MAT2120 Number Theory
Lecture Notes

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1 Divisibility Theory

1.1 September 5th

(Class missed; all principles mentioned in class are written below at September 5th lecture notes.)

1.2 September 10th

1. **Well Ordering Principle**(WOP). Every nonempty set of positive integers has a least element.
2. **Principle of Mathematical Induction**. Let S be a set of positive integers. If S satisfies the following two conditions
   (a) $1 \in S$
   (b) $n \in S \Rightarrow n+1 \in S$
   then S is the set of all positive integers.
3. **Archimedian property**. $\forall a, b \in \mathbb{N}$, then $\exists n \in \mathbb{N}$ such that $na > b$.

Remark 1. $1 \Leftrightarrow 2 \Rightarrow 3$.

Definition 1. *If $a, b \in \mathbb{Z}$, then a* ***divides*** *b, denoted by $a \mid b$, if $c \in \mathbb{Z}$ such that $b = ac$.*

We write $a \nmid b$ if a does not divide b.

Theorem 1 (The Division Algorithm). *If $a, b \in \mathbb{Z}$, $b > 0$, then there are unique integers q and r such that*

$$a = bq + r$$

where $0 \leq r < b$.

Proof. Consider

$$S = \{a - bk \mid k \in \mathbb{Z}\}.$$

Let T be the set of all nonnegative integers in S. Since $T \neq \emptyset$, it follows the WOP, thus T has a least element of $r = a - bq$, and it is clear that $r \geq 0$.

We will claim that $r < b$. Suppose $r \geq b$. then

$$\begin{aligned} r &> r - b \\ &= a - bq - b \\ &= a - (q+1)b \geq 0. \end{aligned}$$

This contradicts to the choice of r: which is that r is the minimum element of S. Hence, $r < b$.

We will claim that q and r are unique. Suppose that $a = bq_1 + r_1 = bq_2 + r_2$, where $0 \leq r_1, r_2 < b$. Note that

$$\begin{aligned} 0 &= b(q_1 - q_2) + (r_1 - r_2) \\ \Rightarrow r_2 = r_1 &= b(q_1 - q_2), \end{aligned}$$

thus $b \mid (r_2 - r_1)$.

Since $0 \leq r_1, r_2 < b$, we have $-1 < r_2 - r_1 < b$. Thus $r_2 - r_1 = 0$, i. e. $r_1 = r_2$. Since $bq_1 + r_1 = bq_2 + r_2$, $q_1 = q_2$. □

Remark 2. 1. If $a, b \in \mathbb{Z}$, $b \neq 0$ then $a = bq + r$, where $0 \leq r < |b|$.
2. If $f(x) = g(x)q(x) + r(x)$, then $0 \leq \deg r(x) < \deg g(x)$.

Theorem 2 (Greatest Common Divisor). *Suppose $a, b \in \mathbb{Z}$, where $a \neq 0$ and $b \neq 0$. Then $\exists! d \in \mathbb{Z}$ satisfying the followings:*

1. $d > 0$.
2. $d \mid a$, $d \mid b$.
3. $k \mid a$, $k \mid b \Rightarrow k \mid d$.

Proof. By WOP, we may choose d to be the least positive integer of the form[1]

$$d = ax + by \qquad x, y \in \mathbb{Z}.$$

It is clear that $d > 0$, and if $k \mid a$, $k \mid d$ then $k \mid (ax + by) = d$.

Note that by the division algorithm, $\exists t, u \in \mathbb{Z}$ such that $a = dt + u$ where $0 \leq u < d$. Then

$$\begin{aligned} dt + u &= (ax + by)t + u \\ &= axt + byt + u, \end{aligned}$$

and so

$$a(1 - xt) + b(-yt) = u.$$

Since $u < d$, it follows the minimality of d that $u = 0$, thus $d \mid a$. Similarily we can show that $d \mid b$.

Remains to show the uniqueness of d. Suppose that d' satifies above conditions. then $d \mid d'$ and $d' \mid d$. Hence $d = d'$, because $d, d' > 0$.

Definition 2 (Greatest Common Divisor). $a, b \in \mathbb{Z}$, $a \neq 0$, $b \neq 0$.

The unique positive integer d given by the theorem above is called the ***greatest common divisor*** *of a and b. It is denoted by* $\gcd(a, b)$*, or* (a, b).

[1] Consider $S = \{ax + by \mid x, y \in \mathbb{Z}\} \supset T = \{s \in S \mid s > 0\} \neq \emptyset$. By WOP, $\exists d \in T$.

Remark 3. 1. $(a,0) = |a|$, $(0,0) := 0$.
2. $7 = (14, 21)$.

Theorem 3. *For any $m \in \mathbb{Z}$,*

$$m\mathbb{Z} := \{mx \mid x \in \mathbb{Z}\}$$

is closed under $+$ and $-$.[2]

Conversely, if a nonempty subset S of $\mathbb{Z}$ is closed under $+$ and $-$, then $\exists! m \geq 0 \in \mathbb{Z}$ such that $S = m\mathbb{Z}$.

1.3 September 19th

Proof. $S \subset \mathbb{Z} \Rightarrow S = m\mathbb{Z}$

Since $S \neq \emptyset$, $\exists a \in S$

Since S is closed under $+$, $-$, $0 \in S$. We may assume that $S \neq \{0\}$. (if $S = \{0\}$, then $S = 0 \cdot \mathbb{Z}$)

Take any $n \in S$. Then $0 - n = -n \in S$. Thus we may also assume that S has a positive integer.

In all, WLOG[3], we may assume that S has a positive integer.

$A = B \Rightarrow A \subset B$ and $B \subset A$
$A \subset B \Rightarrow$ if $x \in A$ then $x \in B$

By WOP, S has a least positive integer m. We want to show that $S = m\mathbb{Z}$.

1. $m\mathbb{Z} \subset S$
   $m \in S$ and S is closed under $+$, $-$. So S must have all multiples of m.
2. $S \subset m\mathbb{Z}$
   Take any $a \in S$. By division algorithm, $\exists q, r \in \mathbb{Z}$ such that $a = qm + r$ where $0 \leq r < m$. Since $mq \in S$ and $a \in S$,

$$r = a - mq \in S$$

[3] Without loss of generality

. Thus $r = 0$ by the minimality of m. Hence $a = mq \in m\mathbb{Z}$.

Remains to show the uniqueness of m. Suppose $m\mathbb{Z} = S = m'\mathbb{Z}$. Then $m = \pm m'$. Since $m, m' > 0$, $m = m'$.

Theorem 4. *Let $d = (a, b)$. Then $d = ax + by$ for some $x, y \in \mathbb{Z}$ and $\{ax + by \mid x, y \in \mathbb{Z}\}$ is the set of all multiples of d. i. e. $a\mathbb{Z} + b\mathbb{Z} = \{ax + by \mid x, y \in \mathbb{Z}\}$.*

Proof. We knew that $d = ax + by$ for some $x, y \in \mathbb{Z}$. (by the theorem in the last class)

Define $S := a\mathbb{Z} + b\mathbb{Z}$. Then $a\mathbb{Z} \subset S$ and $b\mathbb{Z} \subset S$. Since S is closed under $+$, $-$, it follows the previous theorem that

$$\exists m \geq 0 \in \mathbb{Z} \text{ such that } S = m\mathbb{Z}.$$

We want to show that $m = d$. Since $a, b \in S = m\mathbb{Z}$, $m \mid a$, $m \mid b$. If $e \mid a$ and $e \mid b$, then $e \mid m$. ($\because m + as + bt$ for some $s, t \in \mathbb{Z}$)

By the definition of GCD, $m = d$.

Remark 4. The GCD of a and b (not both 0) is the least positive integer that is a linear combination of a and b.

Theorem 5 (Euclidean Algorithm). $a, b \in \mathbb{Z}$, $a \neq 0$. *Using the division algorithm,*

$$b = aq_1 + r_1, \text{ where } 0 < r_1 < |a|.$$

If $r_1 = 0$, terminate process.

Repeating process,

$$
\begin{aligned}
a &= r_1 q_2 + r_2 & 0 < r_2 < r_1 \\
r_1 &= r_2 q_3 + r_3 & 0 < r_3 < r_2 \\
&\vdots \\
r_{n-2} &= r_{n-1} q_n + r_n & 0 < r_n < r_{n-1} \\
r_{n-1} &= r_n q_{n+1}
\end{aligned}
$$

Then $(a, b) = r_n$.

Proof. Clearly, $r_n > 0$. Note that

$$
\begin{aligned}
r_n \mid r_{n-1}, r_n \mid r_n &\Rightarrow r_n \mid r_{n-2} \\
r_n \mid r_{n-2}, r_n \mid r_{n-1} &\Rightarrow r_n \mid r_{n-3} \\
&\vdots \\
r_n \mid r_1, r_n \mid r_2 &\Rightarrow r_n \mid a \\
r_n \mid a, r_n \mid r_1 &\Rightarrow r_n \mid b
\end{aligned}
$$

Note also that if

$$
\begin{aligned}
k \mid a, k \mid b &\Rightarrow k \mid r_1 \\
k \mid r_1, k \mid a &\Rightarrow k \mid r_2 \\
&\vdots \\
k \mid r_n, k \mid r_{n-1} &\Rightarrow k \mid r_n
\end{aligned}
$$

Hence we conclude that $r_n = (a, b)$.

Proof (Alternate proof).

$$
b = aq + r \Rightarrow (a, b) = (a, r) \qquad r = a(-q) + b, b = aq + r
$$

Note that $e \mid a, e \mid b$ iff $e \mid r, e \mid a$. Thus $(a,b) \mid (a,b)$ and $(a,k) \mid (a,b)$.

Hence $(a,b) = (a,r)$, since $(a,b) > 0$ and $(a,k) > 0$. Therefore we can see that

$$(a,b) = (a,r) = (r_1, r_2) = \cdots = (r_{n-1}, r_n).$$

Example

$$\begin{aligned}
(68, 710) &= 2 \\
710 &= 68 \cdot 10 + 30 \\
68 &= 30 \cdot 2 + 8 \\
30 &= 8 \cdot 3 + 6 \\
8 &= 6 \cdot 1 + 2 \\
6 &= 2 \cdot 3
\end{aligned}$$

$$\begin{aligned}
2 &= 8 - 6 \cdot 1 \\
&= 8 - (30 - 8 \cdot 3) \\
&= 8 \cdot 4 + 30 \cdot (-1) \\
&= (68 - 30 \cdot 2) \cdot 4 + 30 \cdot (-1) \\
&= 68 \cdot 4 + 30 \cdot (-1) \\
&= 68 \cdot 4 + (710 - 68 \cdot 10) \cdot (-9) \\
&= 68 \cdot 94 + 710 \cdot (-9)
\end{aligned}$$

> **Definition 3 (Diophantine Equation).** *A **Diophantine equation** is a polynomial equation that allows two or more variables to take integer values only.*

e. g.

$$\begin{aligned}
ax + by &= c \\
x^n + y^n &= z^n \\
x^2 - dy^2 &= 1
\end{aligned}$$

Theorem 6. *$a \neq 0$, $b \neq 0$.*

1. *The equation $ax + by = c$ has integer solutions if and only if $(a, b) \mid c$.*
2. *Suppose that $(a, b) \mid c$. Then the general solution of the equation $ax + by = c$ has form the of*
$$\left\{x_0 + \frac{b}{(a, b)}t,\, y_0 - \frac{a}{(a, b)}t\right\}$$
where $t \in \mathbb{Z}$ and (x_0, y_0) is an arbitrary solution of the equation.

General solution for $y'' - 4y' + 3y = 0$?
$\Rightarrow c_1 e^x + c_2 e^{3x}$
– 2 bases

1.4 September 24th

Proof. Note that

$$\begin{aligned} &a \mid b,\, a \mid c \Rightarrow a \mid (bx + cy) \qquad \forall x, y \in \mathbb{Z} \\ &m \mid ab,\, (m, a) = 1 \Rightarrow m \mid b \qquad \because (m, a) = 1,\, \exists s, t \in \mathbb{Z} \qquad as + mt = 1 \end{aligned}$$

Then $bas + bmt = b$.

Since $m \mid ab$, it follows that $m \mid b$.

1. $(\Rightarrow)$ $(a, b) \mid a$, $(a, b) \mid b \Rightarrow (a, b) \mid (ax + by) = c$
   $(\Leftarrow)$ Let $(a, b) = d$ and $c = c_1 d$. Then $\exists s, t \in \mathbb{Z}$ such that $as + bt = d$. thus
   $$\begin{aligned} c = c_1 d &= c_1 (as + bt) \\ &= ac_1 s + bc_1 t \end{aligned}$$
   hence $(c_1 s, c_1 t)$ is a solution.

2. Note that

$$
\begin{aligned}
& a\left(x_0+\frac{b}{d}t\right)+b\left(y_0-\frac{a}{d}t\right) \\
=& ax_0+\frac{ab}{d}t+by_0-\frac{ba}{d}t \\
=& ax_0+by_0=c
\end{aligned}
$$

Suppose that (x, y) is an arbitrary solution of $ax+by=c$. Since $ax+by=c=ax_0+by_0$, we have

$$a\left(x-x_0\right)=b\left(y_0-y\right).$$

Let $a=a_1d$, $b=b_1d$, where $d=(a, b)$. Then

$$a_1\left(x-x_0\right)=b_1\left(y_0-y\right).$$

Since $(a, b)=1$, $b_1 \mid (x-x_0)$. Then $\exists t \in \mathbb{Z}$ such that $x-x_0=b_1t$, and similarily $y_0-y=a_1t$. Hence

$$x=x_0+\frac{b}{(a, b)}t, y=y_0-\frac{a}{(a, b)}t.$$

Example

$$710x+68y=6$$

[4] Recall

$$
\begin{gathered}
710\cdot(-9)+68\cdot 94=2 \\
710\cdot(-9\times 3)+68\cdot(94\times 3)=2\times 3=6
\end{gathered}
$$

Hence

$$
\begin{aligned}
& x=-27+\frac{68}{2}t=-27+34t \\
& x=282-\frac{710}{2}t=282-355t
\end{aligned}
$$

[4] Maybe an eaxm problem?

Definition 4 (Least Common Multiple). *The **least common multiple** of two nonzero integers a and b, denoted* $[a, b]$ *or* $\operatorname{lcm}(a, b)$ *is the integer l satisfying the followings:*

1. $l > 0$.
2. $a \mid l$, $b \mid l$.
3. $a \mid c$, $b \mid c \Rightarrow l \mid c$.

Theorem 7. *For* $a \neq 0, b \neq 0 \in \mathbb{Z}$, $[a, b]$ *uniquely exists. Moreover,* $a\mathbb{Z} \cap b\mathbb{Z} = [a, b]\mathbb{Z}$.

Proof. Let $S = a\mathbb{Z} \cap b\mathbb{Z}$. Since $ab \in S$, $S \neq \emptyset$. Clearly, S is closed under $+, -$.

By theorem, $\exists l$ such that $S = l\mathbb{Z}$.

We want to show that $l = [a, b]$. Since $l \in S$, $a \mid l$, $b \mid l$. If $a \mid c$, $b \mid c$, then $c \in S = l\mathbb{Z}$ and $l \mid c$.

Remains to show the uniqueness of l. Suppose l_1 and l_2 are both the LCMs of a and b. Then $l_1 \mid l_2$ and $l_2 \mid l_1$. By (2), (3), $l_1 = l_2$, since $l_1 > 0$, $l_2 > 0$.

Remark 5.

$$(a, b)\mathbb{Z} = a\mathbb{Z} + b\mathbb{Z} = \{ax + by \mid x, y \in \mathbb{Z}\}$$

Recall

1. $(0, 0) := 0$
2. $(a, 0) := |a|$
3. $[0, 0] := 0$
4. $[a, 0] := 0$

Theorem 8. *For $a > 0, b > 0 \in \mathbb{Z}$,*

$$(a, b)\,[a, b] = ab.$$

Proof. (Proof left for homework – due September 26th.)

Theorem 9. *Let b be a positive integer with $b > 1$. Then every positive integer n can be expressed in unique form of*

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b^1 + a_0$$

where $a_i \in \mathbb{Z}$, $0 \leq a_i \leq b-1$ for $i = 0, 1, \cdots, k$ and $a_k \neq 0$.

$b \Rightarrow$ base.

Proof. We use the division algorithm. (Proof left for homework – due September 26th.)

Definition 5 (Prime Numbers). *A **prime** is an integer p such that*

1. $p > 1$.
2. $a \mid p \Rightarrow a = \pm 1$ *or* $\pm p$.

Remark 6. p is prime.

1. $\forall a \in \mathbb{Z}$, $(a, p) = 1$ or $(a, p) = p$. (iff p is prime)
2. $p \mid ab \Rightarrow p \mid a$ or $p \mid b$. (iff p is prime)

Theorem 10 (Infinitude of Primes). *There exists infinitely many primes.*

Proof (Euclid's).

Lemma 1. *Every positive integer $n \geq 2$ has a prime factor.*

Proof. Consider the set $S = \{m \mid m \text{ is a divisor of } n\}$. Then $S \neq \emptyset$.

By WOP, $\exists$ least positive integer $p \in S$. Note that every divisor of p is also a divisor of n. Thus p is a prime number by the minimality of p.

Suppose there exists finitely many primes

$$p_1, p_2, \cdots, p_k.$$

Let

$$n := p_1 p_2 \times \cdots \times p_k.$$

Then $n > 1$ and $\exists$ prime p such that $p \mid n$ by Lemma 1.

Thus $p = p_i$ for some $1 \leq i \leq k$, hence $p \mid p_1 p_2 \times \cdots \times p_k$, thus

$$p \mid (n - p_1 p_2 \times \cdots \times p_k) \Rightarrow p \mid 1.$$

Which is a contradiction to the definition of prime numbers. Thus there exists infinitely many primes.

Theorem 11. *There are arbitrary large gaps between successive primes. i. e. For any positive integer n, there exists at least n consecutive composite positive integers.*

Proof. Consider n consecutive integers

$$(n+1)!+2,\ (n+1)!+3,\ \cdots,\ (n+1)!+(n+1)\,.$$

For $2 \le j \le n+1$, it is clear that $j \mid (n+1)!$. Thus $j \mid ((n+1)!+j)$.

Hence $\exists n$ consecutive integers which are all composites.

> **Definition 6 (Mersenne Primes).** *A* ***Mersenne prime*** *is a Mersenne number*[5] *which is also prime.*

e. g. $M_2 = 2^2 - 1 = 3$, $M_3 = 2^3 - 1 = 7$, $M_5 = 2^5 - 1 = 31$, $M_7 = 2^7 - 1 = 127$, $\cdots$ but $M_{11} = 2^{11} - 1 = 2047 = 23 \times 89$

1.5 September 26th

It can be seen that

1. If $2^n - 1$ is prime, then n is prime.
2. If a and p are positive integers such that $a^p - 1$ is prime, then $a = 2$ or $p = 1$. [6]

The converse of 1. does not hold. (e. g. $2^{11} - 1 = 23 \times 89$)

Question Are there infinitely many Mersenne primes? $\Rightarrow$ yet unknown! **O**nly **G**od **K**nows

Remark 7. Using Mersenne numbers and some theorem of groups[7], we can show the infinitude of primes.

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[6] Proof exists at Wikipedia

[7] Lagrange theorem

Example $2^{11213}-1$ is prime (1963)

$2^{82589933}-1$ is prime (2018)

Definition 7 (Fermat Primes). *A **Fermat prime** is a Fermat number*[8] *which is also prime.*

e.g. $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$: the only known Fermat primes.

Theorem 12. *If* 2^m+1 *is an odd prime, then* m *is a power of 2.*

Proof. If m is a positive integer and is not a power of 2, then

$$m = rs$$

where $1 \leq r, s < m$ and s is odd. Note that for any $n \in \mathbb{Z}^+$,

$$(a-b) \mid \left(a^l - b^l\right).$$

Put $a = 2^r$, $b = -1$, $l = s$. Then

$$(2^r+1) \mid (2^{rs}+1) \Rightarrow (2^r+1) \mid (2^m+1).$$

Since $1 < 2^r+1 < 2^m+1$, it follows that 2^m+1 is not prime. $\rightarrow\leftarrow$

Theorem 13. *A regular polygon of n sides can be constructed using an unmarked ruler and compass if and only if*

$$n = 2^m \qquad or \qquad n = 2^r p_1 p_2 \times \cdots \times p_k$$

where $m \geq 2$, $r \geq 0$ *and* $p_1, p_2, \cdots, p_k$ *are distinct Fermat primes.*

e. g.

$$
\begin{aligned}
3 &= 2^{2^0}+1 && : \text{constructive} \\
5 &= 2^{2^1}+1 && : \text{constructive} \\
7 & && : \text{not constructive} \\
17 &= 2^{2^2}+1 && : \text{constructive}
\end{aligned}
$$

Theorem 14.

$$(F_m, F_n) = 1$$

if $m \neq n \in \mathbb{Z}^+ \cup \{0\}$.

Proof. **Claim** $F_n = F_0F_1 \times \cdots \times F_{n-1} + 2$ where $n \geq 1$.

$n = 1$. $F_1 = 5$; $F_0 + 2 = 3 + 2 = 5$.
$n = 2$. $F_2 = 17$; $F_0F_1 + 2 = 3 \times 5 + 2 = 17$.
Inductive step. Assume that the claim is true for $s \leq k$. Then

$$
\begin{aligned}
& F_0F_1 \times \cdots \times F_k + 2 \\
=& (F_0F_1 \times \cdots \times F_{k-1})\, F_k + 2 \\
=& (F_k + 2)\, F_k + 2 \\
=& F_k^2 - 2F_k + 2 \\
=& (F_k - 1)^2 + 1 \\
=& 2^{2^{k+1}} + 1 = F_{k+1}.
\end{aligned}
$$

Note that for $i = 0, 1, \cdots, n-1$,

$$F_n \div F_i = (F_0F_1 \times \cdots \times F_{n-1} + 2) \div F_i$$

leaves the remainder of 2. i. e. $F_n = qF_i + 2$.

Thus if $m \mid F_n$, then $m \mid 2$, and so $m = 1$ or $m = 2$. Since F_n and F_i are odd, it follows that $m = 1$.

Corollary 1. *There are infinitely many primes.*

Proof. It follows immediately by the following statements.

1. $\{F_n \mid n \geq 0\}$ is an infinite set.
2. F_n has a prime factor of p_n.
3. $(F_m, F_n) = 1$ if $m \neq n$.

Remark 8. 1. Fermat conjectured all Fermat numbers are primes, but it's not true:

$$F_5 = 4294967297 = 641 \times 6700417.$$

2. Open questions remains:
   (a) Are there infinitely many Fermat primes?
   (b) Are there infinitely many composite Fermat numbers?
   (c) Is it true that F_n is composite for all $n > 4$?

Theorem 15 (Prime Number Theorem). *If*

$$\pi(x) := (\textit{number of primes less than or equal to } x)$$

Then

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1.$$

e. g. $\pi(10) = 4$.

It was conjectured by Gauss and Legendre; proved by Hadamad and Poisson independently using complex analysis.

Theorem 16. *If n is a positive composite integer, then n has a prime factor not exceeding $\sqrt{n}$.*

i. e. $\exists$ prime factor p such that $p \mid n$ and $p \leq \sqrt{n}$.

Corollary 2. *If n has no prime factors not exceeding $\sqrt{n}$, then n is prime.*

Proof (by the contrapositive of the theorem above). (Proof left for students.)

Theorem 17 (Fundamental Theorem of Arithmetic). *Let $n > 1$ be an integer. Then n can be expressed as a product of prime factors in an unique way, except for the order of factors. i. e. $\mathbb{Z}$ is an unique factorization domain*[9].

Proof. (Using WOP; see the book.)

2 Congrugences

2.1 October 1st

Definition 8. $m \in \mathbb{Z}^+$, $a, b \in \mathbb{Z}$

a is ***congrugent*** *to b modulo m if $m \mid (a-b)$.*

Theorem 18. *1.* $a \equiv a \pmod{m}$

2. $a \equiv b \Rightarrow b \equiv a$

3. $a \equiv b, b \equiv c \Rightarrow a \equiv c$

4. $a \equiv b, c \equiv d \Rightarrow a \pm c \equiv b \pm d, ac \equiv bd.$

$4\frac{1}{2}$. $1 \le i \le n$. *Then* $a_i \equiv b_i \Rightarrow \sum_1^n a_i \equiv \sum_1^n b_i, \prod_1^n a_i \equiv \prod_1^n b_i$

5. *Let* $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_n x^n$, *where* $a_i, b_i \in \mathbb{Z}$. *Suppose* $a_i \equiv b_i \pmod{m}$. *If* $a \equiv b$, *then* $f(a) \equiv g(b)$.

Example 1. $10 \equiv 1 \pmod 3$.

$10 \equiv 1 \pmod 9$.

$10 \equiv -1 \pmod{11}$.

Let $a = a_n \cdot 10^n + \cdots + a_1 \cdot 10 + a_0$. Then

$$\begin{aligned} a &\equiv a_0 + a_1 + \cdots + a_n \pmod 3 \\ &\equiv a_0 + a_1 + \cdots + a_n \pmod 9 \\ &\equiv a_0 - a_1 + \cdots + (-1)^n a_n \pmod{11} \end{aligned}$$

$\because$ If $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, then

$$\begin{aligned} f(10) &\equiv f(1) \pmod 3 \\ f(10) &\equiv f(1) \pmod 9 \\ f(10) &\equiv f(-1) \pmod{11} \end{aligned}$$

e. g.

$$\begin{aligned} 26384 &\equiv 2+6+3+8+4 \equiv 2 \pmod 3 \\ 26384 &\equiv 2+6+3+8+4 \equiv 5 \pmod 9 \\ 26384 &\equiv 2-6+3-8+4 \equiv 6 \pmod{11} \end{aligned}$$

Example 2. $41 \mid \left(2^{20} - 1\right)$?

Note that

$$2^5 \equiv -9 \pmod{41}.$$

Thus

$$\begin{aligned} \left(2^5\right)^4 &\equiv (-9)^4 \\ &\equiv 81 \times 81 \end{aligned}$$

Since $81 \equiv -1 \pmod{41}$, $81 \times 81 \equiv 1 \pmod{41}$. Hence

$$\begin{aligned} 2^{20} - 1 &\equiv \left(2^5 - 4\right) - 1 \\ &\equiv (-9)^4 - 1 \\ &\equiv 1 - 1 \equiv 0 \pmod{41}. \end{aligned}$$

Note that $7 \times 2 \equiv 4 \times 2 \pmod{6}$, but $7 \not\equiv 4 \pmod{6}$, also $7 \equiv 4 \pmod{3}$.

Theorem 19. *$a, b, c \in \mathbb{Z}$, $m \in \mathbb{Z}^+$, $d = (c, m)$.*

If $ac \equiv bc \pmod{m}$, then $a \equiv b \pmod{\frac{m}{d}}$.

Proof. Since $ac \equiv bc \pmod{m}$,

$$m \mid (ac - bc).$$

Thus $\exists k \in \mathbb{Z}$ such that $c(a-b) = km$, and so

$$\frac{c}{d}(a-b) = k\frac{m}{d}.$$

Since $\left(\frac{c}{d}, \frac{m}{d}\right) = 1$, it follows that

$$\frac{m}{d} \mid (a-b).$$

□

Question. $2^{1137} \equiv ? \pmod{17}$

Theorem 20. *Let $m \in \mathbb{Z}^+$. For any $a \in \mathbb{Z}$, $\exists! r \in \mathbb{Z}$ such that*

$$a \equiv r \pmod{m}$$

where $0 \leq r \leq m-1$.

Proof. Use the division algorithm.

Definition 9. *A* ***complete system of residues*** *modulo m is the set of integers such that every integers is congrugent modulo m to exactly one integer of the set.*

e. g.

1. $\{0, 1, 2, \cdots, m-1\}$ is a complete system of residues modulo m.[10]
2. If m is odd, $\left\{-\frac{m-1}{2}, -\frac{m-3}{2}, \cdots, -1, 0, 1, \cdots, \frac{m-3}{2}, \frac{m-1}{2}\right\}$ is also a complete system of residues modulo m.

Theorem 21. *If $\{r_1, r_2, \cdots, r_m\}$ is a complete system of residues modulo m and if $a \in \mathbb{Z}^+$ with $\underline{(a, m) = 1}$, then for any integer b,*

$$\{ar_1 + b, ar_2 + b, \cdots, ar_m + b\}$$

is a complete system of residues modulo m.

e. g. $m = 4 \Rightarrow \{0, 1, 2, 3\}, \{0, 3, 6, 9\}, \{1, 2, 3, 4\}, \cdots$

but $\{0, 2, 4, 6\}$ is not a complete system of residues modulo 4.

Proof. Note that a set of m incongrugent integers modulo m will always form a complete system of residues modulo m.

Thus it suffices to show that no two integers $ar_1 + b, \cdots, ar_m + b$ are congrugent modulo m.

Suppose that

$$ar_j + b \equiv ar_k + b.$$

[10] The least nonnegative residues modulo m

then

$$ar_j \equiv ar_k.$$

Since $(a, m) = 1$, $r_j \equiv r_k$. Hence $j = k$. $\square$

Theorem 22. *$a, b \in \mathbb{Z}^+$, $m \in \mathbb{Z}^+$, $d = (a, m)$.*

If $d \nmid b$, then $ax \equiv b \pmod{m}$ has no solutions.

If $d \mid b$, then $ax \equiv b \pmod{m}$ has exactly d incongrugent solutions modulo m as follows:

$$x = x_0 + \frac{m}{d}t \qquad t = 0, 1, 2, \cdots, d-1$$

where x_0 is a particular solution of $ax \equiv b \pmod{m}$.

Example 3. $9x \equiv 12 \pmod{15}$?

Note that $(9, 15) = 3 \mid 12$, by theorem, $\exists$ exactly 3 incongrugent solutions modulo 15.

To find a particular solution, consider $9x + 15y = 12$. Note that

$$\begin{aligned} 15 &= 9 \times 1 + 6 \\ 9 &= 6 \times 1 + 3 \\ 6 &= 3 \times 2 + 0 \\ 3 &= 9 - 6 = 9 \times 2 - 15. \end{aligned}$$

Thus $9 \times 8 + 15 \times (-4) = 12$.

Hence the general solution is given by

$$\begin{aligned} x &= x_0 \equiv 8 \pmod{15} \\ x &= x_0 + \frac{15}{3} \times 1 \equiv 13 \pmod{15} \\ x &= x_0 + \frac{15}{3} \times 2 = 18 \equiv 3 \pmod{15}. \end{aligned}$$

Proof. (Proof left for homework – due October 3rd.)

Remark 9. Consider $ax \equiv 1 \pmod{m}$. By the previous theorem, $\exists$ solutions of this congrugence if and only if $(a, m) = 1$.

Definition 10. $a \in \mathbb{Z}$, $m \in \mathbb{Z}^+$, $(a, m) = 1$.

*A solution of $ax \equiv 1 \pmod{m}$ is called an **inverse** of a modulo m.*

e. g. $7x \equiv 1 \pmod{31} \Rightarrow x = 9 \pmod{31}$. Thus 9 and all integers congrugent to 9 are inverses of 7 modulo 31.

e. g. $7x \equiv 22 \pmod{31} \Rightarrow 9 \times 7x \equiv 9 \times 22 \pmod{31} \Rightarrow 1 \times x \equiv 12 \pmod{31}$

Remark 10. $\mathbb{Z}_n^* = \{\overline{a} \in \mathbb{Z}_m \mid (a, m) = 1\}$. $(\mathbb{Z}_n^*, *)$ is a group.

$\mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$

e. g. $\mathbb{Z}_8^* = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$

2.2 October 8th

Definition 11 (Euler ϕ Function). *Let $n \in \mathbb{Z}^+$. The **Euler ϕ–function** $\phi(n)$ is defined to be the count of positive integers not exceeding n which are relatively prime to n.*

e. g. $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(8) = 4$, $\phi(12) = 4$

In general, if p is prime, then $\phi(p) = p - 1$.

Question. How to compute $\phi(n)$? Goal: $\phi(mn) = \phi(m)\phi(n)$ if $(m, n) = 1$, i. e. ϕ is multiplicative.

> **Definition 12 (Reduced Residue System).** *A* ***reduced residue system*** *modulo n is a set of $\phi(n)$ integers such that each element of the set is relatively prime to n and no two distinct elements of the set are congrugent modulo n.*

e. g. $n = 8 \Rightarrow \{1, 3, 5, 7\}$: a reduced residue system modulo 8.

Lemma 2. *If $\left\{r_1, r_2, \cdots, r_{\phi(n)}\right\}$ is a reduced residue system modulo n and if $a \in \mathbb{Z}^+$ with $(a, n) = 1$ then $\left\{ar_1, ar_2, \cdots, ar_{\phi(n)}\right\}$ is also a reduced residue system modulo n.*

Only multiplication holds; addition does not hold.

Proof. (See the textbook.)

Theorem 23 (Euler's Theorem). *If $m \in \mathbb{Z}^+$ and $a \in \mathbb{Z}$ with $(a, m) = 1$ then*

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

Proof. Let $\left\{r_1, r_2, \cdots, r_{\phi(m)}\right\}$ be a reduced residue system modulo m. Since $(a, m) = 1$, the set $\left\{ar_1, ar_2, \cdots, ar_{\phi(m)}\right\}$ is a reduced residue system modulo m by Lemma.

Then

$$ar_1 \times ar_2 \times \cdots \times ar_{\phi(m)} \equiv r_1 \times r_2 \times \cdots \times r_{\phi(m)} \pmod{m}$$

and so

$$a^{\phi(m)} \times r_1 \times r_2 \times \cdots \times r_{\phi(m)} = r_1 \times r_2 \times \cdots \times r_{\phi(m)} \pmod{m}.$$

Hence $a^{\phi(m)} \equiv 1 \pmod{m}$.[11] $\square$

Corollary 3 (Fermat's Little Theorem). *If p is prime and $p \nmid a$ ($\Rightarrow (a, p) = 1$), then*

$$a^{p-1} \equiv 1 \pmod{p}.$$

[11] Note that $\left(r_1 r_2 \times \cdots \times r_{\phi(m)}, m\right) = 1$

Corollary 4. *Let p : prime. Then*

$$a^p \equiv a \pmod{p}.$$

Proof. If $a \equiv 0 \pmod{p}$, then $a^p \equiv 0 \equiv a \pmod{p}$.

If $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$ thus $a^{p-1} \equiv a \pmod{p}$. □

Example 4. $2^{1137} \pmod{17}$?

By Euler's theorem, $2^{16} \equiv 1 \pmod{17}$. Thus

$$2^{1137} = \left(2^{16}\right)^{71} \cdot 2 \equiv 1 \cdot 2 \equiv 2 \pmod{17}.$$

Example 5. Show that 117 is not a prime.

Suppose 117 is prime. then

$$2^{117} \equiv 2 \pmod{117}.$$

Note that

$$2^7 \equiv 128 \equiv 11 \pmod{117}.$$

Thus

$$\begin{aligned} 2^{117} &\equiv \left(2^7\right)^{16} \cdot 2^5 \\ &\equiv 11^{16} \cdot 2^5 \\ &\equiv 121^8 \cdot 2^5 \\ &\equiv 4^8 \cdot 2^5 \\ &\equiv 2^{21} \equiv 11^3 \not\equiv 2 \pmod{17}. \end{aligned}$$

Example 6. Solve $x^{35} + 5x^{19} + 11x^3 \equiv 0 \pmod{17}$.

By Fermat's little theorem,

$$x^{17} \equiv x \pmod{17}.$$

Then

$$x^{35} = x\left(x^{17}\right)^2 \equiv x^3$$
$$x^{19} = x^2\left(x^{17}\right) \equiv x^3$$

Thus

$$x^{35} + 5x^{19} + 11x^3 \equiv (1+5+11)x^3 \equiv 0 \cdot x^3 \equiv 0 \pmod{17}.$$

Hence x can be any integer.

Theorem 24 (Wilson's Theorem). *If p is a prime, then*

$$(p-1)! \equiv -1 \pmod{p}.$$

Was conjectured by Wilson; and proved by Lagrange.

Lemma 3. *Let p be prime. a is self-invertible modulo p, i. e. $a \cdot a \equiv 1 \pmod{p}$, if and only if $a \equiv \pm 1 \pmod{p}$.*

Proof (of lemma). ($\Leftarrow$) It's trivial.

($\Rightarrow$) Note that

$$a^2 \equiv 1 \pmod{p}$$

and so $p \mid (a-1)(a+1)$.

Since p is prime, $p \mid (a-1)$ or $p \mid (a+1)$. Thus $a \equiv 1$ or $a \equiv -1 \pmod{p}$. $\square$

Proof (of theorem). If $p = 2$, then $(p-1)! = 1 \equiv -1 \pmod{2}$.

Consider for $p > 2$. Note that $\{1, 2, \cdots, p-1\}$ is a reduced residue system modulo p. By lemma, 1 and $p-1$ are self-invertible. Thus we can group the remaining $p-3$ residues $\frac{p-3}{2}$ pair of inverses a and b such that $ab \equiv 1 \pmod{p}$.

Hence

$$\begin{aligned}(p-1)! &= 1\cdot[2\cdot 3\times\cdots\times(p-2)]\,(p-1)\\ &\equiv 1\cdot 1\times\cdots\times 1\,(p-1)\\ &\equiv p-1\equiv -1 \pmod{p}.\end{aligned}$$

□

e. g. $(6-1)!+1=121\not\equiv 0 \pmod{6}$, thus 6 is not prime.

In fact, the converse of Wilson's theorem also holds, but is inefficient to test primality.

Theorem 25. *If $n\in\mathbb{Z}^+$ and*

$$(n-1)!\equiv -1 \pmod{m},$$

then n is prime.

Proof. Suppose that n is composite. Then $n=ab$ where $1<a<n$ and $1<b<n$. Since $a<n$, $a \mid (n-1)!$. Since $(n-1)\equiv -1 \pmod{n}$,

$$n \mid [(n-1)!+1].$$

Thus $a \mid [(n-1)!+1]$, hence $a \mid 1$, which is a contradiction.

Remark 11. p is prime if and only if $(p-1)!\equiv -1 \pmod{p}$, and also $(p-2)!\equiv 1 \pmod{p}$.

Applications of Euler's and Wilson's theorem.

1. p is odd prime. Then

$$[1\cdot 3\cdot 5\times\cdots\times(p-2)]^2\equiv[2\cdot 4\cdot 6\times\cdots\times(p-1)]^2\equiv(-1)^{\frac{p+1}{2}} \pmod{p}.$$

2. p is odd prime. Then $x^2\equiv -1 \pmod{p}$ has a solution if and only if $p\equiv 1 \pmod{4}$.

2.3 October 10th

Proof. 1. As x runs through $\frac{p-1}{2}$ even integers from 2 to $p-1$, then $p-x$ runs through odd integers from $p-2$ down to 1. Then

$$(2\cdot 4\cdot 6\times\cdots\times(p-1))\equiv(-1)^{\frac{p-1}{2}}(1\cdot 3\cdot 5\times\cdots\times(p-2))\pmod p$$

and so

$$(2\cdot 4\cdot 6\times\cdots\times(p-1))^2\equiv(1\cdot 3\cdot 5\times\cdots\times(p-2))^2\pmod p.$$

By Wilson's theorem,

$$-1\equiv(p-1)!=(1\cdot 3\cdot 5\times\cdots\times(p-2))(2\cdot 4\cdot 6\times\cdots\times(p-1))\pmod p.$$

Thus

$$(-1)^{\frac{p-1}{2}}(1\cdot 3\cdot 5\times\cdots\times(p-2))^2\equiv -1\pmod p,$$

hence

$$(1\cdot 3\cdot 5\times\cdots\times(p-2))^2\equiv(-1)^{\frac{p+1}{2}}\pmod p.$$

□

2. ($\Rightarrow$) Suppose $x_0^2\equiv -1\pmod p$ for some $x\in\mathbb{Z}$. Then

$$x_0^{p-1}=\left(x_0^2\right)^{\frac{p-1}{2}}\equiv(-1)^{\frac{p-1}{2}}.$$

On the other hand, by Euler's theorem, $x_0^{p-1}\equiv 1\pmod p$. [12] Thus $(-1)^{\frac{p-1}{2}}\equiv 1\pmod p$; i. e.

$$p\mid\left[1-(-1)^{\frac{p-1}{2}}\right].$$

Hence $1-(-1)^{\frac{p-1}{2}}=0$. [13] Therefore $\frac{p-1}{2}$ is even and so $p\equiv 1\pmod 4$.

($\Leftarrow$) Note that

$$\begin{aligned}(p-1)! &= \left(1\cdot 2\cdot 3\times\cdots\times\frac{p-1}{2}\right)\left((p-1)(p-2)(p-3)\times\cdots\times\frac{p+1}{2}\right)\\ &\equiv \left(1\cdot 2\cdot 3\times\cdots\times\frac{p-1}{2}\right)\left((-1)(-2)(-3)\times\cdots\times\frac{-(p-1)}{2}\right)\pmod p\\ &\equiv (-1)^{\frac{p-1}{2}}\times 1^2\cdot 2^2\cdot 3^2\times\cdots\times\left(\frac{p+1}{2}\right)^2\pmod p\end{aligned}$$

---

[12] Note that $x_0^2\equiv -1\pmod p$, thus $(x_0,p)\mid 1$, and so $(x_0,p)=1$; i. e. $p\nmid x_0$.

[13] If $1-(-1)^{\frac{p-1}{2}}=2\neq 0$, then $p\mid 2$. $\rightarrow\leftarrow$

Thus

$$-1 \equiv (p-1)! \equiv \left(1 \cdot 2 \cdot 3 \times \cdots \times \frac{p-1}{2}\right)^2 \pmod{p}.$$

Put $x_0 = 1 \cdot 2 \cdot 3 \times \cdots \times \frac{p-1}{2}$. Then $x_0^2 \equiv -1 \pmod{p}$. □

Theorem 26. *Let p be a prime number and $e \in \mathbb{Z}^+$. Then*

$$\phi\left(p^e\right) = p^e - p^{e-1}.$$

Proof. Note that

$$\begin{aligned}\phi\left(p^2\right) &= (\text{the number of positive integers} \le p^e \text{ which are relatively prime to } p^e)\\ &= p^e - (\text{the number of positive integers} \le p^e \text{ which are NOT relatively prime to } p^e)\end{aligned}$$

while the positive integers $\le p^e$ which are NOT relatively prime to p^e are

$$p, 2p, 3p, \cdots, \left(p^{e-1}\right)p.$$

□

Remark 12. 1. $\phi\left(p^e\right) = p^e - p^{e-1} = p^e\left(1 - \frac{1}{p}\right)$.
2. Let $n = p_1^{e_1} p_2^{e_2} \times \cdots \times p_k^{e_k}$. Then

$$\begin{aligned}\phi(n) &= \phi\left(p_1^{e_1}\right)\phi\left(p_2^{e_2}\right) \times \cdots \times \phi\left(p_k^{e_k}\right)\\ &= p_1^{e_1}\left(1 - \frac{1}{p_1}\right) p_2^{e_2}\left(1 - \frac{1}{p_2}\right) \times \cdots \times p_k^{e_k}\left(1 - \frac{1}{p_k}\right)\\ &= \left[p_1^{e_1} p_2^{e_2} \times \cdots \times p_k^{e_k}\right] \times \left[\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \times \cdots \times \left(1 - \frac{1}{p_k}\right)\right]\\ &= n\left[\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \times \cdots \times \left(1 - \frac{1}{p_k}\right)\right].\end{aligned}$$

Note 1. $m = 4, n = 7$ ($(m,n) = 1$)

$$\phi(mn) = \phi(28) = 12 = 2 \times 6 = \phi(4)\,\phi(7).$$

Lemma 4. *If* $m, n \in \mathbb{Z}^+$, $r \in \mathbb{Z}$, $(m, n) = 1$, *then the integers* $r, m+r, 2m+r, \cdots, (m-1)m+r$ *are congrufent to* $0, 1, 2, \cdots, n-1$ *modulo n.*

Proof. Suffies to show that no two integers in the list are congrugent modulo n.

Suppose that $km + r \equiv lm + r \pmod{n}$ where $0 \leq k, l < n$. Then $km \equiv lm \pmod{n}$. Since $(m, n) = 1$, hence $k \equiv l \pmod{n}$. Since $0 \leq k, l < n$, $k = l$. □

Theorem 27. $\phi(mn) = \phi(m)\phi(n)$ *if* $(m, n) = 1$.

Proof. Consider

$$\begin{array}{ccccc} 1 & m+1 & 2m+1 & \cdots & (n-1)m+1 \\ 2 & m+2 & 2m+2 & \cdots & (n-1)m+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m & 2m & 3m & \cdots & nm \end{array}$$

Let $r \leq m$ be a positive integer with $(r, m) > 1$. Let $d = (r, m)$. Then $d \mid r$, $d \mid m$, and so $d \mid (km + r)$ for any $k \in \mathbb{Z}$; i. e. d is a factor of every element in the r^{th} row.

Thus no element in the r^{th} row is relatively prime to m and hence to mn if $(r, m) > 1$. Hence, there are $\phi(m)$ rows satisfying $(r, m) = 1$.

Consider now the r^{th} row where $(r, m) = 1$.

$$r, m+r, 2m+r, \cdots, (n-1)m+r$$

By Lemma, exactly $\phi(n)$ elements in the r^{th} row are relatively prime to n, and hence to mn. Hence we conclude that $\phi(mn) = \phi(m)\phi(n)$ if $(m, n) = 1$. □

Note 2. $n = 28$, $d = n$. $C_d :=$ (the class of positive integers $m \leq n$ satisfying $(m, n) = d$). Then

$$C_1 = \{1, 3, 5, 9, 11, 13, 15, 17, 19, 23, 25, 27\} \qquad 12 = \phi(28) = \phi\left(\frac{28}{1}\right)$$

$$C_2 = \{2, 6, 10, 18, 22, 26\} \qquad 6 = \phi(14) = \phi\left(\frac{28}{2}\right)$$

$$C_4 = \{4, 8, 12, 20, 24, 28\} \qquad 6 = \phi(7) = \phi\left(\frac{28}{4}\right)$$

$$C_7 = \{7, 21\} \qquad 2 = \phi(4) = \phi\left(\frac{28}{7}\right)$$

$$C_{14} = \{14\} \qquad 1 = \phi(2) = \phi\left(\frac{28}{14}\right)$$

$$C_{28} = \{28\} \qquad 1 = \phi(1) = \phi\left(\frac{28}{28}\right)$$

$$12 + 6 + 6 + 2 + 1 + 1 = 28.$$

Theorem 28. *For $n \in \mathbb{Z}^+$,*

$$n = \sum_{d \mid n} \phi(d) = \sum_{d \mid n} \phi\left(\frac{n}{d}\right).$$

Proof. Let $m \in \mathbb{Z}^+$ such that $m \leq n$. Then $m \in C_d$ if and only if $(m, n) = d$, if and only if $\left(\frac{m}{d}, \frac{n}{d}\right) = 1$.

Thus the number of positive integers $\leq \frac{n}{d}$ which are relatively prime to $\frac{n}{d}$ is equlal to the number og elements m in C_d. Hence each class C_d has $\phi\left(\frac{n}{d}\right)$ elements.

Since there is a class corresponding to elery factor d of n and every integer $m \leq n$ belongs to exactly one class, it follows that the sum of the count of elements in various classes is n; i. e. $\sum_{d \mid n} \phi\left(\frac{n}{d}\right) = n$.

As d runs over the divisors of n, so does $\frac{n}{d}$. Hence $\sum_{d \mid n} \phi(d) = n$. $\square$

Theorem 29 (Chinese Remainder Theorem). *Let $m_1, m_2, \cdots, m_r$ are pairwise relatively prime positive integers. Then the system of congrugences*

$$\begin{cases} x \equiv a_1 & \pmod{m_1} \\ x \equiv a_2 & \pmod{m_2} \\ & \vdots \\ x \equiv a_r & \pmod{m_r} \end{cases}$$

where $a_i \in \mathbb{Z}$, has a unique solution modulo $M = m_1 m_2 \times \cdots \times m_r$.

Proof. (Proof left for homework – due October 15th.)

Example 7. 1.

$$\begin{cases} x \equiv 1 & \pmod{4} \\ x \equiv 3 & \pmod{5} \\ x \equiv 2 & \pmod{7} \end{cases}$$

$$\begin{aligned} 35 \times \underline{?}_3 &\equiv 1 && \pmod{4} \\ 28 \times \underline{?}_2 &\equiv 3 && \pmod{5} \\ 20 \times \underline{?}_6 &\equiv 2 && \pmod{7} \end{aligned}$$

Note that

$$\begin{aligned} M = 4 \cdot 5 \cdot 7 &= 35 \cdot 4 = M_1 m_1 \\ &= 28 \cdot 5 = M_2 m_2 \\ &= 20 \cdot 7 = M_3 m_3 \end{aligned}$$

thus $x = 1 \cdot 35 \cdot 3 + 3 \cdot 28 \cdot 2 + 2 \cdot 20 \cdot 6 = 93 \pmod{140}$

2.

$$\begin{cases} 8x \equiv 4 & \pmod{14} \\ 5x \equiv 3 & \pmod{11} \end{cases}$$

$$\Leftrightarrow \begin{cases} 4x \equiv 2 & \pmod{7} \\ 5x \equiv 3 & \pmod{11} \end{cases}$$

$$\Leftrightarrow \begin{cases} x \equiv 4 & \pmod{7} \\ x \equiv 5 & \pmod{11} \end{cases}$$

By CRT, $x = 4 \cdot 11 \cdot 2 + 5 \cdot 7 \cdot 8 \equiv 368 \equiv 60 \pmod{77}$

Note that $x \equiv 60 \pmod{77} \Leftrightarrow x \equiv 60, x \equiv 137 \pmod{154}$.

3 Primitive Roots

3.1 October 15th

Recall By Euler, $(a, m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$. Thus $\exists$ at least one positive integer x such that $a^x \equiv 1 \pmod{m}$. By WOP, $\exists$ a least posivite integer x satisfying $a^x \equiv 1 \pmod{m}$.

> **Definition 13.** $a, m \in \mathbb{Z}^+$, $(a, m) = 1$. *The least positive integer x such that $a^x \equiv 1 \pmod{m}$ is called the* ***order*** *of a modulo m.*
>
> *We denote this as* $\mathrm{order}_m a$*, or* $\mathrm{ord}_m a$.

e. g. $\mathrm{ord}_7 2 = 3$, $\mathrm{ord}_7 3 = 6$

Remark 13. 1. $a \equiv b \pmod{m}$, then $\mathrm{ord}_m a = \mathrm{ord}_m b$. ($\because b^{\mathrm{ord}_m a} \equiv a^{\mathrm{ord}_m a} \equiv 1 \Rightarrow \mathrm{ord}_m b \leq \mathrm{ord}_m a$)

2. Suppose $(a, m) \neq 1$. Then $a^x \equiv 1 \pmod{m}$ has no solution. Thus $a^k \not\equiv 1 \pmod{m}$ $\forall k \in \mathbb{Z}^+$.

Theorem 30. $(a, m) = 1$. *A positive integer x is a solution of $a^x \equiv 1 \pmod{m}$ if and only if* $\operatorname{ord}_m a \mid x$.

Proof. ($\Rightarrow$) By division algorithm,

$$x = q \operatorname{ord}_m a + r \qquad 0 \le r < \operatorname{ord}_m a.$$

Then

$$\begin{aligned} a^x &= a^{q \operatorname{ord}_m a + r} \\ &= \left(a^{\operatorname{ord}_m a}\right)^q a^r \\ &\equiv a^r \pmod{m}. \end{aligned}$$

Since $a^x \equiv 1$, $a^r \equiv 1$. Since $0 \le r < \operatorname{ord}_m a$, it follows that $r = 0$. Hence $a = q \operatorname{ord}_m a$ and so $\operatorname{ord}_m a \mid x$.

($\Leftarrow$) Since $\operatorname{ord}_m a \mid x$, $x = k \operatorname{ord}_m a$ for some $k \in \mathbb{Z}^+$. Then $x^a \equiv x^{k \operatorname{ord}_m a} \equiv \left(a^{\operatorname{ord}_m a}\right)^k \equiv 1^k \equiv 1 \pmod{m}$.

Corollary 5.

$$\begin{aligned} &(a, m) = 1 \\ &\Rightarrow \operatorname{ord}_m a \mid \phi(m). \end{aligned}$$

e. g. $\operatorname{ord}_{17} 5 = 16$, $\phi(17) = 16$.

Recall $m = 7$, then $\operatorname{ord}_7 2 = 3$, $\operatorname{ord}_7 3 = 6$.

$m = 12$, then $\phi(12) = 4$: so there is no positive integer a such that $\operatorname{ord}_m a = 4$.

Definition 14 (Primitive root). $r, m \in \mathbb{Z}^+$ *and* $(r, m) = 1$. *If* $\operatorname{ord}_m r = \phi(m)$, *then r is called a* ***primitive root*** *modulo m.*

e. g.

1. 3 is a primitive root modulo 7.
2. There are no primitive roots modulo 12.

Theorem 31. $(r, m) \in \mathbb{Z}^+$, $(r, m) = 1$. *If r is a primitive root modulo m, then the integers $r, r^2, \cdots, r^{\phi(m)}$ form a reduced residue system modulo m.*

e. g. 2 is a primitive root modulo 9; $\phi(9) = 6$.

$$
\begin{aligned}
2 &\equiv 2 \\
2^2 &\equiv 4 \\
2^3 &\equiv 8 \\
2^4 &\equiv 7 \\
2^5 &\equiv 5 \\
2^6 &\equiv 1
\end{aligned}
$$

Proof. Suffices to show that the first $\phi(m)$ powers of r are all relatively prime to m and that no two are congrugent modulo m.

Since $(r, m) = 1$, $\left(r^k, m\right) = 1$ for any $k \in \mathbb{Z}^+$. Thus $r, r^2, \cdots, r^{\phi(m)}$ are all relatively prime to m.

Assume that $r^i \equiv r^j \pmod{m}$. Since $1 \leq i, j \leq \phi(m)$, we have $i = j$, since $i \equiv j \pmod{\phi(m)}$ by the next theorem.

Theorem 32. $a, m \in \mathbb{Z}^+$, $(a, m) = 1$. $a^i \equiv a^j \pmod{m}$ *if and only if* $i \equiv j \pmod{\mathrm{ord}_m a}$ *where* $i, j \in \mathbb{Z}^+ \cup \{0\}$.

Proof. ($\Rightarrow$) Suppose $a^i \equiv a^j \pmod{m}$ where $i \geq j$. Since $(a, m) = 1$, $\left(a^j, m\right) = 1$. Then

$$a^j a^{i-j} \equiv a^i \equiv a^j \pmod{m}.$$

Since $\left(a^j, m\right) = 1$, $a^{i-j} \equiv 1 \pmod{m}$. Thus $\operatorname{ord}_m a \mid (i-j)$, therefore $i \equiv j \pmod{\operatorname{ord}_m a}$.

($\Leftarrow$) Proof left for students.

Theorem 33. $r, m \in \mathbb{Z}^+$, $(r, m) = 1$. *Suppose r is a primitive root modulo m. Then* r^n *is also a primitive root modulo m if and only if* $(n, \phi(m)) = 1$.

Corollary 6. *If a positive integer m has a primitive root, then it has a total of* $\underline{\phi(\phi(m))}$ *incongrugent primitive roots.*

e. g. $m = 11$

By Corollary, 11 has $\phi(\phi(11)) = 4$ incongrugent primitive roots – of 2, 6, 7, 8.

Lemma 5. *If* $\operatorname{ord}_m a = t$, *then*

$$\operatorname{ord}_m(a^u) = \frac{\operatorname{ord}_m a}{(\operatorname{ord}_m a, u)} = \frac{t}{(t, u)}.$$

Proof (of lemma). Let $s := \operatorname{ord}_m(a^u)$ and $v := (t, u)$. Then $t = t_1 v$, $u = u_1 v$ where $(t_1, u_1) = 1$.

Note that

$$(a^u)^{t_1} \equiv (a^{uv})^{t_1} \equiv \left(a^t\right)^{u_1} \equiv 1^{u_1} \equiv 1.$$

Thus $s \mid t_1$.

On the other handm since $1 \equiv (a^u)^s = a^{us}$, we have $t \mid us$. Then $t = \underline{t_1 v} \mid us = \underline{u_1 v s}$, and so, $t_1 \mid u_1 s$.

Since $(t_1, u_1) = 1$, $t_1 \mid s$. Hence $s = t_1 = \frac{t}{v} = \frac{t}{(t,u)}$.

Proof (of theorem). By Lemma,

$$\begin{aligned}\operatorname{ord}_m\left(r^n\right) &= \frac{\operatorname{ord}_m r}{\left(\operatorname{ord}_m r, n\right)} \\ &= \frac{\phi\left(m\right)}{\left(\phi\left(m\right), n\right)}.\end{aligned}$$

End of midterm.

4 Index (or Discrete Logarithm)

4.1 October 15th

Note 3. Let r be a primitive root modulo m. Then $\left\{r, r^2, \cdots, r^{\phi(m)}\right\}$ is a reduced residue system.

Thus if a is an integer such that $(a, m) = 1$, then $\exists!$ integer x with $1 \leq x \leq \phi(m)$ such that $r^x \equiv a \pmod{m}$.

4.2 October 17th

(Class missed.)