Def^n :- Let $F(t)$ be a function of t specified for $t > 0$. Then the Laplace transform of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$, is defined by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^{\infty} e^{-st} F(t)\, dt \quad \ldots\ldots (1)$$

provided that the integral (1) exists. 's' is a parameter which may be real or complex number.

The Laplace transform of $F(t)$ is said to exist if the integral (1) converges for some values of s; otherwise it does not exist.

Laplace Transform of some Elementary functions.

(1) $\mathcal{L}\{1\}$.

Sol^n :- By definition of Laplace transform we have

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st}\, dt$$

$$= \lim_{P \to \infty} \int_0^{P} e^{-st}\, dt$$

$$= \lim_{P \to \infty} \left[-\frac{1}{s} e^{-st} \right]_0^P$$

$$= \lim_{P \to \infty} \left[-\frac{1}{s} e^{-sP} + \frac{1}{s} \right]$$

$$= \frac{1}{s} \quad \text{if } s > 0$$

(ii) Find $\mathcal{L}\{t\}$.

Soln By defn of Laplace transform, we have -

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}.t\,dt$$

$$= \lim_{P\to\infty}\int_0^P e^{-st}\,t\,dt$$

$$= \lim_{P\to\infty}\left[\frac{-te^{-st}}{s}\right]_0^P + \lim_{P\to\infty}\frac{1}{s}\int_0^P e^{-st}\,dt$$

$$= \lim_{P\to\infty}\frac{-P}{s\,e^{sP}} + 0 - \lim_{P\to\infty}\frac{1}{s^2}\left[e^{-st}\right]_0^P$$

$$= 0 - \lim_{P\to\infty}\frac{1}{s^2}\left[\frac{1}{e^{sP}} - 1\right]$$

$$= \frac{1}{s^2} \quad \text{if } s>0.$$

(iii) Find $\mathcal{L}\{t^n\}$

Soln By defn of Laplace transform, we have

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\,t^n\,dt$$

Putting, $st = x$

$\therefore s\,dt = dx$

$\Rightarrow dt = \frac{1}{s}dx$

Also, $t^n = \frac{x^n}{s^n}$

$$= \int_0^\infty e^{-x}\,\frac{x^n}{s^n}\,\frac{dx}{s}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x}\,x^n\,dx$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x}\,x^{(n+1)-1}\,dx$$

$$= \frac{1}{s^{n+1}}\,\Gamma(n+1) \quad \text{[by using Gamma function]}$$

$$= \frac{n!}{s^{n+1}}, \quad \text{where, } n = 0, 1, 2, \ldots$$

1) Find $\mathcal{L}\{e^{at}\}$.

Soln By defn of Laplace transform, we have

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at}\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\, dt$$

$$= -\frac{1}{(s-a)} \left[e^{-(s-a)t} \right]_0^{\infty}$$

$$= -\frac{1}{(s-a)} (0-1)$$

$$= \frac{1}{s-a} \quad \text{if } s>a.$$

Find $\mathcal{L}\{\sin at\}$.

Soln By definition of Laplace transform, we have –

$$\mathcal{L}\{\sin at\} = \int_0^{\infty} e^{-st} \sin at\, dt$$

$$= \lim_{P\to\infty} \int_0^{P} e^{-st} \sin at\, dt$$

$$= \lim_{P\to\infty} \left[\frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2} \right]_0^{P}$$

$$= \lim_{P\to\infty} \left[\frac{e^{-sP}(-s\sin aP - a\cos aP)}{s^2+a^2} + \frac{a}{s^2+a^2} \right]$$

$$= 0 + \frac{a}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2} \quad \text{if } s>0.$$

(vi) Find $\mathcal{L}\{\cos at\}$.

sol^n By definition of Laplace tranform of, we have –

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \lim_{P\to\infty} \int_0^P e^{-st} \cos at \, dt$$

$$= \lim_{P\to\infty} \left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2} \right]_0^P$$

$$= \lim_{P\to\infty} \left[\frac{e^{-sP}(-s\cos aP + a\sin aP)}{s^2+a^2} + \frac{s}{s^2+a^2} \right]$$

$$= 0 + \frac{s}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2} \quad \text{if } s>0$$

* Similarly, $\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st} e^{iat} \, dt$

$$= \int_0^\infty e^{-(s-ia)t} \, dt$$

$$= \frac{1}{-(s-ia)} \left[e^{-(s-ia)t} \right]_0^\infty$$

$$= \frac{1}{s-ia}$$

$$\Rightarrow \mathcal{L}\{\cos at + i\sin at\} = \frac{s+ia}{s^2+a^2}$$

Now equating real and imaginary pats from both sides, we have

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \quad \text{and} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}.$$

* $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$ if $s>|a|$, $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$ if $s>|a|$.

Properties of Laplace transform.

(A) Linearity Property :-

The Laplace transform is a Linear transform. i.e. If α_1 and α_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transform $f_1(s)$ and $f_2(s)$ respectively, then -

$$\mathcal{L}\{\alpha_1 F_1(t) + \alpha_2 F_2(t)\} = \alpha_1 \mathcal{L}\{F_1(t)\} + \alpha_2 \mathcal{L}\{F_2(t)\}.$$

<u>example</u>: Find the Laplace transform of

$$e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t.$$

<u>Soln</u> Applying Linearity property, we have -

$$\mathcal{L}\{e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t\}$$

$$= \mathcal{L}\{e^{4t}\} + 4\mathcal{L}\{t^3\} - 2\mathcal{L}\{\sin 3t\} + 3\mathcal{L}\{\cos 5t\}.$$

$$= \frac{1}{s-4} + 4 \cdot \frac{3!}{s^4} - 2 \cdot \frac{3}{s^2+3^2} + 3 \cdot \frac{s}{s^2+5^2}.$$

$$= \frac{1}{s-4} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+25}$$

B) First translation (or shifting) property :—

Theorem :- If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at} F(t)\} = f(s-a).$$

Proof:- We have

$$\mathcal{L}\{e^{at}F(t)\} = \int_0^\infty e^{-st}\, e^{at}\, F(t)\, dt$$
$$= \int_0^\infty e^{-(s-a)t}\, F(t)\, dt$$
$$= f(s-a).$$

example:- Find the Laplace transform of $e^{5t}t^3$.

Sol^n We have $\mathcal{L}\{t^3\} = \dfrac{3!}{s^4} = \dfrac{6}{s^4} = f(s)$ (say)

Then by using first shifting property, we have –

$$\mathcal{L}\{e^{5t}t^3\} = f(s-5)$$
$$= \frac{6}{(s-5)^4}.$$

c) Second translation (or shifting) property :–

Theorem :- If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a. \end{cases}$

then $\mathcal{L}\{G(t)\} = e^{-sa}f(s)$.

proof:- We have –

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st}\, G(t)\, dt$$
$$= \int_0^a e^{-st}\, G(t)\, dt + \int_a^\infty e^{-st}\, G(t)\, dt$$
$$= \int_0^a e^{-st}\,(0)\, dt + \int_a^\infty e^{-st}\, F(t-a)\, dt$$
$$= 0 + \int_a^\infty e^{-st}\, F(t-a)\, dt \quad \text{---} (1)$$

Let $t - a = u$ $\therefore dt = du$.

limits: $\left.\begin{matrix} t = a \\ u = 0 \end{matrix}\right\}$ $\left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$.

Thus from ①, we have-

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-s(u+a)} F(u)\, du .$$

$$= e^{-sa} \int_0^\infty e^{-su} F(u)\, du .$$

$$= e^{-sa} f(s) .$$

2) Change of Scale Property:-

Theorem:- If $L\{F(t)\} = f(s)$, then $\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$.

Proof:- We have,

$$\mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at)\, dt \quad \ldots\ldots ①$$

Let $at = u$

$\Rightarrow t = \frac{u}{a}$ $\therefore dt = \frac{du}{a}$.

Thus from ①, we have-

Limits: $\left.\begin{matrix} t = a \\ u = 0 \end{matrix}\right\}$ $\left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$.

$$\mathcal{L}\{F(at)\} = \int_0^\infty e^{-\frac{su}{a}} F(u) \frac{du}{a} .$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right) u} F(u)\, du$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right) .$$

Example: Using change of scale property, find the Laplace transform of $\cos at$.

Sol^n We know that
$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1} = f(s) \quad [\text{say}]$$

$$\therefore \mathcal{L}\{\cos at\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$$= \frac{1}{a} \frac{s/a}{(s/a)^2+1} = \frac{s}{s^2+a^2}.$$

(F) Laplace transform of Derivatives:—

Theorem :- If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$.

proof:- By def^n of Laplace transform, we have —
$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt.$$

$$\therefore \mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt$$

$$= \left[e^{-st} F(t)\right]_0^\infty + s \int_0^\infty e^{-st} F(t)\, dt.$$

$$= 0 - F(0) + sf(s)$$

$$= sf(s) - F(0).$$

Theorem :- If $\mathcal{L}\{F(t)\} = f(s)$, then
$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0).$$

Proof:- By defn of Laplace transform, we have
$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s).$$

$$\therefore \mathcal{L}\{F''(t)\} = \int_0^\infty e^{-st} F''(t)\,dt$$

$$= \left[e^{-st} F'(t)\right]_0^\infty + \int_0^\infty s e^{-st} F'(t)\,dt .$$

$$= 0 - F'(0) + s\int_0^\infty e^{-st} F'(t)\,dt$$

$$= -F'(0) + s\,\mathcal{L}\{F'(t)\}$$

$$= -F'(0) + s\left[sf(s) - F(0)\right]$$

$$= s^2 f(s) - sF(0) - F'(0) .$$

Continuing the process, we can prove that [see last page for proof]

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - s^{n-3}F''(0) - \cdots - F^{n-1}(0)$$

F) Laplace transform of Integrals :—

<u>Theorem</u> :- If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$.

<u>Proof</u>. Let $G(t) = \int_0^t F(u)\,du$.

Then $G'(t) = \frac{d}{dt}\left[\int_0^t F(u)\,du\right] = F(t)$

and $G(0) = \int_0^0 F(u)\,du = 0$

Now $\mathcal{L}\{G'(t)\} = s\,\mathcal{L}\{G(t)\} - G(0)$.

$$\Rightarrow \mathcal{L}\{F(t)\} = s\,\mathcal{L}\{G(t)\} - 0 .$$

$$\Rightarrow f(s) = s\,\mathcal{L}\left\{\int_0^t F(u)\,du\right\} .$$

$$\therefore \mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s} .$$

(G) Derivatives of Laplace transform (Multiplication by powers of t) :-

Theorem :- If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^{(n)}(s)$$

where, $n = 1, 2, 3, \cdots$

Proof:- We have

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt .$$

Then by Leibnitz's rule for differentiating under the integral sign,

$$\frac{df}{ds} = f'(s) = \frac{d}{ds}\int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} F(t)\, dt .$$

$$= \int_0^\infty - t e^{-st} F(t)\, dt$$

$$= -\int_0^\infty e^{-st} \{t F(t)\}\, dt$$

$$= -\mathcal{L}\{t F(t)\} .$$

Thus $\mathcal{L}\{t F(t)\} = -\frac{df}{ds} = -f'(s)$ ...... (1)

which proves the theorem for $n = 1$.

To establish the theorem in general, we use mathematical induction. Assumming that the theorem is true for $n = K$

i.e, $\int_0^\infty e^{-st}\{t^K F(t)\} dt = (-1)^K f^{(K)}(s)$ ...... (2)

Then $\frac{d}{ds}\int_0^\infty e^{-st}\{t^k F(t)\}\,dt = (-1)^k f^{(k+1)}(s)$.

$\Rightarrow -\int_0^\infty e^{-st}\{t^{k+1} F(t)\}\,dt = (-1)^k f^{(k+1)}(s)$ [by Leibnitz's rule]

$\Rightarrow \int_0^\infty e^{-st}\{t^{k+1} F(t)\}\,dt = (-1)^{k+1} f^{(k+1)}(s)$ .....(3)

It follows that if (2) is true i,e if the theorem holds for $n=k$ then (3) is true i,e the theorem holds for $n=k+1$.

But by (1) the theorem is true for $n=1$. Hence it is true for $n=1+1=2$ and $n=2+1=3$ and so on.

Hence the theorem is true for all positive integral values of n.

<u>Example:-</u> Find $\mathcal{L}\{t^2 \cos at\}$.

<u>Soln</u> We know that $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} = f(s)$ (say)

$$\therefore \mathcal{L}\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{2s^3-6a^2 s}{(s^2+a^2)^3}.$$

(H) Division by t :—

Theorem:- If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$.

Proof: Let $G(t) = \frac{F(t)}{t}$

$$\Rightarrow F(t) = t\,G(t)$$

$$\Rightarrow \mathcal{L}\{F(t)\} = \mathcal{L}\{t\,G(t)\}$$

$$\Rightarrow f(s) = (-1)\frac{d}{ds}\mathcal{L}\{G(t)\}.$$

$$\Rightarrow -f(s) = \frac{d}{ds}\mathcal{L}\{G(t)\}.$$

Integrating both sides w.r.to s from s to ∞,

$$\left[\mathcal{L}\{G(t)\}\right]_s^\infty = -\int_s^\infty f(s)\,ds.$$

$$\Rightarrow -\left[\mathcal{L}\{G(t)\}\right]_s^\infty = \int_s^\infty f(s)\,ds.$$

$$\Rightarrow -\lim_{s\to\infty}\mathcal{L}\{G(t)\} + \mathcal{L}\{G(t)\} = \int_s^\infty f(s)\,ds$$

$$\left[\because \lim_{s\to s}\mathcal{L}\{G(t)\} = \mathcal{L}\{G(t)\}\right]$$

$$\Rightarrow -\lim_{s\to\infty}\int_0^\infty e^{-st}G(t)\,dt + \mathcal{L}\{G(t)\} = \int_s^\infty f(s)\,ds.$$

$$\Rightarrow -\lim_{s\to\infty} g(s) + \mathcal{L}\{G(t)\} = \int_s^\infty f(s)\,ds$$

$$\Rightarrow 0 + \mathcal{L}\{G(t)\} = \int_s^\infty f(s)\,ds.$$

Theorem: If $\mathcal{L}\{G(t)\} = g(s)$ then $\lim_{s\to\infty} g(s) = 0$

$$\therefore \mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\,ds$$

$$= \int_s^\infty f(u)\,du.$$

(I) Periodic functions:—

Theorem:- If $F(t)$ has period $T > 0$, then —

$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}.$$

Proof:- If $F(t)$ is a periodic function with period T, then

$$F(t+T) = F(t).$$

Now by definition of Laplace transform, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^T e^{-st} F(t)\, dt + \int_T^{2T} e^{-st} F(t)\, dt + \int_{2T}^{3T} e^{-st} F(t)\, dt + \cdots\cdots$$

In the second integral let $t = u+T$, in the third integral let $t = u+2T$ etc. Then we have —

$$\mathcal{L}\{F(t)\} = \int_0^T e^{-su} F(u)\, du + \int_0^T e^{-s(u+T)} F(u+T)\, du + \int_0^T e^{-s(u+2T)} F(u+2T)\, du + \cdots$$

$$= \int_0^T e^{-su} F(u)\, du + e^{-sT}\int_0^T e^{-su} F(u)\, du + e^{-2sT}\int_0^T e^{-su} F(u)\, du + \cdots$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \cdots\right)\int_0^T e^{-su} F(u)\, du.$$

$\left[\because 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}, |r| < 1\right]$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-su} F(u)\, du.$$

$$= \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}$$ [Proved]

Ex:- Find the Laplace transform of the function

$$F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}.$$

Soln If $F(t)$ has period $T > 0$ then

$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\,dt}{1 - e^{-sT}}$$

In the given problem $F(t)$ has period $T = 2\pi$

$$\therefore \mathcal{L}\{F(t)\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} F(t)\,dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} F(t)\,dt + \int_{\pi}^{2\pi} e^{-st} F(t)\,dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt + \int_{\pi}^{2\pi} e^{-st} \cdot 0\,dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\frac{e^{-st}(-s\sin t - \cos t)}{s^2 + 1} \right]_0^{\pi} + 0$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\frac{e^{-\pi s} + 1}{s^2 + 1} \right]$$

$$= \frac{1}{(1 - e^{-\pi s})(s^2 + 1)}$$

Example: If $F(t) = t^2$, $0 < t < 2$ and $F(t+2) = F(t)$. Then Find $\mathcal{L}\{F(t)\}$.

Ans: $\dfrac{-(4s^2 + 4s + 2)e^{-2s} + 2}{s^3(1 - e^{-2s})}$

Table of Laplace transforms of some elementary functions:—

	$F(t) = \mathcal{L}^{-1}\{f(s)\}$	$\mathcal{L}\{F(t)\} = f(s)$
①	1	$\frac{1}{s}, \; s>0$
②	t	$\frac{1}{s^2}, \; s>0$
③	$t^n, \; n=0,1,2\ldots$	$\frac{n!}{s^{n+1}}, \; s>0$
④	e^{at}	$\frac{1}{s-a}, \; s>a$
⑤	$\sin at$	$\frac{a}{s^2+a^2}, \; s>0$
⑥	$\cos at$	$\frac{s}{s^2+a^2}, \; s>0$
⑦	$\sinh at$	$\frac{a}{s^2-a^2}, \; s>\lvert a\rvert$
⑧	$\cosh at$	$\frac{s}{s^2-a^2}, \; s>\lvert a\rvert$
⑨	$t \sin at$	$\frac{2as}{(s^2+a^2)^2}, \; s>0$
⑩	$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}, \; s>0$
⑪	$t^n e^{at} \; (n=1,2\ldots)$	$\frac{n!}{(s-a)^{n+1}}, \; s>a$
⑫	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
⑬	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
⑭	$e^{at} \sinh bt$	$\frac{b}{(s-a)^2-b^2}$
⑮	$e^{at} \cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$

Inverse Laplace Transform

Def^n :- If $f(s)$ is the Laplace transform of a function $F(t)$, i.e if $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called the inverse Laplace transform of the function $f(s)$ and is symbolically written as

$$F(t) = \mathcal{L}^{-1}\{f(s)\}$$

where $\mathcal{L}^{-1}$ is called the inverse Laplace transformation operator.

Some important properties of the inverse Laplace transform :-

A) Linearity property:-

Theorem:- If $\mathcal{L}\{F_1(t)\} = f_1(s)$ and $\mathcal{L}\{F_2(t)\} = f_2(s)$ and α_1, α_2 are any two constants, then

$$\mathcal{L}^{-1}\{\alpha_1 f_1(s) + \alpha_2 f_2(s)\} = \alpha_1 \mathcal{L}^{-1}\{f_1(s)\} + \alpha_2 \mathcal{L}^{-1}\{f_2(s)\}$$

(B) First translation (or shifting property) :-

Theorem:- If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t).$$

Proof:- By def^n of Laplace transform we have —

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt.$$

$$\therefore f(s-a) = \int_0^\infty e^{-(s-a)t} F(t)\,dt$$

$$= \int_0^\infty e^{-st} \{e^{at} F(t)\}\,dt$$

$$\Rightarrow f(s-a) = \mathcal{L}\{e^{at} F(t)\}$$

$$\therefore \mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t).$$

(C) Theorem :- Second translation (or shifting property) :-

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{e^{-as} f(s)\} = G(t) \quad , \text{where } G(t) = \begin{cases} F(t-a) & , t > a \\ 0 & , t < a. \end{cases}$$

(D) Change of Scale property :-

Theorem :- If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k} F\left(\frac{t}{k}\right).$$

(E) Inverse Laplace Transform of Derivatives :-

Theorem :- If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then.

$$\mathcal{L}^{-1}\{f^{n}(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n} f(s)\right\} = (-1)^n t^n F(t)$$

where, $n = 1, 2, 3, \ldots$

(F) Inverse Laplace transform of integrals :-

Theorem :- If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\int_s^\infty f(u)\,du\right\} = \frac{F(t)}{t}.$$

Multiplication by s^n :-

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$, then

$$\mathcal{L}^{-1}\{s^n f(s)\} = F^n(t).$$

Division by s :-

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du.$$

Example:

Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+4)}\right\}$

Solⁿ By the above theorem we have -

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2} = F(t)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \frac{\sin 2u}{2}\,du.$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2u\right]_0^t$$

$$= \frac{1}{4}\left(1 - \cos 2t\right].$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \int_0^t \frac{1}{4}(1-\cos 2u)\,du$$

$$= \frac{1}{4}\left[u - \frac{1}{2}\sin 2u\right]_0^t$$

$$= \frac{1}{4}t - \frac{1}{8}\sin 2t .$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+4)}\right\} = \int_0^t \left(\frac{1}{4}u - \frac{1}{8}\sin 2u\right) du .$$

$$= \left[\frac{1}{8}u^2 + \frac{1}{16}\cos 2u\right]_0^t .$$

$$= \frac{1}{8}(t^2-0) + \frac{1}{16}(\cos 2t - 1)$$

$$= \frac{1}{8}t^2 + \frac{1}{16}\cos 2t - \frac{1}{16}$$

Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$.

Sol^n Let $f(s) = \frac{1}{s^2+a^2} = \mathcal{L}\{F(t)\}$.

$$\Rightarrow \mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a} = F(t) \quad \cdots\cdots (1)$$

Again,

$$f'(s) = \frac{d}{ds}\left(\frac{1}{s^2+a^2}\right) = \frac{-2s}{(s^2+a^2)^2}$$

$$\Rightarrow \frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right) = -\frac{1}{2}f'(s) .$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\{f'(s)\}$$

$$= -\frac{1}{2}\{-t\,F(t)\} \quad \text{[using Theorem (E)]}$$

$$= \frac{1}{2}t\,F(t)$$

$$= \frac{1}{2}t \cdot \frac{\sin at}{}\quad \text{[by (1)]}$$

Convolution

Let $F(t)$ and $G(t)$ be two functions of a class A, then the onvolution of the two functions $F(t)$ and $G(t)$ denoted by $F * G$ is defined by the relation

$$F * G = \int_0^t F(u)\, G(t-u)\, du .$$

$$= \int_0^t G(u)\, F(t-u)\, du .$$

This relation $F * G$ is also called the resultant or Faltung of F and G.

Properties:-

(i) $F * G$ is commutative i.e, $F * G = G * F$

(ii) $F * G$ is associative. i.e, $(F * G) * H = F * (G * H)$.

(iii) $F * G$ is distributive w.r.to addition.

i.e, $F * (G + H) = F * G + F * H$.

Convolution Theorem:-

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then —

$$\mathcal{L}^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$$

$$= \int_0^t G(u)\, F(t-u)\, du .$$

Proof :-

We have

$$f(s) = \int_0^{\infty} e^{-st} F(t)\, dt \quad \cdots\cdots (1)$$

and $$g(s) = \int_0^{\infty} e^{-s\xi} G(\xi)\, d\xi \quad \cdots (2)$$

Multiplying both sides of (2) by $f(s)$,

$$\therefore f(s)\, g(s) = \int_0^{\infty} e^{-s\xi} f(s)\, G(\xi)\, d\xi \quad \cdots (3)$$

But we know that $L^{-1}\{e^{-as} f(s)\} = F(t-a)\, u(t-a)$

$$\mathcal{L}^{-1}\{e^{-s\xi} f(s)\} = F(t-\xi)\, u(t-\xi) \quad \cdots (4)$$

where, ~~is~~ $u(t-\xi)$ is unit step function (Heaviside's unit function)

where, $$u(t-\xi) = \begin{cases} 0, & 0 \leq t < \xi \\ 1, & t \geq \xi . \end{cases} \quad \cdots (5)$$

From (4), we have –

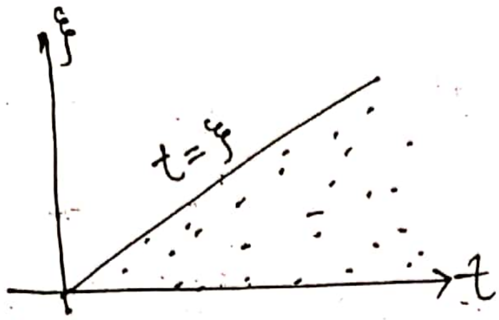
$$e^{-s\xi} f(s) = \mathcal{L}\{F(t-\xi)\, u(t-\xi)\}$$

$$\Rightarrow e^{-s\xi} f(s) = \int_0^{\infty} e^{-st} F(t-\xi)\, u(t-\xi)\, dt \quad \cdots (6)$$

From (3) and (6), we have –

$$f(s)\, g(s) = \int_0^{\infty}\int_0^{\infty} e^{-st} G(\xi)\, F(t-\xi)\, \underline{u(t-\xi)}\, dt\, d\xi .$$

$$= \int_0^{\infty}\int_{t=\xi}^{\infty} e^{-st} G(\xi)\, F(t-\xi)\, dt\, d\xi \quad \cdots\cdots (7)$$



The integration in the $t\xi$ - plane covers the shaded region.
The elements are summed from $t = \xi$ to $t = \infty$ and then from $\xi = 0$ to $\xi = \infty$.

Changing the order of integration, the elements are summed from $\xi = 0$ to $\xi = t$ and from $t = 0$ to $t = \infty$, we get

$$⑦ \Rightarrow f(s)\, g(s) = \int_0^\infty \int_0^t e^{-st}\, G(\xi)\, F(t-\xi)\, d\xi\, dt$$

$$= \int_0^\infty e^{-st} \left\{ \int_0^t G(\xi)\, F(t-\xi)\, d\xi \right\} dt$$

$$= \mathcal{L}\left\{ \int_0^t G(\xi)\, F(t-\xi)\, d\xi \right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{ f(s)\, g(s) \} = \int_0^t G(\xi)\, F(t-\xi)\, d\xi .$$

<u>Theorem</u>:-

$$\mathcal{L}^{-1}\{ e^{-as} f(s) \} = F(t-a)\, u(t-a)$$

Example: Evalute $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ by use of the convolution Theorem

Sol^n.

We can write $\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}$

~~Then since~~ $\cancel{\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}}$

Let $f(s) = \frac{s}{s^2+a^2}$ and $g(s) = \frac{1}{s^2+a^2}$

Since $F(t) = \mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

and $G(t) = \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$.

Therefore, by using the convolution theorem –

$\mathcal{L}^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$, we have –

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au \cdot \frac{\sin a(t-u)}{a}\, du$$

$$= \frac{1}{a}\int_0^t \cos au\,(\sin at \cos au - \cos at \sin au)\, du.$$

$$=$$

$$= \frac{1}{a}\sin at \int_0^t \cos^2 au\, du - \frac{1}{a}\cos at \int_0^t \sin au \cos au\, du$$

$$= \frac{1}{2a}\sin at \int_0^t (1+\cos 2au)\, du - \frac{1}{2a}\cos at \int_0^t \sin 2au\, du.$$

$$= \frac{\sin at}{2a}\left[u + \frac{\sin 2au}{2a}\right]_0^t + \frac{\cos at}{2a}\left[\frac{\cos 2au}{2a}\right]_0^t$$

$$= \frac{\sin at}{2a}\left[t + \frac{\sin 2at}{2a} - 0\right] + \frac{\cos at}{4a^2}\left(\cos 2at - 1\right)$$

$$= \frac{t\sin at}{2a} + \frac{\sin at}{2a} \cdot \frac{2\sin at\cos at}{2a} - \frac{\cos at}{4a^2} \cdot 2\sin^2 at$$

$$= \frac{t\sin at}{2a} + \frac{\sin^2 at\cos at}{2a^2} - \frac{\sin^2 at\cos at}{2a^2}$$

$$= \frac{t\sin at}{2a}$$

Example.:

(By using Convolution Theorem)

Evaluate

(i) $\mathcal{L}^{-1}\left\{\frac{3}{s^2(s+2)}\right\}$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$ Ans:- $te^{-t} + 2e^{-t} + t - 2$

Heaviside's expansion formula

Statement:

Let $P(s)$ and $Q(s)$ be polynomials in s where degree of $P(s)$ < degree of $Q(s)$. If $Q(s)$ has n distinct zeros α_K, $K = 1, 2, \ldots, n$ that is

$$Q(s) = (s-\alpha_1)(s-\alpha_2)\cdots\cdots(s-\alpha_n).$$

then
$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{K=1}^{n} \frac{P(\alpha_K)}{Q'(\alpha_K)} e^{\alpha_K t}$$

$$= \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \cdots\cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)} e^{\alpha_n t}$$

Proof:

Since $P(s)$ is a polynomial of degree less than that of $Q(s)$ and $Q(s)$ has distinct zeros $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$, we can write according to the method of partial fractions —

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \cdots + \frac{A_K}{s-\alpha_K} + \cdots\cdots + \frac{A_n}{s-\alpha_n} \quad \text{—— (1)}$$

Multiplying both sides by $s-\alpha_K$,

$$\frac{P(s)}{Q(s)}(s-\alpha_K) = \frac{A_1(s-\alpha_K)}{s-\alpha_1} + \frac{A_2(s-\alpha_K)}{s-\alpha_2} + \cdots + A_K + \cdots + \frac{A_n(s-\alpha_K)}{s-\alpha_n}$$

Taking $s \to \alpha_K$ on both sides and using L' Hospital's rule, we have

$$\lim_{s \to \alpha_K} \frac{P(s)}{Q(s)}(s-\alpha_K) = 0 + 0 + \cdots + A_K + \cdots + 0$$

$$\Rightarrow A_K = \lim_{s \to \alpha_K} \frac{P(s)}{Q(s)}(s-\alpha_K)$$

$$= \lim_{s \to \alpha_K} P(s) \lim_{s \to \alpha_K} \frac{s-\alpha_K}{Q(s)} \quad \left[\tfrac{0}{0} \text{ form}\right]$$

$$= P(\alpha_K) \lim_{s \to \alpha_K} \frac{1}{Q'(s)} \quad [\text{By L' Hospital's rule}]$$

$$= \frac{P(\alpha_K)}{Q'(\alpha_K)}$$

$$\therefore A_1 = \frac{P(\alpha_1)}{Q'(\alpha_1)}, \quad A_2 = \frac{P(\alpha_2)}{Q'(\alpha_2)}, \quad \cdots, \quad A_n = \frac{P(\alpha_n)}{Q'(\alpha_n)}$$

From ① $\Rightarrow$

$$\frac{P(s)}{Q(s)} = \frac{P(\alpha_1)}{Q'(\alpha_1)} \cdot \frac{1}{s-\alpha_1} + \frac{P(\alpha_2)}{Q'(\alpha_2)} \cdot \frac{1}{s-\alpha_2} + \cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)} \cdot \frac{1}{s-\alpha_n} \quad \text{②}$$

Now taking the inverse Laplace transform of both sides of ②

we get

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(\alpha_1)}{Q'(\alpha_1)} L^{-1}\left\{\frac{1}{s-\alpha_1}\right\} + \frac{P(\alpha_2)}{Q'(\alpha_2)} L^{-1}\left\{\frac{1}{s-\alpha_2}\right\} + \cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)} L^{-1}\left\{\frac{1}{s-\alpha_n}\right\}$$

$$= \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)} e^{\alpha_n t}$$

$$= \sum_{K=1}^{n} \frac{P(\alpha_K)}{Q'(\alpha_K)} e^{\alpha_K t}$$

Example :

Evaluate $\mathcal{L}^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Sol^n

Here, $P(s) = 2s^2-4$, $Q(s) = (s+1)(s-2)(s-3)$

$= (s+1)(s^2-5s+6)$

$= s^3-5s^2+6s+s^2-5s+6$

$= s^3-4s^2+s+6$

$\therefore Q'(s) = 3s^2-8s+1$

$$\frac{P(s)}{Q(s)} = \frac{P(-1)}{Q'(-1)}\cdot\frac{1}{s+1} + \frac{P(2)}{Q'(2)}\cdot\frac{1}{s-2} + \frac{P(3)}{Q'(3)}\,\frac{1}{s-3}$$

$$= \frac{-2}{12}\cdot\frac{1}{s+1} + \frac{4}{-3}\,\frac{1}{s-2} + \frac{14}{4}\cdot\frac{1}{s-3}$$

Heaviside's expansion formula –

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = -\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$$

Ans. $\frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$

Applications of Laplace transforms to Differential Equations

(A) Application of Laplace transform in solving ordinary linear differential equations with constant co-efficients :-

① Solve the following differential equation by using Laplace transform :

$$Y'' + 2Y' + 5Y = e^{-t}\sin t$$

$$Y(0) = 0$$

$$Y'(0) = 1$$

Sol^n The given differential equation is

$$Y'' + 2Y' + 5Y = e^{-t}\sin t \quad \ldots ①$$

Taking the Laplace transform of both sides of ① and using the given condition we get -

$$\mathcal{L}\{Y''\} + 2\mathcal{L}\{Y'\} + 5\mathcal{L}\{Y\} = \mathcal{L}\{e^{-t}\sin t\}$$

$$\Rightarrow s^2 y(s) - sY(0) - Y'(0) + 2\{sy(s) - Y(0)\} + 5y(s) = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow s^2 y(s) - 1 + 2sy(s) + 5y(s) = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow (s^2+2s+5)\,y(s) = 1 + \frac{1}{(s+1)^2+1}$$

$$\Rightarrow y(s) = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s^2+2s+2)}$$

$$= \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} \quad \ldots ②$$

Now

$$\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5} \quad \text{.....(3)}$$

or, $s^2+2s+3 = (As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2)$

or, $s^2+2s+3 = (A+C)s^3+(2A+B+2C+D)s^2+(5A+2B+2C+2D)s+5B+2D$

Equating the co-efficients of s^3, s^2, s and the constant terms from both sides, we get –

$$A+C=0, \quad 2A+B+2C+D=1$$

$$5A+2B+2C+2D=2$$

$$5B+2D=3.$$

Solving these equations for A, B, C, D we get

$$A=0, \quad B=\frac{1}{3}, \quad C=0, \quad D=\frac{2}{3}.$$

$$(3) \Rightarrow \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5}$$

Therefore,

$(2) \Rightarrow$

$$y(s) = \frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5}$$

$$= \frac{1}{3}\cdot\frac{1}{(s+1)^2+1} + \frac{2}{3}\cdot\frac{1}{(s+1)^2+2^2}$$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$\Rightarrow Y = Y(t) = \frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\frac{\sin 2t}{2}$$

$$\therefore Y = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

Solve the following differential equation by using Laplace transform :—

$$Y''' - 3Y'' + 3Y' - Y = t^2 e^t.$$

$$Y(0) = 1, \quad Y'(0) = 0, \quad Y''(0) = -2$$

Soln

The given differential equation is

$$Y''' - 3Y'' + 3Y' - Y = t^2 e^t \quad \ldots\ldots (1)$$

Taking the Laplace transform of both sides of (1) and using the given conditions, we get

$$\mathcal{L}\{Y'''\} - 3\mathcal{L}\{Y''\} + 3\mathcal{L}\{Y'\} - \mathcal{L}\{Y\} = \mathcal{L}\{t^2e^t\}.$$

$$\Rightarrow s^3 y - s^2 Y(0) - sY'(0) - Y''(0) - 3\{s^2 y - sY(0) - Y'(0)\}$$
$$+ 3\{sy - Y(0)\} - y = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right)$$

$$\Rightarrow s^3 y - s^2 + 2 - 3s^2 y + 3s + 3sy - 3 - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow (s^3 - 3s^2 + 3s - 1)y = 1 - 3s + s^2 + \frac{2}{(s-1)^3}$$

$$\Rightarrow y = \frac{1 - 3s + s^2}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\Rightarrow \mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^6}\right\}.$$

$$\Rightarrow Y = e^t - te^t - \frac{1}{2}t^2e^t + \frac{1}{60}t^5e^t$$

1. $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$

2. $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{\ldots}$

Formula:

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = t^n e^{at}$$

Application of Laplace transform in solving ordinary linear differential equations with variable co-efficients.

Solve the following differential equation by using the Laplace transform :-

$$Y'' - tY' + Y = 1$$
$$Y(0) = 1$$
$$Y'(0) = 2 .$$

The given differential equation is

$$Y'' - tY' + Y = 1 \quad \text{----} (1)$$

Taking Laplace transform of both sides of (1) and using the given conditions, we get

$$\mathcal{L}\{Y''\} - \mathcal{L}\{tY'\} + \mathcal{L}\{Y\} = \mathcal{L}\{1\}$$

$$\Rightarrow s^2 y - sY(0) - Y'(0) + \frac{d}{ds}\{sy - Y(0)\} + y = \frac{1}{s}$$

$$\Rightarrow s^2 y - s - 2 + sy' + y - 0 + y = \frac{1}{s}$$

$$\Rightarrow sy' + (s^2+2)y = s + 2 + \frac{1}{s}$$

$$\Rightarrow y' + \left(\frac{s^2+2}{s}\right)y = 1 + \frac{2}{s} + \frac{1}{s^2}$$

$$\Rightarrow \frac{dy}{ds} + \left(s + \frac{2}{s}\right)y = 1 + \frac{2}{s} + \frac{1}{s^2} \quad \text{----} (2)$$

which is a first order diff. eqn.

$$\text{I.F} = e^{\int (s + \frac{2}{s}) ds}$$

$$= e^{\frac{s^2}{2} + 2\log s}$$

$$= e^{s^2/2} \cdot e^{\log s^2}$$

$$= s^2 e^{s^2/2}$$

∴ solution of ② is –

$$s^2 e^{s^2/2} \cdot y = \int \left(1 + \frac{2}{s} + \frac{1}{s^2}\right) s^2 e^{s^2/2}\, ds$$

or, $$y = \frac{1}{s^2} e^{-s^2/2} \int (s^2 + 2s + 1)\, e^{s^2/2}\, ds$$

$$= \frac{1}{s^2} e^{-s^2/2} \left[\int s^2 e^{s^2/2} ds + 2\int s e^{s^2/2} ds + \int e^{s^2/2} \cdot 1\, ds \right]$$

$$= \frac{1}{s^2} e^{-s^2/2} \left[\int s^2 e^{s^2/2} ds + 2 e^{s^2/2} + s e^{s^2/2} - \int e^{s^2/2} \cdot \frac{1}{2} \cdot 2s \cdot s\, ds \right]$$

$$= \frac{1}{s^2} e^{-s^2/2} \left[\int s^2 e^{s^2/2} ds + 2 e^{s^2/2} + s e^{s^2/2} - \int s^2 e^{s^2/2} ds \right]$$

$$= \frac{1}{s^2} e^{-s^2/2} \left[s e^{s^2/2} + 2 e^{s^2/2} + c \right]$$

$$= \frac{1}{s} + \frac{2}{s^2} + \frac{c}{s^2} e^{-s^2/2}$$

$$= \frac{1}{s} + \frac{2}{s^2} + \frac{c}{s^2}\left(1 - \frac{1}{2}s^2 + \frac{1}{8}s^4 - \cdots\cdots\right)$$

$$= \frac{1}{s} + \frac{c+2}{s^2} - c\left(\frac{1}{2} - \frac{1}{8}s^2 + \cdots\right)$$

Taking the inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + (c+2)\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - c\mathcal{L}^{-1}\left\{\frac{1}{2} - \frac{1}{8}s^2 + \cdots\right\}$$

$$\Rightarrow Y = 1 + (c+2)t - c \cdot 0 \qquad \left[\because \mathcal{L}^{-1}\{s^k\} = 0 \text{ for } k = 1, 2, 3 \cdots\right]$$

$$\Rightarrow Y = 1 + (c+2)t \quad \cdots\cdots ③$$

$$\therefore Y'(t) = 0 + c + 2$$

$$\Rightarrow Y'(0) = c + 2$$

$$\Rightarrow 2 = c + 2 \qquad [\because Y'(0) = 2]$$

$$\therefore c = 0$$

∴ From ③ $Y = 1 + 2t$ which is the reqd. solution

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Simultaneous Ordinary Diff. Equations.

Solve
$$\frac{dX}{dt} = 2X - 3Y \quad \ldots\ldots (1)$$
$$\frac{dY}{dt} = Y - 2X \quad \ldots\ldots (2)$$
subject to $X(0) = 8$, $Y(0) = 3$.

Sol^n Taking Laplace transform on both sides of (1) and (2) we get.

$$\mathcal{L}\left\{\frac{dX}{dt}\right\} = \mathcal{L}\{2X - 3Y\}$$
$$\Rightarrow sx - X(0) = 2\mathcal{L}\{X\} - 3\mathcal{L}\{Y\}$$
$$\Rightarrow sx - 8 = 2x - 3y$$
$$\Rightarrow (s-2)x + 3y = 8 \quad \ldots\ldots (3)$$

and $\mathcal{L}\left\{\frac{dY}{dt}\right\} = \mathcal{L}\{Y\} - 2\mathcal{L}\{X\}$
$$\Rightarrow sy - Y(0) = y - 2x$$
$$\Rightarrow sy - 3 = y - 2x$$
$$\Rightarrow 2x + (s-1)y = 3 \quad \ldots\ldots (4)$$

Solving (3) & (4) simultaneously,

$$x = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$$

$$y = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4}$$

$\therefore X = \mathcal{L}^{-1}\{x\} = 5e^{-t} + 3e^{4t}$

$Y = \mathcal{L}^{-1}\{y\} = 5e^{-t} - 2e^{4t}$

Exercise (Schaum's Series)

1. Solve $Y'' + Y = t$; $Y(0) = 1$, $Y'(0) = -2$

   Ans: $Y = t + \cos t - 3\sin t$

2. Solve $Y'' - 3Y' + 2Y = 4e^{2t}$; $Y(0) = -3$, $Y'(0) = 5$

   Ans: $Y = -7e^{t} + 4e^{2t} + 4te^{2t}$

3. Solve $Y''' - 3Y'' + 3Y' - Y = t^2 e^t$

   $Y(0) = 1$, $Y'(0) = 0$, $Y''(0) = -2$

   Ans: $Y = e^t - te^t - \frac{t^2 e^t}{2} + \frac{t^5 e^t}{60}$

4. Solve $Y'' + 9Y = \cos 2t$, $Y(0) = 1$, $Y(\frac{\pi}{2}) = -1$

   Ans: $Y = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t$

5. Solve $Y'' - Y' - 2Y = t^2$

   $Y(0) = 1$, $Y'(0) = 3$

   Ans: $Y = \frac{17}{12}e^{2t} + \frac{1}{3}e^{-t} - \frac{3}{4} + \frac{1}{2}t - \frac{1}{2}t^2$

f $\mathcal{L}\{F(t)\} = f(s)$, then prove that

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - sF^{n-2}(0) - F^{n-1}(0)$$

Given that $\mathcal{L}\{F(t)\} = f(s)$

e shall prove the following theorem by the method of induction.

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - sF^{n-2}(0) - F^{n-1}(0) \quad \ldots (1)$$

the definition of Laplace transform, we have –

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\,dt$$

$$= \left[e^{-st}F(t)\right]_0^\infty + s\int_0^\infty e^{-st}F(t)\,dt$$

$$= 0 - e^0 F(0) + s\,\mathcal{L}\{F(t)\}$$

$$= s f(s) - F(0)$$

Hence the theorem is true for $n = 1$.

et the theorem is true for $n = m$.

i,e, $$\mathcal{L}\{F^m(t)\} = s^m f(s) - s^{m-1}F(0) - s^{m-2}F'(0) - \cdots - sF^{m-2}(0) - F^{m-1}(0) \quad \ldots (3)$$

Now $$\mathcal{L}\{F^m(t)\} = \int_0^\infty e^{-st} F^m(t)\,dt$$

$$= \left[F^m(t)\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty F^{m+1}(t)\frac{e^{-st}}{-s}\,dt$$

$$= -\frac{1}{s}\left[e^{-st}F^m(t)\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}F^{m+1}(t)\,dt$$

$$= -\frac{1}{s}\left(0 - e^0 F^m(0)\right) + \frac{1}{s}\mathcal{L}\{F^{m+1}(t)\}$$

$$\therefore \mathcal{L}\{F^m(t)\} = \frac{1}{s}F^m(0) + \frac{1}{s}\mathcal{L}\{F^{m+1}(t)\} \quad \ldots (4)$$

Now from (3) and (4), we have

$$F^m(0) + \frac{1}{s}\mathcal{L}\{F^{m+1}(t)\} = s^m f(s) - s^{m-1}F(0) - s^{m-2}F'(0) - \cdots - sF^{m-2}(0) - F^{m-1}(0)$$

$$\Rightarrow F^m(0) + \mathcal{L}\{F^{m+1}(t)\} = s^{m+1}f(s) - s^m F(0) - s^{m-1}F'(0) - \cdots - s^2F^{m-2}(0) - sF^{m-1}(0)$$

$$\therefore \mathcal{L}\{F^{m+1}(t)\} = s^{m+1}f(s) - s^m F(0) - s^{m-1}F'(0) - \cdots - sF^{m-1}(0) - F^m(0)$$

so If the theorem is true for $n=m$, then it is true for $n=m+1$.

But the theorem was proved for $n=1$. Therefore the theorem is true for $n=1+1=2$, $n=2+1=3$, $n=3+1=4$ ... etc.

Hence the theorem is true for all positive integral n.

i.e. $\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - sF^{n-2}(0) - F^{n-1}(0)$.