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Project Report

Mixed Element Formulation of the Stokes Equations

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Contents

1 Strong Form of Stokes Equations

The Stokes equations are obtained in the limit of low Reynolds' number flows of the Navier-Stokes equations, when the convective processes are weaker than the rest of the processes. The strong formulation of the Stokes problem in two dimensions will be presented below.

Find the velocity $u = (u_1, u_2)$ and the pressure 'p' such that,

$$-\Delta\mathbf{u} + \nabla\mathbf{p} = \mathbf{f} \quad \text{in } \Omega \tag{1}$$

$$\nabla.\mathbf{u} = 0 \quad \text{in } \Omega \tag{2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \tag{3}$$

where Ω is a bounded domain in R^2 with boundary $\partial\Omega$.

2 Description of Mixed Element Formulation

In the stokes' problem, we have two dependent variables to solve for, *i.e.*, (u_1, u_2)(velocity) and p(pressure). One way to approach such a problem is to try and eliminate one of the dependent variables by differentiation. This will introduce extra constraints on the problem and will also increase the order of the resulting partial differential equation. In our case, we can differentiate the x-momentum with respect to 'y' and the y-momentum equation with respect to 'x' and use the continuity equation to eliminate pressure variable. This will result in a PDE of higher order which can be solved using the standard FEM techniques. But in this case, we need to use a function space that contains functions with higher order continuity, which is really rare. So, in order to use function spaces of lower order continuity, we use a method called the 'Mixed Element Method', where two different function spaces will be used to approximate each of the dependent variables. The next step is to choose a pair of function spaces that can be used to approximate the variables and formulate the matrix-vector equations. As it turns out, one can't choose any random combinations of function spaces. An intuitive explanation(without proof) will be given in the next section.

2.1 Inf-Sup Condition

In order to solve the Stokes' Equations using the Mixed Element method, we need to choose a pair of function spaces that can be used to approximate the two dependent variables. While choosing the function space for a PDE with a single dependent variable we need to see if a certain set of conditions imposed on the defined norms and functionals of the space are satisfied. These conditions are used as a part of the *Lax-Milgram Theorem* to prove the existence and uniqueness of the solution. Boundedness, Symmetry and Ellipticity(Coercivity) are the general conditions that are usually imposed on the functionals defined. Of particular interest is the ellipticity(coercivity) condition, that is an integral part of the *Lax-Milgram Theorem* which has been stated below.

Lax-Milgram Theorem: Suppose that H is a Hilbert space over the field of real numbers, with inner product $(.,.)_H$ and the induced norm $||.||_H$ defined by $||v||_H^2 = (v, v)_H$. Suppose further that *a(.,.)* is a bilinear functional on $H \times H$, l is a linear functional on H, and the following additional properties hold:

- The bilinear functional 'a' is coercive(elliptic), *i.e.*, there exits a positive real number c_a such that

$$a(v, v) \geq c_a ||v||_H^2 \qquad \forall \text{ v in H} \tag{4}$$

- The bilinear functional 'a' is bounded, *i.e.*, there exists a positive real number C_a such that

$$|a(v, w)| \leq C_a ||w||_H ||v||_H \qquad \forall \text{ v,w in H} \tag{5}$$

- The linear functional 'l' is bounded, *i.e.*, there exists a positive real number C_l such that

$$|l(v)| \leq C_l ||v||_H \qquad \forall \text{ v in H} \tag{6}$$

Then there exists a unique $u \in H$ such that $a(u, v) = l(v)$ for all $v \in H$.
We see that, the first condition necessary is the coercivity of the bilinear functional. As we will see in the next section, the weak formulation of a problem using the mixed element method will usually involve a bilinear functional that is defined on the cartesian product of the two different Hilbert Spaces(that is defined by us) instead of one Hibert Space. So, the inf-sup condition is a generalised coercivity condition. This condition(without the proof) has been given below.
If $b(.,.)$ is a bilinear functional defined on $X \times M$, then the inf-sup condition states that,

$$\exists c_b > 0 \text{ s.t. } c_b \leq inf_{q \in M-\{0\}}\, sup_{w \in X-\{0\}} \frac{b(w, q)}{||w||_X ||q||_M} \tag{7}$$

3 Variational/Weak formulation of the Stokes Problem

In this section, the weak form of the Stokes equations in relation to Mixed Element formulation have been derived in detail. First, the weak form is stated and then the derivation will be presented.

Weak Formulation:
Consider the spaces,

$$V = \left[H_0^1(\Omega)\right]^2 = \left\{v = (v_1, v_2) : v_i \in H_0^1(\Omega), i = 1, 2\right\} \tag{8}$$

$$H = \left\{q \in L_2(\Omega) : \int_\Omega q\, dx \equiv 0\right\} \tag{9}$$

Here, $H_0^1(\Omega)$ is a Hilbert Space with the following definition,

$$H_0^1(\Omega) = \{\text{v and it's first derivatives} \in L_2(\Omega) : v = 0 \text{ on } \partial\Omega\} \tag{10}$$

$L_2(\Omega)$ is defined as,

$$L_2(\Omega) = \left\{v : \text{v is defined on } \Omega \text{ and } \int_\Omega v^2\, dx < \infty\right\} \tag{11}$$

Now, we can state the weak form of the Stokes problem:

Find $(\mathbf{u}, \text{p}) \in V \times H$ such that,

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla.\mathbf{v}) = (f, \mathbf{v}) \qquad \forall\, v \in V \tag{12}$$

$$(q, \nabla.\mathbf{u}) = 0 \qquad \forall\, q \in H \tag{13}$$

where (.,.) denotes the L_2-inner products defined by,

$$(\nabla \mathbf{w}, \nabla \mathbf{v}) = \sum_{i=1}^{2} \int_{\Omega} \nabla w_i . \nabla v_i \, dx \tag{14}$$

$$(f, \mathbf{v}) = \sum_{i=1}^{2} \int_{\Omega} f_i v_i \, dx \tag{15}$$

3.1 Finite Dimensional Weak Formulation

We replace the infinite dimensional spaces by the subspaces V_h and H_h - the finite dimensional analogues to obtain the finite dimensional weak formulation.

Find $(u_h, p_h) \in V_h \times H_h$ such that,

$$(\nabla u_h, \nabla \mathbf{v}) - (p_h, \nabla.\mathbf{v}) = (f, \mathbf{v}) \qquad \forall\, v \in V_h \tag{16}$$

$$(q, \nabla.u_h) = 0 \qquad \forall\, q \in H_h \tag{17}$$

The inner products have the same definition as the infinite dimensional spaces.

The domain(Ω), which is a subset of R^2, is divided into a triangulation $T_h = \{K\}$ of non-overlapping rectangles.

Based on the description of the choice of space-combinations given in the previous section, we choose the Q_2-Q_0 combination. This combination has been chosen due to the ease in computing the inner-products of the continuity equation while assembling the stiffness matrix.

The definition of the Q_2 and Q_0 spaces are as follows,

Q_2 is the set of bi-quadratic functions on each rectangle(K).

$$Q_2(K) = \left\{ v : v(x_1, x_2) = \sum_{i,j=0}^{2} a_{ij} x_1^i x_2^j, (x_1, x_2) \in K, \text{where the } a_{ij} \in \mathbf{R} \right\}.$$

Q_0 is the set of constant functions on each rectangle of the triangulation(T_h).

$Q_0(K) = \{v : v(x_1, x_2) = a_{00}, (x_1, x_2) \in K, \text{where the } a_{00} \in \mathbf{R}\}$.

Now, we define the contents of the finite dimensional sub-spaces, V_h and H_h.

$V_h = \{v \in V : v|_K \in [Q_2(K)]^2, \forall K \in T_h\}$.

$H_h = \{q \in H : q|_K \in Q_0(K), \forall K \in T_h\}$.

3.1.1 Problem Formulation:

Let $\{\phi_1, \ldots, \phi_n\}$ be the bases for V_h for the global nodes $1, 2, 3, \ldots, n$(Here, these are the number of nodes for the velocity variable).

Let $\{\psi_1, \ldots, \psi_m\}$ be the bases for H_h for the global nodes $1, 2, 3, \ldots, m$(Here, these are the number of nodes for the pressure variable).

The basic procedure is to first solve for u_h using the finite dimensional continuity equation and use it in the finite dimensional momentum equation to solve for p_h.

$q = \sum\limits_{i=1}^{m} c_i\psi_i$

c_i are arbitrary degrees of freedom for test function 'q'.

$p_h = \sum\limits_{i=1}^{m} \theta_i\psi_i$

θ_i are the degrees of freedom for p_n. They are defined on the center of the rectangular elements of the triangulation belonging to the space H_h.

$u_h = \sum\limits_{j=1}^{n} \chi_j\phi_j$

χ_j are the vector degrees of freedom for the velocity nodes.

$v = \sum\limits_{j=1}^{n-n_d} d_j\phi_j$

d_j are the vector degrees of freedom for the test function in the velocity space. It has to be pointed out that the upper limit of the velocity space test function will be less than that of the total number of velocity nodes because of the dirichlet boundary conditions whose degrees of freedom we already know(n_d).

The objective is to solve the set of equations to obtain the values of the following vectors.

$$\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix}_{n\times 1} \qquad \chi_i = \begin{bmatrix} \chi_{i1} \\ \chi_{i2} \end{bmatrix} \qquad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{bmatrix}_{m\times 1} \tag{18}$$

First, we consider the weak form of the continuity equation.

$$(q, \nabla.u_h) = 0 \tag{19}$$

Writing q as a linear combination of the test functions of H_h as given above.

$$\Rightarrow \left(\sum_{i=1}^{m} c_i\psi_i, \nabla.u_h \right) = 0 \tag{20}$$

$$\Rightarrow \sum_{i=1}^{m} c_i(\psi_i, \nabla.u_h) = 0 \tag{21}$$

Since, c_i are arbitrary, each of it's coefficients must be equal to zero. So, we will have a set of m equations as follows.

$$(\psi_i, \nabla.u_h) = 0 \qquad \forall \quad i = 1, 2, \ldots, m \tag{22}$$

Now, we write u_h as a linear combination of the test functions of V_h.

$$\left(\psi_i, \nabla.\sum_{j=1}^{n}(\chi_j \phi_j)\right) = 0 \qquad \forall \quad i = 1, 2, \ldots, m \tag{23}$$

$$\Rightarrow \sum_{j=1}^{n}(\psi_i, \nabla.(\chi_j \phi_j)) = 0 \qquad \forall \quad i = 1, 2, \ldots, m \tag{24}$$

Since χ_j is a vector degree of freedom, we expand the $\nabla.(\chi_j \phi_j)$ part of the above equation for the sake of clarity.

$$\nabla.(\chi_j \phi_j) = \chi_{j1}\phi_{j,1} + \chi_{j2}\phi_{j,2} = \{\chi_j\}.\{\nabla \phi_j\} \tag{25}$$

$$\sum_{j=1}^{n}(\psi_i, \{\chi_j\}.\{\nabla \phi_j\}) = 0 \qquad \forall \quad i = 1, 2, \ldots, m \tag{26}$$

$$\Rightarrow \sum_{j=1}^{n}(\psi_i, \{\nabla \phi_j\}).\{\chi_j\} = 0 \qquad \forall \quad i = 1, 2, \ldots, m \tag{27}$$

$$(\psi_i, \{\nabla \phi_j\}) = \begin{bmatrix} (\psi_i, \phi_{j,1}) \\ (\psi_i, \phi_{j,2}) \end{bmatrix} \tag{28}$$

A rectangular matrix will result from the above set of m equations as given below.

$$B_1^T \chi = 0 \tag{29}$$

Here, $[B_{1ij}]^T_{m \times 2n}$ and $[\chi_j]_{2n \times 1}$. i goes from 1 to m and j goes from 1 to $2n$.

Now, we perform a similar operation to the weak form of the momentum equation.

Consider the term, $(\nabla u_h, \nabla v)$

$$(\nabla u_h, \nabla v) = \left(\nabla u_h, \nabla(\sum_{j=1}^{n-n_d} d_j \phi_j)\right) \tag{30}$$

$$\Rightarrow (\nabla u_h, \nabla v) = \sum_{j=1}^{n-n_d} (\nabla u_h, \nabla(d_j \phi_j)) \tag{31}$$

$$\Rightarrow (\nabla u_h, \nabla v) = \sum_{j=1}^{n-n_d} \int_\Omega \sum_{i=1}^{2} d_{ji}(\nabla u_{hi}).(\nabla \phi_j)\mathrm{dx} \tag{32}$$

$$\Rightarrow (\nabla u_h, \nabla v) = \sum_{j=1}^{n-n_d} \sum_{i=1}^{2} d_{ji} \int_\Omega (\nabla u_{hi}).(\nabla \phi_j)\mathrm{dx} \tag{33}$$

$$\Rightarrow (\nabla u_h, \nabla v) = \sum_{j=1}^{n-n_d} \begin{bmatrix} \int_\Omega (\nabla u_{h1}).(\nabla \phi_j)\mathrm{dx} \\ \int_\Omega (\nabla u_{h2}).(\nabla \phi_j)\mathrm{dx} \end{bmatrix}^T . \begin{bmatrix} d_{j1} \\ d_{j2} \end{bmatrix} \tag{34}$$

Consider the second term, $(p_h, \nabla v)$

$$(p_h, \nabla v) = \left(p_h, \nabla(\sum_{j=i}^{n-n_d} d_j\phi_j)\right) \tag{35}$$

$$\Rightarrow (p_h, \nabla v) = \sum_{j=i}^{n-n_d} (p_h, \nabla(d_j\phi_j)) \tag{36}$$

$$\Rightarrow (p_h, \nabla v) = \sum_{j=i}^{n-n_d} (p_h, \{d_j\}.\{\nabla\phi_j\}) \tag{37}$$

$$\Rightarrow (p_h, \nabla v) = \sum_{j=i}^{n-n_d} (p_h, \nabla\phi_j).\{d_j\} \tag{38}$$

Here,

$$(p_h, \nabla\phi_j) = \begin{bmatrix} (p_h, \phi_{j,1}) \\ (p_h, \phi_{j,2}) \end{bmatrix}^T \cdot \begin{bmatrix} d_{j1} \\ d_{j2} \end{bmatrix} \tag{39}$$

Finally, we consider the term on the right hand side, (f, v)

$$(f, v) = \sum_{j=1}^{n-n_d} (f, d_j\phi_j) \tag{40}$$

$$\Rightarrow (f, v) = \sum_{j=1}^{n-n_d} \int_\Omega \sum_{i=1}^{2} f_i d_{ji}\phi_j \mathrm{dx} \tag{41}$$

$$\Rightarrow (f, v) = \sum_{j=1}^{n-n_d} \sum_{i=1}^{2} d_{ij} \int_\Omega f_i\phi_j \mathrm{dx} \tag{42}$$

$$\Rightarrow (f, v) = \sum_{j=1}^{n-n_d} (f, \phi_j).\{d_j\} \tag{43}$$

Now, we substitute all of the above evaluated terms in the main equation,

$$\sum_{j=1}^{n-n_d} \begin{bmatrix} \int_\Omega (\nabla u_{h1}).(\nabla\phi_j)\mathrm{dx} \\ \int_\Omega (\nabla u_{h2}).(\nabla\phi_j)\mathrm{dx} \end{bmatrix}^T \cdot \begin{bmatrix} d_{j1} \\ d_{j2} \end{bmatrix} - \begin{bmatrix} (p_h, \phi_{j,1}) \\ (p_h, \phi_{j,2}) \end{bmatrix}^T \cdot \begin{bmatrix} d_{j1} \\ d_{j2} \end{bmatrix} = \sum_{j=1}^{n-n_d} \begin{bmatrix} (f_1, \phi_j) \\ (f_2, \phi_j) \end{bmatrix}^T \cdot \begin{bmatrix} d_{j1} \\ d_{j2} \end{bmatrix} \tag{44}$$

Since, $\{d_j\}$ are arbitrary degrees of freedom, we can write the following,

$$\begin{bmatrix} \int_\Omega (\nabla u_{h1}).(\nabla\phi_j)\mathrm{dx} \\ \\ \int_\Omega (\nabla u_{h2}).(\nabla\phi_j)\mathrm{dx} \end{bmatrix}^T - \begin{bmatrix} (p_h, \phi_{j,1}) \\ \\ (p_h, \phi_{j,2}) \end{bmatrix}^T = \begin{bmatrix} (f_1, \phi_j) \\ \\ (f_2, \phi_j) \end{bmatrix}^T$$

$$\forall j = 1, 2, ..., n - n_d$$

Now we expand the u_h and p_h terms in terms of their degrees of freedom and test functions.

$$\Rightarrow \begin{bmatrix} \sum_{k=1}^{n} \chi_{k1} \int_\Omega (\nabla\phi_k).(\nabla\phi_j)\mathrm{dx} \\ \\ \sum_{k=1}^{n} \chi_{k2} \int_\Omega (\nabla\phi_k).(\nabla\phi_j)\mathrm{dx} \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^{m} \theta_i(\psi_i, \phi_{j,1}) \\ \\ \sum_{i=1}^{m} \theta_i(\psi_i, \phi_{j,2}) \end{bmatrix} = \begin{bmatrix} (f_1, \phi_j) \\ \\ (f_2, \phi_j) \end{bmatrix}$$

$$\Rightarrow \sum_{k=1}^{n} \left(\int_\Omega (\nabla\phi_j).(\nabla\phi_k)\mathrm{dx} \right) \begin{bmatrix} \chi_{k1} \\ \\ \chi_{k2} \end{bmatrix} - \sum_{i=1}^{m} \theta_i \begin{bmatrix} (\psi_i, \phi_{j,1}) \\ \\ (\psi_i, \phi_{j,2}) \end{bmatrix} = \begin{bmatrix} (f_1, \phi_j) \\ \\ (f_2, \phi_j) \end{bmatrix}$$
$$\forall j = 1, 2, 3, \ldots, n - n_d$$

The above expressions can be collectively written as a system of linear algebraic equations of the form,

$$A_1\chi - B\theta = F_1 \tag{45}$$

$$A_{1jk} = \int_\Omega (\nabla\phi_j).(\nabla\phi_k)\,[\mathbf{I}_2] \tag{46}$$

$[\mathbf{I}_2]$ is a 2×2 Identity matrix.

$$B_{ij} = (\psi_j, \nabla\phi_i) \tag{47}$$

$[A_1]_{2n\times 2n}$

$[B]_{2(n-n_d)\times m}$

$F_j = (f, \phi_j)$, is a column vector of size $2(n - n_d) \times 1$

Now, we remove the columns from $[A_1]$ and $[B_1]^T$ corresponding to the dirichlet boundary conditions and the respective values of χ to get the $[A]$ and $[B]$ matrices.Then we take the removed columns to the right hand side and merge them with the forcing function column vector.The resulting equations can now be solved for χ and θ.

$$A\chi - B\theta = F \tag{48}$$

$$B^T\chi = D \tag{49}$$

In order to solve the system, we use certain perturbation method as follows.

Consider ϵ to be a very small constant. We add $\epsilon\theta$ to the second equation to get,

$$\epsilon\theta + B^T\chi = D \tag{50}$$

Multiplying with B and dividing by ϵ on both sides, we get,

$$B\theta + \frac{1}{\epsilon}BB^T\chi = \frac{BD}{\epsilon} \tag{51}$$

Adding the above equation and the set of system of equations, we get the following, which can be solved for χ.

$$(A + \frac{1}{\epsilon}BB^T)\chi = F + \frac{BD}{\epsilon} \tag{52}$$

After solving for χ, we can use it in the perturbed equation to get the value of theta.

$$\theta = \frac{1}{\epsilon}(D - B^T\chi) \tag{53}$$

Finally, we do the calculations that will determine the pressure correction values.

$$\int_\Omega p_h \, dx = 0 \tag{54}$$

$$\Rightarrow \sum_{i=1}^{m} \theta_i \int_\Omega \psi_i \, dx = 0 \tag{55}$$

Since, $\psi_i = 1$ for our case of functional space, the expression reduces to,

$$Area(\Omega) \times \sum_{i=1}^{m} \theta_i = 0 \tag{56}$$

Since, area of our domain is non-zero and constant,

$$\sum_{i=1}^{m} \theta_i = 0 \tag{57}$$

3.2 Matrix Element Evaluations

In this section, we will compute A matrix. For Q_2 space we consider, basis functions as follows

$$A_{1\,jk} = \int_\Omega (\nabla\phi_j).(\nabla\phi_k) \quad [I_2]$$

$$\phi_j = ab(x-\alpha_1)(x-\beta 1)(y-\gamma_1)(y-\delta_1)$$
$$\phi_k = ab(x-\alpha_2)(x-\beta 2)(y-\gamma_2)(y-\delta_2)$$

$$\nabla\phi_j = ab(2x-(\alpha_1+\beta_1))(y-\gamma_1)(y-\delta_1)]\hat{\imath} + [ab(2y-(\gamma_1+\delta_1))(x-\alpha_1)(x-\beta_1)]\hat{\jmath}$$
$$\nabla\phi_k = [cd(2x-(\alpha_2+\beta_2))(y-\gamma_2)(y-\delta_2)]\hat{\imath} + [cd(2y-(\gamma_2+\delta_2))(x-\alpha_2)(x-\beta_2)]\hat{\jmath}$$

$$\nabla\phi_j.\nabla\phi_k = \underbrace{ab[(2x-(\alpha_1+\beta_1)(2x-(\alpha_2+\beta_2)]}_{X_1}\underbrace{cd(y-\gamma_1)(y-\delta_1)(y-\gamma_2)(y-\delta_2)}_{Y_1}$$
$$+\underbrace{ab[(2y-(\gamma_1+\delta_1))(2y-(\gamma_2+\delta_2)]}_{Y_2}\underbrace{cd(x-\alpha_1)(x-\alpha_2)(x-\beta_1)(x-\beta_2)}_{X_2}$$

$$ab = \frac{1}{(x_j-\alpha_1)(x_j-\beta_1)(y_j-\gamma_1)(y_j-\delta_1)}$$
$$cd = \frac{1}{(x_k-\alpha_2)(x_k-\beta_2)(y_k-\gamma_2)(y_k-\delta_2)}$$

where α_1 and β_1 are the x coordinates of the neighbouring elemental nodes in the x direction of global node j. γ_1 and δ_1 are the y coordinates of the neighbouring elemental nodes in the y direction of global node j.

α_2 and β_2 are the x coordinates of the neighbouring elemental nodes in the x direction of global node k.

γ_2 and δ_2 are the y coordinates of the neighbouring elemental nodes in the y direction of global node k.

x_j and y_j are the x and y co-oridnates of the global node j, similarly x_k and y_k are the x and y co-ordinates of the global node k.

We can now segregate X_1, X_2, Y_1, Y_2 and perform change of variables to standardize the integration limits from -1 to 1. This will give us the various acal functions as mentioned in the code. Finally, these functions are integrated using three point Gauss-Legendre quadrature to find $A_{1(ij)}$

Similarly, $B^T_{1(ij)}$ and $F_{1(ij)}$is found by integrating basis functions using three point Gauss-Legendre quadrature.

The rows of A_1 matrix and the columns of A_1, B_1, F_1 matrices corresponding to the Dirichlet degrees of freedom are eliminated to get A,B and F matrices.

4 Algorithm Development

1. Node to Coordinate Matrix : This matrix has two columns every row corresponds to the global node number, the first column represents the X co-ordinates and the second one represents the Y-Co-ordinate.

2. Connectivity matrix: In this matrix, the row corresponds to the element number, each column number corresponds to the local node number and the value corresponds to the global node number.

3. Integration Matrix : This is a three dimensional matrix, for every i and j the third dimension represents the elements of integration for A_{ij}.

4. LocalNumber1 : For every (i,j) the third dimension corresponds to the local node numbers of global node i in the elements common to global nodes i and j.

5. LocalNumber2: For every (i,j) the third dimension corresponds to the local node numbers of global node j in the elements common to global nodes i and j.

6. These matrices, were used to find the elements of integration and solve for the value of A_{ij} using Gaussian quadrature.

7. IntegrationB and localnumberB are used to find the elements of integration and solve for the value of B_{ij} using Gaussian quadrature.

8. DiriN: this is a column matrix which stores the Dirichlet global nodes.

9. Sorted is a column matrix in which all the Dirichlet global nodes are in descending order.

10. Theta and X are the matrices which stores the solution of Pressure and velocity respectively.

11. Successive Over Relaxation method is used for solving the final system of linear equations in $\{\chi\}$.

5 Final Matrix Solver Algorithm- Successive Over-relaxation Method

This is one of the widely used iterative methods of solving a linear system of algebraic equations. It is a modification of the well-known Gauss-Siedel Method. The expression for calculating the new iteration terms will now contain a weighting coefficient(commonly called the relaxation parameter - ω) to accelerate the convergence of the iterative process.
Let $x_i^\star$ be the i_{th} component of the solution obtained using the Gauss-Seidel procedure, then the $(n+1)^{th}$ iteration is computed using the formula,

$$x_i^{n+1} = (1-\omega)x_i^n + \omega x_i^\star \tag{58}$$

Replacing $x_i^\star$ with the general i_th equation,

$$x_i^{n+1} = x_i^n + \omega\left[\frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x^{n+1_j} - \sum_{j=i+1}^{N} a_{ij}x_i^n\right) - x_i^n\right] \tag{59}$$

The algorithm implemented in the code has been given below.

- Input the load and stiffness matrix. Specify the relaxation factor(ω). Initialise the initial guess.
- Compute a new vector from the initial guess using the Gauss-Seidel formula.
- Compute the relative error and begin a loop for computing the subsequent iterations that will terminate after the relative error falls below a threshold value.

6 Problem Statement and Results

Here, we show the dimensions of the domain and give the boundary conditions.

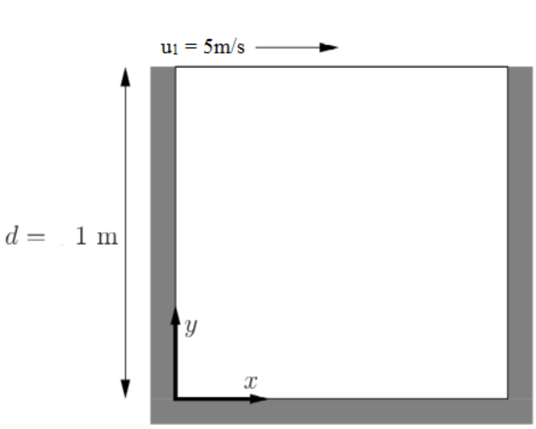

Figure 1: The computational domain of the problem. The grey shaded areas are the walls with no-slip and zero-penetration conditions.

The velocity and pressure profiles of the solved problem with increasing number of elements is shown below.

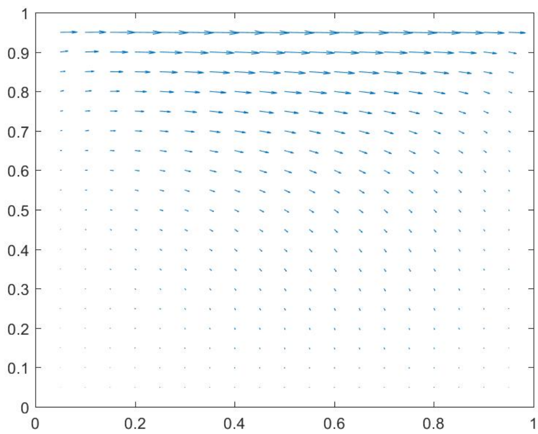

Figure 2: Velocity Field
10×10 Elements

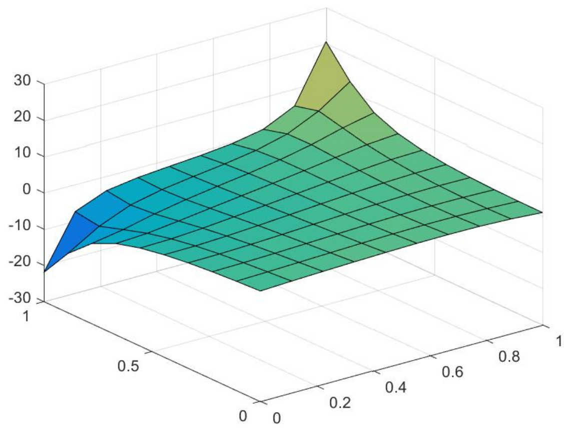

Figure 3: Pressure Field
10×10 Elements

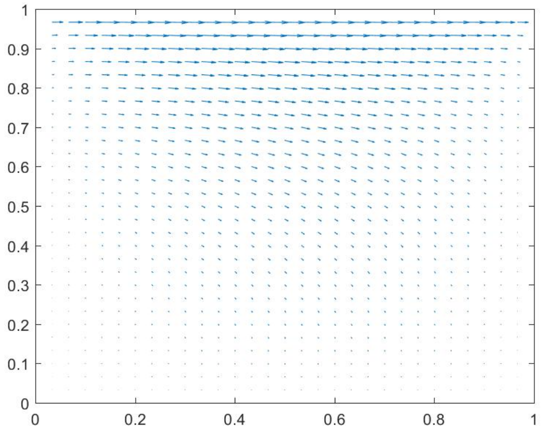

Figure 4: Velocity Field
15 × 15 Elements

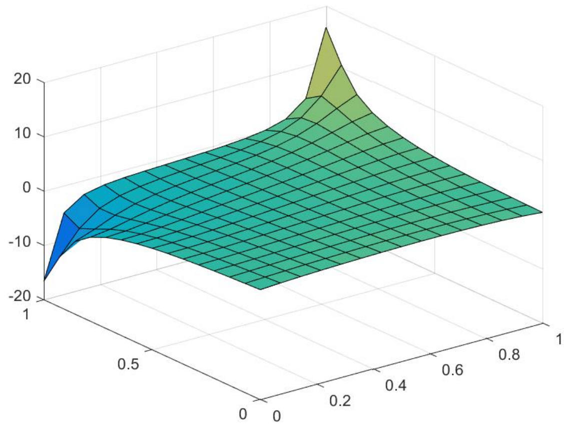

Figure 5: Pressure Field
15 × 15 Elements

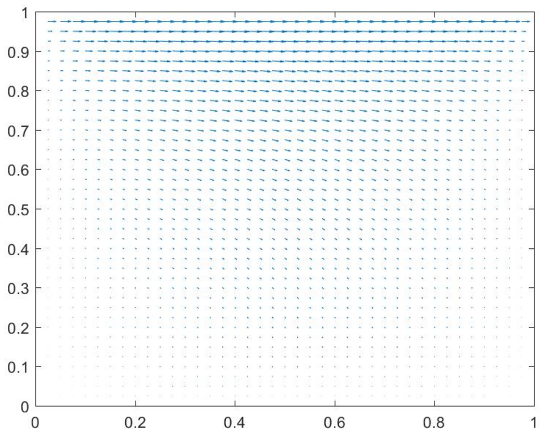

Figure 6: Velocity Field
20 × 20 Elements

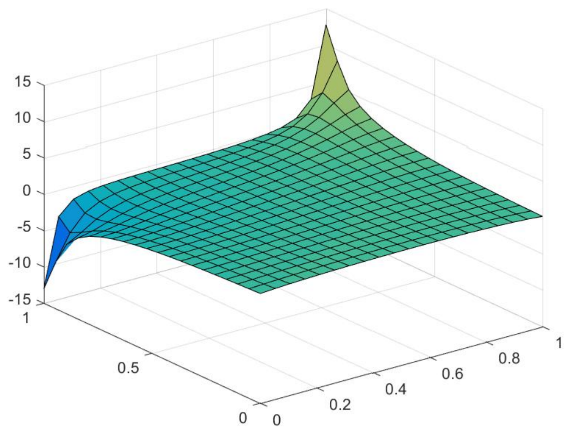

Figure 7: Pressure Field
20 × 20 Elements

6.1 OpenFOAM results

Here, we present the results of a similar case(same physical constant values, boundary conditions and initial conditions) that has been solved in the open-source Finite Volume Method solver - OpenFOAM.

Although the velocity fields are similar to look at, we can see a re-circulation zone in the case of the FVM case because in the FEM case, we have only solved for the pressure gradient and the viscous terms of the navier stokes equations. But in the FVM case, the full navier-stokes equations have been solved, which includes the convective terms which have contributed to the extra momentum transport and hence have resulted in a complete re-circulation zone.

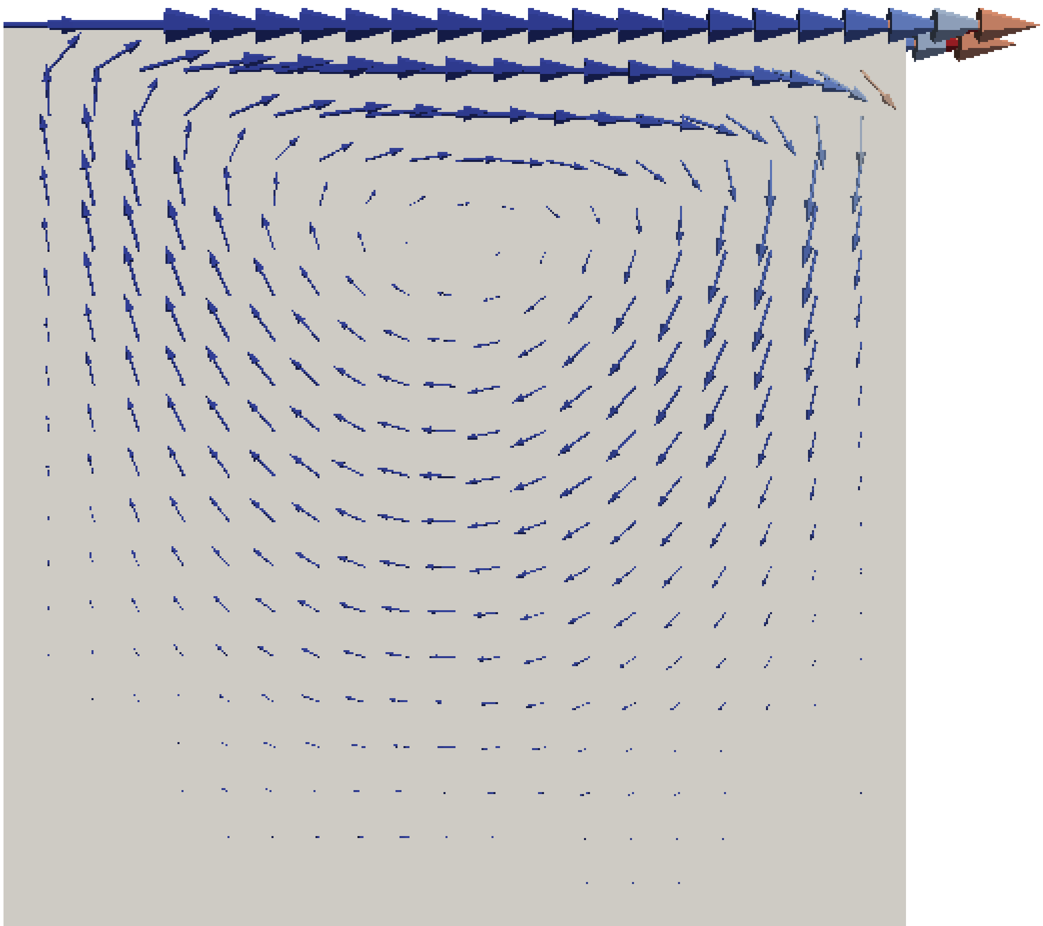

Figure 8: Velocity field of the problem that has been solved using Finite Volume Method.