MAT2120 Number Theory
Problems IV

Suhyun Park (20181634)

Department of Computer Science and Engineering, Sogang University

1. Let d be a positive integer. Show that the simple continued fraction of $\sqrt{d^2+1}$ is $\left[d;\overline{2d}\right]$, and find the simple continued fraction of $\sqrt{101}$.

Solution. Since $\left\lfloor\sqrt{d^2+1}\right\rfloor = d$, the first term is given by d.

Subtracting d from $\sqrt{d^2+1}$ gives

$$\begin{aligned}\sqrt{d^2+1}-d &= \frac{\left(\sqrt{d^2+1}\right)^2-d^2}{\sqrt{d^2+1}+d}\\ &= \frac{1}{\sqrt{d^2+1}+d},\end{aligned}$$

hence the second term is given by $2d$.

Repeating this process by subtracting $2d$ from $\sqrt{d^2+1}+d$ gives $\sqrt{d^2+1}-d$, which is same with above result; thus $\sqrt{d^2+1}=\left[d;\overline{2d}\right]$, and therefore $\sqrt{101}=\left[10;\overline{20}\right]$.

2. Show that the simple continued fraction of $\sqrt{d}$, where d is a positive integer, has period length 1 if and only if $d=a^2+1$, where a is a nonnegative integer.

Proof. ($\Rightarrow$) Suppose the period of the simple continued fraction of $\sqrt{d}$ is 1. Then we can see that $\sqrt{d}=\left[a;\overline{2a}\right]$.

Let $x=\left[2a;\overline{2a}\right]$. Then $\left[a;\overline{2a}\right]=[a;x]$. Note that

$$x=[2a;x]=2a+\frac{1}{x},$$

thus

$$\begin{aligned} x^2 - 2ax - 1 &= 0 \\ \therefore x &= a + \sqrt{a^2+1}. \end{aligned}$$

Hence

$$\begin{aligned} \sqrt{d} &= [a; x] \\ &= a + \frac{1}{x} \\ &= a + \frac{1}{a+\sqrt{a^2+1}} \\ &= \sqrt{a^2+1} \\ \therefore d &= a^2+1. \end{aligned}$$

($\Leftarrow$) Proved in Problem 1.

3. Find the least positive solutions in integers of $x^2 - 29y^2 = -1$.

Solution. Note that $\sqrt{29} = \left[5; \overline{2, 1, 1, 2, 10}\right]$. The convergents h_n and k_n are

n	-2	-1	0	1	2	3	4	5	$\cdots$
a_n	–	–	5	2	1	1	2	10	$\cdots$
h_n	0	1	5	11	16	27	**70**	727	$\cdots$
k_n	1	0	1	2	3	5	**13**	135	$\cdots$
$h_n^2 - 29k_n^2$	-1	1	-4	5	-5	4	$-\mathbf{1}$	4	$\cdots$

Thus the minimal solution is given by $(x, y) = (70, 13)$.

4. Show that if p is prime and $x^p + y^p = z^p$, then $p \mid (x + y - z)$.

Proof. Recall that, by Fermat, for $\forall a \in \mathbb{Z}$:

$$a^p \equiv a \pmod{p}.$$

Thus

$$\begin{aligned} x^p + y^p &\equiv z^p \pmod{p} \\ \Rightarrow x + y &\equiv z \pmod{p} \\ \therefore p &\mid (x + y - z). \end{aligned}$$

□

5. Determine all right triangles with sides of integral length whose areas equal their perimeters.

Solution. Let a and b be the lengths of the sides. We have the relation of

$$\frac{ab}{2} = a + b + \sqrt{a^2 + b^2},$$

hence

$$\begin{aligned} ab &= 2a + 2b + 2\sqrt{a^2 + b^2} \\ \Rightarrow (ab - 2a - 2b)^2 &= 4a^2 + 4b^2 \\ \Rightarrow a^2b^2 - 4a^2b - 4ab^2 + 8ab &= 0 \\ \Rightarrow ab - 4a - 4b + 8 &= 0 \qquad \because ab \neq 0 \\ \Rightarrow a &= 4 \cdot \frac{b-2}{b-4}. \end{aligned}$$

If $c := b - 4$, then

$$a = 4 + \frac{8}{c},$$

thus if a is an integer, then $c = \pm 1, \pm 2, \pm 4$ or ± 8; hence $b = 2, 3, 5, 6, 8$ or 12. Calculating for each b gives

b	2	3	5	6	8	12
a	0	−4	12	8	6	5

hence there exists only two triangles, $(3, 4, 5)$ and $(5, 12, 13)$, which its areas equal their perimeters.

6. Use the fact that 2 is not a congruent number to show that $\sqrt{2}$ is irrational.

Proof. Suppose the right triangle with sides 2, 2, and $2\sqrt{2}$. Suppose $\sqrt{2} \in \mathbb{Q}$. then $2 \cdot 2 \cdot \frac{1}{2} = 2$ is congruent, but it is not, hence $\sqrt{2} \notin \mathbb{Q}$. □

7. Show that if (x, y, z) is a Pythagorean triple, then xyz is divisible by 60.

Proof. Since (x, y, z) is a Pythagorean triple, for $r, s \in \mathbb{Z}^+$ and $r + s \equiv 1 \pmod 2$, let

$$x = r^2 - s^2 \quad y = 2rs \quad z = r^2 + s^2,$$

which gives

$$xyz = 2rs\left(r^2 - s^2\right)\left(r^2 + s^2\right).$$

To prove that $60 \mid xyz$, it suffices to prove that $3 \mid xyz$, $4 \mid xyz$ and $5 \mid xyz$.

($4 \mid xyz$) Since $r + s \equiv 1 \pmod 2$, r or s is even; therefore $4 \mid 2rs \Rightarrow 4 \mid xyz$.

($3 \mid xyz$) If $3 \mid r$, it is trivial. Otherwise if $3 \nmid r$ and $3 \nmid s$, by Euler, $r^{\phi(3)} \equiv 1 \pmod 3 \Rightarrow r^2 - 1 \equiv 0 \pmod 3$ and $s^2 - 1 \equiv 0 \pmod 3$; hence $3 \mid \left[\left(r^2 - 1\right) - \left(s^2 - 1\right)\right] = \left(r^2 - s^2\right)$.

($5 \mid xyz$) Similarity, if $5 \mid r$, it is trivial. Otherwise if $5 \nmid r$, by Euler, $5 \mid \left(r^4 - s^4\right)$. □