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Problem 2

(a) $a_0 = 1$ | $a_0 = \{\ \}$

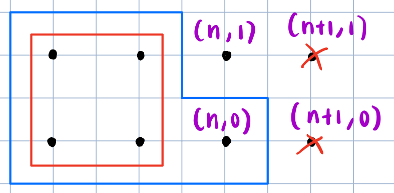
$a_1 = 3$ | $a_1 = \{\ \}, \{(0,0)\}, \{(0,1)\}$

$b_0 = 1$ | $b_0 = \{(0,0)\}$

$b_1 = 2$ | $b_1 = \{(1,0)\}, \{(0,1), (1,0)\}$

(b) $a_{n+1} = a_n + 2b_n$

Here is L_{n+1}, ladder of order $n+1$,



a_{n+1} = number of independent sets in L_{n+1} which include neither $(n+1,0)$ and $(n+1,1)$

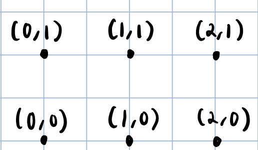
To find a_{n+1} recurrence relations with a_n and b_n, we can look at L_{n+1} as L_n with 2 new vertices $(a_{n+1},0)$ and $(a_{n+1},1)$.
Since a_{n+1} is no. of independent sets in L_{n+1} which include neither $(n+1,0)$ and $(n+1,1)$,
it is basically c_n, the total number of independent sets in L_n

a_n include all independent sets in L_n, in which the independent set must not include vertices $(n,0)$ and $(n,1)$.
It is all independent sets in the red box.

Hence, in order to find independent sets containing $(n,0)$ and $(n,1)$, we will look at b_n.

b_n include all independent sets in L_n and each independent set must include $(n,0)$ but not $(n,1)$
Since the independent sets in b_n must contains $(n,0)$, we know that the independent set included in b_n does not overlap with those in a_n (as a_n's independent set must not include $(n,0)$)

However, we are still missing the number of independent sets that include $(n,1)$.
Let the number of independent sets including $(n,1)$ be b_n'.
We can find out $b_n' = b_n$ by drawing a simple example of L_2.



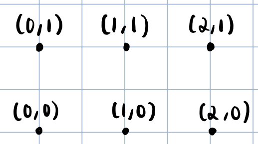
$b_2 = \{(2,0)\}, \{(2,0),(0,1)\}, \{(2,0),(1,1)\}, \{(2,0),(0,0)\}$
$\{(2,0),(1,1),(0,0)\}$

$b_2' = \{(2,1)\}, \{(2,1),(1,0)\}, \{(2,1),(0,0)\}, \{(2,1),(0,1)\}$
$\{(2,1),(1,0),(0,1)\}$

$\therefore |b_2'| = |b_2|$

Hence $a_{n+1} = a_n + 2b_n$

$a_{n+1} = a_n + 2b_n$ can be shown using example too,



$a_2 = \{\ \}, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}$

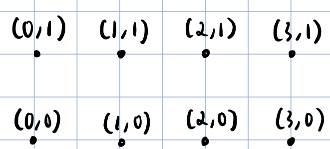
$\{(0,0),(1,1)\}, \{(0,1),(1,0)\} = 7$

$b_2 = \{(2,0)\}, \{(2,0),(0,1)\}, \{(2,0),(1,1)\}, \{(2,0),(0,0)\}$

$\{(2,0),(1,1),(0,0)\} = 5$

$b_2' = \{(2,1)\}, \{(2,1),(1,0)\}, \{(2,1),(0,0)\}, \{(2,1),(0,1)\}$

$\{(2,1),(1,0),(0,1)\} = 5$



$a_3 = \{\ \}, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}, \{(2,0)\}, \{(2,1)\},$

$\{(0,0),(1,1)\}, \{(0,0),(2,1)\}, \{(0,0),(2,0)\},$

$\{(0,1),(1,0)\}, \{(0,1),(2,0)\}, \{(0,1),(2,1)\}$

$\{(1,0),(2,1)\}, \{(1,1),(2,0)\},$

$\{(0,0),(1,1),(2,0)\}, \{(0,1),(1,0),(2,1)\} = 17$

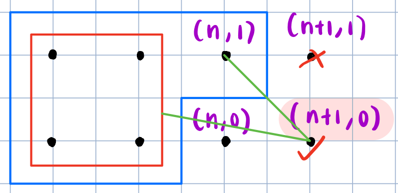
$a_3 = a_2 + 2b_2$

$17 = 7 + 2(5)$

$17 = 17$, which is true

$b_{n+1} = b_n + a_n$

Here is L_{n+1}, ladder of order $n+1$,



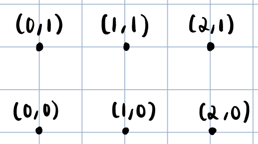
b_{n+1} = number of independent sets in L_{n+1} which include $(n+1, 0)$ but not $(n+1, 1)$.
In other word, it is the number of all independent sets in L_{n+1} in which each independent set must include $(n+1, 0)$.
(we can ignore the condition of not including $(n+1, 1)$, since if $(n+1, 0)$ must be included, $(n+1, 1)$ will not be included as it is adjacent to $(n+1, 0)$).

To find b_{n+1} recurrence relations with a_n and b_n, we can look at L_{n+1} as L_n with 2 new vertices $(a_{n+1}, 0)$ and $(a_{n+1}, 1)$.

a_n include all independent sets in L_n, the independent set must not include vertices $(n, 0)$ and $(n, 1)$.
It is the all independent sets in the red box.
if we add vertice $(n+1, 0)$ into the independent sets included in a_n, these independent sets make parts of b_{n+1}

However, the red box does not take into consideration $(n, 0)$ and $(n, 1)$.
Since b_{n+1} must include $(n+1, 0)$, the $(n, 0)$ vertice cannot in included in b_{n+1} as it is adjacent to $(n+1, 0)$.
Previously we have already established that number of independent sets that include $(n, 1)$ is equal to b_n.
Hence, if we add vertice $(n+1, 0)$ into the independent sets that include $(n, 1)$, which has the same amount as b_n, it covers the remaining part of b_{n+1}.

$b_{n+1} = a_n + b_n$ can be shown using example too,



$$a_2 = \{\ \}, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}$$

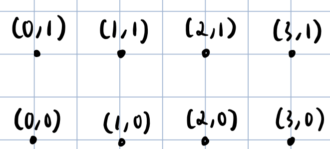
$$\{(0,0),(1,1)\}, \{(0,1),(1,0)\} = 7$$

$$b_2 = \{(2,0)\}, \{(2,0),(0,1)\}, \{(2,0),(1,1)\}, \{(2,0),(0,0)\}$$

$$\{(2,0),(1,1),(0,0)\} = 5$$

$$b_2' = \{(2,1)\}, \{(2,1),(1,0)\}, \{(2,1),(0,0)\}, \{(2,1),(0,1)\}$$

$$\{(2,1),(1,0),(0,1)\}$$



$$b_3 = \{(3,0)\}, \{(3,0),(2,1)\}, \{(3,0),(1,1)\}, \{(3,0),(0,1)\}$$

$$\{(3,0),(1,0)\}, \{(3,0),(0,0)\}$$

$$\{(3,0),(2,1),(1,0)\}, \{(3,0),(2,1),(0,0)\}, \{(3,0),(2,1),(0,1)\}$$

$$\{(3,0),(1,0),(0,1)\},$$

$$\{(3,0),(1,1),(0,0)\},$$

$$\{(3,0),(2,1),(1,0),(0,1)\} = 12$$

$b_3 = a_2 + b_2$
$12 = 7 + 5$
$12 = 12$, which is true

(c) Let $P(n)$ be the statement that $a_n \leq \sqrt{2}(\sqrt{2}+1)^n$ and $b_n \leq (\sqrt{2}+1)^n$ for $n \geq 0$

Inductive basis :

When $n=0$, we know that $a_0 = 1$ and $b_0 = 1$

$1 \leq \sqrt{2}(\sqrt{2}+1)^0$ and $1 \leq (\sqrt{2}+1)^0$

$1 \leq \sqrt{2}$ and $1 \leq 1$

$\therefore P(n)$ is true

Inductive step :

Assume that $P(n)$ is true for $n=k$, where k is some integer ≥ 0

↳ ie $a_k \leq \sqrt{2}(\sqrt{2}+1)^k$ and $b_k \leq (\sqrt{2}+1)^k$

we have to prove that $P(n)$ is true for $n=k+1$

↳ prove that $P(k) \rightarrow P(k+1)$

↳ ie prove that $a_{k+1} \leq \sqrt{2}(\sqrt{2}+1)^{k+1}$ and $b_{k+1} \leq (\sqrt{2}+1)^{k+1}$

$a_{k+1} \leq \sqrt{2}(\sqrt{2}+1)^{k+1}$

From question 2(b), we know that $a_{k+1} = a_k + 2b_k$, hence

$a_k + 2b_k \leq \sqrt{2}(\sqrt{2}+1)^{k+1}$

with the assumption that $a_k \leq \sqrt{2}(\sqrt{2}+1)^k$ and $b_k \leq (\sqrt{2}+1)^k$ is true

$\sqrt{2}(\sqrt{2}+1)^k + 2(\sqrt{2}+1)^k \leq \sqrt{2}(\sqrt{2}+1)^{k+1}$

$(\sqrt{2}+2)(\sqrt{2}+1)^k \leq \sqrt{2}(\sqrt{2}+1)^k(\sqrt{2}+1)$

$(\sqrt{2}+2)(\sqrt{2}+1)^k \leq (\sqrt{2}+2)(\sqrt{2}+1)^k$

$(\sqrt{2}+1)^k \leq (\sqrt{2}+1)^k$ //

$b_{k+1} \leq (\sqrt{2}+1)^{k+1}$

From question 2(b), we know that $b_{k+1} = a_k + b_k$, hence

$a_k + b_k \leq (\sqrt{2}+1)^{k+1}$

with the assumption that $a_k \leq \sqrt{2}(\sqrt{2}+1)^k$ and $b_k \leq (\sqrt{2}+1)^k$ is true

$\sqrt{2}(\sqrt{2}+1)^k + (\sqrt{2}+1)^k \leq (\sqrt{2}+1)^{k+1}$

$(\sqrt{2}+1)(\sqrt{2}+1)^k \leq (\sqrt{2}+1)^k(\sqrt{2}+1)$

$(\sqrt{2}+1)^k \leq (\sqrt{2}+1)^k$ //

Conclusion

by Principle of Mathematical induction, it is true $a_n \leq \sqrt{2}(\sqrt{2}+1)^n$ and $b_n \leq (\sqrt{2}+1)^n$ for $n \geq 0$.

(d) Since a_{n+1} is no. of independent sets in L_{n+1} which include neither $(n+1, 0)$ and $(n+1, 1)$, it is basically c_n, the total number of independent sets in L_n

$$c_n = a_{n+1}$$

In part (c), we proved that $a_{n+1} \leq \sqrt{2}(\sqrt{2}+1)^{n+1}$
Hence,

$$c_n \leq \sqrt{2}(\sqrt{2}+1)^{n+1}$$

$\therefore$ $\sqrt{2}(\sqrt{2}+1)^{n+1}$ is a good closed-form upper bound for c_n.