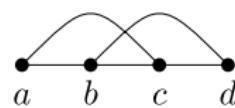
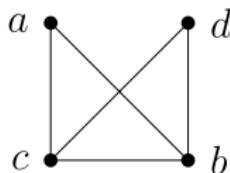
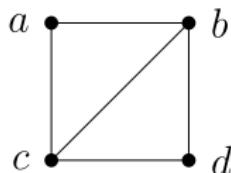
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- A set of *vertices*
- A set of *edges* = relation between vertices

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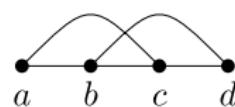
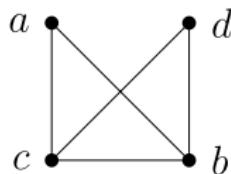
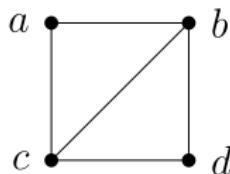
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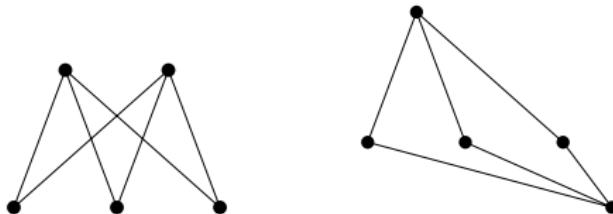
**Applications of graph theory:** Computer science (networks), linguistics, physics and chemistry, biology, social sciences, etc.

## 1. Planar graphs

A graph is called ***planar*** if it can be drawn on the plane with no edge crossings.

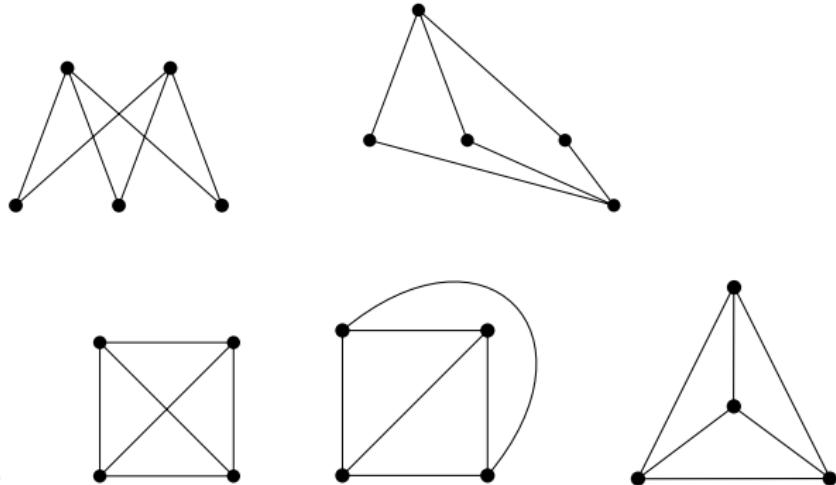
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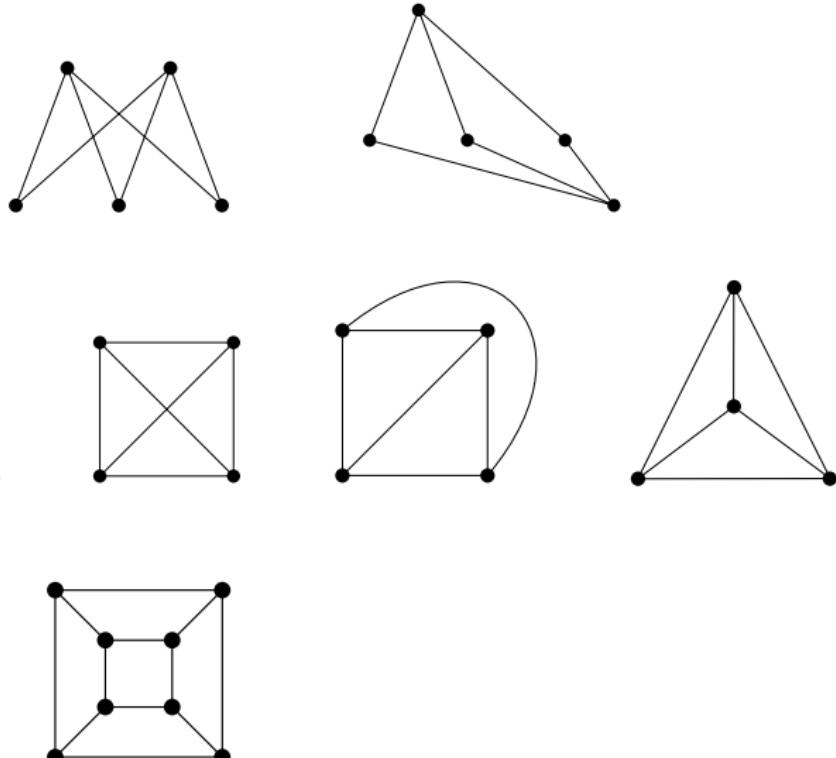
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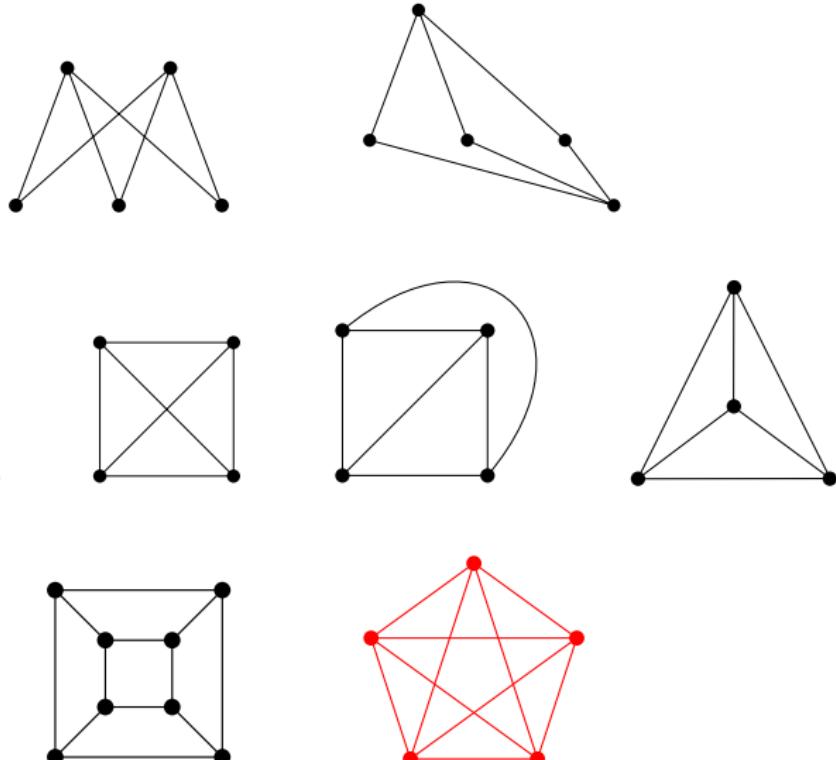
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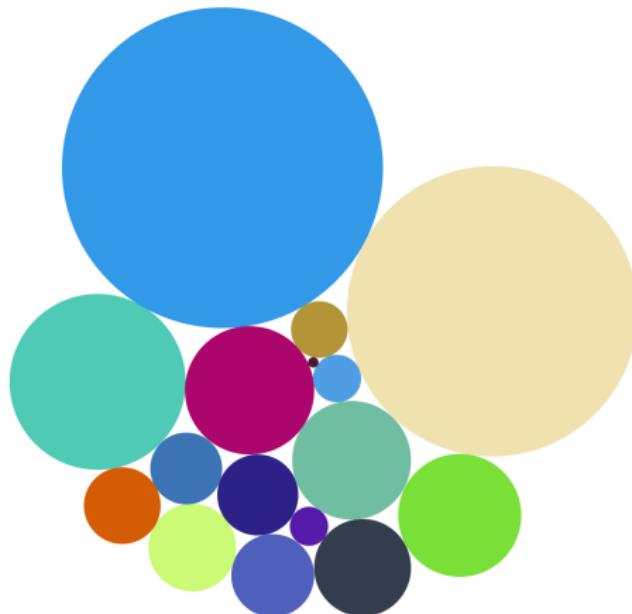
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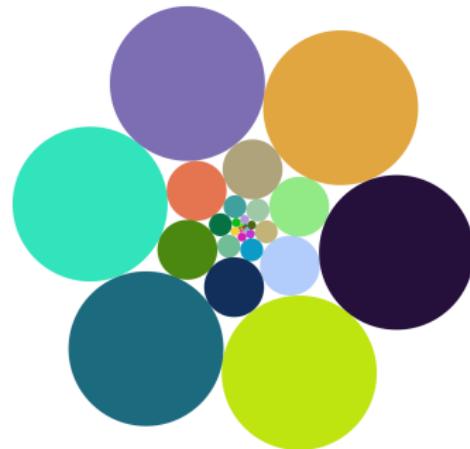
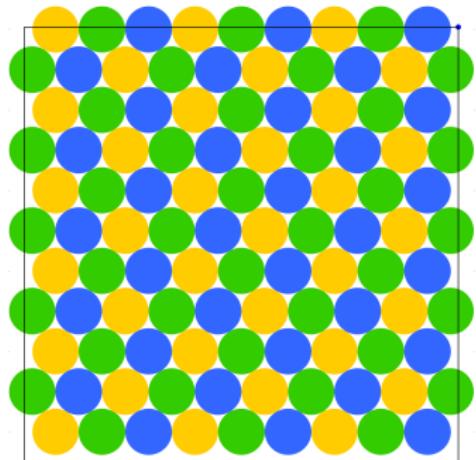


## 2. Circle packings

A ***circle packing*** is a collection of circles that are either disjoint or tangent.

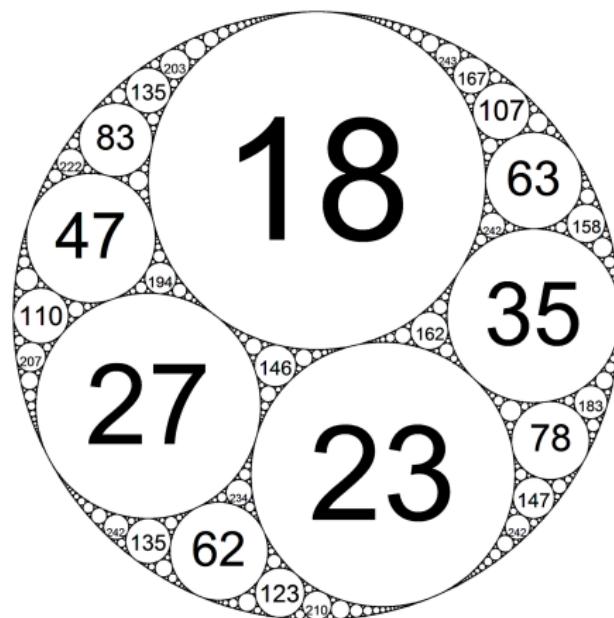


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**Apollonian gasket:**



The curvatures (inverse radii) of four mutually tangent circles satisfy:  
$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2$$

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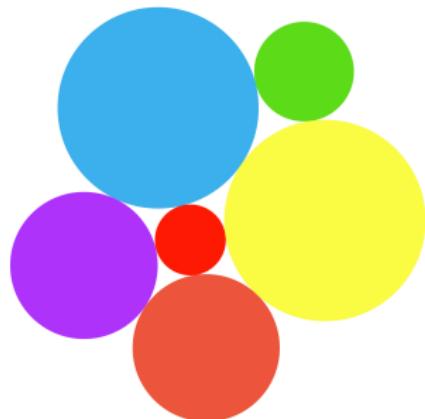
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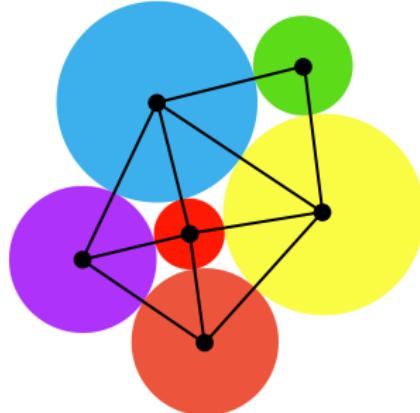
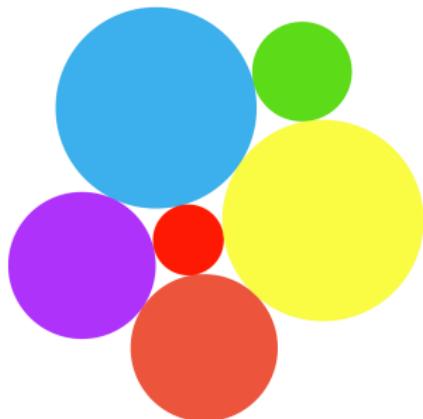
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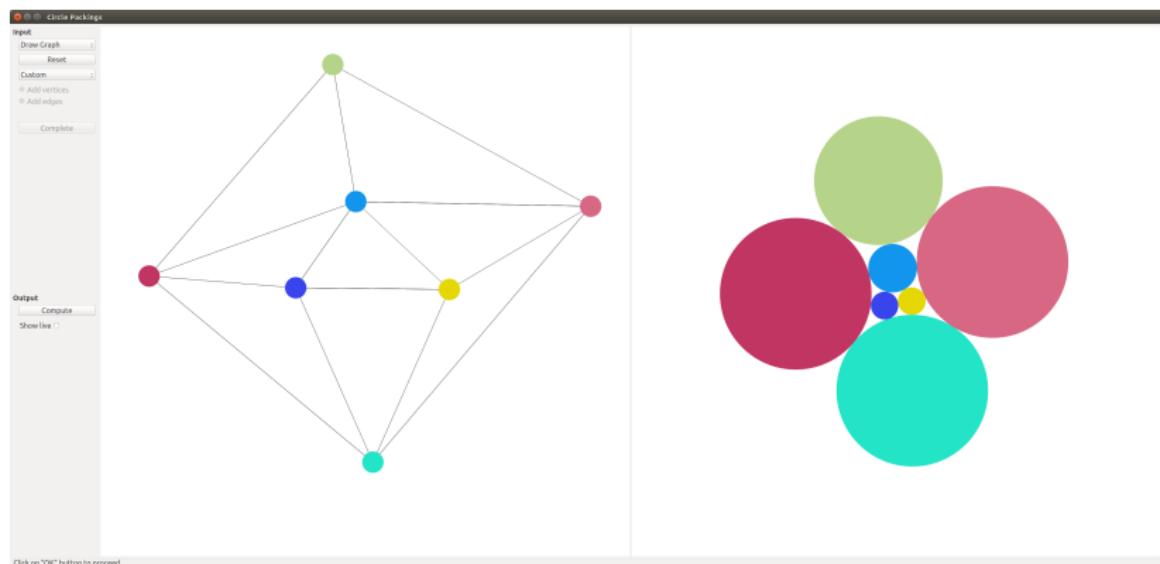
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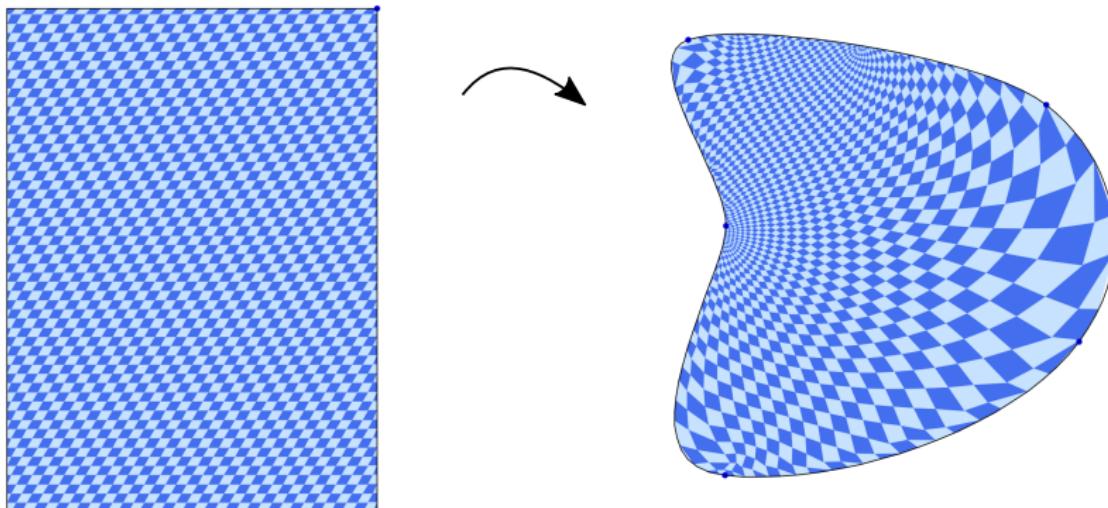


### 3. Conformal maps

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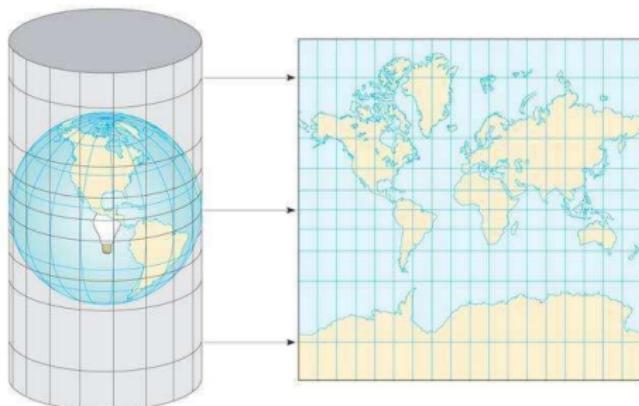


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Conformal maps are intensely studied by mathematicians and have many important real-world applications.

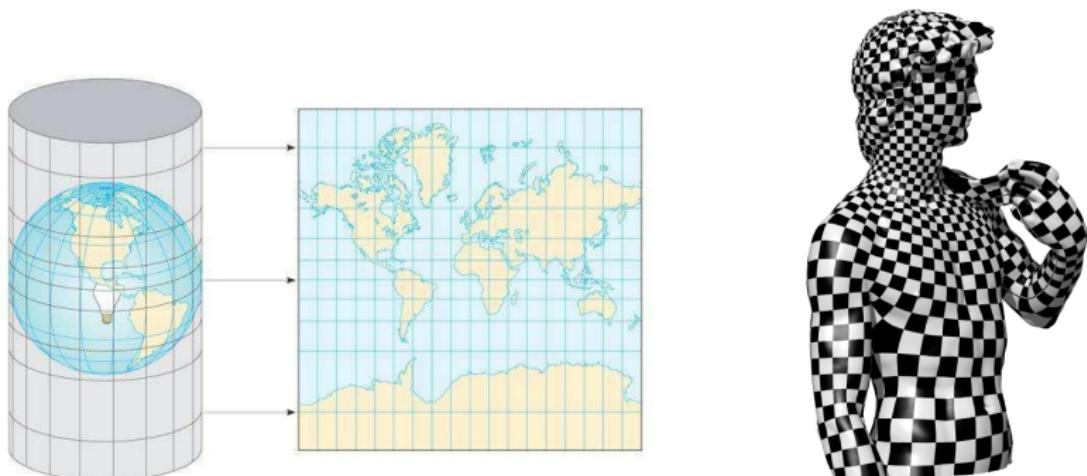
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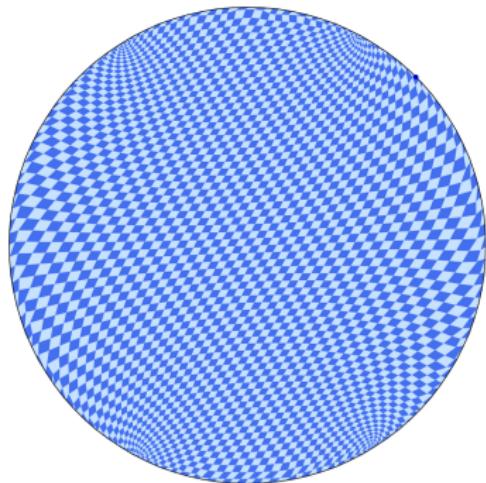
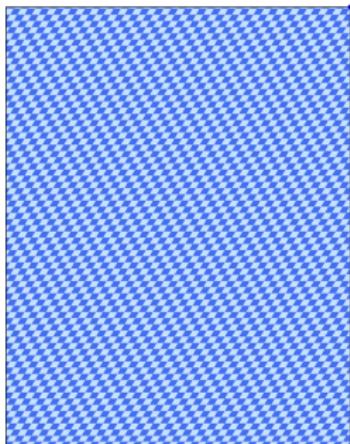
**Theorem (Riemann conformal mapping theorem)**

*Given any two simply-connected (no holes) regions  $R_1$  and  $R_2$  in the plane, there exists a unique conformal map  $f: R_1 \rightarrow R_2$ .*

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**Idea** (Thurston): Circle packings provide discrete conformal structures.